

A GLOBAL TORELLI THEOREM FOR CALABI-YAU MANIFOLDS

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ABSTRACT. We describe the proof that the period map from the Torelli space of Calabi-Yau manifolds to the classifying space of polarized Hodge structures is an embedding. The proof is based on the constructions of holomorphic affine structure on the Teichmüller space and Hodge metric completion of the Torelli space. A canonical global holomorphic section of the holomorphic $(n, 0)$ class on the Teichmüller space is constructed.

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1. INTRODUCTION

In this paper we study the global properties of the period map from the Torelli space of Calabi-Yau manifolds to the classifying space of polarized Hodge structures. Our method is based on the existence of a holomorphic affine structure on the Teichmüller space, and the construction of a completion space of the Torelli space with a compatible affine structure on it.

Although our method works for more general cases, for simplicity we will restrict our discussion to Calabi-Yau manifolds. More precisely a compact projective manifold M of complex dimension n with $n \geq 3$ is called Calabi-Yau in this paper, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$. We fix a lattice Λ with an

pairing Q_0 , where Λ is isomorphic to $H^n(M_0, \mathbb{Z})/\text{Tor}$ for some fixed Calabi–Yau manifold M_0 , and Q_0 is the intersection pairing. A polarized and marked Calabi-Yau manifold is a triple consisting of a Calabi-Yau manifold M , an ample line bundle L over M and a marking γ defined as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q).$$

Let \mathcal{Z}_m be a smooth component containing M of the moduli space of polarized Calabi–Yau manifolds with level m structure with $m \geq 3$, which is constructed by Popp, Viehweg, and Szendrői, for example in Section 2 of [30]. We define the Teichmüller space of Calabi–Yau manifolds to be the universal cover of \mathcal{Z}_m , which will be proved to be independent of the choice of m . The Teichmüller space will be denoted by \mathcal{T} .

The Torelli space \mathcal{T}' is defined to be a connected component of the moduli space of polarized and marked Calabi-Yau manifolds containing M . The Torelli space is also a smooth cover of \mathcal{Z}_m . Therefore both \mathcal{T} and \mathcal{T}' are connected smooth complex manifolds of complex dimension

$$\dim_{\mathbb{C}} \mathcal{T}' = h^{n-1,1}(M) = N,$$

where $h^{n-1,1}(M) = \dim_{\mathbb{C}} H^{n-1,1}(M)$ with $H^{n-1,1}(M)$ the $(n-1, 1)$ -Dolbeault cohomology group of M . See Section 2.2 for details.

There is a universal family $\mathcal{X}_{\mathcal{Z}_m} \rightarrow \mathcal{Z}_m$ over \mathcal{Z}_m constructed in [30]. Then we can pull back the universal family $\mathcal{X}_{\mathcal{Z}_m} \rightarrow \mathcal{Z}_m$ via the covering maps $\mathcal{T} \rightarrow \mathcal{Z}_m$ and $\mathcal{T}' \rightarrow \mathcal{Z}_m$ to get universal families $\varphi : \mathcal{U} \rightarrow \mathcal{T}$ and $\varphi' : \mathcal{U}' \rightarrow \mathcal{T}'$ respectively.

Let D be the classifying space of polarized Hodge structures of the weight n primitive cohomology of M . The period map $\Phi' : \mathcal{T}' \rightarrow D$ assigns to each point in \mathcal{T}' the corresponding Hodge structure of the fiber. The main result of this paper is the proof of the following global Torelli theorem:

Theorem 1.1. *The period map $\Phi' : \mathcal{T}' \rightarrow D$ is injective.*

The main idea of the proof of the above theorem is the construction of affine structures on the Teichmüller space \mathcal{T} and the Hodge metric completion space \mathcal{T}^H of the Torelli space \mathcal{T}' .

This paper is organized as follows. In Section 2 we review the definition of the classifying space of polarized Hodge structures and briefly describe the construction of the Torelli space of polarized and marked Calabi-Yau manifolds and its basic properties. In Section 3 we review the geometry of the local deformation theory of complex structures, especially in the case of Calabi-Yau manifolds.

In Section 4 we construct a holomorphic affine structure on the Teichmüller space. In Section 5, we introduce the Hodge metric completion space of the Torelli space that we constructed in [18]. We show that the Hodge metric completion space \mathcal{T}_m^H are holomorphic affine manifolds. In Section 6, we show that \mathcal{T}_m^H is independent on the level m structure, and for simplicity we denote \mathcal{T}_m^H by \mathcal{T}^H , which contains the Torelli space as an open dense submanifold. One may refer to [18] for more details about \mathcal{T}^H . Based on the completeness and holomorphic affineness of \mathcal{T}^H and the local Torelli theorem, we prove the injectivity of the period map $\Phi' : \mathcal{T}' \rightarrow D$. In Section 7, we extend the local canonical sections of the holomorphic $(n, 0)$ -classes, constructed in Section 3, to the global canonical sections on the Hodge completion space \mathcal{T}^H .

In Appendix A we review some facts about the structures of the period domain D which are known to experts but important to our discussions. We mainly follow Section 3 in [25] to discuss the classifying space D , its compact dual \check{D} , the tangent space of D , and the tangent space of \check{D} as quotients of Lie groups and Lie algebras respectively. Moreover, we describe the relation between the period domain and the unipotent Lie group $N_+ = \exp(\mathfrak{n}_+)$, which is the corresponding Lie group of the nilpotent Lie algebra. We also describe the matrix representation of elements in those Lie groups and Lie algebras if we fix a base point in D and an adapted basis for the Hodge decomposition of the base point.

Now we briefly recall the history of the Torelli problem. The idea to study the periods of abelian varieties on Riemann surfaces goes back to Riemann. In 1914 Torelli asked whether two curves are isomorphic if they have the same periods. See [33] for detail. In [38] Weil reformulated the Torelli problem as follows: Suppose for two Riemann surfaces, there exists an isomorphism of their Jacobians which preserves the canonical polarization of the Jacobians, is it true that the two Riemann surfaces are isomorphic. Andreotti proved Weil's version of the Torelli problem in [1].

Another important achievement about the Torelli problem, conjectured by Weil in [39], was the proof of the global Torelli Theorem for K3 surfaces, essentially due to Shafarevich and Piatetski-Shapiro in [24]. Andreotti's proof is based on specific geometric properties of Riemann surfaces. The approach of Shafarevich and Piatetski-Shapiro is based on the arithmeticity of the mapping class group of a K3 surface. It implies that the special K3 surfaces, the Kummer surfaces, are everywhere dense subset in the moduli of K3 surfaces. Shafarevich and Piatetski-Shapiro observed that the period map has degree one on the set of Kummer surfaces, which implies the global Torelli theorem.

The literature about the Torelli problem is enormous. Many authors made very substantial contributions to the general Torelli problem. We believe that it is impossible to give a complete list of all the achievements in this area and its applications. In [34] Verbitsky used an approach similar to ours in his proof of the global Torelli theorem for hyperkähler manifolds.

This paper is a revised and shortened version of our joint paper with Andrey Todorov who passed away in March 2012. By using methods we recently developed, we correct an error in an earlier version of this paper in proving the existence of affine structure on the Teichmüller space, and introduce the Torelli space and its Hodge metric completion which are the most natural spaces for studying period maps and the global Torelli problem. To avoid overlaps with the other papers we have written recently on this topic, in this paper we only sketch the main ideas of our proofs, and most details are given in [18]. We sincerely apologize for the confusion caused by our careless mistake and the unexpected death of Professor Andrey Todorov.

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2. THE PERIOD MAP

In this section, we review the definitions and some basic results about period domain, Teichmüller space and Torelli space, as well as the period maps on them. In Section 2.1, we recall the definition and some basic properties of period domain. In Section 2.2, we discuss the construction of the Teichmüller space and Torelli space of Calabi-Yau manifolds based on the works of Popp [23], Viehweg [35] and Szendroi [30] on the moduli spaces of Calabi-Yau manifolds. In Section 2.3, we define the period maps from the Teichmüller space and Torelli space to the period domain. We remark that most of the results in this section are standard and can be found from the literature in the subjects.

2.1. Classifying space of polarized Hodge structure. We first review the construction of the classifying space of polarized Hodge structures, which is also called the period domain. We refer the reader to Section 3 of [25] for details.

This paper mainly considers polarized and marked Calabi-Yau manifolds. A pair (M, L) consisting of a Calabi-Yau manifold M of complex dimension n and an ample line bundle L over M is called a polarized Calabi-Yau manifold. By abuse of notation the Chern class of L will also be denoted by L and $L \in H^2(M, \mathbb{Z})$. We use $h^n = \dim_{\mathbb{C}} H^n(M, \mathbb{C})$ to denote the Betti number.

We fix a lattice Λ with a pairing Q_0 , where Λ is isomorphic to $H^n(M_0, \mathbb{Z})/\text{Tor}$ for some Calabi-Yau manifold M_0 and Q_0 is defined by the cup-product. For a polarized Calabi-Yau manifold (M, L) , we define a marking γ as an isometry of the lattices

$$(1) \quad \gamma : (\Lambda, Q_0) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q).$$

Definition 2.1. *Let the pair (M, L) be a polarized Calabi-Yau manifold, we call the triple (M, L, γ) a polarized and marked Calabi-Yau manifold.*

For a polarized and marked Calabi-Yau manifold (M, L, γ) with background smooth manifold X , the marking γ identifies $H_n(M, \mathbb{Z})/\text{Tor}$ isometrically to the fixed lattice Λ . This gives us a canonical identification of the middle dimensional de Rham cohomology of M to that of the background manifold X ,

$$H^n(M) \cong H^n(X)$$

where the coefficient ring can be \mathbb{Q} , \mathbb{R} or \mathbb{C} .

Since the polarization L is an integer class, it defines a map

$$L : H^n(X, \mathbb{Q}) \rightarrow H^{n+2}(X, \mathbb{Q})$$

given by $A \mapsto L \wedge A$ for any $A \in H^n(X, \mathbb{Q})$. We denote by $H_{pr}^n(X) = \ker(L)$ the primitive cohomology groups where, again, the coefficient ring is \mathbb{Q} , \mathbb{R} or \mathbb{C} . We let $H_{pr}^{k, n-k}(M) = H^{k, n-k}(M) \cap H_{pr}^n(M, \mathbb{C})$ and denote its dimension by $h^{k, n-k}$. We have the Hodge decomposition

$$(2) \quad H_{pr}^n(M, \mathbb{C}) = H_{pr}^{n,0}(M) \oplus \cdots \oplus H_{pr}^{0,n}(M).$$

It is easy to see that for a polarized Calabi-Yau manifold, since $H^2(M, \mathcal{O}_M) = 0$, we have

$$H_{pr}^{n,0}(M) = H^{n,0}(M), \quad H_{pr}^{n-1,1}(M) = H^{n-1,1}(M).$$

The Poincaré bilinear form Q on $H_{pr}^n(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d -closed n -forms u, v on X . The bilinear form Q is symmetric if n is even and is skew-symmetric if n is odd. Furthermore, Q is non-degenerate and can be extended to $H_{pr}^n(X, \mathbb{C})$ bilinearly, and satisfies the Hodge-Riemann relations

$$(3) \quad Q(H_{pr}^{k, n-k}(M), H_{pr}^{l, n-l}(M)) = 0 \quad \text{unless } k+l = n, \quad \text{and}$$

$$(4) \quad (\sqrt{-1})^{2k-n} Q(v, \bar{v}) > 0 \quad \text{for } v \in H_{pr}^{k, n-k}(M) \setminus \{0\}.$$

