Positivity and vanishing theorems of ample vector bundles

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Abstract

In this paper, we study the Nakano-positivity and dual-Nakano-positivity of certain adjoint vector bundles associated to ample vector bundles. As applications, we get new vanishing theorems about ample vector bundles. For examples, we prove that if E is an ample vector bundle over a compact complex manifold S, then S^kE is Nakano-positive and dual-Nakano-positive for large k. For $\mathbb{P}^n$, we show that $(S^kT\mathbb{P}^n, S^kh_{FS})$ is Nakano-positive and dual-Nakano-positive for any $k \geq 2$ where h_{FS} is the standard Fubini-Study metric.

1 Introduction

Let E be a holomorphic vector bundle with a Hermitian metric h. Nakano in [29] introduced an analytic notion of positivity using the curvature of (E, h), and now it is called Nakano positivity. Griffiths in [15] defined Griffiths positivity of (E, h). On a Hermitian line bundle, these two concepts are the same. In general, Griffiths positivity is weaker than Nakano positivity. On the other hand, Hartshorne in [17] defined the ampleness of a vector bundle over a projective manifold. A vector bundle E is said to be ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample over $\mathbb{P}(E^*)$.

For a line bundle, it is well known that the ampleness of the bundle is equivalent to the Griffiths positivity of it. In [15], Griffiths conjectured that this equivalence is also valid for vector bundles, i.e., E is an ample vector bundle if and only if E carries a Griffiths-positive metric. It is well known if E admits a Griffiths-positive metric, then $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ has a Griffiths-positive metric(see Proposition 2.2). Finding a Griffiths-positive metric on an ample vector bundle seems to be very difficult but hopeful. In [7], Campana and Flenner gave an affirmative answer to the Griffiths conjecture when the base S is a projective curve, see also [38]. In [36], Siu and Yau proved the Frankel conjecture that every compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to the projective space. The positivity of holomorphic bisectional curvature is the same as Griffiths positivity of the holomorphic tangent bundle. On the other hand, S. Mori([26]) proved the Hartshorne conjecture that any algebraic manifold with ample tangent vector bundle is biholomorphic to the projective space.

In this paper, we consider the existence of positive metrics on ample vector bundles. It is well-known that metrics with good curvature properties are bridges between complex algebraic geometry and complex analytic geometry. Various vanishing theorems about ample vector bundles can be found in [10], [31], [25], [34], [23] and [22]. In this paper we take a different approach, we will construct Nakano-positive and dual-Nakano-positive metrics on various vector bundles associated to ample vector bundles.

Let E be a vector bundle over a compact complex manifold S and F a line bundle over S. Let r be the rank of E and n be the complex dimension of S. In the following we briefly describe our main results.

Theorem 1.1. *If $S^{r+k}E \otimes \det E^* \otimes F$ is (semi-)ample over S, then $S^kE \otimes F$ is (semi-)Nakano-positive and (semi-)dual-Nakano-positive.*

Here we make no assumption on E and we allow E to be negative(see Corollary 4.6). As pointed out by Professor Berndtsson that the Nakano positive part of Theorem 1.1 may also be derived from the result of [2]. For definitions about Nakano-positivity, dual-Nakano-positivity and ampleness, see **Section 2**. As applications of this theorem, we get the following results:

Theorem 1.2. *Let E be an ample vector bundle over S.*

(I) *If F is a* ***nef line bundle****, then there exists $k_0 = k_0(S, E)$ such that $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$. In particular, S^kE is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

(II) *If F is an arbitrary* ***vector bundle****, then there exists $k_0 = k_0(S, E, F)$ such that for any $k \geq k_0$, $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

Moreover, if the Hermitian vector bundle (E, h) is Griffiths-positive, then for large k, (S^kE, S^kh) is Nakano-positive and dual-Nakano-positive.

In general, when k is large enough, S^kE is very ample. Here we get its Nakano-positivity and dual-Nakano-positivity, which is much stronger than very ampleness. We can see the big gap between ampleness and Nakano-positivity from the following vanishing theorems.

Corollary 1.3. *Let E be an ample vector bundle over S.*

(I) *If F is a nef line bundle, then there exists $k_0 = k_0(S, E)$ such that*

$$H^{p,q}(S, S^kE \otimes F) = 0 \tag{1.1}$$

for any $q \geq 1$ and $p \geq 0$.

(II) *If F is an arbitrary vector bundle, then there exists $k_0 = k_0(S, E, F)$ such that for any $k \geq k_0$,*

$$H^{p,q}(S, S^kE \otimes F) = 0 \tag{1.2}$$

for any $q \geq 1$ and $p \geq 0$.

In particular, $H^{n,n}(S, S^kE) = 0$ for any $k \geq 1$.

Remark 1.4. **(1)** In proving vanishing theorems of ample vector bundles, a powerful technique is the Leray-Borel-Le Potier spectral sequence. By using this technique, one can get an isomorphism between Dolbeault cohomology groups $H^{p,q}(S, *)$ about vector bundles and cohomology group $H^{p,q}(\mathbb{P}(E^*), *)$ about line bundles. From the Akizuki-Kodaira-Nakano vanishing theorems for ample line bundles, one can get vanishing theorems of $H^{p,q}(S, *)$. But the lower bound of $p + q$ depends on the rank r of E. If the rank r is large enough, then the vanishing theorems deduced by this method are not effective. For example, the famous Le Potier's vanishing theorem: if E is an ample vector bundle over a compact complex manifold S, then $H^{p,q}(S, E) = 0$ if $p + q \geq n + r$. If $r > n$, then there are no information about Cohomology groups. For more details, see [10], [31], [25], [23] and [22].

(2) On a Griffiths-positive vector bundle E, it is easy to see that $H^{n,n}(E) = 0$. In general we can not get more information, for example, $H^{n,n-1}(\mathbb{P}^n, T\mathbb{P}^n) = \mathbb{C}$.

Corollary 1.5. *If E is an ample vector bundle and F a nef line bundle, or E is a nef vector bundle and F an ample line bundle, then we have*

(I) *$S^k E \otimes \det E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$.*

(II) *If the rank r of E is greater than 1, then $S^m E^* \otimes (\det E)^t \otimes F$ is Nakano-positive and dual-Nakano-positive if $t \geq r + m - 1$.*

(III) *If $S^{r+1} E \otimes \det E^*$ is ample, then E is Nakano-positive and dual-Nakano-positive, so it is Griffiths-positive.*

Remark 1.6. (1) For part (I), similar results were discussed in [2], [3] [4], [27], [28] and [33] recently.

(2) If E is Griffiths-positive, J-P. Demailly and H. Skoda proved that $E \otimes \det E$ and $E^* \otimes (\det E)^r$ are Nakano-positive if $r > 1$(see [12]). For more precise argument, see Theorem 7.3.

(3) Using Nakano-positivity and dual-Nakano-positivity of vector bundles, we can get vanishing theorems of types $H^{n,q}$ and $H^{q,n}$ with $q \geq 1$. See Theorem 6.2 and also [15], [10], [31], [25], [23] and [22].

(4) If E has a Griffiths-positive metric, then the naturally induced metrics on those vector bundles are exactly Nakano-positive and dual-Nakano-positive, see Theorem 7.3.

Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$ and $S^k h_{FS}$ the induced metric on $S^k T\mathbb{P}^n$ by Veronese mapping. Let $n \geq 2$. It is easy to see that $T\mathbb{P}^n$ does not admit a Nakano-positive metric. In particular $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive. However, $(S^k T\mathbb{P}^n, S^k h_{FS})$ is Nakano-positive for any $k \geq 2$.

Corollary 1.7. *Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$, then*

(I) *$(S^{n+1} T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^{n+1} h_{FS} \otimes \det(h_{FS})^{-1})$ is semi-Griffiths-positive for $n \geq 2$.*

(II) *$(S^k T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^k h_{FS} \otimes \det(h_{FS})^{-1})$ is Griffiths-positive for $k \geq n + 2$.*

(III) *$(S^k T\mathbb{P}^n, S^k h_{FS})$ is Nakano-positive and dual-Nakano-positive for any $k \geq 2$.*

This corollary can be viewed as an important evidence of positivity of some adjoint vector bundles, namely, vector bundles of type $S^k E \otimes (\det E)^\ell \otimes K_S$.

