

NEW RESULTS OF INTERSECTION NUMBERS ON MODULI SPACES OF CURVES

KEFENG LIU AND HAO XU

ABSTRACT. We present a series of new results we obtained recently about the intersection numbers of tautological classes on moduli spaces of curves, including a simple formula of the n -point functions for Witten's τ classes, an effective recursion formula to compute higher Weil-Petersson volumes, several new recursion formulae of intersection numbers and our proof of a conjecture of Itzykson and Zuber concerning denominators of intersection numbers. We also present Virasoro and KdV properties of generating functions of general mixed κ and ψ intersections.

Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let ψ_i be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the i -th marked point. Let λ_i be the i -th Chern class of the Hodge bundle \mathbb{E} , whose fiber over each pointed stable curves is $H^0(C, \omega_C)$.

We also have the κ classes originally defined by Mumford [1], Morita [2] and Miller [3]. A more natural variation was later given by Arbarello-Cornalba [4]. It is known that the κ and ψ classes generate the tautological cohomology ring of the moduli spaces, and most of the known cohomology classes are tautological.

The following intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \prod_{j \geq 1} \kappa_j^{b_j} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{j \geq 1} \kappa_j^{b_j}.$$

are called the higher Weil-Petersson volumes [5]. These are important invariants of moduli spaces of curves.

In 1990, Witten [6] made the remarkable conjecture that the generating function of intersection numbers of ψ classes on moduli spaces are governed by KdV hierarchy. Witten's conjecture (first proved by Kontsevich [7]) is among the deepest known properties of moduli spaces of curves and motivated a surge of subsequent developments.

The intersection theory of tautological classes on the moduli space of curves is a very important subject and has close connections to string theory, quantum gravity and many branches of mathematics.

The n -point functions for intersection numbers

Definition 1. We call the following generating function

$$F(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_j = 3g - 3 + n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^n x_j^{d_j}$$

the n -point function.

Consider the following “normalized” n -point function

$$G(x_1, \dots, x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right) \cdot F(x_1, \dots, x_n).$$

Starting from 1-point function $G(x) = \frac{1}{x^2}$, we can obtain any n -point function recursively by the following theorem.

Theorem 2. [8] For $n \geq 2$,

$$G(x_1, \dots, x_n) = \sum_{r, s \geq 0} \frac{(2r + n - 3)!!}{4^s (2r + 2s + n - 1)!!} P_r(x_1, \dots, x_n) \Delta(x_1, \dots, x_n)^s,$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\begin{aligned} \Delta(x_1, \dots, x_n) &= \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3}, \\ P_r(x_1, \dots, x_n) &= \left(\frac{1}{2 \sum_{j=1}^n x_j} \sum_{\underline{n}=I \amalg J} \left(\sum_{i \in I} x_i \right)^2 \left(\sum_{i \in J} x_i \right)^2 G(x_I) G(x_J) \right)_{3r+n-3} \\ &= \frac{1}{2 \sum_{j=1}^n x_j} \sum_{\underline{n}=I \amalg J} \left(\sum_{i \in I} x_i \right)^2 \left(\sum_{i \in J} x_i \right)^2 \sum_{r'=0}^r G_{r'}(x_I) G_{r-r'}(x_J), \end{aligned}$$

where $I, J \neq \emptyset$, $\underline{n} = \{1, 2, \dots, n\}$ and $G_g(x_I)$ denotes the degree $3g + |I| - 3$ homogeneous component of the normalized $|I|$ -point function $G(x_{k_1}, \dots, x_{k_{|I|}})$, where $k_j \in I$.

Thus we have an elementary and more efficient algorithm to calculate all intersection numbers of ψ classes other than the celebrated Witten-Kontsevich theorem.

Since $P_0(x, y) = \frac{1}{x+y}$, $P_r(x, y) = 0$ for $r > 0$ and

$$P_r(x, y, z) = \frac{r!}{2^r (2r+1)!} \frac{(xy)^r (x+y)^{r+1} + (yz)^r (y+z)^{r+1} + (zx)^r (z+x)^{r+1}}{x+y+z},$$

we recover Dijkgraaf’s 2-point function and Zagier’s 3-point function obtained more than ten years ago.

There is another slightly different formula of the n -point functions. When $n = 3$, this has also been obtained by Zagier.

