

RECURSION FORMULAE OF HIGHER WEIL-PETERSSON VOLUMES

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ABSTRACT. In this paper we study effective recursion formulae for computing intersection numbers of mixed ψ and κ classes on moduli spaces of curves. By using the celebrated Witten-Kontsevich theorem, we generalize Mulase-Safnuk form of Mirzakhani's recursion and prove a recursion formula of higher Weil-Petersson volumes. We also present recursion formulae to compute intersection pairings in the tautological rings of moduli spaces of curves.

1. INTRODUCTION

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n -pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point,

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π , and by D_1, \dots, D_n the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf, we have the following tautological classes on moduli spaces of curves.

$$\begin{aligned} \psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\ \kappa_i &= \pi_* \left(c_1 \left(\omega_\pi \left(\sum D_i \right) \right)^{i+1} \right) \\ \lambda_l &= c_l(\pi_*(\omega_\pi)), \quad 1 \leq l \leq g. \end{aligned}$$

The classes κ_i were first defined by Mumford [21] on $\overline{\mathcal{M}}_g$. Their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [1, 2]. Before that time, the classes κ_i were defined as $\pi_*(c_1(\omega_\pi)^{i+1})$. Arbarello-Cornalba's definition turned out to be the correct one especially from the point of view of the restrictions to the boundary strata.

We are interested in the following intersection numbers

$$\langle \kappa_{b_1} \cdots \kappa_{b_k} \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1} \cdots \kappa_{b_k} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $\sum b_j + \sum d_j = 3g - 3 + n$. When $d_1 = \cdots = d_n = 0$, these intersection numbers are called the higher Weil-Petersson volumes of moduli spaces of curves.

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [1], Faber [7] and Kaufmann-Manin-Zagier [13] into a beautiful combinatorial formalism. Faber has a wonderful maple program computing these intersection numbers.

In a series of innovative papers [18, 19], Mirzakhani obtained a beautiful recursion formula of the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces. As discussed by Mulase and Safnuk in [20, 23], Mirzakhani's recursion formula is equivalent to the following enlightening recursion relation of intersection numbers.

$$(2k_1 + 1)!! \langle \kappa_1^{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g$$

$$\begin{aligned}
&= \sum_{j=2}^n \sum_{l=0}^{k_0} \frac{k_0!}{(k_0-l)!} \frac{(2(l+k_1+k_j)-1)!!}{(2k_j-1)!!} \beta_l \langle \kappa_1^{k_0-l} \tau_{k_1+k_j+l-1} \prod_{i \neq 1, j} \tau_{k_i} \rangle_g \\
&+ \frac{1}{2} \sum_{l=0}^k \sum_{d_1+d_2=l+k_1-2} \frac{k_0!}{(k_0-l)!} (2d_1+1)!!(2d_2+1)!! \beta_l \langle \kappa_1^{k_0-l} \tau_{d_1} \tau_{d_2} \prod_{i \neq 1} \tau_{k_i} \rangle_{g-1} \\
&+ \frac{1}{2} \sum_{\substack{m_0+n_0=k_0-l \\ I \amalg J=\{2, \dots, n\}}} \sum_{l=0}^{k_0} \sum_{d_1+d_2=l+k_1-2} \frac{k_0!}{m_0!n_0!} (2d_1+1)!!(2d_2+1)!! \beta_l \\
&\quad \times \langle \kappa_1^{m_0} \tau_{d_1} \prod_{i \in I} \tau_{k_i} \rangle_{g'} \langle \kappa_1^{n_0} \tau_{d_2} \prod_{i \in J} \tau_{k_i} \rangle_{g-g'},
\end{aligned}$$

where

$$\beta_l = (2^{2l+1} - 4) \frac{\zeta(2l)}{(2\pi^2)^l} = (-1)^{l-1} 2^l (2^{2l} - 2) \frac{B_{2l}}{(2l)!}.$$

In a previous paper [16], it is shown that the Witten-Kontsevich theorem implies the Mulase-Safnuk form of Mirzakhani's recursion formula. Its relationship with matrix integrals has been studied by Eynard and Orantin [5, 6].

More discussions about computations of Weil-Petersson or higher Weil-Petersson volumes can be found in the papers [10, 12, 13, 17, 22, 24, 26, 27].

Now we fix notation as in [13]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i . Denote by δ_a the sequence with 1 at the a -th place and zeros elsewhere.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$, and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$\begin{aligned}
|\mathbf{m}| &:= \sum_{i \geq 1} im(i), \quad \|\mathbf{m}\| := \sum_{i \geq 1} m(i), \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i \geq 1} m(i)!, \\
\binom{\mathbf{m}}{\mathbf{t}} &:= \prod_{i \geq 1} \binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a}_1, \dots, \mathbf{a}_n} := \prod_{i \geq 1} \binom{m(i)}{a_1(i), \dots, a_n(i)}.
\end{aligned}$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i \geq 1} \kappa_i^{b(i)}.$$

Theorem 1.1. *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$. Then*

$$\begin{aligned}
(1) \quad &\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1+2|\mathbf{L}|+1)!!}{(2|\mathbf{L}|+1)!!} \langle \kappa(\mathbf{L}') \tau_{d_1+|\mathbf{L}|} \prod_{j=2}^n \tau_{d_j} \rangle_g \\
&= \sum_{j=2}^n \frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{r+s=|d_1|-2} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
&+ \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \amalg J=\{2, \dots, n\}}} \sum_{r+s=d_1-2} \binom{\mathbf{b}}{\mathbf{e}} (2r+1)!!(2s+1)!! \\
&\quad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.
\end{aligned}$$

Theorem 1.2. *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$. Then*

$$\begin{aligned}
 (2) \quad & (2d_1 + 1)!! \langle \kappa(\mathbf{b}) \tau_{d_1} \cdots \tau_{d_n} \rangle_g \\
 &= \sum_{j=2}^n \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{L}') \tau_{|\mathbf{L}| + d_1 + d_j - 1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &+ \frac{1}{2} \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{L}') \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \rangle_{g-1} \\
 &+ \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{e} + \mathbf{f} = \mathbf{b} \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r + 1)!! (2s + 1)!! \\
 &\qquad \qquad \qquad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
 \end{aligned}$$

where the constants $\alpha_{\mathbf{L}}$ are determined recursively from the following formula

$$\sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!} = 0, \quad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \mathbf{L}' \neq \mathbf{0}}} \frac{(-1)^{|\mathbf{L}'| - 1} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!}, \quad \mathbf{b} \neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.

