

Boltje-Maisch Resolutions of Specht Modules

Xingyu Dai^{a,(1)}, Fang Li^{a,(2)}, Kefeng Liu^{a,b(3)}

^aCenter of Mathematical Sciences, Zhejiang University, Zhejiang 310027, China

^bDepartment of Mathematics, University of California, Los Angeles, USA

E-mail: (1) daixingyu12@126.com; (2) fangli@zju.edu.cn; (3) liu@math.ucla.edu

Abstract

In [5], Boltje and Maisch found a permutation complex of Specht modules in representation theory of Hecke algebras, which is the same as the Boltje-Hartmann complex appeared in the representation theory of symmetric groups and general linear groups. In this paper we prove the exactness of Boltje-Maisch complex in the dominant weight case.

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1. Introduction

In module category of the group algebra $R\mathfrak{S}_r$ over an arbitrary commutative ring R , Hartmann and Boltje constructed a finite chain complex in [4] for any composition λ of a positive integer r . Almost all factors are constructed by restricted subsets of homomorphisms between permutation modules except the last one, which is the dual of Specht module S^λ . Partial exactness results of position -1 and 0 about this complex were already achieved in Hartmann and Boltje's work [4] and a full proof of the exactness was obtained recently in [24] with help of Bar resolution in homology theory and the construction using the Schur functor. In [1] [2] [3] [8] [9] [23] [26] [30] [27] [31] and etc., some other permutation resolutions of Specht modules have been established.

In [5], the construction of the chain complex of $R\mathfrak{S}_r$ was lifted to a chain complex of modules of the Iwahori-Hecke algebra \mathcal{H}_r with an integral domain R and the original chain complex reproduced by the specialization $q = 1$ and moreover. This work and its method are useful for our discussion here.

The construction of this complex was completely combinatorial and characteristic free. It was conjectured that this chain complex is exact whenever λ is a partition. Also, some partial exactness has been found in [6]. In this paper, we find a way to prove this conjecture. In order to prove its exactness in the dominant weight case, we follow the method of [24], which constructs a bar resolution in the Borel subalgebra case and transformed it into the module category of q -Schur algebras by induced Functors.

The paper is organized as follows. In section 2, we find an ideal sequence $J_0 \supseteq J_1 \supseteq \cdots \supseteq 0$, and use J_0 and J_1 to construct a bar resolution of the module in the representations of

Borel subalgebra $S_R^+(n, r)$, just as Ana Paula Santana and Ivan Yudin did in the case of symmetric groups, as in [24]. However, since the proof of vanishing theorem in [27] failed for the case of Iwahori Hecke algebra's. In section 3, we use different tools to prove the module R_λ is $S_R(n, r) \otimes_{S_R^+(n, r)}$ -acyclic, which was introduced in [29]. After that, we reach the main results Theorem 4.7 and Theorem 5.3 which give the positive answer of exactness of *Boltje-Maisch* given in [5].

2. Notations and Quoted Results

2.1. Combinatorics. [24]

For any natural number m we denote by \bar{n} the set $\{1, \dots, n\}$. Given a finite set X , for each map μ from X to no negative integers \mathbb{N}_0 , we can define its length $|\mu| := \sum_{x \in X} \mu_x$, where we realize the map μ as $\mu(x) = \mu_x$ for any $x \in X$. Then, we can write a subset $\Lambda(X; r)$ as $\{\mu : X \rightarrow \mathbb{N}_0 \mid |\mu| = r\}$, and get a map from the set X^r to $\Lambda(X; r)$ as following:

$$wt : X^r \rightarrow \Lambda(X; r)$$

$$\text{with } wt(u)_x := \#\{s \mid u_s = x, s = 1, \dots, r\}.$$

There are we specific cases of the definitions given above. First, we denote $I(n, r)$ as the set \bar{n}^r . The elements of $I(n, r)$ are usually called *multi-indices* in other's work as [24], which can be denoted by bold letters such as \mathbf{i} , where $\mathbf{i} := (i_1, \dots, i_r)$ for $1 \leq i_k \leq n$ and $1 \leq k \leq r$. Also, we can denote the set $\bar{n} \times \bar{n}$ alternatively as $\{(i, j) \mid 1 \leq i, j \leq n\}$. Similarly, we can identify the set $(\bar{n} \times \bar{n})^r$ and $I(n, r) \times I(n, r)$ via the map $((i_1, j_1), \dots, (i_r, j_r)) \mapsto ((i_1, \dots, i_r), (j_1, \dots, j_r))$ without any confusions.

For convenience, from now on sets $\Lambda(\bar{n}; r)$ and $\Lambda(\bar{n} \times \bar{n}; r)$ are denoted by $\Lambda(n, r)$ and $\Lambda(n, n; r)$, respectively. We can notice that the elements of $\Lambda(n, r)$ are the same as compositions of r into n parts, which also has same notation as $\Lambda(n, r)$ in many works as in [21]. On the set $I(n, r)$ we defined the ordering \leq by

$$\mathbf{i} \leq \mathbf{j} \iff i_1 \leq j_1, i_2 \leq j_2, \dots, i_r \leq j_r,$$

and write $\mathbf{i} < \mathbf{j}$ if $\mathbf{i} \leq \mathbf{j}$ and $\mathbf{i} \neq \mathbf{j}$.

2.2. Symmetric groups and Iwahori-Hecke algebras. A *composition* λ of r is a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $|\lambda| = \sum_i \lambda_i = r$. Moreover, there is a partial order \trianglelefteq (resp. \trianglerighteq) within compositions of r as: we denote $\lambda \trianglelefteq \mu$ when $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ (resp. $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$) for all $1 \leq k \leq n$.

