Algebraicity of the image of period map

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Abstract

We prove that the image of period map is algebraic, as conjectured by Griffiths.

0   Introduction

In this paper we study the period maps from geometry. More precisely we have an algebraic family $f : \mathfrak{X} \to S_0$ of polarized algebraic manifolds over a quasi-projective manifold S_0. The period map

$$\Phi_0 : S_0 \to \Gamma_0 \backslash D,$$

assigns any point $q \in S_0$, modulo the action of the monodromy group Γ_0, the Hodge structure of the n-th primitive cohomology group $H^n_{pr}(X_q, \mathbb{C})$ of the fiber $X_q = f^{-1}(q)$. Here D denotes the period domain of the polarized Hodge structures on $H = H^n_{pr}(X_q, \mathbb{C})$.

Let $G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$ be the automorphism group of the integral primitive cohomology $H_{\mathbb{Z}} = H^n_{pr}(X_q, \mathbb{Z})$ preserving the Poincaré bilinear form Q. Recall that the monodromy group $\Gamma_0 \subseteq G_{\mathbb{Z}}$ is the image of the monodromy representation in $G_{\mathbb{Z}}$ of the fundamental group of S_0.

Let $\overline{S_0}$ be a compactification of S_0 such that $\overline{S_0}$ is a projective manifold, and $\overline{S_0} \setminus S_0$ is a divisor with simple normal crossings. Let $S'_0 \supseteq S_0$ be the maximal subset of $\overline{S_0}$ to which the period map $\Phi_0 : S_0 \to \Gamma_0 \backslash D$ extends and let

$$\Phi'_0 : S'_0 \to \Gamma_0 \backslash D$$

be the extended period map as given by Griffiths [9], which, following [30], we call the Griffiths extension of the period map Φ_0. Then one has the

commutative diagram

$$\Phi_0$$
$$S_0 \xrightarrow{\ i\ } S_0' \xrightarrow{\ \Phi_0'\ } \Gamma_0 \backslash D.$$

with $i : S_0 \to S_0'$ the inclusion map. Griffiths in [9] proved that Φ_0' is a proper map, therefore $\Phi_0'(S_0')$ is a closed analytic variety with

$$\Phi_0(S_0) \subset \Phi_0'(S_0')$$

a Zariski open subset. He conjectured that $\Phi_0(S_0)$ is an algebraic variety in [10].

The following theorem, which is the main result in this paper, confirms this conjecture.

Theorem 0.1. *The images $\Phi_0(S_0)$ and $\Phi_0'(S_0')$ are algebraic varieties, more precisely they are quasi-projective.*

We remark that there are some partial results by Sommese in [29] and [30] related to this conjecture.

As a standard method, we first lift the period map to a finite cover S of S_0, such that the lifted period map

$$\Phi : S \to \Gamma \backslash D$$

has torsion-free monodromy group $\Gamma \subset \Gamma_0$ and the Griffiths extension

$$\Phi' : S' \to \Gamma \backslash D$$

is locally liftable. See Lemma IV-A in [30] and its proof for details about this construction. Let $\mathcal{T}'$ be the universal cover of S' to which we can lift the period map to get a holomorphic map

$$\widetilde{\Phi}' : \mathcal{T}' \to D.$$

Our method to prove this conjecture is to make substantial uses of the generalized Riemann existence theorem of Grothendieck which is reviewed in the appendix of this paper, the boundedness in certain complex Euclidean

space of the image of $\widetilde{\Phi}'$ as proved in [21], together with the geometric structure of the period domain and the Griffiths transversality of the period map.

We first state the following generalized Riemann existence theorem of Grothendieck which will be repeatedly used in our arguments.

Generalized Riemann existence theorem: *Let X be a quasi-projective variety. Let Y be a complex analytic space, together with a finite étale cover $f : Y \to X$. Then there is a unique algebraic structure on Y such that $f : Y \to X$ is a morphism of algebraic varieties.*

Recall that finite étale cover, or finite étale map, is a notion in algebraic geometry. In complex analytic geometry it corresponds to a finite and surjective holomorphic map between complex analytic varieties which is locally biholomorphic. As defined in [8], a finite holomorphic map between two complex analytic varieties is a proper holomorphic map with finite fibers. In the appendix of this paper, we list several different versions of the generalized Riemann existence theorems.

Although not used in this paper, a slightly stronger version of the generalized Riemann existence theorem, together with a sketch of its proof, is also given in the appendix of this paper, in which the map f is only required to be a finite holomorphic map which is weaker than to be finite étale.

The following is an outline of our proof. First recall that the period domain D and its compact dual $\check{D}$ can be realized as quotients of real Lie groups $D = G_{\mathbb{R}}/V$ and complex Lie groups $\check{D} = G_{\mathbb{C}}/B$ respectively, where the compact subgroup $V = B \cap G_{\mathbb{R}}$.

The Hodge structure at a fixed point o in D induces a Hodge structure of weight zero on the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ as

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} | \; XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}, \; \forall \; r\}$$

where $H_p^{i,n-i} = H_{pr}^{i,n-i}(X_p)$ for $0 \leq i \leq n$. See Section 1 for the detail of the above notations. Then the Lie algebra of B is $\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}$ and the corresponding holomorphic tangent spaces

$$\mathrm{T}_o^{1,0}\check{D} \simeq \mathrm{T}_o^{1,0}D$$

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of $\check{D}$ and D at the base point $o \in D$ are naturally isomorphic to

$$\mathfrak{g}/\mathfrak{b} \simeq \oplus_{k \geq 1} \mathfrak{g}^{-k,k} \triangleq \mathfrak{n}_+.$$

Clearly we can identify the nilpotent Lie subalgebra $\mathfrak{n}_+$ to the complex Euclidean space $\mathrm{T}_o^{1,0}D$ with induced inner product from the homogenous metric on D at o . We denote the corresponding unipotent group by

$$N_+ = \exp(\mathfrak{n}_+)$$

which is considered as a complex Euclidean space with induced Euclidean metric from $\mathfrak{n}_+$. Since $N_+ \cap B = \{\mathrm{Id}\}$, we can identify the unipotent group $N_+ \subseteq G_{\mathbb{C}}$ to its orbit $N_+(o) \subseteq \check{D}$ so that the notation $N_+ \subseteq \check{D}$ is meaningful in this sense.

Next we consider the subspace of $\mathfrak{n}_+$,

$$\mathfrak{p}_+ = \oplus_{k \text{ odd}, \, k \geq 1} \mathfrak{g}^{-k,k} \subset \mathfrak{n}_+.$$

Note that $\mathfrak{p}_+$ can be considered as the horizontal tangent subspace at the base point o of the natural projection

$$\pi : D \to G_{\mathbb{R}}/K$$

as discussed in page 261 of [12]. Let

$$\exp(\mathfrak{p}_+) \subseteq N_+$$

which is considered as a complex Euclidean subspace of N_+, and

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

be the induced projection map. In [21], by using the method of Harish-Chandra in proving his embedding theorem of Hermitian symmetric spaces as bounded symmetric domains in complex Euclidean spaces, we proved that the intersection

$$\exp(\mathfrak{p}_+) \cap D$$

is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+) \subseteq N_+$ with respect to the induced Euclidean metric.

Since the subset $D \setminus (N_+ \cap D) = D \cap (\check{D} \setminus N_+)$ is a proper analytic subvariety of D, the projection P_+ extends to a holomorphic map

$$P_+ : D \to \exp(\mathfrak{p}_+) \cap D.$$

Let us take any discrete subgroup $\Pi \subseteq G_{\mathbb{R}}$ which acts on D from the left. As discussed in [9], page 156, the quotient space $\Pi \backslash D$ is a complex analytic orbifold, since the action of Π is properly discontinuous.

We will show that there is an induced holomorphic action of any subgroup $\Pi \subset G_{\mathbb{R}}$ on $\exp(\mathfrak{p}_+) \cap D \subseteq D$, such that P_+ is equivariant with respect to the action of Π. So for any discrete subgroup $\Pi \subset G_{\mathbb{R}}$, the projection P_+ descends to a holomorphic map on the quotient spaces,

$$P_+^{\Pi} : \Pi \backslash D \to \Pi \backslash (\exp(\mathfrak{p}_+) \cap D).$$

In this paper, we will use the cases when the group Π is Γ, Γ_0, Λ and $G_{\mathbb{Z}}$, where Γ and Λ are certain normal and torsion-free subgroups of finite index of Γ_0 and $G_{\mathbb{Z}}$ respectively, as well as $G_{\mathbb{R}}$ and certain uniform discrete subgroup in $G_{\mathbb{R}}$.

Consider the natural projection maps

$$q_0 : \Gamma_0 \backslash D \to G_{\mathbb{Z}} \backslash D \quad \text{and} \quad q' : \Gamma \backslash D \to G_{\mathbb{Z}} \backslash D.$$

Define the period maps

$$\Psi_0 : S_0 \to G_{\mathbb{Z}} \backslash D \quad \text{and} \quad \Psi : S \to G_{\mathbb{Z}} \backslash D,$$

by composing the period maps Φ_0 and Φ with q_0 and q' respectively,

$$\Psi_0 = q_0 \circ \Phi_0 \quad \text{and} \quad \Psi = q' \circ \Phi.$$

Let

$$\Psi_0' : S_0' \to G_{\mathbb{Z}} \backslash D \quad \text{and} \quad \Psi' : S' \to G_{\mathbb{Z}} \backslash D.$$

be their corresponding Griffiths extensions. Note that the extended period map $\Psi_0' : S_0' \to G_{\mathbb{Z}} \backslash D$ is not necessarily locally liftable. As a standard method, we apply the construction as in Lemma IV-A and its proof of [30], to take a normal and torsion-free subgroup Λ of finite index in $G_{\mathbb{Z}}$, such that

the period maps Ψ_Λ and Ψ'_Λ from S and S' respectively fit into the following commutative diagrams

$$
\begin{array}{ccccc}
S & \xrightarrow{\;i\;} & S' & \xrightarrow{\;\Psi'_\Lambda\;} & \Lambda\backslash D \\
\downarrow & & \downarrow & & \downarrow \\
S_0 & \xrightarrow{\;i\;} & S'_0 & \xrightarrow{\;\Psi'_0\;} & G_{\mathbb{Z}}\backslash D.
\end{array}
$$

Our first step is to show that the induced projection map P_+^Λ restricted to the image of the corresponding period map, is a finite étale cover, and then to apply the generalized Riemann existence theorem of Grothendieck.

In fact, as proved in [21] and explained above, $\exp(\mathfrak{p}_+)\cap D$ is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+)$. Therefore we can take a torsion-free discrete subgroup Σ' in $G_{\mathbb{R}}$ such that $\Sigma'\backslash G_{\mathbb{R}}$ is compact. Then the quotient space

$$\Sigma'\backslash(\exp(\mathfrak{p}_+)\cap D)$$

is compact. See, for examples, page 163 of [9], or Theorem 2 of [20] and the main results in [2] and [25] for the existence of such a uniform discrete subgroup of $G_{\mathbb{R}}$.

From the theorem of Siegel or its extension by Bailey [1], [24], we deduce that the quotient space, $\Sigma'\backslash(\exp(\mathfrak{p}_+)\cap D)$ is a projective manifold with ample canonical line bundle, and that $\exp(\mathfrak{p}_+)\cap D$ is a bounded domain of holomorphy.

