RECURSIONS AND ASYMPTOTICS OF INTERSECTION NUMBERS

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Abstract. We establish the asymptotic expansion of certain integrals of ψ classes on moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$ when either the g or n goes to infinity. Our main tools are cut-join type recursion formulae from the Witten-Kontsevich theorem as well as asymptotics of solutions to the first Painlevé equation. We also raise a conjecture on large genus asymptotics for n-point functions of ψ classes and partially verify the positivity of coefficients in generalized Mirzakhani's formula of higher Weil-Petersson volumes.

Contents

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable n-pointed genus g complex algebraic curves and $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ the morphism that forgets the last marked point. Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π, and by $D_1, \ldots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf, we shall consider integrals of the following tautological classes:

$$\psi_i = c_1(\sigma_i^*(\omega_\pi)), \quad 1 \le i \le n,$$
$$\kappa_i = \pi_*\left(c_1\left(\omega_\pi\left(\sum D_i\right)\right)^{i+1}\right), \quad i \ge 0,$$

on $\overline{\mathcal{M}}_{g,n}$, where $\kappa_0 = 2g-2+n$. The κ classes were first defined on $\overline{\mathcal{M}}_g$ by Mumford [35], its extension to $\overline{\mathcal{M}}_{g,n}$ is due to Arbarello-Cornalba [1]. More background material can be found in [44].

Wolpert [46] showed that $\kappa_1 = \omega_{WP}/(2\pi^2)$, where ω_{WP} is the Weil-Petersson Kähler form. Thus Weil-Petersson volumes are equal to the intersection numbers

$$V_{g,n} = \frac{1}{(3g-3+n)!}\int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}.$$

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It is well-known that integrals of κ and ψ classes are equivalent to each other through explicit combinatorial identities (cf. [1, 19]).

The celebrated Witten-Kontsevich theorem [20, 45] shows that integrals of ψ classes on $\overline{\mathcal{M}}_{g,n}$ are governed by the KdV hierarchy. By using a generalization of McShane's identity in hyperbolic geometry, Mirzakhani [28] obtained a remarkable recursive integral formula of Weil-Petersson volumes of moduli spaces of bordered hyperbolic surfaces. In [34], Mirzakhani's formula was shown to be equivalent to a more explicit Virasoro constraint condition for the mixed integral of ψ and κ_1 classes, which was generalized in [21, 22] to higher degree κ classes. Eynard and Orantin [12] showed that Mirzakhani's recursion formula fits in with the Eynard-Orantin recursion formalism whose spectral curve is the sine curve discovered in [34].

Recently Mirzakhani and Zograf [31] made a breakthrough on large genus asymptotics of Weil-Petersson volumes. Their work is based on an earlier paper of Mirzakhani [30], who brought new ideas to bear on the problem: (i) One should consider the normalized intersection numbers involving both ψ and κ classes; (ii) The terms corresponding to reducible boundary components of $\overline{\mathcal{M}}_{g,n}$ in the cut-join recursions are of lower order in g.

In this paper, we study asymptotics of integrals of pure ψ classes, which appear naturally in the asymptotics of Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten invariants, graph enumerations and 2D gravity. Our main technique is the manipulation of various recursion formulas arising from Witten-Kontsevich theorem, e.g., DVV recursion formula, recursion formula of n-point functions and Mirzakhani recursion formula.

The paper is organized as follows: In §2, we raise a conjecture about large genus asymptotics of the n-point function and give a proof when $n = 2$. In §3, we review the recent work of asymptotics of Weil-Petersson volumes; we also partially verify the positivity of coefficients $\alpha_{\mathbf{L}}$ in a recursion formula of higher Weil-Petersson volumes. In §4, we discuss intersection numbers in the framework of Eynard-Orantin theory and several identities involving $\alpha_{\mathbf{L}}$. In §5, we apply asymptotics of solutions to the first Painlevé equation to establish large genus asymptotic expansion of ψ class integrals $\langle\tau_{d_1}\cdots\tau_{d_n}\tau_2^{3g-3+n-|\mathbf{d}|}\rangle_g$. In §6, we apply DVV formula to establish asymptotic expansion of ψ class integrals $\langle\tau_{d_1}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_g$ when k goes to infinity.

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2. Witten-Kontsevich theorem and integrals of ψ classes

We adopt Witten's notation

$$\langle\tau_{d_1}\cdots\tau_{d_n}\kappa_{a_1}\cdots\kappa_{a_m}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{a_1}\cdots\kappa_{a_m}. \tag{1}$$

For convenience, we denote the normalized tau function as

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}} := \prod_{i=1}^n (2d_i+1)!!\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g. \tag{2}$$

The celebrated Witten-Kontsevich theorem [45, 20] can be equivalently formulated as the following DVV formula [6].

$$\begin{aligned}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}} &= \sum_{j=2}^n (2d_j+1)\langle\tau_{d_2}\cdots\tau_{d_j+d_1-1}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}} \\ &+\frac{1}{2}\sum_{r+s=d_1-2}\langle\tau_r\tau_s\tau_{d_2}\cdots\tau_{d_n}\rangle_{g-1}^{\mathbf{w}}+\frac{1}{2}\sum_{r+s=d_1-2}\sum_{\{2,\cdots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}^{\mathbf{w}}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}^{\mathbf{w}},\end{aligned} \tag{3}$$

which is equivalent to the Virasoro constraint.

When $d_1 = 0$ or 1 in (3), we get the string and dilaton equations respectively

$$\langle\tau_0\tau_{d_2}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}} = \sum_{j=2}^n (2d_j+1)\langle\tau_{d_2}\cdots\tau_{d_j-1}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}}, \tag{4}$$

$$\langle\tau_1\tau_{d_2}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}} = 3(2g-3+n)\langle\tau_{d_2}\cdots\tau_{d_n}\rangle_g^{\mathbf{w}}. \tag{5}$$

Definition 2.1. The following generating function

$$F(x_1,\cdots,x_n) = \sum_{g=0}^\infty F_g(x_1,\cdots,x_n) = \sum_{g=0}^\infty \sum_{\sum d_i=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{i=1}^n x_i^{d_i}$$

is called the n-point function.

The following recursive formula was obtained by integrating the first KdV equation of the Witten-Kontsevich theorem.

$$\begin{aligned}(2g+n-1)\langle\tau_0\prod_{j=1}^n\tau_{d_j}\rangle_g \\ = \frac{1}{12}\langle\tau_0^4\prod_{j=1}^n\tau_{d_j}\rangle_{g-1}+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\langle\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},\end{aligned} \tag{6}$$

which is equivalent to a recursive formula of n-point functions (cf. [25]),

$$F(x_1,\ldots,x_n) = \sum_{r,s\geq 0}\frac{(2r+n-3)!!}{12^s(2r+2s+n-1)!!}S_r(x_1,\ldots,x_n)\left(\sum_{j=1}^n x_j\right)^{3s}, \tag{7}$$

where $n\geq 2$ and S_r is a homogeneous symmetric polynomial of degree $3r+n-3$,

$$\begin{aligned}S_r(x_1,\ldots,x_n) &= \left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2F(x_I)F(x_J)\right)_{3r+n-3} \\ &= \frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2\sum_{r'=0}^r F_{r'}(x_I)F_{r-r'}(x_J),\end{aligned}$$

where $\underline{n} = \{1,2,\ldots,n\}$ and $I, J\neq\emptyset$.

The following closed formulae of one and two-point functions are respectively due to Witten and Dijkgraaf,

$$F(x) = \frac{1}{x^2} \exp\left(\frac{x^3}{24}\right),$$

$$F(x,y) = \frac{1}{x+y} \exp\left(\frac{x^3}{24} + \frac{y^3}{24}\right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2}xy(x+y)\right)^k.$$

The usefulness of n-point functions was noticed by Faber in his pioneering work [14] on tautological rings of moduli spaces of curves. In [47], Zagier obtained several remarkable closed formulae for the three-point function. In [37], Okounkov proved an analytic formula of the n-point function in terms of n-dimensional error-function-type integrals. In [23, 25], the recursion formula (7) was used to give a direct proof of Faber's intersection number conjecture.

Lemma 2.2. *Let* $E(x_1, \dots, x_n) = \sum_{g=0}^{\infty} 12^g (2g+n-1)!! F_g(x_1, \dots, x_n)$. *Then*

$$E(x) = \frac{1}{x^2(1-x^3)}, \tag{8}$$

$$E(x,y) = \frac{1}{(x+y)(1-(x+y)^3)\sqrt{1-(x^3+y^3)}}. \tag{9}$$

Proof. (8) follows easily from $F_g(x) = x^{3g-2}/(24^g g!)$.

From (7) and

$$S_r(x,y) = \frac{(x^3+y^3)^r}{(x+y)24^r r!}, \tag{10}$$

we could get

$$\begin{aligned} E(x,y) &= \sum_{g=0}^{\infty} 12^g (2g+1)!! F_g(x,y) \\ &= \sum_{r,s\geq 0} 12^r (2r-1)!! S_r(x,y)(x+y)^{3s} \\ &= \frac{1}{(x+y)(1-(x+y)^3)} \sum_{r\geq 0} 12^r (2r-1)!! \cdot \frac{(x^3+y^3)^r}{24^r r!} \\ &= \frac{1}{(x+y)((1-(x+y)^3))\sqrt{1-(x^3+y^3)}}, \end{aligned}$$

which proves (9). □

Lemma 2.2 was inspired by the following remarkable formula of Zagier [47],

$$\sum_{g=0}^{\infty} 4^g (2g+1)!! F_g(x,y,z) = \frac{\arctan\left(\frac{\sqrt{(x+y+z)^3 xyz}}{1-\frac{1}{3}(x^3+y^3+z^3)+xyz} \sqrt{\frac{1-\frac{1}{3}(x^3+y^3+z^3)}{1-\frac{1}{3}(x+y+z)^3}}\right)}{\sqrt{(x+y+z)^3 xyz\left(1-\frac{1}{3}(x+y+z)^3\right)}}.$$

The reason that we used slightly different normalization coefficients in Lemma 2.2 is due to (7), which implies

$$E(x_1, \dots, x_n) = \frac{1}{(1-\sum_{j=1}^n x_j)^3} \sum_{r=0}^{\infty} 12^r (2r+n-3)!! S_r(x_1, \dots, x_n). \tag{11}$$

It is not clear whether one can write the above equation into a closed-form expression of $E(x_1,\dots,x_n)$ for arbitrary $n\geq 3$, maybe with different choices of normalization coefficients.

