Quantum Correction and the Moduli Spaces of Calabi-Yau Manifolds

Kefeng Liu, Changyong Yin

Abstract

We define the quantum correction of the Teichmüller space $\mathcal{T}$ of Calabi-Yau manifolds. Under the assumption of no weak quantum correction, we prove that the Teichmüller space $\mathcal{T}$ is a locally symmetric space with the Weil-Petersson metric. For Calabi-Yau threefolds, we show that no strong quantum correction is equivalent to that, with the Hodge metric, the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$. Finally, for Hyperkähler manifold of dimension $2n \geq 4$, we find both locally and globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space of polarized Hyperkähler manifolds.

Contents

1   Introduction

In this paper we study the Teichmüller space of Calabi-Yau and Hyperkähler manifolds. Recall that a compact projective manifold X of complex dimension n with $n \geq 3$ is called a Calabi-Yau manifold, if it has a trivial canonical bundle and satisfies $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. A compact and simply connected projective manifold of complex dimension $2n \geq 4$ is called Hyperkähler, if it admits a nondegenerate holomorphic $(2,0)$-form.

In the first part, we study Calabi-Yau manifolds. A polarized and marked Calabi-Yau manifold is a triple (X, L, γ) of a Calabi-Yau manifold X, an ample line bundle L over X and a basis γ of the integral middle homology group modulo torsion, $H_n(X, \mathbb{Z})/\mathrm{Tor}$. We will denote the moduli space of a polarized and marked Calabi-Yau manifold X by $\mathcal{T}$. Actually, the Teichmüller space $\mathcal{T}$ is precisely the universal cover of the smooth moduli space $\mathcal{Z}_m$ of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$, which is constructed by Popp, Viehweg and Szendröi, for example in [S99, Section 2]. Here, A basis of the quotient space $(H_n(X, \mathbb{Z})/\mathrm{Tor})/m(H_n(X, \mathbb{Z})/\mathrm{Tor})$ is called a level m structure with $m \geq 3$ on the polarized Calabi–Yau manifold (X, L). The versal family $\mathcal{X} \to \mathcal{T}$ of polarized and marked Calabi-Yau manifolds is the pull-back of the versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$, see [S99]. Therefore, $\mathcal{T}$ is a connected and simply connected smooth complex manifold. There has been very active studies about when a horizontal subvariety of a Griffiths period domain is a globally Herimitian symmmetric space and to classify the possible variations of Hodge structure parameterized by a globally Hermitian symmetric space. The reader can refer [F-L13] and its reference. In this paper, we will characterize a class of Calabi-Yau manifolds whose Teichmüller spaces are locally Hermitian symmetric spaces or globally Herimitian symmetric spaces, by studying the quantum correction as motivated by mirror symmetry from String Theory.

Fix $p \in \mathcal{T}$, let X be the corresponding Calabi-Yau manifold in the versal family and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be an orthonormal basis of harmonic Beltrami differentials with respect to the Calabi-Yau metric. Then we can construct a smooth family of Betrami differentials

$$\Phi(t) = \sum_{|I| \geq 1} t^I \varphi_I,$$

which describe the deformation of complex structures around $p \in \mathcal{T}$. Our essential idea is to consider the strong quantum correction at point $p \in \mathcal{T}$, which can be simply described as the following identity of cohomology classes,

$$[\Xi(t)] = [\Omega^c(t)] - [\exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega],$$

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where Ω is a holomorphic $(n,0)$-form over X and $\Omega^c(t)$ is the canonical family of holomorphic $(n,0)$-forms in a neighborhood of $p \in \mathcal{T}$. And the weak quantum correction at point $p \in \mathcal{T}$ is defined as the lowest order expansion of $[\Xi(t)]$ with respect to T, i.e.,

$$[\Xi(t)]_1 = \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega].$$

When $n = 3$, i.e., for Calabi-Yau threefolds, no strong quantum correction implies no quantum correction of the Yukawa coupling in physics literatures, the reader can refer to [B-C-O-V94] for details about quantum correction of the Yukawa coupling.

We first prove the following result which characterizes the Teichmüller space $\mathcal{T}$ of Calabi-Yau manifolds without quantum correction,

Theorem 1.1. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds. If there is no weak quantum correction at any point $p \in \mathcal{T}$, i.e., $[\Xi(t)]_1 = 0$, then $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric.*

Here, locally Hermitian symmetric property is equivalent to $\nabla R = 0$, which is not necessarily complete. Moreover, for polarized and marked Calabi-Yau threefolds, we found the following equivalent condition for no strong quantum correction at any point $p \in \mathcal{T}$.

Theorem 1.2. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau threefolds and $\Phi : \mathcal{T} \to D$ be the period map. Then the following are equivalent:*

1. *$\mathcal{T}$ has no strong quantum correction at any point $p \in \mathcal{T}$;*

2. *With respect to the Hodge metric, the image $\Phi(\mathcal{T})$ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$, which is also a totally geodesic submanifold of the period domain D.*

In the second part, we study Hyperkähler manifolds. Let $\mathcal{T}$ be the Teichmüller space of polarized Hyperkähler manifolds. By explicitly computing the Taylor expansions of the canonical families of $(2,0)$ and $(2n,0)$-classes at any point $p \in \mathcal{T}$, we show that they have no strong quantum correction at any point $p \in \mathcal{T}$. Therefore the Teichmüller spaces of polarized Hyperkähler manifolds are locally Hermitian symmetric with the Weil-Petersson metric, which is proved without using the local Torelli theorem for Hyperkähler manifolds. We then show that these local expansions are actually global expansions on the Teichmüller spaces. More precisely, let X be a Hyperkähler manifold with $\dim_{\mathbb{C}} X = 2n$ and $\Omega^{2,0}$ be a nowhere vanishing $(2,0)$-form over X, then we can actually construct a globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space $\mathcal{T}$.

Theorem 1.3. *Fix $p \in \mathcal{T}$, let X be the corresponding Hyperkähler manifold in the versal family and $\Omega^{2,0}$ be a nowhere vanishing $(2,0)$-form over X, then, in a neighborhood U of p, there exists the local families of $(2,0)$ and $(2n,0)$-classes defined by the canonical families $[\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]$ and*

$[e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}]$. *Furthermore their expansions are actually globally defined over the Teichmüller space $\mathcal{T}$, i.e.,*

$$[\Omega^{2,0}] + \sum_{i=1}^{N}[\varphi_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i=1}^{N}[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}]t_i t_j \in H^{2,0}(X_t),$$

$$\left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N}\left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right]t_i$$

$$+\frac{1}{k!}\sum_{k=1}^{2n}\left(\sum_{1 \le i_1 \le \ldots \le i_k \le N}[\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \wedge^n \Omega^{2,0}]t_{i_1} t_{i_2} \cdots t_{i_k}\right) \in H^{2n,0}(X_t),$$

are globally defined over $\mathcal{T}$.

This paper is organized as follows. In Section 2, we briefly review the construction of the Teichmüller space of polarized and marked Calabi-Yau manifolds, the local deformation theory of Calabi-Yau manifolds and the construction of the canonical family of $(n,0)$-forms. In Section 3, the definition and criteria of Hermitian symmetric space are introduced. Also, we define the quantum correction of the Teichmüller space $\mathcal{T}$, which originally comes from physics literatures. In Section 4, we review the Weil-Petersson metric over the Teichmüller space $\mathcal{T}$, and find a local formula for the covariant derivatives of the curvature tensor $\nabla_r R$ and $\nabla_{\bar{r}} R$ in the flat affine coordinate t. Under the assumption of no weak quantum correction at any point $p \in \mathcal{T}$, we prove that the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric by computing the local formulas of $\nabla_r R$ and $\nabla_{\bar{r}} R$. We remark that the results in Sections 2.2 to 4 actually all hold for both Calabi-Yau and Hyperkähler manifolds. In Section 5, for Calabi-Yau three-folds, we show that no strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map is an open submanifold of a globally Hermitian symmetric space with the same dimension as $\mathcal{T}$. In Section 6, we construct a globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space $\mathcal{T}$ of polarized Hyperkähler manifolds with $\dim_{\mathbb{C}} X = 2n$.

2 Locally Geometric Structure of the Moduli Space

In Section 2.1, we review the construction of the Teichmülller space of polarized and marked Calabi–Yau manifolds based on the works of Popp [P77], Viehweg [V95] and Szendröi [S99]. In Section 2.2 and Section 2.3, the smooth family of Betrami differentials $\Phi(t)$ and the canonical family of $(n,0)$-forms over the deformation space of Calabi-Yau manifolds was introduced. The results in Section 2.2 and Section 2.3 also hold for polarized Hyperkähler manifolds.

2.1 The Construction of the Teichmüller Space

In this section, we briefly review the construction of the Teichmüller space of polarized and marked Calabi-Yau manifolds and its basic properties. For the concept of Kuranishi family of compact complex manifolds, we refer to [S-U02, Pages 8-10], [P77, Page 94] or [V95, Page 19] for equivalent definitions and more details. If a complex analytic family $\pi : \mathcal{X} \to S$ of compact complex manifolds is complete at each point of S and versal at the point $0 \in S$, then the family $\pi : \mathcal{X} \to S$ is called the Kuranishi family of the complex maniflod $X = \pi^{-1}(0)$. The base space S is called the Kuranishi space. If the family is complete at each point of a neighbourhood of $0 \in S$ and versal at 0, then this family is called a local Kuranishi family at $0 \in S$. In particular, by definition, if the family is versal at each point of S, then it is local Kuranishi at each point of S.

A polarized and marked Calabi-Yau manifold is a triple of a Calabi-Yau manifold X, and an ample line bundle L over X and a basis of the integral middle homology group modulo torsion, $H_n(X, \mathbb{Z})/\text{Tor}$. A basis of the quotient space $(H_n(X, \mathbb{Z})/\text{Tor})/m(H_n(X, \mathbb{Z})/\text{Tor})$ is called a level m structure on the polarized Calabi–Yau manifold with $m \geq 3$. For deformation of a polarized Calabi-Yau manifold with level m structure, we have the following theorem, which is a reformulation of [S99, Theorem 2.2]. One can also refer [P77] and [V95] for more details about the construction of the moduli space of Calabi-Yau manifolds.

Theorem 2.1. *Let $m \geq 3$ and (X, L) be a polarized Calabi-Yau manifold with level m structure, then there exists a quasi-projective complex manifold Z_m with a versal family of Calabi-Yau maniflods,*

$$\mathcal{X}_{Z_m} \longrightarrow Z_m, \tag{2.1}$$

containing X as a fiber, and polarized by an ample line bundle $\mathcal{L}_{Z_m}$ on the versal family $\mathcal{X}_{Z_m}$.

Define the Teichmüller space $\mathcal{T}_L(X)$ to be the universal cover of the base space Z_m of the versal family above,

$$\pi : \mathcal{T}_L(X) \longrightarrow Z_m$$

and the family

$$\mathcal{X} \longrightarrow \mathcal{T}_L(X)$$

to be the pull-back of the family (2.1) by the projection π. For simplicity, we will denote $\mathcal{T}_L(X)$ by $\mathcal{T}$. And the Teichmüller space has the following property:

Proposition 2.2. *The Teichmüller space $\mathcal{T}$ is a simply connected smooth complex manifold, and the family*

$$\mathcal{X} \longrightarrow \mathcal{T} \tag{2.2}$$

containing X as a fiber, is local Kuranishi at each point of the Teichmüller space $\mathcal{T}$.

Note that the Teichmüller space $\mathcal{T}$ does not depend on the choice of level m. In fact, let m_1, m_2 be two different positive integers, $\mathcal{X}_1 \to \mathcal{T}_1$ and $\mathcal{X}_2 \to \mathcal{T}_2$ are two versal families

constructed via level m_1 and level m_2 respectively as above, both of which contain X as a fiber. By using the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are simply connected and the definition of versal families, we have a biholomorphic map $f : \mathcal{T}_1 \to \mathcal{T}_2$, such that the versal family $\mathcal{X}_1 \to \mathcal{T}_1$ is the pull-back of the versal family $\mathcal{X}_2 \to \mathcal{T}_2$ by the map f. Thus these two families are isomorphic to each other. One can check that the Teichmïler space defined above is precisely the moduli space of marked and polarized Calabi-Yau manifolds. See [C-G-L13] for more details.

