

MODULAR INVARIANCE AND CHARACTERISTIC NUMBERS

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ABSTRACT. We prove that a general miraculous cancellation formula, the divisibility of certain characteristic numbers and some other topological results are consequences of the modular invariance of elliptic operators on loop space.

1. MOTIVATIONS

In [AW], a gravitational anomaly cancellation formula was derived from direct computations. See also [GS] and [GSW]. This is essentially a formula relating the L -class to the \hat{A} -class and a twisted \hat{A} -class of a 12-dimensional manifold. More precisely, let M be a smooth manifold of dimension 12, then the miraculous cancellation formula is

$$L(M) = 8\hat{A}(M, T) - 32\hat{A}(M)$$

where $T = TM$ denotes the tangent bundle of M and the equality holds at the top degree of each differential form. Here recall that, if we use $\{\pm x_j\}$ to denote the formal Chern roots of $TM \otimes C$, then

$$L(M) = \prod_j \frac{x_j}{\tanh x_j/2}, \quad \hat{A}(M) = \prod_j \frac{x_j/2}{\sinh x_j/2}, \quad \text{and}$$

$$\hat{A}(M, T) = \hat{A}(M) \text{ch}T \quad \text{with} \quad \text{ch}T = \sum_j e^{x_j} + e^{-x_j}.$$

By using this formula, Zhang [Z] derived an analytical version of Ochanine's Rochlin congruence formula for 12-dimensional manifolds. He achieved this by considering the adiabatic limit of the η -invariant of a circle bundle over a characteristic submanifold of M . In [LW], it was shown that a general formula of such type implies that the generalized Rochlin invariant is a spectral invariant.

On the other hand, Hirzebruch [H] and Landweber [L] used elliptic genus to prove Ochanine's result: the signature of an $8k+4$ -dimensional compact smooth spin manifold is divisible by 16. Actually Hirzebruch derived a general formula in [H] relating the signature to the indices of

certain twisted Dirac operators of a compact smooth spin manifold of any dimension.

In this paper, using the modular invariance of certain elliptic operators on loop space or the general elliptic genera, we derive a more general miraculous cancellation formula about the characteristic classes of a smooth manifold and a real vector bundle on it. This formula includes all of the above results and are much more explicit. It is actually a formula of differential forms. Combining with the index formula of [APS] for manifold with boundary and the holonomy formula of determinant line bundle, we are able to show that several seemingly unrelated topological results (in Corollaries 1-6) are connected together by this general cancellation formula.

In the following we will state the results in Section 2 and prove them in Section 3. In Section 4 we discuss some generalizations of the cancellation formula. Also given in this section is a formula relating the generalized Rochlin invariant to the holonomies of certain determinant line bundles. Section 5 contains general discussions about other applications of modular invariance in topology and some comments on the relationship between loop space, double loop space and cohomology theory.

Our main results in this paper grew out of discussions with W. Zhang. The main idea is essentially due to Hirzebruch [H] and Landweber [L]. My sole contribution is to emphasize the role of modular invariance which reflects many beautiful properties of the topology of manifold and its loop space. Note that we have shown in [Liu1] that modular invariance also implies the rigidity of many naturally derived elliptic operators from loop space. In a joint paper with W. Zhang [LZ], combining the results of [Z] and the cancellation formula in this paper, we will derive the analytical expressions of the Finashin's invariant [F] and some other topological invariants.

2. RESULTS

Let M be a dimension $8k+4$ smooth manifold and V be a rank $2l$ real vector bundle on M . We introduce two elements in $K(M)[[q^{\frac{1}{2}}]]$ which consists of formal power series in q with coefficients in the K -group of M .