Denote $\alpha(l, 0, 0, \dots)$ by α_l , we recover Mirzakhani's recursion formula with

$$\alpha_l = l! \beta_l = (-1)^{l-1} (2^{2l} - 2) \frac{B_{2l}}{(2l - 1)!!}.$$

We also have

$$\alpha(\delta_l) = \frac{1}{(2l + 1)!!}.$$

Setting $\mathbf{b} = \mathbf{0}$, we get the Witten-Kontsevich theorem [25, 14] in the form of DVV recursion relation [4].

Note that Theorems 1.1 and 1.2 hold only for $n \geq 1$. If $n = 0$, i.e. for higher Weil-Petersson volumes of $\overline{\mathcal{M}}_g$, we may apply the following formula first (see Proposition 3.1).

$$(3) \quad \langle \kappa(\mathbf{b}) \rangle_g = \frac{1}{2g - 2} \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}| + 1} \kappa(\mathbf{L}') \rangle_g.$$

So we can use Theorems 1.1 and 1.2 to compute any intersection numbers of ψ and κ classes recursively with the three initial values

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \quad \langle \tau_0^3 \rangle_0 = 1, \quad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a maple program [28] implementing Theorems 1.1 and 1.2.

In the arguments of Mirzakhani, Mulase and Safnuk, they use Wolpert's formula [26]

$$\kappa_1 = \frac{1}{2\pi^2} \omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. Since Wolpert's formula has no counterpart for higher degree κ classes, there is no a priori reason that Theorem 1.2 shall be true.

We are led to Theorem 1.2 also by the discovery that ψ and κ classes are compatible, namely recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes, where κ_1 plays no special role. This fact is equivalent to a relation of generating functions in Theorem 4.4.

For $\mathbf{b} \in N^\infty$, we denote by $V_{g,n}(\kappa(\mathbf{b}))$ the higher Weil-Petersson volume

$$\langle \tau_0^n \kappa(\mathbf{b}) \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b}).$$

Let $V_g(\kappa(\mathbf{b}))$ denote $V_{g,0}(\kappa(\mathbf{b}))$.

Higher Weil-Petersson volumes were extensively studied in the paper [13]. The authors found an explicit expression (see Lemma 2.2 below) of $V_{g,n}(\kappa(\mathbf{b}))$ in terms of integrals of ψ classes. In genus zero, they obtained more nice results about generating functions of $V_{0,n}(\kappa(\mathbf{b}))$ and raised the question whether their methods may be generalized to higher genera.

Although we feel it is difficult to generalize Kaufmann-Manin-Zagier's results to higher genera, we did find an effective recursion formula between $V_{g,n}(\kappa(\mathbf{b}))$ valid for all g and n , based on our previous work on integrals of ψ classes. The results are contained in the following two theorems.

Theorem 1.3. *Let $\mathbf{b} \in N^\infty$ and $n \geq 1$. Then*

$$(4) \quad (2g - 1 + \|\mathbf{b}\|)V_{g,n}(\kappa(\mathbf{b})) = \frac{1}{12}V_{g-1,n+3}(\kappa(\mathbf{b})) - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\| \geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,n}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L} \neq \mathbf{0}, \mathbf{L}' \neq \mathbf{0}}} \sum_{r+s=n-1} \binom{\mathbf{b}}{\mathbf{L}} \binom{n-1}{r} V_{g',r+2}(\kappa(\mathbf{L}))V_{g-g',s+2}(\kappa(\mathbf{L}')).$$

Theorem 1.3 is an effective formula for computing higher Weil-Petersson volumes recursively by induction on g and $\|\mathbf{b}\|$, with the initial values

$$V_{0,3}(1) = 1 \quad \text{and} \quad V_{0,n}(\kappa(\boldsymbol{\delta}_{n-3})) = 1, \quad n \geq 4,$$

where $\boldsymbol{\delta}_a$ denotes the sequence with 1 at the a -th place and zeros elsewhere.

Theorem 1.4. *Let $g \geq 2$ and $\mathbf{b} \in N^\infty$. Then*

$$(5) \quad ((2g-1)(2g-2) + (4g-3)\|\mathbf{b}\| + \|\mathbf{b}\|^2)V_g(\kappa(\mathbf{b})) = 5 \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,1}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|+1}) \\ - \frac{1}{6} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g-1,3}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) - \sum_{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} V_{g',1}(\kappa_{|\mathbf{L}|}\kappa(\mathbf{e}))V_{g-g',2}(\kappa(\mathbf{f})) \\ - (2g-1 + \|\mathbf{b}\|) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\| \geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_g(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\| \geq 2}} \binom{\mathbf{b}}{\mathbf{L}} \sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}} \binom{\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}{\mathbf{e}} V_g(\kappa(\mathbf{e})\kappa_{|\mathbf{f}|}).$$

By induction on $\|\mathbf{b}\|$, Theorem 1.4 reduces the computation of $V_g(\kappa(\mathbf{b}))$ to the cases of $V_{g,n}(\kappa(\mathbf{b}))$ for $n \geq 1$, which have been computed by Theorem 1.3. Therefore Theorems 1.3 and 1.4 completely determine higher Weil-Petersson volumes of moduli spaces of curves.

The virtue of the above recursions is that they do not involve ψ classes. So if one wants to compute only higher Weil-Petersson volumes, the above recursions are more efficient both in speed and memory use, especially when we use ‘‘option remember’’ in a maple program.

On the other hand, we know that intersection numbers of mixed ψ and κ classes can be expressed by intersection numbers of pure κ classes [1].

In Section 2, we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.3 and 1.4. In Section 4, we prove that the generating functions of intersection numbers involving general κ and ψ classes satisfy Virasoro constraints and the KdV hierarchy. In Section 5, we consider recursions of Hodge integrals with λ classes.

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2. PROOFS OF THEOREMS 1.1 AND 1.2

The following elementary lemma is crucial to our proof.

