THE FLABBY CLASS GROUP OF A FINITE CYCLIC GROUP

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ABSTRACT. This is an expository paper on work of Endo and Miyata which leads to a computation of the flabby class group of a finite cyclic group.

1. Introduction

The aim of this paper is to give a proof of the calculation of the flabby class group of a finite cyclic group due to Endo and Miyata [2]. In the next section I will recall the definition and some basic facts about this group. In the final section I will give some examples to show that the invertibility conditions used by Endo and Miyata cannot be removed. I would like to thank M.–c. Kang for useful comments and for showing me his results on some related problems.

2. Some basic results

For convenience I will repeat here the results given in [6, §8] omitting the proofs. Let π be a finite group and let $\mathcal{L}_\pi$ be the class of torsion free finitely generated $\mathbb{Z}\pi$–modules. As usual we refer to such modules as π–lattices. Let $\mathcal{P}erm_\pi$ be the subclass of permutation modules, those having a $\mathbb{Z}$–base permuted by π. Let $\mathcal{P}_\pi$ be the class of invertible modules i.e. direct summands of permutation modules. Define $\mathcal{F}_\pi$ to be the class of π-lattices such that $\mathrm{Ext}^1_{\mathbb{Z}\pi}(F,P)=0$ for all permutation modules P and therefore for all invertible modules P. We refer to the elements F of $\mathcal{F}_\pi$ as flabby (or flasque) modules. Define $\mathcal{C}_\pi$ to be the class of π-lattices such that $\mathrm{Ext}^1_{\mathbb{Z}\pi}(P,C)=0$ for all permutation modules P and therefore for all invertible modules P. We refer to the elements C of $\mathcal{C}_\pi$ as coflabby (or coflasque) modules. Define $M^*=\mathrm{Hom}(M,\mathbb{Z})$. This interchanges $\mathcal{C}_\pi$ and $\mathcal{F}_\pi$ and preserves $\mathcal{P}_\pi$. The notation $H^n(\pi,M)$ will always refer to the Tate cohomology theory [1, Ch. XII] unless otherwise specified.

Lemma 2.1. *A π–lattice M lies in $\mathcal{F}_\pi$ if and only if $H^{-1}(\pi',M)=0$ for all subgroups π' of π and it lies in $\mathcal{C}_\pi$ if and only if $H^1(\pi',M)=0$ for all subgroups π' of π*

Remark 2.2. Since $H^i(\pi',M)\to H^i({\pi'}_p,M)$ is injective on p–torsion where ${\pi'}_p$ is a Sylow p–subgroup of π', it is sufficient to assume $H^i(\pi',M)=0$ for p–subgroups π' in order that $H^i(\pi',M)=0$ for all subgroups π'

Lemma 2.3. $\mathcal{P}_\pi\subseteq\mathcal{C}_\pi\cap\mathcal{F}_\pi$.

Lemma 2.4. *For any π–lattice M there are short exact sequences $0\to M\to P\to F\to 0$ and $0\to C\to Q\to M\to 0$ where P and Q are permutation modules, $F\in\mathcal{F}_\pi$, and $C\in\mathcal{C}_\pi$.*

Corollary 2.5. *Let M be a π–lattice. The following are equivalent:*

(1) *M is invertible.*
(2) $\operatorname{Ext}^1_{\mathbb{Z}\pi}(F, M) = 0$ *for all F in $\mathcal{F}_\pi$.*
(3) $\operatorname{Ext}^1_{\mathbb{Z}\pi}(M, C) = 0$ *for all C in $\mathcal{C}_\pi$.*

Definition 2.6. Define an equivalence relation on $\mathcal{F}_\pi$ by $F_1 \sim F_2$ if and only if we have $F_1 \oplus P_1 \approx F_2 \oplus P_2$ for permutation modules P_1 and P_2. Let F_π be the set of equivalence classes of $F \in \mathcal{F}_\pi$. It is a monoid under direct sum. I will refer to F_π as the flabby class monoid of π.

If M is a π–lattice, choose an exact sequence $0 \to M \to P \to F \to 0$ as in Lemma 2.4 and let $\rho(M)$ be the class $[F]$ of F in F_π.

Lemma 2.7. *$\rho(M)$ is well defined.*

Lemma 2.8. *Let M and N be π–lattices. Then $\rho(M) = \rho(N)$ if and only if there are exact sequences $0 \to M \to E \to P \to 0$ and $0 \to N \to E \to Q \to 0$ with P and Q permutation modules.*

Definition 2.9. If M is a π–module I will write $(M)_0$ for $M/t(M)$ where $t(M)$ is the torsion submodule of M.

Let R be a Dedekind ring and let $\theta : \mathbb{Z}\pi \to R$ be a ring homomorphism. Define $c_\theta : F_\pi \to C(R)$ by sending $[F]$ to $(R \otimes_{\mathbb{Z}\pi} F)_0$. This is a well defined homomorphism of monoids. If π is abelian and A is the integral closure of $\mathbb{Z}\pi$ in $\mathbb{Q}\pi$ then A is a product of Dedekind rings so we get a map $c : F_\pi \to C(A)$. Our object is to prove the following theorem.

Theorem 2.10 (Endo and Miyata)**.** *If π is a finite cyclic group then $c : F_\pi \to C(A)$ is an isomorphism.*

Except for the examples in the final section, the results discussed in this paper are due to Endo and Miyata.