$$\begin{aligned}\Theta_1(M, V) &= \otimes_{n=1}^{\infty} S_{q^n}(TM - \dim M) \otimes_{m=1}^{\infty} \Lambda_{q^m}(V - \dim V), \\ \Theta_2(M, V) &= \otimes_{n=1}^{\infty} S_{q^n}(TM - \dim M) \otimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(V - \dim V)\end{aligned}$$

where $q = e^{2\pi i\tau}$ with $\tau \in H$, the upper half plane, is a parameter. Recall that for an indeterminant t ,

$$\Lambda_t(V) = 1 + tV + t^2\Lambda^2V + \dots, \quad S_t(V) = 1 + tV + t^2S^2V + \dots$$

are two operations in $K(M)[[t]]$. They have relations

$$S_t(V) = \frac{1}{\Lambda_{-t}(V)}, \quad \Lambda_t(V - W) = \Lambda_t(V)S_{-t}(W).$$

We can formally expand $\Theta_1(M, V)$ and $\Theta_2(M, V)$ into Fourier series in q

$$\begin{aligned} \Theta_1(M, V) &= A_0 + A_1q + \dots, \\ \Theta_2(M, V) &= B_0 + B_1q^{\frac{1}{2}} + \dots \end{aligned}$$

where the A_j 's and B_j 's are elements in $K(M)$. Let $\{\pm 2\pi iy_j\}$ be the formal Chern roots of $V \otimes C$. If V is spin and $\Delta(V)$ is the spinor bundle of V , one knows that the Chern character of $\Delta(V)$ is given by

$$\text{ch}\Delta(V) = \prod_{j=1}^l (e^{\pi iy_j} + e^{-\pi iy_j}).$$

In the following, we do not assume that V is spin, but still formally use $\text{ch}\Delta(V)$ for the short hand notation of $\prod_{j=1}^l (e^{\pi iy_j} + e^{-\pi iy_j})$ which is a well-defined cohomology class on M . Let p_1 denote the first Pontryagin class. Our main results include the following theorem:

Theorem 1. *If $p_1(V) = p_1(M)$, then*

$$\hat{A}(M)\text{ch}\Delta(V) = 2^{l+2k+1} \sum_{j=0}^k 2^{-6j} b_j$$

where the b_j 's are integral linear combinations of the $\hat{A}(M)\text{ch}B_j$'s.

Here still we only take the top degree terms of both sides. More generally we have that for any A_j as in the Fourier expansion of $\Theta_1(M, V)$, the top degree term of

$$\hat{A}(M)\text{ch}\Delta(V)\text{ch}A_j$$

is an integral linear combination of the top degree terms of the $\hat{A}(M)\text{ch}B_j$'s. All of the b_j 's can be computed explicitly, for example

$$b_0 = -\hat{A}(M), \quad b_1 = \hat{A}(M)\text{ch}V + (24(2k+1) - 2l)\hat{A}(M).$$

Take $V = TM$, then $\hat{A}(M)\text{ch}\Delta(V) = L(M)$. We get, at the top degree,

$$L(M) = 2^3 \sum_{j=0}^k 2^{6k-6j} b_j.$$

When $\dim M = 12$, one recovers the miraculous cancellation formula of [AW].

For a dimension $8k$ manifold M we have similar cancellation formula

$$L(M) = \sum_{j=0}^k 2^{6k-6j} b_j.$$

One can also express the top degree of $\hat{A}(M)$ in terms of those of the twisted L -classes.

Now assume M is compact. As an easy corollary of Theorem 1, we have the following

Corollary 1. *If M and V are spin and $p_1(V) = p_1(M)$, then*

$$\text{Ind } D \otimes \Delta(V) = 2^{l+2k+1} \cdot \sum_{j=0}^k 2^{-6j} b_j$$

where D is the Dirac operator on M .

The b_j 's are integral linear combinations of the $\text{Ind } D \otimes B_j$'s. This gives

Corollary 2. *If M and V are spin with $\dim V \geq \dim M$ and $p_1(V) = p_1(M)$, then $\text{Ind } D \otimes \Delta(V) \equiv 0 \pmod{16}$, especially $\text{sign}(M) \equiv 0 \pmod{16}$.*

When $V = TM$ Corollaries 1 and 2 were derived in [H]. See also [L].