The n-point function appears in several asymptotic formulae of enumerative geometry, such as: the leading term of Mirzakhai's volume polynomial of Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces [29], the highest degree term of Gromov-Witten invariants of projective spaces [36], and the following limit of Hurwitz numbers $H_{g,\mu}$ (cf. [38]):

$$F_g(\mu_1,\dots,\mu_n)=\lim_{N\to\infty}\frac{(2\pi)^{n/2}|\mathrm{Aut}(\mu)|\prod_{i=1}^n\mu_i^{1/2}}{N^{3g-3+n/2}}\frac{H_{g,N\mu}}{e^{N\mu}(2g-2+|\mu|+n)!},$$

where $\mu=(\mu_1,\dots,\mu_n)$ is any given partition and $|\mu|=\mu_1+\cdots+\mu_n$.

In view of these connections, we formulate a conjectural large genus asymptotics of $F_g(x_1,\dots,x_n)$ shall be interesting. In fact, by (7), we have

$$\begin{aligned}&\frac{12^g(2g+n-1)!!}{(x_1+\cdots+x_n)^{3g-3+n}}F_g(x_1,\dots,x_n)\\ =&\frac{12^g(2g+n-1)!!}{(x_1+\cdots+x_n)^{3g-3+n}}\sum_{r=0}^g\frac{(2r+n-3)!!}{12^{g-r}(2g+n-1)!!}S_r(x_1,\dots,x_n)\left(\sum_{j=1}^n x_j\right)^{3s}\\ =&\sum_{r=0}^g 12^r(2r+n-3)!!\frac{S_r(x_1,\dots,x_n)}{(\sum_{j=1}^n x_j)^{3r-3+n}}.\end{aligned}$$

Now let

$$C(x_1,\dots,x_n)=\sum_{r=0}^\infty 12^r(2r+n-3)!!\frac{S_r(x_1,\dots,x_n)}{(\sum_{j=1}^n x_j)^{3r-3+n}}. \tag{12}$$

We conjecture that the series in the right-hand side of the above equation is convergent for any positive real numbers $x_j>0$, $\forall 1\leq j\leq n$.

Conjecture 2.3. *Fix a set of positive real numbers $x_j>0$, $\forall 1\leq j\leq n$. Then there exist functions $C(x_1,\dots,x_n)>0$ independent of g such that as $g\to\infty$,*

$$F_g(x_1,\dots,x_n)\sim C(x_1,\dots,x_n)\frac{(x_1+\cdots+x_n)^{3g-3+n}}{12^g(2g+n-1)!!}, \tag{13}$$

where $a_1(g)\sim a_2(g)$ means $\lim_{g\to\infty}\frac{a_1(g)}{a_2(g)}=1$.

The above conjecture holds trivially when $n=1$. Now we prove it for $n=2$.

Proposition 2.4. *Let $x,y>0$. Then as $g\to\infty$,*

$$F_g(x,y)\sim\frac{x+y}{\sqrt{3xy}}\cdot\frac{(x+y)^{3g-1}}{12^g(2g+1)!!}. \tag{14}$$

Proof. Let

$$f_g(x,y)=\frac{12^g(2g+1)!!}{(x+y)^{3g-1}}F_g(x,y).$$

Then by (7) and (10), we get

$$f_g(x,y)=\sum_{k=0}^g\frac{(2k-1)!!}{2^kk!}\left(\frac{x^3+y^3}{(x+y)^3}\right)^k,$$

which implies

$$\lim_{g\to\infty} f_g(x,y) = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \left(\frac{x^3+y^3}{(x+y)^3}\right)^k = \frac{1}{\sqrt{1-\frac{x^3+y^3}{(x+y)^3}}} = \frac{x+y}{\sqrt{3xy}},$$

i.e. $C(x,y) = \frac{x+y}{\sqrt{3xy}}$. □

Remark 2.5. In [24, §5], we observed that integrals of ψ classes satisfy multinomial-type property, i.e. $\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_g \le \langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_n}\rangle_g$ when $d_1 < d_2$. This is consistent with Conjecture 2.3.

3. Weil-Petersson volumes

As mentioned above, the starting point of using recursion formulae to study large genus asypmtotics of Weil-Petersson volumes is Mirzakhani's insight [29, 30] that one should consider *normalized* intersection numbers:

$$[\tau_{d_1}\cdots\tau_{d_n}]_{g,n} = \frac{\prod_{i=1}^n (2d_i+1)!!\, 4^{|\mathbf{d}|}(2\pi^2)^{d_0}}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_1^{d_0}, \tag{15}$$

where $|\mathbf{d}| = d_1+\cdots+d_n \le 3g-3+n$ and $d_0 = 3g-3+n-|\mathbf{d}|$. Note that $V_{g,n} = [\tau_0,\cdots\tau_0]_{g,n}$ is the Weil-Peterson volume of $\overline{\mathcal{M}}_{g,n}$.

Mirzakhani [28] proved a recursion formula for Weil-Peterson volumes of moduli spaces of bordered Riemann surfaces. The following equivalent form of Mirzakhani's formula was derived by Mulase and Safnuk [34] (cf. also [41, 21, 12]).

$$\begin{aligned}
(16)\quad [\tau_{d_1},\ldots,\tau_{d_n}]_{g,n} &= 8\sum_{j=2}^{n}\sum_{L=0}^{d_0}(2d_j+1)a_L[\tau_{d_1+d_j+L-1}\prod_{i\neq 1,j}\tau_{d_i}]_{g,n-1}\\
&+16\sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L[\tau_{k_1}\tau_{k_2}\prod_{i\neq 1}\tau_{d_i}]_{g-1,n+1}\\
&+16\sum_{\substack{I\amalg J=\{2,\ldots,n\}\\ 0\le g'\le g}}\sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L\,[\tau_{k_1}\prod_{i\in I}\tau_{d_i}]_{g',|I|+1}\times[\tau_{k_2}\prod_{i\in J}\tau_{d_i}]_{g-g',|J|+1}.
\end{aligned}$$

Here $a_L = \zeta(2L)(1-2^{1-2L})$.

Mulase and Safnuk [34] also proved the following inversion to the formula (16),

$$\begin{aligned}
(17)\quad \sum_{L=0}^{d_0}\frac{(-\pi^2)^L}{4(2L+1)!}[\tau_{d_1+L},\ldots,\tau_{d_n}]_{g,n} &= \sum_{j=2}^{n}(2d_j+1)[\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}]_{g,n-1}\\
&+\sum_{k_1+k_2=d_1-2}[\tau_{k_1}\tau_{k_2}\prod_{i\neq 1}\tau_{d_i}]_{g-1,n+1}\\
&+\sum_{\substack{I\amalg J=\{2,\ldots,n\}\\ 0\le g'\le g}}\sum_{k_1+k_2=d_1-2}[\tau_{k_1}\prod_{i\in I}\tau_{d_i}]_{g',|I|+1}\times[\tau_{k_2}\prod_{i\in J}\tau_{d_i}]_{g-g',|J|+1}.
\end{aligned}$$

Motivated by a question of Mirzakhani, Zograf [50] made the following conjecture on large genus asymptotic expansion of $V_{g,n}$ based on numerical data.

Conjecture 3.1 (Zograf)**.** *For any fixed $n \geq 0$, as $g \to \infty$,*

$$V_{g,n} = (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{1}{\sqrt{g\pi}}\left(1+\frac{c_n}{g}+O\left(\frac{1}{g^2}\right)\right), \tag{18}$$

where c_n is a constant independent of g.

Note that the asymptotic expansion of $V_{g,n}$ for fixed g and large n has been completely solved by Manin and Zograf [31]. Recently, Mirzakhani and Zograf [31] proved the following complete asymptotic expansion of Weil-Petersson volumes as n fixed and $g \to \infty$,

$$V_{g,n} = C\frac{(4\pi^2)^{2g+n-3}(2g-3+n)!}{\sqrt{g}}\left(1+\frac{c_n^{(1)}}{g}+\frac{c_n^{(k)}}{g^k}+\dots\right), \tag{19}$$

where $0 < C < \infty$ is a universal constant and each term $c_n^{(i)}$ is a polynomial in n of degree $2i$, which reduces the proof of Zograf's conjecture (cf. (18)) to that of $C = 1/\sqrt{\pi}$.

The following weaker estimate of $V_{g,n}$ was originally proved with the joint effort of Penner [39], Grushevsky [15], and Schumacher-Trapani [42].

Theorem 3.2. *There is a constant C independent of g such that*

$$\left(\frac{1}{C}\right)^g (2g)! < V_{g,n} < C^g(2g)! \tag{20}$$

for fixed n and large g.

A short proof of the above theorem was given in [26, §2], which used (16) and some recursion formulae from [5, 21], together with a technical result on the asymptotics of solutions to the first Painlevé equation (cf. §5).

Now we introduce some notation from [19]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i. Denote by $\boldsymbol{\delta}_a$ the sequence with 1 at the a-th place and zeros elsewhere. Let $\mathbf{m}, \mathbf{L}, \mathbf{a_1}, \dots, \mathbf{a_n} \in N^\infty$. Then

$$|\mathbf{m}| := \sum_{i\geq 1} im(i), \quad ||\mathbf{m}|| := \sum_{i\geq 1} m(i), \quad \mathbf{m}! := \prod_{i\geq 1} m(i)!, \quad \kappa(\mathbf{b}) := \prod_{i\geq 1}\kappa_i^{b(i)},$$

$$\binom{\mathbf{m}}{\mathbf{L}} := \prod_{i\geq 1}\binom{m(i)}{L(i)}, \quad \binom{\mathbf{m}}{\mathbf{a_1}, \dots, \mathbf{a_n}} := \prod_{i\geq 1}\binom{m(i)}{a_1(i), \dots, a_n(i)}.$$

Extensive studies of intersection numbers involving higher degree κ classes can be found in [4, 19, 22, 40]. The following generalization of (16) was proved in [21, 22]. It is equivalent to a recursion formula of generating functions proved by Eynard [8] (cf. Prop. 4.4).

Theorem 3.3. *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$. Then*

$$\begin{aligned}(21)\quad & (2d_1+1)!!\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\ &= \sum_{j=2}^{n}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g \\ &+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{L}')\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\rangle_{g-1}\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}(2r+1)!!(2s+1)!!$$

$$\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where the constants $\alpha_{\mathbf{L}}$ are determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}=0,\qquad \mathbf{b}\neq 0 \tag{22}$$

with the initial value $\alpha_{\mathbf{0}}=1$.

We conjecture that $\alpha_{\mathbf{L}}$ is always positive, which is crucial if one want to study the large genus asymptotics of higher Weil-Petersson volumes using (21).

Conjecture 3.4. *For any $\mathbf{L}\in N^{\infty}$, $\alpha_{\mathbf{L}}>0$.*

Below we give a partial answer to the above conjecture.

