

# Levi-Civita Ricci-flat metrics on compact complex manifolds

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Abstract. In this paper, we study the geometry of compact complex manifolds with Levi-Civita Ricci-flat metrics and prove that compact complex surfaces admitting Levi-Civita Ricci-flat metrics are Kähler Calabi-Yau surfaces or Hopf surfaces.

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## 1. Introduction

Einstein manifolds and Einstein metrics are fundamental topics in math physics and differential geometry. It is well-known that, the background Riemannian metric of a Kähler-Einstein metric is Einstein. However, when the ambient complex manifolds are not Kähler, the relationships between Hermitian metrics and their background Riemannian metrics are complicated and somewhat mysterious. Let  $(X, h)$  be a Hermitian manifold and  $g$  be the background Riemannian metric. On the Hermitian holomorphic tangent bundle  $(T^{1,0}X, h)$ , there are two typical *metric compatible connections*:

- (A) the Chern connection  $\nabla$ , i.e. the unique connection  $\nabla$  compatible with the Hermitian metric and also the complex structure  $\bar{\partial}$ ;
- (B) the Levi-Civita connection  $\nabla^{\text{LC}}$ , i.e. the restriction of the *complexified Levi-Civita connection* on  $T_{\mathbb{C}}X$  to the holomorphic tangent bundle  $T^{1,0}X$ .

The Chern connection is the key object in complex geometry and the Levi-Civita connection  $\nabla^{\text{LC}}$  is a representative of the Riemannian geometry. It is well-known that when  $(X, h)$  is not Kähler,  $\nabla$  and  $\nabla^{\text{LC}}$  are not the same. The complex geometry of the Chern connection is extensively investigated in the literatures by using various methods (e.g. [3, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 25, 26]).

In [13], we introduced the first Aeppli-Chern classes for holomorphic line bundles. Let  $L \rightarrow X$  be a holomorphic line bundle over  $X$ . The first Aeppli-Chern class is defined as

$$c_1^{\text{AC}}(L) = [-\sqrt{-1}\partial\bar{\partial}\log h]_{\text{A}} \in H_{\text{A}}^{1,1}(X)$$

where  $h$  is an arbitrary smooth Hermitian metric on  $L$  and the Aeppli cohomology is

$$H_{\text{A}}^{p,q}(X) := \frac{\text{Ker}\partial\bar{\partial} \cap \Omega^{p,q}(X)}{\text{Im}\partial \cap \Omega^{p,q}(X) + \text{Im}\bar{\partial} \cap \Omega^{p,q}(X)}.$$

For a complex manifold  $X$ ,  $c_1^{\text{AC}}(X)$  is defined to be  $c_1^{\text{AC}}(K_X^{-1})$  where  $K_X^{-1}$  is the anti-canonical bundle of  $X$ . Note that, for a Hermitian line bundle  $(L, h)$ , the classes  $c_1(L)$  and  $c_1^{\text{AC}}(L)$  have the same (1, 1)-form representative  $\Theta^h = -\sqrt{-1}\partial\bar{\partial}\log h$  in different cohomological classes.

It is well-known that on a Hermitian manifold  $(X, \omega)$ , the first Chern Ricci curvature

$$\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \det(\omega)$$

represents the first Chern class  $c_1(X)$ . As an analog, we proved in [13, Theorem 1.1] that the first Levi-Civita Ricci curvature  $\mathfrak{Ric}(\omega)$  represents the first Aeppli-Chern class  $c_1^{\text{AC}}(X)$ . Hence, it is very natural to study (non-Kähler) Calabi-Yau manifolds by using the first Aeppli-Chern class  $c_1^{\text{AC}}(X)$  and the first Levi-Civita Ricci curvature  $\mathfrak{Ric}(\omega)$ .

The classification of various Ricci-flat manifolds are important topics in differential geometry. The following result is fundamental and well-known, and we refer to the nice paper [19] of V. Tosatti for discussions on Bott-Chern classes and Chern Ricci-flat metrics.

**Theorem 1.1.** *Let  $X$  be a compact complex surface. Suppose  $X$  admits a Chern Ricci-flat Hermitian metric  $\omega$ , i.e.  $\text{Ric}(\omega) = 0$ . Then  $X$  is minimal and it is exactly one of the following*

- (1) *Enriques surfaces;*
- (2) *bi-elliptic surfaces;*
- (3) *K3 surfaces;*
- (4) *2-tori;*
- (5) *Kodaira surfaces.*

As shown in [13], the Levi-Civita Ricci-flat condition  $\mathfrak{Ric}(\omega) = 0$  is equivalent to

$$(1.1) \quad \text{Ric}(\omega) = \frac{1}{2}(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega),$$

where  $\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n)$  is the Chern Ricci curvature. The equation (1.1) is not of Monge-Ampère type since there are also non-elliptic terms on the right hand side. As it is well-known, it is particularly challenging to solve such equations. By

using conformal methods, functional analysis and *explicit constructions*, we obtain the following result analogous to Theorem 1.1, which also generalizes the previous results in [14].

**Theorem 1.2.** *Let  $X$  be a compact complex surface. Suppose  $X$  admits a Levi-Civita Ricci-flat Hermitian metric  $\omega$ , i.e.  $\mathfrak{Ric}(\omega) = 0$ . Then  $X$  is minimal. Moreover, it lies in one of the following*

- (1) *Enriques surfaces;*
- (2) *bi-elliptic surfaces;*
- (3) *K3 surfaces;*
- (4) *2-tori;*
- (5) *Hopf surfaces.*

**Remark 1.3.** Note that, Enriques surfaces, bi-elliptic surfaces, K3 surfaces and 2-tori are Kähler Calabi-Yau surfaces. It is obvious that the Kähler Calabi-Yau metrics are Levi-Civita Ricci-flat. However, by using Yau's theorem, there exist many non-Kähler Levi-Civita Ricci-flat metrics on each Kähler Calabi-Yau manifold.

**Remark 1.4.** It is easy to see that, for a Kodaira surface  $X$ , it has  $c_1^{\text{BC}}(X) = c_1(X) = c_1^{\text{AC}}(X) = 0$ . By Theorem 1.1, it has a Chern Ricci-flat metric. However, we can see from Theorem 1.2 that it can **not** support a Levi-Civita Ricci-flat metric.

**Remark 1.5.** It is well-known, it is very difficult to write down explicitly Ricci-flat metrics. In this paper, we obtain Levi-Civita Ricci-flat metrics on Hopf surfaces by explicit constructions. It also worths to point out that we only construct Levi-Civita Ricci-flat metrics on Hopf surfaces of class 1 (see Theorem 1.6). We conjecture that all Hopf surfaces can support Levi-Civita Ricci-flat metrics. On the other hand, every Hopf surface  $X$  is a non-Kähler Calabi-Yau manifold, i.e.  $c_1(X) = 0 \in H^2(X, \mathbb{R})$ . However,  $X$  can not support a Chern Ricci-flat Hermitian metric, i.e. a Hermitian metric  $\omega$  with  $\text{Ric}(\omega) = 0$ . In this view point, the existence of Levi-Civita Ricci-flat Hermitian metrics on Hopf surfaces is quite exceptional.