Recall the definition of the Rochlin invariant. Given a compact smooth dimension $8k + 3$ manifold N with spin structure w , which is the spin boundary of a spin manifold M with spin structure W . The Rochlin invariant $R(N, w)$ is defined to be

$$R(N, w) = \text{sign}(M) \pmod{16}.$$

Ochanine's theorem implies that $R(N, w)$ is well-defined. Let us formally write

$$b_j = \hat{A}(M)\text{ch}\beta_j$$

with β_j the integral linear combinations of the B_j 's. Another corollary of Theorem 1 is

Corollary 3. *The Rochlin invariant of (N, w) is a spectral invariant of N and is given by the explicit formula*

$$R(N, w) \equiv -\eta(\Delta) + \sum_{j=0}^k 2^{6k-6j+2}(\eta(\beta_j) + h_{\beta_j}) \pmod{16}.$$

Here $\eta(\Delta)$ and $\eta(\beta_j)$ are the η -invariant associated to the signature operator $d_s = D \otimes \Delta(M)$ and $D \otimes \beta_j$ respectively. h_{β_j} is the complex dimension of the kernel of the Dirac operator on N twisted by the restriction of β_j to N . They depend on the geometry of (N, w) . See Section 2 for their precise definitions.

For an oriented smooth compact manifold M of dimension $8k + 4$, let F be a submanifold which is the Poincaré dual of the second Stiefel-Whitney class $w_2(M)$. Denote by $F \cdot F$ the self-intersection of F . Let D_F denote the Dirac operator on F , β_j^F be the restriction of β_j to F . In [LZ], combining the cancellation formula in Theorem 1 and the results in [Z], we have proved the following analytic version of the generalized Rochlin congruence formula of Finashin [F]:

Corollary 4.

$$\frac{\text{sign}(M) + \text{sign}(F \cdot F)}{8} \equiv \sum_{j=0}^k 2^{6k-6j-1}(\eta(\beta_j) + h_{\beta_j}) \pmod{2}.$$

Here $\eta(\beta_j)$ and h_{β_j} are the corresponding η -invariant and the dimension of the kernel of certain twisted Dirac operator on F by the restriction of β_j to F . The proof of Corollary 4 and some other more general results in this direction can be found in [LZ].

For the divisibility of the twisted signature and the relation between the generalized Rochlin invariant and the holonomies of determinant line bundles, see Section 4.

3. PROOFS

The proof of Theorem 1 needs the Jacobi theta-functions [Ch] which are

$$\begin{aligned}\theta(v, \tau) &= 2q^{\frac{1}{8}} \sin \pi v \prod_{j=1}^{\infty} (1 - q^j)(1 - e^{2\pi i v} q^j)(1 - e^{-2\pi i v} q^j), \\ \theta_1(v, \tau) &= 2q^{\frac{1}{8}} \cos \pi v \prod_{j=1}^{\infty} (1 - q^j)(1 + e^{2\pi i v} q^j)(1 + e^{-2\pi i v} q^j), \\ \theta_2(v, \tau) &= \prod_{j=1}^{\infty} (1 - q^j)(1 - e^{2\pi i v} q^{j-\frac{1}{2}})(1 - e^{-2\pi i v} q^{j-\frac{1}{2}}), \text{ and} \\ \theta_3(v, \tau) &= \prod_{j=1}^{\infty} (1 - q^j)(1 + e^{2\pi i v} q^{j-\frac{1}{2}})(1 + e^{-2\pi i v} q^{j-\frac{1}{2}}).\end{aligned}$$

These are holomorphic functions for $(v, \tau) \in C \times H$ where C is the complex plane and H is the upper half plane. Up to some complex constants they have the following remarkable transformation formulas

$$\begin{aligned}\theta\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= i\tau^{\frac{1}{2}} e^{-\frac{v^2}{\tau}} \theta(v, \tau) \quad , \quad \theta(v, \tau + 1) = \theta(v, \tau); \\ \theta_1\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \tau^{\frac{1}{2}} e^{-\frac{v^2}{\tau}} \theta_2(v, \tau) \quad , \quad \theta_1(v, \tau + 1) = \theta_1(v, \tau); \\ \theta_2\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \tau^{\frac{1}{2}} e^{-\frac{v^2}{\tau}} \theta_1(v, \tau) \quad , \quad \theta_2(v, \tau + 1) = \theta_3(v, \tau); \\ \theta_3\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \tau^{\frac{1}{2}} e^{-\frac{v^2}{\tau}} \theta_3(v, \tau) \quad , \quad \theta_3(v, \tau + 1) = \theta_2(v, \tau).\end{aligned}$$