The above Hodge decomposition of $H_{pr}^n(M, \mathbb{C})$ can also be described via the Hodge filtration. Let $f^k = \sum_{i=k}^n h^{i, n-i}$, and

$$F^k = F^k(M) = H_{pr}^{n,0}(M) \oplus \cdots \oplus H_{pr}^{k, n-k}(M)$$

from which we have the decreasing filtration

$$H_{pr}^n(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n.$$

We know that

$$(5) \quad \dim_{\mathbb{C}} F^k = f^k,$$

$$(6) \quad H_{pr}^n(X, \mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and}$$

$$(7) \quad H_{pr}^{k, n-k}(M) = F^k \cap \overline{F^{n-k}}.$$

In term of the Hodge filtration $F^n \subset \cdots \subset F^0 = H_{pr}^n(M, \mathbb{C})$, the Hodge-Riemann relations can be written as

$$(8) \quad Q(F^k, F^{n-k+1}) = 0, \quad \text{and}$$

$$(9) \quad Q(Cv, \bar{v}) > 0 \quad \text{if } v \neq 0,$$

where C is the Weil operator given by $Cv = (\sqrt{-1})^{2k-n} v$ when $v \in H_{pr}^{k, n-k}(M)$. The classifying space D for polarized Hodge structures with data (5) is the space of all such Hodge filtrations

$$D = \{F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (5), (8) \text{ and } (9) \text{ hold}\}.$$

The compact dual \check{D} of D is

$$\check{D} = \{F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (5) \text{ and } (8) \text{ hold}\}.$$

The classifying space $D \subset \check{D}$ is an open set. We note that the conditions (5), (8) and (9) imply the identity (6). From the definition of classifying space we naturally get the Hodge bundles on \check{D} by associating to each point in \check{D} the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. Without confusion we will also denote by F^k the bundle with F^k as fiber, for each $0 \leq k \leq n$.

We remark that one may refer to Appendix A for more detailed descriptions of the classifying space D , its compact dual \check{D} , and the tangent space at each point of them, by viewing them as quotients of Lie groups and Lie algebras.

2.2. Constructions of moduli, Teichmüller, and Torelli spaces. In this subsection we briefly describe the construction of the Teichmüller space and Torelli space of Calabi-Yau manifolds and discuss their basic properties.

We need the concept of analytic family (versal or universal) of compact complex manifolds, we refer to page 8-10 in [27], page 94 in [23] or page 19 in [35], for equivalent definitions and more details.

Let (M, L) be a polarized Calabi-Yau manifold. Recall that a marking of (M, L) is defined as an isometry

$$\gamma : (\Lambda, Q_0) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q).$$

For any integer $m \geq 3$, we follow the definition of Szendrői [30] to define an m -equivalent relation of two markings on (M, L) by

$$\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \text{Id} \in m \cdot \text{End}(H^n(M, \mathbb{Z})/\text{Tor}),$$

and denote $[\gamma]_m$ to be the set of all the equivalent classes of γ . Then we call $[\gamma]_m$ a level m structure on the polarized Calabi-Yau manifold. For moduli space of polarized Calabi-Yau manifold with level m structure, we have the following theorem, which is a reformulation of Theorem 2.2 in [30], we just take the statement we need in this paper. One can also look at [23] and [35] for more details about the construction of moduli spaces of Calabi-Yau manifolds.

Theorem 2.2. *Let $m \geq 3$ and M be polarized Calabi-Yau manifold with level m structure, then there exists a quasi-projective complex manifold \mathcal{Z}_m with a universal family of Calabi-Yau manifolds,*

$$(10) \quad \mathcal{X}_{\mathcal{Z}_m} \rightarrow \mathcal{Z}_m,$$

containing M as a fiber, and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

Let (M, L) be a polarized Calabi-Yau manifold. We define \mathcal{T} to be the universal cover of \mathcal{Z}_m with the covering map

$$\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$$

and the family

$$\mathcal{U} \rightarrow \mathcal{T}$$

to be the pull-back of the family (10) by the covering map π_m .

Proposition 2.3. *The Teichmüller space \mathcal{T} is a simply connected smooth complex manifold, and the family*

$$(11) \quad \mathcal{U} \rightarrow \mathcal{T}$$

containing M as a fiber, is a universal family.

Proof. For the first part, because \mathcal{Z}_m is smooth complex manifold, the universal cover of \mathcal{Z}_m is simply connected smooth complex manifold. For the second part, we know that the family (10) is universal family at each point of \mathcal{Z}_m , and π is locally bi-holomorphic, thus the pull-back family via π is also universal, as the universal property of analytic families is local. \square

Note that the Teichmüller space \mathcal{T} does not depend on the choice of m . In fact let m_1, m_2 be two different integers, $\mathcal{U}_1 \rightarrow \mathcal{T}_1$ and $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ are two versal families constructed via level m_1 and level m_2 respectively as above, both of which contain M as a fiber. By using the fact that \mathcal{T}_1 and \mathcal{T}_2 are simply connected and the definition of universal family, we have a bi-holomorphic map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$, such that the universal family $\mathcal{U}_1 \rightarrow \mathcal{T}_1$ is the pull back of the versal family $\mathcal{U}_2 \rightarrow \mathcal{T}_2$ by f . Thus these two families are isomorphic to each other.

Now we introduce the definition of *Torelli space* \mathcal{T}' . For this we need to consider the equivalence classes of triples (M, L, γ) which is called a polarized and marked Calabi-Yau manifold with polarization L and marking γ . To be more precise, two triples (M, L, γ) and (M', L', γ') is called equivalent, if there exists a bi-holomorphic map $f : M \rightarrow M'$ with

$$f^*L' = L, \quad f^*\gamma' = \gamma,$$

where $f^*\gamma'$ is given by $\gamma' : (\Lambda, Q_0) \rightarrow (H^n(M', \mathbb{Z})/\text{Tor}, Q)$ composed with

$$f^* : (H^n(M', \mathbb{Z})/\text{Tor}, Q) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q),$$

then we define $[M, L, \gamma] = [M', L', \gamma'] \in \mathcal{T}'$. More precisely we define the Torelli space as follows.

Definition 2.4. *The Torelli space \mathcal{T}' of Calabi–Yau manifolds is the connected component of the moduli space of equivalence classes of polarized and marked Calabi–Yau manifolds, which contains (M, L) .*

There is a natural covering map $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ by mapping (M, L, γ) to $(M, L, [\gamma]_m)$. From this we see easily that \mathcal{T}' is a smooth and connected complex manifold. We can also get a pull-back universal family $\varphi' : \mathcal{U}' \rightarrow \mathcal{T}'$ on the Torelli space \mathcal{T}' via the covering map π'_m .

Recall that the Teichmüller space \mathcal{T} is defined to be the universal covering space of \mathcal{Z}_m with covering map $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$. Then we can lift π_m via the covering map $\pi'_m : \mathcal{T}' \rightarrow \mathcal{Z}_m$ to get a covering map $\pi : \mathcal{T} \rightarrow \mathcal{T}'$, such that the following diagram commutes.

$$(12) \quad \begin{array}{ccc} \mathcal{T} & & \\ \downarrow \pi_m & \searrow \pi & \\ & & \mathcal{T}' \\ & \swarrow \pi'_m & \\ \mathcal{Z}_m & & \end{array} .$$

We remark that our method of proving the global Torelli theorem for polarized and marked Calabi-Yau manifold applies without change to the case of more general projective manifolds with trivial canonical line bundle, including polarized and marked hyperkähler manifolds and K3 surfaces. In these cases the Teichmüller spaces can be embedded inside symmetric domains of non-compact type, so naturally have global holomorphic affine flat coordinates.

2.3. The period map. We are now ready to define the period maps from the the Teichmüller space and Torelli space to the period domain.

Let us denote the period map on the moduli space by $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \rightarrow D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D . Since the period map $\Phi_{\mathcal{Z}_m}$ is locally liftable, we have the lifted period map $\Phi : \mathcal{T} \rightarrow D$ on the universal cover \mathcal{T} of \mathcal{Z}_m such the following diagram commutes

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & D \\ \downarrow \pi_m & & \downarrow \pi \\ \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma \end{array}$$

For any point $q \in \mathcal{T}'$, let (M_q, L_q, γ_q) be the fiber of family $\pi : \mathcal{U}' \rightarrow \mathcal{T}'$, which is a polarized and marked Calabi-Yau manifold. Since on the Torelli space we have fixed lattice (Λ_0, Q_0) , we can use this to identify $(H^n(M_q, \mathbb{C}), Q)$ isometrically for the fiber of any point q in \mathcal{T}' , and thus get a canonical trivial bundle $H^n(M_p, \mathbb{C}) \times \mathcal{T}'$ with $p \in \mathcal{T}'$ the fixed point. We have similar identifications for $H^n(M, \mathbb{Q})$ and $H^n(M, \mathbb{Z})$.

The period map from \mathcal{T}' to D is defined by assigning each point $q \in \mathcal{T}'$ the Hodge structure on M_p ,

$$\Phi' : \mathcal{T}' \rightarrow D$$

with $\Phi'(q) = \gamma_q^{-1}(\{F^n(M_q) \subset \dots \subset F^0(M_q)\})$ for any $q \in \mathcal{T}'$, where (M_q, L_q, γ_q) is the fiber over q . We denote $F^k(M_q)$ by F_q^k for convenience.

In a summary we have defined the period maps on the Teichmüller space and Torelli space. Moreover they fit into the following commutative diagram

$$(13) \quad \begin{array}{ccccc} \mathcal{T} & \xrightarrow{\Phi} & D & & \\ \downarrow \pi_m & \searrow \pi & \nearrow \Phi' & & \downarrow \pi_D \\ & \mathcal{T}' & & & \\ \downarrow \pi'_m & \swarrow \pi'_m & & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma, & & \end{array}$$

where the maps π_m , π'_m and π are all natural covering maps between the corresponding spaces as in (12).

Period map has several good properties, we refer the reader to Chapter 10 in [36] for details. Among them the most important is the following Griffiths transversality. Let $\Phi : \mathcal{T} \rightarrow D$ be the period map on the Teichmüller space, then Φ is holomorphic map and for any $q \in \mathcal{T}$ and $v \in T_q^{1,0}\mathcal{T}$, the tangent map satisfies,

$$(14) \quad \Phi_*(v) \in \bigoplus_{k=1}^n \text{Hom}(F_q^k/F_q^{k+1}, F_q^{k-1}/F_q^k),$$

where $F^{n+1} = 0$, or equivalently

$$\Phi_*(v) \in \bigoplus_{k=0}^n \text{Hom}(F_q^k, F_q^{k-1}/F_q^k).$$

This is called the Griffiths transversality.

In [11], Griffiths and Schmid studied the Hodge metric on the period domain D . We denote it by h . In particular, this Hodge metric is a complete homogeneous metric. By local Torelli theorem for Calabi–Yau manifolds, we know that the period maps $\Phi_{\mathcal{Z}_m}$, Φ and Φ' are locally injective. Thus it follows from [11] that the pull-back of h by $\Phi_{\mathcal{Z}_m}$, Φ and Φ' on \mathcal{Z}_m , \mathcal{T} and \mathcal{T}' respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics the *Hodge metrics*, and they are all denoted by h .