A *partition* of a finite set $X=\{1,2,\ldots,\ell\}$ into k parts is a collection $\pi=\{A_1,A_2,\ldots,A_k\}$ of subsets of X such that (i) $A_i\neq\emptyset$ for each i; (ii) $A_i\cap A_j=\emptyset$ if $i\neq j$; (iii) $A_1\cup\cdots\cup A_k=X$.

We denote by $\mathscr{P}(X,k)$ the set of all partitions of X into k parts. We know that $|\mathscr{P}(X,k)|$ is given by $S(\ell,k)$, the Stirling number of the second kind. In particular, $S(\ell,1)=1$ and $S(\ell,\ell-1)=\binom{\ell}{2}$.

By (22), we have for $b\neq 0$,

$$\alpha_{\mathbf{b}}=\mathbf{b}!\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}'\neq\mathbf{0}}}\frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} \tag{23}$$

$$=\sum_{k=1}^{||\mathbf{b}||}\sum_{\substack{\mathbf{L}_1+\cdots+\mathbf{L}_k=\mathbf{b}\\ \mathbf{L}_i\neq\mathbf{0}}}\binom{\mathbf{b}}{\mathbf{L}_1,\ldots,\mathbf{L}_k}\frac{(-1)^{||\mathbf{b}||-k}}{\prod_{i=1}^{k}(2|\mathbf{L}_i|+1)!!}.$$

Let $\mathbf{b}=\boldsymbol{\delta}_{p_1}+\cdots+\boldsymbol{\delta}_{p_\ell}\in N^{\infty}$ and $\pi=\{A_1,\ldots,A_k\}$ be a partition of $X=\{1,\ldots,\ell\}$ into k parts. Define $p(\pi,\mathbf{b})=\prod_{j=1}^{k}(2\sum_{i\in A_j}p_i+1)!!$. Then (23) implies

$$\alpha_{\mathbf{b}}=\sum_{k=1}^{\ell}\sum_{\pi\in\mathscr{P}(X,k)}\frac{(-1)^{\ell-k}k!}{p(\pi,\mathbf{b})}. \tag{24}$$

Proposition 3.5. *For any $\mathbf{b}\in N^{\infty}$ with $||\mathbf{b}||\leq 4$, we have $\alpha_{\mathbf{b}}>0$.*

Proof. First note that for any $i,j\geq 1$, we have $(2i+2j+1)!!\geq\frac{5}{3}(2i+1)!!(2j+1)!!$.

(i) When $\mathbf{b}=\boldsymbol{\delta}_i$, we have $\alpha_{\mathbf{b}}=1/(2i+1)!!>0$.

(ii) When $\mathbf{b}=\boldsymbol{\delta}_i+\boldsymbol{\delta}_j$, we have

$$\alpha_{\mathbf{b}}=\frac{2}{(2i+1)!!(2j+1)!!}-\frac{1}{(2i+2j+1)!!}>0.$$

(iii) When $\mathbf{b}=\boldsymbol{\delta}_i+\boldsymbol{\delta}_j+\boldsymbol{\delta}_k$, we have

$$\alpha_{\mathbf{b}}=\frac{6}{(2i+1)!!(2j+1)!!(2k+1)!!}-\frac{2}{(2i+2j+1)!!(2k+1)!!}$$
$$-\frac{2}{(2i+2j+1)!!(2k+1)!!}-\frac{2}{(2i+2j+1)!!(2k+1)!!}+\frac{1}{(2i+2j+2k+1)!!}>0.$$

(iv) When $||\mathbf{b}|| = 4$ and $X = \{1, 2, 3, 4\}$, we have

$$\sum_{\pi \in \mathscr{P}(X,3)} \frac{3!}{p(\pi, \mathbf{b})} \leq \frac{3}{5} \frac{6 \cdot S(4,3)}{\prod_{j=1}^{4}(2p_j+1)!!} < \frac{4!}{\prod_{j=1}^{4}(2p_j+1)!!},$$

which obviously implies that $\alpha_{\mathbf{b}} > 0$. □

From the above proof, it is easy to see that for any $\mathbf{b} \in N^\infty$ with $||\mathbf{b}|| = \ell > 0$, there exists an integer $C_\ell > 0$ such that $\alpha_{\mathbf{b}} > 0$ whenever $b(i) = 0$, $\forall i \leq C_\ell$.

4. Eynard-Orantin theory

We will outline the mathematical definition for the Eynard-Orantin theory [11], which provides a powerful unifying tool for many enumerative problems in geometry. We refer the readers to [2, 3, 7, 32, 33] for more detailed expositions and recent developments.

A spectral curve is a quadruple of data

$$\mathcal{S} = (C, x, y, B),$$

where C is a plane curve of genus 0, x, y are two analytic function on C and $B(z, z')$ is the *Bergman kernel*, i.e. a symmetric differential on C and behaves like

$$B(z, z') \underset{z \to z'}{\sim} \frac{dz \otimes dz'}{(z - z')^2} + O(1).$$

We require dx, dy have only simple zeros and $(x, y) : C \to \mathbb{C}^2$ is an immersion. A *branch point* is a zero of dx.

Given a spectral curve $\mathcal{S} = (C, x, y, B)$, the symmetric meromorphic n-differential $W_n^{(g)}(\mathcal{S}, z_1, \ldots, z_n)$ is defined by

$$W_1^{(0)}(z) = y(z)dx(z), \quad W_2^{(0)}(z, z') = B(z, z')$$

and when $2g - 2 + n \geq 0$

$$W_n^{(g)}(z_1, z_2 \ldots, z_n) = \sum_a \operatorname*{Res}_{z \to a} K(z_1, z) \Bigg[W_{n+1}^{(g-1)}(z, \bar{z}, z_2 \ldots, z_n) + \sum_{\substack{g_1 + g_2 = g \\ I \coprod J = \{2, \ldots, n\}}}^{\text{no } W_1^{(0)} \text{ terms}} W_{1+|I|}^{(g_1)}(z, z_I) W_{1+|J|}^{(g_2)}(\bar{z}, z_J) \Bigg], \tag{25}$$

where a runs over all branch points of C, $\bar{z}$ is determined by $x(\bar{z}) = x(z)$ around a neighborhood of a and the recursion kernel is given by

$$K(z_1, z) = \frac{\int_{z'=\bar{z}}^{z} B(z_1, z')}{2(y(z) - y(\bar{z}))dx(z)}.$$

The free energy invariants $F_g(S)$ is given by the dilaton equation

$$F_g(\mathcal{S}) = W_0^{(g)} = \frac{1}{2 - 2g} \sum_a \operatorname*{Res}_{z \to a} W_1^{(g)}(z) \Phi(z),$$

where $\Phi(z)$ is defined near the branch point a by $d\Phi = ydx$.

The free energy $F_{g,n}(z_1,\dots,z_n)$ is defined to be the primitive of $W_n^{(g)}$:

$$d^{\otimes n}F_{g,n}(z_1,\dots,z_n) = W_n^{(g)}(z_1,\dots,z_n).$$

The following theorem is a key result used in Eynard's proof [9, 10] that for arbitrary spectral curves, $W_n^{(g)}(z_1,\dots,z_n)$ can be explicitly expressed as a universal formula involving intersection numbers of mixed ψ and κ classes, as well as Eyard-Orantin's proof [13] of the BKMP conjecture of a topological recursion for open Gromov-Witten invariants of toric Calabi-Yau 3-folds.

Theorem 4.1 (Eynard [9])**.** *If $\mathcal{S}$ is the deformed Airy curve $y = \sum_k t_{k+2}\, x^{k/2}$, i.e. more precisely $\mathcal{S} = (\mathbb{C}, x(z) = z^2, y(z) = \sum_k t_{k+2}\, z^k, B(z,z') = dz \otimes dz'/(z-z')^2)$, one has for $2g-2+n>0$*

$$W_n^{(g)}(z_1,\dots,z_n) = (-2)^{2-2g-n} \times \sum_{d_1+\cdots+d_n \le 3g-3+n} \prod_{i=1}^n \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \left\langle \prod_{i=1}^n \psi_i^{d_i}\ e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n}, \tag{26}$$

where the dual times $\tilde{t}_k$ are defined by

$$e^{-\sum_k \tilde{t}_k\, u^k} = \sum_k (2k+1)!!\ t_{2k+3}\, u^k. \tag{27}$$

In particular for $g \ge 2$,

$$F_g = 2^{2-2g} \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,0}. \tag{28}$$

Without loss of generality, we may assume $t_3 = 1$, hence $\tilde{t}_0 = 0$. Given $\mathbf{L} \in N^\infty$, we denote $\tilde{t}^{\mathbf{L}} = \prod_{i\ge1} \tilde{t}_i^{L(i)}$.

Lemma 4.2. *Let $\alpha_{\mathbf{L}}$ be the constant in Theorem 3.3. Then*

$$\frac{1}{\sum_{k\ge0} t_{2k+3}} = \sum_{\mathbf{L}\in N^\infty} \frac{\alpha_{\mathbf{L}}}{\mathbf{L}!} \tilde{t}^{\mathbf{L}}. \tag{29}$$

Proof. By (27), we have

$$t_{2k+3} = \sum_{\substack{\mathbf{L}'\in N^\infty \\ |\mathbf{L}'|=k}} (-1)^{||\mathbf{L}'||} \frac{\tilde{t}^{\mathbf{L}'}}{(2|\mathbf{L}'|+1)!!\, \mathbf{L}'!}.$$

Then the lemma follows from the definition of $\alpha_{\mathbf{L}}$. □

Remark 4.3. If we take

$$\sum_{k\ge0} \tilde{t}_k u^k = \ln(1-u) = -\sum_{k=1}^\infty \frac{u^k}{k}, \quad |u|<1,$$

then we have $\tilde{t}_k = -1/k$, $t_{2k+3} = 1/(2k+1)!!$, $k \ge 1$. So (29) becomes

$$\sum_{\mathbf{L}\in N^\infty} \frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!} \prod_{j\ge1} \frac{1}{j^{L(j)}} = \frac{1}{\sum_{k=0}^\infty \frac{1}{(2k+1)!!}} = \frac{1}{\sqrt{2e}\int_0^{\frac{\sqrt{2}}{2}} e^{-t^2}dt} \approx 0.7088.$$

Similarly, if we specify $\sum_{k\geq 0}\tilde{t}_k u^k$ to be the functions $-\ln(1-u), -\ln(1+u)$ and $\ln(1+u)$ respectively, we get the following series

$$\sum_{\mathbf{L}\in N^\infty}\frac{\alpha_{\mathbf{L}}}{\mathbf{L}!}\prod_{j\geq 1}\frac{1}{j^{L(j)}}=\frac{3}{2},\qquad \sum_{\mathbf{L}\in N^\infty}\frac{(-1)^{|\mathbf{L}|}\alpha_{\mathbf{L}}}{\mathbf{L}!}\prod_{j\geq 1}\frac{1}{j^{L(j)}}=\frac{3}{4},$$

$$\sum_{\mathbf{L}\in N^\infty}\frac{(-1)^{|\mathbf{L}|+||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!}\prod_{j\geq 1}\frac{1}{j^{L(j)}}=\frac{1}{\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)!!}}=\frac{\sqrt{e}}{\sqrt{2}\int_0^{\frac{\sqrt{2}}{2}}e^{t^2}dt}\approx 1.3797.$$

The following result is known to experts (cf. [2, 7, 48, 49]). We give a proof for reader's convenience.