These transformation formulas which are simple consequences of the Poisson summation formula will be the key of our argument. Let

$$\begin{aligned}\Gamma_0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \pmod{2} \right\}, \\ \Gamma^0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid b \equiv 0 \pmod{2} \right\}\end{aligned}$$

be two modular subgroups of $SL_2(Z)$. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the two generators of $SL_2(Z)$. Their actions on H are given by

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1.$$

Recall that a modular form over a modular subgroup Γ is a holomorphic function $f(\tau)$ on H which, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, satisfies the transformation formula

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau)$$

where $\chi : \Gamma \rightarrow C^*$ is a character of Γ and k is called the weight of f . We also assume f is holomorphic at $\tau = i\infty$.

Obviously, at $v = 0$, $\theta_j(0, \tau)$ are modular forms of weight $\frac{1}{2}$ and

$$\theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0}$$

is a modular form of weight $\frac{3}{2}$.

With the notations as above we then have the following

Lemma 1. *Assume $p_1(V) = p_1(M)$, then $P_1(\tau) = \hat{A}(M) ch\Delta(V) ch\Theta_1(M, V)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$; $P_2(\tau) = \hat{A}(M) ch\Theta_2(M, V)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$.*

Proof: Let $\{\pm 2\pi i y_\nu\}$ and $\{\pm 2\pi i x_j\}$ be the corresponding formal Chern roots of $V \otimes C$ and $TM \otimes C$. In terms of the theta-functions, we have

$$P_1(\tau) = 2^l \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \right) \prod_{\nu=1}^l \frac{\theta_1(y_\nu, \tau)}{\theta_1(0, \tau)},$$

$$P_2(\tau) = \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \right) \prod_{\nu=1}^l \frac{\theta_2(y_\nu, \tau)}{\theta_2(0, \tau)}.$$

Apply the transformation formulas of the theta-functions, we have

$$P_1\left(-\frac{1}{\tau}\right) = 2^l \tau^{4k+2} P_2(\tau), \quad P_1(\tau + 1) = P_1(\tau)$$

where for the first equality, we need the condition $p_1(V) = p_1(M)$.

It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST , while the generators of $\Gamma^0(2)$ are STS, T^2STS from which the lemma easily follows. ♠

Write $\theta_j = \theta_j(0, \tau)$. We introduce some explicit modular forms

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4) \quad , \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4) \quad , \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4.$$

They have the following Fourier expansions in q

$$\begin{aligned}\delta_1(\tau) &= \frac{1}{4} + 6q + \cdots, & \varepsilon_1(\tau) &= \frac{1}{16} - q + \cdots, \\ \delta_2(\tau) &= -\frac{1}{8} - 3q^{\frac{1}{2}} + \cdots, & \varepsilon_2(\tau) &= q^{\frac{1}{2}} + \cdots,\end{aligned}$$

where \cdots are the higher degree terms all of which are of integral coefficients. Let $M(\Gamma)$ denote the rings of modular forms over Γ with integral Fourier coefficients. We have

Lemma 2. δ_1, δ_2 are modular forms of weight 2 and $\varepsilon_1, \varepsilon_2$ are modular forms of weight 4, and furthermore $M(\Gamma^0(2)) = Z[8\delta_2(\tau), \varepsilon_2(\tau)]$.

The proof is quite easy. In fact the δ_2 and ε_2 generate a graded polynomial ring which has dimension $1 + [\frac{k}{2}]$ in degree $2k$. On the other hand one has a well-known upper bound for this dimension: $1 + \frac{k}{6}[SL_2(Z) : \Gamma^0(2)]$ which is $1 + \frac{k}{2}$. Also note that the leading terms of $8\delta_2, \varepsilon_2$ in the lemma have coefficients 1 which immediately gives the integrality. The modularity follows from the transformation formulas of the theta-functions and the fact that $\Gamma^0(2)$ is generated by STS and T^2STS .