Let \mathfrak{S}_r denote the symmetric group of all permutations of $1, \dots, r$ with Coxeter generators $s_i := (i, i + 1)$, and \mathfrak{S}_λ the Young subgroup corresponding to the composition λ of r . Thus, we have

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\mathbf{a}} = \mathfrak{S}_{\{1, \dots, a_1\}} \times \mathfrak{S}_{\{a_1+1, \dots, a_2\}} \times \dots \times \mathfrak{S}_{\{a_{n-1}+1, \dots, a_n\}},$$

where $\mathbf{a} = [a_0, a_1, \dots, a_n]$ with $a_0 = 0$ and $a_i = \lambda_1 + \dots + \lambda_i$ for all $i = 1, \dots, n$. We denote by \mathcal{D}_λ the set of distinguished representatives of right \mathfrak{S}_λ -cosets and write $\mathcal{D}_{\lambda\mu} := \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$, which is the set of distinguished representatives of double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu$.

As usual one identifies composition λ with *Young diagram* and we say that λ is the *shape* of the corresponding Young diagram. For example, we can represent the partition $(3, 2)$ as $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array}$. A λ -tableau is a filling of the r boxes of the Young diagram of λ of the numbers $1, 2, \dots, r$. We denote the set of λ -tableaux by $\mathcal{T}(\lambda)$ and usually denote an element of $\mathcal{T}(\lambda)$ as \mathfrak{t} .

The group \mathfrak{S}_r acts from the right on $\mathcal{T}(\lambda)$ by simply applying an element $w \in \mathfrak{S}_r$ to the entries of the tableau $\mathfrak{t} \in \mathcal{T}(\lambda)$. This action is free and transitive, and it yields a bijection

$$\mathfrak{S}_r \xrightarrow{\sim} \mathcal{T}(\lambda), \quad w \mapsto t^\lambda w.$$

A λ -tableau \mathfrak{t} is called *row-standard* if its entries are increasing in each row from left to right. The row-standard tableaux form a subset $\mathcal{T}^{rs}(\lambda)$ of $\mathcal{T}(\lambda)$. Two λ -tableaux \mathfrak{t}_1 and \mathfrak{t}_2 are called *row-equivalent* if \mathfrak{t}_1 and \mathfrak{t}_2 can arise from each other by rearranging elements within each row. We denote the row-equivalent class of \mathfrak{t} by $\{\mathfrak{t}\}$ and the set of row equivalent classes by $\overline{\mathcal{T}}(\lambda)$. One has the canonical bijections

$$(2.1) \quad \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{T}^{rs}(\lambda) \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda)$$

given by $d \mapsto t^\lambda d$ and $\mathfrak{t} \mapsto \{\mathfrak{t}\}$.

Definition 2.1. *Let R be a commutative domain with 1 and let q be a unitary element of R . The Iwahori-Hecke algebra $\mathcal{H}_r = \mathcal{H}_{R,q}(\mathfrak{S}_r)$ of \mathfrak{S}_r is the unital associative R -algebra with generators T_1, T_2, \dots, T_{r-1} and relations:*

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, & \text{for } i = 1, 2, \dots, r-1, \\ T_i T_j &= T_j T_i, & \text{for } 1 \leq i < j-1 \leq r-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & \text{for } i = 1, 2, \dots, r-2. \end{aligned}$$

\mathcal{H}_r is a free R -module with a finite R -basis $\{T_w | w \in \mathfrak{S}_r\}$. T_w is defined as $T_{i_1} T_{i_2} \dots T_{i_s}$ if w has a reduced presentation $w = s_{i_1} s_{i_2} \dots s_{i_s}$. Then, we put $x_\mu := \sum_{w \in \mathfrak{S}_\mu} T_w$ for any Young subgroup \mathfrak{S}_μ , which is an element in \mathcal{H}_r , and define M^μ to be the right \mathfrak{S}_r -module $x_\mu \mathcal{H}_r$.

Definition 2.2. *Fix a non-negative integer r , the q -Schur algebra is the endomorphism algebra*

$$S_R(n, r) = \text{End}_{\mathcal{H}_r} \left(\bigoplus_{\mu \in \Lambda(n, r)} M^\mu \right) = \bigoplus_{\lambda, \mu \in \Lambda(n, r)} \text{Hom}_{\mathcal{H}_r}(M^\mu, M^\lambda).$$

For any $\lambda, \mu \in \Lambda(n, r)$, each element d of the set $\mathcal{D}_{\lambda\mu} := \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ is a representative of the double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu$. By [6], $\mathcal{D}_{\lambda\mu}$ forms an R -basis for $\text{Hom}_{\mathcal{H}_r}(M^\mu, M^\lambda)$, where $\psi_{\lambda\mu}^d$ is defined by

$$(2.2) \quad \psi_{\lambda\mu}^d(x_\mu) = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w = x_\lambda \sum_{e \in \mathcal{D}_\nu \cap \mathfrak{S}_\mu} T_{de} = x_\lambda T_d \sum_{e \in \mathcal{D}_\nu \cap \mathfrak{S}_\mu} T_e,$$

for any parameter $d \in \mathcal{D}_{\lambda\mu}$, and where $\nu \in \Lambda(n, r)$ is determined by Young subgroup $\mathfrak{S}_\nu := d^{-1} \mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$.

We can identify the set $\mathcal{D}_{\lambda\mu}$ with a combinatorial set $\mathcal{T}(\lambda, \mu)$, which is call a set of generalized tableaux of λ with content μ . Each element $T \in \mathcal{T}(\lambda, \mu)$ can be realized as Young diagram of composition λ whose boxes are filled with r positive integers where μ_1 entries equal to 1, μ_2 entries equal to 2, etc.

We denote T_μ^λ by the generalized tableau whose boxes are filled in the natural order, which is the stabilizer of left group action $(wT)(i) := T(iw)$. If two generalized tableaux T_1, T_2 arised form each other by rearranging the entries within the rows, We call they *row-equivalent*, i.e., $T_2 = T_1 w$ for some $w \in \mathfrak{S}_\lambda$. We denote the set of row equivalent classes by $\overline{\mathcal{T}}(\lambda, \mu)$. Furthermore, we can achieve a bijection $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda, \mu)$, $\mathfrak{S}_\lambda w \mathfrak{S}_\mu \mapsto w T_\mu^\lambda$.

