A REMARK ON WEIGHTED BERGMAN KERNELS ON ORBIFOLDS

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Abstract. In this note, we explain that Ross-Thomas' result [4, Theorem 1.7] on the weighted Bergman kernels on orbifolds can be directly deduced from our previous result [1]. This result plays an important role in the companion paper [5] to prove an orbifold version of Donaldson Theorem.

In two very interesting papers [4, 5], Ross-Thomas describe a notion of ampleness for line bundles on Kähler orbifolds with cyclic quotient singularities which is related to embeddings in weighted projective spaces. They then apply the results in [4] to prove an orbifold version of Donaldson Theorem [5]. Namely, the existence of an orbifold Kähler metric with constant scalar curvature implies certain stability condition for the orbifold. In these papers, the result [4, Theorem 1.7] on the asymptotic expansion of Bergman kernels plays a crucial role.

In this note, we explain how to directly derive Ross-Thomas' result [4, Theorem 1.7] from Dai-Liu-Ma [1, (5.25)], provided Ross-Thomas condition [4, (1.8)] on c_i holds. Since in [1, §5], we state our results for general symplectic orbifolds, in what follows, we will just use the version from [2, Theorem 5.4.11], where Ma-Marinescu wrote them in detail for Kähler orbifolds. We will use freely the notation in [2, §5.4]. We assume also the auxiliary vector bundle E therein is $\mathbb{C}$.

Let (X, J, ω) be a compact n-dimensional Kähler orbifold with complex structure J, and with singular set $X_{\rm sing}$. Let (L, h^L) be a holomorphic Hermitian proper orbifold line bundle on X. Let ∇^L be the holomorphic Hermitian connections on (L, h^L) with curvature $R^L = (\nabla^L)^2$.

We assume that (L, h^L, ∇^L) is a prequantum line bundle, i.e.,

$$R^L = -2\pi\sqrt{-1}\,\omega. \tag{0.1}$$

Let $g^{TX} = \omega(\cdot, J\cdot)$ be the Riemannian metric on X induced by ω. Let ∇^{TX} be the Levi-Civita connection on (X, g^{TX}). We denote by $R^{TX} = (\nabla^{TX})^2$ the curvature, by r^X the scalar curvature of ∇^{TX}. For $x \in X$, set $d(x, X_{\rm sing}) := \inf_{y\in X_{\rm sing}} d(x, y)$ the distance from x to $X_{\rm sing}$.

For $p \in \mathbb{N}$, the Bergman kernel $P_p(x,x')$ $(x,x' \in X)$ is the smooth kernel of the orthogonal projection from $\mathscr{C}^\infty(X,L^p)$ onto $H^0(X,L^p)$, with respect to the Riemannian volume form $dv_X(x')$.

Theorem 0.1 ([1, Theorem 1.4], [2, Theorem 5.4.10])**.** *There exist smooth coefficients $\boldsymbol{b}_r(x) \in \mathscr{C}^\infty(X)$ which are polynomials in R^{TX}, and its derivatives with order $\leqslant 2r-2$ at x and $C_0 > 0$ such that for any $k,l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $M \in \mathbb{N}$ with*

$$\begin{aligned}(0.2)\quad \Big|\frac{1}{p^n}P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x)p^{-r}\Big|_{\mathscr{C}^l} \\ \leqslant C_{k,l}\Big(p^{-k-1} + p^{l/2}(1+\sqrt{p}\,d(x,X_{\rm sing}))^M e^{-\sqrt{C_0 p}\,d(x,X_{\rm sing})}\Big),\end{aligned}$$

for any $x \in X$, $p \in \mathbb{N}^*$. Moreover

$$(0.3)\qquad \boldsymbol{b}_0 = 1, \quad \boldsymbol{b}_1 = \frac{1}{8\pi} r^X.$$

In local coordinates, there is a more precise form [1, (5.25)], see also [2, Theorem 5.4.11]. Let $\{x_i\}_{i=1}^I \subset X_{\rm sing}$. For each point x_i we consider corresponding local charts $(G_{x_i}, \widetilde{U}_{x_i}) \to U_{x_i}$ with $\widetilde{U}_{x_i} \subset \mathbb{C}^n$, such that $0 \in \widetilde{U}_{x_i}$ is the inverse image of $x_i \in U_{x_i}$, and 0 is a fixed point of the finite stabilizer group G_{x_i} at x_i, which acts $\mathbb{C}$-linearly and effectively on $\mathbb{C}^n$ (cf. [2, Lemma 5.4.3]). We assume moreover that

$$B^{\widetilde{U}_{x_i}}(0,2\varepsilon) \subset \widetilde{U}_{x_i}, \text{ and } X_{\rm sing} \subset W := \cup_{i=1}^I B^{\widetilde{U}_{x_i}}(0,\frac{1}{4}\varepsilon)/G_{x_i}.$$