A compact complex surface  $X$  is called a Hopf surface if its universal covering is analytically isomorphic to  $\mathbb{C}^2 \setminus \{0\}$ . Its fundamental group  $\pi_1(X)$  is a finite extension of an infinite cyclic group generated by a biholomorphic contraction which takes the form  $(z, w) \rightarrow (az, bw + \lambda z^m)$  where  $a, b, \lambda \in \mathbb{C}$ ,  $|a| \geq |b| > 1$ ,  $m \in \mathbb{N}^*$  and  $\lambda(a - b^m) = 0$ . There are two different cases:

- (I) the Hopf surface  $H_{a,b}$  of class 1 if  $\lambda = 0$ ;
- (II) the Hopf surface  $H_{a,b,\lambda,m}$  of class 0 if  $\lambda \neq 0$  and  $a = b^m$ .

Let  $H_{a,b} = \mathbb{C}^2 \setminus \{0\} / \sim$  where  $(z, w) \sim (az, bw)$  and  $|a| \geq |b| > 1$ . We set  $k_1 = \log |a|$  and  $k_2 = \log |b|$ . Define a real smooth function  $\Phi(z, w) = e^{\frac{k_1+k_2}{2\pi}\theta}$  where  $\theta(z, w)$  is a real smooth function defined by  $|z|^2 e^{-\frac{k_1\theta}{\pi}} + |w|^2 e^{-\frac{k_2\theta}{\pi}} = 1$ . We construct **explicitly** Levi-Civita Ricci-flat metrics on  $H_{a,b}$  by perturbations and conformal changes.

**Theorem 1.6.** *On the Hopf surface  $H_{a,b}$  of class 1, the Hermitian metric*

$$(1.2) \quad \omega = \Delta^3 \left( \frac{1}{\Phi} \sqrt{-1} \partial \bar{\partial} \Phi - \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \log \Phi \right)$$

*is Levi-Civita Ricci-flat, i.e.  $\mathfrak{Ric}(\omega) = 0$ , where*

$$\Delta = \alpha |z|^2 \Phi^{-\alpha} + (2 - \alpha) |w|^2 \Phi^{\alpha-2} \quad \text{and} \quad \alpha = \frac{2k_1}{k_1 + k_2}.$$

**Remark 1.7.** If  $a = b$ ,  $H_{a,a}$  is exactly the usual diagonal Hopf surface. In this case, the Levi-Civita Ricci-flat metric constructed in Theorem 1.6 is the same as that constructed in [13, Theorem 6.2] or [14, Theorem 7.3].

It is well-known that, on a compact Kähler manifold  $X$ , the Kähler Ricci-flat metrics are all Einstein flat metrics.

**Question 1.8.** On a compact complex manifold  $X$ , does there exist some Levi-Civita Ricci-flat (non-Kähler) Hermitian metric such that the background Riemannian metric is Einstein?

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## 2. Preliminaries

**2.1. Chern connection on complex manifolds.** Let  $(X, \omega_g)$  be a compact Hermitian manifold. There exists a unique connection  $\nabla$  on the holomorphic tangent bundle  $T^{1,0}X$  which is compatible with the Hermitian metric and also the complex structure of  $X$ . This connection  $\nabla$  is called the Chern connection. The Chern connection  $\nabla$  on  $(T^{1,0}X, \omega_g)$  has curvature components

$$(2.1) \quad R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}.$$

The (first) Chern-Ricci form  $\text{Ric}(\omega_g)$  of  $(X, \omega_g)$  has components

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \bar{z}^j}$$

which also represents the first Chern class  $c_1(X)$  of the complex manifold  $X$ . The Chern scalar curvature  $s_C$  of  $(X, \omega_g)$  is given by

$$(2.2) \quad s_C = \text{tr}_{\omega_g} \text{Ric}(\omega_g) = g^{i\bar{j}} R_{i\bar{j}}.$$

The total Chern scalar curvature of  $\omega_g$  is

$$(2.3) \quad \int_X s_C \omega_g^n = n \int \text{Ric}(\omega_g) \wedge \omega_g^{n-1},$$

where  $n$  is the complex dimension of  $X$ .

**2.2. Bott-Chern classes and Aeppli classes.** The Bott-Chern cohomology and the Aeppli cohomology on a compact complex manifold  $X$  are given by

$$H_{\text{BC}}^{p,q}(X) := \frac{\text{Ker}d \cap \Omega^{p,q}(X)}{\text{Im}\partial\bar{\partial} \cap \Omega^{p,q}(X)} \quad \text{and} \quad H_{\text{A}}^{p,q}(X) := \frac{\text{Ker}\partial\bar{\partial} \cap \Omega^{p,q}(X)}{\text{Im}\partial \cap \Omega^{p,q}(X) + \text{Im}\bar{\partial} \cap \Omega^{p,q}(X)}.$$

Let  $\text{Pic}(X)$  be the set of holomorphic line bundles over  $X$ . As similar as the first Chern class map  $c_1 : \text{Pic}(X) \rightarrow H_{\mathbb{R}}^{1,1}(X)$ , there is a *first Aeppli-Chern class* map

$$(2.4) \quad c_1^{\text{AC}} : \text{Pic}(X) \rightarrow H_{\mathbb{A}}^{1,1}(X).$$

Given any holomorphic line bundle  $L \rightarrow X$  and any Hermitian metric  $h$  on  $L$ , its curvature form  $\Theta_h$  is locally given by  $-\sqrt{-1}\partial\bar{\partial}\log h$ . We define  $c_1^{\text{AC}}(L)$  to be the class of  $\Theta_h$  in  $H_{\mathbb{A}}^{1,1}(X)$ . For a complex manifold  $X$ ,  $c_1^{\text{AC}}(X)$  is defined to be  $c_1^{\text{AC}}(K_X^{-1})$  where  $K_X^{-1}$  is the anti-canonical line bundle. The first Bott-Chern class  $c_1^{\text{BC}}(X)$  can be defined similiary.

**2.3. The Levi-Civita connection on the holomorphic tangent bundle.** Let's recall some elementary settings (e.g. [13, Section 2]). Let  $(M, g, \nabla)$  be a  $2n$ -dimensional Riemannian manifold with the Levi-Civita connection  $\nabla$ . The tangent bundle of  $M$  is also denoted by  $T_{\mathbb{R}}M$ . Let  $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C}$  be the complexification. We can extend the metric  $g$  and the Levi-Civita connection  $\nabla$  to  $T_{\mathbb{C}}M$  in the  $\mathbb{C}$ -linear way. Let  $(M, g, J)$  be an almost Hermitian manifold, i.e.,  $J : T_{\mathbb{R}}M \rightarrow T_{\mathbb{R}}M$  with  $J^2 = -1$ , and for any  $X, Y \in T_{\mathbb{R}}M$ ,  $g(JX, JY) = g(X, Y)$ . The Nijenhuis tensor  $N_J : \Gamma(M, T_{\mathbb{R}}M) \times \Gamma(M, T_{\mathbb{R}}M) \rightarrow \Gamma(M, T_{\mathbb{R}}M)$  is defined as

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

The almost complex structure  $J$  is called *integrable* if  $N_J \equiv 0$  and then we call  $(M, g, J)$  a Hermitian manifold. We can also extend  $J$  to  $T_{\mathbb{C}}M$  in the  $\mathbb{C}$ -linear way. Hence for any  $X, Y \in T_{\mathbb{C}}M$ , we still have  $g(JX, JY) = g(X, Y)$ . By Newlander-Nirenberg's theorem, there exists a real coordinate system  $\{x^i, x^I\}$  such that  $z^i = x^i + \sqrt{-1}x^I$  are local holomorphic coordinates on  $M$ . Moreover, we have  $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$  where

$$T^{1,0}M = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\} \quad \text{and} \quad T^{0,1}M = \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\}.$$