A generalized tableau $T \in \mathcal{T}(\lambda, \mu)$ is said to be *row-semistandard* if its entries are in the natural order from left to right. Write by $\mathcal{T}^{rs}(\lambda, \mu)$ to denote the set of all such tableau. Every row-equivalent class possesses a unique row-semistandard element. It follows that $\mathcal{T}^{rs} \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda, \mu), T \mapsto \{T\}$, is a bijection. Therefore, we have reached a canonical bijection:

$$(2.3) \quad \mathcal{D}_{\lambda\mu} \xrightarrow{\sim} \mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda, \mu) \xleftarrow{\sim} \mathcal{T}^{rs}(\lambda, \mu)$$

Moreover, Theorem 4.7 of [21] shows that $\text{Hom}_{\mathcal{H}}(M^\mu, M^\lambda)$ is free as an R -module with basis $\{\psi_{\lambda\mu}^d | d \in \mathcal{D}_{\lambda\mu}\}$. The q -Schur algebra $S_R(n, r)$ also can be written as a free R -module

$$\bigoplus_{\substack{\lambda, \mu \in \Lambda(n, r) \\ d \in \mathcal{D}_{\lambda\mu}}} R\psi_{\lambda\mu}^d.$$

2.3. Quantized enveloping algebras. In order to show the vanishing theorem in next section, we need some notations about quantum groups here. Recall the definition of the quantized enveloping algebra in the version given by Kashiwara.

Let $A = (a_{i,j})_{i,j \in I}$ be a finite-type *Cartan matrix*. Which means: fix a sequence of positive integers $(d_i)_i \in I$ such that $d_i a_{i,j} = d_j a_{j,i}$ for all $i, j \in I$. In addition, we need it satisfy the structure of root datum as in [18]:

(i) A perfect pairing $\langle, \rangle: P^* \times P \rightarrow \mathbb{Z}$, where P and $P^* = \text{Hom}(P, \mathbb{Z})$ are finitely generated free \mathbb{Z} -modules.

(ii) Linearly independent subsets $\{\alpha_i | i \in I\}$ of P and $\{\alpha_i^\vee | i \in I\}$ of P^+ , satisfying $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for all i and j .

P and P^* are called the lattices of *weight* and *coweights*, respectively; the α_i are the *simple roots*, the α_i^\vee are the *simple coroot*. The *dominance order* on P is defined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ can be written as a sum of simple roots. A weight λ is *dominant* (resp. *antidominant*) if all $\langle \alpha_i^\vee, \lambda \rangle$ are nonnegative (resp. nonpositive).

Definition 2.3. Let U be the $\mathbb{Q}(q)$ -algebra with generators e_i, f_i, q^h , $1 \leq i \leq n$, $h \in P^*$, and relations

$$\begin{aligned} q^0 &= 1, & q^h q^{h'} &= q^{h+h'}, \\ q^h e_i &= q^{\langle h, \alpha_i \rangle} e_i q^h, \\ q^h f_i &= q^{-\langle h, \alpha_i \rangle} f_i q^h, \\ [e_i, f_j] &= \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

$$\sum_{l=0}^a (-1)^l e_i^{(l)} e_j e_i^{(a-l)} = \sum_{l=0}^a (-1)^l f_i^{(l)} f_j f_i^{(a-l)} = 0,$$

where $i \neq j$, $a = 1 - a_{i,j}$.

For convenience, we have set the following abbreviations in above relations: $t_i = q^{d_i \alpha_i^\vee}$, $q_i = q^{d_i}$, $e_i^{(l)} = e_i^l / [l]_i!$, $f_i^{(l)} = f_i^l / [l]_i!$. The subscript i in $[l]_i!$ means that the q in the definition of $[l]!$ is replaced by q_i .

Take subset $J \subseteq I$. Set three set which consist of elements of U :

$$\begin{aligned} \mathcal{E}_J &= \{e_i^{(s)} | s \geq 0, i \in J\}. \\ \mathcal{F}_J &= \{f_i^{(s)} | s \geq 0, i \in J\}. \\ \mathcal{H} &= \{q^h, \begin{bmatrix} q^h; 0 \\ s \end{bmatrix}\} \quad \text{where} \quad \begin{bmatrix} x; c \\ t \end{bmatrix} = \prod_{s=1}^t \frac{xq^{(c-s+1)} - x^{-1}q^{-(c-s+1)}}{q^s - q^{-s}}. \end{aligned}$$

We write \mathcal{A} for the ring $\mathbb{Z}[q, q^{-1}]$ of integral Laurent polynomials in the indeterminate q . Let $U_{\mathcal{A}}^-(J)$, $U_{\mathcal{A}}^+(J)$ be the \mathcal{A} -subalgebras of U generated respectively by \mathcal{F}_J , \mathcal{E}_J . Similarly, we can define these subalgebra structure over any arbitrary commutative ring R and denote it by $U_R^+(J)$ or $U_R^-(J)$, if there exist a with ring homomorphism from \mathcal{A} to R , especially $\mathbb{Q}(q)$ for example.

For convenience, We usually omit the subscript when the algebras are defined over $\mathbb{Q}(q)$. Then, we use the following abbreviations: $U_{\mathcal{A}}^b = U_{\mathcal{A}}^-(I)$, $U_{\mathcal{A}}^\# = U_{\mathcal{A}}^+(I)$, and define $U^b, U^\#$ (resp. $U_R^b, U_R^\#$) similarly with the ground ring $\mathbb{Q}(q)$ (resp. R).

Write $U_R\text{-Int}$ for the category of all integrable U_R -modules V . It is known as in Lusztig [19] that $U\text{-Int}$ is semisimple with simple modules $\Delta(\lambda)$, $\lambda \in P^+$, where $\Delta(\lambda)$ is the unique maximal integrable quotient of *Verma module* $M(\lambda)$. See [28], We usually call it *Weyl module* of highest weight λ .

There is a $\mathbb{Q}(q)$ -algebra anti-automorphism $u \mapsto u^\tau$ of U given by

$$e_i^\tau = f_i, \quad f_i^\tau = e_i, \quad (q^h)^\tau = q^h.$$

If $V \in U_R\text{-Int}$, its *contravariant dual* V° is the linear dual $\text{Hom}_R(V, R)$, with its natural right U_R -action transferred to the left via τ . And, we denote some new objects in category $U_R\text{-Int}$ as $\nabla_R(\lambda) = \Delta_R(\lambda)^\circ$.