Theorem 1.8. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then $S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n - r, 0\}$. Moreover, the lower bound is sharp.*

In general, $\det E \otimes K_S$ is not an ample line bundle, for example, $(S, E) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$, so the (dual-)Nakano-positivity in the above theorem is stronger than the (dual-)Nakano-positivity of $S^k E \otimes \det E$.

Theorem 1.9. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then $E \otimes (\det E)^k \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n+1-r, 2\}$. Moreover, the lower bound is sharp.*

In the case $n+1-r > 2$, that is $1 < r < n-1$, the vector bundle $K_S \otimes (\det E)^{n-r}$ can be a negative line bundle, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2})$. But $E \otimes \det E \otimes K_S \otimes (\det E)^{n-r}$ is still Nakano-positive and dual-Nakano-positive.

Although the lower bounds of k in Theorem 1.8 and Theorem 1.9 are not effective when $r > n$, in considering vanishing theorems of ample vector bundles, it becomes quite interesting when $r \leq n$ as we explained before.

Theorem 1.10. *Let E be an ample vector bundles over an n-dimensional compact complex manifold S of rank r and L be any nef line bundle on S.*

(I) *If $r > 1$, then*

$$H^{n,q}(S, S^k E \otimes (\det E)^2 \otimes K_S \otimes L) = H^{q,n}(S, S^k E \otimes (\det E)^2 \otimes K_S \otimes L) = 0 \qquad (1.3)$$

for any $q \geq 1$ and $k \geq \max\{n-r, 0\}$.

(II) *If $r > 1$, then*

$$H^{n,q}(S, E \otimes (\det E)^k \otimes K_S \otimes L) = H^{q,n}(S, E \otimes (\det E)^k \otimes K_S \otimes L) = 0 \qquad (1.4)$$

for any $q \geq 1$ and $k \geq \max\{n+1-r, 2\}$.

Using theorem 1.1 and by induction we obtained the following theorem,

Proposition 1.11. *Assume that $S^{t+kr}E \otimes L$ is ample where r is the rank of E. Then*

$$H^{n,q}(SM, S^t E \otimes (\det E)^k \otimes L) = H^{q,n}(SM, S^t E \otimes (\det E)^k \otimes L) = 0$$

for any $q \geq 1$.

Remark 1.12. Theorem 1.1 allows us to do induction to deduce more positivity results. For example, if $S^m E \otimes F$ is ample, then $S^{m-r} E \otimes \det E \otimes L$ is (dual-)Nakano-positive and so it is ample. Using Theorem 1.1 again, we get $S^{m-2r} \otimes (\det E)^2 \otimes L$ is Nakano-positive and dual-Nakano-positive. Finally, we get $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive, if $m = t + kr$ for some $0 \leq t < r$. It is obvious that the (dual-)Nakano-positivity turns stronger and stronger under induction. This explains why a lot of vanishing theorems involve a power of $\det E$.

2 Background material

Let E be a holomorphic vector bundle over a compact complex manifold S and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}$ be the local holomorphic coordinates on S and $\{e_\alpha\}_{\alpha=1}^r$ be the local frames of E. The curvature tensor $R^\nabla \in \Gamma(S, \Lambda^2 T^*S \otimes E^* \otimes E)$ has the form

$$R^\nabla = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}\alpha}^{\gamma} dz^i \wedge d\overline{z}^j \otimes e^\alpha \otimes e_\gamma \tag{2.1}$$

where $R_{i\bar{j}\alpha}^{\gamma} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{2.2}$$

A Hermitian vector bundle (E, h) is said to be Griffiths-positive, if for any nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0 \tag{2.3}$$

(E, h) is said to be Nakano-positive, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0 \tag{2.4}$$

(E, h) is said to be dual-Nakano-positive, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0 \tag{2.5}$$

The notions of semi-positivity could be defined similarly. We say E is (semi-)Nakano-positive, if it admits a (semi-)Nakano-positive metric. A line bundle L is said to be nef, if for any irriducible curve $C \subset S$,

$$\int_C c_1(L) \geq 0$$

For a comprehensive description of positivity and related topics, see [9], [15], [34] and [38].

Let E be a Hermitian vector bundle of rank r over a compact complex manifold S, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle on the projective bundle $\mathbb{P}(E^*)$ and π the canonical projection $\mathbb{P}(E^*) \longrightarrow S$. By definition([17]), E is an ample vector bundle over S if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample line bundle over $\mathbb{P}(E^*)$. We say E is semi-ample if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is semi-ample. E is said to be nef, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is nef. To simplify the notations we will denote $\mathbb{P}(E^*)$ by X and the fiber $\pi^{-1}(\{s\})$ by X_s.

Let $(e_1, \cdots, e_r)$ be the local holomorphic frames with respect to a given trivialization on E and the dual frames on E^* are denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \cdots, W_r)$. There is a canonical section e_{L^*} of L^*, i.e., $e_{L^*} \in \Gamma(X, L^*)$:

$$e_{L^*} = \sum_{\alpha=1}^r W_\alpha e^\alpha \tag{2.6}$$

The dual section of it is denoted by $e_L \in \Gamma(X, L)$. Using this non-vanishing section, we can define a Hermitian metric h^L on L if there exists a Hermitian metric h^E on E. More precisely, if $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h^E with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then

$$h^L = \frac{1}{h^E(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta} \tag{2.7}$$

Proposition 2.1. *The curvature of* (L, h^L) *is*

$$R^{h^L} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h^L = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}}W_\alpha\overline{W}_\beta\right) \tag{2.8}$$

where ∂ *and* $\overline{\partial}$ *are operators on the total space* $\mathbb{P}(E^*)$.

By this proposition, we get the following well-known result:

Proposition 2.2. *If* (E, h^E) *is a Griffiths-positive vector bundle, then* E *is ample.*

Proof. By Kodaira embedding theorem, L is ample if and only if there exists a positive Hermitian metric on L. We will show that the induced metric h^L (see 2.7) is positive. Now we fix a point $p \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $s = \pi(p)$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of E around s such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}}z^i\overline{z}^j + O(|z|^3) \tag{2.9}$$

Without loss generality, we assume p is $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$, then

$$R^{h^L}(p) = \frac{\sqrt{-1}}{2\pi}\left(\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{r}R_{i\overline{j}\alpha\overline{\beta}}\frac{a_\beta\overline{a}_\alpha}{|a|^2}dz^i\wedge d\overline{z}^j + \sum_{A,B=1}^{r-1}\left(1-\frac{a_B\overline{a}_A}{|a|^2}\right)dw^A\wedge d\overline{w}^B\right) \tag{2.10}$$

where $|a|^2 = \sum\limits_{\alpha=1}^{r}|a_\alpha|^2$. If R^E is Griffith positive, then

$$\left(\sum_{\alpha,\beta=1}^{r}R_{i\overline{j}\alpha\overline{\beta}}\frac{a_\beta\overline{a}_\alpha}{|a|^2}\right)$$

is a Hermitian positive $n\times n$ matrix. Consequently, $R^{h^L}(p)$ is a Hermitian positive $(1,1)$ form on $\mathbb{P}(E^*)$, i.e. h^L is a positive Hermitian metric. □

The following linear algebraic lemma will be used in Theorem 4.4.