3. Some more useful facts

Lemma 3.1. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of π–lattices with M'' invertible. Then $\rho(M) = \rho(M') + \rho(M'')$ in F_π.*

In section 7 I will show that the hypothesis that M'' is invertible cannot be omitted in general.

Proof. Choose a sequence $0 \to M \to P \to F \to 0$ with P permutation and F flabby. Factoring out M' we get $0 \to M'' \to P/M' \to F \to 0$. This splits since F is flabby and M'' is invertible. Therefore $P/M' \approx F \oplus M''$. It follows that P/M' is flabby since F and M'' are. The sequence $0 \to M' \to P \to P/M' \to 0$ shows that $\rho(M') = [P/M'] = [F] + [M'']$. Since M'' is invertible we can write $M'' \oplus L = Q$ where Q is permutation and the sequence $0 \to M'' \to Q \to L \to 0$ shows that $\rho(M'') = [L]$. Therefore $\rho(M) = [F] = [F] + [Q] = [F] + [M''] + [L] = \rho(M') + \rho(M'')$. □

Lemma 3.2. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be a short exact sequence of $\mathbb{Z}\pi$–modules. If it splits over a Sylow p–subgroup π_p for each prime p then it splits over π.*

Proof. Let $f_p : M'' \to M$ be a π_p–homomorphism such that $jf_p = 1$. Let $\pi = \sqcup \sigma_\nu \pi_p$ be a left coset decomposition and let $g_p(x) = \sum \sigma_\nu f_p(\sigma_\nu^{-1} x)$. Then g_p is a π–homomorphism and $jg_p = |\pi : \pi_p|$. Choose $a_p \in \mathbb{Z}$ such that $\sum a_p |\pi : \pi_p| = 1$. Then $h = \sum a_p g_p$ is the required splitting. □

Lemma 3.3. *If M is a π–lattice which is invertible over each Sylow subgroup π_p then M is invertible.*

Proof. Choose an exact sequence $0 \to M \to P \to F \to 0$ as in Lemma 2.4 with F flabby over π. By Lemma 2.1, F is flabby over all subgroups of π. Therefore the sequence splits over all Sylow subgroups and therefore splits by Lemma 3.2. □

4. Cyclic Groups

We now discuss some results which hold for cyclic groups and more generally for groups whose Sylow subgroups are cyclic.

Lemma 4.1. *If all Sylow subgroups of π are cyclic then $\mathcal{C}_\pi = \mathcal{F}_\pi$*

Proof. By Remark 2.2 it is enough to check this for cyclic groups since all p–subgroups are cyclic. Since the cohomology of a finite cyclic group is periodic with period 2 [1, Ch. XII], we have $H^1(\pi', M) = 0$ if and only if $H^{-1}(\pi', M) = 0$ for cyclic π'. □

Lemma 4.2. *If $f, g \in \mathbb{Z}[x]$ are non–zero then the sequence $0 \to \mathbb{Z}[x]/(f) \to \mathbb{Z}[x]/(fg) \to \mathbb{Z}[x]/(g) \to 0$ is exact.*

Lemma 4.3. *Let π be a finite cyclic p–group of order n with generator x. If M is a finitely generated torsion free module over $\mathbb{Z}\pi/\Phi_n(x) = \mathbb{Z}[\zeta_n]$ then $\rho(M)$ is invertible.*

Proof. Let $n = pq$ and let $\pi'' = \pi/ < x^q >$. The factorization $x^n - 1 = \Phi_n(x)(x^q - 1)$ shows that the sequence $0 \to \mathbb{Z}[\zeta_n] \to \mathbb{Z}\pi \to \mathbb{Z}\pi'' \to 0$ is exact. It follows that $\rho(\mathbb{Z}[\zeta_n]) = [\mathbb{Z}\pi''] = 0$. Since M is projective over $\mathbb{Z}[\zeta_n]$ we can write $M \oplus N = \mathbb{Z}[\zeta_n]^r$ so $\rho(M) + \rho(N) = r\rho(\mathbb{Z}[\zeta_n]) = 0$ showing that $\rho(M)$ is invertible. □

It is clear that an element $[F]$ of F_π has an inverse if and only if F is invertible. Therefore the following theorem characterizes the groups π for which F_π is a group.

Theorem 4.4. *Let π be a finite group. Then $\mathcal{P}_\pi = \mathcal{F}_\pi$ if and only if all Sylow subgroups are cyclic.*

