A REMARK ON MIRZAKHANI'S ASYMPTOTIC FORMULAE

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Abstract. In this note, we answer a question of Mirzakhani on asymptotic behavior of the one-point volume polynomial of moduli spaces of curves. We also present some applications of Mirzakhani's asymptotic formulae of Weil-Petersson volumes.

1. Introduction

We will follow Mirzakhani's notation in [Mir2]. For $\mathbf{d} = (d_1, \cdots, d_n)$ with d_i non-negative integers and $|\mathbf{d}| = d_1 + \cdots + d_n < 3g - 3 + n$, let $d_0 = 3g - 3 + n - |\mathbf{d}|$ and define

$$[\tau_{d_1} \cdots \tau_{d_n}]_{g,n} = \frac{\prod_{i=1}^n (2d_i + 1)!! 2^{2|\mathbf{d}|} (2\pi^2)^{d_0}}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_1^{d_0}, \tag{1}$$

where $\kappa_1 = \omega/2\pi^2$ is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$ [AC]. Note that $V_{g,n} = [\tau_0, \cdots \tau_0]_{g,n}$ is the Weil-Peterson volume of $\overline{\mathcal{M}}_{g,n}$.

Mirzakhani's volume polynomial is given by

$$V_{g,n}(2L) = \sum_{|\mathbf{d}| \le 3g-3+n} [\tau_{d_1} \cdots \tau_{d_n}]_{g,n} \frac{L_1^{2d_1}}{(2d_1 + 1)!} \cdots \frac{L_n^{2d_n}}{(2d_n + 1)!}.$$

Let $S_{g,n}$ be an oriented surface of genus g with n boundary components. Let $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ be the moduli space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1, \ldots, L_n$. Then we know that the Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(L_1, \ldots, L_n))$ equals $V_{g,n}(L_1, \ldots, L_n)$.

In particular, when $n = 1$, Mirzakhani's volume polynomial can be written as

$$V_g(2L) = \sum_{k=0}^{3g-2} \frac{a_{g,k}}{(2k+1)!} L^{2k},$$

where $a_{g,k} = [\tau_k]_{g,1}$ are rational multiples of powers of π.

$$a_{g,k} = \frac{(2k+1)!! 2^{3g-2+2k} \pi^{6g-4-2k}}{(3g-2-k)!} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^k \kappa_1^{3g-2-k}. \tag{2}$$

Let γ be a separating simple closed curve on S_g and $S_g(\gamma) = S_{g_1,1} \times S_{g_2,1}$ the surface obtained by cutting S_g along γ. Then for any $L > 0$, we have

$$\mathrm{Vol}(\mathcal{M}(S_g(\gamma), \ell_\gamma = L)) = V_{g_1}(L) \cdot V_{g_2}(L), \tag{3}$$

where $\mathcal{M}(S_g(\gamma), \ell_\gamma = L)$ is the moduli space of hyperbolic structures on $S_g(\gamma)$ with the length of γ equal to L.

There are many works on the computation of Weil-Petersson volumes (e.g. [Fa, Gr, KMZ, MZ, Pe, ST, Wo, Zo]). In a recent paper [Mir2], Mirzakhani proved some interesting estimates on the asymptotics of Weil-Petersson volumes and found important applications in the geometry of random hyperbolic surfaces. In particular, Mirzakhani proved the following asymptotic relations of the coefficients of the one-point volume polynomial.

Theorem 1.1. **(Mirzakhani** [Mir2]**)** *For given* $i \ge 0$.

$$\lim_{g\to\infty} \frac{a_{g,i+1}}{a_{g,i}} = 1, \qquad \lim_{g\to\infty} \frac{a_{g,3g-2}}{a_{g,0}} = 0.$$

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Mirzakhani asked what is the asymptotics of $a_{g,k}/a_{g,k+1}$ for an arbitrary k (which can grow with g). The following result gives a partial answer to Mirzakhani's question.

Theorem 1.2. *For any given $k \geq 0$, there is a large genus asymptotic expansion*

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} = \frac{\pi^{2k}}{5^k k!}\left(1 + \frac{b_{1,k}}{g} + \frac{b_{2,k}}{g^2} + \cdots\right). \tag{4}$$

We have $b_{1,k} = \frac{1}{14}k^2 - \frac{4}{7}k$, $\forall k \geq 0$. In fact, for any given $k \geq 0$, the series in the bracket of (4) is a rational function of g.