Let $\widetilde{U}_{x_i}^g$ be the fixed point set of $g \in G_{x_i}$ in $\widetilde{U}_{x_i}$, and let $\widetilde{N}_{x_i,g}$ be the normal bundle of $\widetilde{U}_{x_i}^g$ in $\widetilde{U}_{x_i}$. For each $g \in G_{x_i}$, the exponential map $\widetilde{N}_{x_i,g,\widetilde{x}} \ni Y \to \exp_{\widetilde{x}}^{\widetilde{U}_{x_i}}(Y)$ identifies a neighborhood of $\widetilde{U}_{x_i}^g$ with $\widetilde{W}_{x_i,g} = \{Y \in \widetilde{N}_{x_i,g}, |Y| \leqslant \varepsilon\}$. We identify $L|_{\widetilde{W}_{x_i,g}}$ with $L|_{\widetilde{U}_{x_i}^g}$ by using the parallel transport along the above exponential map. Then the g-action on $L|_{\widetilde{W}_{x_i,g}}$ is the multiplication by $e^{i\theta_g}$, and θ_g is locally constant on $\widetilde{U}_{x_i}^g$.

Let $\nabla^{\widetilde{N}_{x_i,g}}$ be the connection on $\widetilde{N}_{x_i,g}$ induced by the Levi-Civita connection via projection. We trivialize $\widetilde{N}_{x_i,g} \simeq \widetilde{U}_{x_i}^g \times \mathbb{C}^{n_g}$ by the parallel transport along the curve $[0,1] \ni t \to t\widetilde{Z}_{1,g}$ for $\widetilde{Z}_{1,g} \in \widetilde{U}_{x_i}^g$, which identifies also the metric on $\widetilde{N}_{x_i,g}$ with the canonical metric on $\mathbb{C}^{n_g}$. If $\widetilde{Z} \in \widetilde{W}_{x_i,g}$, we will write $\widetilde{Z} = (\widetilde{Z}_{1,g}, \widetilde{Z}_{2,g})$ with $\widetilde{Z}_{1,g} \in \widetilde{U}_{x_i}^g$, $\widetilde{Z}_{2,g} \in \mathbb{C}^{n_g}$. We will denote by Z the corresponding point on the orbifold.

Theorem 0.2 ([1, (5.25)], [2, Theorem 5.4.11]). *On $\widetilde{U}_{x_i}$ as above, there exist polynomials $\mathscr{K}_{r,\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})$ in $\widetilde{Z}_{2,g}$ of degree $\leqslant 3r$, of the same parity as r, whose coefficients are polynomials in R^{TX} and its derivatives of order $\leqslant r-2$, and a constant $C_0 > 0$ such that for any $k, l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $N \in \mathbb{N}$ such that*

$$\begin{aligned}(0.4)\quad &\Bigg| \frac{1}{p^n} P_p(\widetilde{Z}, \widetilde{Z}) - \sum_{r=0}^{k} \boldsymbol{b}_r(\widetilde{Z}) p^{-r} \\ &- \sum_{r=0}^{2k} p^{-\frac{r}{2}} \sum_{1 \neq g \in G_{x_0}} e^{i\theta_g p} \mathscr{K}_{r,\widetilde{Z}_{1,g}}(\sqrt{p}\widetilde{Z}_{2,g}) e^{-2\pi p \langle (1-g^{-1})\widetilde{z}_{2,g}, \overline{\widetilde{z}_{2,g}} \rangle} \Bigg|_{\mathscr{C}^l} \\ &\leqslant C_{k,l} \left(p^{-k-1} + p^{-k+\frac{l-1}{2}} \left(1 + \sqrt{p} d(Z, X_{\rm sing})\right)^N e^{-\sqrt{C_0\, p}\, d(Z, X_{\rm sing})} \right),\end{aligned}$$

for any $|\widetilde{Z}| \leqslant \varepsilon/2$, $p \in \mathbb{N}$, with $\boldsymbol{b}_r(\widetilde{Z})$ as in Theorem 0.1 and $\mathscr{K}_{0,\widetilde{Z}_{1,g}} = 1$.