Hodge bundles over \mathcal{T} are the pull-back of the Hodge bundles over D through the period map. For convenience, we still denote them by F^k , for each $0 \leq k \leq n$. We will also denote by P_p^k the projection from $H^n(M, \mathbb{C})$ to F_p^k with respect to the Hodge filtration at M_p , and $P_p^{n-k,k}$ the projection from $H^n(M, \mathbb{C})$ to $H^{n-k,k}(M_p)$ with respect to the Hodge decomposition at M_p .

With all of these preparations, we are ready to state precisely the main theorem of this paper.

Theorem 2.5. *The period map Φ' constructed above is an embedding,*

$$\Phi' : \mathcal{T}' \hookrightarrow D.$$

We will outline the main ideas of proofs of the above theorem, and refer the reader to [18] for details of the arguments and more results.

3. LOCAL GEOMETRIC STRUCTURE OF THE TEICHMÜLLER SPACE

In this section, we review the local deformation theory of polarized Calabi-Yau manifolds, which will be needed for the construction of the global holomorphic affine structure and global holomorphic sections of the Hodge bundle F^n on the Teichmüller space in Section 7. This section can be skipped if the reader is only interested in the proof of the Theorem 2.5.

In Section 3.1, we briefly review the basic local deformation theory of complex structures. In Section 3.2, we recall the local Kuranishi deformation theory of Calabi-Yau manifolds, which depends on the Calabi-Yau metric in a substantial way. In Section 3.3, we describe a local family of the canonical holomorphic $(n, 0)$ -forms as a section of the Hodge bundle F^n over the local deformation space of Calabi-Yau manifolds, from which we obtain an expansion of the family of holomorphic $(n, 0)$ -classes as given in Theorem 3.6.

Most of the results in this section are standard now in the literatures, and can be found in [17], [32], and [31]. For reader's convenience, we also briefly review some arguments. We remark that one may use a more algebraic approach to Theorem 3.6 by using the local Torelli theorem and the Griffiths transversality.

3.1. Local deformation of complex structure. Let X be a smooth manifold of dimension $\dim_{\mathbb{R}} X = 2n$ and let J be an integrable complex structure on X . We denote by $M = (X, J)$ the corresponding complex manifold, and $\partial, \bar{\partial}$ the corresponding differential operators on M .

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be a $T^{1,0}M$ -valued smooth $(0, 1)$ -form. For any point $x \in M$, and any local holomorphic coordinate chart (U, z_1, \dots, z_n) around x . Let us express

$\varphi = \varphi_j^i d\bar{z}_j \otimes \frac{\partial}{\partial z_i} = \varphi^i \partial_i$, where $\varphi^i = \varphi_j^i d\bar{z}_j$ and $\partial_i = \frac{\partial}{\partial z_i}$ for simplicity. Here we use the standard convention to sum over the repeated indices. We can view φ as a map

$$\varphi : \Omega^{1,0}(M) \rightarrow \Omega^{0,1}(M)$$

such that locally we have

$$\varphi(dz_i) = \varphi^i \quad \text{for } 1 \leq i \leq n.$$

We use φ to describe deformation of complex structures. Let

$$\Omega_\varphi^{1,0}(x) = \text{span}_{\mathbb{C}}\{dz_1 + \varphi(dz_1), \dots, dz_n + \varphi(dz_n)\}, \quad \text{and}$$

$$\Omega_\varphi^{0,1}(x) = \text{span}_{\mathbb{C}}\{d\bar{z}_1 + \bar{\varphi}(d\bar{z}_1), \dots, d\bar{z}_n + \bar{\varphi}(d\bar{z}_n)\},$$

if $\Omega_\varphi^{1,0}(x) \cap \Omega_\varphi^{0,1}(x) = 0$ for any x , then we can define a new almost complex structure J_φ by letting $\Omega_\varphi^{1,0}(x)$ and $\Omega_\varphi^{0,1}(x)$ be the eigenspaces of $J_\varphi(x)$ with respect to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively, and we call such φ a Beltrami differential.

It was proved in [22] that the almost complex structure J_φ is integrable if and only if

$$(15) \quad \bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi].$$

If (15) holds, we will call φ an integrable Beltrami differential and denote by M_φ the corresponding complex manifold. Please see Chapter 4 in [17] for more details about the deformation of complex structures.

Let us recall the notation for contractions and Lie bracket of Beltrami differentials. Let (U, z_1, \dots, z_n) be the local coordinate chart defined above, and $\Omega = f dz_1 \wedge \dots \wedge dz_n$ be a smooth $(n, 0)$ -form on M , and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be a Beltrami differential. We define

$$\varphi \lrcorner \Omega = \sum_i (-1)^{i-1} f \varphi^i \wedge dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n.$$

For Beltrami differentials $\varphi, \psi \in A^{0,1}(M, T^{1,0}M)$, with $\varphi = \varphi^i \partial_i$ and $\psi = \psi^k \partial_k$, recall that the Lie bracket is defined as

$$[\varphi, \psi] = \sum_{i,k} (\varphi^i \wedge \partial_i \psi^k + \psi^i \wedge \partial_i \varphi^k) \otimes \partial_k,$$

where $\partial_i \varphi^k = \frac{\partial \varphi_l^k}{\partial z_i} d\bar{z}_l$ and $\partial_i \psi^k = \frac{\partial \psi_l^k}{\partial z_i} d\bar{z}_l$.

For k Beltrami differentials $\varphi_1, \dots, \varphi_k \in A^{0,1}(M, T^{1,0}M)$, with $\varphi_\alpha = \varphi_\alpha^i \partial_i$ and $1 \leq \alpha \leq k$, we define

$$\varphi_1 \wedge \dots \wedge \varphi_k = \sum_{i_1 < \dots < i_k} \left(\sum_{\sigma \in S_k} \varphi_{\sigma(1)}^{i_1} \wedge \dots \wedge \varphi_{\sigma(k)}^{i_k} \right) \otimes (\partial_{i_1} \wedge \dots \wedge \partial_{i_k}),$$

where S_k is the symmetric group of k elements. Especially we have

$$\wedge^k \varphi = k! \sum_{i_1 < \dots < i_k} (\varphi^{i_1} \wedge \dots \wedge \varphi^{i_k}) \otimes (\partial_{i_1} \wedge \dots \wedge \partial_{i_k}).$$

Then we define the contraction,

$$\begin{aligned} & (\varphi_1 \wedge \cdots \wedge \varphi_k) \lrcorner \Omega = \varphi_1 \lrcorner (\varphi_2 \lrcorner (\cdots \lrcorner (\varphi_k \lrcorner \Omega))) \\ & = \sum_{I=(i_1, \dots, i_k) \in A_k} (-1)^{|I| + \frac{(k-1)(k-2)}{2}} f \left(\sum_{\sigma \in S_k} \varphi_{\sigma(1)}^{i_1} \wedge \cdots \wedge \varphi_{\sigma(k)}^{i_k} \right) \wedge dz_{I^c}, \end{aligned}$$

where A_k is the index set

$$A_k = \{(i_1, \dots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Here for each $I = (i_1, \dots, i_k) \in A_k$, we let $|I| = i_1 + \cdots + i_k$ and $dz_{I^c} = dz_{j_1} \wedge \cdots \wedge dz_{j_{n-k}}$ where $j_1 < \cdots < j_{n-k}$ and $j_\alpha \neq i_\beta$ for any α, β . With the above notations, for any Beltrami differentials $\varphi, \psi \in A^{0,1}(M, T^{1,0}M)$ one has the following identity which was proved in [31], [32],

$$(16) \quad \partial((\varphi \wedge \psi) \lrcorner \Omega) = -[\varphi, \psi] \lrcorner \Omega + \varphi \lrcorner \partial(\psi \lrcorner \Omega) + \psi \lrcorner \partial(\varphi \lrcorner \Omega).$$

The following notation will be needed in the construction of the local canonical family of holomorphic $(n, 0)$ -classes.

$$(17) \quad e^\varphi \lrcorner \Omega = \sum_{k \geq 0} \frac{1}{k!} \wedge^k \varphi \lrcorner \Omega.$$

By direct computation, we see that $e^\varphi \lrcorner \Omega = f(dz_1 + \varphi(dz_1)) \wedge \cdots \wedge (dz_n + \varphi(dz_n))$ is a smooth $(n, 0)$ -form on M_φ .

3.2. Local deformation of Calabi-Yau manifold. For a point $p \in \mathcal{T}$, we denote by (M_p, L) the corresponding polarized and marked Calabi-Yau manifold as the fiber over p . Yau's solution of the Calabi conjecture implies that there exists a unique Calabi-Yau metric h_p on M_p , and the imaginary part $\omega_p = \text{Im } h_p \in L$ is the corresponding Kähler form. First by using the Calabi-Yau metric we have the following lemma,

Lemma 3.1. *Let Ω_p be a nowhere vanishing holomorphic $(n, 0)$ -form on M_p such that*

$$(18) \quad \left(\frac{\sqrt{-1}}{2} \right)^n (-1)^{\frac{n(n-1)}{2}} \Omega_p \wedge \overline{\Omega_p} = \omega_p^n.$$

Then the map $\iota : A^{0,1}(M, T^{1,0}M) \rightarrow A^{n-1,1}(M)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega_p$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by ω_p . Furthermore, ι preserves the Hodge decomposition.

Let us briefly recall the proof. We can pick local coordinates z_1, \dots, z_n on M such that $\Omega_p = dz_1 \wedge \cdots \wedge dz_n$ locally and $\omega_p = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, then the condition (18) implies that $\det[g_{i\bar{j}}] = 1$. The lemma follows from direct computations.

Let $\partial_{M_p}, \bar{\partial}_{M_p}, \bar{\partial}_{M_p}^*, \square_{M_p}, G_{M_p}$, and \mathbb{H}_{M_p} be the corresponding operators in the Hodge theory on M_p , where $\bar{\partial}_{M_p}^*$ is the adjoint operator of $\bar{\partial}_{M_p}$, \square_{M_p} the Laplace operator, and G_{M_p} the corresponding Green operator. We let \mathbb{H}_{M_p} denote the harmonic projection onto the kernel of \square_{M_p} . We also denote by $\mathbb{H}^{p,q}(M_p, E)$ the harmonic (p, q) -forms with value in a holomorphic vector bundle E on M_p .

By using the Calabi-Yau metric we have a more precise description of the local deformation of a polarized Calabi-Yau manifold. First from Hodge theory, we have the following identification

$$T_p^{1,0}\mathcal{T} \cong \mathbb{H}^{0,1} \left(M_p, T_{M_p}^{1,0} \right).$$

From Kuranishi theory we have the following local convergent power series expansion of the Beltrami differentials, which is now well-known as the Bogomolov-Tian-Todorov theorem.

Theorem 3.2. *Let $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1} \left(M_p, T_{M_p}^{1,0} \right)$ be a basis. Then there is a unique power series*

$$(19) \quad \varphi(\tau) = \sum_{i=1}^N \tau_i \varphi_i + \sum_{|I| \geq 2} \tau^I \varphi_I$$

which converges for $|\tau| < \varepsilon$ small. Here $I = (i_1, \dots, i_N)$ is a multi-index, $\tau^I = \tau_1^{i_1} \dots \tau_N^{i_N}$ and $\varphi_I \in A^{0,1} \left(M_p, T_{M_p}^{1,0} \right)$. Furthermore, the family of Beltrami differentials $\varphi(\tau)$ satisfy the following conditions:

$$(20) \quad \begin{aligned} \bar{\partial}_{M_p} \varphi(\tau) &= \frac{1}{2} [\varphi(\tau), \varphi(\tau)], \\ \bar{\partial}_{M_p}^* \varphi(\tau) &= 0, \\ \varphi_I \lrcorner \Omega_p &= \partial_{M_p} \psi_I, \end{aligned}$$

for each $|I| \geq 2$ where $\psi_I \in A^{n-2,1}(M_p)$ are smooth $(n-2,1)$ -forms. By shrinking ε we can pick each ψ_I appropriately such that $\sum_{|I| \geq 2} \tau^I \psi_I$ converges for $|\tau| < \varepsilon$.