Since  $T^{1,0}M$  is a subbundle of  $T_{\mathbb{C}}M$ , there is an induced connection  $\nabla^{\text{LC}}$  on the holomorphic tangent bundle  $T^{1,0}M$  given by

$$(2.5) \quad \nabla^{\text{LC}} = \pi \circ \nabla : \Gamma(M, T^{1,0}M) \xrightarrow{\nabla} \Gamma(M, T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M) \xrightarrow{\pi} \Gamma(M, T_{\mathbb{C}}M \otimes T^{1,0}M).$$

Let  $h = (h_{i\bar{j}})$  be the corresponding Hermitian metric on  $T^{1,0}M$  induced by  $(M, g, J)$ . It is obvious that  $\nabla^{\text{LC}}$  is a metric compatible connection on the Hermitian holomorphic vector bundle  $(T^{1,0}M, h)$ , and we call  $\nabla^{\text{LC}}$  the *Levi-Civita connection* on the

complex manifold  $M$ . It is obvious that,  $\nabla^{\text{LC}}$  is determined by the following relations

$$(2.6) \quad \nabla_{\frac{\partial}{\partial z^i}}^{\text{LC}} \frac{\partial}{\partial z^k} := \Gamma_{ik}^p \frac{\partial}{\partial z^p} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial \bar{z}^j}}^{\text{LC}} \frac{\partial}{\partial z^k} := \Gamma_{jk}^p \frac{\partial}{\partial z^p}$$

where

$$(2.7) \quad \Gamma_{ij}^k = \frac{1}{2} h^{k\bar{\ell}} \left( \frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j} \right), \quad \text{and} \quad \Gamma_{ij}^k = \frac{1}{2} h^{k\bar{\ell}} \left( \frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^{\ell}} \right).$$

The curvature tensor  $\mathfrak{R} \in \Gamma(M, \Lambda^2 T_{\mathbb{C}} M \otimes T^{*1,0} M \otimes T^{1,0} M)$  of  $\nabla^{\text{LC}}$  is given by

$$\mathfrak{R}(X, Y)s = \nabla_X^{\text{LC}} \nabla_Y^{\text{LC}} s - \nabla_Y^{\text{LC}} \nabla_X^{\text{LC}} s - \nabla_{[X, Y]}^{\text{LC}} s$$

for any  $X, Y \in T_{\mathbb{C}} M$  and  $s \in T^{1,0} M$ . A straightforward computation shows that the curvature tensor  $\mathfrak{R}$  has  $(1, 1)$  components

$$(2.8) \quad \mathfrak{R}_{i\bar{j}k}^{\ell} = - \left( \frac{\partial \Gamma_{ik}^{\ell}}{\partial \bar{z}^j} - \frac{\partial \Gamma_{jk}^{\ell}}{\partial z^i} + \Gamma_{ik}^s \Gamma_{js}^{\ell} - \Gamma_{jk}^s \Gamma_{si}^{\ell} \right).$$

The (first) Levi-Civita Ricci curvature  $\mathfrak{Ric}(\omega_h)$  of  $(T^{1,0} M, \omega_h, \nabla^{\text{LC}})$  is

$$(2.9) \quad \mathfrak{Ric}(\omega_h) = \sqrt{-1} \mathfrak{R}_{i\bar{j}}^{(1)} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}_{i\bar{j}}^{(1)} = \mathfrak{R}_{i\bar{j}k}^k.$$

The Levi-Civita scalar curvature  $s_{\text{LC}}$  of  $\nabla^{\text{LC}}$  on  $T^{1,0} M$  is

$$(2.10) \quad s_{\text{LC}} = h^{i\bar{j}} h^{k\bar{\ell}} \mathfrak{R}_{i\bar{j}k\bar{\ell}}.$$

**2.4. Special manifolds.** Let  $X$  be a compact complex manifold with complex dimension  $n \geq 2$ . A Hermitian metric  $\omega_g$  is called a Gauduchon metric if  $\partial \bar{\partial} \omega_g^{n-1} = 0$ . It is proved by Gauduchon ([9]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to scaling). A Hermitian metric  $\omega_g$  is called a balanced metric if  $d\omega_g^{n-1} = 0$  or equivalently  $d^* \omega_g = 0$ . On a compact complex surface, a balanced metric is also Kähler, i.e.  $d\omega_g = 0$ . It is well-known many Hermitian manifolds can not support balanced metrics, e.g. Hopf surface  $\mathbb{S}^3 \times \mathbb{S}^1$ . It is also obvious that balanced metrics are Gauduchon.

### 3. Geometry of the Levi-Civita connections

**3.1. Some computational formulas.** In this subsection, we recall some elementary and well-known computational lemmas on Hermitian manifolds.

**Lemma 3.1.** *Let  $(X, \omega)$  be a compact Hermitian manifold and  $\omega = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$ .*

$$(3.1) \quad \partial^* \omega = -\sqrt{-1} \Lambda (\bar{\partial} \omega) = -2\sqrt{-1} \Gamma_{jk}^k d\bar{z}^j \quad \text{and} \quad \bar{\partial}^* \omega = \sqrt{-1} \Lambda (\partial \omega) = 2\sqrt{-1} \Gamma_{ik}^k dz^i.$$

*Proof.* By the well-known Bochner formula (e.g. [12]),

$$[\bar{\partial}^*, L] = \sqrt{-1}(\partial + \tau)$$

where  $\tau = [\Lambda, \partial\omega]$ , we see  $\bar{\partial}^* \omega = \sqrt{-1}\Lambda(\partial\omega) = 2\sqrt{-1}\overline{\Gamma_{ik}^k} dz^i$ .  $\square$

Let  $T$  be the torsion tensor of the Hermitian metric  $\omega$ , i.e.

$$(3.2) \quad T_{ij}^k = h^{k\bar{l}} \left( \frac{\partial h_{j\bar{l}}}{\partial z^i} - \frac{\partial h_{i\bar{l}}}{\partial z^j} \right).$$

**Corollary 3.2.** [14, Corollary 4.2] *Let  $(X, \omega)$  be a compact Hermitian manifold. Let  $s$  be the Riemannian scalar curvature of the background Riemannian metric  $\omega$ . Then*

$$(3.3) \quad s = 2s_C + \left( \langle \partial\bar{\partial}^* \omega + \bar{\partial}\bar{\partial}^* \omega, \omega \rangle - 2|\partial^* \omega|^2 \right) - \frac{1}{2}|T|^2,$$

$$(3.4) \quad s_{\text{LC}} = s_C - \frac{1}{2} \langle \partial\bar{\partial}^* \omega + \bar{\partial}\bar{\partial}^* \omega, \omega \rangle = s_C - \langle \partial\bar{\partial}^* \omega, \omega \rangle.$$

**3.2. The first Aeppli-Chern class and Levi-Civita connections.** The following result is obtained in [13, Theorem 1.2](see also [14, Theorem 4.1]).