Moreover from [24] and [33], we know that the Bergman metric on the domain $\exp(\mathfrak{p}_+)\cap D$ is a complete canonical metric and invariant under the group of automorphisms of the bounded domain $\exp(\mathfrak{p}_+)\cap D$, which contains the Lie group $G_{\mathbb{R}}$. Furthermore, let Λ be a normal and torsion-free subgroup of finite index in $G_{\mathbb{Z}}$ which is an arithmetic subgroup of $G_{\mathbb{R}}$. By using the $G_{\mathbb{R}}$-invariance of the Bergman metric, we will prove that the volume of the quotient manifold

$$\Lambda\backslash(\exp(\mathfrak{p}_+)\cap D)$$

with the Bergman metric is finite. From Corollary 2 in [33], we deduce that $\Lambda\backslash(\exp(\mathfrak{p}_+)\cap D)$ is quasi-projective.

Next we prove that the restriction of the projection map P_+^Λ to the image

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$\Psi'_\Lambda(S')$,
$$P_+^\Lambda : \ \Psi'_\Lambda(S') \to P_+^\Lambda(\Psi'_\Lambda(S')) \subset \Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$$

is a finite étale cover. Therefore $P_+^\Lambda(\Psi'_\Lambda(S'))$ is a closed analytic subset of the quasi-projective variety $\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$, and hence it is quasi-projective. By applying the generalized Riemann existence theorem, we deduce that $\Psi'_\Lambda(S')$ is also quasi-projective.

Then we consider the normal and torsion-free subgroup of finite index Γ in Γ_0, where $\Gamma = \Gamma_0 \cap \Lambda$. Let

$$q : \Gamma\backslash D \to \Lambda\backslash D$$

be the natural projection map. It is easy to see that q is a covering map. We prove that the restriction of q,

$$q : \ \Phi'(S') \to q(\Phi'(S')) = \Psi'_\Lambda(S')$$

is a finite étale cover by a direct application of the properness of the extended period maps as proved by Griffiths in [9]. By applying the generalized Riemann existence theorem again, we conclude that $\Phi'(S')$ is quasi-projective, and q is a morphism of algebraic varieties. Since $\Phi(S)$ is a Zariski open subset of $\Phi'(S')$, we have that $\Phi(S)$ is quasi-projective.

Finally we consider general monodromy group and the corresponding period map $\Phi_0 : S_0 \to \Gamma_0\backslash D$, and its Griffiths extension

$$\Phi'_0 : \ S'_0 \to \Gamma_0\backslash D.$$

It is easy to show that $\Phi'_0(S'_0)$ is the quotient of the quasi-projective variety $\Phi'(S')$ by the finite quotient group $\Gamma\backslash\Gamma_0$. From Corollary 3.46 in [31] which asserts that the quotient of quasi-projective variety by a finite group is quasi-projective, we get the algebraicity of $\Phi'_0(S'_0)$. Since $\Phi_0(S_0)$ is a Zariski open subvariety of $\Phi'_0(S'_0)$, it is quasi-projective, which proves Theorem 0.1.

From our proof one can see that an ample line bundle on $\Phi'_0(S'_0)$ and $\Phi_0(S_0)$ is induced by the canonical line bundle of $\exp(\mathfrak{p}_+) \cap D$ which is invariant under the action of its automorphism groups. Indeed, from [24], [1] and [33], we see that the projective embeddings of the quasi-projective manifold $\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$ and the quasi-projective variety $G_\mathbb{Z}\backslash(\exp(\mathfrak{p}_+) \cap D)$ are given by multiples of their canonical line bundles.

This paper is organized as follows. After reviewing the basic facts from variation of Hodge structure, period map and period domain in Section 1, we recall, in Section 2, some results from [21], in particular the boundedness of the image of the lifted period map

$$\widetilde{\Phi}' : \ \mathcal{T}' \to N_+ \cap D$$

in the complex Euclidean space N_+, as well as the boundedness of $\exp(\mathfrak{p}_+) \cap D$ which was proved by following Harish-Chandra's argument to prove that any Hermitian symmetric space can be embedded in complex Euclidean space as a bounded symmetric domain.

In Section 3 we define the holomorphic action of any subgroup of $G_{\mathbb{R}}$ on $\exp(\mathfrak{p}_+) \cap D$ and prove the equivariance of the projection P_+ with respect to the actions on D, and discuss several natural properties of the action. Section 4 contains applications of the famous Siegel theorem about quotient spaces of bounded domains and its generalizations. Several basic results about quotient spaces and maps of discrete subgroup actions we defined is proved in Section 5. In Section 6 we prove the quasi-projectivity of the images of the period maps when the monodromy group is either Λ or $G_{\mathbb{Z}}$. In Section 7 and Section 8 we prove the quasi-projectivity of the images of the period maps for torsion-free and general monodromy groups respectively.

In the appendix, for reader's convenience we collect several versions of the generalized Riemann existence theorem.

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1   Period domain and period map

In this section we briefly review some basic facts of the period domain and period maps which are needed for our discussions. Most of these materials can be found in [3], [10] or [11].

Let $H_{\mathbb{Z}}$ be a fixed lattice and $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification. Let n be a positive integer, and Q a bilinear form on $H_{\mathbb{Z}}$ which is symmetric if n is even and skew-symmetric if n is odd. Let $h^{i,n-i}$, $0 \le i \le n$, be integers such that $\sum_{i=0}^{n} h^{i,n-i} = \dim_{\mathbb{C}} H$.

The period domain D is the set of all the collections of the filtrations $H = F^0 \supset F^1 \supset \cdots \supset F^n$, such that

$$\dim_{\mathbb{C}} F^i = f^i, \tag{1}$$
$$H = F^i \oplus \overline{F^{n-i+1}}, \text{ for } 0 \leq i \leq n,$$

where $f^i = h^{n,0} + \cdots h^{i,n-i}$, and on which Q satisfies the Hodge-Riemann bilinear relations in the form of Hodge filtrations

$$Q\left(F^i, F^{n-i+1}\right) = 0; \tag{2}$$
$$Q\left(Cv, \overline{v}\right) > 0 \text{ if } v \neq 0, \tag{3}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in F^k \cap \overline{F^{n-k}}$.

Let (X, L) be a polarized manifold with $\dim_{\mathbb{C}} X = n$, which means that X is a projective manifold and L is an ample line bundle on X. For simplicity we use the same notation L to denote the first Chern class of L which acts on the cohomology groups by wedge product. Then the n-th primitive cohomology groups $H^n_{pr}(X, \mathbb{C})$ of X is defined by

$$H^n_{pr}(X, \mathbb{C}) = \ker\{L : H^n(X, \mathbb{C}) \to H^{n+2}(X, \mathbb{C})\}.$$

Given an algebraic family $f : \mathfrak{X} \to S_0$ of polarized algebraic manifolds over a quasi-projective base manifold S_0, we can define a period map

$$\Phi_0 : S_0 \to \Gamma_0 \backslash D,$$

such that for any $q \in S_0$, the point $\Phi_0(q)$, modulo certain action of the monodromy group Γ_0, represents the Hodge structure of the n-th primitive cohomology group $H^n_{pr}(X_q, \mathbb{C})$ of the fiber $X_q = f^{-1}(q)$. Here

$$H \simeq H^n_{pr}(X_q, \mathbb{C})$$

for any $q \in S_0$. Recall that the monodromy group Γ_0, or global monodromy group, is the image of the monodromy representation of $\pi_1(S_0)$ in

$$G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q),$$

the automorphism group of

$$H_{\mathbb{Z}} \simeq H^n_{pr}(X_q, \mathbb{Z})$$

preserving Q.

Since a period map is locally liftable, we can lift the period map to

$$\widetilde{\Phi} : \mathcal{T} \to D$$

by taking the universal cover $\mathcal{T}$ of S_0 such that the diagram

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \widetilde{\Phi}\ } & D \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
S_0 & \xrightarrow{\ \Phi\ } & \Gamma_0 \backslash D
\end{array}
\qquad (4)
$$

is commutative.

Now we fix a point p in $\mathcal{T}$ and its image $o = \widetilde{\Phi}(p)$ as the reference points or base points, and denote the Hodge decomposition corresponding to the point $o = \widetilde{\Phi}(p)$ as

$$H^n_{pr}(X_p, \mathbb{C}) = H^{n,0}_p \oplus H^{n-1,1}_p \oplus \cdots \oplus H^{1,n-1}_p \oplus H^{0,n}_p,$$

where $H^{i,n-i}_p = H^{i,n-i}_{pr}(X_p)$ for $0 \le i \le n$, and the Hodge filtration by

$$H^n_{pr}(X_p, \mathbb{C}) = F^0_p \supset F^1_p \supset \cdots \supset F^n_p$$

with $F^i_p = H^{n,0}_p \oplus \cdots \oplus H^{i,n-i}_p$ for $0 \le i \le n$.

Let $H_{\mathbb{F}} = H^n_{pr}(X, \mathbb{F})$, where $\mathbb{F}$ can be chosen as $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Then $H = H_{\mathbb{C}}$ under this notation. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

the corresponding real Lie group

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

and the discrete subgroups

$$G_{\mathbb{Q}} = \{g \in GL(H_{\mathbb{Q}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{Q}}\}.$$

$$G_{\mathbb{Z}} = \{g \in GL(H_{\mathbb{Z}}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{Z}}\}.$$

The Lie group $G_{\mathbb{C}}$ acts on $\check{D}$ transitively, so does $G_{\mathbb{R}}$ on D. The stabilizer of $G_{\mathbb{C}}$ on $\check{D}$ at the base point o is

$$B = \{g \in G_{\mathbb{C}} | g F_p^k = F_p^k, \ 0 \le k \le n\},$$

and the one of $G_{\mathbb{R}}$ on D is $V = B \cap G_{\mathbb{R}}$. Thus we can realize $\check{D}$, D as

$$\check{D} = G_{\mathbb{C}}/B, \text{ and } D = G_{\mathbb{R}}/V,$$

so that $\check{D}$ is an algebraic manifold and $D \subseteq \check{D}$ is an open complex submanifold.

The natural projection

$$\pi : \ D = G_{\mathbb{R}}/V \to G_{\mathbb{R}}/K,$$

which is not holomorphic in general, will be frequently used in the following sections. The period domain is called non-classical if one of the following two cases holds: (i) $G_{\mathbb{R}}/K$ is not Hermitian symmetric; (ii) the natural projection π is not holomorphic. See [4], for example, for details of classical and non-classical period domains.

Note that from the definition of the homogeneous metric on D as given in [12] which we call the Hodge metric, π is a Riemannian submersion with respect to the natural homogeneous metrics on D and $G_{\mathbb{R}}/K$. See Section 2 in [19] for more discussions about this.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ is

$$\mathfrak{g} = \{X \in \text{End}(H_{\mathbb{C}}) | \ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real subalgebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} | \ X H_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$$

is the Lie algebra of $G_{\mathbb{R}}$. Note that $\mathfrak{g}$ is a simple complex Lie algebra and contains $\mathfrak{g}_0$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$.

Let $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then $\mathfrak{g}_c$ is also a real form of $\mathfrak{g}$. Let us denote the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_c$ by τ_c, and the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$ by τ_0.