Theorem 1.2 will be proved in Section 2. Now we present a numerical test of (4). Denote by $Q_{k,g}$ the ratio of the left-hand side and the truncated right-hand side of (4).

$$Q_{k,g} = \frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} \cdot \frac{5^k k!}{\pi^{2k}} \Big/ \left(1 + \frac{b_{1,k}}{g}\right). \tag{5}$$

Then we can see from Table 1 that $Q_{k,g}$ tends to 1 as g goes to infinity.

TABLE 1. Values of $Q_{k,g}$ (keep 6 decimal places)

k	$g = 20$	$g = 40$	$g = 60$	$g = 80$	$g = 100$
1	1.000438	1.000106	1.000047	1.000026	1.000016
2	1.001334	1.000326	1.000144	1.000080	1.000051
3	1.002300	1.000563	1.000248	1.000139	1.000089
4	1.003090	1.000759	1.000335	1.000188	1.000120

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2. Asymptotics of intersection numbers

In this section, we use Witten's notation

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m}. \tag{6}$$

For convenience, we denote the normalized tau function as

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g^{\mathbf{w}} := \prod_{i=1}^n (2d_i + 1)!! \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g. \tag{7}$$

We have the following forms of the celebrated Witten-Kontsevich theorem [Wi, Ko]. The first one is called the DVV formula (see [DVV])

$$\begin{aligned}(2d_1+1)!!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = & \sum_{j=2}^n \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \tau_{d_2} \cdots \tau_{d_j+d_1-1} \cdots \tau_{d_n} \rangle_g \\ & + \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\ & + \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \sum_{\{2,\cdots,n\}=I \coprod J} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}\end{aligned} \tag{8}$$

which is equivalent to the Virasoro constraint.

We also have the following recursive formula from integrating the first KdV equation of the Witten-Kontsevich theorem (see Proposition 3.3 in [LX1])

$$(2g+n-1)\langle \tau_0 \prod_{j=1}^n \tau_{d_j} \rangle_g = \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^n \tau_{d_j} \rangle_{g-1} + \frac{1}{2} \sum_{\underline{n}=I \coprod J} \langle \tau_0^2 \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \tag{9}$$

Definition 2.1. The following generating function

$$F(x_1, \cdots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_i = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} x_i^{d_i}$$

is called the n-point function.

In particular, we have Witten's one-point function

$$F(x) = \frac{1}{x^2} \exp\left(\frac{x^3}{24}\right),$$

which is equivalent to $\langle \tau_{3g-2} \rangle_g = 1/(24^g g!)$.

The two-point function has a simple explicit form due to Dijkgraaf (see [Fa2])

$$F(x_1, x_2) = \frac{1}{x_1 + x_2} \exp\left(\frac{x_1^3}{24} + \frac{x_2^3}{24}\right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2} x_1 x_2 (x_1 + x_2)\right)^k.$$

A general study of the n-point function can be found in [LX3].

From Dijkgraaf's two-points function, it is not difficult to see that

$$\begin{aligned} \lim_{g\to\infty} \frac{\langle \tau_k \tau_{3g-1-k} \rangle_k}{g^k \langle \tau_{3g-2} \rangle_g} &= \lim_{g\to\infty} \frac{k!}{24^{g-k}(2k+1)!2^k(g-k)!} \cdot \frac{24^g \cdot g!}{g^k} \\ &= \frac{k! 24^k}{(2k+1)!2^k} \\ &= \frac{6^k}{(2k+1)!!}. \end{aligned}$$

In fact, we have the following more general result.

Proposition 2.2. *For any fixed set* $\mathbf{d} = (d_1, \ldots, d_n)$ *of non-negative integers, the limit of the following quantity*

$$C(d_1, \cdots, d_n; g) = \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g}{(6g)^{|\mathbf{d}|} \langle \tau_{3g-2} \rangle_g} \prod_{i=1}^{n} (2d_i + 1)!! \tag{10}$$

exists and we have $\lim_{g\to\infty} C(d_1, \ldots, d_n; g) = 1$.

Proof. We use induction on $|\mathbf{d}|$. When $d_1 = \cdots = d_n = 0$, it is obviously true by the string equation.

From (9) and the string equation, we have that for any $\mathbf{k} = (k_1, \ldots, k_m)$ with $|\mathbf{k}| < |\mathbf{d}|$,

$$\begin{aligned} \langle \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-5+m-|\mathbf{d}|} \rangle_{g-1} &\le \langle \tau_0^4 \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_{g-1} \\ &\le 12(2g+m) \langle \tau_0 \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_g \\ &= O\left(g \cdot \langle \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_g\right). \end{aligned} \tag{11}$$

Here $f_1(g) = O(f_2(g))$ means there exists a constant $C > 0$ independent of g such that

$$f_1(g) \le C f_2(g).$$

Note that the last equation in (11) is obtained by induction, since $|\mathbf{k}| < |\mathbf{d}|$.