**Theorem 3.3.** *Let  $(X, \omega)$  be a compact Hermitian manifold. Then the first Levi-Civita Ricci form  $\mathfrak{Ric}(\omega)$  represents the first Aeppli-Chern class  $c_1^{\text{AC}}(X)$  in  $H_A^{1,1}(X)$ . Moreover, we have the Ricci curvature relation*

$$(3.5) \quad \mathfrak{Ric}(\omega) = \text{Ric}(\omega) - \frac{1}{2}(\partial\bar{\partial}^* \omega + \bar{\partial}\bar{\partial}^* \omega).$$

**Lemma 3.4.** *Let  $(X, \omega)$  be a compact Hermitian manifold with complex dimension  $n$ . Suppose  $f \in C^\infty(X, \mathbb{R})$  and  $\omega_f = e^f \omega$ . Then we have*

$$(3.6) \quad \bar{\partial}_f^* \omega_f = \bar{\partial}^* \omega + \sqrt{-1}(n-1)\partial f \quad \text{and} \quad \bar{\partial}\bar{\partial}_f^* \omega_f = \bar{\partial}\bar{\partial}^* \omega - \sqrt{-1}(n-1)\partial\bar{\partial}f.$$

where  $\bar{\partial}^*, \bar{\partial}_f^*$  are the adjoint operators with respect to the metric  $\omega$  and  $\omega_f$  respectively. In particular, we have

$$\mathfrak{Ric}(e^f \omega) = \mathfrak{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}f.$$

**Definition 3.5.** The Kodaira dimension  $\kappa(L)$  of a line bundle  $L$  is defined to be

$$\kappa(L) := \limsup_{m \rightarrow +\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^{\otimes m})}{\log m}$$

and the Kodaira dimension  $\kappa(X)$  of  $X$  is defined as  $\kappa(X) := \kappa(K_X)$  where the logarithm of zero is defined to be  $-\infty$ .

**Theorem 3.6.** *Let  $(X, \omega)$  be a compact Hermitian manifold of complex dimension. Let  $\omega_f = e^f \omega$  be the Gauduchon metric in the conformal class of  $\omega$ . Then we have*

$$(3.7) \quad \int_X s_f \cdot \omega_f^n = n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \int_X e^{(n-1)f} \cdot s_{\text{LC}} \cdot \omega^n + n \|\bar{\partial}_f^* \omega_f\|_{\omega_f}^2.$$

*Proof.* By Lemma 3.4 and Theorem 3.3, we have

$$\begin{aligned} \operatorname{Ric}(\omega_f) - \frac{\partial\bar{\partial}_f^*\omega_f + \bar{\partial}\bar{\partial}_f^*\omega_f}{2} &= \operatorname{Ric}(\omega) - \frac{\partial\bar{\partial}^*\omega + \bar{\partial}\bar{\partial}^*\omega}{2} - \sqrt{-1}\partial\bar{\partial}f \\ &= \Re\operatorname{ic}(\omega) - \sqrt{-1}\partial\bar{\partial}f. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_X \operatorname{Ric}(\omega_f) \wedge \omega_f^{n-1} &= \int_X \left( \Re\operatorname{ic}(\omega) - \sqrt{-1}\partial\bar{\partial}f + \frac{\partial\bar{\partial}_f^*\omega_f + \bar{\partial}\bar{\partial}_f^*\omega_f}{2} \right) \wedge \omega_f^{n-1} \\ &= \int_X \Re\operatorname{ic}(\omega) \wedge \omega_f^{n-1} + \frac{1}{2} \left( \|\bar{\partial}_f^*\omega_f\|_{\omega_f}^2 + \|\partial_f^*\omega_f\|_{\omega_f}^2 \right) \\ (3.8) \quad &= \frac{1}{n} \int_X e^{(n-1)f} \cdot s_{\text{LC}} \cdot \omega^n + \|\bar{\partial}_f^*\omega_f\|_{\omega_f}^2. \end{aligned}$$

□

**Theorem 3.7.** *Let  $X$  be a compact complex manifold. Suppose  $\omega$  is a Hermitian metric with  $s_{\text{LC}} \geq 0$ . Then either*

- (1)  $\kappa(X) = -\infty$ ; or
- (2)  $\kappa(X) = 0$  and  $(X, \omega)$  is conformally balanced with  $K_X$  a holomorphic torsion, i.e.  $K_X^{\otimes m} = \mathcal{O}_X$  for some  $m \in \mathbb{Z}^+$ .

*Proof.* Let  $\omega_f = e^f \omega$  be the Gauduchon metric in the conformal class of  $\omega$ . Then by formula (3.7), the total Chern scalar curvature of  $\omega_f$  is

$$(3.9) \quad \int_X s_f \cdot \omega_f^n = n \int_X \operatorname{Ric}(\omega_f) \wedge \omega_f^{n-1} \geq n \|\bar{\partial}_f^*\omega_f\|_{\omega_f}^2$$

since the Levi-Civita scalar curvature  $s_{\text{LC}} \geq 0$ . Suppose  $\bar{\partial}_f^*\omega_f \neq 0$ , then

$$\int_X s_f \cdot \omega_f^n > 0.$$

By [25, Corollary 3.3], we have  $\kappa(X) = -\infty$ . On the other hand, if  $\bar{\partial}_f^*\omega_f = 0$ , i.e.  $(X, \omega)$  is conformally balanced. Then the total Chern scalar curvature of the Gauduchon metric

$$\int_X s_f \cdot \omega_f^n \geq 0.$$

Then by [25, Theorem 1.4], we have  $\kappa(X) = -\infty$  or  $\kappa(X) = 0$ , and when  $\kappa(X) = 0$ ,  $K_X$  is a holomorphic torsion. □



## 4. Compact complex manifolds with Levi-Civita Ricci-flat metrics

Let's recall that, a Levi-Civita Ricci-flat metric is a Hermitian metric satisfying  $\Re\text{ic}(\omega) = 0$ , or equivalently, by formula (3.5)

$$(4.1) \quad \text{Ric}(\omega) = \frac{\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega}{2}.$$

It is easy to see that

**Corollary 4.1.** *Let  $X$  be a compact complex manifold. Then*

$$c_1^{\text{BC}}(X) = 0 \implies c_1(X) = 0 \implies c_1^{\text{AC}}(X) = 0.$$

The first obstruction for the existence of Levi-Civita Ricci-flat Hermitian metric is the top first Chern number:

**Corollary 4.2.** *Suppose  $c_1^{\text{AC}}(X) = 0$ , then the top intersection number  $c_1^n(X) = 0$ . In particular, if  $X$  has a Levi-Civita Ricci-flat Hermitian metric  $\omega$ , then  $c_1^n(X) = 0$ .*

*Proof.* By definition, if  $c_1^{\text{AC}}(X) = 0$ , then

$$\text{Ric}(\omega) = \bar{\partial}A + \partial B$$

where  $A$  is a  $(1,0)$ -form and  $B$  is a  $(0,1)$ -form. Hence

$$c_1^n(X) = \int_X (\text{Ric}(\omega))^n = \int_X (\text{Ric}(\omega))^{n-1} \wedge (\bar{\partial}A + \partial B) = 0$$

since  $\text{Ric}(\omega)$  is both  $\partial$  and  $\bar{\partial}$ -closed.  $\square$

**Theorem 4.3.** *Let  $X$  be a compact complex manifold with  $\kappa(X) = -\infty$ . If  $X$  has a Levi-Civita Ricci-flat Hermitian metric  $\omega$ , then  $X$  must be a non-Kähler manifold.*

*Proof.* Let  $\omega$  be a Hermitian metric with  $\Re\text{ic}(\omega) = 0$ . By formula (3.5), we have

$$(4.2) \quad \text{Ric}(\omega) = \frac{\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega}{2}.$$