Given a function $f(p, x)$ in $p \in \mathbb{N}$ and $x \in X$, we write $f = \mathcal{O}_{\mathscr{C}^j}(p^l)$ if the $\mathscr{C}^j$-norm of f is uniformly bounded by $C\, p^l$.

Theorem 0.3. *Let (X, ω) be a compact n-dimensional Kähler orbifold with cyclic quotient singularities (i.e., the stabilizer group G_x is a cyclic group for any $x \in X$), and L be a proper orbifold line bundle on X equipped with a Hermitian metric h^L whose curvature form is $-2\pi\sqrt{-1}\,\omega$, such that for any $x \in X$, the stabilizer group G_x acts on $L_{\widetilde{x}}$ as $\mathbb{Z}_{|G_x|}$-order cyclic group. Fix $N \geq 0$, and $r \geq 0$ and suppose c_i are a finite number of positive constants chosen so that if X has an orbifold point of order m then*

$$(0.5)\quad \frac{1}{m} \sum_i i^k c_i = \sum_{i \equiv u \,mod\, m} i^k c_i \quad \text{for all } u \text{ and all } k = 0, \cdots, N + r.$$

Then the function

$$(0.6)\qquad B_p^{\rm orb}(x) := \sum_i c_i P_{p+i}(x, x).$$

admits a global $\mathscr{C}^{2r}$-expansion of order N. That is, there exist smooth functions $b_0, \cdots, b_N$ on X such that

$$(0.7)\qquad B_p^{\rm orb} = \sum_{j=0}^{N} b_j p^{n-j} + \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}).$$

Furthermore, b_j are universal polynomials in the constants c_i and the derivatives of ω; in particular

$$b_0 = \sum_i c_i, \quad b_1 = \sum_i c_i\Big(n\, i + \frac{1}{8\pi} r^X\Big). \tag{0.8}$$

Remark 0.4. *Theorem 0.3 recovers* [4, Theorem 1.7] *of Ross-Thomas, where the remainder estimate is $\mathcal{O}_{\mathscr{C}^r}(p^{n-N-1})$.*

We improve here their remainder estimate to $\mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$ and we get Theorem 0.3 directly from Theorems 0.1, 0.2.

Remark 0.5. *By Ma-Marinescu* [3, (3.30), Remark 3.10], [2, Theorem 4.1.3, Remark 5.4.13], *Theorem 0.3 generalizes to any J-invariant metric g^{TX} on TX. Set $\Theta := g^{TX}(J\cdot,\cdot)$. The only change is that the coefficients in the expansion become*

$$b_0 = \frac{\omega^n}{\Theta^n}\sum_i c_i,\ b_1 = \frac{\omega^n}{\Theta^n}\sum_i c_i\left[n\, i + \frac{r^X_\omega}{8\pi} - \frac{1}{4\pi}\Delta_\omega \log\left(\frac{\omega^n}{\Theta^n}\right)\right], \tag{0.9}$$

where r^X_ω, Δ_ω are the scalar curvature and the Bochner Laplacian associated to $g^{TX}_\omega = \omega(\cdot, J\cdot)$. Moreover, (0.7) *can be taken to be uniform as (h^L, g^{TX}) runs over a compact set.*

Proof of Theorem 0.3. Recall that now G is a cyclic group of order m. Let ζ be a generator of G. From the local condition for orbi-ample line bundles, ζ acts on L_{x_i} as a primitive m-th root of unity λ. Thus in (0.4), $e^{i\theta_{\zeta^u}} = \lambda^u$. For $u \in \{1, \cdots, m-1\}$, set

$$\begin{aligned} \eta_u &= e^{-2\pi\langle(1-\zeta^{-u})\widetilde{z}_{2,\zeta^u}, \overline{\widetilde{z}}_{2,\zeta^u}\rangle}, \\ S_u(\widetilde{Z}) &= \sum_i c_i \sum_{j=0}^{2N+2r+1} (p+i)^{n-\frac{j}{2}} \mathscr{K}_{j,\widetilde{Z}_{1,\zeta^u}}(\sqrt{p+i}\,\widetilde{Z}_{2,\zeta^u})\lambda^{u(p+i)}\eta_u^{p+i}, \\ \mathcal{S}_2 &= \sum_{u=1}^{m-1} S_u, \qquad \mathcal{S}_1 = \sum_i c_i \sum_{j=0}^{N+r} \boldsymbol{b}_j(\widetilde{Z})(p+i)^{n-j}. \end{aligned} \tag{0.10}$$

Here $Z = z + \overline{z}$, and $z = \sum_i z_i \frac{\partial}{\partial z_i}$, $\overline{z} = \sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$ when we consider them as vector fields, and $\left|\frac{\partial}{\partial z_i}\right|^2 = \left|\frac{\partial}{\partial \overline{z}_i}\right|^2 = \frac{1}{2}$. Similarly for $\widetilde{Z}$ (and those with subscripts).