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On the Lie algebra $\mathfrak{g}$ we can put a Hodge structure of weight zero by

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} \mid X H_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}, \ \forall r\}.$$

By the definition of B, the Lie algebra $\mathfrak{b}$ of B has the form $\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}$. Then the Lie algebra $\mathfrak{v}_0$ of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra

$$\mathfrak{n}_+ := \oplus_{k \geq 1} \mathfrak{g}^{-k,k}.$$

Then one gets the isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. Let us denote the corresponding unipotent Lie group to be

$$N_+ = \exp(\mathfrak{n}_+).$$

We remark that the elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; the elements in B can be realized as nonsingular block upper triangular matrices. If $c, c' \in N_+$ such that $cB = c'B$ in $\check{D}$, then

$$c'^{-1}c \in N_+ \cap B = \{I\},$$

i.e. $c = c'$. This means that the matrix representation in N_+ of the unipotent orbit $N_+(o)$ is unique. Therefore with the fixed base point $o \in \check{D}$, we can identify N_+ with its unipotent orbit $N_+(o)$ in $\check{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$. Therefore our notation $N_+ \subseteq \check{D}$ is meaningful. In particular, when the base point o is in D, we have $N_+ \cap D \subseteq D$.

The horizontal tangent subbundle $\mathrm{T}_h^{1,0}D$ can be constructed as the associated bundle of the principle bundle

$$V \to G_{\mathbb{R}} \to D$$

with the adjoint representation of V on the space $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$. Let $\mathscr{F}^k$, $0 \leq k \leq n$ be the Hodge bundles on D with fibers $\mathscr{F}_s^k = F_s^k$ for any $s \in D$.

As another interpretation of the horizontal bundle in terms of the Hodge bundles $\mathscr{F}^k \to \check{D}$, $0 \leq k \leq n$, one has

$$\mathrm{T}_h^{1,0}\check{D} \simeq \mathrm{T}^{1,0}\check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(\mathscr{F}^k/\mathscr{F}^{k+1}, \mathscr{F}^{k-1}/\mathscr{F}^k). \tag{5}$$

A holomorphic map $\Psi : M \to \check{D}$ of a complex manifold M into $\check{D}$ is called horizontal, if the tangent map $d\Psi : \mathrm{T}^{1,0}M \to \mathrm{T}^{1,0}\check{D}$ takes values in $\mathrm{T}_h^{1,0}\check{D}$. The period map $\widetilde{\Phi} : \mathcal{T} \to D$ is horizontal due to Griffiths transversality.

Let us come back to the setup in [10]. By Hironaka's resolution of singularity theorem, we know that S_0 admits a compactification $\overline{S_0}$, such that $\overline{S_0}$ is smooth and projective, and $\overline{S_0} \setminus S_0$ is a divisor with simple normal crossings. Let $S_0' \supseteq S_0$ be the maximal subset of $\overline{S_0}$ to which the period map $\Phi_0 : S_0 \to \Gamma_0 \backslash D$ extends, and let

$$\Phi_0' : S_0' \to \Gamma_0 \backslash D$$

be the extended map introduced by Griffiths [9]. We will call Φ_0' the Griffiths extension of the period map Φ_0.

Then one has the commutative diagram

$$\begin{array}{ccc} & \Phi_0 & \\ S_0 \xrightarrow{\ i\ } & S_0' \xrightarrow{\ \Phi_0'\ } & \Gamma_0 \backslash D. \end{array}$$

with $i : S_0 \to S_0'$ the inclusion map. The following result is due to Griffiths, see [9], Theorem 9.6 and Proposition 9.11, in which he also proved that $\Phi_0'(S_0')$ is an analytic variety. See also Lemma 1.4 in [21].

Lemma 1.1. *S_0' is a Zariski open subset of $\overline{S_0}$ with the complex codimension of $\overline{S_0} \setminus S_0'$ at least one, such that $S_0' \setminus S_0$ contains the points in $\overline{S_0} \setminus S_0$, around which the Picard-Lefschetz transformations are finite. Moreover the extended period map $\Phi_0' : S_0' \to \Gamma_0 \backslash D$ is a proper holomorphic map.*

Note that in this case, the proper map Φ_0' is not necessarily locally liftable. To go further we need the construction as given in Lemma IV-A and its proof of [30] which is standard now in studying period maps.

Let $\Gamma \subseteq \Gamma_0$ be a normal and torsion-free subgroup of the monodromy group of Γ_0, which is of finite index in Γ_0. In fact, Γ can be taken as the intersection $\Gamma_0 \cap \Lambda$, where $\Lambda \subseteq G_\mathbb{Z}$ is a normal and torsion-free subgroup of finite index which will be introduced in Section 3.

By Lemma IV-A in [30], we have a finite cover S of S_0, where S is also quasi-projective, such that the period map Φ_0 lifts to a period map

$$\Phi : S \to \Gamma \backslash D,$$

which satisfies the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \Gamma \backslash D \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow{\Phi_0} & \Gamma_0 \backslash D. \end{array}$$

Moreover, there exists a smooth projective manifold $\overline{S}$ with a holomorphic surjection $\overline{\pi} : \overline{S} \to \overline{S_0}$, and a Zariski open subset $S' \subset \overline{S}$ with $\overline{\pi}$ restricting to a proper map $\overline{\pi} : S' \to S'_0$, such that the period map Φ extends to a proper holomorphic map

$$\Phi' : S' \to \Gamma \backslash D,$$

which satisfies the following commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & \Gamma \backslash D \\ \downarrow & & \downarrow & & \downarrow \\ S_0 & \xrightarrow{i} & S'_0 & \xrightarrow{\Phi'_0} & \Gamma_0 \backslash D. \end{array}$$

In fact, S and S' can be considered as the corresponding fiber products constructed from the covering

$$r_\Gamma : \Gamma \backslash D \to \Gamma_0 \backslash D$$

and the corresponding period maps,

$$\Phi_0 : S_0 \to \Gamma_0 \backslash D, \text{ and } \Phi'_0 : S'_0 \to \Gamma_0 \backslash D$$

respectively.

Since Γ is torsion-free, we have the following corollary of Lemma 1.1.

Corollary 1.2. *S' is a Zariski open subset of $\overline{S}$ with the complex codimension of $\overline{S} \setminus S'$ at least one, such that $S' \setminus S$ contains the points in $\overline{S} \setminus S$, around which the Picard-Lefschetz transformations are trivial . Moreover the extended period map $\Phi' : S' \to \Gamma \backslash D$ is a proper holomorphic map.*

Note that in this case, since Γ is torsion-free, the extended period map Φ' is proper and locally liftable, furthermore it is easy to show that it is still horizontal as proved in Lemma 1.6 of [21], hence it is also a period map. Let $\mathcal{T}'$ be the universal cover of S' with the covering map $\pi' : \mathcal{T}' \to S'$. We then have the following commutative diagram

$$
\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}' & \xrightarrow{\widetilde{\Phi}'} & D \\
\downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & \Gamma \backslash D,
\end{array}
\tag{6}
$$

where $i_{\mathcal{T}}$ is the lifting of $i \circ \pi_S$ with respect to the covering map

$$
\pi' : \mathcal{T}' \to S'
$$

and $\widetilde{\Phi}'$ is the lifting of $\Phi' \circ \pi'$ with respect to the covering map

$$
\pi : D \to \Gamma \backslash D.
$$

There are different choices of $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$, it is shown in [21], as a elementary topological fact, that we can choose $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that $\widetilde{\Phi} = \widetilde{\Phi}' \circ i_{\mathcal{T}}$, and that i_T and $\widetilde{\Phi}'$ are holomorphic maps.

Without loss of generality, we can assume that S_0 and S are irreducible. From the basic results a) and b) in page 171 of the book of Grauert-Remmert [8], we know that as analytic varieties, S_0, $\mathcal{T}_0$, S, S', $\mathcal{T}$, $\mathcal{T}'$ and their images under the corresponding period maps, as well as other complex varieties we will use in the following sections are all irreducible.

2 Boundedness of the image of the period map

In this section we review the result about the boundedness of the open set $\exp(\mathfrak{p}_+) \cap D$ in the complex Euclidean space $\exp(\mathfrak{p}_+)$. This boundedness

result is proved in [21] by using the argument of Harish-Chandra in his proof of the embedding theorem of Hermitian symmetric spaces as bounded symmetric domains inside complex Euclidean spaces. Some related results about the geometric structure of the period domain and the projection map P_+ we defined will also be discussed.

Recall that we have fixed the base points $p \in \mathcal{T}$ and $o = \widetilde{\Phi}(p) \in D$. Then N_+ can be viewed as a subset in $\check{D}$ by identifying it to its orbit $N_+(o)$ in $\check{D}$. Lemma 1.2 in [21] proves that N_+ is Zariski open in $\check{D}$.

At the base point $\widetilde{\Phi}(p) = o \in N_+ \cap D$, the tangent spaces satisfy

$$\mathrm{T}_o^{1,0} N_+ = \mathrm{T}_o^{1,0} D \simeq \mathfrak{n}_+,$$

and the exponential map

$$\exp : \mathfrak{n}_+ \to N_+$$

is an isomorphism. There is a natural homogeneous metric on D induced from the Killing form as studied in [12], which we will call the Hodge metric. The Hodge metric on $\mathrm{T}_o^{1,0} D$ induces an Euclidean metric on N_+ such that the map $\exp : \mathfrak{n}_+ \to N_+$ is an isometry.

Let

$$\mathfrak{p}_+ = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}) = \mathfrak{p} \cap \mathfrak{n}_+ \subseteq \mathfrak{n}_+$$

denote a subspace of $\mathrm{T}_o^{1,0} D \simeq \mathfrak{n}_+$, and $\mathfrak{p}_+$ can be viewed as an Euclidean subspace of $\mathfrak{n}_+$. Indeed we have

$$\mathfrak{p}_+ = \oplus_{k \text{ odd}, k \geq 1} \mathfrak{g}^{-k,k} \subset \mathfrak{n}_+ = \oplus_{k \geq 1} \mathfrak{g}^{-k,k}.$$

Similarly $\exp(\mathfrak{p}_+)$ can be viewed as an Euclidean subspace of N_+ with the induced metric from N_+.

Define the projection map

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

by

$$P_+ = \exp \circ p_+ \circ \exp^{-1} \tag{7}$$

where $\exp^{-1} : N_+ \to \mathfrak{n}_+$ is the inverse of the isometry $\exp : \mathfrak{n}_+ \to N_+$, and

$$p_+ : \mathfrak{n}_+ \to \mathfrak{p}_+$$

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is the projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $\mathfrak{p}_+$.

The following result was proved in [21] as Corollary 4.3 by using the argument of Harish-Chandra to prove his famous embedding theorem of Hermitian symmetric spaces as bounded symmetric domains in complex Euclidean spaces.

Lemma 2.1. *The complex manifold* $\exp(\mathfrak{p}_+) \cap D$ *is a bounded domain in* $\exp(\mathfrak{p}_+)$ *with respect to the Euclidean metric on* $\exp(\mathfrak{p}_+) \subseteq N_+$.

In fact, by the definition of $\exp(\mathfrak{p}_+) \cap D$, for any $s \in \exp(\mathfrak{p}_+) \cap D$ there is a unique $X \in \mathfrak{p}_0$, such that $\exp(X) \cdot \overline{o} = s$, as well as a unique $Y \in \mathfrak{p}_+$ such that $\exp(Y) \cdot \overline{o} = s$, where $\overline{o} = P_+(o)$ is the base point in $\exp(\mathfrak{p}_+) \cap D$. Here $\exp(X) \cdot \overline{o}$ and $\exp(Y) \cdot \overline{o}$ denote the corresponding left translations of the base point $\overline{o}$. We refer the reader to [21] for details.