Note that  $\text{Ric}(\omega)$  is  $\partial$ -closed and  $\bar{\partial}$ -closed, and so we have

$$\bar{\partial}\partial\partial^*\omega = 0$$

Suppose  $X$  is a Kähler manifold, then by  $\partial\bar{\partial}$ -Lemma on  $X$ , the  $\bar{\partial}$ -closed and  $\partial$ -exact  $(1,1)$ -form  $\partial\partial^*\omega$  is  $\partial\bar{\partial}$ -exact, i.e. there exists a smooth function  $\varphi$  such that

$$\partial\partial^*\omega = \partial\bar{\partial}\varphi.$$

Therefore,

$$\text{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}F$$

where  $F = -\frac{\bar{\varphi}-\varphi}{2\sqrt{-1}} \in C^\infty(X, \mathbb{R})$ . It is obvious that the Hermitian metric  $e^{\frac{F}{n}}\omega$  is Chern Ricci-flat, i.e.  $c_1^{\text{BC}}(X) = c_1(X) = 0$  is unitary flat. Hence  $X$  is a Kähler Calabi-Yau manifold. In particular,  $\kappa(X) = 0$ . This is a contradiction.  $\square$

**Remark 4.4.** On a compact Kähler Calabi-Yau manifold  $X$  with  $\dim_{\mathbb{C}} X = n \geq 2$ , the Levi-Civita Ricci-flat metrics are not necessarily Kähler. Indeed, let  $\omega_{\text{CY}}$  be a Calabi-Yau Kähler metric on  $X$ . Then for any non constant smooth function  $f \in C^\infty(X, \mathbb{R})$ , we can construct a non-Kähler Levi-Civita Ricci-flat metric. By Yau's theorem, there exists a Kähler metric  $\omega_0$  such that

$$\omega_0^n = e^{-f} \omega_{\text{CY}}^n.$$

Let  $\omega = e^f \omega_0$ . Then  $\omega$  is a non-Kähler metric with Levi-Civita Ricci-flat curvature. Indeed,

$$\begin{aligned} \mathfrak{Ric}(\omega) &= \text{Ric}(\omega) - \frac{\partial\bar{\partial}^*\omega + \bar{\partial}\partial^*\omega}{2} \\ &= \text{Ric}(\omega_0) - n\sqrt{-1}\partial\bar{\partial}f - \frac{\partial\bar{\partial}_0^*\omega_0 + \bar{\partial}\partial_0^*\omega_0}{2} + (n-1)\sqrt{-1}\partial\bar{\partial}f \\ &= \text{Ric}(\omega_{\text{CY}}) + \sqrt{-1}\partial\bar{\partial}f - n\sqrt{-1}\partial\bar{\partial}f - \frac{\partial\bar{\partial}_0^*\omega_0 + \bar{\partial}\partial_0^*\omega_0}{2} + (n-1)\sqrt{-1}\partial\bar{\partial}f \\ &= 0, \end{aligned}$$

where we use Lemma 3.4 in the second identity.

Theorem 4.3 has the following variant:

**Corollary 4.5.** *Let  $X$  be a compact Kähler manifold. If  $\kappa(X) = -\infty$ , then  $X$  has no Levi-Civita Ricci-flat Hermitian metric.*

As an application, we obtain

**Theorem 4.6.** *Let  $X$  be a compact complex surface with  $\kappa(X) \geq 0$ . Suppose  $X$  admits a Hermitian metric with  $s_{\text{LC}} \geq 0$ . Then  $X$  is a minimal Kähler surface of Calabi-Yau type, i.e.  $X$  is exactly one of the following*

- (1) an Enriques surface;
- (2) a bi-elliptic surface;
- (3) a K3 surface;
- (4) a torus.

*Proof.* By Theorem 3.7, we know  $\kappa(X) = 0$  and the canonical line bundle  $K_X$  is a holomorphic torsion, i.e.  $K_X^{\otimes m} = \mathcal{O}_X$  for some  $m \in \mathbb{Z}^+$ . Since  $\dim X = 2$ , by Theorem 3.7 again,  $X$  is a balanced surface and so it is Kähler. It is easy to see that,  $X$  is minimal. According to the Kodaira-Enriques' classification,  $X$  is either an Enriques surface, a bi-elliptic surface, a K3 surface or a torus. All these surfaces are Kähler surfaces of Calabi-Yau type, and all Kähler Calabi-Yau metrics are Levi-Civita Ricci-flat.  $\square$

## 5. The proof of Theorem 1.2

In this section, we prove Theorem 1.2, i.e.

**Theorem 5.1.** *Let  $X$  be a compact complex surface. Suppose  $X$  admits a Levi-Civita Ricci-flat Hermitian metric  $\omega$ . Then  $X$  is minimal. Moreover, it lies in one of the following*

- (1) *Enriques surfaces;*
- (2) *bi-elliptic surfaces;*
- (3) *K3 surfaces;*
- (4) *2-tori;*
- (5) *Hopf surfaces.*

*Proof.* Let  $(X, \omega)$  be a compact complex surface with Levi-Civita Ricci-flat metric  $\omega$ . Then we have  $c_1^{\text{AC}}(X) = 0$  and  $s_{\text{LC}} = 0$ . By Theorem 3.7,  $\kappa(X) = -\infty$  or  $\kappa(X) = 0$ . We shall show  $X$  is a minimal surface.

Suppose  $\kappa(X) = 0$ , by Theorem 4.6, we know  $X$  is a minimal Kähler Calabi-Yau surface, i.e.  $X$  is exactly one of the following

- (1) a Enriques surface;
- (2) a bi-elliptic surface;
- (3) a K3 surface;
- (4) a torus.

Suppose  $\kappa(X) = -\infty$ . Let  $X_{\min}$  be the minimal model of  $X$ , then  $X_{\min}$  lies in one of the following classes:

- (1) minimal rational surfaces;
- (2) ruled surfaces of genus  $g \geq 1$ ;
- (3) surface of class  $\text{VII}_0$ .

If  $X_{\min}$  is in (1) or (2), we know  $X$  is projective. Since  $\kappa(X) = -\infty$ , by Corollary 4.5,  $X$  has no Levi-Civita Ricci-flat metric. Hence  $X_{\min}$  is not in (1) or (2). Suppose  $X_{\min}$  lies in (3), i.e. of class  $\text{VII}_0$ . A class  $\text{VII}_0$  surface is a minimal compact complex surface with  $b_1 = 1$  and  $\kappa(X) = -\infty$ . It is well-known that the first Betti number  $b_1$  of compact complex surfaces are invariant under blowing-ups, i.e.  $b_1(X) = 1$ . By [2, Theorem 2.7 on p.139], we know

$$b_1(X) = h^{1,0}(X) + h^{0,1}(X), \quad \text{and} \quad h^{1,0}(X) \leq h^{0,1}(X)$$

hence  $h^{0,1}(X) = 1$ . Since  $\kappa(X) = \kappa(X_{\min}) = -\infty$ , by Serre duality, we have

$$h^{0,2}(X) = h^{2,0}(X) = h^0(X, K_X) = 0.$$

Therefore, by the Euler-Poincaré characteristic formula, we get

$$\chi(\mathcal{O}_X) = 1 - h^{0,1}(X) + h^{0,2}(X) = 0.$$

On the other hand, by the Noether-Riemann-Roch formula,

$$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X)) = 0,$$

we obtain

$$c_2(X) = -c_1^2(X).$$

Note also that  $c_2(X)$  is the Euler characteristic  $e(X)$  of  $X$ , i.e.

$$c_2(X) = e(X) = 2 - 2b_1(X) + b_2(X) = b_2(X)$$

and so  $c_1^2(X) = -b_2(X) \leq 0$ . Suppose  $X$  has a Levi-Civita Ricci-flat Hermitian metric, then we have  $c_1^{\text{AC}}(X) = 0$ . By Corollary 4.2, we have  $c_1^2(X) = 0$ . Therefore  $b_2(X) = 0$ . It is well-known that, blowing-ups increase the second Betti number at least by 1, hence we have  $X = X_{\min}$ . We complete the proof of the statement that: if a compact complex surface admits a Levi-Civita Ricci-flat metric, then it is a minimal surface.