Proof. We know that $\mathcal{P}_\pi \subseteq \mathcal{F}_\pi$ in any case by Lemma 2.3. Suppose all Sylow subgroups of π are cyclic. If M is flabby over π it is flabby over all subgroups of π by Lemma 2.1. By Lemma 3.3 it is enough to show M invertible over the Sylow subgroups so we can assume that π is a cyclic p–group. Write $|\pi| = n$ and $n = pq$ where n and q are powers of p. Let x generate π, let $M' = \{m \in M | \Phi_n(x)m = 0\}$, and write $0 \to M' \to M \to M'' \to 0$. Clearly M'' is torsion–free. Since $x^n - 1 = \Phi_n(x)(x^q - 1)$, $x^q - 1$ annihilates M'' which is therefore a lattice over $\pi'' = \pi/ < x^q >$. We can assume by induction that the theorem holds for π'', the case $n = 1$ being trivial. We claim that M'' is flabby over π i.e. $H^{-1}(\pi', M'') = 0$ for all subgroups π' of π. This is clear for $\pi' = 1$. If $\pi' \neq 1$ then $M'^{\pi'} = 0$ since it is annihilated by $\Phi_n(x)$ and by $x^d - 1$ where $\pi' =< x^d >$, $d < n$, $\gcd(\Phi^n, x^d - 1) = 1$ over $\mathbb{Q}$ and M' is torsion free. It follows that $\widehat{H}^0(\pi', M') = 0$ and the exact cohomology sequence gives $0 = H^{-1}(\pi', M) \to H^{-1}(\pi', M'') \to \widehat{H}^0(\pi', M') = 0$ showing that M'' is flabby over π. It follows that M'' is also flabby over $\pi'' = \pi/ < x^q >$ since a sequence $0 \to P \to E \to M'' \to 0$ over π'' with P permutation has the same properties over π and therefore splits over π and so over π''. Induction now shows that M'' is invertible over π'' and therefore also over π. By Lemma 3.1 we now have $\rho(M) = \rho(M') + \rho(M'')$. Now $\rho(M'') = -[M'']$ is invertible and so is $\rho(M')$ by Lemma 4.3. Therefore $\rho(M)$ is also invertible. Choose a sequence $0 \to M \to P \to F \to 0$ with P permutation and F flabby so that $\rho(M) = [F]$ and therefore F is invertible. By Lemma 4.1, M is coflabby so the sequence splits giving $M \oplus F \approx P$ and showing that M is invertible.

For the converse let I be the augmentation ideal of $\mathbb{Z}\pi$. We will show that if $\rho(I^*)$ is invertible then all Sylow subgroups are cyclic. Let n be the order of π. The cohomology sequence of $0 \to I \to \mathbb{Z}\pi \to \mathbb{Z} \to 0$ shows that $H^1(\pi, I) = \mathbb{Z}/n\mathbb{Z}$. Choose a sequence $0 \to I^* \to P \to F \to 0$ with P permutation and F flabby so that $\rho(I^*) = [F]$. Then $0 \to F^* \to P^* \to I \to 0$ gives $0 = H^1(\pi, P^*) \to H^1(\pi, I) \to H^2(\pi, F^*)$ showing that $H^2(\pi, F^*)$ has an element of order n. If $\rho(I^*)$ is invertible then so is F and therefore so is F^*. Write $F^* \oplus L = Q$ where Q is permutation. Then $H^2(\pi, Q)$ has an element of order n. Let $Q = \oplus \mathbb{Z}\pi/\pi_i$. Then $H^2(\pi, Q) = \oplus H^2(\pi, \mathbb{Z}\pi/\pi_i) = \oplus H^2(\pi_i, \mathbb{Z})$. Let $r = \mathrm{ord}_p(n)$. Some $H^2(\pi_i, \mathbb{Z})$ must have an element of order p^r so the next lemma shows that the Sylow subgroup of π_i is cyclic of order p^r and therefore the same is true of π. □

Lemma 4.5. *Let π be a group of order n, let p be a prime and let q be the highest power of p dividing n. If $H^2(\pi, \mathbb{Z})$ has an element of order divisible by q then the Sylow p–subgroup of π is cyclic of order q.*

Proof. The sequence $0 \to \mathbb{Z} \xrightarrow{q} \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z} \to 0$ gives $0 = H^1(\pi, \mathbb{Z}) \to H^1(\pi, \mathbb{Z}/q\mathbb{Z}) \to H^2(\pi, \mathbb{Z}) \xrightarrow{q} H^2(\pi, \mathbb{Z})$. It follows that $H^1(\pi, \mathbb{Z}/q\mathbb{Z}) = \mathrm{Hom}(\pi, \mathbb{Z}/q\mathbb{Z})$ has an element of order q so there is a map of π onto $\mathbb{Z}/q\mathbb{Z}$. This clearly implies the result. □

5. Devissage

The aim of this section is to prove a devissage theorem which will be the main tool in the proof of the main theorem. The only property of $\rho : \mathcal{L}_\pi \to F_\pi$ which is needed is that of Lemma 3.1 so I will state the theorem more generally for a map $\phi : \mathcal{L}_\pi \to G$ assigning an element of an abelian group G to each $\mathbb{Z}\pi$–lattice and satisfying the following property: If $0 \to M' \to M \to M'' \to 0$ is exact with M'' invertible then $\phi(M) = \phi(M') + \phi(M'')$. For any abelian group M I will write $(M)_0$ for $M/t(M)$ where $t(M)$ is the torsion submodule of M.

Theorem 5.1. *Let $\pi =< x : x^N = 1 >$ be a cyclic group of order N with generator x and let $n|N$. Let M be an invertible $\mathbb{Z}\pi$–lattice. Then*

$$\phi(M/(x^n - 1)M) = \sum_{d|n} \phi((M/\Phi_d(x)M)_0)$$

In section 7 I will show that the hypothesis that M is invertible cannot be omitted in general. By the Möbius inversion formula, this theorem is equivalent to the following result.