Indeed, we have,

$$X = T_0(Y + \tau_0(Y))$$

for certain real number T_0, which gives one-to-one correspondence from $\mathfrak{p}_0$ into $\mathfrak{p}_+$, and therefore induces a diffeomorphism

$$\exp(\mathfrak{p}_0) \to \exp(\mathfrak{p}_+) \cap D.$$

More precisely, after fixing the base point $\overline{o}$ in $\exp(\mathfrak{p}_+) \cap D$, and viewing $\exp(X) \in \exp(\mathfrak{p}_0)$ as a point in

$$\exp(\mathfrak{p}_+) \subset \check{D} = G_{\mathbb{C}}/B,$$

we have

$$\exp(X) = \exp(Y)(\operatorname{mod} B)$$

as in the explicit computations in Harish-Chandra's proof of his embedding theorem. This is equivalent to the statement that

$$\exp(X) \cdot \overline{o} = \exp(Y) \cdot \overline{o} = s$$

in $\exp(\mathfrak{p}_+) \cap D$.

For a more detailed description of the above explicit correspondence from $\mathfrak{p}_0$ to $\mathfrak{p}_+$ in proving the Harish-Chandra embedding, please see Lemma 7.11 in

pages 390 – 391 in [17], page 95-97 of [23], or the discussion in pages 463-466 in [32].

Consider the projection map

$$\pi : D \to G_{\mathbb{R}}/K.$$

We recall the proof of the following lemma from [21],

Lemma 2.2. *When restricted to* $\exp(\mathfrak{p}_+) \cap D \subseteq D$, π *is given by the diffeomorphism*

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\simeq} G_{\mathbb{R}}/K,$$

and the diagram

$$
\begin{array}{ccc}
N_+ \cap D & \xrightarrow{\ \pi\ } & G_{\mathbb{R}}/K \\
\ \downarrow{\scriptstyle P_+} & \nearrow{\scriptstyle \pi_+} & \\
\exp(\mathfrak{p}_+) \cap D & &
\end{array}
$$

is commutative.

Proof. As discussed above, the underlying real manifold of $\exp(\mathfrak{p}_+) \cap D$ is diffeomorphic to $G_{\mathbb{R}}/K \simeq \exp(\mathfrak{p}_0)$. This diffeomorphism is given explicitly by identifying the point $\exp(X) \in \exp(\mathfrak{p}_0)$ to the point $\exp(Y) \in \exp(\mathfrak{p}_+) \cap D$, where $X = T_0(Y + \tau_0(Y))$ for certain real number T_0.

On the other hand, since the natural projection map

$$\pi : D \to G_{\mathbb{R}}/K$$

is a Riemannian submersion, the real geodesic

$$c(t) = \exp(tX) \subset \exp(\mathfrak{p}_+) \cap D$$

with $X \in \mathfrak{p}_0$ connecting the based point o and any point $z \in \exp(\mathfrak{p}_+) \cap D$ is the horizontal lift of the geodesic $\pi(c(t))$ in $G_{\mathbb{R}}/K$. This is a basic fact in Riemmanian submersion as given in, for example, Corollary 26.12 in page 339 of [22].

Hence the natural projection $\pi : D \to G_{\mathbb{R}}/K$ maps $c(t)$ isometrically to $\pi(c(t))$. In particular π maps $\exp(X)$ in $\exp(\mathfrak{p}_+) \cap D$ to the corresponding point $\exp(X)$ in

$$G_{\mathbb{R}}/K \simeq \exp(\mathfrak{p}_0).$$

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In general we can write

$$\exp(tX) = \exp(T(t)Y)(\mathrm{mod}\, B)$$

where $T(t)$ is a monotone real valued function of t, as follows from the proof of Harish-Chandra.

From this, one sees that the projection π, when restricted to the underlying real manifold of $\exp(\mathfrak{p}_+) \cap D$, is given by the diffeomorphism

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\simeq} G_{\mathbb{R}}/K$$

with

$$\pi_+(\exp(Y)) = \exp(X).$$

From the above discussion we get that the diagram

$$
\begin{array}{ccc}
N_+ \cap D & \xrightarrow{\ \pi\ } & G_{\mathbb{R}}/K \\
{\scriptstyle P_+}\downarrow & \nearrow {\scriptstyle \pi_+} & \\
\exp(\mathfrak{p}_+) \cap D & &
\end{array}
,
$$

is commutative. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

In [21] we proved that $\widetilde{\Phi}'(\mathcal{T}') \subset N_+ \cap D$ is a closed subvariety. Furthermore we have the following proposition stated as Corollary 4.3 in [21] which follows directly from Theorem 3.6 in [21].

Proposition 2.3. *The restriction of the projection map*

$$P_+ : \widetilde{\Phi}'(\mathcal{T}') \to P_+(\widetilde{\Phi}'(\mathcal{T}'))$$

is a finite holomorphic map.

As defined in [8], a finite holomorphic map is a proper, finite, and possibly ramified covering map onto its image which is also called an analytic covering.

3   The action of the subgroups of $G_{\mathbb{R}}$

The main purpose of this section is to introduce the natural action of any subgroup $\Pi \subseteq G_{\mathbb{R}}$ on $\exp(\mathfrak{p}_+) \cap D$ which is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+)$. This construction will allow us to apply the result of Siegel that the smooth compact quotient space of a bounded domain must be a projective manifold as discussed in [20], and its various generalizations in [1], or [24] and [33] for quasi-projective quotients.

Our applications will use the cases when Π is discrete subgroup of $G_{\mathbb{R}}$, and when Π is $G_{\mathbb{R}}$ itself.

Consider any discrete subgroup $\Pi \subset G_{\mathbb{R}}$ which acts on D and on $G_{\mathbb{R}}/K$ from the left. Since Π acts on the left and both V and K act on right, we have the following commutative diagram of maps,

$$
\begin{array}{ccc}
D & \xrightarrow{\ \ p\ \ } & \Pi \backslash D \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^{\Pi}} \\
G_{\mathbb{R}}/K & \xrightarrow{\ \ p^K\ \ } & \Pi \backslash G_{\mathbb{R}}/K,
\end{array}
$$

where p and p^K denote the corresponding quotient maps, and π^{Π} is the induced quotient map from π.

We will define an induced action of Π on $\exp(\mathfrak{p}_+) \cap D$. First, we need the following lemma.

Lemma 3.1. *The projection map*

$$ P_+ : \ N_+ \cap D \to exp(\mathfrak{p}_+) \cap D $$

extends to

$$ P_+ : \ D \to exp(\mathfrak{p}_+) \cap D. $$

Proof. As proved in Lemma 1.2 and discussed in detail in the proof of Proposition 1.3 of [21], $\check{D} \setminus N_+$ is a proper analytic subvariety of $\check{D}$, so $D \setminus (N_+ \cap D) = D \cap (\check{D} \setminus N_+)$ is a proper analytic subvariety of D.

Since $\exp(\mathfrak{p}_+) \cap D \subseteq \exp(\mathfrak{p}_+)$ is a bounded domain in complex Euclidean space as proved in Lemma 2.1, from the Riemann extension theorem we get the extension of P_+. $\qquad\square$

With the above lemma, we can define the action of any subgroup $\Pi \subset G_{\mathbb{R}}$ on $\exp(\mathfrak{p}_+) \cap D$ as follows:

Definition 3.2. *We define the induced Π-action on $exp(\mathfrak{p}_+) \cap D$ to be*

$$\widetilde{\gamma} z = P_+(\gamma z),$$

for any $z \in exp(\mathfrak{p}_+) \cap D$ and $\gamma \in \Pi$.

From the definition, it is clear that the action defined above is a holomorphic action. We will show that the projection map P_+ is equivariant with respect to the Π-action on D and the above defined action on $\exp(\mathfrak{p}_+) \cap D$. More precisely we have the following lemma,

Lemma 3.3. *The map $P_+ : D \to exp(\mathfrak{p}_+) \cap D$ is equivariant with respect to the Π-actions on D and on $exp(\mathfrak{p}_+) \cap D$, i.e.*

$$P_+(\gamma z) = \widetilde{\gamma} P_+(z),$$

for any $z \in D$ and $\gamma \in \Pi$.

Proof. We only need to show that, for any $z \in D$,

$$P_+(\gamma z) = P_+(\gamma P_+(z)) = \widetilde{\gamma} P_+(z).$$

The second identity is by definition of $\widetilde{\gamma}$, while for the first identity, we note that

$$\pi(P_+(\gamma z)) = \pi(\gamma z) = \gamma \pi(z) = \gamma \pi(P_+(z)) = \pi(\gamma P_+(z)) = \pi(P_+(\gamma P_+(z)))$$

by the commutativity of the projection

$$\pi : D \to G_{\mathbb{R}}/K$$

with the Π-action on D and on $G_{\mathbb{R}}/K$, and the commutativity of P_+ and π as given in Lemma 2.2.

This implies that z and $P_+(z)$ lie in the same fiber $\pi^{-1}(z')$ where $z' = \pi(z)$, and γz and $P_+(\gamma z)$ also lie in the same fiber

$$\pi^{-1}(\gamma z') = \gamma \pi^{-1}(z')$$

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of the projection π.

Since $P_+(\gamma z)$ and $P_+(\gamma P_+(z))$ both lie in $\exp(\mathfrak{p}_+) \cap D$ and

$$\pi = \pi_+ : \exp(\mathfrak{p}_+) \cap D \to G_\mathbb{R}/K$$

is a diffeomprphism, this gives that

$$P_+(\gamma z) = P_+(\gamma P_+(z))$$

as needed. □

This lemma implies that

$$P_+ : D \to \exp(\mathfrak{p}_+) \cap D$$

is equivariant map with respect to the Π-action, so when Π is a discrete subgroup, P_+ descends to a holomorphic map

$$P_+^\Pi : \Pi\backslash D \to \Pi\backslash(\exp(\mathfrak{p}_+) \cap D),$$

such that the following diagram is commutative

$$
\begin{array}{ccc}
D & \xrightarrow{\ P_+\ } & \exp(\mathfrak{p}_+) \cap D \\
\Big\downarrow{\scriptstyle \pi_D} & & \Big\downarrow{\scriptstyle \widetilde{p}} \\
\Pi\backslash D & \xrightarrow{\ P_+^\Pi\ } & \Pi\backslash(\exp(\mathfrak{p}_+) \cap D).
\end{array}
\tag{8}
$$

As a corollary we have the following lemma.

Lemma 3.4. *The diffeomorphism*

$$\pi_+ : exp(\mathfrak{p}_+) \cap D \to G_\mathbb{R}/K$$

is equivariant with the actions of Π. When $\Pi \subset G_\mathbb{R}$ is a torsion-free discrete subgroup, we have the following diffeomorphism of smooth manifolds,

$$\Pi\backslash(exp(\mathfrak{p}_+) \cap D) \simeq \Pi\backslash G_\mathbb{R}/K.$$

Proof. From the definition, we have, for any $z \in \exp(\mathfrak{p}_+) \cap D$,

$$\pi_+(\widetilde{\gamma}z) = \pi(P_+(\gamma z)) = \pi(\gamma z) = \gamma \pi_+(z),$$

which implies that π_+ is equivariant with respect to the Π-action on $G_{\mathbb{R}}/K$ and the induced action on $\exp(\mathfrak{p}_+) \cap D$. So we have the following commutative diagram for the Π-action on $\exp(\mathfrak{p}_+) \cap D$ and on $G_{\mathbb{R}}/K$,

$$\begin{array}{ccc} \exp(\mathfrak{p}_+) \cap D & \xrightarrow{\pi_+} & G_{\mathbb{R}}/K \\ \downarrow{\widetilde{\gamma}} & & \downarrow{\gamma} \\ \exp(\mathfrak{p}_+) \cap D & \xrightarrow{\pi_+} & G_{\mathbb{R}}/K, \end{array}$$

for any $\gamma \in \Pi$.