3. Coordinate ring and bar resolutions

Definition 3.1. [11] For a commutative ring R , let $R[M_n(q)]$ be the associative algebra over R generated by X_{ij} with $1 \leq i, j \leq n$ such that

$$(3.1) \quad \begin{cases} X_{ij}X_{ik} = qX_{ik}X_{ij}, & \text{if } j > k, \\ X_{ji}X_{ki} = X_{ki}X_{ji}, & \text{if } j > k, \\ X_{ij}X_{rs} = q^{-1}X_{rs}X_{ij}, & \text{if } i > r, j < s, \\ X_{ij}X_{rs} - X_{rs}X_{ij} = (q^{-1} - 1)X_{is}X_{rj}, & \text{if } i < r, j < s \end{cases}$$

As an R -module, $R[M_n(q)]$ has a basis $\{\prod_{ij} X_{ij}^{t_{ij}} | t_{ij} \in \mathbb{Z}^+\}$, where the products are formed with respect to any fixed order of the X_{ij} 's. Let $A_q(n, r)$ be the r th homogeneous component of $R[M_n(q)]$. Then $A_q(n, r)$ has a basis

$$\{X_{\lambda\mu}^d := X_{\mathbf{i}_\lambda d, \mathbf{i}_\mu} | \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda\mu}\},$$

where $\mathcal{D}_{\lambda\mu}$ denotes the set of distinguished representatives for $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_\mu$ (see Definition 2.2), and $X_{\mathbf{i}\mathbf{j}} = X_{i_1, j_1} X_{i_2, j_2} \cdots X_{i_r, j_r}$ if $\mathbf{i} = (i_1, \dots, i_r)$ and $\mathbf{j} = (j_1, \dots, j_r)$. Denote by $A_q(n, r)^*$ the linear dual of $A_q(n, r)$. Then, by [11]

$$\varphi : \text{End}_{\mathcal{H}_r} \left(\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_r \right) \cong A_q(n, r)^*,$$

where the natural basis for q -Schur algebra $\text{End}_{\mathcal{H}_r} \left(\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_r \right)$ is given as follows:

For $\lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda\mu}$, if we identify $\psi_{\lambda\mu}^d$ with its images under the isomorphism above. The basis $\{\psi_{\lambda\mu}^d\}$ is the dual of the basis $\{X_{\lambda\mu}^d := X_{\mathbf{i}_\lambda d, \mathbf{i}_\mu} | \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda\mu}\}$ for $A_q(n, r)$. Moreover by [11], we have $\varphi(\psi_{\lambda\mu}^d)(X_{\rho\nu}^{d_1}) = \delta_{\lambda\rho} \delta_{\mu\nu} \delta_{d, d_1}$. Sometimes we denote the basis $\{\varphi(\psi_{\lambda\mu}^d) | \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda\mu}\} \subset A_q(n, r)^*$ as $\{\psi_{\mathbf{i}_\lambda d, \mathbf{i}_\mu} | \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda\mu}\}$.

Remark 3.2. Combination correspondence:

By using the notations of section 2.1, we can identify the following three sets:

$$(3.2) \quad \Xi : \bigsqcup_{\lambda, \mu \in \Lambda(n, r)} \mathcal{D}_{\lambda\mu} \rightarrow I(n, r) \times I(n, r) / \sim \rightarrow \Lambda(n, n; r)$$

$$(3.3) \quad d \in \mathcal{D}_{\lambda\mu} \mapsto \overline{(\mathbf{i}_\lambda d, \mathbf{i}_\mu)} \mapsto wt(\mathbf{i}_\lambda d, \mathbf{i}_\mu)$$

Where we put $I(n, r) \times I(n, r)$ as a quotient set of $I(n, r) \times I(n, r)$ with following relation

$$(\mathbf{i}, \mathbf{j}) \sim (\mathbf{i}', \mathbf{j}') \iff wt(\mathbf{i}, \mathbf{j}) = (\mathbf{i}', \mathbf{j}'),$$

and the map wt is defined in subsection 2.1.

Remark 3.3. The elements of $\Lambda(n, n; r)$ can be realized as $n \times n$ matrices of non-negative integers $(\omega_{st})_{s, t}$ with $1 \leq s, t \leq n$ such that $\sum_{s, t=1}^n \omega_{st} = r$.

Denote by $\Lambda^s(n, n; r)$ the subset of $\Lambda(n, n; r)$ as

$$\Lambda^s(n, n; r) = \{\omega \in \Lambda(n, n; r) | \omega_{ij} = 0 \forall i > j, \sum_{1 \leq k \leq l \leq n} (l - k) \omega_{kl} \geq s\}.$$

Under the identification in 3.2, we simply find that

- (1) (\mathbf{i}, \mathbf{j}) satisfies that $\mathbf{i} \geq \mathbf{j} \iff Put(\omega_{ij}) = wt(\mathbf{i}, \mathbf{j})$, which means $\omega_{ij} = 0 \forall i > j$.
- (2) Put $(\omega_{ij}) = wt(\mathbf{i}_\lambda d, \mathbf{j}_\mu)$. There exists a pair i, j such that $\omega_{ij} \neq 0 \implies \lambda \neq \mu$.