2.2   Local Deformation of Calabi-Yau Manifolds

Fix $p \in \mathcal{T}$, we denote the corresponding polarized Calabi-Yau manifold in the versal family by (X, L). Yau's solution of Calabi conjecture assigns a unique Calabi-Yau metric g on X, and its imaginary part $\omega = \mathrm{Im}\, g \in L$ is the corresponding Kähler form. Under the Calabi-Yau metric g, we have the following lemma which follows from the Calabi-Yau theorem directly,

Lemma 2.3. *Let Ω be a nowhere vanishing holomorphic $(n, 0)$-form on X such that*

$$(\frac{\sqrt{-1}}{2})^n (-1)^{\frac{n(n-1)}{2}} \Omega \wedge \overline{\Omega} = \omega^n.$$

Then the map $\iota : A^{0,1}(X, T^{1,0}X) \to A^{n-1,1}(X)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by the Calabi-Yau metric g. Furthermore, the map ι preserves the Hodge decomposition.

Under the Calabi-Yau metric g, we have a precise description of the local deformation of the polarized Calabi-Yau manifolds. By the Hodge theory, we have the following identification

$$T_p^{1,0}\mathcal{T} \cong \mathbb{H}^{0,1}(X, T^{1,0}X),$$

where X is the corresponding fiber in the versal family $\mathcal{X} \to \mathcal{T}$. By the Kodaira-Spencer-Kuranishi theory, we have the following convergent power series expansion of the Betrami differentials, which is now well-known as the Bogomolov-Tian-Todorov Theorem [B78, T87, T89].

Theorem 2.4. *Let X be a Calabi-Yau manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be a basis. Then for any nontrivial holomorphic $(n, 0)$-form Ω on X, we can construct a smooth power series of Betrami differentials as follows*

$$\Phi(t) = \sum_{|I| \geq 1} t^I \varphi_I = \sum_{\substack{\nu_1 + \cdots + \nu_N \geq 1, \\ each\ \nu_i \geq 0, i = 1, 2, \cdots, N}} \varphi_{\nu_1 \cdots \nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0}), \qquad (2.3)$$

where $\varphi_{0 \cdots \nu_i \cdots 0} = \varphi_i$. This power series has the following properties:

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1) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$, *the integrability condition;*
2) $\overline{\partial}^{*}\varphi_I = 0$ *for each multi-index I with $|I| \geq 1$;*
3) $\varphi_I \lrcorner \Omega$ *is ∂-exact for each I with $|I| \geq 2$.*
4) *it converges when $|t| < \epsilon$.*

For the convergent radius, the reader can refer to [L-R-Y14, Theorem 4.4]. This theorem will be used to define the local flat affine coordinates $\{t_1, \cdots, t_N\}$ around the point $p \in \mathcal{T}$, for a given orthonormal basis $\{\varphi_1, \cdots, \varphi_N\}$ of $\mathbb{H}^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric over X.

2.3 Canonical Family of $(n, 0)$-Classes

Based on the construction of the smooth family $\Phi(t)$ of Beltrami differentials in Theorem 2.4, we can construct a canonical family of holomorphic $(n, 0)$-forms on the deformation spaces of Calabi-Yau manifolds. Here we just list the results we need, the reader can refer [L-R-Y14, Section 5.1] for details.

Let X be an n-dimensional Calabi-Yau manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ a basis where $N = \dim \mathbb{H}^{0,1}(X, T^{1,0}X)$. As constructed in Theorem 2.4, there exists a smooth family of Beltrami differentials in the following form

$$\Phi(t) = \sum_{i=1}^{N} \varphi_i t_i + \sum_{|I| \geq 2} \varphi_I t^I = \sum_{\nu_1 + \cdots + \nu_N \geq 1} \varphi_{\nu_1 \cdots \nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0})$$

for $t \in \mathbb{C}^N$ with $|t| < \epsilon$. It is easy to check that the map

$$e^{\Phi(t)} \lrcorner : \ A^0(X, K_X) \to A^0(X_t, K_{X_t}) \tag{2.4}$$

is a well-defined linear isomorphism.

Proposition 2.5. *For any smooth $(n, 0)$-form $\Omega \in A^{n,0}(X)$, the section $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ is holomorphic with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t if and only if*

$$\overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega) = 0. \tag{2.5}$$

Proof. This is a direct consequence of the following formula, which is [L-R-Y14, Corollary 3.5],

$$e^{-\Phi(t)} \lrcorner d \left(e^{\Phi(t)} \lrcorner \Omega\right) = \overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega).$$

In fact, the operator d can be decomposed as $d = \overline{\partial}_t + \partial_t$, where $\overline{\partial}_t$ and ∂_t denote the $(0, 1)$-part and $(1, 0)$-part of d, with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t. Note that $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ and so

$$\partial_t(e^{\Phi(t)} \lrcorner \Omega) = 0.$$

Hence,

$$e^{-\Phi(t)} \lrcorner \overline{\partial}_t \left(e^{\Phi(t)} \lrcorner \Omega\right) = \overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega),$$

which implies the assertion. $\qquad \square$

Theorem 2.6. *Let Ω be a nontrivial holomorphic $(n,0)$-form on the Calabi-Yau manifold X and $X_t = (X_t, J_{\Phi(t)})$ be the deformation of X induced by the smooth family $\Phi(t)$ of Beltrami differentials on X as constructed in Theorem 2.4. Then, for $|t| < \epsilon$,*

$$\Omega^c(t) := e^{\Phi(t)} \lrcorner \Omega \tag{2.6}$$

defines a canonical family of holomorphic $(n,0)$-forms on X_t which depends on t holomorphically.

Proof. Since Ω is holomorphic, and $\Phi(t)$ is smooth, by Proposition 2.5, we only need to show that

$$\partial(\Phi(t) \lrcorner \Omega) = 0$$

in the distribution sense. In fact, for any test form η on X,

$$(\Phi(t) \lrcorner \Omega, \partial^* \eta) = \lim_{k \to \infty} \left(\left(\sum_{|I| \le k} \varphi_I t^I \right) \lrcorner \Omega, \partial^* \eta \right) = \lim_{k \to \infty} \left(\sum_{i=1}^{N} t_i \varphi_i \lrcorner \Omega + \sum_{2 \le |I| \le k} t^I \partial \psi_I, \eta \right) = 0,$$

as $\varphi_i \lrcorner \Omega$, $1 \le i \le N$ are harmonic and $\varphi_I \lrcorner \Omega = \partial \psi_I$ are ∂-exact for $|I| \ge 2$ by Theorem 2.4. $\qquad\square$

Corollary 2.7. *Let $\Omega^c(t) := e^{\Phi(t)} \lrcorner \Omega$ be the canonical family of holomorphic $(n,0)$-forms as constructed in Theorem 2.6. Then for $|t| < \epsilon$, there holds the following expansion of $[\Omega^c(t)]$ in cohomology classes,*

$$[\Omega^c(t)] = [\Omega] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega] t_i + O(|t|^2), \tag{2.7}$$

where $O(|t|^2)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(X)$ of order at least 2 in t.

3 Hermitian Symmetric Space and Quantum Correction

In Section 3.1, we review the definitions of locally Hermitian symmetric spaces and globally Hermitian symmetric spaces. In Section 3.2, we define the quantum correction over the Teichmüller space of polarized and marked Calabi-Yau manifolds, which originally comes from the quantum correction of Yukawa coupling in the Kodaira-Spencer theory developed in [B-C-O-V94]. The definition of quantum correction also applies to polarized Hyperkähler manifolds.

3.1 Hermitian Symmetric Space

First let us review some basic definitions of symmetric spaces, the reader can refer to [K-N69, Chapter 11] or [Z00, Chapter 3] for details. Let N be a Riemannian manifold,

$p \in N$, and $r_p > 0$ the injective radius at point p. Consider the diffeomorphism s_p from the geodesic ball $B_{r_p}(p)$ onto N defined by

$$s_p(\exp_p(X)) = \exp_p(-X), \quad \forall X \in B_{r_p}(0) \subset T_pN. \tag{3.1}$$

The map s_p is called the geodesic symmetry at p. It has p as an isolated fixed point, and $(s_p)_{*p} = -id$. In general, it is not an isometry.

Definition 3.1. A Riemannian manifold N is called a locally Riemannian symmetric space, if for any point $p \in N$, the geodesic symmetry s_p is an isometry on $B_{r_p}(p)$. N is called a globally Riemannian symmetric space if, for any point $p \in N$, there exists an isometry in its isometry group $I(N)$ whose restriction on $B_{r_p}(p)$ is s_p.

Clearly, globally Riemannian symmetric spaces are locally Riemannian symmetric spaces. Applying the theorem of Cartan-Ambrose-Hicks [C46, A56, H59, H66] to the map s_p and the isometry $I = -id$ at T_pN, we immediately get the following lemma

Lemma 3.2. *A Riemannian manifold N is a locally Riemannian symmetric space if and only if $\triangledown R = 0$, i.e., the curvature tensor is parallel. Also, if a locally Riemannian symmetric space is complete and simply-connected, then it is a globally Riemannian symmetric space. Two locally Riemannian symmetric spaces are locally isometric if they have the same curvature at one point.*

Now, let us consider the complex case,

Definition 3.3. A Hermitian manifold N is a locally Hermitian symmetric space if, for any point $p \in N$, $s_p : \exp_p(X) \to \exp_p(-X), \forall X \in T_pN$ is a local automorphism around p of N, i.e., s_p leaves its Levi-Civita connection ∇ and complex structure J invariant. It is called a globally Hermitian symmetric space if it is connected and for any point $p \in N$ there exists an involutive automorphism s_p of N with p as an isolated fixed point.

Similarly, in terms of the curvature tensor, we have the following characterization of locally Hermitian symmetric spaces.

Theorem 3.4. *A Hermitian manifold is a locally Hermitian symmetric space if and only if*

$$\nabla R = 0 = \nabla J. \tag{3.2}$$

where ∇ is the Levi-Civita connection associated to the underlying Riemannian metric.

Corollary 3.5. *Let N be a Kähler manifold , if N is a locally Riemannian symmetric space, then N is a locally Hermitian symmetric space.*

For the Riemannican curvature tensor, Nomizu and Ozeki [N-O62] and later Nomizu, without assuming completeness, proved the following proposition.

Proposition 3.6. *(Nomizu and Ozeki [N-O62], Nomizu) For a Riemannian manifold (N, g), if $\nabla^k R = 0$ for some $k \geq 1$, then $\nabla R = 0$.*

3.2 Quantum Correction

In this section, we will define the quantum correction over the Teichmüller space of polarized and marked Calabi-Yau manifolds. Our motivation for quantum correction comes from the Kodaira-Spencer theory developed in [B-C-O-V94, Chapter 5].