Proposition 4.4. *The Eynard-Orantin recursion formula* (25) *for the deformed Airy curve* $\{x(z)=z^2, y(z)=\sum_k t_{k+2}z^k\}$ *is equivalent to the recursion formula of mixed* ψ *and* κ *classes in Theorem 3.3.*

Proof. The unique branch point is $z=0$ and the recursion kernel equals

$$K(z_1,z)=\frac{\frac{1}{z_1-z'}\Big|_{z'=-z}^{z}}{8\sum_{k\geq 0}t_{2k+3}z^{2k+2}}\frac{dz_1}{dz}=\frac{1}{4(z_1-z)(z_1+z)\sum_{k\geq 0}t_{2k+3}z^{2k+1}}\frac{dz_1}{dz}.$$

For any fixed set $(d_1,\dots,d_n)$ of non-negative integers and $\mathbf{b}\in N^\infty$ with $|\mathbf{b}|+\sum_{j=1}^n d_j=3g-3+n$, the coefficient of

$$(-2)^{2-2g-n}\frac{1}{\mathbf{b}!}\prod_{i=1}^n\frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}$$

in $W_n^{(g)}(z_1,\dots,z_n)$ equals $\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g$ by (26). On the other hand side, the right-hand side of (25) is the summation of the following three terms.

$$\operatorname*{Res}_{z\to 0}K(z_1,z)W_{n+1}^{(g-1)}(z,-z,z_2\dots,z_n),\tag{30}$$

$$\operatorname*{Res}_{z\to 0}K(z_1,z)\sum_{j=2}^n\Big(W_2^{(0)}(z,z_j)W_n^{(g)}(-z,z_2,\dots,\widehat{z_j},\dots,z_n)+W_2^{(0)}(-z,z_j)W_n^{(g)}(z,z_2,\dots,\widehat{z_j},\dots,z_n)\Big),\tag{31}$$

$$\operatorname*{Res}_{z\to 0}K(z_1,z)\sum_{\substack{g_1+g_2=g\\ I\coprod J=\{2,\dots,n\}}}^{\text{stable}}W_{1+|I|}^{(g_1)}(z,z_I)W_{1+|J|}^{(g_2)}(-z,z_J).\tag{32}$$

To prove that the coefficients of (30) give the second term in the right-hand side of (21), we need only prove that for any given $r,s\geq 0$,

$$\operatorname*{Res}_{z\to 0}\frac{1}{(z_1^2-z^2)\sum_{k\geq 0}t_{2k+3}z^{2k+1}\cdot z^{2r+2s+4}}=\sum_{\mathbf{L}\in N^\infty}\frac{\alpha_{\mathbf{L}}}{\mathbf{L}!}\tilde{t}^{\mathbf{L}}\frac{1}{z_1^{2r+2s+6-2|\mathbf{L}|}}.$$

This identity follows from Lemma 4.2.

To prove that the coefficients of (31) give the first term in the right-hand side of (21), we need only prove that for any given $1\leq j\leq n$ and $r\geq 0$, the coefficient of $1/z_j^{2d_j+2}$ in

$$\operatorname*{Res}_{z\to 0}\left(\frac{1}{(z_j-z)^2}+\frac{1}{(z_j+z)^2}\right)\frac{1}{2(z_1^2-z^2)\sum_{k\geq 0}t_{2k+3}z^{2k+1}\cdot z^{2r+2}}$$

$$= \operatorname*{Res}_{z\to 0} \frac{z_j^2+z^2}{z_j^2\left(1-\frac{z}{z_j}\right)^2 (z_1^2-z^2)\sum_{k\geq 0} t_{2k+3} z^{2k+1}\cdot z^{2r+2}}$$

is equal to

$$\sum_{\mathbf{L}\in N^\infty} \frac{\alpha_{\mathbf{L}}}{\mathbf{L}!}\tilde{t}^{\mathbf{L}} \frac{1}{z_1^{2r+2-2|\mathbf{L}|-2d_j}},$$

which again follows from Lemma 4.2.

Finally it is easy to see that the coefficients of (32) give the third term in the right-hand side of (21). □

5. Large g asymptotics of integrals of ψ classes

By Faber-Kauffmann-Manin-Zagier's formula [19]

$$\langle \prod_{j=1}^n \tau_{d_j}\kappa_1^m\rangle_g = \sum_{p=1}^m \frac{(-1)^{m-p}}{p!} \sum_{\substack{m_1+\cdots+m_p=m\\ m_i>0}} \binom{m}{m_1,\ldots,m_p} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^p \tau_{m_j+1}\rangle_g,$$

the asymptotics of integrals of ψ classes should be helpful in understanding the asysmptotics of Weil-Petersson volumes. The following result was proved by an induction argument using (3) and (6) (cf. [26, §3]).

Proposition 5.1 ([26]). *For any fixed set $\mathbf{d}=(d_1,\ldots,d_n)$ of non-negative integers, we have the large g asymptotic expansion*

$$\frac{24^g g! \prod_{i=1}^n (2d_i+1)!!\langle \tau_{d_1}\cdots\tau_{d_n}\tau_{3g-2+n-|\mathbf{d}|}\rangle_g}{(6g)^{|\mathbf{d}|}} = 1 + \frac{C_1(d_1,\ldots,d_n)}{g} + \frac{C_2(d_1,\ldots,d_n)}{g^2} + \cdots, \tag{33}$$

where the left-hand side is a polynomial in $1/g$ with degree no more than $|\mathbf{d}|$ and each $C_r(d_1,\ldots,d_n)$ is a polynomial in $|\mathbf{d}|$ and n.

Consider the following recursion relation

$$\alpha_{k+1} = k^2\alpha_k + \sum_{m=2}^{k-1} \alpha_m\alpha_{k+1-m}, \quad k\geq 2, \tag{34}$$

one may check directly (cf. [18]) that if we put $\alpha_0=-\frac{1}{2}$, $\alpha_1=\frac{1}{50}$, $\alpha_2=\frac{49}{2500}$ and α_k, $k\geq 3$ are recursively given by (34), then the formal series

$$y = -\sqrt{\frac{2}{3}}\sum_{k=0}^\infty \left(\frac{25}{8\sqrt{6}}\right)^k \alpha_k x^{\frac{1-5k}{2}}$$

is a solution of the first Painlevé equation: $d^2y/dx^2 = 6y^2 - x$. The proof of the following asymptotic expansion of α_k is due to Joshi and Kitaev [18].

Theorem 5.2 ([18, 43]). *When $0<\alpha_2\leq \frac{1}{4}$, the solution of the recursion relation* (34) *has an asymptotic expansion*

$$\alpha_k = c(\alpha_2)(k-1)!^2\left(1+\delta_k\right), \tag{35}$$

where $c(\alpha_2) > 0$ is independent of k. In particular, we have

$$c(49/2500) = \frac{1}{4\pi^2}\sqrt{\frac{3}{5}}. \tag{36}$$

The correction term δ_k can be expanded as

$$\delta_k = \sum_{l=2}^{\infty} \frac{\eta_l (k-\gamma_l)}{\prod_{m=1}^{l}(k-m)^2}, \quad k \to \infty. \tag{37}$$

In particular, $\eta_2 = -\frac{2}{3}\alpha_2$, $\gamma_2 = 3$, $\eta_3 = -\frac{32}{15}\alpha_2$, $\gamma_3 = \frac{9}{2} + \frac{5}{48}\alpha_2$.

Proof. (sketch) Define $p_k = \alpha_k/((k-1)!)^2$, then the recursion (34) becomes

$$p_{k+1} = p_k + \sum_{m=2}^{k-1} p_m p_{k+1-m} \left(\frac{(k-m)!(m-1)!}{k!} \right)^2. \tag{38}$$

It is obvious that the sequence p_k is increasing. In fact, it is also upperbounded by (see [18] for a proof)

$$\frac{1}{2\ln 2 - 1} - \sqrt{\frac{1}{(2\ln 2-1)^2} - \frac{2p_2}{2\ln 2 - 1}}.$$

It follows that $c(\alpha_2) = \lim_{k\to\infty} p_k$ is finite.

The existence of the asymptotic expansion (37) follows from an estimate of the quadratic term in (38). See [18] for details. For a proof of (36), see [43]. □

Remark 5.3. By work of [16], the condition $\alpha_2 \le \frac{1}{4}$ in Theorem 5.2 can be weakened. Equation (37) implies that $\delta_k = O(1/k^3)$.

The following lemma gives a recursion formula for the coefficients of the asymptotic expansion of δ_k.

Lemma 5.4. *Let $\alpha_2 > 0$. Then the coefficients in the asymptotic expansion*

$$\alpha_k = c(\alpha_2)(k-1)!^2 \left(1 + \frac{\lambda_1}{k} + \frac{\lambda_2}{k^2} + \frac{\lambda_3}{k^3} + \cdots \right), \quad k \to \infty \tag{39}$$

satisfy the recursion

$$\begin{aligned} -n\lambda_n = &\sum_{i=3}^{n-1} (-1)^{n-i} \binom{n}{i-1} \lambda_i \\ &+ \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} 2\alpha_i \sum_{\substack{m_1+\cdots+m_{i-1}=n+1-2i \\ m_p \ge 0}} \prod_{j=1}^{i-1} (m_j+1) j^{m_j} \\ + \sum_{i=2}^{\lfloor \frac{n-2}{2} \rfloor} 2\alpha_i &\sum_{j=3}^{n+1-2i} \sum_{\substack{m_1+\cdots+m_{i-1} \\ =n+1-2i-j \\ m_p\ge 0}} \binom{j+1+m_{i-1}}{j+1} (i-1)^{m_{i-1}} \prod_{l=1}^{i-2} (m_l+1) l^{m_l} \lambda_j. \end{aligned} \tag{40}$$

In particular, $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\frac{2}{3}\alpha_2$, $\lambda_4 = -2\alpha_2$, $\lambda_5 = -\frac{82}{15}\alpha_2$.