There are three classes of surfaces of  $\text{VII}_0$ :

- class  $\text{VII}_0$  surfaces with  $b_2 > 0$ ;
- Inoue surfaces: a class  $\text{VII}_0$  surface has  $b_2 = 0$  and contains no curves;
- Hopf surfaces: its universal covering is  $\mathbb{C}^2 - \{0\}$ , or equivalently a class  $\text{VII}_0$  surface has  $b_2 = 0$  and contains a curve.

(1). A class  $\text{VII}_0$  surface  $X$  with  $b_2 > 0$  has no Levi-Civita Ricci-flat metrics. Indeed, by a similar computation as before, we know  $c_1^2(X) = -b_2 < 0$  which contradicts to Corollary 4.2.

(2). On an Inoue surface  $X$ , there is no Levi-Civita Ricci-flat Hermitian metrics. This is essentially proved in [14, Theorem 7.2]. For the reader's convenience, we include a sketched proof here. It is well-known ([10]) that an Inoue surface is a quotient of  $\mathbb{H} \times \mathbb{C}$  by a properly discontinuous group of affine transformations where  $\mathbb{H}$  is the upper half-plane. There are three types of Inoue surfaces:

- (A) Inoue surfaces  $S_M$ . Let  $M$  be a matrix in  $\text{SL}_3(\mathbb{Z})$  admitting one real eigenvalue  $\alpha > 1$  and two complex conjugate eigenvalues  $\beta \neq \bar{\beta}$ . Let  $(a_1, a_2, a_3)$  be a real eigenvector of  $M$  corresponding to  $\alpha$  and let  $(b_1, b_2, b_3)$  be an eigenvector of  $M$  corresponding to  $\beta$ . Then  $X = S_M$  is the quotient of  $\mathbb{H} \times \mathbb{C}$  by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, \beta z), \quad g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3.$$

- (B) Inoue surfaces  $X = S_{N,p,q,r;t}^+$  are defined as the quotient of  $\mathbb{H} \times \mathbb{C}$  by the group of affine automorphisms generated by

$$g_0(w, z) = (\alpha w, z + t), \quad g_i(w, z) = (w + a_i, z + b_i w + c_i), \quad i = 1, 2$$

$$g_3(w, z) = \left( w, z + \frac{b_1 a_2 - b_2 a_1}{r} \right),$$

where  $(a_1, a_2)$  and  $(b_1, b_2)$  are the eigenvectors of some matrix  $N \in \mathrm{SL}_2(\mathbb{Z})$  admitting real eigenvalues  $\alpha > 1$ ,  $\alpha^{-1}$ . Moreover  $t \in \mathbb{C}$  and  $p, q, r (r \neq 0)$  are integers, and  $(c_1, c_2)$  depends on  $(a_i, b_i), p, q, r$ .

- (C) Inoue surfaces  $X = S_{N,p,q,r;t}^-$  have unramified double cover which are Inoue surfaces of type  $S_{N,p,q,r;t}^+$ .

Suppose to the contrary that there exists a Levi-Civita Ricci-flat Hermitian metric  $\omega$  on the Inoue surface  $X$ . Let  $\omega_f = e^f \omega$  be the Gauduchon metric in the conformal class of  $\omega$ , then by formula (3.7), the total Chern scalar curvature of  $\omega_f$  is

$$(5.1) \quad \int_X s_f \cdot \omega_f^2 = 2 \int_X \mathrm{Ric}(\omega_f) \wedge \omega_f = 2 \|\bar{\partial}_f^* \omega_f\|_{\omega_f}^2 \geq 0.$$

We shall show that on each Inoue surface, there exists a smooth Gauduchon metric with non-positive but not identically zero first Chern-Ricci curvature. Indeed, it is easy to see that the metric  $h^{-1} = [\mathrm{Im}(w)]^{-1} (dw \wedge dz) \otimes (d\bar{w} \wedge d\bar{z})$  (resp.  $h^{-1} = [\mathrm{Im}(w)]^{-2} (dw \wedge dz) \otimes (d\bar{w} \wedge d\bar{z})$ ) is a globally defined Hermitian metric on the anti-canonical bundle of  $S_M$  (resp.  $S_{N,p,q,r;t}^+$ ) (e.g. [4, Section 6]). Hence, the Chern Ricci curvature of  $S_M$  is

$$-\sqrt{-1} \partial \bar{\partial} \log h^{-1} = \sqrt{-1} \partial \bar{\partial} \log [\mathrm{Im}(w)] = -\frac{\sqrt{-1}}{4} \frac{dw \wedge d\bar{w}}{[\mathrm{Im}(w)]^2},$$

which also represents  $c_1^{\mathrm{BC}}(X)$ . By Theorem [17, Theorem 1.3], there exists a Gauduchon metric  $\omega_G$  with

$$\mathrm{Ric}(\omega_G) = -\frac{\sqrt{-1}}{4} \frac{dw \wedge d\bar{w}}{[\mathrm{Im}(w)]^2} \leq 0.$$

Hence, for any Gauduchon metric  $\omega$ , one has

$$\int_X \mathrm{Ric}(\omega) \wedge \omega = \int_X \mathrm{Ric}(\omega_G) \wedge \omega < 0$$

which is a contradiction to (5.1). We can deduce similar contradictions for  $S_{N,p,q,r;t}^\pm$ .

(3). A compact complex surface  $X$  is called a Hopf surface if its universal covering is analytically isomorphic to  $\mathbb{C}^2 \setminus \{0\}$ . It has been prove by Kodaira that its fundamental group  $\pi_1(X)$  is a finite extension of an infinite cyclic group generated by a biholomorphic contraction which takes the form

$$(5.2) \quad (z, w) \rightarrow (az, bw + \lambda z^m)$$

where  $a, b, \lambda \in \mathbb{C}$ ,  $|a| \geq |b| > 1$ ,  $m \in \mathbb{N}^*$  and  $\lambda(a - b^m) = 0$ . Hence, there are two different cases:

- (I) the Hopf surface  $H_{a,b}$  of class 1 if  $\lambda = 0$ ;
- (II) the Hopf surface  $H_{a,b,\lambda,m}$  of class 0 if  $\lambda \neq 0$  and  $a = b^m$ .