The physical fields of the Kodaira-Spencer theory are differential forms of type $(0,1)$ on X with coefficients $(1,0)$-vectors, i.e., sections $\psi \in C^\infty(X, T^{*0,1}X \otimes T^{1,0}X)$ with

$$\partial(\psi \lrcorner \Omega) = 0,$$

where Ω is a nowhere vanishing $(n,0)$-form normalized as in Lemma 2.3. Then the Kodaira-Spencer action is given as follows

$$\lambda^2 S(\psi, \varphi|p) = \frac{1}{2} \int_X \psi \lrcorner \Omega \wedge \frac{1}{\partial}\overline{\partial}(\psi \lrcorner \Omega) + \frac{1}{6} \int_X ((\psi + \varphi) \wedge (\psi + \varphi)) \lrcorner \Omega \wedge (\psi + \varphi) \lrcorner \Omega,$$

where λ is the coupling constant. The Euler-Lagrange equation of this action is

$$\overline{\partial}(\varphi \lrcorner \Omega) + \frac{1}{2}\partial((\psi + \varphi) \wedge (\psi + \varphi)) \lrcorner \Omega = 0.$$

They also discussed that the Kodaira-Spencer action is a closed string theory action at least up to cubic order. In a properly regularized Kodaira-Spencer theory, the partition function should satisfy

$$e^{W(\lambda, \varphi|t, \overline{t})} = \int D\psi e^{S(\lambda, \varphi|t, \overline{t})}. \tag{3.3}$$

The effective action $W(\lambda, \varphi|t, \overline{t})$ is physically computed in [B-C-O-V94] in the flat affine coordinate $t = (t_1, t_2, \cdots, t_N)$. The term $W_0(\lambda, \varphi|t, \overline{t})$ in front of λ^{-2} satisfies

$$W_0(\lambda, \varphi|t, \overline{t}) = \lambda^2 S_0(\varphi, \psi|t, \overline{t}), \tag{3.4}$$

where $\psi(t)$ and $W_0(\lambda, \varphi|t, \overline{t})$ satisfy

$$\frac{\partial \psi(t)}{\partial t_i}\Big|_{t=0} = \varphi_i, \quad \overline{\partial}(\psi \lrcorner \Omega) + \frac{1}{2}((\varphi + \psi(t)) \wedge (\varphi + \psi(t))) \lrcorner \Omega = 0 \tag{3.5}$$

and

$$\frac{\partial^3 W_0(\varphi|t, \overline{t})}{\partial t_i \partial t_j \partial t_k} = C_{ijk}(t_1, t_2, \cdots, t_N). \tag{3.6}$$

$W_0(\varphi|t, \overline{t})$ may be viewed as the effective action for the massless modes φ from which the massive modes will be integrated out. It is quite amazing that integrating the massive modes has only the effectivity of taking derivatives of the Yukawa coupling. For example the four point function give rise to $\nabla_l C_{ijk}$, the five point function to $\nabla_s \nabla_l C_{ijk}$ and the six

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point function to $\nabla_r \nabla_s \nabla_l C_{ijk}$. Thus all the discussion suggests us to define the quantum correction of the Yukawa coupling as

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N}. \tag{3.7}$$

On the other hand, besides the canonical family of holomorphic $(n, 0)$-forms, we can define the classic canonical family as

$$\Omega^{cc}(t) = \exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega \in A^n(X, \mathbb{C}). \tag{3.8}$$

Proposition 3.7. *Fix $p \in \mathcal{T}$, let X be the corresponding fiber in the versal family $\mathcal{X} \to \mathcal{T}$. Let Ω be a nontrivial holomorphic $(n, 0)$-form over X and $\{\varphi_i\}_{i=1}^{N}$ be an orthonormal basis of $\mathbb{H}^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. If the cohomology class*

$$[\Xi(t)] = [\Omega^c(t)] - [\Omega^{cc}(t)] = [\Omega^c(t)] - [\exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega] = 0,$$

then the quantum correction of the Yukawa coupling vanishes, i.e.,

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N} = 0.$$

Moreover, $\sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] = 0$ if and only if the first order quantum correction of the Yukawa coupling

$$\sum_{s_1+\cdots+s_N=1} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N} = 0.$$

Proof. From the definition of Yukawa coupling, with the flat affine coordinate $t = (t_1, \cdots, t_N)$, we have

$$\begin{aligned} C_{ijk}(t_1, t_2, \cdots, t_N) &= \int_X \Omega^c(t) \wedge \frac{\partial^3 \Omega^c(t)}{\partial t_i \partial t_j \partial t_k} \\ &= \int_X \Omega^c(t) \wedge \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega + \Xi(t) \right), \end{aligned}$$

where $\Xi(t) = \Omega^c(t) - \exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega$. Direct computation shows that

$$\frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega \right) = \varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega,$$

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thus the term

$$\int_X \Omega^c(t) \wedge \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega \right)$$

has order zero with respect to t. Thus the quantum correction of Yukawa coupling satisfies

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N} = \int_X \Omega^c(t) \wedge \Xi(t).$$

Therefore, $[\Xi(t)] = 0$ implies that the quantum correction of the Yukawa coupling vanishes. Moreover, we have

$$\sum_{s_1+\cdots+s_N=1} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N} = 0;$$

$$\Longleftrightarrow \int_X \Omega \wedge \varphi_i \lrcorner \varphi_j \lrcorner \varphi_I \lrcorner \Omega + \int_X \varphi_i \lrcorner \Omega \wedge \varphi_j \lrcorner \varphi_I \lrcorner \Omega = 0 \quad \text{for any} \quad 1 \le i,j \le N \quad \text{and} \quad |I| = 2;$$

$$\Longleftrightarrow \int_X \varphi_i \lrcorner \Omega \wedge \varphi_j \lrcorner \varphi_I \lrcorner \Omega = 0, \quad \text{for any} \quad 1 \le i,j \le N \quad \text{and} \quad |I| = 2;$$

$$\left(\text{as } \varphi_i \lrcorner \Omega \wedge \varphi_j \lrcorner \varphi_I \lrcorner \Omega = \Omega \wedge \varphi_i \lrcorner \varphi_j \lrcorner \varphi_I \lrcorner \Omega \right);$$

$$\Longleftrightarrow \mathbb{H}(\varphi_j \lrcorner \varphi_I \lrcorner \Omega) = 0, \quad \text{for any} \quad 1 \le j \le N \quad \text{and} \quad |I| = 2;$$

$$\left(\text{as } \{[\varphi_i \lrcorner \Omega]\}_{i=1}^{N} \text{ is a basis of } \mathbb{H}^{2,1}(X) \right);$$

$$\Longleftrightarrow \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] = 0.$$

$\square$

Lemma 3.8. *Under the conditions as Proposition 3.7, the form $\Xi(t)$ are identically zero, i.e., $\Xi(t) = 0$ if and only if $[\varphi_i, \varphi_j] = 0$ for all $1 \le i, j \le N$. And, for $|t| < \epsilon$, there holds the following expansion of $[\Xi(t)]$ in cohomology classes,*

$$[\Xi(t)] = \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] + O(|t|^4). \tag{3.9}$$

where $O(|t|^4)$ denotes the terms of order at least 4 in t.

Proof. From the construction of the smooth family 2.3 of Beltrami differentials, see [M-K71, page 162] or [T89], we have

$$\varphi_K = -\frac{1}{2} \overline{\partial}^* G(\sum_{I+J=K} [\varphi_I, \varphi_J]) \quad \text{for} \quad |K| \ge 2. \tag{3.10}$$

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Thus we have

$$\Xi(t) = 0 \iff \left(\exp\left(\sum_{|I| \geq 1} \varphi_I t^I \right) - \exp\left(\sum_{i=1}^{N} \varphi_i t_i \right) \right) \lrcorner \Omega = 0$$

$$\iff \sum_{|I| \geq 1} \varphi_I t^I = \sum_{i=1}^{N} \varphi_i t_i$$

$$\iff \varphi_I = 0 \quad \text{for} \quad |I| \geq 2$$

$$\iff [\varphi_i, \varphi_j] = 0 \quad \text{for} \quad 1 \leq i, j \leq N \quad \text{by Formula (3.10)}.$$

Moreover, by the property that $\varphi_I \lrcorner \Omega = \partial \psi_I$ for $|I| \geq 2$ by Theorem 2.4, the cohomology class of the quantum correction satisfies

$$
\begin{aligned}
[\Xi(t)] &= \left[\left(\exp\left(\sum_{|I| \geq 1} \varphi_I t^I \right) - \exp\left(\sum_{i=1}^{N} \varphi_i t_i \right) \right) \lrcorner \Omega \right] \\
&= \left[\sum_{i,j=1}^{N} (\varphi_{ij} \lrcorner \Omega) t_i t_j + \sum_{i,j,k=1}^{N} (\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega + \varphi_{ijk} \lrcorner \Omega) t_i t_j t_k + O(|t|^4) \right] \\
&= \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] + O(|t|^4).
\end{aligned}
$$

$\square$

Thus the lowest order quantum correction has the form $\sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega]$, so we have the following definition,

Definition 3.9. We define the cohomology class $[\Xi(t)]$ to be the strong quantum correction at point $p \in \mathcal{T}$ and the cohomology class

$$[\Xi(t)]_1 = \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega],$$

to be the weak quantum correction at point $p \in \mathcal{T}$. If $[\Xi(t)]_1 = 0$, we will say that there is no weak quantum correction at point $p \in \mathcal{T}$. Moreover, if $[\Xi(t)] = 0$, we will say that there is no strong quantum correction at point $p \in \mathcal{T}$.

Remark 3.10. For Calabi-Yau threefolds, by Proposition 3.7, no strong quantum correction at point $p \in \mathcal{T}$ implies that there is no quantum correction of Yukawa coupling at point $p \in \mathcal{T}$. Moreover, no weak quantum correction at point $p \in \mathcal{T}$ is equivalent to that the first order quantum correction of Yukawa coupling at $p \in \mathcal{T}$ vanishes.

4 Quantum Correction and the Weil-Petersson Metric

In Section 4.1 and Section 4.2, we review the Weil-Petersson metric over the Teichmüller space $\mathcal{T}$ and derive a local formula for $\nabla_r R$ and $\nabla_{\bar{r}} R$ in the flat affine coordinate. In Section 4.3, under the assumption of no weak quantum correction at any point $p \in \mathcal{T}$, we prove that $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric by using the formula of $\nabla_r R$ and $\nabla_{\bar{r}} R$. The results in this section also hold for polarized Hyperkähler manifolds.

4.1 The Weil-Petersson Geometry

The local Kuranishi family of polarized Calabi-Yau manifolds $\pi : \mathcal{X} \to S$ is unobstructed by the Bogomolov-Tian-Todorov theorem [T87, T89]. One can assign the unique Ricci-flat or Calabi-Yau metric $g(s)$ on the fiber $X := X_s$ in the polarization Kähler class [Y78]. Then, on the fiber X, the Kodaira-Spencer theory gives rise to an injective map $\rho : T_s S \longrightarrow H^1(X, T^{1,0}X) \cong \mathbb{H}^{0,1}(X, T^{1,0}X)$, the space of harmonic representatives. The metric $g(s)$ induces a metric on $A^{0,1}(X, T^{1,0}X)$. The reader may also refer to [W03] for the discussion. For $v, w \in T_s S$, one then defines the Weil-Petersson metric on S by

$$g_{WP}(v, w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)} dvol_g(s). \tag{4.1}$$

Let $\dim X = n$, by the fact that the global holomorphic $(n, 0)$-form $\Omega := \Omega(s)$ is flat with respect to $g(s)$, it can be shown [T87] that

$$g_{WP}(v, w) = -\frac{\widetilde{Q}(i(v)\Omega, \overline{i(w)\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}. \tag{4.2}$$

Here, for convenience, we write $\widetilde{Q}(\cdot, \cdot) = (\sqrt{-1})^n Q(\cdot, \cdot)$, where Q is the intersection product. Therefore, $\widetilde{Q}$ has alternating signs in the successive primitive cohomology groups $H_{pr}^{p,q} \subset H^{p,q}$, $p + q = n$, which we simply denote by $H^{p,q}$ for convenience.

The formula (4.2) implies that the natural map $H^1(X, T^{1,0}X) \longrightarrow \mathrm{Hom}(H^{n,0}, H^{n-1,1})$ via the interior product $v \mapsto v \lrcorner \Omega$ is an isometry from the tangent space $T_s S$ to $(H^{n,0})^* \otimes H^{n-1,1}$. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain. Let F^n denote the Hodge bundle induced by $H^{n,0}$. A simple calculation in formal Hodge theory shows that

$$\omega_{WP} = Ric_{\widetilde{Q}}(F^n) = -\partial\overline{\partial} \log \widetilde{Q}(\Omega, \overline{\Omega}) = -\frac{\widetilde{Q}(\partial_i \Omega, \overline{\partial_j \Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})} + \frac{\widetilde{Q}(\partial_i \Omega, \overline{\Omega})\, \widetilde{Q}(\Omega, \overline{\partial_j \Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})^2}, \tag{4.3}$$

where ω_{WP} is the 2-form associated to g_{WP}. In particular, g_{WP} is Kähler and is independent of the choice of Ω. In fact, g_{WP} is also independent of the choice of the polarization.

Next, we define

$$K_i = -\partial_i \log \widetilde{Q}(\Omega, \overline{\Omega}) = -\frac{\partial_i \Omega, \overline{\Omega}}{\widetilde{Q}(\Omega, \overline{\Omega})}$$

and

$$D_i \Omega = \partial_i \Omega + K_i \Omega$$

for $1 \le i \le N$. And it is easy to check that $D_i \Omega$ is the projection of $\partial_i \Omega$ into $H^{n-1,1}$ with respect to the quadratic form $\widetilde{Q}(\cdot, \cdot)$. And if we denote the Christoffel symbol of the Weil-Petersson metric by Γ_{ij}^k, it is easy to check that

$$D_j D_i \Omega = \partial_j D_i \Omega - \Gamma_{ij}^k D_k \Omega + K_j D_i \Omega,$$

is the projection of $\partial_j D_j \Omega$ into $H^{n-2,2}$.