Note that if Π is a torsion-free discrete subgroup of $G_{\mathbb{R}}$, then its action on $\exp(\mathfrak{p}_+) \cap D$ is free, and the quotient space $\Pi \backslash (\exp(\mathfrak{p}_+) \cap D)$ is a complex manifold.

In fact, if $z \in \exp(\mathfrak{p}_+) \cap D$ is such that $\widetilde{\gamma}z = z$, then we have

$$\pi \widetilde{\gamma}(z) = \gamma \pi(z) = \pi(z)$$

which implies that $z' = \pi(z)$ is fixed by γ in $G_{\mathbb{R}}/K$ which is a contradiction to the condition that Π is torsion-free, which implies that its action on $G_{\mathbb{R}}/K$ is free.

Therefore, when Π is discrete and torsion-free, the diffeomorphism π_+ descends to a diffeomorphism of smooth manifolds,

$$\pi_+^{\Pi} : \Pi \backslash (\exp(\mathfrak{p}_+) \cap D) \to \Pi \backslash G_{\mathbb{R}}/K,$$

such that the following commutative diagram holds,

$$\begin{array}{ccc} \exp(\mathfrak{p}_+) \cap D & \xrightarrow{\pi_+} & G_{\mathbb{R}}/K \\ \downarrow{\widetilde{p}} & & \downarrow{p^K} \\ \Pi \backslash (\exp(\mathfrak{p}_+) \cap D) & \xrightarrow{\pi_+^{\Pi}} & \Pi \backslash G_{\mathbb{R}}/K \end{array}$$

where

$$p : \exp(\mathfrak{p}_+) \cap D \to \Pi \backslash (\exp(\mathfrak{p}_+) \cap D), \ \ p^K : G_{\mathbb{R}}/K \to \Pi \backslash G_{\mathbb{R}}/K$$

denote the corresponding quotient maps by the action Π. $\square$

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4 Algebraicity of quotient spaces

In this section we prove the algebraicity of some quotient spaces of

$$\exp(\mathfrak{p}_+) \cap D$$

by certain discrete subgroups of $G_{\mathbb{R}}$. Since $\exp(\mathfrak{p}_+) \cap D$ is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+)$, the results in this section can be considered as direct applications of the famous theorem of Siegel as discussed in [20] and its various generalizations as given in [1], [24], and[33].

As is well-known, we have a uniform discrete subgroup Σ of $G_{\mathbb{R}}$ such that $\Sigma \backslash G_{\mathbb{R}}$ is compact. See related discussion by Griffiths in page 163 of [9], or the main results in [2] and [25]

Following the discussion in page 167 of [20], as another well-known fact, we can take a normal and torsion-free subgroup Σ' of finite index in Σ such that $\Sigma' \backslash G_{\mathbb{R}}$ is a compact manifold. From this we see that both $\Sigma' \backslash D$ and $\Sigma' \backslash G_{\mathbb{R}} / K$ are compact manifolds. As a corollary of Lemma 3.4, we know that

$$\Sigma' \backslash (\exp(\mathfrak{p}_+) \cap D),$$

which is diffeomorphic to $\Sigma' \backslash G_{\mathbb{R}} / K$, is a compact complex manifold.

Lemma 4.1. *The quotient manifold $\Sigma' \backslash (exp(\mathfrak{p}_+) \cap D)$ is a projective manifold, and the domain $exp(\mathfrak{p}_+) \cap D$ is a bounded domain of holomorphy with complete Bergman metric in the complex Euclidean space $exp(\mathfrak{p}_+)$.*

Proof. By the famous theorem of Siegel as discussed in [1], [20] and [24], we know that a bounded domain in a complex Euclidean space covers a compact complex manifold, if and only if the domain is a bounded domain of holomorphy with complete Bergman metric, and the quotient space is a projective manifold. See also Theorem 1 in [33]. □

Note that, as pointed out in Section 2 of [24], by a famous result of Mok-Yau, there exists a complete Kähler-Einstein metric on any bounded domain of holomorphy. Both the Bergman metric and the Kähler-Einstein metric on $\exp(\mathfrak{p}_+) \cap D$ are canonical metrics, they are invariant under the action of the group of automorphisms of $\exp(\mathfrak{p}_+) \cap D$. See Section 2 of [24] and Theorem 1 of [33] for more details about this.

In the following discussion, we will use the Bergman metric g_B on the bounded domain of holomorphy, $\exp(\mathfrak{p}_+) \cap D$. See Section 3 of [24] or Theorem 1 in [33]. As mentioned above, the Bergman metric g_B on $\exp(\mathfrak{p}_+) \cap D$ is a canonical metric that is invariant under the group of automorphisms of $\exp(\mathfrak{p}_+) \cap D$ which contains $G_\mathbb{R}$ as proved in Lemma 3.4. Therefore it induces well-defined measure on the quotient spaces of $\exp(\mathfrak{p}_+) \cap D$.

With the above preparations, we can prove the following lemma. Here recall that

$$\Lambda \subset G_\mathbb{Z}$$

is a normal and torsion-free subgroup of finite index in $G_\mathbb{Z}$.

Lemma 4.2. *The quotient space $G_\mathbb{Z} \backslash (exp(\mathfrak{p}_+) \cap D)$ is a quasi-projective variety, and the quotient manifold $\Lambda \backslash (exp(\mathfrak{p}_+) \cap D)$ is a quasi-projective manifold.*

Proof. By the result of Borel and Harish-Chandra [2], see also Theorem 2 in [20], we know that the volume of the quotient space

$$\Lambda \backslash G_\mathbb{R}$$

is finite with the metric induced by the $G_\mathbb{R}$-invariant metric on $G_\mathbb{R}$. Therefore the volume of

$$\Lambda \backslash G_\mathbb{R} / K$$

is finite with the metric induced by the $G_\mathbb{R}$-invariant metric $g_{G_\mathbb{R}/K}$ on $G_\mathbb{R}/K$.

Consider the $G_\mathbb{R}$-equivariant diffeomorphism

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \to G_\mathbb{R}/K$$

in Lemma 3.4. Then the pull-back metric $(\pi_+^{-1})^* g_B$ via the $G_\mathbb{R}$-equivariant diffeomorphism π_+^{-1} is also $G_\mathbb{R}$-invariant on $G_\mathbb{R}/K$, where g_B denotes the Bergmann metric on $\exp(\mathfrak{p}_+) \cap D$.

Let v_B and $v_{G_\mathbb{R}/K}$ denote the volume forms of the corresponding $G_\mathbb{R}$-invariant metrics g_B on $\exp(\mathfrak{p}_+) \cap D$ and $g_{G_\mathbb{R}/K}$ on $G_\mathbb{R}/K$ respectively. Then we have

$$(\pi_+^{-1})^* v_B = c\, v_{G_\mathbb{R}/K}$$

on $G_\mathbb{R}/K$, for certain constant $c > 0$, which implies that

$$v_B = c\, \pi_+^*(v_{G_\mathbb{R}/K})$$

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on the quotient manifold

$$M = \Lambda\backslash(\exp(\mathfrak{p}_+) \cap D).$$

So by projection formula we have the equality,

$$\int_M v_B = c \int_M \pi_+^*(v_{G_\mathbb{R}/K}) = c \int_{G_\mathbb{R}/K} v_{G_\mathbb{R}/K}.$$

This proves that the volume of the quotient manifold $\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$ with the Bergmann metric g_B is a multiple of the volume of $\Lambda\backslash G_\mathbb{R}/K$ with the $G_\mathbb{R}$-invariant metric $g_{G_\mathbb{R}/K}$, which is finite.

Since the bounded domain $\exp(\mathfrak{p}_+) \cap D$ covers a compact projective manifold as proved in Lemma 4.1, it follows directly from Proposition 1 and Corollary 2 in [33], that the quotient manifold $\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$, which has finite volume with the Bergman metric, is quasi-projective.

From the construction, we know that the quotient group $\Lambda\backslash G_\mathbb{Z}$ is a finite group. Note that the variety

$$G_\mathbb{Z}\backslash(\exp(\mathfrak{p}_+) \cap D)$$

is the quotient of

$$\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$$

by the finite group $\Lambda\backslash G_\mathbb{Z}$. By the second part of Corollary 3.46 in [31], which asserts that the quotient of a quasi-projective variety by a finite group is still quasi-projective, we get that the quotient space $G_\mathbb{Z}\backslash(\exp(\mathfrak{p}_+) \cap D)$ is a quasi-projective variety. $\square$

From [1], [24] or [33], we know that the canonical line bundles of the quasi-projective manifold $\Lambda\backslash(\exp(\mathfrak{p}_+ \cap D))$ or the quasi-projective variety $G_\mathbb{Z}\backslash(\exp(\mathfrak{p}_+ \cap D))$ are ample, since their embeddings in projective spaces are given by sections of multiples of the corresponding canonical line bundle.

5 Topology of quotient spaces

In this section we consider the action of any torsion-free discrete subgroup $\Pi \subset G_\mathbb{R}$ and discuss some basic topological properties of the quotient spaces,

and the induced quotient map of the projection map P_+,

$$P_+^\Pi : \; \Pi \backslash D \to \Pi \backslash (\exp(\mathfrak{p}_+) \cap D)$$

between the quotient spaces, as well as the restricted map of P_+^Π on the image of the corresponding extended period map.

Most of the results should be well-known in basic topology, or are direct consequences of the geometric structure of the period domain and the Griffiths transversality, we include the proofs here for reader's convenience.

Recall that a map is called proper if the inverse image of any compact subset is compact. By definition, a finite holomorphic map in complex analytic geometry is a proper holomorphic map with finite fibers. See [8] for basic results about complex spaces and holomorphic maps between them.

Lemma 5.1. *The extended projection*

$$P_+ : \; D \to \exp(\mathfrak{p}_+) \cap D$$

is a proper map.

Proof. From Lemma 2.2, we have the commutative diagram,

$$
\begin{array}{ccc}
N_+ \cap D & \xrightarrow{\;\;\pi\;\;} & G_\mathbb{R}/K \\
{\scriptstyle P_+}\big\downarrow & \nearrow{\scriptstyle \pi_+} & \\
\exp(\mathfrak{p}_+) \cap D & &
\end{array}
\;,
$$

which induces the following commutative diagram from the extension of P_+,

$$
\begin{array}{ccc}
D & \xrightarrow{\;\;\pi\;\;} & G_\mathbb{R}/K \\
{\scriptstyle P_+}\big\downarrow & \nearrow{\scriptstyle \pi_+} & \\
\exp(\mathfrak{p}_+) \cap D & &
\end{array}
\;.
$$

Since $\pi : D \to G_\mathbb{R}/K$ is a projection of fiber bundle with compact fiber, as a basic fact in general topology as stated in Proposition 3.4 of [6], π is proper map. On the other hand, Lemma 2.2 tells that π_+ is a diffeomorphism, from which we deduce the properness of P_+.