Corollary 5.2. *Under the same conditions we have*

$$\phi((M/\Phi_n(x)M)_0) = \sum_{d|n} \mu(\tfrac{n}{d})\phi(M/(x^d-1)M)$$

where μ is the Möbius function.

Note that $M/(x^n-1)M$ will be torsion free by the following simple observation.

Lemma 5.3. *If π acts on a set X and π' is a normal subgroupof π then $\mathbb{Z}\pi/\pi' \otimes_{\mathbb{Z}\pi} \mathbb{Z}[X] = \mathbb{Z}[X/\pi']$.*

It follows that if M is a permutation lattice over $\mathbb{Z}\pi$ then so is $\mathbb{Z}\pi/\pi' \otimes_{\mathbb{Z}\pi} M$. The same is therefore true for invertible lattices.

We also need the following fact which is easily checked.

Lemma 5.4. *Let $M' \xrightarrow{i} M \to M'' \to 0$ be an exact sequence of abelian groups. Suppose that M'' is torsion free and the kernel of i is torsion. Then $0 \to (M')_0 \to (M)_0 \to M'' \to 0$ is exact.*

Lemma 5.5. *Let π be a cyclic group of order N with generator x. Let M be a $\mathbb{Z}\pi$–module. Let $n|N$ and let $f, g \in \mathbb{Z}[X]$ be such that $fg|x^n-1$. If $M/g(x)M$ is torsion–free then $0 \to (M/f(x)M)_0 \to (M/f(x)g(x)M)_0 \to M/g(x)M \to 0$ is exact.*

Proof. By Lemma 4.2 the sequence $0 \to \mathbb{Z}[X]/(f) \to \mathbb{Z}[X]/(fg) \to \mathbb{Z}[X]/(g) \to 0$ is exact. Since $fg|x^N-1$, this sequence is the same as $0 \to \mathbb{Z}\pi/(f(x)) \to \mathbb{Z}\pi/(f(x)g(x)) \to \mathbb{Z}\pi/(g(x)) \to 0$. Applying $-\otimes_{\mathbb{Z}\pi} M$ shows that $\mathrm{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/g(x), M) \to M/f(x)M \to M/f(x)g(x)M \to M/g(x)M \to 0$ is exact. Since the Tor term is torsion and $M/g(x)M$ is torsion–free, Lemma 5.4 applies. □

We now turn to the proof of Theorem 5.1. Let $p_1, \ldots, p_r$ be the distinct prime divisors of n. Construct a sequence $d_0, d_1, \ldots, d_{2^r-1}$ as follows. Let $d_0 = 1$ and $d_1 = p_1$. If $d_0, d_1, \ldots, d_{2^{s-1}-1}$ have been defined for $s \geq 1$ (using $p_1, \ldots, p_{s-1}$) let $d_\nu = p_s d_{2^s-\nu-1}$ for $2^{s-1} \leq \nu \leq 2^s - 1$. Therefore $d_{2^{s-1}}, \ldots, d_{2^s-1}$ is $p_s d_{2^{s-1}-1}, \ldots, p_s d_1, p_s d_0$. Note that d_ν is squarefree since the p_i are distinct. Also the number of primes dividing d_ν is congruent to ν modulo 2. (If this holds for $\nu \leq 2^{s-1}-1$ then for $2^{s-1} \leq \nu \leq 2^s - 1$, the number of prime divisors of d_ν is 1 more than that of $d_{2^s-\nu-1}$ and so is congruent to $1 + 2^s - \nu - 1 \equiv \nu \mod 2$.) It follows that $\mu(d_\nu) = (-1)^\nu$.

Let $e_\nu = n/d_\nu$ and define $f_k(X) = \prod_{\nu=0}^{k}(X^{e_\nu}-1)^{(-1)^\nu}$ for $k = 0, \ldots, 2^r-1$. We also set $f_{-1}(X) = 1$

Lemma 5.6.

1. *f_k is monic and lies in $\mathbb{Z}[X]$.*
2. *$f_{2k}(X) = f_{2k-1}(X)(X^{e_{2k}}-1) = f_{2k+1}(X)(X^{e_{2k+1}}-1)$ for $0 \leq k \leq 2^{r-1}-1$.*
3. *$f_k|(X^n-1)$, $f_0 = X^n - 1$, and $f_{2^r-1} = \Phi_n(X)$*