Let us expand $\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g$ using (8). From (11) and by induction, we see that the second term in the right-hand side of (8) has the estimate

$$\frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \prod_{i=2}^{n} \tau_{d_i} \tau_{3g-2+n-|\mathbf{d}|} \rangle_{g-1} = O\left(g^{|\mathbf{d}|-1}\right). \tag{12}$$

Similarly, the third term in the right-hand side of (8) has the estimate

$$\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\sum_{\{2,\cdots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\tau_{3g-2+n-|\mathbf{d}|}\rangle_{g-g'}=O\left(g^{|\mathbf{d}|-2}\right). \tag{13}$$

So by induction, we have

$$\begin{aligned}\lim_{g\to\infty}C(d_1,\dots,d_n;g)=&\lim_{g\to\infty}\sum_{j=2}^{n}\frac{(2d_j+1)C(d_2,\cdots,d_j+d_1-1,\cdots d_n;g)}{6g}\\&+\lim_{g\to\infty}\frac{(2d_1+2(3g-2+n-|\mathbf{d}|)-1)!!}{(2(3g-2+n-|\mathbf{d}|)-1)!!}\cdot\frac{C(d_2,\cdots,d_n;g)}{(6g)^{d_1}}\\=&1.\end{aligned}\tag{14}$$

□

Corollary 2.3. *We have the following large genus asymptotic expansion*

$$C(d_1,\dots,d_n;g)=1+\frac{C_1(d_1,\dots,d_n;g)}{g}+\frac{C_2(d_1,\dots,d_n;g)}{g^2}+\cdots, \tag{15}$$

where the coefficients $C_j(d_1,\dots,d_n;g)$ are determined recursively by induction on $|\mathbf{d}|$,

$$\begin{aligned}C(d_1,\dots,d_n;g)=&\frac{1}{6g}\sum_{j=2}^{n}(2d_j+1)C(d_2,\dots,d_j+d_1-1,\dots,d_n;g)\\&+\frac{\prod_{j=1}^{d_1}(g+\frac{2n-2|\mathbf{d}|+2j-5}{6})}{g^{d_1}}C(d_2,\dots,d_n;g)+\frac{(g-1)^{|\mathbf{d}|-2}}{3g^{|\mathbf{d}|-1}}\sum_{r+s=d_1-2}C(r,s,d_2,\dots,d_n;g-1)\\&+\sum_{r+s=d_1-2}\sum_{\{2,\cdots,n\}=I\coprod J}24^{g'}6^{|J|+1-n-3g'}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle^{\mathbf{w}}_{g'}\\&\times\frac{(g-g')^{|J|+1-n+|\mathbf{d}|-3g'}\prod_{j=1}^{g'}(g+1-j)}{g^{|\mathbf{d}|}}C(s,d_J;g-g'),\end{aligned}\tag{16}$$

where d_J denote the set $\{d_i\}_{i\in J}$.

In fact, the expansion $C(d_1,\dots,d_n;g)$ has only finite nonzero terms, i.e. $C_j(d_1,\dots,d_n;g)=0$ when j is large enough.

Proof. The recursive relation follows from the asymptotic expansions of equations (12), (13) and (14). The last assertion will follow from Corollary 2.6. □

Remark 2.4. When $n=0$ or $|\mathbf{d}|=0$, we have

$$C(\emptyset;g)=C(0,\dots,0;g)=1. \tag{17}$$

By the string and dilaton equations, we have

$$C(0,d_2,\dots,d_n;g)=\frac{1}{6g}\sum_{j=2}^{n}(2d_j+1)C(d_2,\dots,d_j-1,\dots,d_n;g)+C(d_2,\dots,d_n;g), \tag{18}$$

$$C(1,d_2,\dots,d_n;g)=(1+\frac{n-2}{2g})C(d_2,\dots,d_n;g). \tag{19}$$

So we may assume $d_i\geq 2,\forall i$ in $C(d_1,\dots,d_n;g)$.

Remark 2.5. In large g expansion, we have

$$\frac{1}{(g-m)^k}=\left(\sum_{i=1}^{\infty}\frac{m^{i-1}}{g^i}\right)^k \tag{20}$$

for any given m.