Lemma 2.3. *If the matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible and D *is invertible, then* $(A - BD^{-1}C)^{-1}$ *exists and*

$$T^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}+D^{-1} \end{pmatrix}$$

Moreover, if T *is positive definite, then* $A - BD^{-1}C$ *is positive definite.*

3 Curvature formulas

Suppose $\widetilde{L} = L^k \otimes \pi^*(F)$ for $k \geq 0$ where F is a line bundle over S. Let h_0 be a Hermitian metric on $\widetilde{L}$ and $\{\omega_s\}_{s\in S}$ a smooth family of Kähler metrics on the the fibers $X_s = \mathbb{P}(E_s^*)$ of X. Let $\{w^A\}_{A=1}^{r-1}$ be the local holomorphic coordinates on the fiber X_s which is generated by the homogeneous coordinates $[W_1, \cdots, W_r]$ on the corresponding trivialization chart. Using these notations, we can write the metric ω_s as

$$\omega_s = \frac{\sqrt{-1}}{2\pi} \sum_{A,B=1}^{r-1} g_{A\overline{B}}(s,w) dw^A \wedge d\overline{w}^B \tag{3.1}$$

It is well known $H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(k))$ can be identified as the space of homogeneous polynomials of degree k in r variables. The sections of $H^0(X_s, \widetilde{L}|_{X_s})$ are of the form $V_\alpha e_L^{\otimes k} \otimes \underline{e}$ where V_α are homogenous polynomials in $\{W_1, \cdots, W_r\}$ of degree k and $\underline{e}$ the base of $\pi^*(F)$ induced by a base e of F. For example, if $\alpha = (\alpha_1, \cdots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r = k$ and α_j are nonnegative integers, then

$$V_\alpha = W_1^{\alpha_1} \cdots W_r^{\alpha_r}. \tag{3.2}$$

Now we set

$$E_\alpha = e_1^{\otimes \alpha_1} \otimes \cdots \otimes e_r^{\otimes \alpha_r} \otimes e \quad \text{and} \quad e_{\widetilde{L}} = e_L^{\otimes k} \otimes \underline{e}$$

which are basis of $S^k E \otimes F$ and $\widetilde{L}$ respectively.

We get a vector bundle whose fibers are $H^0(X_s, \widetilde{L}|_{X_s})$. In fact, this vector bundle is $\widetilde{E} = S^k E \otimes F$. Now we define a Hermitian metric f on $S^k E \otimes F$ by $(\widetilde{L}, h_0)$ and (X_s, ω_s), locally it is

$$f_{\alpha\overline{\beta}} := f(E_\alpha, E_\beta) = \int_{X_s} \langle V_\alpha e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \frac{\omega_s^{r-1}}{(r-1)!} \tag{3.3}$$

Here we regard h_0 locally as a positive function. In this general setting, the Hermitian metric h_0 on $\widetilde{L}$ and Kähler metrics ω_s on the fibers are independent.

Let $(z^1, \cdots, z^n)$ be the local holomorphic coordinates on S. By definition, the curvature tensor of f is

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + \sum_{\gamma,\delta} f^{\gamma\overline{\delta}} \frac{\partial f_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{3.4}$$

In the following, we will compute the curvature of f. Let $T_{X/S}$ be the relative tangent bundle of the fibration $\mathbb{P}(E^*) \to S$, then $g_{A\overline{B}}$ is a metric on $T_{X/S}$ and $\det(g_{A\overline{B}})$ is a metric on $\det(T_{X/S})$. Let $\varphi = -\log(h_0 \det(g_{A\overline{B}}))$ be the local weight of induced Hermitian metric $h_0 \det(g_{A\overline{B}})$ on $\widetilde{L} \otimes \det(T_{X/S})$. In the sequel, we will use the following notations

$$\varphi_i = \frac{\partial \varphi}{\partial z^i}, \ \varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}, \quad \varphi_{A\overline{B}} = \frac{\partial^2 \varphi}{\partial w^A \partial \overline{w}^B}, \quad \varphi_{i\overline{B}} = \frac{\partial^2 \varphi}{\partial z^i \partial \overline{w}^B} \ \text{ and } \ \varphi_{A\bar{j}} = \frac{\partial^2 \varphi}{\partial \overline{z}^j \partial w^A}$$

and $(\varphi^{A\overline{B}})$ is the transpose inverse of the $(r-1) \times (r-1)$ matrix $(\varphi_{A\overline{B}})$,

$$\sum_{B=1}^{r-1} \varphi^{A\overline{B}} \varphi_{C\overline{B}} = \delta_C^A$$

The following lemma can be deduced from the formulas in [32], [39] and [35]. In the case of holomorphic fibration $\mathbb{P}(E^*) \to S$, we could compute it directly, since it is locally flat and each fiber of it is isomorphic to $\mathbb{P}^{r-1}$.

Lemma 3.1. *The first order derivative of $f_{\alpha\overline{\beta}}$ is*

$$\frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} = -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!} = \int_{X_s} \langle -V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \tag{3.5}$$

Proof. By the local expression of ω_s,

$$\frac{\omega_s^{r-1}}{(r-1)!} = \det(g_{A\overline{B}}) dV_{\mathbb{C}^{r-1}}$$

where $dV_{\mathbb{C}^{r-1}}$ is standard volume on $\mathbb{C}^{r-1}$. Therefore

$$f_{\alpha\overline{\beta}} = \int_{X_s} e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}}$$

and the first order derivative is

$$\begin{aligned} \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} &= \int_{X_s} \frac{\partial e^{-\varphi}}{\partial z^i} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\ &= -\int_{X_s} \varphi_i e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\ &= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!} \end{aligned}$$

□

Theorem 3.2. *The curvature tensor of Hermitian metric f on $S^k E \otimes F$ is*

$$R_{i\overline{j}\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \tag{3.6}$$

where

$$P_{i\alpha} = -V_\alpha \varphi_i - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\overline{\delta}} \frac{\partial f_{\alpha\overline{\delta}}}{\partial z^i} \right) \tag{3.7}$$

Proof. For simplicity of notations, we set $A_{i\alpha} = -V_\alpha \varphi_i$. The Hermitian metric 3.3 is also a norm on the space $\Gamma(X_s, \widetilde{L}|_{X_s})$, and it induces an orthogonal projection

$$\widetilde{\pi}_s : \Gamma(X_s, \widetilde{L}|_{X_s}) \longrightarrow H^0(X_s, \widetilde{L}|_{X_s})$$

Using this projection, we could rewrite the first derivative as

$$\begin{aligned} \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} &= \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) + (A_{i\alpha} e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}})), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \end{aligned}$$

since $(A_{i\alpha}e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}))$ is in the orthogonal space of $H^0(X_s, \widetilde{L}|_{X_s})$. By this relation, we could write $\widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}})$ in the basis $\{V_\alpha e_{\widetilde{L}}\}$ of $H^0(X_s, \widetilde{L}|_{X_s})$,

$$\widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}) = \sum_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) (V_\gamma e_{\widetilde{L}}) \tag{3.8}$$

From this identity, we see

$$\int_{X_s} \left\langle \widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}), \widetilde{\pi}_s(A_{j\beta}e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} = \sum_{\gamma,\delta} f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{3.9}$$

Suppose

$$P_{i\alpha} = A_{i\alpha} - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) \tag{3.10}$$

then $A_{i\alpha}e_{\widetilde{L}} = \widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}) + P_{i\alpha}e_{\widetilde{L}}$, that is,

$$\widetilde{\pi}_s(P_{i\alpha}e_{\widetilde{L}}) = 0 \tag{3.11}$$

Similar to Lemma 3.1, we get the second order derivative

$$\begin{aligned}
\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} &= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta \varphi_j e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, A_{j\beta} e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \left\langle P_{i\alpha} e_{\widetilde{L}} + \widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}), P_{j\beta} e_{\widetilde{L}} + \widetilde{\pi}_s(A_{j\beta}e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \left\langle \widetilde{\pi}_s(A_{i\alpha}e_{\widetilde{L}}), \widetilde{\pi}_s(A_{j\beta}e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} + \sum_{\gamma,\delta} f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\bar{\beta}}}{\partial \bar{z}^j}
\end{aligned}$$

By formula 3.4, we get the curvature formula 3.6. □

4 $\overline{\partial}$-estimate and positivity of Hermitian metrics

In this section, we will use $\overline{\partial}$-estimate on polarized Kähler manifolds to analyze the curvature formula in Theorem 3.2,

$$R_{i\bar{j}\alpha\bar{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!}$$

The first term on the right hand side involves the base direction curvature $\varphi_{i\bar{j}}$ of the line bundle $\widetilde{L} \otimes \det(T_{X/S})$. If the line bundle $\widetilde{L} \otimes \det(T_{X/S})$ is positive in the base direction, we can choose (h_0, ω_s) such that φ is positive in the base direction, i.e. $(\varphi_{i\bar{j}})$ is Hermitian positive. We will give a lower bound of the second term by the following Lemma.