Proof. The irreducible factors occurring in f_k are cyclotomic polynomials Φ_h for certain $h|n$. Fix such an h. Since $\mathrm{ord}_{\Phi_h}(x^e-1)$ is 1 if $h|e$ and 0 if not, we see that $\mathrm{ord}_{\Phi_h} f_k = \epsilon_0 + \cdots + \epsilon_k$ where $\epsilon_\nu = (-1)^\nu$ if $h|e_\nu$ or, equivalently, $d_\nu|\frac{n}{h}$ and $\epsilon_\nu = 0$ otherwise. Clearly $\epsilon_0 = 1$. We claim that the sequence $\epsilon_0, \ldots, \epsilon_{2^r-1}$ is either $1, 0, 0, \ldots, 0$ or consists of alternate $+1$'s and -1's with 0's between. Suppose this is true for $\epsilon_0, \ldots, \epsilon_{2^{s-1}-1}$ where $s \geq 1$. If p_s does not divide $\frac{n}{h}$ then d_ν does not divide $\frac{n}{h}$ for $2^{s-1} \leq \nu \leq 2^s - 1$ so $\epsilon_0, \ldots, \epsilon_{2^s-1}$ is $\epsilon_0, \ldots, \epsilon_{2^{s-1}-1}, 0, 0, \ldots, 0$ which has the required form. If p_s does divide $\frac{n}{h}$ then, for $2^{s-1} \le \nu \le 2^s - 1$, d_ν divides $\frac{n}{h}$ if and only if $d_{2^s-\nu-1}$ divides $\frac{n}{h}$. It follows that $\epsilon_\nu = -\epsilon_{2^s-\nu-1}$ so $\epsilon_0, \dots, \epsilon_{2^s-1}$ is $\epsilon_0, \dots, \epsilon_{2^{s-1}-1}, -\epsilon_{2^{s-1}-1}, \dots - \epsilon_1, -\epsilon_0$ which again clearly has the required form. It follows that $\mathrm{ord}_{\Phi_h} f_k$ is either 0 or 1 showing that f_k is a monic polynomial and divides $X^n - 1$. The preceding argument also shows that $\sum_0^{2^s-1} \epsilon_\nu = 0$ if $\sum_0^{2^{s-1}-1} \epsilon_\nu = 0$ or if p_s divides $\frac{n}{h}$. Therefore $\sum_0^{2^r-1} \epsilon_\nu = 0$ unless no p_s divides $\frac{n}{h}$ so that $h = n$. This shows that $f_{2^r-1} = \Phi_n(X)$. The remaining assertions are obvious. □

Now let $\pi =< x : x^N = 1 >$ be a cyclic group of order N with generator x and let $n|N$. Let M be an invertible $\mathbb{Z}\pi$–lattice. Define $M_k = (M/f_k(x)M)_0$, set $M_{-1} = 0$, and let $Q_k = M/(x^{e_k} - 1)M$. Then $M_0 = M/(x^n - 1)M = Q_0$, $M_{2^r-1} = (M/\Phi_n(x)M)_0$ and Q_k is invertible for all k by Lemma 5.3. In particular, Q_k is torsion–free. By Lemma 5.6 and Lemma 5.5 we get short exact sequences $0 \to M_{2k-1} \to M_{2k} \to Q_{2k} \to 0$ and $0 \to M_{2k+1} \to M_{2k} \to Q_{2k+1} \to 0$ for $0 \le k \le 2^{r-1} - 1$. By the additivity property assumed for our function ϕ we get $\phi(M_{2k}) = \phi(M_{2k-1}) + \phi(Q_{2k}) = \phi(M_{2k+1}) + \phi(Q_{2k+1})$. It follows that $\phi(M_{2k+1}) - \phi(M_{2k-1}) = \phi(Q_{2k}) - \phi(Q_{2k+1})$. Summing from $k = 0$ to $k = 2^{r-1} - 1$ and using $M_{-1} = 0$ we get $\phi(M_{2^r-1}) = \sum_{\nu=0}^{2^r-1} (-1)^\nu \phi(Q_\nu)$. Since the d_ν run over all squarefree divisors of n, $\mu(d_\nu) = (-1)^\nu$ and $\mu(d) = 0$ if d is not squarefree we can write this as $\phi(M/\Phi_n(x)M) = \sum_{d|n} \mu(d)\phi(M/(x^{\frac{n}{d}} - 1)M) = \sum_{d|n} \mu(\frac{n}{d})\phi(M/(x^d - 1)M)$. This proves Corollary 5.2 and Theorem 5.1 follows by the Möbius inversion formula.

6. Proof of the main theorem

Let $\pi =< x : x^N = 1 >$ be a cyclic group of order N with generator x. Let A be the integral closure of $\mathbb{Z}\pi$ in $\mathbb{Q}\pi$. Then $A = \prod_{n|N} R_n$ where $R_n = \mathbb{Z}[\zeta_n]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_n)$ and x maps to (ζ_n). The map $c : F_\pi \to C(A) = \prod_{n|N} C(R_d)$ sends $[F]$ to $(R_n \otimes_{\mathbb{Z}\pi} F)_0$. Therefore if $c(F) = 0$ then all $(R_n \otimes_{\mathbb{Z}\pi} F)_0$ are free. Now, by Corollary 5.2, we have $\rho(R_n) = \sum_{d|n} \mu(\frac{n}{d})\rho(\mathbb{Z}\pi/(x^d - 1)) = 0$ since all $\mathbb{Z}\pi/(x^d - 1)$ are permutation modules. Therefore it follows that $\rho((R_n \otimes_{\mathbb{Z}\pi} F)_0) = 0$ since $(R_n \otimes_{\mathbb{Z}\pi} F)_0$ is free over R_n. By Theorem 5.1 we have $\rho(F) = \sum_{n|N} \rho((R_n \otimes_{\mathbb{Z}\pi} F)_0) = 0$. Since F is invertible we can write $F \oplus G = P$ where P is permutation and the sequence $0 \to F \to P \to G \to 0$ shows that $\rho(F) = [G] = -[F]$ and it follows that $[F] = 0$ showing that $c : F_\pi \to C(A)$ is injective.