Thus, we can set two definition of subset as following

$$\Omega^{\geq 0} := \Xi^{-1}(\Lambda^0(n, n; r)) = \bigsqcup_{\lambda, \mu \in \Lambda(n, r)} \{d \in \mathcal{D}_{\lambda\mu} | \mathbf{i}_\lambda d \geq \mathbf{i}_\mu\} := \bigsqcup_{\lambda, \mu \in \Lambda(n, r)} \Omega_{\lambda\mu}^{\geq 0}.$$

$$\Omega^{\geq 1} := \Xi^{-1}(\Lambda^1(n, n; r)) = \{d \in \Omega^{\geq 0} | \lambda, \mu \in \Lambda(n, r), \lambda \neq \mu \text{ and } d \neq 1\}.$$

Similarly, we can defined a sequence of sets as $\Omega^{\succeq} \supseteq \Omega^{\succeq 1} \supseteq \Omega^{\succeq 2} \supseteq \dots \Omega^{\succeq n} \supseteq \dots$, and defined subspaces of $S_R(n, r)$:

$$J_n := \bigoplus_{d \in \Omega^{\succeq n}} R\psi_{\lambda\mu}^d \subseteq S_R^+(n, r) \quad \text{with } d \in \mathcal{D}_{\lambda\mu}.$$

We set $S_R^+(n, r) := J_0$, it is Borel subalgebra of q -Schur algebra. This definition is the same as [12]. $J_1 = \bigoplus_{\psi_{\lambda\mu}^d \neq \psi_{\lambda\lambda}^1} R\psi_{\lambda\mu}^d$ and $J_0 \supseteq J_1 \supseteq J_2 \supseteq \dots$

Moreover, if we define subsets of $\Omega^{\succeq n}$ as $\Omega_{\lambda\mu}^{\succeq n} := \Omega^{\succeq n} \cap \mathcal{D}_{\lambda\mu}$, then, we find trivially that $\bigsqcup_{\lambda, \mu \in \Lambda(n, r)} \Omega_{\lambda\mu}^{\succeq n} = \Omega^{\succeq n}$. Using this notations, we can state the following lemmas.

Lemma 3.4. *If $d_1 \in \Omega_{\lambda\mu}^{\succeq n}$ and $d_2 \in \Omega_{\mu\nu}^{\succeq m}$, then $\psi_{\lambda\mu}^{d_1} \psi_{\mu\nu}^{d_2} = \sum_{d \in \Omega_{\lambda\nu}^{\succeq m+n}} a_d \psi_{\lambda\nu}^d$ for some $a_d \in R$.*

Proof. Suppose $\lambda, \mu \in \Lambda(n, r), d \in \Omega_{\lambda\mu}^{\succeq n} \subseteq \mathcal{D}_{\lambda\mu}$. We claim that $\psi_{\lambda\mu}^d(X_{\mathbf{ij}}) \neq 0$ implies $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I(n, r)$ satisfies $\sum_{1 \leq l \leq k \leq n} (l - k)\omega_{lk} = \sum_{k=1}^r (i_k - j_k) \geq n$.

Indeed, by the definition above, we have hypothesis that $\mathbf{j} = \mathbf{i}_\mu w$ for some $w \in \mathfrak{S}_r$. If $\ell(w) = 0$, i.e., $w = 1$, then $\mathbf{j} = \mathbf{i}_\mu$ and $\mathbf{i} = \mathbf{i}_\lambda d$, which trivially shows $(\mathbf{i}, \mathbf{j}) \in \Omega_{\lambda\mu}^{\succeq n}$.

Assume now $\ell(w) > 0$. Write $w = w't$ with $t = (a, a+1)$ and $\ell(w) = \ell(w') + 1$. Then by definition of \mathbf{i}_μ , we have $j_a > j_{a+1}$. If $i_a \leq i_{a+1}$, then by 3.1, $X_{\mathbf{ij}} = qX_{\mathbf{it}}, \mathbf{jt} = qX_{\mathbf{it}}, \mathbf{i}_\mu w'$. By induction on the length of elements in \mathfrak{S}_r , $\psi_{\lambda\mu}^d(X_{\mathbf{ij}}) = q\psi_{\lambda\mu}^d(X_{\mathbf{it}}, \mathbf{i}_\mu w') \neq 0$ shows $(\mathbf{it}, \mathbf{i}_\mu w')$ satisfy above claim, which trivially implies (\mathbf{i}, \mathbf{j}) satisfy this claim too.

If $i_a > i_{a+1}$, then also by 3.1 relations.

$$\begin{aligned} X_{i_a j_a} X_{i_{a+1} j_{a+1}} &= X_{i_{a+1} j_{a+1}} X_{i_a j_a} - (q^{-1} - 1) X_{i_{a+1} j_a} X_{i_a j_{a+1}} \\ &= X_{i_{a+1} j_{a+1}} X_{i_a j_a} - (1 - q) X_{i_a j_{a+1}} X_{i_{a+1} j_a} \end{aligned}$$

Thus $\psi_{\lambda\mu}^d(X_{\mathbf{ij}}) = \psi_{\lambda\mu}^d(X_{\mathbf{it}, \mathbf{jt}}) - (1 - q)\psi_{\lambda\mu}^d(X_{\mathbf{i}, \mathbf{jt}}) \neq 0$. We have either $\psi_{\lambda\mu}^d(X_{\mathbf{it}, \mathbf{jt}}) \neq 0$ or $\psi_{\lambda\mu}^d(X_{\mathbf{i}, \mathbf{jt}}) \neq 0$. By induction, either $(\mathbf{it}, \mathbf{jt})$ or $(\mathbf{i}, \mathbf{jt})$ satisfy above claim, which trivially shows that (\mathbf{i}, \mathbf{j}) satisfies this claim too.

Using the multiplication rules in $A_q(n, r)^*$, we have following:

$$\begin{aligned} \psi_{\lambda\mu}^{d_1} \cdot \psi_{\mu\nu}^{d_2}(X_{\mathbf{ij}}) &= \langle \psi_{\lambda\mu}^{d_1} \otimes \psi_{\mu\nu}^{d_2}, \Delta(X_{\mathbf{ij}}) \rangle \\ &= \sum_{\mathbf{k} \in I(n, r)} \psi_{\lambda\mu}^{d_1}(X_{\mathbf{ik}}) \psi_{\mu\nu}^{d_2}(X_{\mathbf{kj}}) \neq 0. \end{aligned}$$

So there is $\mathbf{k} \in I(n, r)$ such that $\psi_{\lambda\mu}^{d_1}(X_{\mathbf{ik}}) \neq 0$ and $\psi_{\mu\nu}^{d_2}(X_{\mathbf{kj}}) \neq 0$. Thus, $\sum_{h=1}^r (i_h - k_h) \geq n$

and $\sum_{l=1}^r (k_l - j_l) \geq m$, and hence $\sum_{s=1}^r (i_s - j_s) = \sum (i_h - k_h) - \sum (k_l - j_l) \geq n + m$.