Lemma 4.1. *Let (M^n,ω_g) be a compact Kähler manifold and (L,h) a Hermitian line bundle over M. If there exists a positive constant c such that*

$$Ric(\omega_g)+R^h\geq c\omega_g \tag{4.1}$$

then for any $w\in\Gamma(M,T^{*0,1}M\otimes L)$ such that $\overline{\partial}w=0$, there exists a unique $u\in\Gamma(M,L)$ such that $\overline{\partial}u=w$ and $\widetilde{\pi}(u)=0$ where $\widetilde{\pi}:\Gamma(M,L)\longrightarrow H^0(M,L)$ is the orthogonal projection. Moreover,

$$\int_M |u|_h^2\frac{\omega_g^n}{n!}\leq\frac{1}{c}\int_M |w|^2_{g^*\otimes h}\frac{\omega_g^n}{n!} \tag{4.2}$$

For the first part, condition 4.1 asserts that $K_M^*\otimes L$ is a positive line bundle over M, so by Akizuki-Kodaira-Nakano vanishing theorem(see e.g. [9]),

$$H^{n,1}(M,K_M^*\otimes L)=0$$

that is $H^{0,1}(M,L)=0$. For the second part, since u has minimal energy, the inequality 4.2 follows by standard $\overline{\partial}$-estimate. We refer the reader to [9] or [19].

Now we apply Lemma 4.1 to each fiber (X_s,ω_s) and $(\widetilde{L}|_{X_s},h_0|_{X_s})$. At a fixed point $s\in S$, the fiber direction curvature of the induced metric on line bundle $\widetilde{L}\otimes\det(T_{X/S})$ is

$$-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log(h_0\det(g_{A\overline{B}}))=R^{\widetilde{L}_s^{h_0}}+Ric_F(\omega_s) \tag{4.3}$$

On the other hand

$$-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log(h_0\det(g_{A\overline{B}}))=\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\varphi$$

where $\varphi=-\log(h_0\det(g_{A\overline{B}}))$. So the condition 4.1 turns to be

$$(\varphi_{A\overline{B}})\geq c_s(g_{A\overline{B}}) \tag{4.4}$$

for some positive constant $c_s=c(s)$.

Theorem 4.2. *If $(\varphi_{A\overline{B}})\geq c_s(g_{A\overline{B}})$ at point $s\in S$, then for any*

$$u=\sum_{i,\alpha}u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha\in\Gamma(S,T^{1,0}S\otimes\widetilde{E})$$

with $\widetilde{E}=S^kE\otimes F$, we have the following estimate at point s,

$$R^{\widetilde{E}}(u,u)=\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\overline{u^{j\beta}}\geq\sum_{i,j,\alpha,\beta}\int_{X_s}h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\bar{j}}-\frac{\sum\limits_{A,B=1}^{r-1}g^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\bar{j}}}{c_s}\right)\frac{\omega_s^{r-1}}{(r-1)!} \tag{4.5}$$

Proof. At point $s \in S$, we set

$$P = \sum_{i,\alpha} P_{i\alpha} u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s), \quad K = -\sum_{i,\alpha} V_\alpha \varphi_i u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s)$$

It is obvious that $\overline{\partial}_s P = \overline{\partial}_s K$ where $\overline{\partial}_s$ is $\overline{\partial}$ on the fiber direction. On the other hand, by 3.11, $\widetilde{\pi}_s(P) = 0$. So we could apply Lemma 4.1 and get

$$\int_{X_s} |P|^2_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \leq \frac{1}{c_s} \int_{X_s} |\overline{\partial}_s K|^2_{g_s^* \otimes h_0} \frac{\omega_s^{r-1}}{(r-1)!} \tag{4.6}$$

Since $\overline{\partial}_s K = -\sum\limits_{i,\alpha,B} V_\alpha \varphi_{i\overline{B}} u^{i\alpha} d\overline{z}^B \otimes e_{\widetilde{L}}$,

$$|\overline{\partial}_s K|^2_{g_s^* \otimes h_0} = \sum_{i,j} \sum_{\alpha,\beta} h_0(V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} g^{A\overline{B}} \varphi_{i\overline{B}} \varphi_{A\overline{j}}$$

By inequality 4.6 and Theorem 3.2, we get the estimate 4.5. □

Before proving the main theorem, we need the following lemma:

Lemma 4.3. *If E is a holomorphic vector bundle over a compact complex manifold S with rank r and F is a line bundle over S such that $S^{k+r}E \otimes \det E^* \otimes F$ is a (semi-)ample over S, then there exists a (semi-)positive Hermitian metric λ_0 on $\mathcal{O}_{\mathbb{P}(E^*)}(k) \otimes \pi^*(F) \otimes \det(T_{X/S})$.*

Proof. Let $\widehat{E}$ be $S^{k+r}E \otimes \det(E^*) \otimes F$. It is obvious that $\mathbb{P}(S^{k+r}E^*) = \mathbb{P}(\widehat{E}^*)$. The tautological line bundles of them are related by the following formula

$$\mathcal{O}_{\mathbb{P}(\widehat{E})}(1) = \mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1) \otimes \pi^*_{k+r}(\det E^*) \otimes \pi^*_{k+r}(F) \tag{4.7}$$

where $\pi_{k+r} : \mathbb{P}(S^{k+r}E^*) \to S$ is the canonical projection. Let $v_{k+r} : \mathbb{P}(E^*) \to \mathbb{P}(S^{k+r}E^*)$ be the standard Veronese embedding, then

$$\mathcal{O}_{\mathbb{P}(E^*)}(k+r) = v^*_{k+r}\left(\mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1)\right) \tag{4.8}$$

Similarly, let μ_{k+r} be the induced mapping $\mu_{k+r} : \mathbb{P}(E^*) \longrightarrow \mathbb{P}(\widehat{E})$, then

$$\mu^*_{k+r}\left(\mathcal{O}_{\mathbb{P}(\widehat{E})}(1)\right) = \mathcal{O}_{\mathbb{P}(E^*)}(k+r) \otimes \pi^*(F \otimes \det E^*) \tag{4.9}$$

By the identity

$$K_X = \pi^*(K_S) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(-r) \otimes \pi^*(\det E), \tag{4.10}$$

we have

$$\mu^*_{k+r}\left(\mathcal{O}_{\mathbb{P}(\widehat{E})}(1)\right) = \mathcal{O}_{\mathbb{P}(E^*)}(k) \otimes \pi^*(F) \otimes \det(T_{X/S}) = \widetilde{L} \otimes \det(T_{X/S}) \tag{4.11}$$

By definition if $\widehat{E}$ is (semi-)ample, then $\mathcal{O}_{\mathbb{P}(\widehat{E})}(1)$ is (semi-)ample and so is $\widetilde{L} \otimes \det(T_{X/S})$. By Kodaira embedding theorem, there exists a (semi-)positive Hermitian metric λ_0 on it. □

Theorem 4.4. *If E is a holomorphic vector bundle over a compact complex manifold S with rank r and F is a line bundle over S such that $S^{k+r}E\otimes\det E^*\otimes F$ is a (semi-)ample over S, then there exists a Hermitian metric f on $S^kE\otimes F$ such that $(S^kE\otimes F, f)$ is (semi-)Nakano-positive and (semi-)dual-Nakano-positive.*

Proof. At first, we assume $S^{k+r}E\otimes\det E^*\otimes F$ is ample. Let λ_0 be a positive Hermitian metric on $\widetilde{L}\otimes\det(T_{X/S})$ and set

$$\omega_s = -\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log\lambda_0 = \frac{\sqrt{-1}}{2\pi}\sum_{A,B=1}^{r-1} g_{A\overline{B}}(s,w)dw^A\wedge d\overline{w}^B$$

which is a smooth family of Kähler metrics on the fibers X_s. We get an induced Hermitian metric on $\widetilde{L}$, namely,

$$h_0 = \frac{\lambda_0}{\det(g_{A\overline{B}})}$$

Let f be the Hermitian metric on the vector bundle $S^kE\otimes\det F$ induced by $(\widetilde{L}, h_0)$ and (X_s,ω_s)(see 3.3). In this setting, the weight φ of induced metric on $\widetilde{L}\otimes\det(T_{X/S})$ is

$$\varphi = -\log\left(h_0\det(g_{A\overline{B}})\right) = -\log\lambda_0$$

Hence

$$\left(\varphi_{A\overline{B}}\right) = \left(g_{A\overline{B}}\right) \tag{4.12}$$

In Theorem 4.2, $c_s = 1$ for any $s\in S$, therefore

$$\begin{aligned} R^{\widetilde{E}}(u,u) = \sum_{i,j}\sum_{\alpha,\beta} R_{i\overline{j}\alpha\overline{\beta}}u^{i\alpha}\overline{u^{j\beta}} &\geq \sum_{i,j}\sum_{\alpha,\beta}\int_{X_s} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}} - \sum_{A,B=1}^{r-1} g^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!} \\ &= \sum_{i,j}\sum_{\alpha,\beta}\int_{X_s} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}} - \sum_{A,B=1}^{r-1} \varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!} \end{aligned}$$

for any $u = \sum_{i,\alpha} u^{i\alpha}\frac{\partial}{\partial z^i}\otimes E_\alpha \in \Gamma(S, T^{1,0}S\otimes\widetilde{E})$.