Lemma 2.1. *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_0 \beta_0 = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \beta_{\mathbf{L}'} = 0, \quad \mathbf{b} \neq 0.$$

Then the following two identities are equivalent.

$$\begin{aligned} G(\mathbf{b}, n) &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \\ F(\mathbf{b}, n) &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \end{aligned}$$

Proof. Assume the first identity holds, we have

$$\begin{aligned} \sum_{\mathbf{a}=0}^{\mathbf{b}} \beta_{\mathbf{a}} G(\mathbf{b} - \mathbf{a}, n + |\mathbf{a}|) &= \sum_{\mathbf{a}=0}^{\mathbf{b}} \beta_{\mathbf{a}} \sum_{\mathbf{a}'=0}^{\mathbf{b}-\mathbf{a}} \alpha_{\mathbf{a}'} F(\mathbf{b} - \mathbf{a} - \mathbf{a}', n + |\mathbf{a} + \mathbf{a}'|) \\ &= \sum_{\mathbf{L}=0}^{\mathbf{b}} \sum_{\mathbf{a}+\mathbf{a}'=\mathbf{L}} (\beta_{\mathbf{a}} \alpha_{\mathbf{a}'}) F(\mathbf{b} - \mathbf{L}, n + |\mathbf{L}|) \\ &= \sum_{\mathbf{L}=0}^{\mathbf{b}} \delta_{\mathbf{L},0} F(\mathbf{b} - \mathbf{L}, n + |\mathbf{L}|) \\ &= F(\mathbf{b}, n). \end{aligned}$$

So we have proved the second identity. The proof of the other direction is the same. \square

We also need the following combinatorial formula from [13].

Lemma 2.2. [13] *Let $\mathbf{m} \in N^\infty$.*

$$\left\langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \right\rangle_g = \sum_{k=0}^{|\mathbf{m}|} \frac{(-1)^{|\mathbf{m}|-k}}{k!} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_k \\ \mathbf{m}_i \neq 0}} \binom{\mathbf{m}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{|\mathbf{m}_j|+1} \right\rangle_g$$

$$= \sum_{k \geq 0} \sum_{\substack{\mathbf{m} = a_1 \mathbf{m}_1 + \dots + a_k \mathbf{m}_k \\ \mathbf{m}_i \neq \mathbf{m}_j, i \neq j}} \frac{(-1)^{|\mathbf{m}| - \sum_{i=1}^k a_i}}{\prod_{i=1}^k a_i!} \left(\underbrace{\mathbf{m}_1, \dots, \mathbf{m}_1}_{a_1}, \dots, \underbrace{\mathbf{m}_k, \dots, \mathbf{m}_k}_{a_k} \right) \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{|\mathbf{m}_j|+1} \right\rangle_g$$

where in the last term, these distinct $\{\mathbf{m}_1, \dots, \mathbf{m}_k\}$ are unordered in the summation and a_i are positive integers.

Proof. We only give a sketch. Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \dots \psi_{n+p}^{a_p+1})$ by $R(a_1, \dots, a_p)$, then we have the formula [1]

$$R(a_1, \dots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\text{each cycle } c \text{ of } \sigma} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group \mathbb{S}_p as a product of disjoint cycles.

By a formal combinatorial argument, we get the following inversion result

$$\kappa_{a_1} \dots \kappa_{a_p} = \sum_{k=1}^p \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1, \dots, p\} = S_1 \amalg \dots \amalg S_k \\ S_k \neq \emptyset}} R\left(\sum_{j \in S_1} a_j, \dots, \sum_{j \in S_k} a_j\right),$$

from which Lemma 2.2 follows. \square

Proof of Theorem 1.1

Let LHS and RHS denote the left and right hand side of Theorem 1.1 respectively. By Lemma 2.2 and the Witten-Kontsevich theorem, we get

$$\begin{aligned} & (2d_1 + 1)!! \left\langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \right\rangle_g \\ &= (2d_1 + 1)!! \sum_{k=0}^{|\mathbf{b}|} \frac{(-1)^{|\mathbf{b}| - k}}{k!} \sum_{\substack{\mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} \\ \mathbf{m}_i \neq \mathbf{0}}} \binom{\mathbf{b}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{|\mathbf{m}_j|+1} \right\rangle_g \\ &= \sum_{k=0}^{|\mathbf{b}|} \frac{(-1)^{|\mathbf{b}| - k}}{k!} \sum_{\substack{\mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} \\ \mathbf{m}_i \neq \mathbf{0}}} \binom{\mathbf{b}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \\ & \quad \times \left(\sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \left\langle \tau_{d_1 + d_j - 1} \prod_{i \neq 1, j} \tau_{d_i} \prod_{i=1}^k \tau_{|\mathbf{m}_i|+1} \right\rangle_g \right. \\ & \quad + \sum_{j=1}^k \frac{(2(d_1 + |\mathbf{m}_j|) + 1)!!}{(2|\mathbf{m}_j| + 1)!!} \left\langle \tau_{d_1 + |\mathbf{m}_j|} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{|\mathbf{m}_i|+1} \right\rangle_g \\ & \quad + \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!! (2s+1)!! \left\langle \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{|\mathbf{m}_i|+1} \right\rangle_{g-1} \\ & \quad + \frac{1}{2} \sum_{\substack{I \amalg I' = \{1, \dots, n\} \\ J \amalg J' = \{1, \dots, k\}}} \sum_{r+s=d_1-2} (2r+1)!! (2s+1)!! \\ & \quad \times \left\langle \tau_r \prod_{i \in I} \tau_{d_i} \prod_{i \in I'} \tau_{|\mathbf{m}_i|+1} \right\rangle_{g'} \left\langle \tau_s \prod_{i \in J} \tau_{d_i} \prod_{i \in J'} \tau_{|\mathbf{m}_i|+1} \right\rangle_{g-g'} \Big) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &\quad + \frac{1}{2} \sum_{r+s=|d_1|-2} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
 &\quad + \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \sqcup J = \{2, \dots, n\}}} \sum_{r+s=d_1-2} \binom{\mathbf{b}}{\mathbf{e}} (2r+1)!!(2s+1)!! \\
 &\quad \quad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \\
 &\quad + \sum_{k=0}^{|\mathbf{b}|} \frac{(-1)^{|\mathbf{b}|-k}}{k!} \sum_{\substack{\mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} \\ \mathbf{m}_i \neq \mathbf{0}}} \binom{\mathbf{b}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \\
 &\quad \times \sum_{j=1}^k \frac{(2(d_1 + |\mathbf{m}_j|) + 1)!!}{(2|\mathbf{m}_j| + 1)!!} \langle \tau_{d_1+|\mathbf{m}_j|} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{|\mathbf{m}_i|+1} \rangle_g \\
 &= RHS + \sum_{k \geq 0} \frac{(-1)^{|\mathbf{b}|-k-1}}{(k+1)!} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L} \neq \mathbf{0}}} \sum_{\substack{\mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} - \mathbf{L} \\ \mathbf{m}_i \neq \mathbf{0}}} \binom{\mathbf{b}}{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \\
 &\quad \times (k+1) \frac{(2(d_1 + |\mathbf{L}|) + 1)!!}{(2|\mathbf{L}| + 1)!!} \langle \tau_{d_1+|\mathbf{L}|} \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{|\mathbf{m}_i|+1} \rangle_g \\
 &= RHS - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L} \neq \mathbf{0}}} (-1)^{|\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1 + 2|\mathbf{L}| + 1)!!}{(2|\mathbf{L}| + 1)!!} \langle \kappa(\mathbf{L}') \tau_{d_1+|\mathbf{L}|} \prod_{j=2}^n \tau_{d_j} \rangle_g \\
 &= RHS - LHS + (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.
 \end{aligned}$$