To see that c is onto we consider the map $C(\mathbb{Z}\pi) \to F_\pi$ sending $[P] - [Q]$ in $C(\mathbb{Z}\pi)$ to $[P] - [Q]$ in F_π. Since P is projective, $R_n \otimes_{\mathbb{Z}\pi} P$ is torsion free, being projective over R_n. Therefore the composition $C(\mathbb{Z}\pi) \to F_\pi \to C(A)$ is the canonical map sending $[P] - [Q]$ to $[A \otimes_{\mathbb{Z}\pi} P] - [A \otimes_{\mathbb{Z}\pi} Q]$. Since this map is onto by the next (well–known) lemma it follows that $F_\pi \to C(A)$ is onto and therefore an isomorphism. It also follows that $C(\mathbb{Z}\pi)/\tilde{C}(\mathbb{Z}\pi) \approx F_\pi$ where $\tilde{C}(\mathbb{Z}\pi)$ is the kernel of $C(\mathbb{Z}\pi) \to C(A)$.

Lemma 6.1. *$C(\mathbb{Z}\pi) \to C(A)$ is onto.*

Proof. Let $[P] - [Q]$ lie in $C(A)$. Find a sequence $0 \to P \to Q \to X \to 0$ where X is finite of order prime to the index of $\mathbb{Z}\pi$ in A. Let $0 \to S \to F \to X \to 0$ be a resolution of X over $\mathbb{Z}\pi$ with F free. Then S is projective and $[F] - [S]$ maps to $[P] - [Q]$ in $C(A)$. This follows by tensoring A with $0 \to S \to F \to X \to 0$ over

$\mathbb{Z}\pi$ and applying Schanuel's lemma. Note that $A \otimes_{\mathbb{Z}\pi} X \xrightarrow{\approx} X$ since locally either $\mathbb{Z}\pi = A$ or $X = 0$, and $0 \to A \otimes_{\mathbb{Z}\pi} S \to A \otimes_{\mathbb{Z}\pi} F \to A \otimes_{\mathbb{Z}\pi} X \to 0$ is exact for the same reason. (This also follows from the fact that $A \otimes_{\mathbb{Z}\pi} S$ is torsion free.) □

In the next section we will also need the following observation.

Corollary 6.2. *The map $\rho : C(A) \to F_\pi$ is an isomorphism.*

Proof. Let M be a flabby π–lattice. Then M is invertible. By Theorem 5.1 (with n being the order of π) we see that $\rho(M)$ lies in the image of $\rho : C(A) \to F_\pi$. Since $\rho(M) = -[M]$ in F_π this shows that $\rho : C(A) \to F_\pi$ is onto. Since $C(A) \approx F_\pi$ by Theorem 2.10 it follows that $\rho : C(A) \to F_\pi$ is an isomorphism. □

7. Examples

In this section I will give examples to show that the invertibility conditions in Lemma 3.1 and Theorem 5.1 cannot be omitted.

As above let A be the integral closure of $\mathbb{Z}\pi$ in $\mathbb{Q}\pi$. If π has order n then $A = \prod_{d|n} R_d$ where $R_d = \mathbb{Z}[\zeta_d]$ with $\zeta_d = e^{\frac{2\pi i}{d}}$ and the generator σ of π maps to ζ_d in R_d. The following lemma will be useful.

Lemma 7.1. *Let R be a Dedekind ring and let $A \supseteq R$ be a domain containing R and finite over R. Let I be an ideal of A such that $A = R + I$ i.e. $R/J = A/I$ where $J = R \cap I$. If I is principal then so is J.*

Proof. We can assume $I \neq 0$. Since I is principal, we have an exact sequence $0 \to A \to A \to A/I \to 0$. Regarding this as a sequence over R and noting that A is projective as an R–module we see that $[A/I] = [A] - [A] = 0$ in $K_0(R)$. But $A/I \approx R/J$ so $[R/J] = [R] - [J] = 0$. Since cancellation holds for finitely generated projective modules over a Dedekind ring we have $J \approx R$. □

The following lemma is an extension of the example considered in [5].

Lemma 7.2. *Let q be a prime such that R_q contains a non–principal prime ideal $\mathfrak{p}$ whose norm is a prime p. Let $\mathfrak{P}$ be a prime ideal of R_{pq} extending $\mathfrak{p}$. Then $\mathfrak{P}$ is also non–principal and of norm p so that $R_q/\mathfrak{p} \to R_{pq}/\mathfrak{P}$ is an isomorphism.*

Proof. We have $R_{pq} = R_q[\zeta_p] = R_q[x]/\Phi_p(x)$ so $R_{pq}/\mathfrak{p}R_{pq} = \mathbb{F}_p[x]/\Phi_p(x) = \mathbb{F}_p[x]/(x-1)^{p-1}$. This is local with residue field $\mathbb{F}_p$ so there is a unique prime ideal $\mathfrak{P}$ of R_{pq} lying over $\mathfrak{p}$ and it has residue field $\mathbb{F}_p$ and $\mathfrak{P}$ is not principal otherwise Lemma 7.1 would imply that $\mathfrak{p}$ was principal. □

Remark 7.3. Standard density theorems show that such an ideal $\mathfrak{p}$ will exist whenever R_q has class number $h \neq 1$. An explicit example used in [5] is given by $q = 23$ and $p = 47$. This is proved as follows: Since $p \equiv 1 \mod q$, $\mathbb{F}_p$ contains a primitive q–th root of 1 so R_q has a prime ideal $\mathfrak{p}$ with $R_q/\mathfrak{p} \approx \mathbb{F}_p$. To see that $\mathfrak{p}$ is non–principal let $B = \mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$ which is a subring of R_q. If $\mathfrak{p}$ was principal then Lemma 7.1 would imply that $\mathfrak{p} \cap B$ was principal but it is easy to check that there is no element of B with norm p so this is impossible. An similar argument is given in [5].