Remark 3.3. *The coordinate $\{\tau_1, \dots, \tau_N\}$ depends on the choice of basis $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1} \left(M_p, T_{M_p}^{1,0} \right)$. But one can also determine the coordinate by fixing a basis $\{\eta_0\}$ and $\{\eta_1, \dots, \eta_N\}$ for $H^{n,0}(M_p)$ and $H^{n-1,1}(M_p)$ respectively. In fact, Lemma 3.1 implies that there is a unique choice of $\varphi_1, \dots, \varphi_N$ such that $\eta_k = [\varphi_k \lrcorner \eta_0]$, for each $1 \leq k \leq N$.*

Theorem 3.2 was proved in [32], and in [31] in a form without specifying the Kuranishi gauge, the second and the third condition in (20). This theorem implies that the local deformation of a Calabi-Yau manifold is unobstructed. Here we only mention two important points of its proof. For the convergence of $\sum_{|I| \geq 2} \tau^I \psi_I$, noting that $\partial_{M_p} \psi_I = \varphi_I \lrcorner \Omega_p$ and $\bar{\partial} \varphi_I \lrcorner \Omega_p = 0$, we can pick $\psi_I = \partial_{M_p}^* G(\varphi_I \lrcorner \Omega_p)$. It follows that

$$\|\psi_I\|_{k,\alpha} \leq C(k, \alpha) \|\varphi_I \lrcorner \Omega_p\|_{k-1,\alpha} \leq C'(k, \alpha) \|\varphi_I\|_{k-1,\alpha}.$$

The desired convergence follows from the estimate on φ_I . We note that the convergence of (19) follows from standard elliptic estimate. See [32], or Chapter 4 of [17] for details.

For the third condition in (20), by using the first two conditions in (20), for example we have in the case of $|I| = 2$,

$$(21) \quad \bar{\partial}_{M_p} \varphi_{ij} = [\varphi_i, \varphi_j] \text{ and } \bar{\partial}_{M_p}^* \varphi_{ij} = 0.$$

Then by using formula (16) and Lemma 3.1, we get that

$$[\varphi_i, \varphi_j] \lrcorner \Omega_p = \partial_{M_p} (\varphi_i \wedge \varphi_j \lrcorner \Omega_p)$$

is ∂_{M_p} exact. It follows that $\bar{\partial}_{M_p}(\varphi_{ij} \lrcorner \Omega_p) = (\bar{\partial}_{M_p} \varphi_{ij}) \lrcorner \Omega_p$ is also ∂_{M_p} exact. Then by the $\partial\bar{\partial}$ -lemma we have

$$\bar{\partial}_{M_p}(\varphi_{ij} \lrcorner \Omega_p) = \bar{\partial}_{M_p} \partial_{M_p} \psi_{ij}$$

for some $\psi_{ij} \in A^{n-2,1}$. It follows that

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p} \psi_{ij} + \bar{\partial}_{M_p} \alpha + \beta$$

for some $\alpha \in A^{n-1,0}(M_p)$ and $\beta \in \mathbb{H}^{n-1,1}(M_p)$. By using the condition $\bar{\partial}_{M_p}^* \varphi_{ij} = 0$ and Lemma 3.1, we have

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p} \psi_{ij} + \beta.$$

Because φ_{ij} is not uniquely determined by condition (21), we can choose φ_{ij} such that the harmonic projection $\mathbb{H}(\varphi_{ij}) = 0$. Then by using Lemma 3.1 again, we have

$$\varphi_{ij} \lrcorner \Omega_p = \partial_{M_p} \psi_{ij}.$$

Thus there exists a unique φ_{ij} which satisfies all three conditions in (20). We can then proceed by induction and the same argument as above to show that the third condition in (20) holds for all $|I| \geq 2$. See [32] and [31] for details.

Theorem 3.2 can be used to define the local holomorphic affine flat coordinates $\{\tau_1, \dots, \tau_N\}$ around p , for a given orthonormal basis $\varphi_1, \dots, \varphi_N \in \mathbb{H}^{0,1}(M_p, T_{M_p}^{1,0})$. Sometimes we also denote by M_τ the deformation given by the Beltrami differential $\varphi(\tau)$. This local affine coordinates can be extended to a global one as discussed in Section 7.

3.3. Local canonical section of holomorphic $(n, 0)$ -classes. By using the local deformation theory, in [32] Todorov constructed a canonical local holomorphic section of the line bundle $H^{n,0} = F^n$ over the local deformation space of a Calabi-Yau manifold at the differential form level. We first recall the construction of the holomorphic $(n, 0)$ -forms in [32].

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential and let M_φ be the Calabi-Yau manifold defined by φ . We refer the reader to Section 3.1 for the definition of the contraction $e^\varphi \lrcorner \Omega_p$.

Lemma 3.4. *Let Ω_p be a nowhere vanishing holomorphic $(n, 0)$ -form on M_p and $\{z_1, \dots, z_n\}$ is a local holomorphic coordinate system with respect to J such that*

$$\Omega_p = dz_1 \wedge \dots \wedge dz_n$$

locally. Then the smooth $(n, 0)$ -form

$$\Omega_\varphi = e^\varphi \lrcorner \Omega_p$$

is holomorphic with respect to the complex structure on M_φ if and only if $\partial_{M_p}(\varphi \lrcorner \Omega_p) = 0$.

Proof. The proof in [32] is by direct computations, here we give a simple proof.

Being an $(n, 0)$ -form on M_φ , $e^\varphi \lrcorner \Omega_p$ is holomorphic on M_φ if and only if $d(e^\varphi \lrcorner \Omega_p) = 0$. For any smooth $(n, 0)$ -form Ω_p and Beltrami differential $\varphi \in A^{0,1}(M, T^{1,0}M)$, we have the following formula,

$$d(e^\varphi \lrcorner \Omega_p) = e^\varphi \lrcorner (\bar{\partial}_{M_p} \Omega_p + \partial_{M_p}(\varphi \lrcorner \Omega_p)) + (\bar{\partial}_{M_p} \varphi - \frac{1}{2}[\varphi, \varphi]) \lrcorner (e^\varphi \lrcorner \Omega_p),$$

which can be verified by direct computations. In our case the Beltrami differential φ is integrable, i.e. $\bar{\partial}_{M_p}\varphi - \frac{1}{2}[\varphi, \varphi] = 0$ and Ω_p is holomorphic on M_p . Therefore we have

$$d(e^\varphi \lrcorner \Omega_p) = e^\varphi \lrcorner (\partial_{M_p}(\varphi \lrcorner \Omega_p)),$$

which implies that $e^\varphi \lrcorner \Omega_p$ is holomorphic on M_φ if and only if $\partial_{M_p}(\varphi \lrcorner \Omega_p) = 0$. \square

Now we can construct the canonical family of holomorphic $(n, 0)$ -forms on the local deformation space of Calabi-Yau manifolds.

Proposition 3.5. *We fix on M_p a nowhere vanishing holomorphic $(n, 0)$ -form Ω_p and an orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $\mathbb{H}^1(M_p, T^{1,0}M_p)$. Let $\varphi(\tau)$ be the family of Beltrami differentials given by (20) that defines a local deformation of M_p which we denote by M_τ . Let*

$$(22) \quad \Omega_p^c(\tau) = e^{\varphi(\tau)} \lrcorner \Omega_p.$$

Then $\Omega_p^c(\tau)$ is a well-defined nowhere vanishing holomorphic $(n, 0)$ -form on M_τ and depends on τ holomorphically.

Proof. We call such family the canonical family of holomorphic $(n, 0)$ -forms on the local deformation space of M_p . The fact that $\Omega(\tau)^c$ is a nowhere vanishing holomorphic $(n, 0)$ -form on the fiber M_τ follows from its definition and Lemma 3.4 directly. In fact we only need to check that $\partial_{M_p}(\varphi(\tau) \lrcorner \Omega_p) = 0$. By formulae (19) and (20) we know that

$$\varphi(\tau) \lrcorner \Omega_p = \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \sum_{|I| \geq 2} \tau^I (\varphi_I \lrcorner \Omega_p) = \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \partial_{M_p} \left(\sum_{|I| \geq 2} \tau^I \psi_I \right).$$

Because each φ_i is harmonic, by Lemma 3.1 we know that $\varphi_i \lrcorner \Omega_p$ is also harmonic and thus $\partial_{M_p}(\varphi_i \lrcorner \Omega_p) = 0$. Furthermore, since $\sum_{|I| \geq 2} \tau^I \psi_I$ converges when $|\tau|$ is small, we see that $\partial_{M_p}(\varphi(\tau) \lrcorner \Omega_p) = 0$ from formula (20). The holomorphic dependence of $\Omega_p^c(\tau)$ on τ follows from formula (22) and the fact that $\varphi(\tau)$ depends on τ holomorphically. \square

From Theorem 3.2 and Proposition 3.5 we get the expansion of the deRham cohomology classes of the canonical family of holomorphic $(n, 0)$ -forms. This expansion is closely related to the construction of the holomorphic affine structure on the Teichmüller space. We remark that one may also directly deduce this expansion from the local Torelli theorem for Calabi-Yau manifold and the Griffiths transversality.

Theorem 3.6. *Let $\Omega_p^c(\tau)$ be a canonical family defined by (22). Then we have the following expansion for $|\tau| < \epsilon$ small,*

$$(23) \quad [\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + A(\tau),$$

where $\{[\varphi_1 \lrcorner \Omega_p], \dots, [\varphi_N \lrcorner \Omega_p]\}$ give a basis of $H^{n-1,1}(M_p)$ and $A(\tau) = O(|\tau|^2) \in \bigoplus_{k=2}^n H^{n-k,k}(M_p)$ denotes terms of order at least 2 in τ .

Proof. By Theorem 3.2 and Proposition 3.5 we have

$$\begin{aligned}
(24) \quad \Omega_p^c(\tau) &= \Omega_p + \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \sum_{|I| \geq 2} \tau^I (\varphi_I \lrcorner \Omega_p) + \sum_{k \geq 2} \frac{1}{k!} (\wedge^k \varphi(\tau) \lrcorner \Omega_p) \\
&= \Omega_p + \sum_{i=1}^N \tau_i (\varphi_i \lrcorner \Omega_p) + \partial_{M_p} \left(\sum_{|I| \geq 2} \tau^I \psi_I \right) + a(\tau),
\end{aligned}$$

where

$$(25) \quad a(\tau) = \sum_{k \geq 2} \frac{1}{k!} (\wedge^k \varphi(\tau) \lrcorner \Omega_p) \in \bigoplus_{k \geq 2} A^{n-k, k}(M).$$

By Hodge theory, we have

$$\begin{aligned}
(26) \quad [\Omega_p^c(\tau)] &= [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + \left[\mathbb{H}(\partial_{M_p}(\sum_{|I| \geq 2} \tau^I \psi_I)) \right] + [\mathbb{H}(a(\tau))] \\
&= [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + [\mathbb{H}(a(\tau))].
\end{aligned}$$

Let $A(\tau) = [\mathbb{H}(a(\tau))]$, then (25) shows that $A(\tau) \in \bigoplus_{k=2}^n H^{n-k, k}(M)$ and $A(\tau) = O(|\tau|^2)$ which denotes the terms of order at least 2 in τ . \square

In fact we have the following expansion of the canonical family of $(n, 0)$ -classes up to order 2 in τ ,

$$[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^N \tau_i [\varphi_i \lrcorner \Omega_p] + \frac{1}{2} \sum_{i, j} \tau_i \tau_j [\mathbb{H}(\varphi_i \wedge \varphi_j \lrcorner \Omega_p)] + \Xi(\tau),$$

where $\Xi(\tau) = O(|\tau|^3)$ denotes terms of order at least 3 in τ , and $\Xi(\tau) \in \bigoplus_{k=2}^n H^{n-k, k}(M)$. This will not be needed in this paper.