And we only need to show that $a_d \neq 0$ only if $\mathbf{i}_\lambda \succeq \mathbf{i}_\nu$, which has already been done in [12] by Du and Rui. □

Proposition 3.5. *The subspaces $\{J_m\}$ with $m \in \mathbb{N}_0$ are ideals of $S_R^+(n, r)$. Moreover, nilpotent ideal J_1 is actually the radical of $S_R^+(n, r)$ when R is a field, which is spanned by $\{\psi_{\lambda\mu}^d | \lambda \neq \mu, d \neq 1\}$.*

Proof.

$$\begin{aligned}\psi_{\lambda\mu}^{d_1} \cdot \psi_{\omega\nu}^{d_2}(X_{\mathbf{ij}}) &= \langle \psi_{\lambda\mu}^{d_1} \otimes \psi_{\omega\nu}^{d_2}, \Delta(X_{\mathbf{ij}}) \rangle \\ &= \sum_{k \in I(n, r)} \psi_{\lambda\mu}^{d_1}(X_{\mathbf{ik}}) \psi_{\omega\nu}^{d_2}(X_{\mathbf{kj}}).\end{aligned}$$

Using the formula above, $\psi_{\lambda\mu}^d(X_{\rho\nu}^{d_1}) = \delta_{\lambda\rho} \delta_{\mu\nu} \delta_{d, d_1}$, we can claim that: if that $\mu \neq \omega$, then $\psi_{\lambda\mu}^{d_1} \cdot \psi_{\omega\nu}^{d_2}(X_{\mathbf{ij}}) = 0$. Then by the consequence of Lemma 3.4, we find subspace $J_n := \bigoplus_{d \in \Omega \geq n} R\psi_{\lambda\mu}^d$ is ideal of $S_R^+(n, r)$, since $J_0 J_n \subseteq J_n$ with $J_0 = S_R^+(n, r)$.

Let $L_{n, r} := \bigoplus_{\lambda \in \Lambda(n, r)} R\psi_{\lambda\lambda}^1$, then $L_{n, r}$ is a commutative R -subalgebra of $S_R^+(n, r)$, and $S_R^+(n, r) = L_{n, r} \oplus J_1$. For every $\lambda \in \Lambda(n, r)$ we have a R -free module $R_\lambda := R\psi_{\lambda\lambda}^1$ of rank one. Note that $\psi_{\lambda\lambda}^1$ acts on R_λ as identity, and $\psi_{\mu\mu}^1$, $\mu \neq \lambda$, acts as zero. We will denote in the same way the module over $S_R^+(n, r)$ obtain from R_λ by the natural projection of $S_R^+(n, r)$ on $L_{n, r}$.

Note that if R is a field the algebra $L_{n, r}$ is semi-simple, and so J_1 is the radical of $S_R^+(n, r)$. In this case, $\{R_\lambda | \lambda \in \Lambda(n, r)\}$ is a complete set of pairwise non-isomorphic simple modules over $S_R^+(n, r)$. \square

Remark 3.6. *By the result of 3.5, in the category of rings, we can construct a splitting map $p: S_R^+(n, r) \rightarrow \bigoplus_{\lambda \in \Lambda(n, r)} R\psi_{\lambda\lambda}^1$ of the including map $i: \bigoplus_{\lambda \in \Lambda(n, r)} R\psi_{\lambda\lambda}^1 \hookrightarrow S_R^+(n, r)$.*

From now on, denote $A := S_R^+(n, r)$, $J := \text{rad}(S_R^+(n, r))$ and $S := \bigoplus_{\lambda \in \Lambda(n, r)} R\psi_{\lambda\lambda}^1$. Then, we define a homomorphism of S -bimodules as $\tilde{p}: A \rightarrow J$ with $a \mapsto a - p(a)$. Obviously S is a commutative ring and the restriction of \tilde{p} to J is the identity map.

Definition 3.7. *For every left A -module M , define a complex $\mathbb{C}_*(A, S, M)$ whose factors denoted by $C_k(A, S, M)$, where integer $k \geq -1$, following the notation of in Remark 3.6. We set the several notations:*

$$(3.4) \quad \begin{cases} C_{-1}(A, S, M) = M & k = -1 \\ C_0(A, S, M) = A \otimes M & k = 0 \\ C_k(A, S, M) = A \otimes J^{\otimes k} \otimes M & k > 1 \end{cases}$$

where all the tensor products are taken over S .

Next we define A -module homomorphism $d_{k, j}: C_k(A, S, M) \rightarrow C_{k-1}(A, S, M)$, $0 \leq j \leq k$, and S -module homomorphisms $s_k: C_k(A, S, M) \rightarrow C_{k+1}(A, S, M)$ by:

$$\begin{aligned}d_{0,0}(m) &:= am, \\ d_{k,0}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= aa_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m, \\ d_{k,j}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= a \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k \otimes m, \quad 1 \leq j \leq k-1, \\ d_{k,k}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= a \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k m,\end{aligned}$$

$$\begin{aligned}
s_{-1}(m) &:= e \otimes m, \\
s_k(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= e \otimes \tilde{p}(a) \otimes a_1 \otimes \cdots \otimes a_k \otimes m, 0 \leq k.
\end{aligned}$$

Define an A -module homomorphism $d_k: C_k(A, S, M) \rightarrow C_{k-1}(A, S, M)$ by:

$$d_k := \sum_{t=0}^k (-1)^t d_{k,t}$$

Proposition 3.8. [24] *The sequence $(C_k(A, S, M), d_k)_{k \geq -1}$ is a complex of left A -module. Moreover, we have the relations:*

$$\begin{aligned}
d_0 s_{-1} &= id_{C_{-1}(A, S, M)}, \\
d_{k+1} s_k + s_{k-1} d_k &= id_{C_k(A, S, M)}, 0 \leq k.
\end{aligned}$$

Thus, these maps $\{s_k\}_{k \geq 1}$ give a splitting of $C_*(A, S, M)$ in the category of S -modules, i.e., Prove that id_{C_*} a zero holomorphic chain map. In particular, $(C_k(A, S, M), d_k)_{k \geq -1}$ is exact.