We make $R_{pq}/\mathfrak{P}$ into a $\mathbb{Z}\pi$–module by $\mathbb{Z}\pi \to R_{pq} \to R_{pq}/\mathfrak{P}$ and make $R_q/\mathfrak{p}$ into a $\mathbb{Z}\pi$–module by $\mathbb{Z}\pi \to R_q \to R_q/\mathfrak{p}$. Our examples are based on the following observation.

Lemma 7.4. *Let p and q be as in Lemma 7.2 and let π be cyclic of order pq. Then the natural map $R_q/\mathfrak{p} \to R_{pq}/\mathfrak{P}$ is a $\mathbb{Z}\pi$–isomorphism.*

Proof. We can identify $R_q/\mathfrak{p}$ and $R_{pq}/\mathfrak{P}$ with $\mathbb{F}_p$ so that $R_q \to \mathbb{F}_p$ is the restriction of $R_{pq} \to \mathbb{F}_p$. The map $\mathbb{Z}\pi \to R_{pq} \to R_{pq}/\mathfrak{P}$ sends σ to the image ξ of ζ_{pq} in $\mathbb{F}_p$ while the map $\mathbb{Z}\pi \to R_q \to R_q/\mathfrak{p}$ sends σ to the image η of ζ_q in $\mathbb{F}_p$. Since $\zeta_q = \zeta_{pq}^p$ this shows that $\eta = \xi^p$. Since $\mathbb{F}_p$ satisfies the identity $x^p = x$, we have $\eta = \xi$ so the two maps are the same. □

Now $K_0(A) = \prod_{d|n} K_0(R_d)$. Since R_d is a Dedekind ring, $K_0(R_d) = \mathbb{Z} \oplus C(R_d)$ where the class $(\mathfrak{a})$ in $C(R_d)$ corresponds to $[R_d] - [\mathfrak{a}]$ in $K_0(R_d)$. So $K_0(A) = C(A) \oplus F$ where $F = \prod_{d|n} \mathbb{Z}$ is free abelian and $C(A) = \prod_{d|n} C(R_d)$.

Let $G_0(\mathbb{Z}\pi)$ be the Grothendieck group having generators $[M]$ for all finitely generated π–modules M with relations $[M] = [M'] + [M'']$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$. Define $K_0(A) \to G_0(\mathbb{Z}\pi)$ by sending $[M]$ to $[M]$ with M considered as a $\mathbb{Z}\pi$–module. This gives us a map $C(A) \to G_0(\mathbb{Z}\pi)$.

Theorem 7.5 ([4, Corollary 6.1])**.** *Let p and q be as in Lemma 7.2 and let π be cyclic of order pq. Then $C(A) \to G_0(\mathbb{Z}\pi)$ is not injective.*

Proof. We have $C(A) = C(\mathbb{Z}) \times C(R_p) \times C(R_q) \times C(R_{pq})$. The class of $\mathfrak{P}$ lies in $C(R_{pq})$ while the class of $\mathfrak{p}$ lies in $C(R_q)$. The images of these elements in $G_0(\mathbb{Z}\pi)$ are $[R_{pq}] - [\mathfrak{P}] = [R_{pq}/\mathfrak{P}]$ and $[R_q] - [\mathfrak{p}] = [R_q/\mathfrak{p}]$. By Lemma 7.4, these elements are the same and the lemma follows. □

Corollary 7.6. *Let p and q be as in Lemma 7.2 and let π be cyclic of order pq. Then $\rho : \mathcal{L}_\pi \to F_\pi$ does not satisfy $\rho(M) = \rho(M') + \rho(M'')$ for all short exact sequences of π–lattices.*

Proof. If this condition was satisfied then $\rho : \mathcal{L}_\pi \to F_\pi$ would factor through $G_0(\mathbb{Z}\pi)$ and therefore so would $\rho : C(A) \to F_\pi$. Since $C(A) \to G_0(\mathbb{Z}\pi)$ is not injective, this contradicts Corollary 6.2. □

If f is an endomorphism of a module M we write ${}_fM$ for the set of elements of M annihilated by f. Let π be a finite cyclic group of order n with generator σ. If M is a $\mathbb{Z}\pi$–lattice then ${}_{\Phi_n(\sigma)}M$ is the largest R_n–lattice contained in M and $(M/\Phi_n(\sigma))_0$ is the largest R_n–lattice which is a quotient of M. We give an example to show that these two R_n–lattices need not be isomorphic.