Applying (0.4) for $k = N + r + 1$ we obtain for $|\widetilde{Z}| \leq \varepsilon/2$,

$$\begin{aligned}(0.11)\quad & \Big| B_p^{\mathrm{orb}}(\widetilde{Z}) - \mathcal{S}_1 - \mathcal{S}_2 \Big|_{\mathscr{C}^l} \\ & \leqslant C_l\, p^{n-N-r-2} \left(1 + p^{\frac{l+1}{2}} \left(1 + \sqrt{p} d(Z, X_{\mathrm{sing}})\right)^M e^{-\sqrt{C_0\, p}\, d(Z, X_{\mathrm{sing}})}\right) \\ & + \sum_i c_i (p+i)^{n-N-r-1} \left(\sum_{u=1}^{m-1} \Big| \mathscr{K}_{2N+2r+2, \widetilde{Z}_{1,\zeta^u}}(\sqrt{p+i}\widetilde{Z}_{2,\zeta^u}) \eta_u^{p+i} \Big|_{\mathscr{C}^l} + \Big| \boldsymbol{b}_{N+r+1}(\widetilde{Z}) \Big|_{\mathscr{C}^l} \right).\end{aligned}$$

In what follows, we write for simplicity $\widetilde{Z}_{1,\zeta^u}$ as $Z_{1,u}$ and $\widetilde{Z}_{2,\zeta^u}$ as $Z_{2,u}$. For a function $f(p, Z)$ with $p \in \mathbb{N}$ and $|Z| \leq \varepsilon/2$ we write $f = \mathcal{O}_{\mathscr{C}^j}(g(p, Z))$ if the $\mathscr{C}^j$-norm of f in Z can be uniformly controlled by $C\, |g(p, Z)|$.

Note that $\mathscr{K}_{j,Z_{1,u}}(Z_{2,u})$ is a polynomial in $Z_{2,u}$ with the same parity as j and $\deg \mathscr{K}_{j,Z_{1,u}} \leq 3j$. Denote by $\mathscr{K}_{j,Z_{1,u},l}$ the l-homogeneous part of $\mathscr{K}_{j,Z_{1,u}}$. Then $\mathscr{K}_{j,Z_{1,u},l} = 0$ if l and j are not in the same parity or $l > 3j$. By (0.10),

$$\begin{aligned}(0.12)\quad S_u(Z) &= \sum_i c_i \sum_{j=0}^{2N+2r+1} \sum_l (p+i)^{n-\frac{j-l}{2}} \mathscr{K}_{j,Z_{1,u},l}(Z_{2,u}) \lambda^{u(p+i)} \eta_u^{p+i} \\ &= \lambda^{up} \sum_{j=0}^{2N+2r+1} \Bigg\{ \Big(\sum_{l \geq j-2n} \sum_{q=0}^{n-\frac{j-l}{2}} + \sum_{l < j-2n} \sum_{q=0}^{N+r} \Big) \mathscr{K}_{j,Z_{1,u},l}(\sqrt{p} Z_{2,u}) \\ &\qquad \times\ p^{n-\frac{j}{2}-q} \binom{n-\frac{j-l}{2}}{q} \sum_i c_i i^q \lambda^{ui} \eta_u^{p+i} \Bigg\} + \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}).\end{aligned}$$

Here we used $(p+i)^\gamma = \sum_{q=0}^{N+r} p^{\gamma-q} \binom{\gamma}{q} i^q + \mathcal{O}(p^{\gamma-N-r-1})$ for $\gamma < 0$ and the following relations for $r', r'' \in \mathbb{N}$, $r'' \leq l$,

$$(0.13)\qquad \begin{aligned} \mathscr{K}_{j,Z_{1,u},l}(\sqrt{p} Z_{2,u}) \eta_u^p &= \mathcal{O}_{\mathscr{C}^{r'}}(p^{\frac{r'}{2}} \eta_u^{p/2}), \\ \mathscr{K}_{j,Z_{1,u},l}(\sqrt{p} Z_{2,u}) &= \mathcal{O}_{\mathscr{C}^{r''}}(p^{\frac{l}{2}} |Z_{2,u}|^{l-r''}). \end{aligned}$$