On the other hand λ_0 is a positive Hermitian metric on the line bundle $\widetilde{L}\otimes\det(T_{X/S})$. The curvature form of λ_0 can be represented by a Hermitian positive matrix, namely, the coefficients matrix of Hermitian positive $(1,1)$ form $\sqrt{-1}\partial\overline{\partial}\varphi$ on X. By Lemma 2.3,

$$\left(\varphi_{i\overline{j}} - \sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)$$

is a Hermitian positive $n\times n$ matrix. Since the integrand is nonnegative, $R^{\widetilde{E}}(u,u) = 0$ if and only if

$$\sum_{i,j}\sum_{\alpha,\beta} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}} - \sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\equiv 0 \tag{4.13}$$

on X_s which means $(u^{i\alpha})$ is a zero matrix. In summary, we get

$$R^{\widetilde{E}}(u,u)>0$$

for nonzero u, i.e. the induced metric f on $\widetilde{E}=S^kE\otimes F$ is Nakano-positive.

For the semi-ample case, the proof is similar. Let λ_0 be a semi-positive Hermitian metric on $\widetilde{L}\otimes\det(T_{X/S})$. Since $\mathbb{P}(E^*)$ is Kähler, there exists a Kähler metric ω_X. Now we replace ω_s by

$$\omega_s^{\varepsilon}=-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log\lambda_0+\varepsilon\omega_X|_{X_s} \tag{4.14}$$

for small $\varepsilon\in(0,1)$. So we could repeat the above process and let ε go to 0.

For the dual-Nakano-positivity, by definition

$$\sum_{i,j}\sum_{\alpha,\beta}R_{i\overline{j}\alpha\overline{\beta}}u^{i\beta}\overline{u^{j\alpha}}\geq\sum_{i,j}\sum_{\alpha,\beta}\int_{X_s}h_0(V_\alpha\overline{u^{j\alpha}})\overline{(V_\beta\overline{u^{i\beta}})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!} \tag{4.15}$$

The right hand side equals zero if and only if

$$(V_\alpha\overline{u^{j\alpha}})\overline{(V_\beta\overline{u^{i\beta}})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\equiv 0 \tag{4.16}$$

on X_s which infers $(u^{i\alpha})$ is a zero matrix. The dual-Nakano-positivity of f is proved. □

Corollary 4.5. *If E is an ample vector bundle and F is a nef line bundle, then there exists $k_0=k_0(S,E)$ such that $S^kE\otimes F$ is Nakano-positive and dual-Nakano-positive for any $k\geq k_0$. In particular, S^kE is Nakano-positive and dual-Nakano-positive for $k\geq k_0$.*

Proof. Since for large k, $S^kE\otimes\det E^*$ is ample, so $S^kE\otimes E^*\otimes F$ is ample. By Theorem 4.4, $S^kE\otimes F$ is Nakano-positive and dual-Nakano-positive. In particular, S^kE is Nakano-positive and dual-Nakano-positive for $k\geq k_0$. □

Corollary 4.6. *If E is an ample vector bundle and F is a nef line bundle, or E is a nef vector bundle and F is an ample line bundle,*

(I) *$S^kE\otimes\det E\otimes F$ is Nakano-positive and dual-Nakano-positive for any $k\geq 0$.*

(II) *If the rank r of E is greater than 1, then $S^mE^*\otimes(\det E)^t\otimes F$ is Nakano-positive and dual-Nakano-positive if $t\geq r+m-1$.*

Proof. (**I**) follows by the ampleness of $S^{k+r}E\otimes F=S^{k+r}E\otimes\det E^*\otimes(\det E\otimes F)$.
(**II**) If $r>1$, it is easy to see $E^*\otimes\det E=\wedge^{r-1}E$. By the relation

$$S^{r+m}(E^*\otimes\det E)\otimes(\det E)^{t-r-m+1}\otimes F=S^{r+m}E^*\otimes\det E\otimes(\det E)^t\otimes F \tag{4.17}$$

we can apply Theorem 4.4 to the pair $(E^*,(\det E)^t\otimes F)$ and obtain the Nakano-positivity and dual-Nakano-positivity of $S^mE^*\otimes(\det E)^t\otimes F$ when $t\geq r+m-1$. Let $E=T\mathbb{P}^2$, then $E=E^*\otimes\det E$ is Griffiths-positive but not Nakano-positive. So we need the restriction $t\geq r+m-1$. □

Corollary 4.7. *If $S^{r+1}E\otimes\det E^*$ is ample, then E is Nakano-positive and dual-Nakano-positive and so it is Griffiths-positive.*

5 Nakano-positivity and dual-Nakano-positivity of adjoint vector bundles

The following lemma is due to Fujita ([14]) and Ye-Zhang [40].

Lemma 5.1. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r \geq n+1$, then $\det E \otimes K_S$ is ample except $(S, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1})$.*

Theorem 5.2. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S.*

(I) *If $r > 1$, then $S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n-r, 0\}$.*

(II) *If $r = 1$, then the line bundle $E^{\otimes(n+2)} \otimes K_S$ is Nakano-positive.*

Moreover, the lower bound on k is sharp.

Proof. **(I)** If $r > 1$, then $X = \mathbb{P}(E^*)$ is a $\mathbb{P}^{r-1}$ bundle which is not isomorphic to any projective space. By Lemma 5.1, $\mathcal{O}_{\mathbb{P}(E^*)}(n+r) \otimes K_X$ is ample. That is

$$\mathcal{O}_{\mathbb{P}(E^*)}(n) \otimes \pi^* (K_S \otimes \det E)$$

is ample which is equivalent to the ampleness of $S^n E \otimes (\det E^*) \otimes (\det E)^2 \otimes K_S$. If $k \geq \max\{n-r, 0\}$, $S^{r+k} E \otimes \det E^* \otimes (\det E)^2 \otimes K_S$ is ample, hence by Theorem 4.4, $S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. **(II)** follows from Lemma 5.1. In fact, the vector bundle $\widetilde{E} = E^{\oplus(n+2)}$ is an ample vector bundle of rank $n+2$ and $\det \widetilde{E} = E^{\otimes(n+2)}$. By Lemma 5.1, $\det \widetilde{E} \otimes K_S = E^{\otimes(n+2)} \otimes K_S$ is ample. By Kodaira embedding theorem, it carries a Griffiths-positive.

Here the lower bound $n-r$ is sharp. For any integer $k_0 < n-r$, there exists some ample vector E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. ☐

Remark 5.3. In general, $\det E \otimes K_S$ is not an ample line bundle, so the positivity in case **(I)** is stronger than the positivity of $S^k E \otimes \det E$.

Theorem 5.4. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then $E \otimes (\det E)^k \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n+1-r, 2\}$. Moreover, the lower bound is sharp.*

Proof. If $r \geq n-1$, by Theorem 5.2, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Now we consider $1 < r < n-1$. By ([20], Theorem 2.5), $K_S \otimes (\det E)^{n-r}$ is nef except the case $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. It is easy to check

$$S^{r+1} E \otimes K_S \otimes (\det E)^{n-r}$$

is also ample in that case. By Theorem 4.4, $E \otimes (\det E)^{n+1-r} \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Here the lower bound $n+1-r$ is sharp. For any integer $k_0 < n+1-r$, there exists an ample vector bundle E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. ☐

Remark 5.5. In Theorem 5.2 and 5.4, if $r \geq n$, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. If $E = T\mathbb{P}^n$, then $S^2 E \otimes \det E \otimes K_{\mathbb{P}^n}$ is Nakano-positive and dual-Nakano-positive.