4.2 Property of the Curvature Tensor

To simplify the notation, we abstract the discussion by considering a variations of polarized Hodge structure $H \longrightarrow S$ of weight n with $h^{n,0} = 1$ and a smooth base S. Also, we always assume that it is effectively parametrized in the sense that the infinitesimal period map

$$\Phi_{*,s} : T_s S \longrightarrow \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \oplus \mathrm{Hom}(H^{n-1,1}, H^{n-2,2}) \oplus \cdots \qquad (4.4)$$

be injective in the first piece. Then the Weil-Petersson metric g_{WP} on S is defined by formula (4.2). In our abstract setting, instead of using $H_{pr}^{p,q}$ in the geometric case, we will write $H^{p,q}$ directly for simplicity.

Theorem 4.1. *For a given effectively parametrized ploarized variation of Hodge structure $H \to S$ of weight n with $h^{n,0} = 1$ and smooth S, the Riemannian curvature of the Weil-Petersson metric g_{WP} on S satisfies:*

1. *Its Riemannian curvature tensor is*

$$R_{i\bar{j}k\bar{l}} = g_{ij}g_{kl} + g_{il}g_{kj} - \frac{\widetilde{Q}(D_k D_i \Omega, \overline{D_l D_j \Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}.$$

2. *The covariant derivative of the Riemannian curvature tensor is*

$$\nabla_r R_{i\bar{j}k\bar{l}} = \frac{\widetilde{Q}(\partial_r \partial_k \partial_i \Omega, \overline{D_l D_j \Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})};$$

$$\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}} = \frac{\widetilde{Q}(D_k D_i \Omega, \overline{\partial_r \partial_l \partial_j \Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}.$$

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The main idea of the proof is that, when we use the canonical family of $(n,0)$-classes constructed in Corollary 2.7 to express the Weil-Petersson metric, the flat affine coordinate $t = (t_1, \cdots, t_N)$ is normal at the point $t = 0$. Since the problem is local, we may assume that S is a disk in $\mathbb{C}^N$, where $N = \dim H^{n-1,1}$, around $t = 0$. The first part of the above theorem for the curvature formula of the Weil-Petersson metric is due to Strominger. See [W03].

Proof. Let $\Omega^c(t)$ be the canonical family of holomorphic $(n,0)$-forms constructed in Theorem 2.6, so we have

$$
\begin{aligned}
\Omega^c(t) &= \Omega + \sum_{i=1}^N t_i \varphi_i \lrcorner \Omega + \sum_{|I| \geq 2} t^I \, \varphi_I \lrcorner \Omega + \sum_{k \geq 2} \wedge^k \Phi(t) \lrcorner \Omega \\
&= \Omega + \sum_{i=1}^N t_i \varphi_i \lrcorner \Omega + \frac{1}{2!} \sum_{i,j=1}^N t_i t_j (\varphi_i \lrcorner \varphi_j \lrcorner \Omega + \varphi_{ij} \lrcorner \Omega) \\
&\quad + \frac{1}{3!} \sum_{i,j<k} t_i t_j t_k (\varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega + \varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega + \varphi_{ijk} \lrcorner \Omega) + O(|t|^4) \\
&= a_0 + \sum_{i=1}^N a_i t_i + \cdots + \sum_{|I|=k} a_I t^I + \cdots .
\end{aligned}
$$

And the coefficients satisfy $\widetilde{Q}(a_0, a_0) = 1, \widetilde{Q}(a_i, a_j) = -\delta_{ij}$ and $\widetilde{Q}(a_0, a_i) = \widetilde{Q}(a_0, a_I) = \widetilde{Q}(a_i, a_I) = 0$ for $|I| \geq 2$. For multi-indices I and J, we set $q_{I,\overline{J}} := \widetilde{Q}(a_I, \overline{a}_J)$. Then we have

$$
\begin{aligned}
q(t) :&= \widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)}) \\
&= 1 - \sum_i t_i \overline{t}_i + \sum_{i,j,k,l} \frac{1}{2!2!} q_{ik,\overline{jl}} t_i t_k \overline{t_j t_l} \\
&\quad + \sum_{i,j,k,l,r} \frac{1}{2!3!} q_{ik,\overline{jlr}} t_i t_k \overline{t_j t_l t_r} + \sum_{i,j,k,l,r} \frac{1}{2!3!} q_{ikr,\overline{jl}} t_i t_k t_r \overline{t_j t_k} + O(t^6),
\end{aligned}
$$

where

$$
\begin{aligned}
q_{ik,\overline{jl}} &= \widetilde{Q}(\varphi_i \lrcorner \varphi_k \lrcorner \Omega, \ \varphi_j \lrcorner \varphi_l \lrcorner \Omega), \\
q_{ik,\overline{jlr}} &= \frac{1}{3} \widetilde{Q}(\varphi_i \lrcorner \varphi_k \lrcorner \Omega, \ \varphi_j \lrcorner \varphi_{lr} \lrcorner \Omega + \varphi_l \lrcorner \varphi_{jr} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{jl} \lrcorner \Omega), \\
q_{ikr,\overline{jl}} &= \frac{1}{3} \widetilde{Q}(\varphi_i \lrcorner \varphi_{kr} \lrcorner \Omega + \varphi_k \lrcorner \varphi_{ir} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{ik} \lrcorner \Omega, \ \varphi_j \lrcorner \varphi_l \lrcorner \Omega).
\end{aligned}
$$

Thus, the Weil-Petersson metric can be expressed as

$$
\begin{aligned}
g_{k\bar{l}} &= -\partial_k \partial_{\bar{l}} \log q = q^{-2}(\partial_k q \partial_{\bar{l}} q - q \partial_k \partial_{\bar{l}} q) \\
&= (1 + 2 \sum_i t_i \bar{t}_i + \cdots)[t_l t_{\bar{k}} - (1 - \sum_i t_i \bar{t}_i)(-\delta_{kl} + \sum_{i,j} q_{ik,jl} t_i \bar{t}_j \\
&\quad + \sum_{i,j,r} q_{ik,\overline{jlr}} t_i \overline{t_j t_r} + \sum_{i,j,r} q_{ikr,\overline{jl}} t_i t_r \bar{t}_l + \cdots)] \\
&= \delta_{kl} + \delta_{kl} \sum_i t_i \bar{t}_i + t_l \bar{t}_k - \sum_{i,j} q_{ik,jl} t_i \bar{t}_j + \sum_{i,j,r} q_{ik,\overline{jlr}} t_i \overline{t_j t_r} + \sum_{i,j,r} q_{ikr,\overline{jl}} t_i t_r \bar{t}_l + \cdots .
\end{aligned}
$$

As a result, the Weil-Petersson metric g is already in its geodesic normal form at $t = 0$, so the Christoffel symbols at point $t = 0$ is zero, i.e. $\Gamma^k_{ij}(0) = 0$. So the full curvature tensor at $t = 0$ is given by

$$
R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^2 g_{k\bar{l}}}{\partial t_i \partial \bar{t}_j}(0) = \delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} + q_{ik,\overline{jl}}.
$$

Rewrite this in its tensor form then gives the formula in the theorem.
By using the well-known formula:

$$
\nabla_r R_{i\bar{j}k\bar{l}} = \frac{\partial}{\partial t_r} R_{i\bar{j}k\bar{l}} - \Gamma^q_{ri} R_{q\bar{j}k\bar{l}} - \Gamma^q_{rk} R_{i\bar{j}q\bar{l}},
$$

$$
\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}} = \frac{\partial}{\partial \bar{t}_r} R_{i\bar{j}k\bar{l}} - \overline{\Gamma^q_{rj}} R_{i\bar{q}k\bar{l}} - \overline{\Gamma^q_{rl}} R_{i\bar{j}k\bar{q}},
$$

at the point $t = 0$, we have

$$
\nabla_r R_{i\bar{j}k\bar{l}}(0) = \frac{\partial}{\partial t_r} R_{i\bar{j}k\bar{l}}(0); \quad \nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}(0) = \frac{\partial}{\partial \bar{t}_r} R_{i\bar{j}k\bar{l}}(0).
$$

as $\Gamma^k_{ij}(0) = 0$ for any $1 \leq i, j, k \leq N$. And, from the formula of Riemannian curvature for Kähler manifold

$$
R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial t_k \partial \bar{t}_l} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial t_k} \frac{\partial g_{j\bar{k}}}{\partial \bar{t}_l}, \tag{4.5}
$$

each term of $\nabla_r R_{i\bar{j}k\bar{l}}$ or $\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}$ includes degree 1 term as a factor, except $\frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \bar{t}_j \partial t_r}$, $\frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \bar{t}_j \partial \bar{t}_r}$, thus it is zero at $t = 0$ from the expression of $g_{k\bar{l}}$. So we have

$$
\nabla_r R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \bar{t}_j \partial t_r}(0) = q_{ikr,\overline{jl}},
$$

$$
\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \bar{t}_j \partial \bar{t}_r}(0) = q_{ik,\overline{jlr}}.
$$

Rewrite this in its tensor form, we get the formula. $\qquad\square$

4.3 Quantum Correction and the Weil-Petersson Metric

In this section, we consider the locally Hermitian symmetric property of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau manifolds.

Theorem 4.2. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds. If there is no weak quantum correction at any point $p \in \mathcal{T}$, i.e., $[\Xi(t)]_1 = 0$, then $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric.*

Proof. Fix $p \in \mathcal{T}$, let X be the corresponding fiber in the versal family $\mathcal{X} \to \mathcal{T}$. If there is no weak quantum correction at point $p \in \mathcal{T}$, i.e.,

$$[\Xi(t)]_1 = \sum_{i+j+k=1} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] = 0,$$

then $[\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] + [\varphi_j \lrcorner \varphi_{ik} \lrcorner \Omega] + [\varphi_k \lrcorner \varphi_{ij} \lrcorner \Omega] = 0$ for any $1 \le i,j,k \le N$. So, from Theorem 4.1, we have

$$\nabla_r R_{i\bar{j}k\bar{l}}(p) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial t_r}(p) = q_{ikr,\overline{jl}} = \frac{1}{3}\widetilde{Q}(\varphi_i \lrcorner \varphi_{kr} \lrcorner \Omega + \varphi_k \lrcorner \varphi_{ir} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{ik} \lrcorner \Omega, \ \varphi_j \lrcorner \varphi_l \lrcorner \Omega) = 0,$$

$$\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}(p) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial \overline{t_r}}(p) = q_{ik,\overline{jlr}} = \frac{1}{3}\widetilde{Q}(\varphi_i \lrcorner \varphi_k \lrcorner \Omega, \ \varphi_j \lrcorner \varphi_{lr} \lrcorner \Omega + \varphi_l \lrcorner \varphi_{jr} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{jl} \lrcorner \Omega) = 0,$$

i.e., $\nabla R = 0$. So $\mathcal{T}$ is a locally Hermitian symmetric space by Lemma 3.2. $\square$

On the other hand, by the definition of locally Hermitian symmetric spaces, the following condition can also guarantee the locally Hermitian symmetric property for the Teichmüller space $\mathcal{T}$.

Theorem 4.3. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds and $\Omega^c(t)$ the canonical form constructed in Theorem 2.6. If the Weil-Petersson potential $\widetilde{Q}(\Omega_t^c, \overline{\Omega_t^c})$ only has finite terms, i.e., a polynormial in terms of the flat affine coordinate t, then $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric. Furthermore, if $\mathcal{T}$ is complete, then $\mathcal{T}$ is a globally Hermitian symmetric space with the Weil-Petersson metric.*

Proof. Because of Proposition 3.6, to prove a Kähler manifold is a locally Hermitian symmetric space, we only need to show its curvature tensor satisfies $\nabla^m R = 0$ for some positive integer m. If the Weil-Petersson potential $\widetilde{Q}(\Omega(t), \overline{\Omega}(t))$ only has finite terms, i.e., it is a polynomial of the flat affine coordinate $t = (t_1, t_2, \cdots, t_N)$. Then, in the flat affine coordinate t, the coefficients of the Weil-Petersson metric and its curvature tensor

$$\begin{aligned}
g_{k\bar{l}} &= -\partial_k \partial_{\bar{l}} \log \widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)}) \\
R_{i\bar{j}k\bar{l}} &= \frac{\partial^2 g_{i\bar{j}}}{\partial t_k \partial \overline{t_l}} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial t_k} \frac{\partial g_{p\bar{j}}}{\partial \overline{t_l}}.
\end{aligned}$$

is a polynomial of variable $(t_1, t_2, \cdots, t_N, \overline{t_1}, \overline{t_2}, \cdots, \overline{t_N})$.