In the following, we consider the Hopf surface of class 1. Let  $H_{a,b} = \mathbb{C}^2 \setminus \{0\} / \sim$  where  $(z, w) \sim (az, bw)$  and  $|a| \geq |b| > 1$ . We set  $k_1 = \log |a|$  and  $k_2 = \log |b|$ . Define a real smooth function

$$(5.3) \quad \Phi(z, w) = e^{\frac{k_1+k_2}{2\pi}\theta}$$

where  $\theta(z, w)$  is a real smooth function defined by

$$(5.4) \quad |z|^2 e^{-\frac{k_1\theta}{\pi}} + |w|^2 e^{-\frac{k_2\theta}{\pi}} = 1.$$

This is well-defined since for fixed  $(z, w)$  the function  $t \rightarrow |z|^2 |a|^t + |w|^2 |b|^t$  is strictly increasing with image  $\mathbb{R}_+$ . Let  $\alpha = \frac{2k_1}{k_1+k_2}$  and so  $1 \leq \alpha < 2$ . Then the key equation (5.4) is equivalent to

$$(5.5) \quad |z|^2 \Phi^{-\alpha} + |w|^2 \Phi^{\alpha-2} = 1.$$

It is easy to see that

$$\theta(az, bw) = \theta(z, w) + 2\pi, \quad \text{and} \quad \Phi(az, bw) = |a||b|\Phi(z, w).$$

We define a quantity

$$(5.6) \quad \Delta = \alpha |z|^2 \Phi^{-\alpha} + (2 - \alpha) |w|^2 \Phi^{\alpha-2}.$$

In the next theorem, we construct precisely Levi-Civita Ricci-flat metrics on Hopf surfaces of class 1.  $\square$

**Theorem 5.2.** *On the Hopf surface  $H_{a,b}$  of class 1, the Hermitian metric*

$$(5.7) \quad \omega = \Delta^3 \left( \frac{1}{\Phi} \sqrt{-1} \partial \bar{\partial} \Phi - \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \log \Phi \right)$$

is Levi-Civita Ricci-flat, i.e.  $\mathfrak{Ric}(\omega) = 0$ .

**Remark 5.3.** The proof of Theorem 5.2 is carried out in the next section. We should point out the construction follows from the ideas in [13, Theorem 6.2] and [14, Theorem 7.3]. More precisely, when  $a = b$ , we have  $\alpha = 1$ ,  $\Delta = 1$  and  $\Phi = |z|^2 + |w|^2$ . In this case, the Levi-Civita Ricci-flat metric constructed in Theorem 5.2 is exactly the same as the metrics constructed in [13, Theorem 6.2] and [14, Theorem 7.3].

## 6. The construction of Levi-Civita Ricci-flat metrics on Hopf surfaces of type 1

In this section, we prove Theorem 5.2.

**Lemma 6.1.**  $|z|^2 \Phi^{-\alpha}$  and  $|w|^2 \Phi^{\alpha-2}$  are well-defined on  $H_{a,b}$ .

*Proof.* Indeed,

$$|az|^2 \Phi^{-\alpha}(az, bw) = |a|^2 |a|^{-\alpha} |b|^{-\alpha} \cdot |z|^2 \Phi^{-\alpha}(z, w)$$

and

$$|a|^2|a|^{-\alpha}|b|^{-\alpha} = e^{k_1(2-\alpha)}e^{-k_2\alpha} = 1.$$

Similarly, we can show  $|w|^2\Phi^{2-\alpha}$  is well-defined on  $H_{a,b}$ .  $\square$

**Lemma 6.2.**  $\sqrt{-1}\partial\bar{\partial}\log\Phi$  has a semi-positive matrix representation

$$(6.1) \quad \frac{1}{\Delta^3\Phi^2} \begin{bmatrix} (\alpha-2)^2|w|^2 & \alpha(\alpha-2)\bar{w}z \\ \alpha(\alpha-2)\bar{z}w & \alpha^2|z|^2 \end{bmatrix},$$

and  $\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi$  has a matrix representation

$$(6.2) \quad \frac{1}{\Delta^2\Phi^{2\alpha-2}} \begin{bmatrix} |z|^2 & \bar{w}z\Phi^{2\alpha-2} \\ \bar{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4} \end{bmatrix}.$$

*Proof.* See [24, Appendix].  $\square$

As motivated by [13, Theorem 6.2] and [14, Theorem 7.3], we consider the  $(1,1)$ -form

$$(6.3) \quad \omega_\lambda = \frac{\sqrt{-1}\partial\bar{\partial}\Phi}{\Phi} + \lambda\sqrt{-1}\partial\bar{\partial}\log\Phi.$$

It also takes the form

$$\omega_\lambda = (1+\lambda)\sqrt{-1}\partial\bar{\partial}\log\Phi + \frac{\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi}{\Phi^2}$$

and it has the matrix representation

$$(6.4) \quad \begin{bmatrix} \frac{(1+\lambda)(\alpha-2)^2|w|^2}{\Delta^3\Phi^2} + \frac{|z|^2}{\Delta^2\Phi^{2\alpha}} & \frac{(1+\lambda)\alpha(\alpha-2)\bar{w}z}{\Delta^3\Phi^2} + \frac{\bar{w}z\Phi^{2\alpha-2}}{\Delta^2\Phi^{2\alpha}} \\ \frac{(1+\lambda)\alpha(\alpha-2)\bar{z}w}{\Delta^3\Phi^2} + \frac{\bar{z}w\Phi^{2\alpha-2}}{\Delta^2\Phi^{2\alpha}} & \frac{(1+\lambda)\alpha^2|z|^2}{\Delta^3\Phi^2} + \frac{|w|^2\Phi^{4\alpha-4}}{\Delta^2\Phi^{2\alpha}} \end{bmatrix}.$$

Since  $\det(\sqrt{-1}\partial\bar{\partial}\log\Phi) = \det(\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi) = 0$ , the determinant

$$\begin{aligned} \det(\omega_\lambda) &= \frac{(1+\lambda)(\alpha-2)^2|w|^2}{\Delta^3\Phi^2} \cdot \frac{|w|^2\Phi^{4\alpha-4}}{\Delta^2\Phi^{2\alpha}} + \frac{|z|^2}{\Delta^2\Phi^{2\alpha}} \cdot \frac{(1+\lambda)\alpha^2|z|^2}{\Delta^3\Phi^2} \\ &\quad - \frac{(1+\lambda)\alpha(\alpha-2)\bar{w}z}{\Delta^3\Phi^2} \cdot \frac{\bar{z}w\Phi^{2\alpha-2}}{\Delta^2\Phi^{2\alpha}} - \frac{\bar{w}z\Phi^{2\alpha-2}}{\Delta^2\Phi^{2\alpha}} \cdot \frac{(1+\lambda)\alpha(\alpha-2)\bar{z}w}{\Delta^3\Phi^2} \\ &= \frac{(1+\lambda)|w|^2(\alpha-2)}{\Delta^5} \left( \frac{(\alpha-2)|w|^2}{\Phi^{6-2\alpha}} - \frac{\alpha|z|^2}{\Phi^4} \right) \\ &\quad + \frac{(1+\lambda)\alpha|z|^2}{\Delta^5} \left( \frac{\alpha|z|^2}{\Phi^{2\alpha+2}} - \frac{(\alpha-2)|w|^2}{\Phi^4} \right) \\ &= \frac{1+\lambda}{\Delta^5} \cdot \frac{1}{\Phi^{2+2\alpha}} (\alpha|z|^2 + (2-\alpha)|w|^2\Phi^{2\alpha-2})^2. \end{aligned}$$

By (5.6), we have

$$\alpha|z|^2 + (2-\alpha)|w|^2\Phi^{2\alpha-2} = \Delta\Phi^\alpha$$

and so

$$(6.5) \quad \det(\omega_\lambda) = \frac{1+\lambda}{\Delta^3\Phi^2}.$$

It is easy to see from (6.3) that, when  $\lambda > -1$ ,

$$\omega_\lambda = \frac{\sqrt{-1}\partial\bar{\partial}\Phi}{\Phi} + \lambda\sqrt{-1}\partial\bar{\partial}\log\Phi$$

is a Hermitian metric. Let  $\partial^*$  and  $\bar{\partial}^*$  be the adjoint operators taken with respect to the metric  $\omega_\lambda$ , and  $\Lambda$  be the dual operator of  $\omega_\lambda \wedge \bullet$ .