Problem: Is $S^2 E \otimes \det E \otimes K_S$ Nakano-positive and dual-Nakano-positive when $r \geq n$? If one knows $S^{n+2} E \otimes K_S$ is ample, or equivalently, $\mathcal{O}_{\mathbb{P}(E^*)}(n+2) \otimes \pi^*(K_S)$ is ample, by Theorem 4.4, $S^2 E \otimes \det E \otimes K_S$ is Nakano-positive and dual-Nakano-positive.

6 Vanishing theorems

The following vanishing theorem is dual to Nakano([29])(see Demailly([9])):

Lemma 6.1 (**Nakano**). *Let E be a vector bundle over a compact complex manifold M. If E is Nakano-positive, then $H^{n,q}(M, E) = 0$ for any $q \geq 1$. If E is dual-Nakano-positive, then $H^{q,n}(M, E) = 0$ for any $q \geq 1$.*

Theorem 6.2. *Let $E, E_1, \cdots, E_\ell$ be vector bundles over an n-dimensional compact complex manifold M. The ranks of them are $r, r_1, \cdots, r_\ell$ respectively. Let L be a line bundle on M.*

(I) *If E is ample and L is nef, then there exists k_0 such that for any $k \geq k_0 = k_0(M, E)$,*

$$H^{p,q}(M, S^k E \otimes L) = 0 \tag{6.1}$$

for any $q \geq 1$ and $p \geq 0$.

(II) *If E is ample, L is nef and $r > 1$, then*

$$H^{n,q}(M, S^k E \otimes (\det E)^2 \otimes K_M \otimes L) = H^{q,n}(M, S^k E \otimes (\det E)^2 \otimes K_M \otimes L) = 0 \tag{6.2}$$

for any $q \geq 1$ and $k \geq \max\{n - r, 0\}$.

(IIII) *If E is ample, L is nef and $r > 1$, then*

$$H^{n,q}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = H^{q,n}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = 0 \tag{6.3}$$

for any $q \geq 1$ and $k \geq \max\{n + 1 - r, 2\}$.

(IV) *Let $r > 1$. If E is ample and L is nef, or E is nef and L is ample, then*

$$H^{n,q}(M, S^m E^* \otimes (\det E)^t \otimes L) = H^{q,n}(M, S^m E^* \otimes (\det E)^t \otimes L) = 0 \tag{6.4}$$

for any $q \geq 1$ and $t \geq r + m - 1$.

(V) *If all E_i are ample and L is nef, or, all E_i are nef and L is ample, then any $k_1 \geq 0, \cdots, k_\ell \geq 0$,*

$$\begin{aligned} & H^{n,q}(M, S^{k_1} E_1 \otimes \cdots \otimes S^{k_\ell} E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) \\ = \; & H^{q,n}(M, S^{k_1} E_1 \otimes \cdots \otimes S^{k_\ell} E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) = 0 \end{aligned}$$

for $q \geq 1$.

Proof. For (**II**), (**III**), (**IV**) and (**V**), the vector bundles in consideration are all Nakano-positive and dual-Nakano-positive. The vanishing theorems follow from Lemma 6.1. For **(I)**, by the following Theorem 6.7, there exists $k_0 = k_0(M, E)$ such that $S^k E \otimes \Lambda^{n-p} T^{1,0} M$ is Nakano-positive for any p. On the other hand

$$H^{p,q}(M, S^k E \otimes L) = H^{n,q}(M, S^k E \otimes L \otimes \Lambda^{n-p} T^{1,0} M) \tag{6.5}$$

By Nakano's vanishing theorem, $H^{p,q}(M, S^k E \otimes L) = 0$ for $q \geq 1$ and $p \geq 0$ if $k \geq k_0$. ☐

Remark 6.3. Part (**V**) can be regarded as a generalization vanishing theorems of Griffiths ([15], Theorem G) and Demailly([10], Theorem 0.2).

The following result generalizes Griffiths' vanishing theorem, see also ([23], Corollary 1.5):

Proposition 6.4. *Assume that $S^{t+kr} E \otimes L$ is ample where r is the rank of E. Then*

$$H^{n,q}(M, S^t E \otimes (\det E)^k \otimes L) = H^{q,n}(M, S^t E \otimes (\det E)^k \otimes L) = 0$$

for any $q \geq 1$.

Proof. By Theorem 4.4, $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive. The results follow by Nakano's vanishing theorem. ☐

If L is an ample line bundle over a compact complex manifold M and F is an arbitrary line bundle over M. By comparing the Chern classes, we know there exists a constant m_0 such that $L^{m_0} \otimes F$ is ample and so it is positive. If E is an ample vector bundle and F is an arbitrary vector bundle, it is easy to see $S^k E \otimes F$ is ample for large k. But, in general, we don't know whether an ample vector bundle carries a Griffiths-positive or Nakano-positive metric. In the following, we will construct Nakano-positive and dual-Nakano-positive metrics on various ample vector bundles.

Lemma 6.5. *If L is an ample line bundle over M and F is an arbitrary vector bundle. Then there exists an integral m_0 such that $L^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive.*

Proof. Let h_0 be a positive metric on L and ω be the curvature of h_0 which is also the Kähler metric fixed on M. For any metric g on F, the curvature R^g has a lower bound in the sense

$$\min_{x \in M} \inf_{u \neq 0} \frac{R^g(u(x), u(x))}{|u(x)|^2} \geq -(m_0 - 1) \tag{6.6}$$

where $u \in \Gamma(M, T^{1,0} M \otimes F)$. The curvature of metric $h^{m_0} \otimes g$ on $L^{m_0} \otimes F$ is given by

$$\widehat{R} = m_0 \omega \cdot g + h_0^m \cdot R^g \tag{6.7}$$

Therefore

$$\widehat{R}(v \otimes u, v \otimes u) \geq |u|^2 h_0^{m_0}(v, v)$$

for any $v \in \Gamma(M, L^{m_0})$ and $u \in \Gamma(M, TM \otimes F)$. ☐

Lemma 6.6. *If E is (dual-)Nakano-positive and F is a nef* line *bundle, then $E \otimes F$ is (dual-)Nakano-positive.*

Proof. Fix a Kähler metric on M. Let g be a Nakano-positive metric on E, then there exists $2\varepsilon > 0$ such that

$$R^g(u(x), u(x)) \geq 2\varepsilon |u(x)|^2$$

for any $u \in \Gamma(M, T^{1,0}M \otimes E)$. On the other hand, by a result of [11], there exists a metric h_0 on the nef line bundle F such that

$$R^{h_0} \geq -\varepsilon \omega h_0 \tag{6.8}$$

The curvature of $g \otimes h_0$ on $E \otimes F$ is

$$\widehat{R} = R^g \cdot h_0 + g \cdot R^{h_0}$$

For any $u \in \Gamma(M, T^{1,0}M \otimes E)$ and $v \in \Gamma(M, F)$

$$\widehat{R}(u \otimes v, u \otimes v) \geq \left(R^g(u, u) - \varepsilon |u|^2\right) h_0(v, v) \geq \varepsilon |u|^2 h_0(v, v) \tag{6.9}$$

For dual-Nakano-positivity, the proof is similar. $\square$

Theorem 6.7. *If E is an ample vector bundle over M and F is an arbitrary vector bundle over M, then there exists $k_0 = k_0(M, E, F)$ such that $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

Proof. By Lemma 6.5, there exists m_0 such that $(\det E)^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive. On the other hand, there exists $k_0 = k_0(E, m_0, M)$ such that $\mathcal{O}_{\mathbb{P}(E^*)}(r + k) \otimes \pi^*(\det E^*)^{m_0+1}$ is ample for $k \geq k_0$. It is equivalent to the ampleness of vector bundle $S^{r+k}E \otimes (\det E^*)^{m_0+1}$. By Theorem 4.4, $S^kE \otimes (\det E^*)^{m_0}$ is Nakano-positive and dual-Nakano-positive. Since the tensor product of two (dual-)Nakano-positive vector bundles is (dual-)Nakano-positive, $S^kE \otimes F = (S^kE \otimes (\det E^*)^{m_0}) \otimes ((\det E)^{m_0} \otimes F)$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$. $\square$

Theorem 6.8. *If E is ample over M and F is an arbitrary vector bundle over M, then there exists $k_0 = k_0(M, E, F)$ such that*

$$H^{p,q}(M, S^kE \otimes F) = 0 \tag{6.10}$$

for $q \geq 1$ and $p \geq 0$ if $k \geq k_0$.