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The fact that $\pi : D \to G_{\mathbb{R}}/K$ is proper can also be seen directly as follows. Since $G_{\mathbb{R}}/K$ is contractible, so topologically π is a trivial fiber bundle and D is diffeomorphic to a product

$$G_{\mathbb{R}}/K \times \pi^{-1}(p)$$

where p is a point in $G_{\mathbb{R}}/K$ and $K/V \simeq \pi^{-1}(p)$ denotes a fiber of π. From this, the properness of π is clear, since K/V is compact. $\qquad\square$

Before proceeding further, we first derive the following corollary from the above discussions which should be a standard result in general topology.

Corollary 5.2. *The holomorphic map*

$$P_+^{\Pi} : \Pi\backslash D \to \Pi\backslash(exp(\mathfrak{p}_+) \cap D)$$

is proper.

Proof. We deduce the properness of the projection map P_+^{Π} from the properness of P_+, as a standard fact of general topology.

In fact, let us consider the fundamental domain $F \subset \exp(\mathfrak{p}_+) \cap D$ of the Π-action, which by definition satisfies

$$\exp(\mathfrak{p}_+) \cap D = \bigcup_{\gamma \in \Pi} \gamma F,$$

and $\gamma F \neq F$ if γ is not identity in Π. By Proposition 4.22 of [18], such a fundamental domain F exists for the Π-action on $\exp(\mathfrak{p}_+) \cap D$, since $\exp(\mathfrak{p}_+) \cap D$ is complete with respect to the Bergman metric as given in Lemma 4.1, and Π acts isometrically with respect to the Bergman metric on $\exp(\mathfrak{p}_+) \cap D$.

Also by definition, the restriction of the projection map

$$\widetilde{p} : \exp(\mathfrak{p}_+) \cap D \to \Pi\backslash(\exp(\mathfrak{p}_+) \cap D)$$

to the closure $\overline{F}$ of F,

$$\widetilde{p}|_{\overline{F}} : \overline{F} \to \Pi\backslash(\exp(\mathfrak{p}_+) \cap D)$$

is surjective.

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Let E be a compact subset of $\Pi \backslash (\exp(\mathfrak{p}_+) \cap D)$, and $\widetilde{E} = (\widetilde{p}|_{\overline{F}})^{-1}(E)$ which is clearly a compact subset in $\overline{F}$. Then the preimage $\widetilde{p}^{-1}(E)$ is given by the Π-orbit

$$\widetilde{p}^{-1}(E) = \Pi(\widetilde{E}) = \bigcup_{\gamma \in \Pi} \gamma \widetilde{E}.$$

By Lemma 3.3, we have $P_+^{-1}(\gamma \widetilde{E}) = \gamma(P_+^{-1}(\widetilde{E}))$ for any $\gamma \in \Pi$. Since P_+ is proper, the preimage $P_+^{-1}(\widetilde{E})$ is compact in D. From the commutative diagram (8), we get that

$$(P_+^{\Pi})^{-1}(\widetilde{E}) = \pi_D(\bigcup_{\gamma \in \Pi} P_+^{-1}(\gamma \widetilde{E})) = \pi_D(\bigcup_{\gamma \in \Pi} \gamma P_+^{-1}(\widetilde{E})) = \pi_D(P_+^{-1}(\widetilde{E})),$$

which is compact in $\Pi \backslash D$. $\qquad \square$

Note that the argument in the proof of Corollary 5.2 actually proves the following general fact in basic topology which should be well-known.

Corollary 5.3. *Let X and Y be two complete Riemannian manifolds on which a discrete group Π acts isometrically, properly and discontinuously. Let $f : X \to Y$ be an equivariant proper map, then the induced quotient map on the quotient spaces,*

$$f^{\Pi} : \Pi \backslash X \to \Pi \backslash Y$$

is also proper.

It is interesting to describe another proof of Corollary 5.2 by using Lemma 3.4, from which we show that the properness of the projection map P_+^{Π} follows from certain basic facts in general topology as given, for example, in [6].

Indeed, as discussed above, the projection map π of the fiber bundle with compact fiber

$$\pi : D \to G_{\mathbb{R}}/K$$

is proper. Since the Π-action is properly discontinuous and isometric with respect the natural homogeneous metrics on D and $G_{\mathbb{R}}/K$, Corollary 5.3 tells us that the induced quotient map,

$$\pi_+^{\Pi} : \Pi \backslash D \to \Pi \backslash G_{\mathbb{R}}/K,$$

is proper. In fact, Proposition 3.4 in [6] applies directly to give the properness of π_+^Π.

On the other hand, from the commutative diagram

$$
\begin{array}{ccc}
\Pi\backslash D & \xrightarrow{\ P_+^\Pi\ } & \Pi\backslash(\exp(\mathfrak{p}_+)\cap D) \\
& \searrow{\scriptstyle \pi_+^\Pi} & \ \downarrow{\scriptstyle \simeq} \\
& & \Pi\backslash G_\mathbb{R}/K,
\end{array}
$$

we conclude that the holomorphic map

$$
P_+^\Pi : \ \Pi\backslash D \to \Pi\backslash(\exp(\mathfrak{p}_+)\cap D)
$$

is also proper.

Now we will study the properties of the restricted maps of the proper holomorphic map P_+^Π to the images of the period maps.

First, let us consider the extended period map

$$
\Phi' : \ S' \to \Gamma\backslash D
$$

and its lifting

$$
\widetilde{\Phi}' : \ \mathcal{T}' \to D.
$$

Suppose that the torsion-free discrete subgroup $\Pi \subset G_\mathbb{R}$ contains the torsion-free monodromy group Γ. Then we can define the period map

$$
\Psi'_\Pi : \ S' \to \Pi\backslash D,
$$

by the period map Φ' composed with the natural quotient map

$$
r : \ \Gamma\backslash D \to \Pi\backslash D.
$$

We are ready to prove the following lemma.

Proposition 5.4. *Let the torsion-free discrete subgroup $\Pi \subset G_\mathbb{R}$ contain the torsion-free monodromy group Γ. Then the induced projection map*

$$
P_+^\Pi|_{\Psi'_\Pi(S')} : \ \Psi'_\Pi(S') \to P_+^\Pi(\Psi'_\Pi(S'))
$$

is a finite étale cover.

Proof. From Corollary 2.3, we know that

$$P_+ : \widetilde{\Phi}'(\mathcal{T}') \to P_+(\widetilde{\Phi}'(\mathcal{T}')) = \widetilde{\Phi}'_+(\mathcal{T}')$$

is a finite holomorphic map, so

$$P_+^\Pi : \Psi'_\Pi(S') \to P_+^\Pi(\Psi_\Pi'(S') = \widetilde{p}(\widetilde{\Phi}'_+(\mathcal{T}'))$$

has finite fibers, where

$$\widetilde{p} : \exp(\mathfrak{p}_+) \cap D \to \Pi \backslash (\exp(\mathfrak{p}_+) \cap D)$$

is the projection map.

Also since $\Psi'_\Pi(S')$ is a closed subvariety in $\Pi \backslash D$, and P_+^Π is a closed map, we deduce that the restriction of P_+^Π to $\Psi'_\Pi(S')$,

$$P_+^\Pi : \Psi'_\Pi(S') \to P_+^\Pi(\Psi_\Pi'(S')) = \widetilde{p}(\widetilde{\Phi}'_+(\mathcal{T}')) \subset \Pi \backslash (\exp(\mathfrak{p}_+) \cap D)$$

is also a closed map, therefore a finite map as defined in page 47 of [8], i.e. a proper holomorphic map with finite fibers.

On the other hand, since $\widetilde{\Phi}'(\mathcal{T}')$ is an analytic subvariety of the period domain D, for any $p \in \widetilde{\Phi}'(\mathcal{T}')$ we can find a open neighborhood U of p in D such that

$$U \cap \widetilde{\Phi}'(\mathcal{T}')$$

is an analytic subvariety of U.

Lemma 5.5 to be proved below, which is a direct application of the Griffiths transversality, implies that the natural projection map

$$\pi : D \to G_\mathbb{R}/K$$

is injective on $U \cap \widetilde{\Phi}'(\mathcal{T}')$ if U is small enough. From Lemma 2.2, we have

$$\pi = \pi_+ \circ P_+$$

on $\widetilde{\Phi}'(\mathcal{T}')$, which implies that P_+ is injective on $U \cap \widetilde{\Phi}'(\mathcal{T}')$. Therefore we have

$$P_+|_{\widetilde{\Phi}'(\mathcal{T}')} : \widetilde{\Phi}'(\mathcal{T}') \to P_+(\widetilde{\Phi}'(\mathcal{T}'))$$

is a locally biholomorphic map of analytic varieties.

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From this we deduce that the induced quotient map of $P_+|_{\widetilde{\Phi}'(\mathcal{T}')}$,

$$P_+^{\Pi}|_{\Psi_{\Pi}'(S')} :\ \Psi_{\Pi}'(S') \to P_+^{\Pi}(\Psi_{\Pi}'(S')) = \widetilde{p}(\widetilde{\Phi}'_+(\mathcal{T}')),$$

is a finite, and locally biholomorphic map of analytic varieties, i.e. a finite étale cover. $\square$

Now we prove the following lemma, which is used in the proof of Proposition 5.4. The proof is the same as that of Lemma 3.4 in [21], which is direct consequence of the Griffiths transversality. We include it here for reader's convenience.

Lemma 5.5. *Let the notations be as in the proof of Proposition 5.4. Then the natural projection map*

$$\pi :\ D \to G_{\mathbb{R}}/K$$

is injective on $U \cap \widetilde{\Phi}'(\mathcal{T}')$, if the open neighborhood U of p is small enough.

Proof. Following the notations of Lemma 3.4 in [21], we consider the Whitney stratification

$$U \cap \widetilde{\Phi}'(\mathcal{T}') = \bigcup_{1 \le i \le n} L_i,$$

of the analytic subvariety $U \cap \widetilde{\Phi}'(\mathcal{T}')$, which comes from a filtration by closed analytic subvarieties

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = U \cap \widetilde{\Phi}'(\mathcal{T}'),$$

such that $L_i = X_i \backslash X_{i-1}$ is smooth whenever it is nonempty, for $1 \le i \le n$.

Note that, by the Griffiths transversality, we know that at any point $t \in L_i$, the corresponding real tangent spaces satisfy

$$\mathrm{T}_t L_i \subset \mathrm{T}_{h,t} D \subset \mathrm{T}_{\bar{t}} G_{\mathbb{R}}/K \simeq \mathfrak{p}_0,$$

where $\bar{t} = \pi(t)$ and $\mathrm{T}_h D$ is the real tangent subbundle corresponding to the horizontal holomorphic tangent bundle $\mathrm{T}_h^{1,0} D$ defined in Section 1. Here the inclusion $\mathrm{T}_{h,t} D \subset \mathrm{T}_{\bar{t}} G_{\mathbb{R}}/K$ is induced by the tangent map of π at t. Therefore the tangent map of

$$\pi|_{L_i} :\ L_i \to G_{\mathbb{R}}/K$$

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at $t \in L_i$ is injective, and π is injective in a small neighborhood of t in L_i.

From the above discussion, we see that the lemma is an obvious corollary from the Griffiths transversality, if $U \cap \widetilde{\Phi}'(\mathcal{T}')$ is smooth. The proof for general case is essentially the same, except that we need to use the Whitney stratification of $U \cap \widetilde{\Phi}'(\mathcal{T}')$ and apply the Griffiths transversality on each stratum L_i.