On the other hand, from the proof of Theorem 4.1, the flat affine coordinate t is a normal coordinate at the point $t = 0$. So we have the Christoffel symbols at the point $t = 0$ vanish, i.e., $\Gamma^k_{ij}(0) = 0$. Thus, at the point $t = 0$, the covariant derivative $\nabla_p T = \partial_p T$ for any $(0, m)$-tensor T. Therefore, for a large enough integer m, we have $\nabla^m R(0) = 0$. Thus the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric. $\quad\square$

In particular, we have the following corollary,

Corollary 4.4. *If the canonical family of $(n, 0)$-classes $[\Omega^c(t)]$ constructed in Corollary 2.7 has finite terms, i.e., a polynormial in terms of the flat affine coordinate t, then $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric.*

Proof. The proof follows directly from Theorem 4.3. $\quad\square$

5 Quantum Correction and Calabi-Yau Threefolds

In Section 5.1, we review some basic properties of the period domain from Lie group and Lie algebra point of view. In Section 5.2, for Calabi-Yau threefolds, we show that no strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map is an open submanifold of a globally Hermitian symmetric space with the same dimension as $\mathcal{T}$.

5.1 Period Domain

Let us briefly recall some properties of the period domain from Lie group and Lie algebra point of view. All results in this section are well-known to the experts in the subject. The purpose to give details is to fix notations. One may either skip this section or refer to [G-S69] and [S73] for most of the details.
A pair (X, L) consisting of a Calabi–Yau manifold X of complex dimension n with $n \geq 3$ and an ample line bundle L over X is called a polarized Calabi–Yau manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(X, \mathbb{Z})$. The Poincaré bilinear form Q on $H^n_{pr}(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed n-forms u, v on X. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X, \mathbb{C})$ bilinearly. Let $f^k = \sum_{i=k}^n h^{i,n-i}$ and $F^k = F^k(X) = H^{n,0}_{pr}(X) \oplus \cdots \oplus H^{k,n-k}_{pr}(X)$, from which we have the decreasing filtration $H^n_{pr}(X, \mathbb{C}) = F^0 \supset \cdots \supset F^n$. We know that

$$\dim_{\mathbb{C}} F^k = f^k, \tag{5.1}$$

$$H^n_{pr}(X, \mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and} \quad H^{k,n-k}_{pr}(X) = F^k \cap \overline{F^{n-k}}. \tag{5.2}$$

In terms of the Hodge filtration, then the Hodge-Riemann relations are

$$Q\left(F^k, F^{n-k+1}\right) = 0, \quad \text{and} \tag{5.3}$$
$$Q\left(Cv, \overline{v}\right) > 0 \quad \text{if} \quad v \neq 0, \tag{5.4}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in H_{pr}^{k,n-k}(X)$. The period domain D for polarized Hodge structures with data (5.1) is the space of all such Hodge filtrations

$$D = \left\{ F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (5.1), (5.3) \text{ and } (5.4) \text{ hold} \right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{ F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (5.1) \text{ and } (5.3) \text{ hold} \right\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. For any $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, We call a basis

$$\zeta = \left\{ \zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{f^k+1}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^2}, \cdots, \zeta_{f^1-0}, \zeta_{f^0-1} \right\}$$

of $H^n(X, \mathbb{C})$ an *adapted basis for the given filtration*

$$F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^0$$

if it satisfies $F^k = \text{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. Let us simply denote $H_{\mathbb{C}} = H^n(X, \mathbb{C})$ and $H_{\mathbb{R}} = H^n(X, \mathbb{R})$. The group of the $\mathbb{C}$-rational points is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

which acts on $\check{D}$ transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \to D$. Fix a point $p \in \mathcal{T}$ with the image $O := \Phi(p) = \{F_p^n \subset \cdots \subset F_p^0\} \in D$. The points $p \in \mathcal{T}$ and $O \in D$ may be referred as the base points or the reference points. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_p^k = F_p^k$ for each k. Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} \mid gF_p^k = F_p^k, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D} \quad \text{with} \quad V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_\mathbb{R}/V = G_\mathbb{R}/G_\mathbb{R} \cap B \subseteq G_\mathbb{C}/B$. The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_\mathbb{C}$ can be described as

$$\mathfrak{g} = \{X \in \mathrm{End}(H_\mathbb{C})| \ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_\mathbb{C}\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g}| \ XH_\mathbb{R} \subseteq H_\mathbb{R}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_\mathbb{R} \subseteq G_\mathbb{C}$, $\mathfrak{g}_0$ becomes Lie algebra of $G_\mathbb{R}$. One observes that the reference Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ of $H^n(M, \mathbb{C})$ induces a Hodge structure of weight zero on $\mathrm{End}(H^n(M, \mathbb{C}))$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}|XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}\}.$$

Since the Lie algebra $\mathfrak{b}$ of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}.$$

The Lie algebra $\mathfrak{v}_0$ of V is $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k \geq 1}\mathfrak{g}^{-k,k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. Since D is an open set in $\check{D}$, we have the following relation:

$$T_{O,h}^{1,0}D = T_{O,h}^{1,0}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{5.5}$$

We define the unipotent group $N_+ = \exp(\mathfrak{n}_+)$.

Remark 5.1. With a fixed base point, we can identify N_+ with its unipotent orbit in $\check{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$; that is, $N_+ = N_+(\text{base point}) \cong N_+B/B \subseteq \check{D}$. In particular, when the base point O is in D, we have $N_+ \cap D \subseteq D$. We can also identify a point $\Phi(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\} \in D$ with any fixed adapted basis of the corresponding Hodge filtration, we have matrix representations of elements in the above Lie groups and Lie algebras. For example, elements in N_+ can be realized as non-singular block upper triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block lower triangular matrices.

5.2 Quantum Correction and Calabi-Yau Threefolds

For any point $p \in \mathcal{T}$, let (X_p, L) be the corresponding fiber in the versal family $\mathcal{X} \to \mathcal{T}$, which is a polarized and marked Calabi–Yau manifold. The period map from $\mathcal{T}$ to D is defined by assigning each point $p \in \mathcal{T}$ the Hodge structure on X_p, that is

$$\Phi : \mathcal{T} \to D, \qquad p \mapsto \Phi(p) = \{F^n(X_p) \subset \cdots \subset F^0(X_p)\}.$$

In [G-S69], Griffiths and Schmid studied the Hodge metric over the period domain D. In particular, this Hodge theory is a complete homogenous metric. Consider the period map on the Teichmüller space $\Phi : \mathcal{T} \to D$. By local Torelli theorem for Calabi-Yau manifolds, we know that the period map Φ is locally injective. Thus it follows from [G-S69] that the pull-back of the Hodge metric over D by Φ on $\mathcal{T}$ is well-defined Kähler metrics. We will call the pull-back metric the Hodge metric over Teichmüller space $\mathcal{T}$, denoted by h.

Theorem 5.2. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau threefolds and $\Phi : \mathcal{T} \to D$ be the period map. Then the following conditions are equivalent:*

1. *There is no strong quantum correction at any point $p \in \mathcal{T}$.*

2. *With respect to the Hodge metric, the image $\Phi(\mathcal{T})$ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$, which is also a totally geodesic submanifold of the period domain D.*

This theorem also implies that, under the assumption of no quantum correction at any point $p \in \mathcal{T}$, the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space and its image $\Phi(\mathcal{T})$ under the period map is a totally geodesic submanifold of the period domain D, both with the natural Hodge metrics. Moreover, assuming the global Torelli theorem in [C-G-L13] for Calabi-Yau manifolds, another consequence of this theorem is that the period map Φ embed the Teichmüller space $\mathcal{T}$ as a Zariski-open subset in the totally geodesic submanifold W.

Proof. Fix $p \in \mathcal{T}$, let X be the corresponding Calabi-Yau manifold in the versal family and $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be an orthonormal basis with respect to the Calabi-Yau metric. If Ω is a nowhere vanishing holomorphic $(n, 0)$-form over X and $\eta_i = \varphi_i \lrcorner \Omega \in \mathbb{H}^{2,1}(X)$, then $[\Omega], [\eta_1], \cdots, [\eta_N], [\overline{\eta}_1], \cdots, [\overline{\eta}_N], [\overline{\Omega}]$ is a basis of $H^3(X)$ adapted the Hodge filtration of X. Assume t is the flat affine coordinate around $p \in \mathcal{T}$, then we have $\varphi_i = \kappa(\frac{\partial}{\partial t_i})$, where $\kappa : T_p^{1,0}\mathcal{T} \to H^{0,1}(X, T^{1,0}X) \cong \mathbb{H}^{0,1}(X, T^{1,0}X)$ is the Kodaira-Spencer map. And if we assume

$$\begin{bmatrix} [\varphi_i \lrcorner \eta_1] \\ \vdots \\ [\varphi_i \lrcorner \eta_N] \end{bmatrix} = A_i \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix},$$

for some $N \times N$-matrix A_i, then, from the identity $[\varphi_i \lrcorner \overline{\eta}_j] = \delta_{ij}[\overline{\Omega}]$, we have

$$\begin{bmatrix} [\varphi_i \lrcorner \Omega] \\ [\varphi_i \lrcorner \eta] \\ [\varphi_i \lrcorner \overline{\eta}] \\ [\varphi_i \lrcorner \overline{\Omega}] \end{bmatrix} = \begin{bmatrix} 0 & e_i & & \\ & 0_{N \times N} & A_i & \\ & & 0_{N \times N} & e_i^T \\ & & & 0 \end{bmatrix} \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix} = E_i \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}, \tag{5.6}$$

where $e_i = (0, \cdots, 1, \cdots, 0)$, $[\eta] = \big[[\eta_1], \cdots, [\eta_N]\big]^T$, $[\overline{\eta}] = \big[[\overline{\eta}_1], \cdots, [\overline{\eta}_N]\big]^T$, $[\varphi_i \lrcorner \eta] = \big[[\varphi_i \lrcorner \eta_1], \cdots, [\varphi_i \lrcorner \eta_N]\big]^T$ and $[\varphi_i \lrcorner \overline{\eta}] = \big[[\varphi_i \lrcorner \overline{\eta}_1], \cdots, [\varphi_i \lrcorner \overline{\eta}_N]\big]^T$.