4. AFFINE STRUCTURE ON THE TEICHMÜLLER SPACE

In Section 4.1, we fix a base point $p \in \mathcal{T}$ and introduce the unipotent space $N_+ \subseteq \check{D}$, which is biholomorphic to \mathbb{C}^d . Then we explain that the image $\Phi(\mathcal{T})$ is bounded in $N_+ \cap D$ with respect to the Euclidean metric on N_+ . In Section 4.2, using the property that $\Phi(\mathcal{T}) \subseteq N_+$ where $A \subseteq N_+$ is an Abelian subgroup, we define a holomorphic map $\Psi : \mathcal{T} \rightarrow A \cap D \subset A \simeq \mathbb{C}^N$. Then we use local Torelli theorem to show that Ψ defines a local coordinate chart around each point in \mathcal{T} , and this shows that $\Psi : \mathcal{T} \rightarrow A \cap D$ defines a holomorphic affine structure on \mathcal{T} . In this section we will use some properties of the period domain from Lie group and Lie algebra point of view, which can be found in Appendix A. The detailed proofs of the main results in this section are all contained in [18].

4.1. Boundedness of the period map. Now let us fix the base point $p \in \mathcal{T}$ with $\Phi(p) \in D$. Then according to Remark A.2 in the appendix, N_+ can be viewed as a subset in \check{D} by identifying it with its orbit in \check{D} with base point $\Phi(p)$. Let us also fix an adapted basis $(\eta_0, \dots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. Then we can identify elements in N_+ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. We define

$$\check{\mathcal{T}} = \Phi^{-1}(N_+).$$

At the base point $\Phi(p) = o \in N_+ \cap D$, the tangent space $T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+ \simeq N_+$, then the Hodge metric on $T_o^{1,0}D$ induces an Euclidean metric on N_+ . In the proof of the following theorem, for simplicity we require all the root vectors to be unit vectors with respect to this Euclidean metric.

Theorem 4.1. *The restriction of the period map $\Phi : \check{\mathcal{T}} \rightarrow N_+$ is bounded in N_+ with respect to the Euclidean metric on N_+ .*

The proof of this theorem depends on a slight extension of Harish-Chandra's proof of his famous embedding theorem of the Hermitian symmetric domains as bounded domains in complex Euclidean spaces. One may refer to Lemma 7 and Lemma 8 at pp. 582–583 in [13], Proposition 7.4 at pp. 385 and Ch VIII §7 at pp. 382–396 in [14], Proposition 1 at pp. 91 and Proof of Theorem 1 at pp. 95–97 in [21], and Lemma 2.2.12 at pp. 141–142 and §5.4 in [40] for more details. See [18] for the detail of the proof of the above theorem.

According to the definition of $\check{\mathcal{T}}$, it is not hard to conclude the following lemma, one may refer to [18] for the complete proof.

Lemma 4.2. *The subset $\check{\mathcal{T}}$ is an open dense submanifold in \mathcal{T} , and $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} with $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$.*

Corollary 4.3. *The image of $\Phi : \mathcal{T} \rightarrow D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+ .*

Proof. According to Lemma 4.2, $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of \mathcal{T} and the complex codimension of $\mathcal{T} \setminus \check{\mathcal{T}}$ is at least one; by Theorem 4.1, the holomorphic map $\Phi : \check{\mathcal{T}} \rightarrow N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\Phi' : \mathcal{T} \rightarrow N_+ \cap D$ such that $\Phi'|_{\check{\mathcal{T}}} = \Phi|_{\check{\mathcal{T}}}$. Since as holomorphic maps, Φ' and Φ agree on the open subset $\check{\mathcal{T}}$, they must be the same on the entire \mathcal{T} . Therefore, the image of Φ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on N_+ . As a consequence, we also get $\mathcal{T} = \check{\mathcal{T}} = \Phi^{-1}(N_+)$. \square

4.2. Holomorphic affine structure on the Teichmüller space. We first review the definition of complex affine manifolds. One may refer to page 215 of [20] for more details.

Definition 4.4. *Let M be a complex manifold of complex dimension n . If there is a coordinate cover $\{(U_i, \varphi_i); i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on \mathbb{C}^n whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i); i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M .*

There is a canonical Euclidean subspace $A \subset N_+$ given by $\mathfrak{a} = \Phi_*(T_p^{1,0}\mathcal{T}) \subset \mathfrak{n}_+$, the abelian subalgebra, and $A = \exp(\mathfrak{a})$ the corresponding Lie group. We then define the

holomorphic map

$$\Psi : \check{\mathcal{T}} \rightarrow A \cap D,$$

by $\Psi = P \circ \Phi|_{\check{\mathcal{T}}}$, where P is the projection map from $N_+ \cap D$ to $A \cap D$. By Theorem 4.3, we can extend the holomorphic map $\Psi : \check{\mathcal{T}} \rightarrow A \cap D$ over \mathcal{T} as

$$\Psi : \mathcal{T} \rightarrow A \cap D,$$

such that $\Psi = P \circ \Phi$.

By the local Torelli theorem for Calabi–Yau manifolds which implies that Ψ is nondegenerate, and the definition of holomorphic affine structure, we can prove with the main theorem of this section. Again we refer the reader to [18] for the detail of its proof.

Theorem 4.5. *The holomorphic map $\Psi : \mathcal{T} \rightarrow A \cap D$ defines a local coordinate around each point $q \in \mathcal{T}$. Thus the map Ψ itself gives a global holomorphic coordinate for \mathcal{T} with the transition maps all identity maps. In particular, the global holomorphic coordinate $\Psi : \mathcal{T} \rightarrow A \cap D \subset A \simeq \mathbb{C}^N$ defines a holomorphic affine structure on \mathcal{T} . Therefore, \mathcal{T} is a complex affine manifold.*

Note that this affine structure on \mathcal{T} depends on the choice of the base point p . Affine structures on \mathcal{T} defined in this ways by fixing different base point may not be compatible with each other.

5. HODGE METRIC COMPLETION OF THE TORELLI SPACE

This section contains a review of our extension of the affine structure to the Hodge metric completion \mathcal{T}^H of the Torelli space \mathcal{T} , as well as its consequences including a proof of the global Torelli theorem for polarized and marked Calabi–Yau manifolds on the Torelli space.

Recall that in Section 2.3, we have introduced the Hodge metric on the period domain D and its pull backs on \mathcal{Z}_m , \mathcal{T} and \mathcal{T}' via the period maps from the corresponding spaces. By the work of Viehweg in [35], we know that \mathcal{Z}_m is quasi-projective and consequently we can find a smooth projective compactification $\overline{\mathcal{Z}}_m$ such that \mathcal{Z}_m is Zariski open in $\overline{\mathcal{Z}}_m$ and the complement $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor of normal crossings. Therefore, \mathcal{Z}_m is dense and open in $\overline{\mathcal{Z}}_m$ with the complex codimension of the complement $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ at least one. Moreover as $\overline{\mathcal{Z}}_m$ is a compact space, it is a complete space.

Let us now take \mathcal{Z}_m^H to be the completion of \mathcal{Z}_m with respect to the Hodge metric. Then \mathcal{Z}_m^H is the smallest complete space with respect to the Hodge metric that contains \mathcal{Z}_m . Although the compact space $\overline{\mathcal{Z}}_m$ may not be unique, the Hodge metric completion space \mathcal{Z}_m^H is unique up to isometry, and is identified to the Griffiths extension as given in Theorem 9.5 and Theorem 9.6 in [9]. In particular, $\mathcal{Z}_m^H \subseteq \overline{\mathcal{Z}}_m$ and thus the complex codimension of the complement $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is at least one. By Lemma 2.7 in [18], we conclude the following useful result.

- The Hodge metric completion \mathcal{Z}_m^H is a dense and open smooth submanifold in $\overline{\mathcal{Z}}_m$ with $\text{codim}_{\mathbb{C}}(\mathcal{Z}_m^H \setminus \mathcal{Z}_m) \geq 1$, and $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ consists of those points in $\overline{\mathcal{Z}}_m$ around which the so-called Picard-Lefschetz transformations are trivial. Moreover the extended period map $\Phi_{\mathcal{Z}_m}^H : \mathcal{Z}_m^H \rightarrow D/\Gamma$ is proper and holomorphic.

The proof of this conclusion uses the Griffiths extension of the period map as well as the basic definitions of metric completion.

We denote by \mathcal{T}_m^H the universal cover of \mathcal{Z}_m^H , which is now a connected and simply connected complete smooth complex manifold. By using elementary topology argument, we show that the following commutative diagram holds:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m}^H} & D/\Gamma, \end{array}$$

where i is the inclusion map, i_m is a lifting of $i \circ \pi_m$, and Φ_m^H is a lifting of $\Phi_{\mathcal{Z}_m}^H \circ \pi_m^H$, and we fix a suitable choice of i_m and Φ_m^H such that $\Phi = \Phi_m^H \circ i_m$.

Let us denote $\mathcal{T}_m := i_m(\mathcal{T})$ and the restriction map $\Phi_m = \Phi_m^H|_{\mathcal{T}_m}$, then we also have $\Phi = \Phi_m \circ i_m$. Moreover, it is easy to see that Φ_m is also bounded by Corollary 4.3. By using argument of basic algebraic topology, we prove that the image \mathcal{T}_m equals to the preimage $(\pi_m^H)^{-1}(\mathcal{Z}_m)$, and moreover $i_m : \mathcal{T} \rightarrow \mathcal{T}_m$ is a covering map. See Proposition 2.8 of [18] for the details. Therefore, \mathcal{T}_m is a connected open complex submanifold in \mathcal{T}_m^H and $\text{codim}_{\mathbb{C}}(\mathcal{T}_m^H \setminus \mathcal{T}_m) \geq 1$. It is easy to see that $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is an analytic subvariety of \mathcal{Z}_m^H , and hence the set $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is also an analytic subvariety of \mathcal{T}_m^H .

Recall that we have fixed a base point $p \in \mathcal{T}$ and an adapted basis $\{\eta_0, \dots, \eta_{m-1}\}$ for the Hodge decomposition of the base point $\Phi(p) \in D$. With the fixed base point in D , we can identify N_+ with its unipotent orbit in D . Then applying the Riemann extension theorem to the bounded map $\Phi_m : \mathcal{T}_m \rightarrow N_+ \cap D$, we obtain the following lemma.