In the third equation, only the quadratic term needs a careful verification. So we have proved $RHS = LHS$.

We will see that Theorem 1.2 follows from Theorem 1.1 and Lemma 2.1.

Proof of Theorem 1.2

Let

$$F(\mathbf{b}, d_1) = \frac{(2d_1 + 1)!!}{\mathbf{b}!} \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g$$

and

$$\begin{aligned}
 G(\mathbf{b}, d_1) &= \sum_{j=2}^n \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{\mathbf{b}!(2d_j - 1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &\quad + \frac{1}{2} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{\mathbf{b}!} \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \rangle_{g-1}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \amalg J=\{2,\dots,n\}}} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{\mathbf{e}!\mathbf{f}!} \\
& \qquad \qquad \qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.
\end{aligned}$$

Note that Theorem 1.1 is just

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{\|\mathbf{L}\|}}{\mathbf{L}!(2\|\mathbf{L}\|+1)!!} F(\mathbf{L}', d_1 + \|\mathbf{L}\|) = G(\mathbf{b}, d_1).$$

By Lemma 2.1, we have

$$F(\mathbf{b}, d_1) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{\alpha_{\mathbf{L}}}{\mathbf{L}!} G(\mathbf{L}', d_1 + \|\mathbf{L}\|),$$

which is just the result we want.

3. HIGHER WEIL-PETERSSON VOLUMES

By applying Lemma 2.2 as in the proof of Theorem 1.1, we may generalize recursions of pure ψ classes to recursions including both ψ and κ classes.

First we have the following generalization of the string and dilation equations.

Proposition 3.1. *For $\mathbf{b} \in N^\infty$, $n \geq 0$ and $d_j \geq 0$,*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{\|\mathbf{L}\|} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = \sum_{j=1}^n \langle \tau_{d_{j-1}} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \rangle_g,$$

and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{\|\mathbf{L}\|+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g-2+n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.$$

Proof. The first identity follows by taking $d_1 = 0$ in Theorem 1.1. For the second identity, we have

$$\begin{aligned}
& \langle \prod_{j=1}^n \tau_{d_j} \tau_1 \kappa(\mathbf{b}) \rangle_g \\
& = \sum_{k \geq 0} \sum_{\substack{\mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} \\ \mathbf{m}_i \neq 0}} \frac{(-1)^{\|\mathbf{b}\|-k}}{k!} \binom{\mathbf{b}}{\mathbf{m}_1, \dots, \mathbf{m}_k} \langle \tau_1 \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{\|\mathbf{m}_j\|+1} \rangle_g \\
& = (2g+n-2) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g \\
& + \sum_{k \geq 0} \sum_{\substack{\mathbf{L} + \mathbf{m}_1 + \dots + \mathbf{m}_k = \mathbf{b} \\ \mathbf{L} \neq 0, \mathbf{m}_i \neq 0}} \frac{(-1)^{\|\mathbf{b}\|-k-1}}{k!} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{m}_1, \dots, \mathbf{m}_k} \langle \tau_{\|\mathbf{L}\|+1} \prod_{j=1}^k \tau_{\|\mathbf{m}_j\|+1} \prod_{j=1}^n \tau_{d_j} \rangle_g.
\end{aligned}$$

Subtracting the last term from each side, we have proved the second identity. \square

For the particular case $\mathbf{b} = (m, 0, 0, \dots)$, Proposition 3.1 has been proved by Norman Do and Norbury [3] in their study of the intermediary moduli spaces consisting of hyperbolic surfaces with a cone point of a specified angle.

We need the following results from [1].

Lemma 3.2. *Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism that forgets the last marked point.*

- i) $\pi_{n*}(\psi_1^{a_1} \cdots \psi_{n-1}^{a_{n-1}} \psi_n^{a_n+1}) = \psi_1^{a_1} \cdots \psi_{n-1}^{a_{n-1}} \kappa_{a_n}$ for $a_j \geq 0$;
- ii) $\kappa_a = \pi_{n+1}^*(\kappa_a) + \psi_{n+1}^a$ on $\overline{\mathcal{M}}_{g,n+1}$;
- iii) $\kappa_0 = 2g - 2 + n$ on $\overline{\mathcal{M}}_{g,n}$.

We have the following generalization of a recursion formula from the Witten-Kontsevich theorem corresponding to the first equation in the KdV hierarchy (see Theorem 1.2 of [15]).