Proof. For any given $m \geq 1$, substituting (39) into (34) and dividing by $c(\alpha_2)k!^2$, we get

$$1+\sum_{i=1}^{\infty}\frac{\lambda_i}{(k+1)^i}=1+\sum_{i=1}^{\infty}\frac{\lambda_i}{k^i}+\sum_{i=2}^{m}\frac{2\alpha_i\left(1+\sum_{j=1}^{\infty}\frac{\lambda_j}{(k-i+1)^j}\right)}{k^2(k-1)^2\cdots(k-i+1)^2}+O\left(\frac{1}{k^{2m+2}}\right).$$

The remainder of the quadratic term in (34) can be estimated by using (55).

By comparing the coefficient of $\frac{1}{k^2}$, we get $-\lambda_1+\lambda_2=\lambda_2$, i.e. $\lambda_1=0$.

By comparing the coefficient of $\frac{1}{k^3}$, we get $-2\lambda_2+\lambda_3=\lambda_3$, i.e. $\lambda_2=0$.

In general, by comparing the coefficient of $\frac{1}{k^{n+1}}$, $n\geq 3$, we get

$$\begin{aligned}\lambda_{n+1}+\sum_{i=3}^{n}\lambda_i\left[\frac{1}{(1+1/k)^i}\right]_{k^{-(n+1-i)}}&=\lambda_{n+1}+\sum_{i=2}^{\lfloor\frac{n+1}{2}\rfloor}2\alpha_i\left[\prod_{j=1}^{i-1}\frac{1}{(1-j/k)^2}\right]_{k^{-(n+1-2i)}}\\&+\sum_{i=2}^{\lfloor\frac{n-2}{2}\rfloor}2\alpha_i\sum_{j=3}^{n+1-2i}\lambda_j\left[\frac{1}{(1-\frac{i-1}{k})^{j+2}}\prod_{l=1}^{i-2}\frac{1}{(1-l/k)^2}\right]_{k^{-(n+1-2i-j)}},\end{aligned}$$

which can be further simplified by using the binomial identity

$$\binom{-a-1}{b}=\binom{a+b}{b}(-1)^b,\quad a,b\geq 0.$$

In particular, when $i\geq 1$, $b\geq 0$, we have

$$\binom{-n}{1}=-n,\qquad \binom{-i}{n+1-i}=(-1)^{n+1-i}\binom{n}{i-1},$$
$$(-1)^b\binom{-2}{b}=b+1,\qquad (-1)^b\binom{-(j+2)}{b}=\binom{j+1+b}{b}.$$

So (40) follows immediately. □

Corollary 5.5. *Let $n\geq 0$. Then λ_n is a polynomial in α_2 of degree $\lfloor n/3\rfloor$.*

Proof. It can be proved by an inductive argument using (40). Note that α_k is a polynomial in α_2 of order $\lfloor k/2\rfloor$. □

It was proved by Itzykson and Zuber [17] that up to a normalization coefficient, the intersection numbers $\langle\tau_2^{3g-3}\rangle_g$ is a solution of the recursion relation (34). We give a more direct proof using (6), which is essentially the same as [51, Prop. 4.2].

Lemma 5.6. ([17]) *For $g\geq 2$, define*

$$\alpha_g=\left(\frac{24}{25}\right)^g\frac{(5g-5)(5g-3)}{(3g-3)!2^{g+1}}\langle\tau_2^{3g-3}\rangle_g. \tag{41}$$

Then α_g is a solution of the recursion relation (34) *with $\alpha_2=49/2500$.*

Proof. When $g\geq 2$, we have

$$\begin{aligned}\langle\tau_0^k\tau_2^{3g-3+k}\rangle_g&=(3g-3+k)\langle\tau_1\tau_0^{k-1}\tau_2^{3g-4+k}\rangle_g\\&=(3g-3+k)(5g+2k-7)\langle\tau_0^{k-1}\tau_2^{3g-4+k}\rangle_g\\&=\prod_{i=1}^{k}(3g-3+i)\prod_{i=1}^{k}(5g-7+2i)\langle\tau_2^{3g-3}\rangle_g.\end{aligned}\tag{42}$$

When $g = 1$, we have $\langle\tau_0^k\tau_2^k\rangle_1 = 2^{k-1}k!(k-1)!/24$. Taking all $d_j = 2$ in (6) with $g \geq 3$ and using the above equations, we get

$$\begin{aligned}
&(3g-2)(5g-3)(5g-5)\langle\tau_2^{3g-3}\rangle_g \\
&= \frac{1}{12}(3g-2)(3g-3)(3g-4)(3g-5)(5g-4)(5g-6)(5g-8)(5g-10)\langle\tau_2^{3g-6}\rangle_{g-1} \\
&\quad + \frac{1}{6}\binom{3g-2}{2}(3g-4)(3g-5)(5g-8)(5g-10)\langle\tau_2^{3g-6}\rangle_{g-1} \\
&\quad + \frac{1}{2}\sum_{h=2}^{g-2}\binom{3g-2}{3h-1}(3h-1)(3h-2)(5h-3)(5h-5)\langle\tau_2^{3h-3}\rangle_h \\
&\quad \times (3g-3h-1)(3g-3h-2)(5g-5h-3)(5g-5h-5)\langle\tau_2^{3g-3h-3}\rangle_{g-h}.
\end{aligned}$$

Substituting $t_g = (5g-5)(5g-3)\langle\tau_2^{3g-3}\rangle_g/(3g-3)!$ to the above equation,

$$\begin{aligned}
t_{g+1} &= \frac{1}{12}(5g+1)(5g-1)t_g + \frac{1}{12}t_g + \frac{1}{2}\sum_{h=2}^{g-1} t_h t_{g+1-h} \\
&= \frac{25g^2}{12}t_g + \frac{1}{2}\sum_{h=2}^{g-1} t_h t_{g+1-h},
\end{aligned}$$

which implies that when setting $\alpha_g = (24/25)^g t_g/2^{g+1}$, we get

$$\alpha_{g+1} = g^2\alpha_g + \sum_{h=2}^{g-1}\alpha_h\alpha_{g+1-h}, \quad g \geq 2. \tag{43}$$

as claimed. □

Corollary 5.7. *The large genus asymptotic expansion of* $\langle\tau_2^{3g-3}\rangle_g$ *is given by*

$$\langle\tau_2^{3g-3}\rangle_g = \left(\frac{25}{24}\right)^g \frac{2^{g-1}\sqrt{3/5}(3g-3)!((g-1)!)^2}{\pi^2(5g-5)(5g-3)} \times \left(1 - \frac{49}{3750g^3} - \frac{49}{1250g^4} + \cdots\right). \tag{44}$$

Proof. It follows from Theorem 5.2 and Lemma 5.6. □

Next we study the asymptotic expansion of $\langle\tau_{d_1}\cdots\tau_{d_n}\tau_2^{3g-3+n-|\mathbf{d}|}\rangle_g$ as $g \to \infty$.

Proposition 5.8. *For any fixed set* $\mathbf{d} = (d_1, \ldots, d_n)$ *of non-negative integers, let* $t = |\mathbf{d}| - 2n$ *and* $p = 3g-3+n-|\mathbf{d}|$. *Define*

$$Z_g(d_1, \ldots, d_n) = (15g)^t \frac{\langle\tau_{d_1}\cdots\tau_{d_n}\tau_2^p\rangle_g^{\mathbf{w}}}{\langle\tau_2^{3g-3}\rangle_g^{\mathbf{w}}}. \tag{45}$$

Then $\lim_{g\to\infty} Z_g(d_1, \ldots, d_n) = 1$.

Proof. Equation (57) implies that

$$Z_g(0, d_2, \ldots, d_n) = \frac{1}{15g}\sum_{j=2}^{n}(2d_j+1)Z_g(d_2, \ldots, d_j-1, \ldots, d_n) + Z_g(d_2, \ldots, d_n)\left(1 + O\left(\frac{1}{g}\right)\right). \tag{46}$$

Equation (56) implies that

$$Z_g(1, d_2, \cdots, d_n) = Z_g(d_2, \cdots, d_n)\left(1 + O\left(\frac{1}{g}\right)\right). \tag{47}$$

From (46) and (47), we see that both the string and dilaton equations are compatible with $\lim_{g\to\infty} Z_g(d_1, \ldots, d_n) = 1$, so we may assume $d_j \geq 2$. We will proceed by induction on n and t. By the DVV formula,

$$\begin{aligned}
\langle \tau_{d_1} \cdots \tau_{d_n} \tau_2^p \rangle_g^{\mathbf{w}} &= \sum_{i=2}^{n} (2d_i + 1) \langle \tau_{d_i + d_1 - 1} \prod_{j \neq 1, i} \tau_{d_j} \tau_2^p \rangle_g^{\mathbf{w}} \\
&+ 5p \cdot \langle \tau_{d_1+1} \tau_{d_2} \cdots \tau_{d_n} \tau_2^{p-1} \rangle_g^{\mathbf{w}} + \frac{1}{2} \sum_{r+s=d_1-2} \langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_{g-1}^{\mathbf{w}} \\
&+ \frac{1}{2} \sum_{\substack{r+s=d_1-2 \\ \{2,\cdots,n\}=I\coprod J}} \sum_{g'=0}^{g} \binom{p}{p'} \langle \tau_r \prod_{i\in I} \tau_{d_i} \tau_2^{p'} \rangle_{g'}^{\mathbf{w}} \langle \tau_s \prod_{i \in J} \tau_{d_i} \tau_2^{p-p'} \rangle_{g-g'}^{\mathbf{w}},
\end{aligned} \tag{48}$$

where $p' = 3g' - 2 + |I| - \sum_{i\in I} d_i - r$.

Multiplying both sides of (48) by $(15g)^{|\mathbf{d}|-2n} / \langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}$, we will prove that the third and fourth terms in the right-hand side of (48) belong to $o(1)$ when g goes to infinity.

From (44), we have

$$\langle \tau_2^{3g-6} \rangle_{g-1} = O\left(\frac{\langle \tau_2^{3g-3} \rangle_g}{g^5}\right). \tag{49}$$

For the third term in the right-hand side of (48), we have

$$\begin{aligned}
(15g)^{|\mathbf{d}|-2n} &\frac{\langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_{g-1}^{\mathbf{w}}}{\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}} \\
&= O\left(\frac{(15g)^4}{g^5} \cdot (15g)^{|\mathbf{d}|-2n-4} \frac{\langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_{g-1}^{\mathbf{w}}}{\langle \tau_2^{3g-6} \rangle_{g-1}^{\mathbf{w}}}\right) \\
&= O\left(\frac{(15g)^4}{g^5} Z_{g-1}(r, s, d_2, \ldots, d_n)\right) = o(1).
\end{aligned} \tag{50}$$

The last equation is obtained by induction, since $r + s + \sum_{i=2}^{n} d_i - 2(n+1) < \sum_{i=1}^{n} d_i - 2n$.