From Theorem 2.1.2 of [27], we know that the tangent bundle

$$\mathrm{T}(U \cap \widetilde{\Phi}'(\mathcal{T}'))$$

of the stratified space $U \cap \widetilde{\Phi}'(\mathcal{T}')$ is a stratified space with a smooth structure, such that the projection

$$\mathrm{T}(U \cap \widetilde{\Phi}'(\mathcal{T}')) \to U \cap \widetilde{\Phi}'(\mathcal{T}')$$

is smooth and a morphism of stratified spaces.

For any sequence of points $\{p_k\}$ in a stratum L_i converging to a point p in $U \cap \widetilde{\Phi}'(\mathcal{T}')$, the limit of the real tangent spaces,

$$\lim_{k \to \infty} \mathrm{T}_{p_k} L_i = \mathrm{T}_p L_i$$

exists by the Whitney conditions, and is defined as the generalized tangent space at p in page 44 of [7]. Also see the discussion in page 64 of [27].

Denote $\overline{p} = \pi(p)$. With these notations understood, and by the Griffiths transversality, we get the following relations for the corresponding real tangent spaces,

$$\mathrm{T}_p(U \cap \widetilde{\Phi}'(\mathcal{T}')) = \cup_i \mathrm{T}_p L_i \subset (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}) \cap \mathfrak{g}_0 \subset \mathrm{T}_{\overline{p}} G_{\mathbb{R}}/K \simeq \mathfrak{p}_0.$$

This implies that the tangent map of

$$\pi|_{U \cap \widetilde{\Phi}'(\mathcal{T}')} : U \cap \widetilde{\Phi}'(\mathcal{T}') \to G_{\mathbb{R}}/K$$

at p is injective in the sense of stratified space, or equivalently it is injective on each $\mathrm{T}_p L_i$ considered as generalized tangent space.

Therefore we can choose the open neighborhood U of p small enough, such that the restriction of π to $U \cap \widetilde{\Phi}'(\mathcal{T}')$,

$$\pi|_{U \cap \widetilde{\Phi}'(\mathcal{T}')} : U \cap \widetilde{\Phi}'(\mathcal{T}') \to G_{\mathbb{R}}/K,$$

is an injective map in the sense of stratified spaces.

In particular, π is injective on the closure of

$$L_n = X_n \backslash X_{n-1} = (U \cap \widetilde{\Phi}'(\mathcal{T}'))\backslash X_{n-1},$$

which contains $U \cap \widetilde{\Phi}'(\mathcal{T}')$. Therefore we have proved the injectivity of the restriction of π to $U \cap \widetilde{\Phi}'(\mathcal{T}')$. □

6 Algebraicity, the case of $G_{\mathbb{Z}}$

In this section we consider the period map

$$\Psi_0 : S_0 \to G_{\mathbb{Z}}\backslash D$$

and its Griffiths extension

$$\Psi_0' : S_0' \to G_{\mathbb{Z}}\backslash D.$$

Recall that $G_{\mathbb{Z}}\backslash D$ is a complex orbifold, or a normal complex space, by the discussion as given in page 163 of [9], or by [2] and [25].

Note that Ψ_0' is not necessarily locally liftable. As discussed in Section 1, following the construction of Lemma IV-A in [30], we take a normal and torsion-free subgroup Λ of finite index in $G_{\mathbb{Z}}$, and lift the period map to a finite cover S of S_0 to get the period map

$$\Psi_{\Lambda} : S \to \Lambda\backslash D,$$

and its Griffiths extension

$$\Psi_{\Lambda}' : S' \to \Lambda\backslash D,$$

such that the following diagram is commutative

$$
\begin{array}{ccccc}
S & \xrightarrow{i} & S' & \xrightarrow{\Psi_{\Lambda}'} & \Lambda\backslash D \\
\downarrow & & \downarrow & & \downarrow \\
S_0 & \xrightarrow{i} & S_0' & \xrightarrow{\Psi_0'} & G_{\mathbb{Z}}\backslash D.
\end{array}
$$

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In this case, $\Lambda\backslash D$ is a complex manifold, and the extended period map

$$\Psi'_\Lambda : S' \to \Lambda\backslash D$$

is still locally liftable. Taking the universal covers $\mathcal{T}$ of S and $\mathcal{T}'$ of S', we get the lifted period maps $\widetilde{\Phi} : \mathcal{T} \to D$ and $\widetilde{\Phi}' : \mathcal{T}' \to D$, which fit into the following commutative diagram

$$
\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{i_\mathcal{T}} & \mathcal{T}' & \xrightarrow{\widetilde{\Phi}'} & D \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{i} & S' & \xrightarrow{\Psi'_\Lambda} & \Lambda\backslash D \\
\downarrow & & \downarrow & & \downarrow \\
S_0 & \xrightarrow{i} & S'_0 & \xrightarrow{\Psi'_0} & G_\mathbb{Z}\backslash D,
\end{array}
\tag{9}
$$

such that $\widetilde{\Phi} = \widetilde{\Phi}' \circ i_\mathcal{T}$. See Section 1 for the discussion about the existences of the liftings $i_\mathcal{T}$ and $\widetilde{\Phi}'$, which follow from basic general topology as proved in the appendix of [21].

We are ready to give the proof of the following theorem.

Theorem 6.1. *The images $\Psi_\Lambda(S)$ and $\Psi'_\Lambda(S')$ are algebraic, more precisely they are quasi-projective.*

Proof. When restricted to the images of the extended period maps, the projection maps P_+ and P_+^Λ fit into the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{\Phi}'(\mathcal{T}') & \xrightarrow{P_+} & \exp(\mathfrak{p}_+)\cap D \\
\downarrow{\scriptstyle\pi_D} & & \downarrow{\scriptstyle p} \\
\Psi'_\Lambda(S') & \xrightarrow{P_+^\Lambda} & \Lambda\backslash(\exp(\mathfrak{p}_+)\cap D)
\end{array}
$$

which follows from diagram (8).

Let

$$\widetilde{\Phi}'_+ = P_+ \circ \widetilde{\Phi}' : \mathcal{T}' \to \exp(\mathfrak{p}_+)\cap D.$$

The image of P_+^Λ satisfies

$$P_+^\Lambda(\Psi'_\Lambda(S')) = p(\widetilde{\Phi}'_+(\mathcal{T}')) \subset p(\exp(\mathfrak{p}_+)\cap D) = \Lambda\backslash(\exp(\mathfrak{p}_+)\cap D).$$

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Let us summarize the related maps and images in the following commutatihs diagram,

$$\widetilde{\Phi}'(\mathcal{T}') \xrightarrow{\;P_+\;} \widetilde{\Phi}'_+(\mathcal{T}')$$

$$\downarrow{\scriptstyle \pi_D} \qquad\qquad \downarrow{\scriptstyle p}$$

$$\Psi'_\Lambda(S') \xrightarrow{\;P^\Lambda_+\;} P^\Lambda_+(\Psi_\Lambda{}'(S')).$$

By Corollary 5.2, the map

$$P^\Lambda_+ : \; \Lambda\backslash D \to \Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$$

is a proper map. As proved by Griffiths, $\Psi'_\Lambda(S')$ is a closed analytic subvariety in $\Lambda\backslash D$, so

$$P^\Lambda_+(\Psi'_\Lambda(S')) = p(\widetilde{\Phi}'_+(\mathcal{T}'))$$

is a closed analytic subvariety in the quasi-projective manifold $\Lambda\backslash(\exp(\mathfrak{p}_+) \cap D)$, therefore it is quasi-projective.

From Proposition 5.4, we have that

$$P^\Lambda_+|_{\Psi'_\Lambda(S')} : \; \Psi'_\Lambda(S') \to P^\Lambda_+(\Psi'_\Lambda(S'))$$

is a finite étale cover. From Theorem A.1 in the appendix, the generalized Riemann existence theorem of Grothendieck, we get that $\Psi'_\Lambda(S')$ is quasi-projective, or algebraic, and P^Λ_+ is a morphism between quasi-projective varieties.

Since $\Psi_\Lambda(S)$ is a Zariski open subvariety of $\Psi'_\Lambda(S')$, it is quasi-projective.

$\square$

Let us now consider the period map $\Psi_0 : \; S_0 \to G_\mathbb{Z}\backslash D$, and its Griffiths extension

$$\Psi'_0 : \; S'_0 \to G_\mathbb{Z}\backslash D.$$

We prove the following theorem.

Theorem 6.2. *The images $\Psi_0(S_0)$ and $\Psi'_0(S'_0)$ are algebraic, more precisely they are quasi-projective.*

Proof. From commutative diagram (9), we know that

$$\Psi_0'(S_0') = r_\Lambda(\Psi_\Lambda'(S'))$$

where

$$r_\Lambda : \Lambda \backslash D \to G_\mathbb{Z} \backslash D$$

is the natural quotient map by the finite quotient group $\Lambda \backslash G_\mathbb{Z}$.

From this we see that $\Psi_0'(S_0')$ is the quotient of the quasi-projective variety $\Psi_\Lambda'(S')$ by the subgroup of the finite quotient group $\Lambda \backslash G_\mathbb{Z}$ that preserves $\Psi_\Lambda'(S')$. From Corollary 3.46 in [31] which asserts that the quotient of quasi-projective variety by a finite group is quasi-projective, we get the algebraicity of $\Psi_0'(S_0')$.

Since $\Psi_0(S_0)$ is a Zariski open subvariety of $\Psi_0'(S_0')$, it is quasi-projective. $\square$

7   Algebraicity, the torsion-free case

In this section we discuss the case when the monodromy group Γ is a normal and torsion-free subgroup of finite index in the monodromy group Γ_0. We will prove that the images of the corresponding period maps, $\Phi'(S')$ and $\Phi(S)$ in $\Gamma \backslash D$, both are algebraic by using the algebraicity of the variety

$$\Lambda \backslash (\exp(\mathfrak{p}_+) \cap D)$$

and $\Psi_\Lambda'(S')$ as proved in last section, and by applying again the generalized Riemann existence theorem.

Let Λ be a normal and torsion-free subgroup of finite index in $G_\mathbb{Z}$, we take the intersection

$$\Gamma = \Lambda \cap \Gamma_0 \subseteq \Gamma_0$$

which is normal and torsion-free subgroup of finite index in Γ_0.

We define the period map

$$\Psi_\Lambda : S \to \Lambda \backslash D$$

to be the composition of the period map $\Phi : S \to \Gamma \backslash D$ with the natural projection map

$$q : \Gamma \backslash D \to \Lambda \backslash D. \tag{10}$$

Note that Ψ_Λ, which is still horizontal and locally liftable, is a period map as defined by Griffiths in Section 9 of [9].

Since the monodromy group Γ is torsion-free, the Griffiths extension

$$\Phi' : S' \to \Gamma \backslash D$$

is still horizontal and locally liftable.

The extended period map Φ' composed with the projection map q in (10) gives the Griffiths extension of the period map Ψ' as

$$\Psi'_\Lambda : S' \to \Lambda \backslash D.$$

We write them in the diagram,

$$S' \xrightarrow{\Phi'} \Gamma \backslash D \xrightarrow{q} \Lambda \backslash D.$$

Note that Ψ'_Λ is still locally liftable, since the Picard-Lefschetz transformations around the points in $S' \backslash S$ still lie in the monodromy group Γ which is torsion-free, therefore the monodromy around the points in $S' \backslash S$ are trivial. Hence we can lift the period maps Ψ_Λ and Ψ'_Λ to the universal covers $\mathcal{T}$ and $\mathcal{T}'$ respectively, which are respectively the period map and its Griffiths extension

$$\widetilde{\Phi} : \mathcal{T} \to D \text{ and } \widetilde{\Phi}' : \mathcal{T}' \to D$$

as defined in Section 1.