Proposition 3.3. *Let $\mathbf{b} \in N^\infty$ and $n \geq 0$. Then*

$$(6) \quad \langle \tau_0 \tau_1 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g = \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_{g-1} \\ + \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \underline{n} = I \amalg J}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}.$$

Now we give a proof of Theorem 1.3. Let LHS and RHS denote the left and right hand side of Proposition 3.3 respectively. Taking $d_j = 0$ and applying Lemma 3.2, we have

$$LHS = \int_{\overline{\mathcal{M}}_{g,n+1}} \pi_{n+2*} \left(\psi_{n+2} \prod_{i \geq 1} (\pi_{n+2}^* \kappa_i + \psi_{n+2}^i)^{b(i)} \right) \\ = \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^{n+1} \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \rangle_g \\ = ((2g - 1 + n) + \|\mathbf{b}\|) \langle \tau_0^{n+1} \kappa(\mathbf{b}) \rangle_g + \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \|\mathbf{L}'\| \geq 2}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^{n+1} \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \rangle_g$$

and

$$RHS = \frac{1}{12} \langle \tau_0^{n+4} \kappa(\mathbf{b}) \rangle_{g-1} + \frac{1}{2} \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \sum_{r+s=n} \binom{\mathbf{b}}{\mathbf{L}} \binom{n}{r} \langle \tau_0^{r+2} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^{s+2} \kappa(\mathbf{L}') \rangle_{g-g'} \\ = \frac{1}{12} \langle \tau_0^{n+4} \kappa(\mathbf{b}) \rangle_{g-1} + \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \mathbf{L} \neq \mathbf{0}, \mathbf{L}' \neq \mathbf{0}}} \sum_{r+s=n} \binom{\mathbf{b}}{\mathbf{L}} \binom{n}{r} \langle \tau_0^{r+2} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^{s+2} \kappa(\mathbf{L}') \rangle_{g-g'} \\ + n \langle \tau_0^{n+1} \kappa(\mathbf{b}) \rangle_g.$$

So Theorem 1.3 follows from $LHS = RHS$.

By further expanding the term $V_{g-1,n+3}(\kappa(\mathbf{b}))$ in Theorem 1.3, we get

$$V_{g,n}(\kappa(\mathbf{b})) = \delta_{\|\mathbf{b}\|,0} + \frac{1}{24g!} \delta_{\|\mathbf{b}\|,1} + \sum_{h=0}^g \frac{(2h-3 + \|\mathbf{b}\|)!!}{12^{g-h} (2g-1 + \|\mathbf{b}\|)!!} \times \\ \left(\frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \mathbf{L} \neq \mathbf{0}, \mathbf{L}' \neq \mathbf{0}}} \sum_{r+s=n-1+3(g-h)} \binom{\mathbf{b}}{\mathbf{L}} \binom{n-1+3(g-h)}{r} V_{h',r+2}(\kappa(\mathbf{L})) V_{h-h',s+2}(\kappa(\mathbf{L}')) \right)$$

$$- \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\|\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_{h,n+3(g-h)}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}).$$

The following proposition is a generalization of a recursion formula proved in Proposition 2.6 of [15].

Proposition 3.4. *Let $\mathbf{b} \in N^\infty$, $n \geq 0$ and $r \geq 0$. Then*

$$(7) \quad \langle \tau_1 \tau_r \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g = (2r+3) \langle \tau_0 \tau_{r+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g - \frac{1}{6} \langle \tau_0^3 \tau_r \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_{g-1} \\ - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \underline{n}=I \amalg J}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0 \tau_r \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}.$$

Let LHS and RHS denote the left and right hand side of Proposition 3.4 respectively. Taking $r = 1$ and $n = 0$, we have

$$LHS = \int_{\overline{\mathcal{M}}_{g,1}} \pi_{2*} \left(\psi_1 \psi_2 \prod_{i \geq 1} (\pi_2^* \kappa_i + \psi_2^i)^{b(i)} \right) \\ = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1 \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \\ = (\|\mathbf{b}\| + 2g - 1) \int_{\overline{\mathcal{M}}_{g,1}} \psi_1 \kappa(\mathbf{b}) + \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\|\geq 2}} \int_{\overline{\mathcal{M}}_{g,1}} \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \\ = ((2g-1)(2g-2) + (4g-3)\|\mathbf{b}\| + \|\mathbf{b}\|^2) V_g(\kappa(\mathbf{b})) \\ + (2g-1 + \|\mathbf{b}\|) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\|\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_g(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ + \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \|\mathbf{L}'\|\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} \sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}+\delta_{|\mathbf{L}'|}} \binom{\mathbf{L}+\delta_{|\mathbf{L}'|}}{\mathbf{e}} V_g(\kappa(\mathbf{e})\kappa_{|\mathbf{f}'|}).$$

and similarly,

$$RHS = 5 \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,1}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|+1}) - \frac{1}{6} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g-1,3}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ - \sum_{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} V_{g',1}(\kappa_{|\mathbf{L}|}\kappa(\mathbf{e})) V_{g-g',2}(\kappa(\mathbf{f})).$$

So we have proved Theorem 1.4.

4. VIRASORO CONSTRAINTS AND THE KDV HIERARCHY

In this section, we follow the arguments of Mulase and Safnuk [20] to study properties of generating functions of intersections of ψ and κ classes using Theorems 1.1 and 1.2.

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m}, \mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \dots \tau_0^{n_0} \tau_1^{n_1} \dots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

Propositions 3.3 and 3.4 can be reformulated in terms of differential operators.

Proposition 4.1. *Let $r \geq 0$. Then we have*

$$\frac{\partial^2 G}{\partial t_0 \partial t_1} = \frac{1}{12} \frac{\partial^4 G}{\partial t_0^4} + \frac{1}{2} \frac{\partial^2 G}{\partial t_0^2} \frac{\partial^2 G}{\partial t_0^2}$$

and

$$\frac{\partial^2 G}{\partial t_1 \partial t_r} = (2r + 3) \frac{\partial^2 G}{\partial t_0 \partial t_{r+1}} - \frac{1}{6} \frac{\partial^4 G}{\partial t_0^3 \partial t_r} - \frac{\partial^2 G}{\partial t_0 \partial t_r} \frac{\partial^2 G}{\partial t_0^2}.$$

We define $\beta_{\mathbf{L}} = \alpha_{\mathbf{L}} / \mathbf{L}!$ where $\alpha_{\mathbf{L}}$ are the same constants in Theorem 1.2. We introduce the following family of differential operators for $k \geq -1$,

$$(8) \quad \hat{V}_k = -\frac{(2k+3)!!}{2} \frac{\partial}{\partial t_{k+1}} + \delta_{k,-1} \left(\frac{t_0^2}{4} + \frac{s_1}{48} \right) + \frac{\delta_{k,0}}{16} \\ + \frac{1}{2} \sum_{\mathbf{L}} \sum_{j=0}^{\infty} \frac{(2(|\mathbf{L}| + j + k) + 1)!!}{(2j-1)!!} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} t_j \frac{\partial}{\partial t_{|\mathbf{L}|+j+k}} \\ + \frac{1}{4} \sum_{\mathbf{L}} \sum_{\substack{d_1+d_2= \\ |\mathbf{L}|+k-1}} (2d_1+1)!! (2d_2+1)!! \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}}.$$