Let us estimate the fourth term in the right-hand side of (48). Take $\mathbf{a} = (a_1, \ldots, a_m)$ with $m < n$ or $|\mathbf{a}| - 2m < t$, by induction we have

$$\frac{\langle \tau_{a_1} \cdots \tau_{a_m} \tau_2^{3h-3+m-|\mathbf{a}|} \rangle_h^{\mathbf{w}}}{(3h-3+m-|\mathbf{a}|)!} \sim C(\mathbf{a}) \left(\frac{25}{12}\right)^h h^{m-2} (h-1)!^2, \tag{51}$$

where $C(\mathbf{a})$ is a constant independent of h. Take $\mathbf{b} = (b_1, \ldots, b_{m'})$ with $m' < n$ or $|\mathbf{b}| - 2m' < t$, by induction we also have

$$\frac{\langle \tau_{b_1} \cdots \tau_{b_{m'}} \tau_2^{3h-3+m'-|\mathbf{b}|} \rangle_h^{\mathbf{w}}}{(3h-3+m'-|\mathbf{b}|)!} \sim C(\mathbf{b}) \left(\frac{25}{12}\right)^h h^{m'-2} (h-1)!^2 \tag{52}$$

Let $\mathbf{d} = (a_1 + b_1 + 2, a_2, \ldots, a_m, b_2, \ldots, b_{m'})$. Then

$$\frac{\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}}{(15g)^{|\mathbf{d}|-2n}} \sim C(\mathbf{d}) \left(\frac{25}{12}\right)^g g^{m+m'-3}(g-1)!^2(3g-3+n-|\mathbf{d}|)!. \tag{53}$$

Thus in order to prove that the fourth term in the right hand-side of (48), after multiplied by $(15g)^{|\mathbf{d}|-2n}/\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}$, belongs to $o(1)$ when g goes to infinity, we need only prove that when $m, m' \geq 1$,

$$\sum_{h=1}^{g-1} h^{m-2}(h-1)!^2(g-h)^{m'-2}(g-h-1)!^2 = o\left(g^{m+m'-3}(g-1)!^2\right), \tag{54}$$

which in turn follows from

$$\sum_{h=1}^{g-1} \frac{(h-1)!^2(g-h-1)!^2}{(g-1)!^2} = \sum_{h=1}^{g-1} \frac{1}{(g-1)^2\binom{g-2}{h-1}^2} \leq \frac{1}{g-1}. \tag{55}$$

So we proved that only the first two terms in the right-hand side of (48) contribute to the large genus limit of $Z_g(d_1, \ldots, d_n)$.

$$\begin{aligned} Z_g(d_1, \ldots, d_n) = &\frac{1}{15g} \sum_{j=2}^{n} (2d_j+1) Z_g(d_2, \ldots, d_j + d_1 - 1, \ldots, d_n) \\ &+ \frac{5(3g-3+n-|\mathbf{d}|)}{15g} Z_g(d_1+1, d_2, \ldots, d_n) + o(1). \end{aligned}$$

Replacing $d_1 + 1$ by d_1 and leting $g \to \infty$, we obtain $\lim_{g\to\infty} Z_g(d_1, \ldots, d_n) = 1$ by induction. □

Lemma 5.9. *The dilaton and string equations for $Z_g(d_1, \ldots, d_n)$ are*

$$Z_g(1, d_2, \ldots, d_n) = \frac{5g-7+2n-|\mathbf{d}|}{5g} Z_g(d_2, \ldots, d_n), \tag{56}$$

$$\begin{aligned} Z_g(0, d_2, \ldots, d_n) = &\frac{1}{15g} \sum_{j=2}^{n} (2d_j+1) Z_g(d_2, \ldots, d_j - 1, \ldots, d_n) \\ &+ \frac{(3g-3+n-|\mathbf{d}|)(5g-7+2n-|\mathbf{d}|)}{15g^2} Z_g(d_2, \ldots, d_n), \end{aligned} \tag{57}$$

where $|\mathbf{d}| = d_2 + \cdots + d_n$.

Proof. By (5), we have

$$\begin{aligned} Z_g(1, d_2, \ldots, d_n) &= (15g)^{|\mathbf{d}|+1-2n} \frac{\langle \tau_1 \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_g^{\mathbf{w}}}{\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}} \\ &= \frac{3(2g-3+n+p)}{15g} (15g)^{|\mathbf{d}|+2-2n} \frac{\langle \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_g^{\mathbf{w}}}{\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}}, \end{aligned}$$

where $p = 3g - 4 + n - |\mathbf{d}|$, from which (56) follows.

By (4), we have

$$Z_g(0, d_2, \ldots, d_n) = (15g)^{|\mathbf{d}|-2n} \frac{\langle \tau_0 \tau_{d_2} \cdots \tau_{d_n} \tau_2^p \rangle_g^{\mathbf{w}}}{\langle \tau_2^{3g-3} \rangle_g^{\mathbf{w}}}$$

$$= \frac{1}{15g}\sum_{j=2}^{n}(2d_j+1)Z_g(d_2,\dots,d_j-1,\dots,d_n) + \frac{3g-3+n-|\mathbf{d}|}{3g}Z_g(1,d_2,\dots,d_n),$$

which implies (57) through (56). □

Corollary 5.10. *We have* $Z_g(\emptyset)=1$ *and*

$$Z_g(0) = 1-\frac{5}{3g}+\frac{2}{3g^2}, \qquad Z_g(1) = 1-\frac{1}{g},$$
$$Z_g(2,d_1,\dots,d_n) = Z_g(d_1,\dots,d_n).$$

Proof. It is obvious. □

Corollary 5.11. *For any fixed set* $\mathbf{d}=(d_1,\dots,d_n)$ *of non-negative integers, we have*

$$(58)\quad \langle \tau_{d_1}\cdots\tau_{d_n}\tau_2^{3g-3+n-|\mathbf{d}|}\rangle_g \sim \frac{15^n g^{2n-|\mathbf{d}|}}{\prod_{i=1}^n(2d_i+1)!!}\left(\frac{25}{24}\right)^g \frac{2^{g-1}\sqrt{3/5}(3g-3)!((g-1)!)^2}{\pi^2(5g-5)(5g-3)}.$$

Proof. It follows from Proposition 5.8 and Corollary 5.7. □

Theorem 5.12. *For any fixed set* $\mathbf{d}=(d_1,\dots,d_n)$ *of non-negative integers, the coefficients in the asymptotic expansion*

$$(59)\qquad Z_g(d_1,\dots,d_n) = 1+\frac{\beta_1(d_1,\dots,d_n)}{g}+\frac{\beta_2(d_1,\dots,d_n)}{g^2}+\cdots, \quad g\to\infty$$

satisfy the recursion

$$(60)\quad \beta_r(d_1+1,\dots,d_n)$$
$$= \beta_r(d_1,\dots,d_n) - \frac{1}{15}\sum_{j=2}^{n}(2d_j+1)\beta_{r-1}(d_2,\dots,d_j+d_1-1,\dots,d_n)$$
$$- \frac{n-|\mathbf{d}|-3}{3}\beta_{r-1}(d_1+1,d_2\dots,d_n)$$
$$-\frac{2}{15}\sum_{j=0}^{d_1-2}\left[\frac{(1-\frac{1}{g})^{2n+2-|\mathbf{d}|}(1-\frac{3}{5g})\sum_{i=0}^\infty \frac{\beta_i(j,d_1-2-j,d_2,\dots,d_n)}{g^i(1-\frac{1}{g})^i}\sum_{i=0}^\infty\frac{\lambda_i}{g^i(1-\frac{1}{g})^i}}{(1-\frac{4}{3g})(1-\frac{5}{3g})(1-\frac{2}{g})(1-\frac{8}{5g})\sum_{i=0}^\infty\frac{\lambda_i}{g^i}}\right]_{g^{-(r-1)}}$$
$$-\sum_{\substack{j=0\\ \{2,\cdots,n\}=I\coprod J}}^{d_1-2}\sum_h 3^{-2h-|I|}4^h 5^{\sum_{i\in I}d_i+j+2-2|I|-5h}\frac{\langle\tau_j\prod_{i\in I}\tau_{d_i}\tau_2^{p'}\rangle_h^{\mathbf{w}}}{p'!}$$
$$\cdot\left[\frac{(1-\frac{h}{g})^{2|J|+4-d_1+j-\sum_{i\in J}d_i}(1-\frac{1}{g})(1-\frac{3}{5g})\sum_{i=0}^\infty\frac{\beta_i(d_1-2-j,d_J)}{g^i(1-\frac{h}{g})^i}\sum_{i=0}^\infty\frac{\lambda_i}{g^i(1-\frac{h}{g})^i}}{(1-\frac{h+1}{g})(1-\frac{5h+3}{5g})\prod_{i=3}^{3h+2}(1-\frac{i}{3g})\prod_{i=1}^h(1-\frac{i}{g})^2\sum_{i=0}^\infty\frac{\lambda_i}{g^i}}\right.$$
$$\left.\times\prod_{l=-3h+1+|J|-\sum_{i\in J}d_i-d_1+j}^{-3+n-|\mathbf{d}|}(1+\frac{l}{3g})\right]_{g^{-(r-2h-|I|)}},$$

where $\lambda_i=\lambda_i(\frac{49}{2500})$, $p'=3h-2+|I|-\sum_{i\in I}d_i-j$ *and the summation range of* h *is* $\max(0,\lceil\frac{j+\sum_{i\in I}d_i-|I|+2}{3}\rceil)\le h\le\lfloor\frac{r-|I|}{2}\rfloor$. *And* $\beta_0(d_1,\dots,d_n)=1$, $\beta_r(\emptyset)=0$ *when* $r>0$.

Proof. The proof is a tedious but straightforward computation using (48). We omit the details. □

Corollary 5.13. *The dilaton and string equations for* $\beta(d_1,\dots,d_n)$ *are*

$$\beta_r(1,d_2,\dots,d_n)=\beta_r(d_2,\dots,d_n)+\frac{2n-|\mathbf{d}|-7}{5}\beta_{r-1}(d_2,\dots,d_n), \tag{61}$$

$$\begin{aligned}\beta_r(0,d_2,\dots,d_n)&=\frac{1}{15}\sum_{j=2}^{n}(2d_j+1)\beta_{r-1}(d_2,\dots,d_j-1,\dots,d_n)\\&+\beta_r(d_2,\dots,d_n)+\frac{11n-8|\mathbf{d}|-36}{15}\beta_{r-1}(d_2,\dots,d_n)\\&+\frac{(n-|\mathbf{d}|-3)(2n-|\mathbf{d}|-7)}{15}\beta_{r-2}(d_2,\dots,d_n),\end{aligned} \tag{62}$$

where $|\mathbf{d}|=d_2+\cdots+d_n$.