To summarize, the extended period maps Φ' and $\Psi'_\Lambda = q \circ \Phi'$ fits into the commutative diagram,

$$
\begin{array}{ccccc}
S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & \Gamma \backslash D \\
\Big\| = & & \Big\| = & & \Big\downarrow q \\
S & \xrightarrow{i} & S' & \xrightarrow{\Psi'_\Lambda} & \Lambda \backslash D.
\end{array}
$$

From the definition of the period maps Ψ_Λ and Ψ'_Λ, they also fit into the following commutative diagram

$$
\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{i_\mathcal{T}} & \mathcal{T}' & \xrightarrow{\widetilde{\Phi}'} & D \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & \Gamma\backslash D \\
\Vert & & \Vert & & \downarrow{\scriptstyle q} \\
S & \xrightarrow{i} & S' & \xrightarrow{\Psi'_\Lambda} & \Lambda\backslash D.
\end{array}
$$

See the discussion following commutative diagram (9) about the existences of the liftings $i_\mathcal{T}$ and $\widetilde{\Phi}'$ which is proved in the appendix of [21] by an elementary argument in general topology.

The proof of the following lemma uses substantially the properness of the extended period maps as proved by Griffiths.

Lemma 7.1. *The induced map from the quotient map*

$$ q : \Phi'(S') \rightarrow q(\Phi'(S')) = \Psi'_\Lambda(S') $$

is a finite étale cover.

Proof. Since both Γ and Λ are torsion-free, the quotient spaces $\Gamma\backslash D$ and $\Lambda\backslash D$ are smooth, and the quotient map

$$ q : \Gamma\backslash D \rightarrow \Lambda\backslash D $$

is a covering map. Therefore the restriction of the covering map q to the analytic subvariety $\Phi'(S')$, $q|_{\Phi'(S')}$, is locally biholomorphic. Hence the restriction map

$$ q : \Phi'(S') \rightarrow q(\Phi'(S')) = \Psi'_\Lambda(S') $$

is an étale cover.

Now, we only need to show that $q|_{\Phi'(S')}$ is a finite holomorphic map. First, we show that

$$ q : \Phi'(S') \rightarrow q(\Phi'(S')) = \Psi'_\Lambda(S') $$

is a proper map.

In fact, to prove that $q^{-1}(E)$ is a compact subset in $\Phi'(S')$ for any compact subset E in $\Psi'_\Lambda(S')$, we note that

$$(\Psi'_\Lambda)^{-1}(E) = (\Phi')^{-1}(q^{-1}(E)),$$

therefore

$$q^{-1}(E) = \Phi'((\Psi'_\Lambda)^{-1}(E)).$$

Since Φ' is continuous and $(\Psi'_\Lambda)^{-1}(E)$ is a compact subset in S', the image $\Phi'((\Psi'_\Lambda)^{-1}(E))$ is a compact subset in $\Phi'(S')$. From this we deduce that $q^{-1}(E)$ is a compact subset in $\Phi'(S')$. So we have proved that

$$q: \Phi'(S') \to q(\Phi'(S')) = \Psi'_\Lambda(S')$$

is a proper map.

On the other hand, given any point $z \in \Psi'_\Lambda(S')$, $q^{-1}(z)$ is a discrete set, so $q^{-1}(z)$ consists of finite number of points.

Therefore we have proved that $q|_{\Phi'(S')}$ is a finite étale cover onto its image. $\square$

Now we can prove the main result of this section.

Theorem 7.2. *The complex analytic varieties $\Phi(S)$ and $\Phi'(S')$ are algebraic, more precisely they are quasi-projective.*

Proof. By Theorem 6.1, $q(\Phi'(S')) = \Psi'_\Lambda(S')$ is quasi-projective. Since

$$q: \Phi'(S') \to q(\Phi'(S')) = \Psi'_\Lambda(S')$$

is a finite étale cover, by applying Theorem A.1 in the appendix, the generalized Riemann existence theorem of Grothendieck, we know that $\Phi'(S')$ is quasi-projective, such that q is a morphism of quasi-projective varieties.

Since $\Phi(S)$ is a Zariski open subvariety of $\Phi'(S')$, it is quasi-projective. $\square$

8   Algebraicity, general monodromy groups

In this section, we consider the period map for general monodromy group Γ_0,

$$\Phi_0: S_0 \to \Gamma_0 \backslash D,$$

as introduced in Section 1, and its Griffiths extension

$$\Phi_0' : S_0' \to \Gamma_0 \backslash D.$$

We will prove the algebraicity of $\Phi_0(S_0)$ and $\Phi_0'(S_0')$ by using the algebraicity of $\Phi'(S')$.

As in Section 7, we can choose Γ to be a normal and torsion-free subgroup of finite index in the monodromy group Γ_0, by taking

$$\Gamma = \Lambda \cap \Gamma_0 \subseteq \Gamma_0,$$

where Λ is a normal and torsion-free subgroup of finite index of $G_{\mathbb{Z}}$ which always exists by Selberg lemma. See for example, Proposition 4.45 and the discussion before Proposition 4.47 in [18].

Consider the projection map

$$r_\Gamma : \Gamma \backslash D \to \Gamma_0 \backslash D, \tag{11}$$

in the commutative diagram

$$
\begin{array}{ccccc}
S & \xrightarrow{\ i\ } & S' & \xrightarrow{\ \Phi'\ } & \Gamma \backslash D \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle r_\Gamma} \\
S_0 & \xrightarrow{\ i\ } & S_0' & \xrightarrow{\ \Phi_0'\ } & \Gamma_0 \backslash D,
\end{array}
$$

which is as defined in Section 1.

Now we are ready to prove the following theorem.

Theorem 8.1. *The complex analytic varieties $\Phi_0(S_0)$ and $\Phi_0'(S_0')$ are algebraic, more precisely they are quasi-projective.*

Proof. Let $\Gamma \backslash \Gamma_0$ denote the quotient group which is a finite group. The projection map r_Γ in (11) is a quotient map by the finite group

$$\Gamma \backslash \Gamma_0.$$

So we have

$$r_\Gamma(\Phi'(S')) = \Phi_0'(S_0'),$$

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from which we deduce that $\Phi'_0(S'_0)$ is the quotient of $\Phi'(S')$ by the finite group $\Gamma \backslash \Gamma_0$ that preserves $\Phi'(S') \subseteq \Gamma \backslash D$.

From Corollary 3.46 in [31] which asserts that the quotient of quasi-projective variety by a finite group is quasi-projective, we see that the quasi-projectivity of $\Phi'_0(S'_0)$ follows from that of $\Phi'(S')$.

Finally, as $\Phi_0(S_0)$ is Zariski open in $\Phi'_0(S'_0)$, we have the the quasi-projectivity of $\Phi_0(S_0)$, which proves the following theorem. $\qquad\square$

From the above proof and the results in [24], [1] and [33], we see that an ample line bundle on $\Phi_0(S_0)$ and $\Phi'_0(S'_0)$ is induced by the canonical line bundle of $\exp(\mathfrak{p}_+ \cap D)$ which is invariant under its automorphism group. In particular the projective embeddings of the quasi-projective manifold $\Lambda \backslash (\exp(\mathfrak{p}_+ \cap D))$ and the quasi-projective variety $G_{\mathbb{Z}} \backslash (\exp(\mathfrak{p}_+) \cap D)$ are given by sections of the multiples of their canonical line bundles.

A The generalized Riemann existence theorem

This section is a brief summary of several versions of the generalized Riemann existence theorem which is for reader's convenience.

First recall that a finite étale cover, or finite étale map, in the language of complex analytic geometry, corresponds to a finite and surjective holomorphic map between complex analytic varieties which is locally biholomorphic.

Now we restate the generalized Riemann existence theorem in the language of complex analytic geometry as used in this paper, which is due to Grothendieck.

Theorem A.1. *Let X be a quasi-projective variety. Let Y be a complex analytic space, and a finite étale map $f : Y \to X$. Then there is a unique algebraic structure on Y such that $f : Y \to X$ is a morphism of algebraic varieties.*

This theorem is a reformulation of the following theorem of Grothendieck in the language of complex analysis,

Generalized Riemann existence theorem (Grothendieck). (c.f. [13], page 333) *Let X be a scheme locally of finite type over $\mathbb{C}$, and X^{an} be the analytic space associated to X. The functor Ψ, which associates the analytic space X'^{an} to each finite étale cover X' of X, is an equivalence of the category of the finite étale covers of X into the category of the finite étale covers of X^{an}.*

The proof of the above theorem is given in SGA1 by Grothendieck in [13]. See, for example [34], for a concise exposition. This theorem is a generalization of the following classical theorem of Grauert-Remmert.

Generalized Riemann existence theorem (Grauert-Remmert). (c.f. [16], page 442) *Let X be a normal scheme of finite type over $\mathbb{C}$. Let $\mathscr{X}'$ be a normal complex analytic space, together with a finite morphism $f : \mathscr{X}' \to X_h$. (X_h is the analytic space associated to X. We define a finite morphism of analytic spaces to be a proper morphism with finite fibers.) Then there is a unique normal scheme X' and a finite morphism $g : X' \to X$ such that $X'_h \simeq \mathscr{X}'$ and $g_h = f$.*

We would like to sketch a proof of the following slightly more general form of the generalized Riemann existence theorem. This theorem, although not used in this paper, can make the applications easier, since we only need to check the map f to be a finite holomorphic map, i.e. a proper holomorphic map with finite fibers as defined in [8], which is a weaker condition than to be finite étale.

Theorem A.2. *Let X be a quasi-projective variety. Let Y be a complex analytic space, together with a finite holomorphic map $f : Y \to X$. Then there is a unique algebraic structure on Y such that $f : Y \to X$ is a morphism of algebraic varieties.*

Sketch of the proof. Consider a finite holomorphic map,

$$f : Y \to X.$$

We can take the normalizations $\hat{Y}$ and $\hat{X}$ of Y and X respectively and a holomorphic map

$$\hat{f} : \hat{Y} \to \hat{X}$$

which fits in the following commutative diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{f}} & \hat{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X. \end{array}$$

By Sommese's Lemma III-B in [30], we have that $\hat{X}$ is quasi-projective. Since the normalization $\hat{Y}$ ($\hat{X}$ resp.) of Y (X resp.) is a proper map with finite fibers, i.e. a finite holomorphic map. we have, from the above commutative diagram, that

$$\hat{f} : \hat{Y} \to \hat{X}$$

is also a finite holomorphic map.

Then the generalized Riemann existence theorem of Grauert-Remmert as stated above, or the version of Mumford as stated below, implies that $\hat{Y}$ is also quasi-projective. At last, we conclude that Y is quasi-projective, by applying the argument of Viehweg in the proof of Corollary 9.28 in page 303 of [31], which is based on Proposition 2.6.2 of Grothendieck in page 112 of [14]. $\qquad \square$

Generalized Riemann existence theorem (Mumford). (c.f. [26], page 227) *If X is any normal algebraic variety, Y any normal analytic space, and $f : Y \to X$ is a proper holomorphic map with finite fibers, and if there is a Zariski-open set $U \subset X$ such that $f^{-1}(U)$ is dense in Y and the restriction $f : f^{-1}(U) \to U$ is unramified, then Y has one and only one structure of algebraic variety making f into a morphism.*

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