Theorem 4.2. *We have $\hat{V}_k \exp(G) = 0$ for $k \geq -1$ and*

$$[\hat{V}_n, \hat{V}_m] = (n-m) \sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \hat{V}_{n+m+|\mathbf{L}|}.$$

Proof. Note that the termination cases of the recursion formula in Theorem 1.2 are

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \quad \langle \tau_0^3 \rangle_0 = 1, \quad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

So $\hat{V}_k \exp(G) = 0$ for $k \geq -1$ is just a restatement of Theorem 1.2.

One may check directly that

$$[\hat{V}_n, \hat{V}_m] = (n-m) \sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \hat{V}_{n+m+|\mathbf{L}|}.$$

□

The following constants are inverse to $\beta_{\mathbf{L}}$,

$$\gamma_{\mathbf{L}} := \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}! (2|\mathbf{L}| + 1)!!}.$$

Define a new family of differential operators V_k for $k \geq -1$ by

$$(9) \quad V_k = -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!! (2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{16},$$

Theorem 1.1 implies $V_k \exp(G) = 0$. We now prove that the operators V_k satisfy the Virasoro relations

$$[V_n, V_m] = (n - m)V_{n+m}.$$

Introduce new variables

$$T_{2i+1} := \frac{t_i}{(2i+1)!!}, \quad i \geq 0$$

which transform the operators \hat{V}_k into

$$\begin{aligned} \hat{V}_k &= -\frac{1}{2} \frac{\partial}{\partial T_{2k+3}} + \delta_{k,-1} \left(\frac{t_0^2}{4} + \frac{s_1}{48} \right) + \frac{\delta_{k,0}}{16} \\ &\quad + \frac{1}{2} \sum_{\mathbf{L}} \sum_{j=0}^{\infty} (2j+1) \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} T_{2j+1} \frac{\partial}{\partial T_{2(|\mathbf{L}|+j+k)+1}} \\ &\quad + \frac{1}{4} \sum_{\mathbf{L}} \sum_{\substack{d_1+d_2= \\ |\mathbf{L}|+k-1}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial^2}{\partial T_{2d_1+1} \partial T_{2d_2+1}}. \end{aligned}$$

Define operators J_p for $p \in \mathbb{Z}$ by

$$J_p = \begin{cases} (-p)T_{-p} & \text{if } p < 0, \\ \frac{\partial}{\partial T_p} & \text{if } p > 0. \end{cases}$$

Then

$$\hat{V}_k = -\frac{1}{2} J_{2k+3} + \sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} E_{k+|\mathbf{L}|},$$

where

$$E_k = \frac{1}{4} \sum_{p \in \mathbb{Z}} J_{2p+1} J_{2(k-p)-1} + \frac{\delta_{k,0}}{16}.$$

It's not difficult to see that

$$V_k = \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \hat{V}_{k+|\mathbf{L}|} = -\frac{1}{2} \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} J_{2k+2|\mathbf{L}|+3} + E_k.$$

Theorem 4.3. *The operators V_k , $k \geq -1$ satisfy the Virasoro relations*

$$[V_n, V_m] = (n - m)V_{n+m}.$$

Proof. Since

$$\begin{aligned} E_k &= \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ &\quad + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!! (2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{16}. \end{aligned}$$

We can check directly that

$$[E_n, E_m] = (n - m)E_{n+m}, \quad [J_{2k+3}, E_m] = \frac{2k+3}{2} J_{2(k+m)+3}.$$

So we have

$$[V_n, V_m] = \left[-\frac{1}{2} \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} J_{2(n+|\mathbf{L}|)+3} + E_n, -\frac{1}{2} \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} J_{2(m+|\mathbf{L}|)+3} + E_m \right]$$

$$\begin{aligned}
 &= -\frac{1}{2} \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \left([J_{2(n+|\mathbf{L}|)+3}, E_m] + [E_n, J_{2(m+|\mathbf{L}|)+3}] \right) + [E_n, E_m] \\
 &= -\frac{1}{2} \sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} (n-m) J_{2(n+m+|\mathbf{L}|)+3} + (n-m) E_{n+m} \\
 &= (n-m) V_{n+m}.
 \end{aligned}$$

□

Now we recall the KdV hierarchy, which is the following hierarchy of differential equations for $n \geq 1$,

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1},$$

where R_n are polynomials in $U, \partial U / \partial t_0, \partial^2 U / \partial t_0^2, \dots$, which is defined recursively by

$$R_1 = U, \quad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1} \left(\frac{\partial U}{\partial t_0} R_n + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} R_n \right).$$

In particular, it is easy to see that

$$R_2 = \frac{1}{2} U^2 + \frac{1}{12} \frac{\partial^2 U}{\partial t_0^2},$$

so the first equation in the KdV hierarchy is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}.$$

The Witten-Kontsevich theorem [25, 14] states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ -function for the KdV hierarchy, i.e. $\partial^2 F / \partial t_0^2$ obeys all equations in the KdV hierarchy.

Theorem 4.4. *We have*

$$(10) \quad G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

where p_k are polynomials in \mathbf{s} given by

$$p_k = - \sum_{|\mathbf{L}|=k-1} (2|\mathbf{L}|+1)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{|\mathbf{L}|-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of \mathbf{s} , $G(\mathbf{s}, \mathbf{t})$ is a τ -function for the KdV hierarchy.