When $d_1,\dots,d_2\geq 2$, from (16) we can deduce that

$$C_1(d_1,\dots,d_n;g)=-\frac{|\mathbf{d}|^2}{6}+\frac{(n-1)|\mathbf{d}|}{3}+\frac{n^2}{12}-\frac{5n}{12}. \tag{21}$$

In particular,

$$C_1(d_1;g) = -\frac{d_1}{6} - \frac{1}{3},$$
$$C_1(d_1,d_2;g) = -\frac{1}{6}(d_1+d_2)^2 + \frac{1}{3}(d_1+d_2) - \frac{1}{2}.$$

For the full expansion of $C(d_1,\ldots,d_n;g)$, let us look at some examples

$$C(1;g) = C(1,1;g) = 1 - \frac{1}{2g},$$
$$C(2;g) = 1 - \frac{1}{g} + \frac{5}{12g^2},$$
$$C(3;g) = 1 - \frac{11}{6g} + \frac{95}{72g^2} - \frac{35}{72g^3},$$
$$C(2,2;g) = 1 - \frac{11}{6g} + \frac{17}{12g^2} - \frac{7}{12g^3}.$$

In fact, we will see in a moment that the expansion (15) of $C(d_1,\ldots,d_n;g)$ is a polynomial in $1/g$. Let

$$P_{d_1,\ldots,d_n}(g) = (6g)^{|\mathbf{d}|}C(d_1,\ldots,d_n;g). \tag{22}$$

The recursive formula (16) in Corollary 2.3 becomes

$$\begin{aligned} P_{d_1,\ldots,d_n}(g) &= \sum_{j=2}^{n}(2d_j+1)P_{d_2,\ldots,d_j+d_1-1,\ldots,d_n}(g) \\ &+ \prod_{j=1}^{d_1}(6g+2n-2|\mathbf{d}|+2j-5)P_{d_2,\ldots,d_n}(g) + 12g\sum_{r+s=d_1-2}P_{r,s,d_2,\ldots,d_n}(g-1) \\ &+ \sum_{r+s=d_1-2}\ \sum_{\{2,\cdots,n\}=I\coprod J} 24^{g'}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}^{\mathbf{w}}\prod_{j=1}^{g'}(g+1-j)P_{s,d_J}(g-g'), \end{aligned} \tag{23}$$

Corollary 2.6. *For any fixed set $\mathbf{d}=(d_1,\ldots,d_n)$ of non-negative integers,*

$$P_{d_1,\ldots,d_n}(g) = \frac{\langle\tau_{d_1}\cdots\tau_{d_n}\tau_{3g-2+n-|\mathbf{d}|}\rangle_g}{\langle\tau_{3g-2}\rangle_g}\prod_{i=1}^{n}(2d_i+1)!!$$

is a polynomial in $\mathbb{Z}[g]$ with highest-degree term $6^{|\mathbf{d}|}g^{|\mathbf{d}|}$. These polynomials $P_{d_1,\ldots,d_n}(g)$ are determined uniquely by the recursive relation (23) *and $P_\emptyset(g) = P_{0,\ldots,0}(g) = 1$.*

Proof. By Theorem 4.3(iv) and Proposition 4.4 in [LX4], we have

$$24^{g'}g'!\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}^{\mathbf{w}} \in \mathbb{Z}.$$

Since $g'!$ divides $\prod_{j=1}^{g'}(g+1-j)$, it is not difficult to see that $P_{d_1,\ldots,d_n}(g)$ are polynomials with integer coefficients by induction using (23). □

We introduce some notation. Consider the semigroup N^∞ of sequences $\mathbf{m}=(m(1),m(2),\ldots)$ where $m(i)$ are nonnegative integers and $m(i)=0$ for sufficiently large i. We also use $(1^{m(1)}2^{m(2)}\ldots)$ to denote $\mathbf{m}$.

Let $\mathbf{m},\mathbf{a_1},\ldots,\mathbf{a_n}\in N^\infty$, $\mathbf{m}=\sum_{i=1}^n \mathbf{a_i}$.

$$|\mathbf{m}| := \sum_{i\geq 1} im(i) \quad ||\mathbf{m}|| := \sum_{i\geq 1} m(i) \quad \binom{\mathbf{m}}{\mathbf{a_1},\ldots,\mathbf{a_n}} := \prod_{i\geq 1}\binom{m(i)}{a_1(i),\ldots,a_n(i)}.$$

Let $\mathbf{m}\in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{m}) := \prod_{i\geq 1}\kappa_i^{m(i)}.$$

The following remarkable identity was proved in [KMZ].