In order to prove (0.7) it is sufficient to show that for $0 \leq l \leq N + r$, $r' \leq 2r$,

$$(0.14)\qquad w_{l,p} := \sum_i c_i i^l \lambda^{ui} \eta_u^{p+i} = \mathcal{O}_{\mathscr{C}^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{p/2}).$$

In fact, we will prove that $w_{l,p} = \mathcal{O}_{\mathscr{C}^{r'}}(p^{l-N-r-1+\frac{r'}{2}} \eta_u^{(\frac{3}{4}-\frac{r'}{8r})p})$ for $r' \leq 2r$.

Since $dw_{l,p} = \frac{d\eta_u}{\eta_u}(pw_{l,p}+w_{l+1,p})$, and $\frac{d\eta_u}{\eta_u}$ has a term $z_{2,u}$ or $\overline{z}_{2,u}$ which can be absorbed by $\eta_u^{\frac{1}{8r}p}$ to get a factor $p^{-1/2}$, we see by induction that it is sufficient to prove $w_{l,p} = \mathcal{O}_{\mathscr{C}^0}(p^{l-N-r-1}\eta_u^{\frac{3}{4}p})$. To this end, write

$$w_{l,p} = \left[\frac{\sum_i c_i i^l \lambda^{ui}\eta_u^i}{(\eta_u-1)^{N+r-l+1}}\right](\eta_u-1)^{N+r-l+1}\eta_u^p. \tag{0.15}$$

Since λ is a primitive m-th root of unity, $\lambda^u \neq 1$ if $u \in \{1, \cdots, m-1\}$. From [4, Lemma 3.5], under the condition (0.5), the function $\eta \to \sum_i c_i i^l \lambda^{ui}\eta^i$ has a root of order $N+r-l+1$ at $\eta = 1$ and so the term in square brackets is bounded.

For $|z_{2,u}| \leq \varepsilon$, we have by (0.10),

$$|\eta_u - 1| \leq C|z_{2,u}|^2. \tag{0.16}$$

By using (0.10) and (0.16) and the fact that $[0,\infty) \ni x \mapsto x^s e^{-x}$ is bounded for any $s \geq 0$, we get

$$(\eta_u-1)^s\eta_u^{p/4} = \mathcal{O}(p^{-s}) \quad \text{for } s \geq 1. \tag{0.17}$$

Thus, $w_{l,p} = \mathcal{O}_{\mathscr{C}^0}(p^{l-N-r-1}\eta_u^{\frac{3}{4}p})$ and (0.14) follows.

Back in (0.12), for $q > N + r$, the corresponding contribution is certainly $p^{n-\frac{j}{2}-q} \cdot \mathcal{O}_{\mathscr{C}^{2r}}(p^r\eta_u^{p/2}) = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}\eta_u^{p/2})$, by (0.13). On the other hand, if $0 \leq q \leq N+r$, then, by (0.13) and (0.14), the corresponding contribution is $p^{n-\frac{j}{2}-q} \cdot \mathcal{O}_{\mathscr{C}^{2r}}(p^{q-N-r-1+r}(1+\sqrt{p}|Z_{2,u}|)^{6N+6r+3}\eta_u^{p/2}) = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1}\eta_u^{p/4})$ again. Thus $S_u = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$.

From (0.10) and the above argument, $\mathcal{S}_2 = \mathcal{O}_{\mathscr{C}^{2r}}(p^{n-N-1})$. Combining with (0.10), (0.11) and (0.13), we get (0.7) and (0.8). □

References

[1] X. Dai, K. Liu, and X. Ma, On the asymptotic expansion of Bergman kernel, J. Differential Geom. 72(2006), 1–41.

[2] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, vol. 254, Birkhäuser Boston Inc., Boston, MA, 2007, 422 pp.

[3] X. Ma and G. Marinescu, Generalized Bergman kernels on symplectic manifolds, Adv. Math. 217 (2008), no. 4, 1756–1815.

[4] J. Ross and R. Thomas, Weighted Bergman kernels on orbifolds. arXiv:0907.5215, J. Differential Geom. 88(2011), 87–108.

[5] J. Ross and R. Thomas, Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics, J. Differential Geom. 88(2011), 109–160.

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