Now we can prove Theorem 1. By Lemmas 1 and 2 we can write

$$P_2(\tau) = b_0(8\delta_2)^{2k+1} + b_1(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + b_k(8\delta_2)\varepsilon_2^k$$

where the b_j 's are integral linear combinations of the $\hat{A}(M)\text{ch}B_j$'s.

Apply the modular transformation $S : \tau \rightarrow -\frac{1}{\tau}$, we have

$$\begin{aligned}\delta_2\left(-\frac{1}{\tau}\right) &= \tau^2\delta_1(\tau) & , & & \varepsilon_2\left(-\frac{1}{\tau}\right) &= \tau^4\varepsilon_1(\tau), \\ P_2\left(-\frac{1}{\tau}\right) &= 2^{-l}\tau^{4k+2}P_1(\tau).\end{aligned}$$

Therefore

$$P_1(\tau) = 2^l[b_0(8\delta_1)^{2k+1} + b_1(8\delta_1)^{2k-1}\varepsilon_1 + \cdots + b_k(8\delta_1)\varepsilon_1^k].$$

At $q = 0$, $8\delta_1 = 2$ and $\varepsilon_1 = 2^{-4}$. One gets the result by a simple manipulation. The cancellation formula for dimension $8k$ case can be proved in the same way. ♠

One can certainly get more divisibility results of characteristic numbers from the above formulas.

Corollary 1 is an easy consequence of Theorem 1. Since in the case that M and V are spin, all of the b_j 's are integral linear combinations of the $\text{Ind}D \otimes B_j$'s. *Corollary 2* follows from Corollary 1, since in dimension $8k + 4$, each b_j is an even integer.

For Corollary 3, we first recall the definition of η -invariant. Let $(N, w) = \partial(M, W)$ be as in Section 2. Let E be a real vector bundle on

M . Consider a twisted Dirac operator $D \otimes E$ on M . With a suitable choice of metrics on M , N and E , in a neighborhood of N one can write

$$D \otimes E = \sigma\left(\frac{\partial}{\partial u} + D_N \otimes E|_N\right)$$

where D_N is the Dirac operator on N , $E|_N$ is the restriction of E to N , σ is the bundle isomorphism induced by the symbol of $D \otimes E$ and u is the parameter in the normal direction to N . Let $\{\lambda_j\}$ be the non-zero eigenvalues of $D_N \otimes E|_N$ and h_E be the complex dimension of its zero eigenvectors. Then the η -invariant associated to $D \otimes E$, which we denote by $\eta(E)$, is given by evaluating at $s = 0$ of the function

$$\eta(s) = \sum_j \text{sign} \lambda_j \cdot \lambda_j^{-s}.$$

One has the following index formula from [APS]

$$\text{Ind} D \otimes E = \int_M \hat{A}(M) \text{ch} E - \frac{\eta(E) + h_E}{2}.$$

For convenience, we will call $\eta(E)$ the η -invariant associated to $D \otimes E$. One should note that $\eta(E)$ and h_E are actually geometrical invariants of N .

For the *proof of Corollary 3*, we take $V = TM$ in Theorem 1 and apply this formula to $L(M)$ and each $b_j = \hat{A}(M) \text{ch} \beta_j$. For β_j the [APS] formula gives us

$$\text{Ind} D \otimes \beta_j = \int_M \hat{A}(M) \text{ch} \beta_j - \frac{\eta(\beta_j) + h_{\beta_j}}{2}.$$

For $L(M)$ we have

$$\text{sign}(M) = \int_M L(M) - \eta(\Delta)$$

where $\eta(\Delta)$ is the η -invariant associated to the signature operator d_s . Put these two formulas into the equality of Theorem 1 with $V = TM$, we get

$$R(N, w) \equiv -\eta(\Delta) + \sum_{j=0}^k 2^{6k-6j+2} (\eta(\beta_j) + h_{\beta_j}) \pmod{16}.$$

Here we have used the fact that $\text{Ind} D \otimes \beta_j$ is even in dimension $8k + 4$.