$$\langle\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{m})\rangle_g=\sum_{p=0}^{||\mathbf{m}||}\frac{(-1)^{||\mathbf{m}||-p}}{p!}\sum_{\substack{\mathbf{m}=\mathbf{m_1}+\cdots+\mathbf{m_p}\\ \mathbf{m_i}\neq\mathbf{0}}}\binom{\mathbf{m}}{\mathbf{m_1},\dots,\mathbf{m_p}}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{p}\tau_{|\mathbf{m_j}|+1}\rangle_g. \tag{24}$$

Proof of Theorem 1.2. For any $k\geq 1$, by definition we have

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}}=\frac{(6g-3-2k)!!2^{6g-4-2k}(2\pi^2)^k\langle\tau_{3g-2-k}\kappa_1^k\rangle_g/k!}{g^k(6g-3)!!2^{6g-4}\langle\tau_{3g-2}\rangle_g}. \tag{25}$$

Using (24) to expand $\langle\tau_{3g-2-k}\kappa_1^k\rangle_g$ and taking limit as $g\to\infty$, we get by Proposition 2.2

$$\begin{aligned}\lim_{g\to\infty}\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}}&=\lim_{g\to\infty}\frac{(6g-3-2k)!!(2\pi^2)^k\langle\tau_{3g-2-k}\tau_2^k\rangle_g}{g^k(6g-3)!!2^{2k}k!\langle\tau_{3g-2}\rangle_g}\\&=\frac{\pi^{2k}}{5^k k!}\lim_{g\to\infty}\frac{15^k\langle\tau_{3g-2-k}\tau_2^k\rangle_g}{(6g)^{2k}\langle\tau_{3g-2}\rangle_g}\\&=\frac{\pi^{2k}}{5^k k!}\lim_{g\to\infty}C(\underbrace{2,\dots,2}_{k};g)\\&=\frac{\pi^{2k}}{5^k k!}.\end{aligned}$$

So we get the leading term in the right-hand side of (4).

Now we compute the coefficient of $1/g$ in the asymptotic expansion of $a_{g,3g-2-k}/(g^k a_{g,3g-2})$. We have

$$\begin{aligned}\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}}&=\frac{(6g-3-2k)!!\pi^{2k}\left(\langle\tau_{3g-2-k}\tau_2^k\rangle_g-\frac{k(k-1)}{2}\langle\tau_{3g-2-k}\tau_2^{k-2}\tau_3\rangle_g\right)}{g^k(6g-3)!!2^k k!\langle\tau_{3g-2}\rangle_g}+O(1/g^2)\\&=\frac{\pi^{2k}}{5^k k!}\left(\frac{(6g)^k}{\prod_{j=1}^k(6g-2j-1)}C(\underbrace{2,\dots,2}_{k};g)\right.\\&\qquad\left.-\frac{15}{14}k(k-1)\cdot\frac{(6g)^{k-1}}{\prod_{j=1}^k(6g-2j-1)}C(\underbrace{2,\dots,2}_{k-2},3;g)\right)+O(1/g^2).\end{aligned} \tag{26}$$

By (21), we have

$$C_1(\underbrace{2,\dots,2}_{k};g)=\frac{1}{12}k^2-\frac{13}{12}k. \tag{27}$$

Substituting it into (26), the coefficient of $1/g$ in the asymptotic expansion of $a_{g,3g-2-k}/(g^k a_{g,3g-2})$ equals

$$C_1(\underbrace{2,\dots,2}_{k};g)+\sum_{j=1}^{k}\frac{1+2j}{6}-\frac{15}{14}k(k-1)\times\frac{1}{6}=\frac{1}{14}k^2-\frac{4}{7}k. \tag{28}$$

So we get the second term in the right-hand side of (4), namely

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}}=\frac{\pi^{2k}}{5^k k!}\left(1+\left(\frac{1}{14}k^2-\frac{4}{7}k\right)\frac{1}{g}+O(1/g^2)\right). \tag{29}$$

Since there are only finite number of terms in the right-hand side of (24), from the above proof it is not difficult to see that for each $k\geq 1$, the series in the bracket of (29) is a rational function of g. So we conclude the proof of Theorem 1.2.