Proof. By Theorem 6.7, there exists $k_0 = k_0(M, E, F)$ such that $S^kE \otimes F \otimes \Lambda^{n-p}T^{1,0}M$ is Nakano-positive for any p. On the other hand

$$H^{p,q}(M, S^kE \otimes F) = H^{n,q}(M, S^kE \otimes F \otimes \Lambda^{n-p}T^{1,0}M) \tag{6.11}$$

By Nakano vanishing theorem, $H^{p,q}(M, S^kE \otimes F) = 0$ for $q \geq 1$ and $p \geq 0$ if $k \geq k_0$. $\square$

7 Comparison of Griffiths-positive and Nakano-positive metrics

Let (E, h) be a Hermitian vector bundle. In general, it is very difficult to write down the exact curvature formula about $(S^k E, S^k h)$. In this section, we give an algorithm to compute the curvature of $(S^k E, S^k h)$. As applications, we can disprove the Griffiths-positivity and Nakano-positivity of a given metric on $\mathbb{P}^n$.

Let ω_{FS} be the standard Fubini-Study metric on $\mathbb{P}^{n-1}$ and $[W_1, \cdots W_n]$ the homogeneous coordinates on $\mathbb{P}^{n-1}$. If $\alpha = (\alpha_1, \cdots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$, we define the generalized Kronecker δ for multi-index by the following formula

$$\delta_{\alpha\beta} = \sum_{\sigma \in S_k} \prod_{j=1}^{k} \delta_{\alpha_{\sigma(j)}\beta_{\sigma(j)}} \tag{7.1}$$

where S_k is the permutation groups in k symbols.

Lemma 7.1. *If $V_\alpha = W_{\alpha_1} \cdots W_{\alpha_k}$ and $V_\beta = W_{\beta_1} \cdots W_{\beta_k}$, then*

$$\int_{\mathbb{P}^{n-1}} \frac{V_\alpha \overline{V}_\beta}{|W|^{2k}} \frac{\omega_{FS}^{n-1}}{(n-1)!} = \frac{\delta_{\alpha\beta}}{(n+k-1)!} \tag{7.2}$$

For simple index notations,

$$\int_{\mathbb{P}^{n-1}} \frac{W_\alpha \overline{W}_\beta}{|W|^2} \frac{\omega_{FS}^{n-1}}{(n-1)!} = \frac{\delta_{\alpha\beta}}{n!}, \qquad \int_{\mathbb{P}^{n-1}} \frac{W_\alpha \overline{W_\beta} W_\gamma \overline{W_\delta}}{|W|^4} \frac{\omega_{FS}^{n-1}}{(n-1)!} = \frac{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{(n+1)!} \tag{7.3}$$

$$\int_{\mathbb{P}^{n-1}} \frac{W_\alpha \overline{W_\beta} W_\gamma \overline{W_\delta} W_\lambda \overline{W}_\mu}{|W|^6} \frac{\omega_{FS}^{n-1}}{(n-1)!} = \frac{\delta_{\alpha\beta}\left(\delta_{\gamma\delta}\delta_{\lambda\mu} + \delta_{\gamma\mu}\delta_{\lambda\delta}\right) + \delta_{\alpha\delta}\left(\delta_{\gamma\beta}\delta_{\lambda\mu} + \delta_{\gamma\mu}\delta_{\lambda\beta}\right) + \delta_{\alpha\mu}\left(\delta_{\gamma\beta}\delta_{\lambda\delta} + \delta_{\gamma\delta}\delta_{\lambda\beta}\right)}{(n+2)!} \tag{7.4}$$

Let h be a Hermitian metric on E, h^L be the induced metric (see 2.7) on $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let F be a line bundle with Hermitian metric h^F. Naturally, there is a metric $S^k h \otimes h^F$ on the vector bundle $S^k E \otimes F$. On the other hand, we can construct a new metric f on $S^k E \otimes F$ by formula 3.3. There is a **canonical way** to construct it. Let $\widetilde{L} = L^k \otimes \pi^*(F)$. The induced metric on $\widetilde{L}$ is $h_0 = (h^L)^k \otimes \pi^*(h^F)$ and the induced metric on $\det(T_{X/S})$ is $(h^L)^r \otimes \pi^*(\det(h)^{-1})$. These two metrics induce a metric $\lambda_0 = (h^L)^{k+r} \otimes \pi^*\left(\det(h)^{-1}\right)$ on $\widetilde{L} \otimes \det(T_{X/S})$. Now we could polarize each fiber X_s by the curvature of λ_0. By formula 2.8, when the curvature R^{λ_0} is restricted to a fiber X_s, the Kähler form is

$$\omega_s = \frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log \lambda_0 = \frac{(k+r)\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) = (k+r)\omega_{FS}$$

Now we can use $(\widetilde{L}, h_0)$ and (X_s, ω_s) to construct a "new" metric f on $S^k E \otimes F$ by formula 3.3.

Theorem 7.2. *The metric constructed in Theorem 4.4 is*

$$f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F \tag{7.5}$$

Proof. Without loss generality, we choose the normal coordinates for metric h at a fix point $s \in S$. The metric in Theorem 4.4 turns to be

$$f_{\alpha\overline{\beta}} = (k+r)^{r-1} h^F \int_{\mathbb{P}^{r-1}} \frac{V_\alpha \overline{V}_\beta}{|W|^{2k}} \frac{\omega_{FS}^{r-1}}{(r-1)!}$$

Here V_α, V_β are homogeneous monomials of degree k in $W_1, \cdots, W_r$. By Lemma 7.1,

$$f_{\alpha\beta} = \frac{(r+k)^{r-1}}{(r+k-1)!} \delta_{\alpha\beta} h^F$$

that is $f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F$. ☐

Theorem 7.3. *If (E, h) is a Griffiths-positive vector bundle, then*

(I) *$(S^k E \otimes (\det E)^\ell, S^k h \otimes (\det h)^\ell)$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$ and $\ell \geq 1$.*

(II) *There exists $k_0 = k_0(M, E)$ such that $(S^k E, S^k h)$ is Nakano-positive and dual-Nakano-positive.*

Proof. These follow by Theorem 4.4 and Theorem 7.2. ☐

Remark 7.4. The Nakano-positivity of $(E \otimes \det E, h \otimes \det h)$ was firstly proved by Demailly and Skoda in [12] where they used a discrete Fourier transformation method.

Corollary 7.5. *Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$, then*

(I) *$(S^{n+1} T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^{n+1} h_{FS} \otimes \det(h_{FS})^{-1})$ is semi-Griffiths-positive for $n \geq 2$.*

(II) *$(S^k T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^k h_{FS} \otimes \det(h_{FS})^{-1})$ is Griffiths-positive for any $k \geq n+2$.*

(III) *$(S^k T\mathbb{P}^n, S^k h_{FS})$ is Nakano-positive and dual-Nakano-positive for any $k \geq 2$.*

Proof. The curvature of $E = T\mathbb{P}^n$ with respect to the standard Fubini-Study metric h_{FS} is

$$R_{i\overline{j}k\overline{\ell}} = h_{i\overline{j}} h_{k\overline{\ell}} + h_{i\overline{\ell}} h_{k\overline{j}} \tag{7.6}$$

Without loss generality, we assume $h_{i\overline{j}} = \delta_{ij}$ at a fixed point, then

$$R_{i\overline{j}k\overline{\ell}} u^{ik} \overline{u}^{j\ell} = \frac{1}{2} \sum_{j,k} |u^{jk} + u^{kj}|^2 \tag{7.7}$$

which means that (E, h_{FS}) is semi-Nakano-positive but not Nakano-positive. By Theorem 7.2, $(S^{n+1} T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^{n+1} h_{FS} \otimes \det(h_{FS})^{-1})$ is semi-Griffiths-positive. **(II)** follows by the identity

$$S^k T\mathbb{P}^n \otimes K_{\mathbb{P}^n} = S^k (T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(k-n-1)$$

and semi-Griffiths positivity of $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$.
(III) By Theorem 4.4, the canonically induced metric f is Nakano-positive and dual-Nakano-positive. On the other hand, by Theorem 7.2, f is a constant multiply of $S^k h_{FS}$. The lower bound of k follows from **(II)**. ☐