Theorem 3. [8] For $n \geq 2$,

$$F(x_1, \dots, x_n) = \exp\left(\frac{(\sum_{j=1}^n x_j)^3}{24}\right) \sum_{r, s \geq 0} \frac{(-1)^s P_r(x_1, \dots, x_n) \Delta(x_1, \dots, x_n)^s}{8^s (2r + 2s + n - 1) s!}$$

where P_r and Δ are the same polynomials as defined in Theorem 2.

Okounkov [9] obtained an analytic expression of the n -point functions using n -dimensional error-function-type integrals. Brézin and Hikami [10] use correlation functions of GUE ensemble to find explicit formulae of n -point functions.

Recursion formulae of higher Weil-Petersson volumes

We have discovered a general recursion formula of higher Weil-Petersson volumes [12], which is a vast generalization of the Mirzakhani recursion formula [11].

First we fix notations as in [5].

Consider the semigroup N^∞ of sequences $\mathbf{m} = (m_1, m_2, \dots)$ where m_i are non-negative integers and $m_i = 0$ for sufficiently large i .

Let $\mathbf{m}, \mathbf{t}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$, and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i \geq 1} i m_i, \quad \|\mathbf{m}\| := \sum_{i \geq 1} m_i, \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m_i}, \quad \mathbf{m}! := \prod_{i \geq 1} m_i!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i \geq 1} \binom{m_i}{t_i}, \quad \binom{\mathbf{m}}{\mathbf{a}_1, \dots, \mathbf{a}_n} := \prod_{i \geq 1} \binom{m_i}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i \geq 1} \kappa_i^{b_i}.$$

Theorem 4. [12] *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$.*

$$\begin{aligned} & (2d_1 + 1)!! \langle \kappa(\mathbf{b}) \prod_{j=1}^n \tau_{d_j} \rangle_g \\ &= \sum_{j=2}^n \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{L}') \tau_{|\mathbf{L}| + d_1 + d_j - 1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\ &+ \frac{1}{2} \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{L}') \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{e} + \mathbf{f} = \mathbf{b} \\ I \sqcup J = \{2, \dots, n\}}} \sum_{r+s=|\mathbf{L}| + d_1 - 2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r + 1)!! (2s + 1)!! \\ &\quad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \end{aligned}$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} \frac{(-1)^{|\mathbf{L}'|} \alpha_{\mathbf{L}}}{\mathbf{L}'! (2|\mathbf{L}'| + 1)!!} = 0, \quad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \mathbf{L}' \neq \mathbf{0}}} \frac{(-1)^{|\mathbf{L}'| - 1} \alpha_{\mathbf{L}}}{\mathbf{L}'! (2|\mathbf{L}'| + 1)!!}, \quad \mathbf{b} \neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.

The proof of the above theorem is to use Witten-Kontsevich theorem, a combinatorial formula in [5] expressing κ classes by ψ classes and the following elementary but crucial lemma [12].

Lemma 5. *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_{\mathbf{0}}\beta_{\mathbf{0}} = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\beta_{\mathbf{L}'} = 0, \quad \mathbf{b} \neq \mathbf{0}.$$

Then the following two identities are equivalent.

$$\begin{aligned} G(\mathbf{b}, n) &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \\ F(\mathbf{b}, n) &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N} \end{aligned}$$

When $\mathbf{b} = (l, 0, 0, \dots)$, Theorem 4 recovers Mirzakhani's recursion formula of Weil-Petersson volumes for moduli spaces of bordered Riemann surfaces [13, 14, 15, 16].

Theorem 4 also provides an effective algorithm to compute higher Weil-Petersson volumes recursively.

In fact we can use the main formula in [5] to generalize almost all pure ψ intersections to identities of higher Weil-Petersson volumes which share similar structures as Theorem 4. For example, the identities in the following theorem are generalizations of the string and dilaton equations.

Theorem 6. [12] *For $\mathbf{b} \in N^\infty$ and $d_j \geq 0$,*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = \sum_{j=1}^n \langle \tau_{d_j-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \rangle_g,$$

and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g - 2 + n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.$$

Note that Theorem 6 generalizes the results in [17].

New identities of intersection numbers

The next two theorems follow from a detailed study of coefficients of the n -point functions in Theorem 2.