Proof. It follows from Lemma 5.9. □

Lemma 5.14. *(i) Let* $s=\#\{i\mid d_i=0\}$. *Then*

$$\beta_1(d_1,\dots,d_n)=\frac{|\mathbf{d}|^2+11|\mathbf{d}|-4n|\mathbf{d}|}{10}+\frac{2n^2-11n}{5}+\frac{5s-s^2}{30}. \tag{63}$$

(ii) Let $d_i\ge 3$, $\forall 1\le i\le n$. *Then*

$$\begin{aligned}\beta_2(d_1,\dots,d_n)&=\frac{1}{200}|\mathbf{d}|^4+\left(-\frac{1}{25}n+\frac{7}{60}\right)|\mathbf{d}|^3+\left(\frac{3}{25}n^2-\frac{7}{10}n+\frac{143}{200}\right)|\mathbf{d}|^2\\&+\left(-\frac{4}{25}n^3+\frac{7}{5}n^2-\frac{143}{50}n+\frac{169}{300}\right)|\mathbf{d}|+\frac{2}{25}n^4-\frac{14}{15}n^3+\frac{143}{50}n^2-\frac{251}{225}n.\end{aligned}$$

Proof. For (i), first note that by (62),

$$\beta_1(0^n)=\beta_1(0^{n-1})+\frac{11n-36}{15}=\frac{11n^2-61n}{30}.$$

Let $q=\#\{i\ge 2\mid d_i=0\}$. By (60), we have

$$\begin{aligned}\beta_1(d_1+1,\dots,d_n)&=\beta_1(d_1,\dots,d_n)-\frac{1}{15}\sum_{j=2}^{n}(2d_j+1)+\frac{q}{15}\delta_{d_1,0}\\&-\frac{n-|\mathbf{d}|-3}{3}-\frac{2}{15}(d_1-1)-\frac{2}{15}\delta_{d_1,0}\\&=\beta_1(d_1,\dots,d_n)+\frac{|\mathbf{d}|}{5}-\frac{2n-6}{5}+\frac{q-2}{15}\delta_{d_1,0}.\end{aligned}$$

By iteration, we have

$$\begin{aligned}\beta_1(d_1,\dots,d_n)&=\beta_1(0^n)+\frac{1}{5}\sum_{i=1}^{|\mathbf{d}|-1}i-\frac{(2n-6)|\mathbf{d}|}{5}+\sum_{i=s}^{n-1}\frac{i-2}{15}\\&=\frac{11n^2-61n}{30}+\frac{|\mathbf{d}|^2-|\mathbf{d}|}{10}-\frac{(2n-6)|\mathbf{d}|}{5}+\frac{(n+s-5)(n-s)}{30}\\&=\frac{|\mathbf{d}|^2+11|\mathbf{d}|-4n|\mathbf{d}|}{10}+\frac{2n^2-11n}{5}+\frac{5s-s^2}{30},\end{aligned}$$

The proof of (ii) is similar. We omit the details. □

Remark 5.15. Let $d_i \geq 0$ and $r \geq 1$. One could prove from (60) inductively that each $\beta_r(d_1, \dots, d_n)$ is a polynomial in $|\mathbf{d}|$ and n as long as $\min(d_1, \dots, d_n)$ is sufficiently large.

From (60), we computed the first few terms of $Z_g(3)$,

$$(64)\quad Z_g(3) = \frac{7g\langle \tau_3 \tau_2^{3g-5}\rangle_g}{\langle \tau_2^{3g-3}\rangle_g} = 1 + \frac{\beta_1(3)}{g} + \frac{\beta_2(3)}{g^2} + \cdots$$
$$= 1 + \frac{6}{5g} + \frac{127}{90g^2} + \frac{2207}{1350g^3} + \frac{94726}{50625g^4} + \frac{3219853}{1518750g^5} + \cdots .$$

It would be interesting to see whether $Z_g(3)$ is a rational function of g.

For $g \geq 2$, define

$$c_g = \frac{(5g-4)(5g-6)}{(5g-5)!}\langle \tau_3 \tau_2^{3g-5}\rangle_g.$$

In particular, $c_2 = 29/240$.

Let $a_g = \langle \tau_2^{3g-3}\rangle_g/(3g-3)!$. Similar to the proof of Lemma 5.6, we have the following recursion formula which can be used to compute c_g fastly,

$$(65)\quad c_g = \frac{1}{12}(25g^2 - 60g + 36)c_{g-1} - (15g^2 - 27g + 12)a_g$$
$$+ \left(125g^4 - 750g^3 + \frac{13255}{8}g^2 - \frac{19177}{12}g + \frac{1706}{3}\right) a_{g-1}$$
$$+ \sum_{h=2}^{g-2}(5g-5h-3)(5g-5h-5)\Big((30h^2 - 52h + 22)a_h + c_h\Big)a_{g-h},$$

for $g \geq 3$. Denote by $Q_{k,g}$ the error term of order k approximation to $Z_g(3)$.

$$Q_{k,g} = g^k \left(Z_g(3) - \sum_{r=0}^{k} \frac{\beta_r(3)}{g^r} \right),$$

which should goes to 0 as $g \to \infty$ (see Table 1).

Table 1. Values of $Q_{k,g}$ (keep 6 decimal places)

k	$g = 600$	$g = 700$	$g = 800$	$g = 900$	$g = 1000$
0	0.002003	0.001717	0.001502	0.001335	0.001201
1	0.002356	0.002019	0.001766	0.001569	0.001412
2	0.002729	0.002339	0.002046	0.001818	0.001636
3	0.003124	0.002677	0.002342	0.002081	0.001873
4	0.003540	0.003033	0.002653	0.002358	0.002122

6. Large n asymptotics of integrals of ψ classes

In this sectioin, we study the asymptotic expansion of integrals of ψ classes when the number of marked points goes to infinity while the genus g is fixed.

Theorem 6.1. *For any fixed $g \geq 0$ and a set $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers, we have*

$$\lim_{k\to\infty} \frac{\langle \tau_{d_1}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n+1}}{k^{|\mathbf{d}|}} = \frac{1}{24^g g! \prod_{j=1}^n d_j!}. \tag{66}$$

Proof. We use induction on $|\mathbf{d}|$. When $|\mathbf{d}| = 0$, (66) holds by the string equation. We may also assume all $d_j \geq 1$. Then by the DVV formula (3), we have

$$\begin{aligned}
(2d_1+1)!!\langle \tau_{d_1}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n+1} \\
= \sum_{j=2}^n \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!}\langle \tau_{d_j+d_1-1}\prod_{\substack{i=2\\ i\neq j}}^n \tau_{d_i}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n} \\
+ k(2d_1-1)!!\langle\tau_{d_2}\cdots\tau_{d_n}\tau_{d_1-1}\tau_0^{k-1}\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n} \\
+ \frac{(2d_1+6g-5+2k+2n-2|\mathbf{d}|)!!}{(6g-5+2k+2n-2|\mathbf{d}|)!!}\langle\tau_{d_2}\cdots\tau_{d_n}\tau_0^k\tau_{3g-3+k+n-|\mathbf{d}|+d_1}\rangle_{g,k+n} \\
+ \frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_2}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g-1,k+n+2} \\
+ \sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\sum_{j=0}^k\binom{k}{j} \\
\times \sum_{\{2,\cdots,n\}=I\coprod J}\langle\tau_s\tau_{3g-2+k+n-|\mathbf{d}|}\tau_0^{k-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\langle\tau_r\tau_0^j\prod_{i\in I}\tau_{d_i}\rangle_{g'}.
\end{aligned} \tag{67}$$

By induction on $|\mathbf{d}|$, the first and fourth terms in the right-hand side of (67) are of orders $O(k^{|\mathbf{d}|-1})$ and $O(k^{|\mathbf{d}|-2})$ respectively, so they can be omitted. Let us analyze the remaining three terms. For the second term,

$$k(2d_1-1)!!\langle\prod_{j=2}^n\tau_{d_j}\tau_{d_1-1}\tau_0^{k-1}\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n} \sim \frac{d_1(2d_1-1)!!k^{|\mathbf{d}|}}{24^g g!\prod_{j=1}^n d_j!}.$$

For the third term,

$$\frac{(2d_1+6g-5+2k+2n-2|\mathbf{d}|)!!}{(6g-5+2k+2n-2|\mathbf{d}|)!!}\langle\prod_{j=2}^n\tau_{d_j}\tau_0^k\tau_{3g-3+k+n-|\mathbf{d}|+d_1}\rangle_{g,k+n} \sim \frac{2^{d_1}d_1!k^{|\mathbf{d}|}}{24^g g!\prod_{j=1}^n d_j!}.$$

For the last term,

$$\begin{aligned}
\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\sum_{j=0}^k\binom{k}{j} \\
\times\sum_{\{2,\cdots,n\}=I\coprod J}\langle\tau_s\tau_{3g-2+k+n-|\mathbf{d}|}\tau_0^{k-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\langle\tau_r\tau_0^j\prod_{i\in I}\tau_{d_i}\rangle_{g'} \\
\sim \sum_{r=0}^{d_1-2}(2r+1)!!(2d_1-3-2r)!!\binom{k}{r+2}\langle\tau_{d_1-2-j}\tau_{3g-2+k+n-|\mathbf{d}|}\tau_0^{k-r-2}\prod_{i=2}^n\tau_{d_i}\rangle_g\langle\tau_r\tau_0^{r+2}\rangle_0 \\
= \sum_{r=0}^{d_1-2}\frac{(2d_1-3-2r)!!(2r+1)!!d_1!}{(r+2)!(d_1-2-r)!}\cdot\frac{k^{|\mathbf{d}|}}{24^g g!\prod_{j=1}^n d_j!}.
\end{aligned}$$

So (66) would follow if we can prove that

$$d_1(2d_1-1)!!+2^{d_1}d_1!+\sum_{r=0}^{d_1-2}\frac{(2d_1-3-2r)!!(2r+1)!!d_1!}{(r+2)!(d_1-2-r)!}=(2d_1+1)!!.$$

Since $(2n-1)!!=2^n\Gamma(n+\frac{1}{2})/\sqrt{\pi}$ and $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(-\frac{1}{2})=-2\sqrt{\pi}$, the above equation is equivalent to

$$\sum_{r=0}^{n}\binom{n}{r}\Gamma\left(n-r+\frac{3}{2}\right)\Gamma\left(r-\frac{1}{2}\right)=-\pi\Gamma(n+1),\quad n\geq 0. \tag{68}$$