Example 2.7. When $k=1$, we have

$$\frac{a_{g,3g-3}}{g a_{g,3g-2}}=\frac{\pi^2}{5}\cdot\frac{6g}{6g-3}C(2;g)$$

$$= \frac{\pi^2}{5} \cdot \frac{12g^2 - 12g + 5}{6g(2g-1)}$$

$$= \frac{\pi^2}{5} \left(1 - \frac{1}{2g} + \sum_{j=2}^{\infty} \frac{1}{3 \cdot 2^{j-1} g^j} \right).$$

When $k = 2$, we have

$$\frac{a_{g,3g-4}}{g^2 a_{g,3g-2}} = \frac{\pi^4}{50} \left(\frac{(6g)^2}{(6g-3)(6g-5)} C(2,2;g) - \frac{15}{7} \cdot \frac{6g}{(6g-3)(6g-5)} C(3;g) \right)$$

$$= \frac{\pi^4}{50} \cdot \frac{(g-1)(1008g^3 - 1200g^2 + 888g - 175)}{84g^2(2g-1)(6g-5)}$$

$$= \frac{\pi^4}{50} \left(1 - \frac{6}{7g} + \frac{43}{84g^2} + \cdots \right).$$

These equations can be verified in low genera using the following data:

$$a_{1,0} = \frac{\pi^2}{12}, \quad a_{1,1} = \frac{1}{2}, \quad a_{2,0} = \frac{29\pi^8}{192}, \quad a_{2,1} = \frac{169\pi^6}{120}, \quad a_{2,2} = \frac{139\pi^4}{12},$$

$$a_{2,3} = \frac{203\pi^2}{3}, \quad a_{2,4} = 210, \quad a_{3,0} = \frac{9292841\pi^{14}}{4082400}, \quad a_{3,1} = \frac{8497697\pi^{12}}{388800},$$

$$a_{3,2} = \frac{8983379\pi^{10}}{45360}, \quad a_{3,3} = \frac{127189\pi^8}{81}, \quad a_{3,4} = \frac{94418\pi^6}{9},$$

$$a_{3,5} = \frac{166364\pi^4}{3}, \quad a_{3,6} = \frac{616616\pi^2}{3}, \quad a_{3,7} = 400400.$$

Corollary 2.8. *For any* $\mathbf{m} = (m(1), m(2), \dots) \in N^\infty$, *we have the following limit equation involving higher degree* κ *classes*

$$\lim_{g\to\infty} \frac{\langle \prod_{i=1}^n \tau_{d_i} \tau_{3g-2+n-|\mathbf{d}|-|\mathbf{m}|} \kappa(\mathbf{m}) \rangle_g}{(6g)^{|\mathbf{d}|+|\mathbf{m}|+||\mathbf{m}||} \langle \tau_{3g-2} \rangle_g} = \frac{\mathbf{m}!}{||\mathbf{m}||! \prod_{i=1}^n (2d_i+1)!! \prod_{j\geq 1} ((2j+3)!!)^{m(j)}}. \tag{30}$$

Proof. This identity follows directly from Proposition 2.2 and equation (24). □

3. Asymptotics of Weil-Petersson volumes

The large genus asymptotics of Weil-Petersson volumes was conjectured by Zograf based on his numerical experiments [Zo].

Conjecture 3.1. (Zograf) *For any fixed* $n \geq 0$

$$V_{g,n} = (4\pi^2)^{2g+n-3} (2g-3+n)! \frac{1}{\sqrt{g\pi}} \left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right) \right)$$

as $g \to \infty$, *where* c_n *is a constant depending only on* n.

Note that the asymptotic behavior of $V_{g,n}$ for fixed g and large n has been determined by Manin and Zograf [MZ]. Next We recall Mirzakhani's work in [Mir2]. We use the notation introduced in Section 1. For $n \geq 0$, define

$$a_n = \zeta(2n)(1 - 2^{1-2n}).$$

We have the following properties of a_n.

Lemma 3.2. (Mirzakhani [Mir2]**)** $\{a_n\}_{n=1}^\infty$ *is an increasing sequence. Moreover we have* $\lim_{n\to\infty} a_n = 1$, *and*

$$a_{n+1} - a_n \asymp 1/2^{2n}. \tag{31}$$

Here $f_1(n) \asymp f_2(n)$ *means that there exists a constant* $C > 0$ *independent of* n *such that*

$$\frac{1}{C} f_2(n) \leq f_1(n) \leq C f_2(n).$$

We have the following differential form of Mirzakhani's recursion formula [Mir1, MS] (see also [Sa, LX1, LX2, EO]).

$$[\tau_{d_1},\dots,\tau_{d_n}]_{g,n} = 8\left(\sum_{j=2}^{n}\mathcal{A}_{\mathbf{d}}^{j} + \mathcal{B}_{\mathbf{d}} + \mathcal{C}_{\mathbf{d}}\right), \tag{32}$$

where

$$\mathcal{A}_{\mathbf{d}}^{j} = \sum_{L=0}^{d_0}(2d_j+1)\ a_L[\tau_{d_1+d_j+L-1},\prod_{i\neq 1,j}\tau_{d_i}]_{g,n-1}, \tag{33}$$

$$\mathcal{B}_{\mathbf{d}} = \sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L[\tau_{k_1}\tau_{k_2}\prod_{i\neq 1}\tau_{d_i}]_{g-1,n+1}, \tag{34}$$

and

$$\mathcal{C}_{\mathbf{d}} = \sum_{\substack{I\amalg J=\{2,\dots,n\}\\ 0\le g'\le g}}\sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L\ [\tau_{k_1}\prod_{i\in I}\tau_{d_i}]_{g',|I|+1}\times[\tau_{k_2}\prod_{i\in J}\tau_{d_i}]_{g-g',|J|+1}. \tag{35}$$