4. GENERALIZATIONS

Let M and V be as in Section 2. We introduce two more elements in $K(M)[[q^{\frac{1}{2}}]]$.

$$\begin{aligned}\Phi_1(M, V) &= \Theta_1(M, TM) \otimes \otimes_{n=1}^{\infty} \Lambda_{q^n}(V - \dim V), \\ \Phi_2(M, V) &= \Theta_2(M, TM) \otimes \otimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V - \dim V)\end{aligned}$$

where $\Theta_j(M, TM)$ is defined as in Section 2 with $V = TM$.

Similarly introduce two cohomology classes on M :

$$\begin{aligned}Q_1(\tau) &= L(M) \text{ch} \Delta(V) \text{ch} \Phi_1(M, V), \\ Q_2(\tau) &= \hat{A}(M) \text{ch} \Phi_2(M, V).\end{aligned}$$

Express them in terms of the theta-functions, we have

$$\begin{aligned}Q_1(\tau) &= 2^{l+4k+2} \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau) \theta_1(x_j, \tau)}{\theta(x_j, \tau) \theta_1(0, \tau)} \right) \prod_{\nu=1}^l \frac{\theta_1(y_\nu, \tau)}{\theta_1(0, \tau)}, \\ Q_2(\tau) &= \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau) \theta_2(x_j, \tau)}{\theta(x_j, \tau) \theta_2(0, \tau)} \right) \prod_{\nu=1}^l \frac{\theta_2(y_\nu, \tau)}{\theta_2(0, \tau)}.\end{aligned}$$

If $p_1(V) = 0$, then it is easy to see that the top degree terms of these two classes are modular forms over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. Similar method to the proof of Theorem 1 can be used to derive an expression of $L(M) \text{ch} \Delta(V)$ in terms of the integral linear combination of the $\hat{A}(M) \text{ch} D_j$'s where the D_j 's are the Fourier coefficients of $\Phi_2(M, V)$. More explicitly we have

$$L(M) \text{ch} \Delta(V) = 2^{l+3} \sum_{j=0}^k 2^{6k-6j} d_j$$

where each d_j is the integral linear combination of the $\hat{A}(M) \text{ch} D_j$'s. As a corollary, one gets

Corollary 5. *Let V be a rank $2l$ spin vector bundle on a spin manifold M of dimension $8k+4$, if $p_1(V) = 0$, then $\text{Ind } d_s \otimes \Delta(V) \equiv 0 \pmod{2^{l+4}}$.*

We remark that this corollary can also be derived from Corollary 2 by taking $V \oplus TM$ as the V there.

Using Corollary 3, we can relate the Rochlin invariant to the holonomy of certain determinant line bundles. This generalizes the formula in [LMW] to higher dimension. We first recall some basic notations.

Given a smooth family of $8k+2$ -dimensional compact smooth spin manifolds $\pi : Z \rightarrow N$ and a spin vector bundle E on Z , let D_x be the

Dirac operator on the fiber $M_x = \pi^{-1}(x)$. To each x we associate the one dimensional complex vector space

$$(\Lambda^{\max} \ker D_x \otimes E|_{M_x})^* \otimes \Lambda^{\max} \text{coker } D_x \otimes E|_{M_x}$$

which patches together to give a well-defined smooth complex line bundle on N , called determinant line bundle and denoted by $\det D \otimes E$. When E, N and Z are equipped with smooth metrics, then one has the following equality of differential forms [BF]

$$c_1(\det D \otimes E)_Q = \int_{M_x} \hat{A}(M_x) \text{ch} E$$

where the left hand side is the first Chern form with respect to the Quillen metric [BF].

Let M be a dimension $8k + 2$ compact smooth spin manifold, and f be a diffeomorphism of M . Denote by $(M \times S^1)_f$ the mapping torus of f . Recall that it is defined to be

$$M \times [0, 1]/(x, 0) \sim (f(x), 1).$$

Each spin structure w on M naturally induces a spin structure which we still denote by w , on the mapping torus [LMW].