To prove (68), we use $\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}$ and check directly that both sides satisfy the recursion $f(n)=nf(n-1)$. □

For any given $g\geq 0$ and a set $\mathbf{d}=(d_1,\ldots,d_n)$ of non-negative integers, define

$$Y_{k,g}(d_1,\ldots,d_n)=\frac{24^g g!\prod_{j=1}^n d_j!\langle\tau_{d_1}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n+1}}{k^{|\mathbf{d}|}}. \tag{69}$$

Theorem 6.2. *$Y_{k,g}(d_1,\ldots,d_n)$ satisfies the following recursion formula*

$$\begin{aligned}(70)\quad &(2d_1+1)!!Y_{k,g}(d_1,\ldots,d_n)\\ &=\frac{1}{k}\sum_{j=2}^{n}\frac{(2d_1+2d_j-1)!!d_1!d_j!}{(2d_j-1)!!(d_j+d_1-1)!}Y_{k,g}(d_1,\ldots,d_j+d_1-1\ldots,d_n)\\ &+d_1\cdot(2d_1-1)!!\left(1-\frac{1}{k}\right)^{|\mathbf{d}|-1}Y_{k-1,g}(d_1-1,d_2\ldots,d_n)\\ &+2^{d_1}d_1!\prod_{i=1}^{d_1}\left(1+\frac{2d_1+6g+2n-2|\mathbf{d}|-2i-3}{2k}\right)Y_{k,g}(d_2,\ldots,d_n)\\ &+\frac{1}{k^2}\sum_{i=0}^{d_1-2}\frac{(2i+1)!!(2d_1-2i-3)!!\,12g\cdot d_1!}{i!(d_1-2-i)!}Y_{k,g-1}(i,d_1-2-i,d_2,\ldots,d_n)\\ &+\sum_{\substack{j=0\\ \{2,\cdots,n\}=I\coprod J}}^{d_1-2}(2j+1)!!(2d_1-2j-3)!!\sum_h\langle\tau_j\tau_0^p\prod_{i\in I}\tau_{d_i}\rangle_h\frac{24^h d_1!\prod_{i=0}^{h-1}(g-i)\prod_{i\in I}d_i!}{p!(d_1-2-j)!}\\ &\times\frac{1}{k^{3h+|I|}}\left(1-\frac{p}{k}\right)^{d_1-2-j+\sum_{i\in J}d_i}\prod_{i=1}^{p-1}\left(1-\frac{i}{k}\right)Y_{k-p,g-h}(d_1-2-j,d_J),\end{aligned}$$

where $p=j+\sum_{i\in I}d_i-3h+2-|I|$ and the summation range of h is $0\leq h\leq\min(g,\lfloor\frac{j+\sum_{i\in I}d_i+2-|I|}{3}\rfloor)$. Moreover, $Y_{k,g}(d_1,\ldots,d_n)$ is a polynomial in $1/k$.

Proof. The recursion follows by multiplying $\frac{24^g g!\prod_{j=1}^n d_j!}{k^{|\mathbf{d}|}}$ to Equation (67). The last assertion follows from Lemma 6.5. □

Corollary 6.3. *For any given $g\geq 0$ and a set $\mathbf{d}=(d_1,\ldots,d_n)$ of non-negative integers, the coefficients in the asymptotic expansion*

$$Y_{k,g}(d_1,\ldots,d_n)=1+\frac{\eta_{1,g}(d_1,\ldots,d_n)}{k}+\frac{\eta_{2,g}(d_1,\ldots,d_n)}{k^2}+\cdots,\quad k\to\infty \tag{71}$$

satisfy the recursion

$$
\begin{aligned}
(72)\quad & (2d_1+1)!!\eta_{r,g}(d_1,\dots,d_n) \\
&= \sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!d_1!d_j!}{(2d_j-1)!!(d_j+d_1-1)!}\eta_{r-1,g}(d_1,\dots,d_j+d_1-1\dots,d_n) \\
&+ d_1\cdot(2d_1-1)!!\sum_{j=0}^{r}(-1)^{r-j}\binom{|\mathbf{d}|-j-1}{r-j}\eta_{j,g}(d_1-1,d_2\dots,d_n) \\
&+ 2^{d_1}d_1!\sum_{j=0}^{\min(d_1,r)}\left[\prod_{i=1}^{d_1}\left(1+\frac{2d_1+6g+2n-2|\mathbf{d}|-2i-3}{2k}\right)\right]_{k^{-j}}\eta_{r-j,g}(d_2,\dots,d_n) \\
&+ \sum_{i=0}^{d_1-2}\frac{(2i+1)!!(2d_1-2i-3)!!\,12g\cdot d_1!}{i!(d_1-2-i)!}\eta_{r-2,g-1}(i,d_1-2-i,d_2,\dots,d_n) \\
&+ \sum_{\substack{j=0\\ \{2,\cdots,n\}=I\coprod J}}^{d_1-2}(2j+1)!!(2d_1-2j-3)!!\sum_h\langle\tau_j\tau_0^p\prod_{i\in I}\tau_{d_i}\rangle_h\frac{24^h d_1!\prod_{i=0}^{h-1}(g-i)\prod_{i\in I}d_i!}{p!(d_1-2-j)!} \\
&\times\left[\left(1-\frac{p}{k}\right)^{d_1-2-j+\sum_{i\in J}d_i}\prod_{i=1}^{p-1}\left(1-\frac{i}{k}\right)\sum_{i=0}^{\infty}\frac{\eta_{i,g-h}(d_1-2-j,d_J)}{k^i(1-\frac{p}{k})^i}\right]_{k^{-(r-3h-|I|)}},
\end{aligned}
$$

where $p=j+\sum_{i\in I}d_i-3h+2-|I|$ *and the summation range of* h *is* $0\le h\le\min(g,\lfloor\frac{j+\sum_{i\in I}d_i+2-|I|}{3}\rfloor,\lfloor\frac{r-|I|}{3}\rfloor)$.

Proof. It follows from Equation (70). □

Corollary 6.4. *Let* $|\mathbf{d}|=d_2+\cdots+d_n$*. Then*

$$
(73)\qquad Y_{k,g}(1,d_2,\dots,d_n)=\left(1+\frac{2g-2+n}{k}\right)Y_{k,g}(d_2,\dots,d_n),
$$

$$
(74)\qquad Y_{k,g}(0,d_2,\dots,d_n)=\left(1+\frac{1}{k}\right)^{|\mathbf{d}|}Y_{k+1,g}(d_2,\dots,d_j-1,\dots,d_n),
$$

or equivalently in terms of coefficients of the asymptotic expansion,

$$
\begin{aligned}
\eta_{r,g}(1,d_2,\dots,d_n)&=\eta_{r,g}(d_2,\dots,d_n)+(2g-2+n)\eta_{r-1,g}(d_2,\dots,d_n),\\
\eta_{r,g}(0,d_2,\dots,d_n)&=\sum_{j=0}^{r}\binom{|\mathbf{d}|-j}{r-j}\eta_{j,g}(d_2,\dots,d_n).
\end{aligned}
$$

Proof. Equation (73) follows from the dilaton equation and (74) follows from the definition. □

Lemma 6.5. *Given* $d_i\ge 0$*, then*

$$
\begin{aligned}
y_{d_1,\dots,d_n}(k,g)&:=\frac{k^{|\mathbf{d}|}\prod_{j=1}^{n}(2d_j+1)!!}{\prod_{j=1}^{n}d_j!}Y_{k,g}(d_1,\dots,d_n)\\
&=24^g g!\prod_{j=1}^{n}(2d_j+1)!!\langle\tau_{d_1}\cdots\tau_{d_n}\tau_0^k\tau_{3g-2+k+n-|\mathbf{d}|}\rangle_{g,k+n+1}
\end{aligned}
$$

is an integer-valued polynomial in k *and* g *with degree* $|\mathbf{d}|$*, whose highest degree terms in* k *and* g *are respectively* $\frac{\prod_{j=1}^{n}(2d_j+1)!!}{\prod_{j=1}^{n}d_j!}k^{|\mathbf{d}|}$ *and* $(6g)^{|\mathbf{d}|}$.

Proof. We have $y_\emptyset(k,g)=1$ and by (70),

$$
\begin{aligned}
(75)\quad y_{d_1,\dots,d_n}(k,g) &= \sum_{j=2}^{n}(2d_j+1)!!\,y_{d_1,\dots,d_j+d_1-1\dots,d_n}(k,g)\\
&+k\,y_{d_1-1,d_2\dots,d_n}(k-1,g)+\prod_{i=1}^{d_1}(2k+2d_1+6g+2n-2|\mathbf{d}|-2i-3)y_{d_2,\dots,d_n}(k,g)\\
&+\sum_{i=0}^{d_1-2}12g\,y_{i,d_1-2-i,d_2,\dots,d_n}(k,g-1)+\sum_{\substack{j=0\\ \{2,\cdots,n\}=I\coprod J}}^{d_1-2}\ \sum_{h=0}^{\lfloor\frac{j+\sum_{i\in I}d_i+2-|I|}{3}\rfloor}\langle\tau_j\tau_0^p\prod_{i\in I}\tau_{d_i}\rangle_h^{\mathbf{w}}\\
&\times\frac{24^h\prod_{i=0}^{h-1}(g-i)\prod_{i=0}^{p-1}(k-i)}{p!}\,y_{d_1-2-j,d_J}(k-p,g-h),
\end{aligned}
$$

where $p=j+\sum_{i\in I}d_i-3h+2-|I|$. From [24, Thm. 4.3 (iv) and Prop. 4.4], we know

$$24^h h!\cdot\langle\tau_j\tau_0^p\prod_{i\in I}\tau_{d_i}\rangle_h^{\mathbf{w}}\in\mathbb{Z}.$$

We can see inductively from (75) that $y_{d_1,\dots,d_n}(k,g)$ is an integer-valued polynomial in k and g.

For the degree of $y_{d_1,\dots,d_n}(k,g)$, we need only check that in the last term

$$
\begin{aligned}
|\mathbf{d}|-\left(p+h+\sum_{i\in I}d_i+d_1-2-j\right)&=|\mathbf{d}|-(|\mathbf{d}|-2h)\\
&=2h\geq 0
\end{aligned}
$$

The coefficient of $k^{|\mathbf{d}|}$ is obvious. The coefficient of $g^{|\mathbf{d}|}$ follows by induction. □

The above lemma generalized [26, Thm 4.1] (corresponding to the case $k=0$).

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