Definition 3.9. Let ξ be the anti-automorphism of the q -Schur algebra $S_R(n, r)$, which is defined as

$$S_R(n, r) \rightarrow S_R(n, r) \quad \psi_{\lambda\mu}^d \mapsto \psi_{\mu\lambda}^{d^{-1}}$$

It is clear that with definition of $S_R(n, r)$, $S_R^+(n, r)$ and $S_R^-(n, r) := \xi(S_R^+(n, r))$. This anti-automorphism ξ can be restricted as a ring homomorphism $S_R^+(n, r) \rightarrow S_R^-(n, r)$. Following the definition in Appendix 7 of [23], we define a functor \mathcal{J} which is called a contravariant dual functor of the left $S_R(n, r)$ -module category ${}_{S_R(n, r)}\mathbf{mod}$:

$$\begin{aligned}
{}_{S_R(n, r)}\mathbf{mod} &\rightarrow {}_{S_R(n, r)}\mathbf{mod} \\
V &\mapsto V^\otimes
\end{aligned}$$

where the left module structure is defined to satisfy that $(s\theta)(v) = \theta(\xi(s)v)$, for $\theta \in \text{Hom}_R(V, R)$, $s \in S_R(n, r)$, $v \in V$.

Then we may consider the right exact functor

$$F = S_R(n, r) \otimes_{S_R^-(n, r)} -: {}_{S_R^-(n, r)}\mathbf{mod} \rightarrow {}_{S_R(n, r)}\mathbf{mod},$$

and the left exact functor

$$G = \text{Hom}_{S_R^+(n, r)}(S_R(n, r), -): {}_{S_R^+(n, r)}\mathbf{mod} \rightarrow {}_{S_R(n, r)}\mathbf{mod}.$$

Lemma 3.10. *With the notation above, there is a $S_R(n, r)$ -isomorphism*

$$F(V^\otimes) \cong (G(V))^\otimes$$

naturally in $V \in S_R^+(n, r)\text{-mod}$

Proof. In [23](section 7), the author proved a more general result for these algebras. \square

Let $R_\Lambda := \bigoplus_{\lambda \in \Lambda} R\psi_\lambda$, with $\psi_\lambda := \psi_{\lambda\lambda}^1$. Then R_Λ is a commutative R -subalgebra of $S_R^+(n, r)$, and $S_R^+(n, r) = R_\Lambda \oplus J_1$. For every $\lambda \in \Lambda(n, r)$ we have a free R -module module $L_\lambda := R\psi_\lambda$ of rank one over R_Λ . Note that ψ_λ acts on L_λ by identity, and ψ_μ , $\mu \neq \lambda$, acts by zero. We will denote in the same way the module over $S_R^+(n, r)$ obtained from L_λ by inflating along the natural projection of $S_R^+(n, r)$ on R_Λ , i.e., J_1 and ψ_μ ($\mu \neq \lambda$) act on L_λ as zero, and ψ_λ as identity.

Note that if R is a field then algebra R_Λ is semi-simple, and so J_1 is the radical of $S_R^+(n, r)$. In this case $\{L_\lambda | \lambda \in \Lambda(n, r)\}$ is a complete set of pairwise non-isomorphic simple modules over $S_R^+(n, r)$.

For $\lambda \in \Lambda(n, r)$ we denote the resolution $C_*(S_R^+(n, r), R_\Lambda, L_\lambda)$ defined in 3.7 by $\mathbb{C}_*^+(L_\lambda)$, and call it *bar resolution*. The factor $\mathbb{C}_k^+(L_\lambda)$ in resolution $C_*(S_R^+(n, r), R_\Lambda, L_\lambda)$ has following form:

$$(3.5) \quad S_R^+(n, r) \otimes J_1 \otimes \cdots \otimes J_1 \otimes L_\lambda,$$

where all tensor products are over commutative ring L_λ and there are k factors J_1 .

Proposition 3.11. *Let $\nu, \mu \in \Lambda$ and $n \geq 0$. Then $\psi_\nu J_n \psi_\mu = 0$ unless $\nu \triangleright \mu$ (which means $\nu \succeq \mu$ but $\nu \neq \mu$). If $\nu \triangleright \mu$, then*

$$\{\psi_{\nu\mu}^d | d \in \Omega_{\nu\mu}^{\succeq n}\}$$

is an R -basis of the free R -module $\psi_\nu J_n \psi_\mu$.

Proof. From Remark 3.3 it follow that the set $J_n = \bigoplus_{\substack{d \in \Omega_{\theta\eta}^{\succeq n} \\ \theta, \eta \in \Lambda(n, r)}} R\psi_{\theta\eta}^d$. Then, we have $\psi_\nu J_n \psi_\mu = \psi_{\nu\nu}^1 \cdot \left(\bigoplus_{\substack{d \in \Omega_{\theta\eta}^{\succeq n} \\ \theta, \eta \in \Lambda(n, r)}} R\psi_{\theta\eta}^d \right) \cdot \psi_{\mu\mu}^1 = \bigoplus_{d \in \Omega_{\nu\mu}^{\succeq n}} R\psi_{\nu\mu}^d$. The first statement follows from the fact $\Omega_{\nu\mu}^{\succeq n} \subseteq \Omega_{\nu\mu}^{\succeq}$ and $d \in \Omega_{\nu\mu}^{\succeq}$ shows that $\nu \triangleright \mu$. \square

Proposition 3.12. *For all $\lambda \in \Lambda(n, r)$, We have $\mathbb{C}_0^+(L_\lambda) \cong S_R^+(n, r)\psi_\lambda$, and when any $k \geq 1$, factor of $\mathbb{C}_k^+(L_\lambda)$ has the form*

$$\bigoplus_{\mu^{(1)} \triangleright \cdots \triangleright \mu^{(k)} \triangleright \lambda} S_R^+(n, r)\psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda,$$

Proof. First of all, let M be a right R_Λ -module and N a left R_Λ -module. It follows from Corollary 9.3 in [20] that $M \otimes_{R_\Lambda} N \cong \bigoplus_{\lambda \in \Lambda(n, r)} M\psi_\lambda \otimes_R \psi_\lambda N$.