Let $\gamma : S^1 \rightarrow N$ be an immersed circle in N . Pulling back from the family $\pi : Z \rightarrow N$, we get a $8k + 3$ dimensional manifold which is isomorphic to a mapping torus $(M \times S^1)_f$. Let $H(E, w)$ denote the holonomy of the line bundle $\gamma^* \det D \otimes E$ with spin structure w on M around the circle. Scaling the (induced) metric on S^1 by ε^{-2} and let ε go to zero, we have the following holonomy formula [BF]

$$H(E, w) = \lim_{\varepsilon \rightarrow 0} \exp(-\pi i(\eta(E) + h_E)).$$

Given two spin structures w_1, w_2 on M , From Corollary 3 we have

$$R((M \times S^1)_f, w_1) - R((M \times S^1)_f, w_2) = \sum_{j=0}^k 2^{6k-6j+2} \{(\eta^{w_1}(\beta_j) + h_{\beta_j}^{w_1}) - (\eta^{w_2}(\beta_j) + h_{\beta_j}^{w_2})\}$$

where the superscript w_j denotes the invariant of the corresponding spin structure. Therefore the above holonomy formula gives us the following

Corollary 6.

$$\exp\left\{-\frac{\pi i}{4}(R((M \times S^1)_f, w_1) - R((M \times S^1)_f, w_2))\right\} = \prod_{j=0}^k (H(\beta_j, w_1) H(\beta_j, w_2)^{-1})^{2^{6k-6j}}.$$

Note that in this equality, the β_j 's are considered as elements in $K((M \times S^1)_f)$ by restriction. More precisely they are the restriction to $(M \times S^1)_f$ of the corresponding β_j 's on W where $\partial W = (M \times S^1)_f$.

For any oriented compact smooth manifold M with $p_1(M) = 0$, the top degree term of

$$\hat{A}(M) \text{ch} \otimes_{n=1}^{\infty} S_{q^n}(TM - \dim M)$$

is a modular form over $SL_2(Z)$ of weight $k = \frac{1}{2} \dim M$. When M is spin with $p_1(M) = 0$, this cohomology class is the index density of the Dirac operator on the loop space LM . See [W], [Liu1] for the other aspects of this operator. From elementary theory of modular forms, we can easily get the following

Corollary 7. *If $\hat{A}(M) = 0$, $p_1(M) = 0$ and $\dim M < 24$, then the top degree term of*

$$\hat{A}(M) \text{ch} \otimes_{n=1}^{\infty} S_{q^n} TM$$

vanishes.

This corollary applies to the case that M is spin with positive scalar curvature or with S^1 -action. It is interesting to find out the geometric meaning of this corollary.

Similarly one can consider

$$\begin{aligned} \hat{A}(M) \quad \text{ch} \quad (\Delta^+(M) - \Delta^-(M)) \otimes_{n=1}^{\infty} S_{q^n}(TM - \dim M) \\ \text{ch} \quad \otimes (\Delta^+(V) - \Delta^-(V)) \otimes_{m=1}^{\infty} \Lambda_{q^m}(V - \dim V) \end{aligned}$$

which is the index density of the Euler characteristic operator on LM . Here

$$\text{ch}(\Delta^+(V) - \Delta^-(V)) = \prod_{j=1}^l (e^{\pi i y_j} - e^{-\pi i y_j})$$

and $\text{ch}(\Delta^+(M) - \Delta^-(M))$ denotes the corresponding class for TM .

If $p_1(V) = p_1(M)$, its top degree term is a modular form of weight $k - l$ over $SL_2(Z)$. Here $2l = \text{rank} V$ and $2k = \dim M$. When $l > k$ or $k = l$ and the Euler characteristic of V is zero, the top degree term of this class vanishes.

5. DISCUSSIONS

Modular invariance is one of the most fundamental principle in modern mathematical physics. In [Liu1], we proved that many elliptic operators derived from loop space have modular invariance which in turn implies their rigidity. We note that formal application of the Lefschetz fixed point formula on loop space together with modular invariance gives us a lot of information about the index theory on loop space.