Moreover, if we define $A(t) = \sum_{i=1}^{N} t_i A_i$, then it is easy to check that

$$\sum_{i,j=1}^{N} t_i t_j [\varphi_i \lrcorner \varphi_j \lrcorner \Omega] = \sum_{i=1}^{N} t_i (t_1, \cdots, t_N) \begin{bmatrix} [\varphi_i \lrcorner \eta_1] \\ \vdots \\ [\varphi_i \lrcorner \eta_N] \end{bmatrix} = (t_1, \cdots, t_N) A(t) \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix}.$$

And, from the identity $[\varphi_i \lrcorner \overline{\eta}_j] = \delta_{ij} [\overline{\Omega}]$, we know that

$$
\begin{aligned}
\frac{1}{3!} \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega] &= \frac{1}{3!} \left(\sum_{i=1}^{N} t_i \varphi_i \right) \lrcorner \left(\sum_{j,k=1}^{N} t_j t_k [\varphi_j \lrcorner \varphi_k \lrcorner \Omega] \right) \\
&= \frac{1}{3!} (t_1, \cdots, t_N) A(t) \left(\sum_{i=1}^{N} t_i \varphi_i \right) \lrcorner \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix} \\
&= \frac{1}{3!} (t_1, \cdots, t_N) A(t) \begin{bmatrix} t_1 \\ \vdots \\ t_N \end{bmatrix}.
\end{aligned}
$$

Therefore, the classic canonical form

$$\Omega^{cc}(t) = \exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega = \Omega + \sum_{i=1}^{N} t_i \varphi_i \lrcorner \Omega + \frac{1}{2!} \sum_{i,j=1}^{N} t_i t_j \varphi_i \lrcorner \varphi_j \lrcorner \Omega + \frac{1}{3!} \sum_{i,j,k=1}^{N} t_i t_j t_k \varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega,$$

satisfies

$$[\Omega^{cc}(t)] = \left[1, (t_1, \cdots, t_N), \tfrac{1}{2!}(t_1, \cdots, t_N) A(t), \tfrac{1}{3!}(t_1, \cdots, t_N) A(t)(t_1, \cdots, t_N)^T \right] \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}. \quad (5.7)$$

On the other hand, let $O := \Phi(p)$ be the base point or reference point. If we define $E_i := \Phi_*(\frac{\partial}{\partial t_i}) \in \mathfrak{g}^{-1,1}$, then $\mathfrak{a} = \mathrm{span}_{\mathbb{C}} \{E_1, E_2, \cdots, E_N\}$ is an abelian Lie subalgebra, see [C-M-P, Lemma 5.5.1, page 173]. So we can define

$$W := \exp(\sum_{i=1}^{N} \tau_i E_i) \cap D,$$

which is a totally geodesic submanifold of D with respect to the natural Hodge metric. We consider the map $\rho : \mathbb{C}^N \to \check{D}$ given by

$$\rho(\tau) = \exp(\sum_{i=1}^{N} \tau_i E_i)$$

where $\tau = (\tau_1, \cdots, \tau_N)$ is the standard coordinates of $\mathbb{C}^N$. Then the Hodge-Riemann bilinear relations define a bounded domain $\beth \subset \mathbb{C}^N$, which is biholomorphic to W via the map ρ. As the $E_i's$ commute with each other, thus, for any $z, w \in \beth$, we have

$$\rho(z)(\rho(w))^{-1} = \exp(\sum_{i=1}^{N} z_i E_i)\exp(-\sum_{i=1}^{N} w_i E_i) = \exp(\sum_{i=1}^{N}(z_i - w_i)E_i) = I$$
$$\Longleftrightarrow z_i = w_i \quad \text{for} \quad 1 \le i \le N,$$

i.e., ρ is one-to-one, which means that it defines a global coordinate τ over W.

$(1) \Rightarrow (2)$: First we show that there exists a small coordinate chart U_p of $p \in \mathcal{T}$ such that $\Phi(U_p) \subset W$. For any point $q \in U_p$ with flat affine coordinate t, there is a unique upper-triangle matrix $\sigma(t) \in N_+$, i.e., an nonsingular upper triangular block matrices with identity blocks in the diagonal, such that

$$\sigma(t)\left[[\Omega], [\eta], [\overline{\eta}], [\overline{\Omega}]\right]^T \tag{5.8}$$

is a basis of $H^3(X_q)$ adapted to the Hodge filtration at X_q.

Also, we know that $[\Omega^c(t)] = [e^{\Phi(t)}\lrcorner\Omega]$ is a basis of $F^3(X_q)$, so $[\frac{\partial}{\partial t_i}\Omega^c(t)] \in F^2(X_q), 1 \le i \le N$ by Griffiths' transversality. By the assumption that the there is no strong quantum correction at point $p \in \mathcal{T}$ and Formula 5.7, we have

$$[\Omega^c(t)] = [\Omega^{cc}(t)] = \left[1, (t_1, \cdots, t_N), \tfrac{1}{2!}(t_1, \cdots, t_N)A(t), \tfrac{1}{3!}(t_1, \cdots, t_N)A(t)(t_1, \cdots, t_N)^T\right]\begin{bmatrix}[\Omega]\\ [\eta]\\ [\overline{\eta}]\\ [\overline{\Omega}]\end{bmatrix}.$$

And, for any $1 \le i \le N$, we have

$$[\frac{\partial\Omega^c(t)}{\partial t_i}] = [\varphi_i\lrcorner\Omega] + \sum_{j=1}^{N} t_j[\varphi_i\lrcorner\varphi_j\lrcorner\Omega] + \frac{1}{2!}\sum_{j,k=1}^{N} t_j t_k[\varphi_i\lrcorner\varphi_j\lrcorner\varphi_k\lrcorner\Omega]$$

$$= [\eta_i] + (t_1, \cdots, t_N)\begin{bmatrix}\varphi_i\lrcorner\eta_1\\ \vdots\\ \varphi_i\lrcorner\eta_N\end{bmatrix} + \frac{1}{2}(t_1, \cdots, t_N)A(t)\begin{bmatrix}[\varphi_i\lrcorner\overline{\eta}_1]\\ \vdots\\ [\varphi_i\lrcorner\overline{\eta}_N]\end{bmatrix},$$

which implies that

$$\begin{bmatrix}[\frac{\partial\Omega^c(t)}{\partial t_1}]\\ \vdots\\ [\frac{\partial\Omega^c(t)}{\partial t_N}]\end{bmatrix} = [\eta] + A(t)[\overline{\eta}] + \frac{1}{2!}(t_1, \cdots, t_N)A(t)[\overline{\Omega}]$$

$$= \left[0_{n\times 1}, I_{n\times n}, A(t), \tfrac{1}{2}A^T(t)(t_1, \cdots, t_N)^T\right]\begin{bmatrix}[\Omega]\\ [\eta]\\ [\overline{\eta}]\\ [\overline{\Omega}]\end{bmatrix}.$$

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Thus $\{[\Omega^c(t)], [\frac{\partial \Omega^c(t)}{\partial t_1}] \cdots [\frac{\partial \Omega^c(t)}{\partial t_N}]\}$ is a basis of $F^2(X_q)$ as they are linearly independent, which implies that the unique matrix $\sigma(t) \in N_+$ has the form

$$\sigma(t) = \begin{bmatrix} I & (t_1, \cdots, t_N) & \frac{1}{2}(t_1, \cdots, t_N)A(t) & \frac{1}{3!}(t_1, \cdots, t_N)A(t)(t_1, \cdots, t_N)^T \\ & I & A(t) & \frac{1}{2}A^T(t)(t_1, \cdots, t_N)^T \\ & & I & * \\ & & & I \end{bmatrix},$$

where $*$ represents an unknown column vector. Meanwhile, we know $\sigma(t) \in G_{\mathbb{C}}$, i.e.

$$\sigma(T)^T \begin{bmatrix} & & & 1 \\ & & -I_{N \times N} & \\ & I_{N \times N} & & \\ -1 & & & \end{bmatrix} \sigma(t) = \begin{bmatrix} & & & 1 \\ & & -I_{N \times N} & \\ & I_{N \times N} & & \\ -1 & & & \end{bmatrix},$$

which implies that

$$\sigma(t) = \begin{bmatrix} I & (t_1, \cdots, t_N) & \frac{1}{2}(t_1, \cdots, t_N)A(t) & \frac{1}{3!}(t_1, \cdots, t_N)A(t)(t_1, \cdots, t_N)^T \\ & I & A(t) & \frac{1}{2}A^T(t)(t_1, \cdots, t_N)^T \\ & & I & (t_1, \cdots, t_N)^T \\ & & & I \end{bmatrix}.$$

Direct computation shows that

$$\sigma(t) = \exp\left(\begin{bmatrix} 0 & (t_1, \cdots, t_N) & & \\ & 0_{N \times N} & A & \\ & & 0_{N \times N} & (t_1, \cdots, t_N)^T \\ & & & 0 \end{bmatrix}\right) = \exp(\sum_{i=1}^N t_i E_i) \in W, \qquad (5.9)$$

i.e., $\Phi(q) \in W$ for any point $q \in U_p$, which implies that $\Phi(U_p) \subset W$.

Take $q \in \mathcal{T}$ such that $\Phi(q) \in \Phi(\mathcal{T}) \cap W$, then, the same argument for the point $q \in \mathcal{T}$, there exists a neighborhood U_q of q such that $\Phi(U_q) \subset W$, so $\Phi(U_q) \subset \Phi(\mathcal{T}) \cap W$. By the local Torelli theorem for Calabi-Yau manifolds, $\Phi(U_q) \subset \Phi(\mathcal{T}) \cap W$ is an open neighborhood of $\Phi(q)$ as $\mathcal{T}$ and W has the same dimension, thus $\Phi(\mathcal{T}) \cap W \subset \Phi(\mathcal{T})$ is an open subset of $\Phi(\mathcal{T})$. On the other hand, $W = \exp(\sum_{i=1}^N \tau_i E_i) \cap D \subset D$ is a closed subset of the period domain D. The closedness of $W \subset D$ implies that $\Phi(\mathcal{T}) \cap W \subset \Phi(\mathcal{T})$ is also a closed subset. Therefore, $\Phi(\mathcal{T}) \cap W = \Phi(\mathcal{T})$, i.e., $\Phi(\mathcal{T}) \subset W$ as $\Phi(\mathcal{T})$ is connected and not empty.

(2) $\Rightarrow$ (1): Fix $p \in \mathcal{T}$, we have $\Phi(\mathcal{T}) \subset W = \exp(\sum_{i=1}^N \tau_i E_i) \cap D$ with $\tau \in \beth$. If the flat affine coordinate of $q \in U_p$ is t where U_p is a small coordinate chart, then we know the canonical family of $(3,0)$-classes is given by

$$[\Omega^c(t)] = [e^{\Phi(t)} \lrcorner \Omega] \in F^3(X_q).$$

Moreover, from the fact that $\Phi(q) \in W$, there exists $\tau = (\tau_1, \cdots, \tau_N) \in \beth$ such that

$$\Phi(q) = \exp(\sum_{i=1}^N \tau_i E_i),$$

i.e., the image of $q \in U_p$ under the period map Φ is

$$\Phi(q) = \exp\left(\sum_{i=1}^{N} \tau_i E_i\right) \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (\tau_1, \cdots, \tau_N) & \frac{1}{2}(\tau_1, \cdots, \tau_N)A(\tau) & \frac{1}{3!}(\tau_1, \cdots, \tau_N)A(\tau)(\tau_1, \cdots, \tau_N)^T \\ & I_{N \times N} & A(\tau) & \frac{1}{2}A^T(\tau)(\tau_1, \cdots, \tau_N)^T \\ & & I_{N \times N} & (\tau_1, \cdots, \tau_N)^T \\ & & & 1 \end{bmatrix} \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix},$$

which is a basis adapted to the Hodge filtration over X_q. In particular, we know the first element in the basis

$$A_q(t) = \left(1, (\tau_1, \cdots, \tau_N), \frac{1}{2!}(\tau_1, \cdots, \tau_N)A(\tau), \frac{1}{3!}(\tau_1, \cdots, \tau_N)A(\tau)(\tau_1, \cdots, \tau_N)^T\right) \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}$$

$$\in F^3(X_q) = H^{3,0}(X_q).$$

Thus, by the fact that $H^{3,0}(X_q) \cong \mathbb{C}$ and $[\Omega^c(t)] \in H^{3,0}(X_q)$, there exists a constant $\lambda \in \mathbb{C}$ such that

$$A_q(t) = \lambda[\Omega^c(t))].$$

Also, we have

$$\mathrm{Pr}_{H^{3,0}(X)}(A_q(t)) = \mathrm{Pr}_{H^{3,0}(X)}([\Omega^c(t)]) = [\Omega],$$

so $A_q(t) = [\Omega^c(t))]$. Thus

$$\mathrm{Pr}_{H^{2,1}(X)}(\Omega^c(t)) = \mathrm{Pr}_{H^{2,1}(X)}(A_q(t)) = \sum_{i=1}^{N} \tau_i[\eta_i], \tag{5.10}$$

and, from Corollary 2.7,

$$[\Omega^c(t)] = [\Omega] + \sum_{i=1}^{N} t_i[\eta_i] + A(t), \tag{5.11}$$

where $A(t) \in H^{1,2}(X) \oplus H^{0,3}(X)$. Then we have

$$\sum_{i=1}^{N} t_i[\eta_i] = \sum_{i=1}^{N} \tau_i[\eta_i],$$

which implies that $\tau_i = t_i$, $1 \le i \le N$ as $\{[\eta_i]\}_{i=1}^{N}$ is a basis of $H^{2,1}(X)$. Therefore,

$$[\Omega^c(t)] = \left(1, (t_1, \cdots, t_N), \frac{1}{2!}(t_1, \cdots, t_N)A(t), \frac{1}{3!}(t_1, \cdots, t_N)A(t)(t_1, \cdots, t_N)^T\right) \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix} = [\Omega^{cc}(t)],$$

by Formula 5.7, i.e., there is no strong quantum correction at point $p \in \mathcal{T}$. $\square$

6 Compact Hyperkähler Manifolds

In Section 6.1, we review some preliminary results about Hyperkähler manifolds and the period domain of weight 2. In Section 6.2, we construct the canonical families of $(2, 0)$ and $(2n, 0)$-classes by using the canonical families of smooth forms $e^{\Phi(t)} \lrcorner \Omega^{2,0}$ and holomorphic forms $e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}$. In Section 6.3, we prove the expansions of the canonical families of $(2, 0)$ and $(2n, 0)$-classes are actually globally defined over Teichmüller space $\mathcal{T}$.