Lemma 5.1. *The map Φ_m^H is a bounded holomorphic map from \mathcal{T}_m^H to $N_+ \cap D$.*

Now by using the Riemann extension theorem, we can extend the map $\Psi_m : \mathcal{T}_m \rightarrow A \cap D$, which is also bounded by Lemma 5.1, to its extension

$$\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D,$$

such that $\Psi_m^H = P \circ \Phi_m^H$, where P is the projection map from $N_+ \cap D$ to its subspace $A \cap D$. Moreover, we also have $\Psi = P \circ \Phi = P \circ \Phi_m^H \circ i_m = \Psi_m^H \circ i_m$. Then by identifying the tangent bundle of \mathcal{T}_m with Hodge bundles and the property that the holomorphic map Ψ defines the holomorphic affine structure on \mathcal{T} , we can prove the following theorem.

Theorem 5.2. *The holomorphic map $\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$ is a local embedding, therefore it defines a global holomorphic affine structure on \mathcal{T}_m^H .*

Now by the completeness of \mathcal{T}_m^H with Hodge metric, Ψ_m^H is an isometry with Hodge metric at every point of \mathcal{T}_m^H , and a result of Griffiths and Wolf [12], we can conclude that the holomorphic map $\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$ is a covering map. It is easy to show that $A \cap D$ is simply connected, therefore Ψ_m^H must be a biholomorphic map.

Moreover, as $\Psi_m^H = P \circ \Phi_m^H$, where P is the projection map and Φ_m^H is a bounded map, we may conclude the injectivity of Φ_m^H . To conclude, we have the following theorem, and one may refer to [18] for its detailed proof.

Theorem 5.3. *For any $m \geq 3$, the holomorphic map $\Psi_m^H : \mathcal{T}_m^H \rightarrow A \cap D$ is an injection and hence bi-holomorphic. In particular, the completion space \mathcal{T}_m^H is a bounded domain $A \cap D$ in $A \simeq \mathbb{C}^N$. Moreover, the holomorphic map $\Phi_m^H : \mathcal{T}_m^H \rightarrow N_+ \cap D$ is also an injection.*

6. PROOF OF GLOBAL TORELLI ON TORELLI SPACE

In this section we describe the main idea of the proof of the global Torelli theorem on Torelli space. That is to prove that the period map $\Phi' : \mathcal{T}' \rightarrow D$ is an injective holomorphic map. The key step of the proof is to first show that $\mathcal{T}_m = i_m(\mathcal{T})$ is independent of m , and that \mathcal{T}_m is bi-holomorphic to the Torelli space \mathcal{T}' .

To start, one first notes that one direct consequence of Theorem 5.3 is the following proposition.

Proposition 6.1. *For any $m, m' \geq 3$, the complete complex manifolds \mathcal{T}_m^H and $\mathcal{T}_{m'}^H$ are biholomorphic to each other.*

This allows us to introduce the new simplified notations by dropping the level m .

Definition 6.2. *We define the complete complex manifold $\mathcal{T}^H = \mathcal{T}_m^H$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \rightarrow D$ by $\Phi^H = \Phi_m^H$ for any $m \geq 3$. In particular, with these new notations, we have the commutative diagram*

$$(27) \quad \begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow \pi_m & & \downarrow \pi_m^H & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m}^H} & D/\Gamma. \end{array}$$

Let $\mathcal{T}_0 \subset \mathcal{T}^H$ be defined by $\mathcal{T}_0 := i_{\mathcal{T}}(\mathcal{T})$. Since $\mathcal{T}_0 \simeq \mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$ for any $m \geq 3$, $\pi_m^H : \mathcal{T}_0 \rightarrow \mathcal{Z}_m$ is a covering map. Thus the fundamental group of \mathcal{T}_0 is a subgroup of the fundamental group of \mathcal{Z}_m , that is, $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_m)$ for any $m \geq 3$. Moreover, the universal property of the universal covering map $\pi_m : \mathcal{T} \rightarrow \mathcal{Z}_m$ with the identity $i_m \circ \pi_m = \pi_m^H|_{\mathcal{T}_0} \circ i_{\mathcal{T}}$ implies that $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}_0$ is also a covering map, as discussed in Section 5.

On the other hand, let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $m_k < m_{k+1}$ and $m_k | m_{k+1}$ for each $k \geq 1$. From the discussion of Lecture 10 of [23], or Page 5 of [30], there is a natural covering map from $\mathcal{Z}_{m_{k+1}}$ to \mathcal{Z}_{m_k} for each k . Thus the fundamental group $\pi_1(\mathcal{Z}_{m_{k+1}})$ is a subgroup of $\pi_1(\mathcal{Z}_{m_k})$ for each k . Hence, the inverse system of fundamental groups

$$\pi_1(\mathcal{Z}_{m_1}) \longleftarrow \pi_1(\mathcal{Z}_{m_2}) \longleftarrow \cdots \longleftarrow \pi_1(\mathcal{Z}_{m_k}) \longleftarrow \cdots$$

has an inverse limit, which is the fundamental group of the Torelli space \mathcal{T}' . Since $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_{m_k})$ for any k , we have the inclusion $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{T}')$. This implies that \mathcal{T}_0 is a covering of \mathcal{T}' . Let $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ be the covering map. Then the covering map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ together with diagram (12) and (27) implies that the following diagram is commutative

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}_0 & \xrightarrow{\Phi^H|_{\mathcal{T}_0}} & D \\ \downarrow \pi & \searrow & \downarrow \pi_0 & \nearrow \Phi' & \downarrow \pi_D \\ \mathcal{T} & & \mathcal{T}' & & \\ \downarrow \pi_m & \swarrow \pi'_m & \downarrow \pi_m^H|_{\mathcal{T}_0} & & \downarrow \pi_D \\ \mathcal{Z}_m & \xrightarrow{i_m} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m}^H} & D/\Gamma. \end{array}$$

Since $\Phi^H : \mathcal{T}^H \rightarrow D$ is injective, so is the restriction map $\Phi^H|_{\mathcal{T}_0} : \mathcal{T}_0 \rightarrow D$, which implies the injectivity of the map $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ by the relation $\Phi^H|_{\mathcal{T}_0} = \Phi' \circ \pi_0$. Therefore $\pi_0 : \mathcal{T}_0 \rightarrow \mathcal{T}'$ is biholomorphic.

In a summary, we have proved the following proposition. Note that in [18] we give a more elementary proof of this result by directly constructing the map π_0 .

Proposition 6.3. *Let $\mathcal{T}_0 \subset \mathcal{T}^H$ be defined by $\mathcal{T}_0 := i_{\mathcal{T}}(\mathcal{T})$. Then \mathcal{T}_0 is biholomorphic to the Torelli space \mathcal{T}' .*

From Proposition 6.3, we can see that the complete complex manifold \mathcal{T}^H is actually the completion space of the Torelli space \mathcal{T}' with respect to the Hodge metric.

Since the restriction map $\Phi^H|_{\mathcal{T}_0}$ is injective and $\Phi' = \Phi^H|_{\mathcal{T}_0} \circ (\pi_0)^{-1}$, we get the global Torelli theorem for the period map $\Phi' : \mathcal{T}' \rightarrow D$ from the Torelli space to the period domain as follows.

Theorem 6.4 (Global Torelli theorem). *The period map $\Phi' : \mathcal{T}' \rightarrow D$ is injective.*

We refer the reader to [18] for all of the details in the above proofs.

7. GLOBAL CANONICAL SECTIONS OF THE HOLOMORPHIC $(n, 0)$ -CLASSES

In this section we extend the local canonical sections of the holomorphic $(n, 0)$ -classes to the global canonical sections on the Hodge completion space \mathcal{T}^H , by constructing global holomorphic sections of the Hodge bundles $\{F^k\}_{k=0}^n$ over \mathcal{T}^H . Same results hold on the Techmüller space \mathcal{T} by pulling back through the covering map $i_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^H$.

Recall that we have fixed a base point $p \in \mathcal{T}^H$ and an adapted basis $\{\eta_0, \dots, \eta_{m-1}\}$ for the Hodge decomposition of the base point $\Phi^H(p) \in D$. With the fixed base point in D , we can identify the unipotent group N_+ with its unipotent orbit in \check{D} by identifying an element $c \in N_+$ with $[c] = cB$ in \check{D} .

On one hand, as we have fixed an adapted basis $\{\eta_0, \dots, \eta_{m-1}\}$ for the Hodge decomposition of the base point, the elements in $G_{\mathbb{C}}$ can be identified with a subset of the nonsingular block matrices. In particular, the elements in N_+ can be realized as nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. Namely, for any element $\{F_o^k\}_{k=0}^n \in N_+ \subseteq \check{D}$, there exists a unique nonsingular block lower triangular matrices $A(o) \in G_{\mathbb{C}}$ such that $(\eta_0, \dots, \eta_{m-1})A(o)$ is an adapted basis for the Hodge filtration $\{F_o^k\} \in N_+$ that represents this element in N_+ .

Similarly, any elements in B can be realized as nonsingular block upper triangular matrices in $G_{\mathbb{C}}$. Moreover, as $\check{D} = G_{\mathbb{C}}/B$, we have that for any $U \in G_{\mathbb{C}}$, which is a nonsingular block upper triangular matrix, $(\eta_0, \dots, \eta_{m-1})A(o)U$ is also an adapted basis for the Hodge filtration $\{F^k(o)\}_{k=0}^n$. Conversely, if $(\zeta_0, \dots, \zeta_{m-1})$ is an adapted basis for the Hodge filtration $\{F_o^k\}_{k=0}^n$, then there exists a unique $U \in G_{\mathbb{C}}$ such that $(\zeta_0, \dots, \zeta_{m-1}) = (\eta_0, \dots, \eta_{m-1})A(o)U$. For any $q \in \mathcal{T}^H$, let us denote the Hodge filtration at $q \in \mathcal{T}^H$ by $\{F_q^k\}_{k=0}^n$, and we have that $\{F_q^k\}_{k=0}^n \in N_+ \cap D$ by Theorem 5.3. Thus there exists a unique nonsingular block lower triangular matrices $\tilde{A}(q)$ such that $(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)$ is an adapted basis for the Hodge filtration $\{F_q^k\}_{k=0}^n$.

On the other hand, for any adapted basis $\{\zeta_0(q), \dots, \zeta_{m-1}(q)\}$ for the Hodge filtration $\{F_q^k\}_{k=0}^n$ at q , we know that there exists an $m \times m$ transition matrix $A(q)$ such that

$(\zeta_0(q), \dots, \zeta_{m-1}(q)) = (\eta_0, \dots, \eta_{m-1})A(q)$. Moreover, we set the blocks of $A(q)$ as in (32) and denote the (i, j) -th block of $A(q)$ by $A^{i,j}(q)$.

As both $(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)$ and $(\eta_0, \dots, \eta_{m-1})A(q)$ are adapted bases for the Hodge filtration for $\{F_q^k\}_{k=0}^n$, there exists a $U \in G_{\mathbb{C}}$ which is a block nonsingular upper triangular matrix such that $(\eta_0, \dots, \eta_{m-1})\tilde{A}(q)U = (\eta_0, \dots, \eta_{m-1})A(q)$. Therefore, we conclude that

$$(28) \quad \tilde{A}(q)U = A(q).$$

where $\tilde{A}(q)$ is a nonsingular block lower triangular matrix in $G_{\mathbb{C}}$ with all the diagonal blocks equal to identity submatrix, while U is a block upper triangular matrix in $G_{\mathbb{C}}$. However, according to basic linear algebra, we know that a nonsingular matrix $A(q) \in G_{\mathbb{C}}$ have the decomposition of the type in (28) if and only if the principal submatrices $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ are nonsingular for all $0 \leq k \leq n$.