Proof. The change of variables

$$\tilde{t}_i = \begin{cases} t_i & \text{for } i = 0, 1, \\ t_i - \sum_{|\mathbf{L}|=i-1} (2|\mathbf{L}|+1)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} & \text{otherwise,} \end{cases}$$

transforms the operators V_k of (9) into

$$\begin{aligned}
 V_k &= -\frac{1}{2} (2k+3)!! \frac{\partial}{\partial \tilde{t}_{k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} \tilde{t}_j \frac{\partial}{\partial \tilde{t}_{j+k}} \\
 &\quad + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!! (2d_2+1)!! \frac{\partial^2}{\partial \tilde{t}_{d_1} \partial \tilde{t}_{d_2}} + \frac{\delta_{k,-1} \tilde{t}_0^2}{4} + \frac{\delta_{k,0}}{16},
 \end{aligned}$$

which is just the operator obtained by setting $\mathbf{s} = \mathbf{0}$ in \hat{V}_k of (8). Since Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \dots)$ and $F(t_0, t_1, \dots)$, we have for any fixed values of \mathbf{s} ,

$$G(\mathbf{s}, t_0, t_1, t_2, \dots) = F(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots).$$

So we have proved the theorem. \square

Theorem 4.4 can also be proved directly by applying Lemma 2.2, as discussed in [17].

5. TAUTOLOGICAL CONSTANTS OF HODGE INTEGRALS

The results in this section can be applied to study Faber's perfect pairing conjecture [8] and its generalizations.

Let $\mathcal{M}_{g,n}^{rt}$ be the moduli space of "curves with rational tails" (i.e. with dual graph with a vertex of genus g). Let $\mathcal{M}_{g,n}^{ct}$ be the moduli space of "curves of compact type", (i.e. with dual graph with no loops). Hence

$$\mathcal{M}_{g,n}^{rt} \subset \mathcal{M}_{g,n}^{ct} \subset \overline{\mathcal{M}}_{g,n}.$$

Conjecture 5.1. (Faber, Hain, Looijenga, Pandharipande, et al.) *The space $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}_{g,n}^{rt}$, $\mathcal{M}_{g,n}^{ct}$) "behaves like" a complex variety of dimension $D = 3g - 3 + n$ (resp. $g - 2 + n$, $2g - 3 + n$). More precisely, its tautological ring R^* has the following properties.*

- Socle statement: $R^i = 0$ for $i > D$, $R^D \cong \mathbb{Q}$, and
- Perfect pairing statement: for $0 \leq i \leq D$, the natural map $R^i \times R^{D-i} \rightarrow R^D$ is a perfect pairing.

The socle statement has been proved by Graber and Vakil [11]. While the perfect pairing statement is still open.

By the above conjecture, tautological relations in $\mathcal{M}_{g,n}^{rt}$ and $\mathcal{M}_{g,n}^{ct}$ are determined respectively by the following linear functionals, called intersection pairings.

$$\begin{aligned} R^i(\mathcal{M}_{g,n}^{rt}) \times R^{g-2+n-i}(\mathcal{M}_{g,n}^{rt}) &\longrightarrow \mathbb{Q} \\ (u, v) &\longmapsto \int_{\overline{\mathcal{M}}_{g,n}} uv \lambda_g \lambda_{g-1}, \end{aligned}$$

and

$$\begin{aligned} R^i(\mathcal{M}_{g,n}^{ct}) \times R^{2g-3+n-i}(\mathcal{M}_{g,n}^{ct}) &\longrightarrow \mathbb{Q} \\ (u, v) &\longmapsto \int_{\overline{\mathcal{M}}_{g,n}} uv \lambda_g. \end{aligned}$$

Since tautological classes are represented by linear combinations of decorated stable graphs, the computation of intersection pairings will eventually reduce to the following integrals

$$\begin{aligned} &\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1} \cdots \kappa_{b_k} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g \lambda_{g-1}, \\ &\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1} \cdots \kappa_{b_k} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g. \end{aligned}$$

Commonly, one would compute the above integrals by first eliminating κ classes, then applying the $\lambda_g \lambda_{g-1}$ theorem or the λ_g theorem.

Now we present more efficient recursion formulae computing these integrals, their patterns may well give some implications of the perfect pairing conjectures.

From degree 0 Virasoro constraints for a surface, Getzler and Pandharipande [9] obtained the following recursion.

Lemma 5.2. [9] *Let $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\begin{aligned} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \lambda_{g-1} \rangle_g &= \frac{(2d + 2d_0 - 1)!!}{(2d - 1)!!(2d_0 - 1)!!} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \lambda_{g-1} \rangle_g \\ &\quad + \sum_{j=1}^n \frac{(2d + 2d_j - 3)!!}{(2d - 1)!!(2d_j - 3)!!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \mid \lambda_g \lambda_{g-1} \rangle_g \end{aligned}$$

Lemma 5.2 has the following generalization.

Theorem 5.3. *Let $\mathbf{b} \in N^\infty$, $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$. Then*

$$\begin{aligned} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d + 2|\mathbf{L}| - 1)!!}{(2|\mathbf{L}| - 1)!!} \langle \tau_{d+|\mathbf{L}|} \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g \\ = \frac{(2d + 2d_0 - 1)!!}{(2d_0 - 1)!!} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g \\ + \sum_{j=1}^n \frac{(2d + 2d_j - 3)!!}{(2d_j - 3)!!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g \end{aligned}$$

and

$$\begin{aligned} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g \\ = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d + 2d_0 + 2|\mathbf{L}| - 1)!!}{(2d - 1)!!(2d_0 - 1)!!} \langle \tau_{d_0+d+|\mathbf{L}|-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g \\ + \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{j=1}^n \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d + 2d_j + 2|\mathbf{L}| - 3)!!}{(2d - 1)!!(2d_j - 3)!!} \langle \tau_{d_0} \tau_{d_j+d+|\mathbf{L}|-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g \end{aligned}$$

where $\gamma_{\mathbf{L}} \in \mathbb{Q}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \gamma_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| - 1)!!} = 0, \quad \mathbf{b} \neq \mathbf{0},$$

with the initial value $\gamma_{\mathbf{0}} = 1$.

Corollary 5.4. *In Theorem 5.3, we have*

$$\gamma_l = \frac{E_l}{(2l - 1)!!}, \quad \gamma(\underbrace{0, \dots, 0}_l, 1) = \frac{1}{(2l - 1)!!}$$

where E_l are the Euler numbers that satisfy

$$\sec x = \frac{1}{\cos x} = \sum_{k=0}^{\infty} \frac{E_k}{(2k)!} x^{2k} = 1 + \frac{1}{2!} x^2 + \frac{5}{4!} x^4 + \frac{61}{6!} x^6 + \frac{1385}{8!} x^8 + \frac{50521}{10!} x^{10} + \dots$$

Proof. We have

$$\cos(\sqrt{2}x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k - 1)!!} x^{2k},$$

by Theorem 5.3,

$$\sec(\sqrt{2}x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^{2k}.$$

So we get the formula of γ_l . □

The following recursion follows from degree 0 Virasoro constraints for a curve.