Many topological results, such as rigidity and divisibility, are much easier to understand when we look at them while standing on loop space, therefore they are infinite dimensional phenomena. On the other hand many seemingly unrelated topological results discussed above are also intrinsically connected together by the modular invariance of elliptic operators derived from loop space.

Another interesting point is not directly related to modular invariance. It is only a kind of wild speculation. As observed by Witten, the inverse of the \hat{A} -genus is actually the equivariant Euler class of the normal bundle of M in its loop space LM with respect to the natural S^1 -action. Therefore up to certain normalization, the Atiyah-Singer index formula can be formally derived from the Duistermaat-Heckman localization formula on LM . See [A]. We note that, up to certain normalization, the inverse of the loop space \hat{A} -genus

$$q^{-\frac{k}{12}} \hat{A}(M) \text{ch} \otimes \otimes_{n=1}^{\infty} S_{q^n} TM$$

where $2k = \dim M$, is formally equal to the equivariant Euler class of the normal bundle of M in the double loop space LLM with respect to the natural $T = S^1 \times S^1$ -action. Up to certain normalization constant, this is a corollary of the following 'product' formula

$$\prod_{m,n} (x + m\tau + n) = q^{\frac{1}{12}} (e^{\pi i x} - e^{-\pi i x}) \prod_{n=1}^{\infty} (1 - e^{2\pi i x} q^n) (1 - e^{-2\pi i x} q^n).$$

This means that the index formula on loop space, especially elliptic genera, can also be derived from the formal application of the Duistermaat-Heckman formula to the double loop space LLM !

More precisely, one knows that the K -theory Euler class of TM is $\Delta^+(M) - \Delta^-(M)$, while under the Chern character map, we have

$$\text{ch} (\Delta^+(M) - \Delta^-(M)) = \text{the equivariant Euler class of } TLM|_M$$

where $TLM|_M$ denotes the restriction of the tangent bundle of LM to M . Similarly the equivariant K -theory Euler class of $TLM|_M$ is

$$q^{\frac{k}{12}} (\Delta^+(M) - \Delta^-(M)) \otimes_{n=1}^{\infty} \Lambda_{-q^n} TM$$

and its Chern character is

$$q^{\frac{k}{12}} (e^{\pi i x} - e^{-\pi i x}) \prod_{n=1}^{\infty} (1 - e^{2\pi i x} q^n) (1 - e^{-2\pi i x} q^n)$$

which is also the equivariant cohomology Euler class of $TLLM|_M$ with respect to the natural T -action.

Let us use \Leftrightarrow to mean 'correspondence', then modulo certain normalization, we can put the above discussions in the following way.

The equivariant cohomology Euler class of $TLM|_M \Leftrightarrow$
the K -theory Euler class of TM ;

The equivariant cohomology Euler class of $TLLM|_M \Leftrightarrow$
the equivariant K -theory Euler class of $TLM|_M$

Another interesting point is that, under the natural transformation of cohomology theory

$$\text{Ell}^*(M) \rightarrow K(M)[[q]],$$

the Euler class of elliptic cohomology is transformed to the equivariant K -theory Euler class of $TLM|_M$. We remark that the elliptic cohomology here may be slightly different from the one in [L]. Actually the elliptic cohomology in [L] is associated to the signature operator on loop space, while the one here should be associated to the Dirac operator on loop space, which conjecturally should exist.

These discussions make it reasonable to speculate that the equivariant cohomology of LM should correspond to the K -theory of M , while the equivariant cohomology of the double loop space LLM should correspond to the equivariant K -theory of LM which in turn corresponds to the elliptic cohomology of M . We can put these into the following diagram

$$\begin{aligned} H_{S^1}(LM) &\Leftrightarrow K(M) \\ H_T^*(LLM) &\Leftrightarrow K_{S^1}(LM), \\ K_{S^1}(LM) &\Leftrightarrow \text{Ell}^*(M) \Leftrightarrow H_T^*(LLM). \end{aligned}$$

This means that 'looping' M once 'lifts' the equivariant cohomology theory one order higher.

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