Lemma 3.3. *Given $\mathbf{d}=(d_1,\dots,d_n)$ and $g,n\ge 0$, the following recursive formulas hold*

$$[\tau_0\tau_1\prod_{i=1}^{n}\tau_{d_i}]_{g,n+2} = [\tau_0^4\prod_{i=1}^{n}\tau_{d_i}]_{g-1,n+4} + 6\sum_{\substack{g_1+g_2=g\\ \{1,\dots,n\}=I\amalg J}}[\tau_0^2\prod_{i\in I}\tau_{d_i}]_{g_1,|I|+2}[\tau_0^2\prod_{i\in J}\tau_{d_i}]_{g_2,|J|+2}, \tag{36}$$

$$(2g-2+n)[\prod_{i=1}^{n}\tau_{d_i}]_{g,n} = \frac{1}{2}\sum_{L\ge 0}(-1)^L(L+1)\frac{\pi^{2L}}{(2L+3)!}[\tau_{L+1}\prod_{i=1}^{n}\tau_{d_i}]_{g,n+1}, \tag{37}$$

$$\sum_{j=1}^{n}(2d_j+1)[\tau_{d_j-1}\prod_{i\neq j}\tau_{d_i}]_{g,n} = \sum_{L\ge 0}\frac{(-\pi^2)^L}{4(2L+1)!}[\tau_L\prod_{i=1}^{n}\tau_{d_i}]_{g,n+1}. \tag{38}$$

The above three equations in such forms were stated at Section 3 of [Mir2]. Mirzakhani proved the following remarkable asymptotic formulae based on the data computed by Zograf [Zo].

Theorem 3.4. (Mirzakhani [Mir2]**)** *Let $n\ge 0$. Then we have*

$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + O(1/g) \tag{39}$$

and

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g). \tag{40}$$

Following Mirzakhani's notation, denote

$$[\mathbf{x}]_{g,n} := [\tau_{x_1}\dots\tau_{x_n}]_{g,n},$$

where $\mathbf{x}=(x_1,\dots,x_n)$.

Lemma 3.5. (Mirzakhani [Mir2]**)** *In terms of the above notation, for $\mathbf{x}=(x_1,\dots,x_l)$, and $\mathbf{y}=(y_1,\dots,y_m)$, we have*

$$\sum_{g_1+g_2=g}[\mathbf{x}]_{g_1,l}\times[\mathbf{y}]_{g_2,m} = o(V_{g,n-2}), \tag{41}$$

where $n=l+m$.

The above lemma is a weaker form of Lemma 3.3 in [Mir2].

Lemma 3.6. *When $d_1>0$, we have*

$$[\tau_{d_1}\cdots\tau_{d_n}]_{g,n} < [\tau_{d_1-1}\tau_{d_2}\cdots\tau_{d_n}]_{g,n}. \tag{42}$$

Proof. We expand both sides of the inequalities using (32). Since each term in $\mathcal{A}^j_{\mathbf{d}}, \mathcal{B}_{\mathbf{d}}, \mathcal{C}_{\mathbf{d}}$ is positive, by comparing corresponding terms in the expansion, the inequality (42) follows from Lemma 3.2 that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence. □

Corollary 3.7. *For any fixed set* $\mathbf{d} = (d_1, \ldots, d_n)$ *of non-negative integers, we have*

$$[\tau_{d_1} \cdots \tau_{d_n}]_{g,n} \le V_{g,n}. \tag{43}$$

We can now prove the following Zograf's conjecture [Zo] giving large genus ratio of Weil-Peterson volumes and intersection numbers involving ψ-classes. The proof is essentially due to Mirzakhani [Mir2].