Lemma 5.5. [9] *Let $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\begin{aligned} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \rangle_g &= \binom{d+d_0}{d_0} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \rangle_g \\ &+ \sum_{j=1}^n \binom{d_j+d-1}{d_j-1} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \mid \lambda_g \rangle_g, \end{aligned}$$

Lemma 5.5 has the following generalization.

Theorem 5.6. *Let $\mathbf{b} \in N^\infty$, $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\begin{aligned} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} (-1)^{|\mathbf{L}|} \frac{(d+|\mathbf{L}|)!}{|\mathbf{L}|!} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g \\ = \frac{(d+d_0)!}{d_0!} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g \\ + \sum_{j=1}^n \frac{(d_j+d-1)!}{(d_j-1)!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g \end{aligned}$$

and

$$\begin{aligned} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g \\ = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(d+d_0+|\mathbf{L}|)!}{d_0!d!} \langle \tau_{d_0+d+|\mathbf{L}|-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g \\ + \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{j=1}^n \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(d_j+d+|\mathbf{L}|-1)!}{(d_j-1)!d!} \langle \tau_{d_0} \tau_{d_j+d+|\mathbf{L}|-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g \end{aligned}$$

where $\gamma_{\mathbf{L}} \in \mathbb{Q}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \gamma_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! |\mathbf{L}'|!} = 0, \quad \mathbf{b} \neq 0,$$

with the initial value $\gamma_{\mathbf{0}} = 1$.

We recall the definition of the Bessel functions of the first kind. For the Bessel equations of order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

we have the following solutions

$$y = J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{\nu+2k}.$$

These are called Bessel functions of the first kind of order ν .

Corollary 5.7. *In Theorem 5.6, we have*

$$\gamma(\underbrace{0, \dots, 0}_l, 1) = \frac{1}{l!}$$

and γ_l is given by

$$\frac{1}{J_0(\sqrt{4x})} = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = 1 + x + \frac{3/2}{2!} x^2 + \frac{19/6}{3!} x^3 + \frac{211/24}{4!} x^4 + \frac{1217/40}{5!} x^5 + \dots,$$

where J_0 is the Bessel function of the first kind of order zero

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} x^{2k}.$$

Proof. The corollary follows easily from Theorem 5.6 and the following

$$J_0(\sqrt{4x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} x^k.$$

□

It is interesting to notice that the Bessel function of the first kind of order zero also appears in Manin and Zograf's work [17] on asymptotics for Weil-Petersson volumes.

REFERENCES

- [1] E. Arbarello and M. Cornalba, *Combinatorial and Algebraic-Geometric cohomology classes on the Moduli Spaces of Curves*, J. Algebraic Geometry, **5** (1996), 705–709.
- [2] E. Arbarello and M. Cornalba, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Publications Mathématiques de l’IHÉS, 88 (1998), 97–127.
- [3] Norman Do and P. Norbury, *Weil-Petersson volumes and cone surfaces*, to appear in Geometriae Dedicata.
- [4] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Topological strings in $d < 1$* , Nuclear Phys. B 352 (1991), 59–86.
- [5] B. Eynard and N. Orantin, *Weil-Petersson volume of moduli spaces, Mirzakhani’s recursion and matrix models*, math-ph/0705.3600.
- [6] B. Eynard, *Recursion between Mumford volumes of moduli spaces*, math-ph/0706.4403.
- [7] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, in New Trends in Algebraic Geometry (K. Hulek, F. Catanese, C. Peters and M. Reid, eds.), 93–109, Cambridge University Press, 1999.
- [8] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*. In Moduli of curves and abelian varieties, Aspects Math., E33, Vieweg, Braunschweig, Germany, 1999. 109–129.
- [9] E. Getzler, R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle*, Nuclear Phys. B **530** (1998), no. 3, 701–714.
- [10] S. Grushevsky, *An explicit upper bound for Weil-Petersson volumes of the moduli spaces of punctured Riemann surfaces*, Math. Ann. 321 (2001), 1–13.
- [11] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. 30 (2005), 1–37.
- [12] A. Kabanov and T. Kimura, *Intersection Numbers and Rank One Cohomological Field Theories in Genus One*, Commun. Math. Phys. 194 (1998), 651–674.
- [13] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n -pointed curves*, Comm. Math. Phys. 181 (1996), 763–787.
- [14] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. **147** (1992), no. 1, 1–23.
- [15] K. Liu and H. Xu, *An effective recursion formula for computing intersection numbers*, math.AG/0710.5322.
- [16] K. Liu and H. Xu, *Mirzakhani’s recursion formula is equivalent to the Witten-Kontsevich theorem*, to appear in Asterisque.
- [17] Yu. Manin and P. Zograf, *Invertible cohomological field theories and Weil-Petersson volumes*, Ann. Inst. Fourier. 50 (2000), 519–535.
- [18] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. 167 (2007), 179–222.
- [19] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. 20 (2007), 1–23.

- [20] M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, Indiana J. Math. 50 (2008), 189–218.
- [21] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry* (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271-328.
- [22] R. Penner, *Weil-Petersson volumes*, J. Differential Geom. 35 (1992), 559–608.
- [23] B. Safnuk, *Integration on moduli spaces of stable curves through localization*, math.DG/0704.2530.
- [24] G. Schumacher and S. Trapani, *Estimates of Weil- Petersson volumes via effective divisors*, Comm. Math. Phys. 222 (2001), 1–7.
- [25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry, vol.1, (1991) 243–310.
- [26] S. Wolpert, *On the homology of the moduli space of stable curves*, Ann. Math., 118 (1983) 491–523.
- [27] P. Zograf, *An algorithm for computing Weil-Petersson volumes of moduli spaces of curves*, Institut Mittag-Leffler - Preprints 2006/2007.
- [28] *A Maple program to compute higher Weil-Petersson volumes*, available at <http://www.cms.zju.edu.cn/news.asp?id=1214&ColumnName=pdfbook&Version=english>

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