Theorem 7. [8] *We have*

(1) *Let $k > 2g$, $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g + n - k$.*

$$\sum_{\mathbf{z}=I} \sum_{\prod J} \sum_{j=0}^k (-1)^j \langle \tau_j \tau_0^2 \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{k-j} \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} = 0.$$

(2) Let $d_j \geq 1$ and $\sum_{j=1}^n d_j = g + n$.

$$\sum_{\underline{n}=I \amalg J} \sum_{j=0}^{2g} (-1)^j \langle \tau_j \tau_0^2 \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-j} \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} = \frac{(2g+n+1)!}{4^g (2g+1)! \prod_{j=1}^n (2d_j-1)!}.$$

Theorem 8. [18, 8] *We have*

(1) Let $k > 2g$, $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g + n - k - 1$.

$$\sum_{j=0}^k (-1)^j \langle \tau_{k-j} \tau_j \prod_{i=1}^n \tau_{d_i} \rangle_g = 0.$$

(2) Let $d_j \geq 1$ and $\sum_{j=1}^n (d_j - 1) = g - 1$.

$$\sum_{j=0}^{2g} (-1)^j \langle \tau_{2g-j} \tau_j \prod_{i=1}^n \tau_{d_i} \rangle_g = \frac{(2g+n-1)!}{4^g (2g+1)! \prod_{j=1}^n (2d_j-1)!}.$$

In fact, it's easy to see that Theorems 7 and 8 imply each other through the following proposition.

Proposition 9. [8] *Let $d_j \geq 0$ and $\sum_{j=1}^n d_j = g + n$.*

$$\begin{aligned} \sum_{\underline{n}=I \amalg J} \sum_{j=0}^{2g} (-1)^j \left(\langle \tau_j \tau_0^2 \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-j} \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} + \langle \tau_j \tau_{2g-j} \tau_0^2 \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \right) \\ = (2g+n+1) \sum_{j=0}^{2g} (-1)^j \langle \tau_0 \tau_j \tau_{2g-j} \prod_{i=1}^n \tau_{d_i} \rangle_g \end{aligned}$$

Since $\text{ch}_k(\mathbb{E}) = 0$ for $k > 2g$, $\lambda_g \lambda_{g-1} = (-1)^{g-1} (2g-1)! \text{ch}_{2g-1}(\mathbb{E})$, by Mumford's formula [1] of the Chern character of Hodge bundles, it's not difficult to see that Theorem 8 implies the following theorem.

Theorem 10. [18, 8] *Let k be an even number and $k \geq 2g$, $d_j \geq 0$, $\sum_{j=1}^n d_j = 3g + n - k - 2$.*

$$\begin{aligned} (1) \quad \langle \prod_{j=1}^n \tau_{d_j} \tau_k \rangle_g &= \sum_{j=1}^n \langle \tau_{d_j+k-1} \prod_{i \neq j} \tau_{d_i} \rangle_g \\ &\quad - \frac{1}{2} \sum_{\underline{n}=I \amalg J} \sum_{j=0}^{k-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{k-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \end{aligned}$$

Note that when $k = 2g$, the above theorem is equivalent to the following Hodge integral identity [19] (also known as Faber's intersection number conjecture [20])

$$\int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)! |B_{2g}|}{2^{2g-1} (2g)! \prod_{j=1}^n (2d_j-1)!}.$$

where $\sum_{j=1}^n (d_j - 1) = g - 2$ and $d_j \geq 1$.

The above $\lambda_g \lambda_{g-1}$ integral follows from degree 0 Virasoro constraints for \mathbb{P}^2 announced by Givental [21]. However it is very desirable to have a direct proof of identity (1) when $k = 2g$, possibly using our explicit formulae of the n -point functions (see also [22]).

As pointed out in the last section, we can generalize all of the above new recursion formulae of ψ classes to identities of higher Weil-Petersson volumes. For example, we may generalize Proposition 9 and Theorem 10 to the following

Proposition 11. [12] *Let $\mathbf{b} \in N^\infty$, $d_j \geq 0$.*

$$\begin{aligned} & \sum_{j=0}^{2g} (-1)^j \langle \tau_0 \tau_1 \tau_j \tau_{2g-j} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g \\ &= \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \underline{n} = I \amalg J}} \sum_{j=0}^{2g} (-1)^j \binom{\mathbf{b}}{\mathbf{L}} \left(\langle \tau_j \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle \langle \tau_{2g-j} \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle \right. \\ & \quad \left. + \langle \tau_j \tau_{2g-j} \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle \right) \end{aligned}$$

Theorem 12. [12] *Let $\mathbf{b} \in N^\infty$, $M \geq 2g$ be an even number and $d_j \geq 0$.*

$$\begin{aligned} & \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+M} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = \sum_{j=1}^n \langle \tau_{d_j+M-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \rangle_g \\ & \quad - \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ \underline{n} = I \amalg J}} \sum_{j=0}^{M-2} (-1)^j \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_j \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_{M-2-j} \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}. \end{aligned}$$

We also found the following conjectural identity experimentally, which is amazing if compared with Theorems 8 and 10.