Lemma 7.7. *Let π be a finite cyclic group of order n with generator σ. Suppose $X^n - 1 = f(X)g(X)$ in $\mathbb{Z}[X]$ Then ${}_{f(\sigma)}\mathbb{Z}\pi = g(\sigma)\mathbb{Z}\pi$*

Proof. Clearly $g(\sigma)M \subseteq {}_{f(\sigma)}M$ for any $\mathbb{Z}\pi$–module M. Suppose $f(\sigma)h(\sigma) = 0$ in $\mathbb{Z}\pi$. Then $f(X)h(X) = (X^n - 1)k(X) = f(X)g(X)k(X)$ in $\mathbb{Z}[X]$ so $h(X) = g(X)k(X)$ □

Theorem 7.8. *Let p and q be as in Lemma 7.2 and let π be cyclic of order $n = pq$. Then there is a $\mathbb{Z}\pi$–lattice I such that ${}_{\Phi_n(\sigma)}I$ is not isomorphic to $(I/\Phi_n(\sigma)I)_0$.*

Proof. Let I be the kernel of the map $\mathbb{Z}\pi \to R_n \to R_n/\mathfrak{P} = \mathbb{F}_p$. Then $(I/\Phi_n(\sigma)I)_0 \approx \mathfrak{P}$ since the sequence $0 \to I \to \mathbb{Z}\pi \to \mathbb{F}_p \to 0$ gives an exact sequence $I/\Phi_n(\sigma) \to R_n \to \mathbb{F}_p \to 0$ and the kernel of the left hand map is torsion since $\mathbb{Q}I = \mathbb{Q}\mathbb{Z}\pi$. On the other hand, ${}_{\Phi_n(\sigma)}I \approx R_n$ since we have an exact sequence $0 \to {}_{\Phi_n(\sigma)}I \to {}_{\Phi_n(\sigma)}\mathbb{Z}\pi \to \mathbb{F}_p$ and, by Lemma 7.7, ${}_{\Phi_n(\sigma)}\mathbb{Z}\pi = \Psi(\sigma)\mathbb{Z}\pi$ where $\Psi(X) = (X^n -$

$1)/\Phi_n(X)$. By Lemma 7.7, we see that $\Psi(\sigma) : \mathbb{Z}\pi \to \Psi(\sigma)\mathbb{Z}\pi$ has kernel $\Phi_n(\sigma)\mathbb{Z}\pi$ so $\Psi(\sigma)\mathbb{Z}\pi \approx R_n$ and $\Psi(\sigma)$ maps to zero in $\mathbb{F}_p$. The last statement follows from the fact that $\Psi = \Phi_1\Phi_p\Phi_q = (X^q - 1)\Phi_p$ so $\Psi(\zeta_n) = (\zeta_p - 1)\Phi_p(\zeta_n)$ but ζ_p maps to 1 in $\mathbb{F}_p$. □

We now show that the hypothesis that M is invertible cannot be omitted from Theorem 5.1. We use the same I as in the proof of Theorem 7.8.

Theorem 7.9. *Let I be the kernel of the map $\mathbb{Z}\pi \to R_n \to \mathbb{F}_p$. Then*

$$\rho(I) \neq \sum_{d|n} \rho((I/\Phi_d(x)I)_0)$$

Proof. Let π'' be the quotient group of π of order q. The map $\mathbb{Z}\pi \to \mathbb{F}_p$ factors through $\mathbb{Z}\pi''$. Let J be the kernel of the resulting map $\mathbb{Z}\pi'' \to \mathbb{F}_p$. Chasing the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathbb{Z}\pi & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & \mathbb{Z}\pi'' & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \end{array}$$

gives the exact sequence $0 \to I \to \mathbb{Z}\pi \oplus J \to \mathbb{Z}\pi'' \to 0$. By Lemma 3.1 we get $\rho(I) + \rho(\mathbb{Z}\pi'') = \rho(\mathbb{Z}\pi) + \rho(J)$ so that $\rho(I) = \rho(J)$. Now J is projective over $\mathbb{Z}\pi''$ and therefore invertible so by Theorem 5.1 we have $\rho(J) = \sum_{d|n} \rho((J/\Phi_d(\sigma)J)_0)$. Now $J/\Phi_d(\sigma)J$ is torsion for $d = n$ and $d = p$ and $(J/\Phi_1(\sigma)J)_0$ is free over $R_1 = \mathbb{Z}$ so $\rho(J) = \rho((J/\Phi_q(\sigma)J)_0)$. The sequence $0 \to J \to \mathbb{Z}\pi'' \to \mathbb{F}_p \to 0$ gives $J/\Phi_q(\sigma)J \to R_q \to \mathbb{F}_p \to 0$ and the image of the left hand map is $(J/\Phi_q(\sigma)J)_0 = \mathfrak{p}$ so we have $\rho(I) = \rho(J) = \rho(\mathfrak{p})$. If Theorem 5.1 held for I we would have

$$\rho(I) = \sum_{d|n} \rho((I/\Phi_d(\sigma)I)_0).$$

Since $C(A) = \bigoplus_{d|n} C(R_d)$, Corollary 6.2 shows that $F_\pi = \bigoplus_{d|n} \rho C(R_d)$ and the term $\rho((I/\Phi_d(\sigma)I)_0)$ lies in the summand $\rho C(R_d)$. Since $\rho(I) = \rho(\mathfrak{p})$ lies in $\rho C(R_q)$, the other terms must be 0 so $\rho((I/\Phi_n(\sigma)I)_0) = 0$. But in the previous section we showed that $(I/\Phi_n(\sigma)I)_0 \approx \mathfrak{P}$. Since $\mathfrak{P}$ is not principal and $\rho : C(A) \to F_\pi$ is an isomorphism, this is a contradiction, proving the theorem. □

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