Theorem 3.8. *For any fixed* $n > 0$ *and a fixed set* $\mathbf{d} = (d_1, \cdots, d_n)$ *of non-negative integers, we have*

$$\lim_{g\to\infty} \frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1. \tag{44}$$

Proof. We use induction on $|\mathbf{d}|$. We need only prove the following limit equation

$$\lim_{g\to\infty} \left| \frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n}} - 1 \right| = 0 \tag{45}$$

By induction, we may assume

$$\lim_{g\to\infty} \frac{[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1. \tag{46}$$

So in order to prove (45), we need only prove that

$$\lim_{g\to\infty} \left| \frac{[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n} - [\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} \right| = 0. \tag{47}$$

By comparing each term in Mirzakhani's recursion formula (32) for $[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n}$ and $[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}$, this actually follows from (43), (31), Theorem 3.4 and Lemma 3.5. The argument is similar to the proof of Theorem 3.5 in [Mir2]. We omit the details. □

Remark 3.9. We thank Mirzakhani [Mir3] for pointing out that Zograf was able to prove Theorem 3.8 using the method of [MZ].

Lemma 3.10. *When* $3g + n - 2 > 0$*, we have*

$$V_{g,n+1} \le \frac{\pi^2}{6} [\tau_1 \tau_0^n]_{g,n+1}. \tag{48}$$

The equality holds only when $(g, n) = (0, 3)$ *or* $(1, 0)$.

Proof. First note that the coefficients in (38)

$$\left\{ \frac{\pi^{2L}}{4(2L+1)!} \right\}_{L\ge 1}$$

is a decreasing sequence.

From Lemma 3.6, we know $[\tau_L \prod_{i=1}^n \tau_{d_i}]_{g,n+1}$ is a decreasing sequence in L.

Taking all $d_i = 0$ in (38), the left-hand side becomes 0. Writing down the first two terms of the right-hand side, we get

$$\frac{1}{4} V_{g,n+1} - \frac{2\pi^2}{2^4 \cdot 3} [\tau_1 \tau_0^n]_{g,n+1} < 0,$$

which is just (48). □

Remark 3.11. The inequality (48) can also be obtained using Mirzakhani's recursion formula (32). Let $f(x) = \zeta(2x)(1 - 2^{1-2x})$, we can check that $f''(x) < 0$ when $x \ge 1$. This implies that $\{a_{n+1} - a_n\}_{n\ge 1}$ is a decreasing sequence. By Mirzakhani's recursion formula (32), we have

$$V_{g,n+1} - [\tau_1 \tau_0^n]_{g,n+1} \le \frac{a_1 - a_0}{a_1} V_{g,n+1}. \tag{49}$$

Substituting $a_0 = \frac{1}{2}$ and $a_1 = \frac{\pi^2}{12}$, we get

$$[\tau_1\tau_0^n]_{g,n+1} \geq \frac{6}{\pi^2} V_{g,n+1}.$$

Corollary 3.12. *For any $g, n \geq 0$, we have*

$$V_{g,n+1} > 12(2g-2+n)V_{g,n} \quad \text{and} \quad V_{g,n+1} < C(2g-2+n)V_{g,n}, \tag{50}$$

where $C = \frac{20\pi^2}{10-\pi^2} = 1513.794\ldots$.

Proof. It is not difficult to see that the coefficients in (37)

$$\left\{\frac{1}{2}(L+1)\frac{\pi^{2L}}{(2L+3)!}\right\}_{L\geq 0}$$

is a decreasing sequence.

Taking all $d_i = 0$ in (37) and keeping only the first term in the right-hand side, we get

$$(2g-2+n)V_{g,n} \leq \frac{1}{12}[\tau_1\tau_0^n]_{g,n+1} < \frac{1}{12}V_{g,n+1},$$

which is the first inequality in (50).

If we take first two terms in the right-hand side of (37) and apply Lemma 3.10, we get

$$\begin{aligned}(2g-2+n)V_{g,n} &\geq \frac{1}{12}[\tau_1\tau_0^n]_{g,n+1} - \frac{\pi^2}{120}[\tau_2\tau_0^n]_{g,n+1} \\ &> (\frac{1}{12} - \frac{\pi^2}{120})[\tau_1\tau_0^n]_{g,n+1} \\ &\geq \frac{10-\pi^2}{120}\cdot\frac{6}{\pi^2}V_{g,n+1} \\ &= \frac{10-\pi^2}{20\pi^2}V_{g,n+1},\end{aligned}$$

which is the second inequality in (50). □

The inequalities (50) imply that

$$12 \leq \liminf_{g\to\infty} \frac{V_{g,n(g)+1}}{(2g-2+n(g))V_{g,n(g)}} \leq \limsup_{g\to\infty} \frac{V_{g,n(g)+1}}{(2g-2+n(g))V_{g,n(g)}} \leq \frac{20\pi^2}{10-\pi^2},$$

where $n(g) \to \infty$ as $g \to \infty$.

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