Conjecture 13. [18] *Let $g \geq 2$, $d_j \geq 1$, $\sum_{j=1}^n (d_j - 1) = g$.*

$$\begin{aligned} & \frac{(2g-3+n)!}{2^{2g+1}(2g-3)! \prod_{j=1}^n (2d_j-1)!!} = \langle \prod_{j=1}^n \tau_{d_j} \tau_{2g-2} \rangle_g - \sum_{j=1}^n \langle \tau_{d_j+2g-3} \prod_{i \neq j} \tau_{d_i} \rangle_g \\ & \quad + \frac{1}{2} \sum_{\underline{n} = I \amalg J} \sum_{j=0}^{2g-4} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-4-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \end{aligned}$$

Since $(2g-3)! \text{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1} (3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2})$, it's easy to see that the above identity is equivalent to the following identity of Hodge integrals.

Conjecture 14. *Let $g \geq 2$, $d_j \geq 1$, $\sum_{j=1}^n (d_j - 1) = g$.*

$$\begin{aligned} & \frac{2g-2}{|B_{2g-2}|} \left(\langle \prod_{j=1}^n \tau_{d_j} \mid \lambda_{g-1}\lambda_{g-2} \rangle_g - 3 \langle \prod_{j=1}^n \tau_{d_j} \mid \lambda_{g-3}\lambda_g \rangle_g \right) \\ &= \frac{1}{2} \sum_{j=0}^{2g-4} (-1)^j \langle \tau_{2g-4-j} \tau_j \prod_{i=1}^n \tau_{d_i} \rangle_{g-1} + \frac{(2g-3+n)!}{2^{2g+1}(2g-3)! \prod_{j=1}^n (2d_j-1)!!}. \end{aligned}$$

Virasoro constraints and KdV hierarchy

From Theorem 4, we found new Virasoro constraints and KdV hierarchy for generating functions of higher Weil-Petersson volumes which vastly generalize the Witten conjecture and the results of Mulase and Safnuk [15].

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m}, \mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \cdots \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

We introduce the following family of differential operators for $k \geq -1$,

$$\begin{aligned} V_k = & -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ & + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}, \end{aligned}$$

where $\gamma_{\mathbf{L}}$ are defined by

$$\gamma_{\mathbf{L}} = \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!(2|\mathbf{L}|+1)!}.$$

Theorem 15. [12, 15] *We have $V_k \exp(G) = 0$ for $k \geq -1$ and the operators V_k satisfy the Virasoro relations*

$$[V_n, V_m] = (n-m)V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ -function for the KdV hierarchy.

Since Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \dots)$ and $F(t_0, t_1, \dots)$, we have the following theorem.

Theorem 16. [12, 15]

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

where p_k are polynomials in \mathbf{s} given by

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{|\mathbf{L}|-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of \mathbf{s} , $G(\mathbf{s}, \mathbf{t})$ is a τ -function for the KdV hierarchy.

Theorem 16 also generalized results in [23].

Denominators of intersection numbers

Let $\text{denom}(r)$ denotes the denominator of a rational number r in reduced form (coprime numerator and denominator, positive denominator). We define

$$D_{g,n} = \text{lcm} \left\{ \text{denom} \left(\left\langle \prod_{j=1}^n \tau_{d_j} \right\rangle_g \right) \mid \sum_{j=1}^n d_j = 3g - 3 + n \right\}$$

and for $g \geq 2$,

$$\mathcal{D}_g = \text{lcm} \left\{ \text{denom} \left(\int_{\overline{\mathcal{M}}_g} \kappa(\mathbf{b}) \right) \mid |\mathbf{b}| = 3g - 3 \right\}$$

where *lcm* denotes *least common multiple*.

Since denominators of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ all come from orbifold quotient singularities, the divisibility properties of $D_{g,n}$ and \mathcal{D}_g should reflect overall behavior of singularities.

We have the following properties of $D_{g,n}$ and \mathcal{D}_g .