**Lemma 6.3.** *We have*

$$(6.6) \quad \frac{\partial\partial^*\omega_\lambda + \bar{\partial}\bar{\partial}^*\omega_\lambda}{2} = \partial\partial^*\omega_\lambda = \bar{\partial}\bar{\partial}^*\omega_\lambda = \frac{\sqrt{-1}\partial\bar{\partial}\log\Phi}{1+\lambda}.$$

*Proof.* The metric  $\omega_\lambda$  has local matrix representation

$$(h_{i\bar{j}}) = \begin{pmatrix} \frac{1+\lambda}{\Phi}\Phi_{1\bar{1}} - \frac{\lambda\Phi_1\Phi_{\bar{1}}}{\Phi^2} & \frac{1+\lambda}{\Phi}\Phi_{1\bar{2}} - \frac{\lambda\Phi_1\Phi_{\bar{2}}}{\Phi^2} \\ \frac{1+\lambda}{\Phi}\Phi_{2\bar{1}} - \frac{\lambda\Phi_2\Phi_{\bar{1}}}{\Phi^2} & \frac{1+\lambda}{\Phi}\Phi_{2\bar{2}} - \frac{\lambda\Phi_2\Phi_{\bar{2}}}{\Phi^2} \end{pmatrix}$$

and its inverse matrix representation is

$$(h^{i\bar{j}}) = \frac{\Phi^2\Delta^3}{1+\lambda} \begin{pmatrix} \frac{1+\lambda}{\Phi}\Phi_{2\bar{2}} - \frac{\lambda\Phi_2\Phi_{\bar{2}}}{\Phi^2} & -\frac{1+\lambda}{\Phi}\Phi_{1\bar{2}} + \frac{\lambda\Phi_1\Phi_{\bar{2}}}{\Phi^2} \\ -\frac{1+\lambda}{\Phi}\Phi_{2\bar{1}} + \frac{\lambda\Phi_2\Phi_{\bar{1}}}{\Phi^2} & \frac{1+\lambda}{\Phi}\Phi_{1\bar{1}} - \frac{\lambda\Phi_1\Phi_{\bar{1}}}{\Phi^2} \end{pmatrix}.$$

By Lemma 3.1, we have

$$\partial^*\omega_\lambda = -\sqrt{-1}\Lambda\bar{\partial}\omega_\lambda = -2\sqrt{-1}(\Gamma_\lambda)_{i\bar{k}}^k d\bar{z}^i.$$

A straightforward computation shows that

$$\begin{aligned} \frac{\partial h_{j\bar{\ell}}}{\partial\bar{z}^i} &= \frac{\partial}{\partial\bar{z}^i} \left( \frac{1+\lambda}{\Phi}\Phi_{j\bar{\ell}} - \frac{\lambda}{\Phi^2}\Phi_j\Phi_{\bar{\ell}} \right) \\ &= -\frac{1+\lambda}{\Phi^2}\Phi_i\Phi_{j\bar{\ell}} + \frac{1+\lambda}{\Phi}\Phi_{j\bar{\ell}i} + \frac{2\lambda}{\Phi^3}\Phi_i\Phi_j\Phi_{\bar{\ell}} - \frac{\lambda}{\Phi^2}\Phi_{j\bar{i}}\Phi_{\bar{\ell}} - \frac{\lambda}{\Phi^2}\Phi_j\Phi_{\bar{\ell}i}. \end{aligned}$$

Hence, we have

$$\begin{aligned} (\Gamma_\lambda)_{i\bar{j}}^k &= \frac{1}{2}h^{k\bar{\ell}} \left( \frac{\partial h_{j\bar{\ell}}}{\partial\bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial\bar{z}^{\bar{\ell}}} \right) \\ &= \frac{1}{2}h^{k\bar{\ell}} \left( -\frac{1}{\Phi^2}\Phi_{j\bar{\ell}}\Phi_i + \frac{1}{\Phi^2}\Phi_{j\bar{i}}\Phi_{\bar{\ell}} \right) \end{aligned}$$

and

$$\begin{aligned} \partial^*\omega_\lambda &= -2\sqrt{-1}(\Gamma_\lambda)_{i\bar{k}}^k d\bar{z}^i = -\sqrt{-1}h^{k\bar{\ell}} \frac{1}{\Phi^2} (-\Phi_{k\bar{\ell}}\Phi_i + \Phi_{k\bar{i}}\Phi_{\bar{\ell}}) d\bar{z}^i \\ &= -\sqrt{-1}h^{k\bar{\ell}} \frac{1}{\Phi^2} \left( -\Phi_i \left( \frac{\Phi h_{k\bar{\ell}}}{1+\lambda} + \frac{\lambda\Phi_k\Phi_{\bar{\ell}}}{(1+\lambda)\Phi} \right) + \Phi_{\bar{\ell}} \left( \frac{\Phi h_{k\bar{i}}}{1+\lambda} + \frac{\lambda\Phi_k\Phi_{\bar{i}}}{(1+\lambda)\Phi} \right) \right) \\ &= -\sqrt{-1}h^{k\bar{\ell}} \frac{1}{\Phi^2} \left( \frac{-\Phi_i\Phi h_{k\bar{\ell}}}{1+\lambda} + \frac{\Phi_{\bar{\ell}}\Phi h_{k\bar{i}}}{1+\lambda} \right) \\ &= \frac{\Phi_i}{\Phi} \frac{\sqrt{-1}}{1+\lambda} d\bar{z}^i = \frac{\sqrt{-1}}{1+\lambda} \bar{\partial}\log\Phi. \end{aligned}$$

Therefore, we get (6.6).  $\square$



By formulas (3.5), (6.5) and (6.6), we obtain

$$\begin{aligned}\mathfrak{Ric}(\omega_\lambda) &= -\sqrt{-1}\partial\bar{\partial}\det(\omega_\lambda) - \frac{\partial\partial^*\omega_\lambda + \bar{\partial}\bar{\partial}^*\omega_\lambda}{2} \\ &= \left(2 - \frac{1}{1+\lambda}\right)\sqrt{-1}\partial\bar{\partial}\log\Phi + 3\sqrt{-1}\partial\bar{\partial}\log\Delta.\end{aligned}$$

In particular, we can take  $\lambda = -\frac{1}{2}$ , and obtain

$$(6.7) \quad \mathfrak{Ric}(\omega_{-\frac{1}{2}}) = 3\sqrt{-1}\partial\bar{\partial}\log\Delta.$$

**Theorem 6.4.** *Let*

$$\omega = \Delta^3\omega_{-\frac{1}{2}} = \Delta^3\left(\frac{\sqrt{-1}}{\Phi}\partial\bar{\partial}\Phi - \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\log\Phi\right).$$

*Then we have*

$$(6.8) \quad \mathfrak{Ric}(\omega) = 0.$$

*Proof.* By Lemma 3.4 and formula (6.7), we have

$$(6.9) \quad \mathfrak{Ric}(\omega) = \mathfrak{Ric}(\omega_{-\frac{1}{2}}) - \sqrt{-1}\partial\bar{\partial}\log\Delta^3 = 0.$$

□

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