6.1 Preliminary Results

In this section, we will review some preliminary results about Hyperkähler manifolds and the period domain. We define Hyperkähler manifolds as follows,

Definition 6.1. Let X be a compact and simply-connected Kähler manifold of complex dimension $2n \geq 4$ such that there exists a non-zero holomorphic non-degenerate $(2, 0)$-form $\Omega^{2,0}$ on X, unique up to a constant such that $\det(\Omega^{2,0}) \neq 0$ at each point $x \in X$ and $H^1(X, \mathcal{O}_X) = 0$. Then X is called a Hyperkähler manifold.

The conditions on the holomorphic $(2, 0)$-form $\Omega^{2,0}$ imply that $\dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) = 1$. A pair (X, L) consisting of a Hyperkähler manifold X of complex dimension $2n$ with $2n \geq 4$ and an ample line bundle L over X is called a polarized Hyperkähler manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(X, \mathbb{Z})$. Let $\omega = \omega_g$ correspond to the Calabi-Yau metric in the class L, then

$$\mathbb{H}_L^{0,1}(X, T^{1,0}X) = \{\varphi \in \mathbb{H}^{0,1}(X, T^{1,0}X) | [\varphi \lrcorner \omega] = 0\}$$

And we know that if $\varphi \in \mathbb{H}_L^{0,1}(X, T^{1,0}X)$ is harmonic, then $\varphi \lrcorner \omega$ is a harmonic $(0, 2)$-form. Thus we have the identification

$$\mathbb{H}_L^{0,1}(X, T^{1,0}X) = \{\varphi \in \mathbb{H}^{0,1}(X, T^{1,0}X) | \varphi \lrcorner \omega = 0\}$$

Furthermore, the primitive cohomology groups satisfy:

$$\begin{aligned} H_{pr}^{1,1}(X) & \cong \mathbb{H}_{pr}^{1,1}(X) = \{\eta \in H^{1,1}(X) | \eta \wedge \omega^{2n-1} = 0\} \\ H_{pr}^2(X) & = H^{2,0}(X) \oplus H_{pr}^{1,1}(X) \oplus H^{0,2}(X) \end{aligned}$$

The primitive cohomology group $H_{pr}^2(X)$ carry a nondegenerate bilinear form, the so-called Hodge bilinear form

$$Q(\alpha, \beta) = -\int_X \omega^{2n-2} \wedge \alpha \wedge \beta, \qquad \alpha, \beta \in H_{pr}^2(X), \tag{6.1}$$

which is evidently defined over $\mathbb{Q}$.

We consider the decreasing Hodge filtration $H_{pr}^2(M, \mathbb{C}) = F^0 \supset F^1 \supset F^2$ with condition

$$\dim_C F^2 = 1, \quad \dim_{\mathbb{C}} F^1 = b_2 - 2, \quad \dim_{\mathbb{C}} F^0 = b_2 - 1. \tag{6.2}$$

Then the Hodge-Riemann relations becomes

$$Q(F^k, F^{3-k}) = 0, \tag{6.3}$$

$$Q(Cv, \overline{v}) > 0 \quad \text{if } v \neq 0, \tag{6.4}$$

where C is the Weil operator given by $Cv = (\sqrt{-1})^{2k-2}v$ for $v \in H^{k,2-k}_{pr}(M) = F^k \cap \overline{F^{2-k}}$. The period domain D for polarized Hodge structures with data 6.2 is the space of all such Hodge filtrations

$$D = \{F^2 \subset F^1 \subset F^0 = H^2_{pr}(X, \mathbb{C}) | 6.2, \ 6.3 \text{ and } 6.4 \text{ hold}\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \{F^2 \subset F^1 \subset F^0 = H^2_{pr}(X, \mathbb{C}) | 6.2 \text{ and } 6.3 \text{ hold}\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. We may identify the period space D with the Grassmannian of positive 2-plans in $L^\perp$, and this gives us

$$D \cong SO(b_2 - 3, 2)/SO(2) \times SO(b_2 - 3).$$

So the period domain D is a global Hermitian symmetric space.
Let $\mathcal{T} = \mathcal{T}_L$ be the Teichmüller space of the polarized irreducible Hyperkähler manifold (X, L) which is a smooth complex manifold. The following result follows from [V09] or [C-G-L13, C-G-L14],

Theorem 6.2. *The period map*

$$\Phi : \mathcal{T} \to D$$

is injective.

6.2 Local Family of $(2, 0)$- and $(2n, 0)$-Classes

In this section, we derive the expansions of the canonical families of $(2, 0)$ and $(2n, 0)$-calsses $[\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]$ and $[e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}]$, where $\Omega^{2,0}$ is a nowhere vanishing holomorphic $(2, 0)$-form over the Hyperkäher manifold X. First we have the following Bochner's principle for compact Ricci-flat manifolds:

Proposition 6.3. *(Bochner's principle) On a compact Kähler Ricci-flat manifold, any holomorphic tensor field (covariant or contravariant) is parallel.*

The proof rests on the following formula, which follows from a tedious but straightforward computation [B-Y53, p. 142]: if τ is any tensor field,

$$\Delta(\|\tau\|^2) = \|\nabla \tau\|^2.$$

Therefore $\Delta(\|\tau\|^2)$ is nonnegative, hence 0 since its mean value over X is 0 by the Stokes' formula. It follows that τ is parallel.
Then we consider the canonical family of smooth $(2, 0)$-forms $\Omega^{c;2,0}(t) = e^{\Phi(t)} \lrcorner \Omega^{2,0}$ whose harmonic projection has the following expansion,

Theorem 6.4. *Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkähler manifold and $\Omega^{2,0}$ be a nonzero holomorphic nondegenerate $(2,0)$-form over X and $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis $\mathbb{H}_L^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. Then, in a neighborhood U of p, there exists a locally canonical family of smooth $(2,0)$-forms,*

$$\Omega^{c;2,0}(t) = e^{\Phi(t)} \lrcorner \Omega^{2,0},$$

which defines a canonical family of $(2,0)$-classes

$$[\mathbb{H}(\Omega^{c;2,0}(t))] = [\Omega^{2,0}] + \sum_{i=1}^N [\varphi_i \lrcorner \Omega^{2,0}] t_i + \frac{1}{2}\sum_{i=1}^N \left[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right] t_i t_j. \tag{6.5}$$

Proof. Let us consider the canonical family of smooth $(2,0)$-forms

$$e^{\Phi(t)}\lrcorner\Omega^{2,0} = \sum_{i=1}^N \varphi_i \lrcorner \Omega^{2,0} t^i + \frac{1}{2}\sum_{i,j=1}^N \left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0} + \varphi_{ij}\lrcorner\Omega^{2,0}\right) t_i t_j$$

$$+ \sum_{|I|\geq 3}\left(\varphi_I\lrcorner\Omega^{2,0} + \sum_{J+K=I}\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0}\right). \tag{6.6}$$

We claim that

Claim 6.5. *Suppose, for any multi-index J, K with $|J| \geq 2$, the harmonic projection $\mathbb{H}\left(\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0}\right) = c \cdot \Omega^{2,0}$ for some constant c, then we have*

$$\mathbb{H}\left(\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0} \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}\right) = c \cdot \wedge^n\Omega^{2,0} \wedge \wedge^n\overline{\Omega^{2,0}}. \tag{6.7}$$

Proof. By the Hodge decomposition, we have

$$\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0} = c \cdot \overline{\Omega^{2,0}} + d\alpha_1 + d^*\alpha_2.$$

The proof bases on the following facts:

$$(\overline{\partial}\varphi)_{A_p,\overline{\alpha B_q}} = \sum_\alpha \nabla_{\overline{\alpha}}\varphi_{A_p,\overline{B_q}}, \quad (\overline{\partial}^*\varphi)_{A_p,\overline{B_q}} = (-1)^{p+1}\sum_{\alpha,\beta}g^{\overline{\beta}\alpha}\nabla_\alpha\varphi_{A_p,\overline{\beta B_q}}, \tag{6.8}$$

and their conjugate, which can be found in [M-K71]. From the formula

$$\nabla(\alpha \wedge \beta) = \nabla\alpha \wedge \beta + \alpha \wedge \nabla\beta, \tag{6.9}$$

and $\nabla\Omega^{2,0} = 0$, which comes from the Bochner principal 6.3, we have

$$\nabla(\alpha \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) = \nabla\alpha \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}. \tag{6.10}$$

From the formulas 6.8 and their conjugate, we have

$$d(\alpha_1 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) = d\alpha_1 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}},$$
$$d^*(\alpha_2 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) = d^*\alpha_2 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}.$$

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Thus we have

$$\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \wedge \wedge^n \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}$$
$$= c \cdot \wedge^n \Omega^{2,0} \wedge \wedge^n \overline{\Omega^{2,0}} + d(\alpha_1 \wedge \wedge^n \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}) + d^*(\alpha_2 \wedge \wedge^n \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}),$$

which implies our claim. $\qquad\square$

Direct computations show that

$$\left(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^n \Omega^{2,0} = \left(\varphi_J \lrcorner \Omega^{2,0} \right) \wedge \left(\varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^{n-1} \Omega^{2,0} \qquad (6.11)$$

as a smooth $(2n, 2)$-form. Therefore, for any multi-index J, K with $|J| \geq 2$, we have

$$\int_X \left(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \right) \wedge \Omega^{2,0} \wedge \wedge^{n-1} \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}$$
$$= \int_X \left(\varphi_J \lrcorner \Omega^{2,0} \right) \wedge \left(\varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^{n-1} \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}$$
$$= \frac{1}{n} \int_X \left(\varphi_J \lrcorner \wedge^n \Omega^{2,0} \right) \wedge \left(\varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^{n-1} \overline{\Omega^{2,0}}$$
$$= \frac{1}{n} \int_X \partial \psi_J \wedge \left(\varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^{n-1} \overline{\Omega^{2,0}}$$
$$= \frac{1}{n} \int_X \partial (\psi_J \wedge \left(\varphi_K \lrcorner \Omega^{2,0} \right) \wedge \wedge^{n-1} \overline{\Omega^{2,0}}) \qquad (\text{as } \varphi_K \lrcorner \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}} \text{ is d-closed}).$$
$$= 0.$$

On the other hand, by Claim 6.5 and the Stokes' formula, we have

$$0 = \int_X \left(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \right) \wedge \Omega^{2,0} \wedge \wedge^{n-1} \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}} = c \cdot \int_X \wedge^n \Omega^{2,0} \wedge \wedge^n \overline{\Omega^{2,0}}$$

So we have $c = 0$ i.e., $\mathbb{H} \left((\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) \right) = 0$ for any multiple-index J, K with $|J| \geq 2$.
Next, for the $(1, 1)$-form $\varphi_I \lrcorner \Omega^{2,0}$, we claim that

Claim 6.6. 1. $\varphi_i \lrcorner \Omega^{2,0}$ is harmonic for $1 \leq i \leq N$.

2. For any multi-index I with $|I| \geq 2$, $\varphi_I \lrcorner \Omega^{2,0}$ is ∂-exact, which implies that

$$\mathbb{H}(\varphi_I \lrcorner \Omega^{2,0}) = 0.$$

Proof. 1. As φ_i is harmonic and $\Omega^{2,0}$ is parallel with respect to Levi-Civita connection. So, from the formula 6.8 and

$$\nabla(\varphi_i \lrcorner \Omega^{2,0}) = \nabla \varphi_i \lrcorner \Omega^{2,0} + \varphi_i \lrcorner \nabla \Omega^{2,0},$$

we have

$$d(\varphi_i \lrcorner \Omega^{2,0}) = d\varphi_i \lrcorner \Omega^{2,0} = 0,$$
$$d^*(\varphi_i \lrcorner \Omega^{2,0}) = d^* \varphi_i \lrcorner \Omega^{2,0} = 0,$$

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i.e., $\varphi_i \lrcorner \Omega^{2,0}$ is harmonic for $1 \leq i \leq N$.