Proposition 7.6. **(I)** *(E,h) is Griffiths-positive if and only if (S^kE, S^kh) is Griffiths-positive for some $k \geq 1$.*

(II) *If (E,h) is (dual-)Nakano-positive, then (S^kE, S^kh) is (dual-)Nakano-positive for any $k \geq 1$.*

Proof. For the convenience of the reader, we assume $k = 2$ at first. We could choose normal coordinates at a fixed point. Let $\{e_1, \cdots, e_r\}$ be the local basis at that point. The ordered basis of S^2E at that point are $\{e_1 \otimes e_1, e_1 \otimes e_2, \cdots, e_r \otimes e_{r-1}, e_r \otimes e_r\}$. We denote them by $e_{(\alpha,\beta)} = e_\alpha \otimes e_\beta$ with $\alpha \leq \beta$. The curvature tensor S^2h is

$$R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} = R_{i\bar{j}\alpha\bar{\beta}}\delta_{\gamma\delta} + R_{i\bar{j}\gamma\bar{\delta}}\delta_{\alpha\beta} + R_{i\bar{j}\gamma\bar{\beta}}\delta_{\alpha\delta} + R_{i\bar{j}\alpha\bar{\delta}}\delta_{\gamma\beta} \tag{7.8}$$

where $R_{i\bar{j}\alpha\bar{\beta}}$ is the curvature tensor of E. Let $u = \sum\limits_i \sum\limits_{\alpha\leq\gamma} u_{i(\alpha,\gamma)}e_{(\alpha,\gamma)} \in \Gamma(M, T^{1,0}M \otimes S^2E)$. For simplicity of notations, we extend the values of $u_{i(\alpha,\gamma)}$ to all indices (α,γ) by setting $u_{i(\alpha,\gamma)} = 0$ if $\gamma < \alpha$. Therefore

$$\begin{aligned}
\sum_{i,j}\sum_{\substack{\alpha\leq\gamma\\ \beta\leq\delta}} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}}u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\delta)} &= \sum_{i,j}\sum_{\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}}u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\delta)} \\
&= \sum \big(R_{i\bar{j}\alpha\bar{\beta}}u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\gamma)} + R_{i\bar{j}\gamma\bar{\delta}}u_{i(\alpha,\gamma)}\overline{u}_{j(\alpha,\delta)} \\
&\quad + R_{i\bar{j}\gamma\bar{\beta}}u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\alpha)} + R_{i\bar{j}\alpha\bar{\delta}}u_{i(\alpha,\gamma)}\overline{u}_{j(\gamma,\delta)}\big) \\
&= \sum_\gamma \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}}\left(u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)}\right)\overline{\left(u_{j(\beta,\gamma)} + u_{j(\gamma,\beta)}\right)}
\end{aligned} \tag{7.9}$$

Hence (S^2E, S^2h) is Nakano-positive if (E,h) is Nakano-positive. For the general case, we set $A = (\alpha_1, \cdots, \alpha_k)$ and $B = (\beta_1, \cdots, \beta_k)$ with $\alpha_1 \leq \cdots \leq \alpha_k$ and $\beta_1 \leq \cdots \leq \beta_k$. The basis of S^kE are $\{e_A = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}\}$. In Theorem 4.4, all inequalities are identities, so by Lemma 7.1, the curvature tensor of (S^kE, S^kh) is

$$R_{i\bar{j}A\overline{B}} = \sum_{\alpha,\beta=1}^{r}\sum_{s,t=1}^{k} R_{i\bar{j}\alpha\bar{\beta}}\delta_{\alpha\alpha_s}\delta_{\beta\beta_t}\delta_{A_sB_t} \tag{7.10}$$

where $A_s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha_{s+1}, \cdots, \alpha_k)$, $B_t = (\beta_1, \cdots, \beta_{t-1}, \beta_{t+1}, \cdots, \beta_k)$ and $\delta_{A_sB_t}$ is the multi-index delta function(see formula 7.1). By the curvature formula,

$$\sum_{i,j,A,B} R_{i\bar{j}A\overline{B}}u_{iA}\overline{u}_{jB} = \sum_{\alpha_1,\cdots,\alpha_{k-1}}\sum_{\sigma\in S_{k-1}}\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}}V_{i\alpha\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}}\overline{V}_{j\beta\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}} \tag{7.11}$$

where S_{k-1} is the permutation group in $(k-1)$ symbols and

$$V_{i\alpha\alpha_1\cdots\alpha_{k-1}} = \sum_{s=1}^{k} u_{iA^s}, \quad A^s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha, \alpha_{s+1}, \cdots, \alpha_k)$$

With the help of curvature formula 7.10, we could prove Griffiths-positivity and dual-Nakano-positivity of S^kE in a similar way. Here, we use another way to show it. S^kE can be viewed as a quotient bundle of $E^{\otimes k}$. If (E,h) is Griffiths-positive or dual-Nakano-positive, then $(E^{\otimes k}, h^{\otimes k})$ is Griffiths-positive or dual-Nakano-positive and so the quotient bundles are Griffiths-positive or dual-Nakano-positive(see [9]). The induced metrics on quotient bundles are exactly the given ones. $\square$

Remark 7.7. Part (**I**) is an analogue of ampleness: E is ample if and only if S^kE is ample for some $k \geq 1$. The converse of part **(II)** is wrong. We know $(S^2T\mathbb{P}^n, S^2h_{FS})$ is Nakano-positive, but $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive as shown in the following.

Example 7.8. In this example, we will show the Nakano-positivity of $(S^2T\mathbb{P}^2, S^2h_{FS})$ in local coordinates. At a fixed point, we choose the normal coordinates of $T\mathbb{P}^2$. Let $\{e_1, e_2\}$ be the ordered basis of $T\mathbb{P}^2$ at that point. The ordered basis of $S^2T\mathbb{P}^2$ are $e_{(1,1)} = e_1 \otimes e_1, e_{(1,2)} = e_1 \otimes e_2$ and $e_{(2,2)} = e_2 \otimes e_2$. Using the same notation as Proposition 7.6, we set $V_{i\alpha\gamma} = u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)}$ where $u = \sum\limits_i \sum\limits_{\alpha \leq \gamma} u_{i(\alpha,\gamma)} \frac{\partial}{\partial z^i} \otimes e_{(\alpha,\gamma)} \in \Gamma(\mathbb{P}^2, T^{1,0}\mathbb{P}^2 \otimes S^2T\mathbb{P}^2)$. For $\gamma = 1$, the 2×2 matrix $(V_{i\alpha 1})$ has the form

$$T_1 = \begin{pmatrix} 2u_{1(1,1)} & u_{1(1,2)} \\ 2u_{2(1,1)} & u_{2(1,2)} \end{pmatrix}$$

For $\gamma = 2$, the 2×2 matrix $(V_{i\alpha 2})$ is

$$T_2 = \begin{pmatrix} u_{1(1,2)} & 2u_{1(2,2)} \\ u_{2(1,2)} & 2u_{2(2,2)} \end{pmatrix}$$

The total 2×3 matrix $\left(u_{i(\alpha,\beta)}\right)$ is

$$T = \begin{pmatrix} u_{1(1,1)} & u_{1(1,2)} & u_{1(2,2)} \\ u_{2(1,1)} & u_{2(1,2)} & u_{2(2,2)} \end{pmatrix}$$

By formula 7.9 and identity 7.7,

$$\begin{aligned} \sum_{i,j} \sum_{\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\delta)} &= \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 1} \overline{V}_{j\beta 1} + \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 2} \overline{V}_{j\beta 2} \\ &= \frac{1}{2} \sum_{i,\alpha} |V_{i\alpha 1} + V_{\alpha i 1}|^2 + \frac{1}{2} \sum_{i,\alpha} |V_{i\alpha 2} + V_{\alpha i 2}|^2 \end{aligned}$$

It equals zero if and only if T_1 and T_2 are skew-symmetric which means $T \equiv 0$. The Nakano-positivity of $(S^2T\mathbb{P}^2, S^2h_{FS})$ is proved.

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