To conclude, by Theorem 5.3, we have that $\Phi^H(q) \in N_+$ for any $q \in \mathcal{T}^H$. Therefore, for any adapted basis $(\zeta_0(q), \dots, \zeta_{m-1}(q))$, there exists a nonsingular block matrix $A(q) \in G_{\mathbb{C}}$ with $\det[A^{i,j}(q)]_{0 \leq i, j \leq n-k} \neq 0$ for any $0 \leq k \leq n$ such that $(\zeta_0(q), \dots, \zeta_{m-1}(q)) = (\eta_0, \dots, \eta_{m-1})A(q)$. Let $\{F_p^k\}_{k=0}^n$ be the reference Hodge filtration at the base point $p \in \mathcal{T}^H$. For any point $q \in \mathcal{T}^H$ with the corresponding Hodge filtrations $\{F_q^k\}_{k=0}^n$, we define the following maps

$$P_q^k : F_q^k \rightarrow F_p^k \quad \text{for any } 0 \leq k \leq n$$

to be the projection map with respect to the Hodge decomposition at the base point p . With the above notation, we therefore have the following lemma.

Lemma 7.1. *For any point $q \in \mathcal{T}^H$ and $0 \leq k \leq n$, the map $P_q^k : F_q^k \rightarrow F_p^k$ is an isomorphism. Furthermore, P_q^k depends on q holomorphically.*

Proof. We have already fixed $\{\eta_0, \dots, \eta_{m-1}\}$ as an adapted basis for the Hodge decomposition of the Hodge structure at the base point p . Thus it is also the adapted basis for the Hodge filtration $\{F_p^k\}_{k=0}^n$ at the base point. For any point $q \in \mathcal{T}^H$, let $\{\zeta_0, \dots, \zeta_{m-1}\}$ be an adapted basis for the Hodge filtration $\{F_q^k\}_{k=0}^n$ at q . Let $[A^{i,j}(q)]_{0 \leq i, j \leq n} \in G_{\mathbb{C}}$ be the transition matrix between the basis $\{\eta_0, \dots, \eta_{m-1}\}$ and $\{\zeta_0, \dots, \zeta_{m-1}\}$ for the same vector space $H^n(M, \mathbb{C})$. We have showed that $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ is nonsingular for all $0 \leq k \leq n$.

On the other hand, the submatrix $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ is the transition matrix between the bases of F_q^k and F_p^0 for any $0 \leq k \leq n$, that is

$$(\zeta_0(q), \dots, \zeta_{f^{k-1}}(q)) = (\eta_0, \dots, \eta_{m-1})[A^{i,j}(q)]_{0 \leq i, j \leq n-k} \quad \text{for any } 0 \leq k \leq n,$$

where $(\zeta_0(q), \dots, \zeta_{f^{k-1}}(q))$ and $(\eta_0, \dots, \eta_{m-1})$ are the bases for F_q^k and F_p^0 respectively. Thus the matrix of P_q^k with respect to $\{\eta_0, \dots, \eta_{f^{k-1}}\}$ and $\{\zeta_0, \dots, \zeta_{f^{k-1}}\}$ is the first $(n-k+1) \times (n-k+1)$ principal submatrix $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ of $[A^{i,j}(q)]_{0 \leq i, j \leq n}$. Now since $[A^{i,j}(q)]_{0 \leq i, j \leq n-k}$ for any $0 \leq k \leq n$ is nonsingular, we conclude that the map P_q^k is an isomorphism for any $0 \leq k \leq n$.

From our construction, it is clear that the projection P_q^k depends on q holomorphically. \square

Using this lemma, we are ready to construct the global holomorphic sections of Hodge bundles over \mathcal{T}^H . For any $0 \leq k \leq n$, we know that $\{\eta_0, \eta_1, \dots, \eta_{f^{k-1}}\}$ is an adapted

basis of the Hodge decomposition of F_p^k for any $0 \leq k \leq n$. Then we define the sections

$$(29) \quad s_i : \mathcal{T}^H \rightarrow F^k, \quad q \mapsto (P_q^k)^{-1}(\eta_i) \in F_q^k \quad \text{for any } 0 \leq i \leq f^k - 1.$$

Lemma 7.1 implies that $\{(P_q^k)^{-1}(\eta_i)\}_{i=0}^{f^k-1}$ form a basis of F_q^k for any $q \in \mathcal{T}^H$. In fact, we have proved the following theorem for polarized and marked Calabi–Yau manifolds.

Theorem 7.2. *For all $0 \leq k \leq n$, the Hodge bundles F^k over \mathcal{T}^H are trivial bundles, and the trivialization can be obtained by $\{s_i\}_{0 \leq i \leq f^k-1}$ which is defined in (29). In particular, the section $s_0 : \mathcal{T}^H \rightarrow F^n$ is a global nowhere zero section of the Hodge bundle F^n for Calabi–Yau manifolds.*

With the adapted basis at the base point $p \in \mathcal{T}^H$, we can also see $\Phi_*(\mathbb{T}_p^{1,0}(\mathcal{T})) = \mathfrak{a} \subset \mathfrak{n}_+$ as a block lower triangle matrix whose diagonal elements are zero. Moreover by local Torelli theorem for Calabi–Yau manifolds, we can conclude that \mathfrak{a} is isomorphic to its $(1,0)$ -block as vector spaces, see (32) for the definition. Let $(\tau_1, \dots, \tau_N)^T$ be the $(1,0)$ -block of \mathfrak{a} . Since the affine structure on A is induced by $\exp : \mathfrak{a} \rightarrow A$ which is an isomorphism, $(\tau_1, \dots, \tau_N)^T$ also defines a global affine structure on A , and hence on \mathcal{T}^H . We denote it by

$$\tau^H : \mathcal{T}^H \rightarrow \mathbb{C}^N, \quad q \mapsto (\tau_1(q), \dots, \tau_N(q)).$$

Note that from linear algebra, it is easy to see that the $(1,0)$ -block of $A = \exp(\mathfrak{a})$ is still $(\tau_1, \dots, \tau_N)^T$. Hence the affine map defined as above can be constructed as the $(1,0)$ -block of the image of the period map. More precisely, let $P^{1,0} : N_+ \rightarrow \mathbb{C}^N$ to be the projection of the matrices in N_+ onto their $(1,0)$ -blocks. Then the affine map is

$$\tau^H = P^{1,0} \circ \Phi^H : \mathcal{T}^H \rightarrow \mathbb{C}^N.$$

Moreover τ^H is injective, and hence it defines another embedding of \mathcal{T}^H into \mathbb{C}^N .

Remark 7.3. *The global affine coordinate $\tau^H : \mathcal{T}^H \rightarrow \mathbb{C}^N$ extends the local holomorphic flat coordinate around the base point $p \in \mathcal{T}^H$ in Theorem 3.2.*

Using the same notation as in Lemma 3.6, we are ready to prove the following theorem for Calabi–Yau manifolds,

Theorem 7.4. *Choose $[\Omega_p] = \eta_0$, then the section s_0 of F^n is a global holomorphic extension of the local section $[\Omega_p^c(\tau)]$.*

Proof. Because both s_0 and $[\Omega_p^c(\tau)]$ are holomorphic sections of F^n , we only need to show that $s_0|_{U_p} = [\Omega_p^c(\tau)]$. In fact, from the expansion formula (23), we have that for any $q \in U_p$

$$P_q^n([\Omega_p^c(\tau(q))]) = [\Omega_p] = \eta_0.$$

Therefore, $[\Omega_p^c(\tau(q))] = (P_q^n)^{-1}(\eta_0) = s_0(q)$ for any point $q \in U_p$. □

APPENDIX A. CLASSIFYING SPACES AND HODGE STRUCTURES FROM LIE GROUP POINT OF VIEW.

In this appendix, we review some properties of the period domain from Lie group and Lie algebra point of view. All of the results in this section is well-known to the experts in the subject. The purpose to give details is to fix notations. One may either skip this section or refer to [11] and [25] for most of the details.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over \mathbb{Q} . Let us simply denote $H_{\mathbb{C}} = H^n(M, \mathbb{C})$ and $H_{\mathbb{R}} = H^n(M, \mathbb{R})$. The group of the \mathbb{C} -rational points is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

which acts on \check{D} transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \rightarrow D$. Fix a point $p \in \mathcal{T}$ with the image $o := \Phi(p) = \{F_p^n \subset \cdots \subset F_p^0\} \in D$. The points $p \in \mathcal{T}$ and $o \in D$ may be referred as the base points or the reference points. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_p^k = F_p^k$ for each k . Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} \mid gF_p^k = F_p^k, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D} \quad \text{with} \quad V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B$. The Lie algebra \mathfrak{g} of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \text{End}(H_{\mathbb{C}}) \mid Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, \mathfrak{g}_0 becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k, n-k}\}_{k=0}^n$ of $H^n(M, \mathbb{C})$ induces a Hodge structure of weight zero on $\text{End}(H^n(M, \mathbb{C}))$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k, -k} \quad \text{with} \quad \mathfrak{g}^{k, -k} = \{X \in \mathfrak{g} \mid XH_p^{r, n-r} \subseteq H_p^{r+k, n-r-k}\}.$$

Since the Lie algebra \mathfrak{b} of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k, -k}.$$

The Lie algebra \mathfrak{v}_0 of V is $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of \check{D} at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \bigoplus_{k \geq 1} \mathfrak{g}^{-k, k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We take the unipotent group $N_+ = \exp(\mathfrak{n}_+)$.

As $\text{Ad}(g)(\mathfrak{g}^{k, -k})$ is in $\bigoplus_{i \geq k} \mathfrak{g}^{i, -i}$ for each $g \in B$, the sub-Lie algebra $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\text{Ad}(B)$ -invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$ -invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by $T_{o,h}^{1,0}\check{D}$, and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point o , restricted to D , determines a subbundle $T_{o,h}^{1,0}D$ of the holomorphic tangent bundle $T_o^{1,0}D$

of D at the base point. The $G_{\mathbb{C}}$ -invariance of $T_{o,h}^{1,0}\check{D}$ implies the $G_{\mathbb{R}}$ -invariance of $T_{o,h}^{1,0}D$. As another interpretation of this holomorphic horizontal bundle at the base point, one has

$$(30) \quad T_{o,h}^{1,0}\check{D} \simeq T_o^{1,0}\check{D} \cap \bigoplus_{k=1}^n \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k).$$

In [25], Schmid call a holomorphic mapping $\Psi : M \rightarrow \check{D}$ of a complex manifold M into \check{D} *horizontal* if at each point of M , the induced map between the holomorphic tangent spaces takes values in the appropriate fibre $T^{1,0}\check{D}$. It is easy to see that the period map $\Phi : \mathcal{T} \rightarrow D$ is horizontal since $\Phi_*(T_p^{1,0}\mathcal{T}) \subseteq T_{o,h}^{1,0}D$ for any $p \in \mathcal{T}$. Since D is an open set in \check{D} , we have the following relation:

$$(31) \quad T_{o,h}^{1,0}D = T_{o,h}^{1,0}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+.$$

Remark A.1. *With a fixed base point, we can identify N_+ with its unipotent orbit in \check{D} by identifying an element $c \in N_+$ with $[c] = cB$ in \check{D} ; that is, $N_+ = N_+(\text{base point}) \cong N_+B/B \subseteq \check{D}$. In particular, when the base point o is in D , we have $N_+ \cap D \subseteq D$.*

Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. For any $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \{\xi_0, \xi_1, \dots, \xi_N, \dots, \xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}, \dots, \xi_{f^2}, \dots, \xi_{f^1-1}, \xi_{f^0-1}\}$$

of $H^n(M_p, \mathbb{C})$ an *adapted basis for the given Hodge decomposition*

$$H^n(M_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \dots \oplus H_p^{1,n-1} \oplus H_p^{0,n},$$

if it satisfies $H_p^{k,n-k} = \text{Span}_{\mathbb{C}}\{\xi_{f^{k+1}}, \dots, \xi_{f^{k-1}}\}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$. We call a basis

$$\zeta = \{\zeta_0, \zeta_1, \dots, \zeta_N, \dots, \zeta_{f^{k+1}}, \dots, \zeta_{f^{k-1}}, \dots, \zeta_{f^2}, \dots, \zeta_{f^1-0}, \zeta_{f^0-1}\}$$

of $H^n(M_p, \mathbb{C})$ an *adapted basis for the given filtration*

$$F^n \subseteq F^{n-1} \subseteq \dots \subseteq F^0$$

if it satisfies $F^k = \text{Span}_{\mathbb{C}}\{\zeta_0, \dots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$. Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^0$. The blocks of the $m \times m$ matrix T is set as follows: for each $0 \leq \alpha, \beta \leq n$, the (α, β) -th block $T^{\alpha, \beta}$ is

$$(32) \quad T^{\alpha, \beta} = [T_{ij}(\tau)]_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n-1}, f^{-\beta+n+1} \leq j \leq f^{-\beta+n-1}},$$

where T_{ij} is the entries of the matrix T , and f^{n+1} is defined to be zero. In particular, $T = [T^{\alpha, \beta}]$ is called a *block lower triangular matrix* if $T^{\alpha, \beta} = 0$ whenever $\alpha < \beta$.

Remark A.2. *We remark that by fixing a base point, we can identify the above quotient Lie groups or Lie algebras with their orbits in the corresponding quotient Lie algebras or Lie groups. For example, $\mathfrak{n}_+ \cong \mathfrak{g}/\mathfrak{b}$, $\mathfrak{g}^{-1,1} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$, and $N_+ \cong N_+B/B \subseteq \check{D}$. We can also identify a point $\Phi(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \dots \subseteq F_p^0\} \in D$ with its Hodge decomposition $\bigoplus_{k=0}^n H_p^{k,n-k}$, and thus with any fixed adapted basis of the corresponding Hodge decomposition for the base point, we have matrix representations of elements in the above Lie groups and Lie algebras. For example, elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices.*

We shall review and collect some facts about the structure of simple Lie algebra \mathfrak{g} in our case. Again one may refer to [11] and [25] for more details. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the Weil operator, which is defined by

$$\theta(X) = (-1)^p X \quad \text{for } X \in \mathfrak{g}^{p,-p}.$$

Then θ is an involutive automorphism of \mathfrak{g} , and is defined over \mathbb{R} . The $(+1)$ and (-1) eigenspaces of θ will be denoted by \mathfrak{k} and \mathfrak{p} respectively. Moreover, set

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

The fact that θ is an involutive automorphism implies

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let us consider $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then \mathfrak{g}_c is a real form for \mathfrak{g} . Recall that the killing form $B(\cdot, \cdot)$ on \mathfrak{g} is defined by

$$B(X, Y) = \text{Trace}(\text{ad}(X) \circ \text{ad}(Y)) \quad \text{for } X, Y \in \mathfrak{g}.$$

A semisimple Lie algebra is compact if and only if the Killing form is negative definite. Thus it is not hard to check that \mathfrak{g}_c is actually a compact real form of \mathfrak{g} , while \mathfrak{g}_0 is a non-compact real form. Recall that $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$ is the subgroup which corresponds to the subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. Let us denote the connected subgroup $G_c \subseteq G_{\mathbb{C}}$ which corresponds to the subalgebra $\mathfrak{g}_c \subseteq \mathfrak{g}$. Let us denote the complex conjugation of \mathfrak{g} with respect to the compact real form by τ_c , and the complex conjugation of \mathfrak{g} with respect to the compact real form by τ_0 .

The intersection $K = G_c \cap G_{\mathbb{R}}$ is then a compact subgroup of $G_{\mathbb{R}}$, whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}_c$. With the above notations, Schmid showed in [25] that K is a maximal compact subgroup of $G_{\mathbb{R}}$, and it meets every connected component of $G_{\mathbb{R}}$. Moreover, $V = G_{\mathbb{R}} \cap B \subseteq K$. As remarked in §1 in [11] of Griffiths and Schmid, one gets that \mathfrak{v} must have the same rank of \mathfrak{g} as \mathfrak{v} is the intersection of the two parabolic subalgebras \mathfrak{b} and $\tau_c(\mathfrak{b})$. Moreover, \mathfrak{g}_0 and \mathfrak{v}_0 are also of equal rank, since they are real forms of \mathfrak{g} and \mathfrak{v} respectively. Therefore, we have the following proposition.

Proposition A.3. *There exists a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and \mathfrak{h}_0 is also a Cartan subalgebra of \mathfrak{k}_0 .*

Proposition A.3 implies that the simple Lie algebra \mathfrak{g}_0 in our case is a simple Lie algebra of first category as defined in §4 in [28]. In the upcoming part, we will briefly derive the result of a simple Lie algebra of first category in Lemma 3 in [29]. One may also refer to [40] Lemma 2.2.12 at pp. 141-142 for the same result.

Let us still use the above notations of the Lie algebras we consider. By Proposition 4, we can take \mathfrak{h}_0 to be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and \mathfrak{h}_0 is also a Cartan subalgebra of \mathfrak{k}_0 . Let us denote \mathfrak{h} to be the complexification of \mathfrak{h}_0 . Then \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Write $\mathfrak{h}_0^* = \text{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_{\mathbb{R}}^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_{\mathbb{R}}^*$ can be identified with $\mathfrak{h}_{\mathbb{R}} := \sqrt{-1}\mathfrak{h}_0$ by duality using the restriction of the Killing form B of \mathfrak{g} to $\mathfrak{h}_{\mathbb{R}}$. Let $\rho \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$, one can define the following subspace of \mathfrak{g}

$$\mathfrak{g}^{\rho} = \{x \in \mathfrak{g} \mid [h, x] = \rho(h)x \quad \text{for all } h \in \mathfrak{h}\}.$$

An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ is called a root of \mathfrak{g} with respect to \mathfrak{h} if $\mathfrak{g}^{\varphi} \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ denote the space of nonzero \mathfrak{h} -roots. Then each root space

$$\mathfrak{g}^{\varphi} = \{x \in \mathfrak{g} \mid [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$$

belongs to some $\varphi \in \Delta$ is one-dimensional over \mathbb{C} , generated by a root vector e_{φ} .

Since the involution θ is a Lie-algebra automorphism fixing \mathfrak{k} , we have $[h, \theta(e_{\varphi})] = \varphi(h)\theta(e_{\varphi})$ for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_{\varphi})$ is also a root vector belonging to the root φ , so e_{φ} must be an eigenvector of θ . It follows that there is a decomposition of the roots Δ into $\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_{\varphi} \subseteq \mathfrak{k}$ and \mathfrak{p} respectively. The adjoint representation of \mathfrak{h} on \mathfrak{g} determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^{\varphi}.$$

There also exists a Weyl base $\{h_i, 1 \leq i \leq l; e_{\varphi}, \text{ for any } \varphi \in \Delta\}$ with $l = \text{rank}(\mathfrak{g})$ such that $\text{Span}_{\mathbb{C}}\{h_1, \dots, h_l\} = \mathfrak{h}$, $\text{Span}_{\mathbb{C}}\{e_{\varphi}\} = \mathfrak{g}^{\varphi}$ for each $\varphi \in \Delta$, and

$$\tau_c(h_i) = \tau_0(h_i) = -h_i, \quad \text{for any } 1 \leq i \leq l;$$

$$\tau_c(e_{\varphi}) = \tau_0(e_{\varphi}) = -e_{-\varphi} \quad \text{for any } \varphi \in \Delta_{\mathfrak{k}}; \quad \tau_0(e_{\varphi}) = -\tau_c(e_{\varphi}) = e_{\varphi} \quad \text{for any } \varphi \in \Delta_{\mathfrak{p}}.$$

With respect to this Weyl base, we have

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{h}_0 + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}(e_{\varphi} - e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{k}}} \mathbb{R}\sqrt{-1}(e_{\varphi} + e_{-\varphi}); \\ \mathfrak{p}_0 &= \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}(e_{\varphi} + e_{-\varphi}) + \sum_{\varphi \in \Delta_{\mathfrak{p}}} \mathbb{R}\sqrt{-1}(e_{\varphi} - e_{-\varphi}). \end{aligned}$$

Let us now introduce a lexicographic order (cf. pp.41 in [40] or pp.416 in [28]) in the real vector space $\mathfrak{h}_{\mathbb{R}}$ as follows: we fix an ordered basis e_1, \dots, e_l for $\mathfrak{h}_{\mathbb{R}}$. Then for any $h = \sum_{i=1}^l \lambda_i e_i \in \mathfrak{h}_{\mathbb{R}}$, we call $h > 0$ if the first nonzero coefficient is positive, that is, if $\lambda_1 = \dots = \lambda_k = 0, \lambda_{k+1} > 0$ for some $1 \leq k < l$. For any $h, h' \in \mathfrak{h}_{\mathbb{R}}$, we say $h > h'$ if $h - h' > 0$, $h < h'$ if $h - h' < 0$ and $h = h'$ if $h - h' = 0$. Now let us first choose a *maximal* linearly independent subset $S = \{s_1, \dots, s_k\}$ of $\Delta_{\mathfrak{k}}$, and then choose a linearly independent subset $E = \{e_1, \dots, e_{l-k}\}$ of $\Delta_{\mathfrak{p}}$ such that $E \cup S$ is a basis for $\mathfrak{h}_{\mathbb{R}}^*$, where l is the real dimension of $\mathfrak{h}_{\mathbb{R}}^*$. Now we order this basis $E \cup S$ as $\{e_1, \dots, e_{l-k}, s_1, \dots, s_k\}$, namely, we put the noncompact roots in front of the compact ones. Then we define the above lexicographic order in $\mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$. Then we define Δ^{\pm} , $\Delta_{\mathfrak{p}}^{\pm}$, and $\Delta_{\mathfrak{k}}^{\pm}$.

Definition A.4. *Two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$, which is denoted by $\varphi \perp \psi$.*

For the real simple Lie algebra $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ which has a Cartan subalgebra \mathfrak{h}_0 in \mathfrak{k}_0 , the maximal abelian subspace of \mathfrak{p}_0 can be described as in the following lemma, which is a slight extension of a lemma of Harish-Chandra in [13]. One may refer to Lemma 3 in [29] or Lemma 2.2.12 at pp.141–142 in [40] for more details. For reader's convenience we give the detailed proof.

Lemma A.5. *There exists a set of strongly orthogonal noncompact positive roots $\Lambda = \{\varphi_1, \dots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$ such that*

$$\mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R}(e_{\varphi_i} + e_{-\varphi_i})$$

is a maximal abelian subspace in \mathfrak{p}_0 .

This lemma is a slight extension of a lemma of Harish-Chandra in [13]. One may refer to Lemma 3 in [29] or Lemma 2.2.12 at pp.141–142 in [40] for more details. One may also find the detailed proof in [18].

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