2. As $\Omega^{2,0}$ is a nowhere vanishing holomorphic $(2,0)$-form, so we can define $\Omega^{*2,0} \in \Gamma(X, \wedge^2 \Theta)$ by requiring $\langle \Omega^{2,0}, \Omega^{*2,0} \rangle = 1$ pointwise on X. Actually, in a coordinate chart $\{z_1, z_2, \cdots, z_{2n}\}$, we can assume

$$\Omega^{2,0} = \sum_{i,j=1}^{2n} a_{ij} dz_i \wedge dz_j \quad \text{with} \quad a_{ij} = -a_{ji}$$

$$\Omega^{*2,0} = \sum_{i,j=1}^{2n} b_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \quad \text{with} \quad b_{ij} = -b_{ji}.$$

Then, if we define matrices $A = (a_{ij})$ and $B = (b_{ij})$, then $\det(A) \neq 0$ and

$$\langle \Omega^{2,0}, \Omega^{*2,0} \rangle = \sum_{i,j=1}^{2n} a_{ij} b_{ij} = tr(AB^T) = 1,$$

so, locally, the matrix B can be defined by

$$B = \frac{1}{2n} (A^{-1})^T.$$

And it is easy to check that this definition is independent of the local coordinates and $\nabla \Omega^{*2,0} = 0$ by the Bochner's principle 6.3. Then, by Theorem 2.4, we have

$$\varphi_I \lrcorner \wedge^n \Omega^{2,0} = \partial_X \psi_I, \quad |I| \geq 2,$$

which implies that

$$\varphi_I \lrcorner \Omega^{2,0} = \wedge^{n-1} \Omega^{*2,0} \lrcorner (\varphi_I \lrcorner \wedge^n \Omega^{2,0}) = \wedge^{n-1} \Omega^{*2,0} \lrcorner \partial \psi_I = \partial(\wedge^{n-1} \Omega^{*2,0} \lrcorner \psi_I),$$

by formulas 6.8 and 6.9. □

Thus, the harmonic projection of the family of $(2,0)$-forms $e^{\Phi(t)} \lrcorner \Omega^{2,0}$ is given by

$$\mathbb{H}\left(e^{\Phi(t)} \lrcorner \Omega^{2,0}\right) = \Omega^{2,0} + \sum_{i=1}^{N} \varphi_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2} \sum_{i=1}^{N} \left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right) t_i t_j.$$

Theorem 6.4 is proved. □

Corollary 6.7. *Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkähler manifold and $\Omega^{2,0}$ be a non-zero holomorphic non-degenerate $(2,0)$-form over X and $\{\varphi_i\}_{i=1}^{N}$ be an orthonormal basis $\mathbb{H}_L^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. Then, in a neighborhood U of p, there exists a canonical family of holomorphic $(2n, 0)$-forms,*

$$\Omega^c(t) = e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}$$

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which defines a canonical family of $(2n, 0)$-classes

$$[\Omega^c(t)] = \left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N} \left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i + \frac{1}{k!} \sum_{k=1}^{2n} \left(\sum_{1 \le i_1 \le \ldots \le i_k \le N} \left[\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \Omega\right] t_{i_1} t_{i_2} \cdots t_{i_k} \right).$$
(6.12)

In particular, the expansions imply that the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space.

Proof. From the Proposition 2.6, we have the harmonic projection

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) \in \mathbb{H}^{2n,0}(X_t).$$

And from Theorem 6.4, we know that

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0}) \in \mathbb{H}^{2,0}(X_t),$$

therefore, we have

$$\mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})] \in \mathbb{H}^{2n,0}(X_t).$$

Because of $\dim_{\mathbb{C}} \mathbb{H}^{2n,0}(X_t) = 1$, there exists $\lambda \in \mathbb{C}$ such that

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) = \lambda \mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})].$$
(6.13)

On the other hand, we have

$$\mathrm{Pr}_{\mathbb{H}^{2n,0}(X)} \left(\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) \right) = \mathrm{Pr}_{\mathbb{H}^{2n,0}(X)} \left(\mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})] \right) = \wedge^n \Omega^{2,0}.$$
(6.14)

Thus $\lambda = 1$, i.e.,

$$
\begin{aligned}
[\Omega^c(t)] &= [e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}] = \left[\wedge^n \left(\Omega^{2,0} + \sum_{i=1}^{N} \varphi_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2} \sum_{i=1}^{N} \left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0} \right) t_i t_j \right) \right] \\
&= \left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N} \left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i + \frac{1}{k!} \sum_{k=1}^{2n} \left(\sum_{0 \le i_1 \le \ldots \le i_k} \left[\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \Omega\right] t_{i_1} t_{i_2} \cdots t_{i_k} \right),
\end{aligned}
$$

which is a polynomial in terms of the flat affine coordinate $t = (t_1, \cdots, t_N)$. Thus, by the Corollary 4.4, the Teichmüller space $\mathcal{T}$ of polarized Hyperkähler manifolds is a locally Hermitian symmetric space. $\square$

6.3 Global Family of (2,0)- and (2n,0)-Classes

In this section, we will show that the flat affine coordinate t is globally defined over the Teichmüller space $\mathcal{T}$, so do the expansions of the canonical families of $(2, 0)$ and $(2n, 0)$-classes.

Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkäher manifold. Fix a nowhere vanishing holomorphic $(2,0)$-form $\Omega^{2,0}$ over X and a basis of harmonic form $\eta_1, \cdots, \eta_N \in \mathbb{H}^{1,1}_{pr}(X)$. Then $[\Omega^{2,0}], [\eta_1], \cdots, [\eta_N], [\overline{\Omega^{2,0}}]$ is a basis of $H^2_{pr}(X, \mathbb{C})$. We normalize this basis such that $Q([\Omega^{2,0}], [\overline{\Omega^{2,0}}]) = -1$ and $Q([\eta_i], [\eta_j]) = \delta_{ij}$. Moreover, we can pick up a harmonic basis of $\mathbb{H}^{0,1}_L(X, T^{1,0}X)$, denote by $\varphi_1, \cdots, \varphi_N$, such that $\varphi_i \lrcorner \Omega^{2,0} = \eta_i$.

Using the point $O = \Phi(p)$ as the base point or reference pint, we can parameterize the period domain D. Let $e_i = (0, \cdots, 1, \cdots, 0)$ with $1 \leq i \leq N$ be the standard basis of $\mathbb{C}^N$. Here we view e_i as a row vector, then we can define

$$E_i = \begin{bmatrix} 0 & e_i & 0 \\ 0 & 0 & e_i^T \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}^{-1,1}.$$

It follows that $E_i E_j = 0$ if $i \neq j$ and

$$E_i^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that

$$\exp\left(\sum_{i=1}^N t_i E_i\right) = \begin{bmatrix} 1 & (\tau_1, \cdots, \tau_N) & \frac{1}{2}\sum_{i=1}^N \tau_i^2 \\ 0 & I_{N \times N} & (\tau_1, \cdots, \tau_N)^T \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\beth \subset \mathbb{C}^N$ be the domain enclosed by the real hypersurface

$$1 - \sum_{i=1}^N |\tau_i|^2 + \frac{1}{4}|\sum_{i=1}^N \tau_i^2| = 0.$$

Let $\tau = (\tau_1, \cdots, \tau_N)$ be the standard coordinates on $\mathbb{C}^N$, then the map $\rho : \beth \to D$ given by $\rho(t) = \exp(\sum_{i=1}^N \tau_i E_i)$ is a biholormorphic map. This is the Harish-Chandra realization [H-C56] of the period domain D. Moreover, from the the global Torelli theorem (cf. [V09] and [C-G-L13, C-G-L14]) for Hyperkähler manifolds, we know that the map

$$\rho^{-1} \circ \Phi : \mathcal{T} \to \beth, \tag{6.15}$$

is an injective map. So the coordinate $\tau = (\tau_1, \cdots, \tau_N)$ are global coordinates on the Teichmüller space $\mathcal{T}$. We call it the Harish-Chandra coordinate. We know that, in a neighborhood of $p \in \mathcal{T}$, there is another flat affine coordinate t. Actually, these two coordinates coincide, and we have the following theorem:

Theorem 6.8. *Fix $p \in \mathcal{T}$, in a neighborhood U of p, the global Harish-Chandra coordinate τ coincide with the flat affine coordinate t. So the flat affine coordinate t is globally defined and the*

expansions of the canonical families of cohomology classes 6.5 and 6.12, i.e.,

$$[\Omega^{2,0}] + \sum_{i=1}^{N}[\varphi_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i=1}^{N}[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}]t_i t_j \in H^{2,0}(X_t),$$

$$\left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N}\left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i$$

$$+ \frac{1}{k!}\sum_{k=1}^{2n}\left(\sum_{1 \leq i_1 \leq ... \leq i_k \leq N}[\varphi_{i_1} \lrcorner ... \lrcorner \varphi_{i_k} \lrcorner \wedge^n \Omega^{2,0}]t_{i_1}t_{i_2}\cdots t_{i_k}\right) \in H^{2n,0}(X_t).$$

are globally defined over the Teichmüller space $\mathcal{T}$.

Proof. Let t be its flat affine coordinate of $q \in U_p$ where U_p is a small coordinate and τ be its Harish-Chandra coordinate. We only need to show that $t = \tau$. From the definition of τ, we know that

$$\exp(\sum_{i=1}^{N}\tau_i E_i)\begin{bmatrix}[\Omega^{2,0}]\\ [\eta_1]\\ \vdots\\ [\eta_N]\\ [\overline{\Omega^{2,0}}]\end{bmatrix} = \begin{bmatrix}1 & (\tau_1,\cdots,\tau_N) & \frac{1}{2}\sum_{i=1}^{N}\tau_i^2\\ 0 & I_{N\times N} & (\tau_1,\cdots,\tau_N)^T\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}[\Omega^{2,0}]\\ [\eta_1]\\ \vdots\\ [\eta_N]\\ [\overline{\Omega^{2,0}}]\end{bmatrix},$$

is a basis of $H^2(X_q)$ adapted to the Hodge filtration of X_q. Consider the first element in the basis, we have

$$B_q(t) = [\Omega^{2,0}] + \sum_{i=1}^{N}\tau_i[\eta_i] + \frac{1}{2}\sum_{i=1}^{N}\tau_i^2[\overline{\Omega^{2,0}}] \in H^{2,0}(X_q).$$

On the other hand, from Theorem 6.4, we know that

$$[\mathbb{H}(\Omega^{c;2,0}(t))] = [\Omega^{2,0}] + \sum_{i=1}^{N}[\varphi_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i=1}^{N}\left[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right]t_i t_j$$

$$= [\Omega^{2,0}] + \sum_{i=1}^{N}[\eta_i]t_i + \frac{1}{2}\sum_{i=1}^{N}t_i^2[\overline{\Omega^{2,0}}] \in H^{2,0}(X_q),$$

in the flat affine coordinate t. Thus, by the fact that $H^{2,0}(X_q) \cong \mathbb{C}$, there exists a constant $\lambda \in \mathbb{C}$ such that

$$B_q(t) = \lambda[\mathbb{H}(\Omega^{c;2,0}(t))].$$

Moreover, we have

$$[\Omega^{2,0}] = \text{Pr}_{H^{2,0}(X)}(B_q(t)) = \lambda\text{Pr}_{H^{2,0}(X)}([\mathbb{H}(\Omega^{c;2,0}(t)0]) = \lambda[\Omega^{2,0}],$$

so $\lambda = 1$, i.e., $B_q(t) = [\mathbb{H}(\Omega^{c;2,0}(t))]$. Thus we know $t_i = \tau_i$, $1 \leq i \leq N$, as $[\Omega^{2,0}], [\eta_1], \cdots, [\eta_N]$, $[\overline{\Omega^{2,0}}]$ is a basis of $H^2(X)$. $\square$

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CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, CHINA; DEPARTMENT OF MATHEMATICS,UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CA 90095-1555, USA

E-mail address: liu@math.ucla.edu, liu@cms.zju.edu.cn

DEPARTMENT OF MATHEMATICS,UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CA 90095-1555, USA

E-mail address: changyong@math.ucla.edu