Proposition 17. [24] *We have $D_{g,n} \mid D_{g,n+1}$, $D_{g,n} \mid \mathcal{D}_g$ and $\mathcal{D}_g = D_{g,3g-3}$.*

Theorem 18. [24] *For $1 < g' \leq g$, the order of any automorphism group of a Riemann surface of genus g' divides $D_{g,3}$.*

The following corollary of Theorem 18 is a conjecture raised by Itzykson and Zuber [25] in 1992.

Corollary 19. *For $1 < g' \leq g$, the order of any automorphism group of an algebraic curve of genus g' divides \mathcal{D}_g .*

The proof of Theorem 18 needs the following two lemmas (see [24]).

Lemma 20. *If $p \leq g + 1$ is a prime number, then $\text{ord}(p, D_{g,3}) \geq 2$.*

Lemma 21. [26] *Let X be a Riemann Surface of genus $g \geq 2$, then for any prime number p ,*

$$\text{ord}(p, |Aut(X)|) \leq \lfloor \log_p \frac{2pg}{p-1} \rfloor + \text{ord}(p, 2(g-1)).$$

We have also obtained conjectural exact values of \mathcal{D}_g for all g in [18].

REFERENCES

- [1] Mumford D. (1983) in *Arithmetic and Geometry* (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 271–328.
- [2] Morita S. (1987) *Invent. Math.* **90**, 551–577.
- [3] Miller E. (1986) *J. Diff. Geom.* **24**, 1–14.
- [4] Arbarello E. & Cornalba M. (1996) *J. Alg. Geom.* **5**, 705–709.
- [5] Kaufmann R., Manin Yu. & Zagier D. (1996) *Comm. Math. Phys.* **181**, 763–787.
- [6] Witten E. (1991) *Surveys in Differential Geometry*, vol.1, 243–310.
- [7] Kontsevich M. (1992) *Comm. Math. Phys.* **147**, 1–23.
- [8] Liu K. & Xu H. (2007) e-Print Archive, <http://xxx.lanl.gov>, math.AG/0701319.
- [9] Okounkov A. (2002) *Internat. Math. Res. Notices*, 933–957.
- [10] Brézin E. & Hikami S. (2007) e-Print Archive, <http://xxx.lanl.gov>, math-ph/0704.2044.
- [11] Liu K. & Xu H. (2007) e-Print Archive, <http://xxx.lanl.gov>, math.AG/0705.2086
- [12] Liu K. & Xu H. in preparation
- [13] Mirzakhani M. (2007) *Invent. Math.* **167** 179–222.
- [14] Mirzakhani M. (2007) *J. Amer. Math. Soc.* **20**, 1–23.
- [15] Mulase M. and Safnuk B. (2006) e-Print Archive, <http://xxx.lanl.gov>, math.AG/0601194.
- [16] Safnuk B. (2007) e-Print Archive, <http://xxx.lanl.gov>, math.DG/0704.2530.
- [17] Do Norman & Norbury P. (2006) e-Print Archive, <http://xxx.lanl.gov>, math.AG/0603406.
- [18] Liu K. & Xu H. (2006) e-Print Archive, <http://xxx.lanl.gov>, math.AG/0609367.
- [19] Getzler E. & Pandharipande R. (1998) *Nuclear Phys. B* **530**, 701–714.

- [20] Faber C. (1999) In *Moduli of curves and abelian varieties*, Aspects Math., E33, Vieweg, Braunschweig, Germany, 109–129.
- [21] Givental A. (2001) *Mosc. Math. J.* **1**, 551–568, 645.
- [22] Goulden I., Jackson D. & Vakil R. (2006) e-Print Archive, [http://xxx.lanl.gov, math.AG/0611659](http://xxx.lanl.gov/math.AG/0611659).
- [23] Manin Yu. & Zograf P. (2000) *Ann. Inst. Fourier.* **50**, 519–535.
- [24] Liu K. & Xu H. (2006) e-Print Archive, [http://xxx.lanl.gov, math.AG/0608209](http://xxx.lanl.gov/math.AG/0608209).
- [25] Itzykson C. & Zuber J. (1992) *Int. J. Mod. Phys. A* **7**, 5661–5705.
- [26] Harvey W. (1966) *Quart. J. Math.*, **17**, 86–97.

CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, CHINA; DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CA 90095-1555, USA

E-mail address: liu@math.ucla.edu, liu@cms.zju.edu.cn

CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, CHINA

E-mail address: haoxu@cms.zju.edu.cn