Then from the above statement, we can tell that $\mathbb{C}_k^+(L_\lambda)$ is the direct sum of $S_R^+(n, r)$ -modules like:

$$S_R^+(n, r)\psi_{\mu^{(1)}} \otimes \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes \cdots \otimes \psi_{\mu^{(k)}} J_1 \psi_{\mu^{(k+1)}} \otimes \psi_{\mu^{(k+1)}} L_\lambda.$$

where all tensor products are over R and the sum is taken on any sequence $(\mu^{(1)}, \dots, \mu^{(k+1)}) \in (\Lambda(n, r))^{k+1}$. With the property $\psi_{\mu^{(k+1)}} L_\lambda = 0$ unless $\mu^{(k+1)} = \lambda$ and the consequence of Proposition 3.12, the summation is in fact over the sequences $\mu^{(1)} \triangleright \cdots \triangleright \mu^{(k)} \triangleright \lambda$. \square

Proposition 3.13. *Let $\lambda \in \Lambda(n, r)$. Then $\mathbb{C}_*^+(L_\lambda)$ is a projective resolution of the module L_λ over $S_R^+(n, r)$ with finite length .*

Proof. Let N be the length of the maximal strictly decreasing sequence in $(\Lambda(n, r), \triangleright)$. Then $\mathbb{C}_k^+(L_\lambda) = 0$ for $k > N$ by Proposition 3.11 and 3.12. Therefore, we can tell that $\mathbb{C}_*^+(L_\lambda)$ is complex with finite length.

The summand $S_R^+(n, r)\psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda$ of $\mathbb{C}_*^+(L_\lambda)$ is a projective $S_R^+(n, r)$ -module, since $S_R^+(n, r)\psi_{\mu^{(1)}}$ is projective and every $\psi_{\mu^{(i)}} J_1 \psi_{\mu^{(i+1)}}$ is isomorphic to a free R -module with finite rank. Then, we can say that $\mathbb{C}_k^+(L_\lambda)$ is a projective $S_R^+(n, r)$ -module and $\mathbb{C}_*^+(L_\lambda)$ is a projective resolution of the module L_λ over $S_R^+(n, r)$. \square

4. Woodcock's condition and Kempf's Vanishing theorem

In this section we explain a condition which appears in Kashiwara's work [17] and Woodcock's work [17], and explain how the result of Kempf's Vanishing theorem can be applied to prove that $S_R^+(n, r)$ -module L_λ is acyclic for the induction functor $S_R(n, r) \otimes_{S_R^+(n, r)} -$.

Let P be a set and $\lambda \in P$, and let $R(\lambda)$ be a copy of the trivial R -coalgebra. Let 1_λ be the element of $\coprod_{\lambda \in P} R(\lambda)$ whose μ -component is $\delta_{\lambda, \mu}$. Then, for any left (resp. right) comodule (V, ρ) of $\coprod_{\lambda \in P} R(\lambda)$, we have $V = \otimes_{\lambda \in P} {}^\lambda V$ (resp. $\otimes_{\lambda \in P} V^\lambda$), where ${}^\lambda V := \{v \in V | \rho(v) = 1_\lambda \otimes v\}$ (resp. $V^\lambda := \{v \in V | \rho(v) = v \otimes 1_\lambda\}$) is called left (resp. right) *weight space* for weight λ .

Suppose we are given a partial order \leq on a poset P and a subset P^+ of P which is a *locally finite* poset, i.e., for each $\lambda \in P$ there are only finitely many $\mu \in P^+$ with $\mu \leq \lambda$.

Let $(A(\Lambda), \mu_\Gamma^\Lambda)$ be a filtered system of R -coalgebras indexed by the finite ideals in P^+ . Assume that each map $\mu_\Gamma^\Lambda : A(\Gamma) \rightarrow A(\Lambda)$ is injective, and maps $A(\Lambda) \rightarrow \coprod_{\lambda \in P} R(\lambda)$ compatible with μ_Λ^Γ . Put $C = \varinjlim_\Lambda A(\Lambda)$.

Definition 4.1. *we say that $(A(\Lambda), P^+, C)$ satisfy a Woodcock condition: If we have an isomorphism of bicomodules*

$$A(\Lambda)/A(\Lambda \setminus \{\lambda\}) \cong \nabla(\lambda) \otimes \nabla'(\lambda).$$

where $\nabla(\lambda)$ and $\nabla'(\lambda)$ are, respectively, left and right $A(\lambda)$ -comodules satisfying:

- (1) ${}^\lambda \nabla(\lambda) \cong \nabla'(\lambda)^\lambda \cong R(\lambda)$.
- (2) For all $\lambda \in P$, ${}^\mu \nabla(\lambda) \neq 0$ or $\nabla'(\lambda)^\mu \neq 0$ implies $\mu \leq \lambda$.
- (3) $\nabla(\lambda)$ and $\nabla'(\lambda)$ are finitely generated and projective over R .

The most valuable example of Woodcock condition appears in quantized enveloping algebra and the *coordinate algebra* corresponding to it.

Example 4.2. [28] *Let U be a quantized enveloping algebra.*

$A := \{a \in U^* | Ua, aU \text{ are both integrable}\}$ has a $\mathbb{Q}(q)$ -coalgebra structure where the comultiplication is defined by $\Delta(c)(u \otimes v) = c(vu)$ and the counit by $\varepsilon(c) = c(1)$, for any $c \in A$ and $u, v \in U$.

In [28], Woodcock showed that, if Λ is a finite ideal in P^+ and $V \in U\text{-int}$, let $O_\Lambda V$ be the largest submodule V' of V such that ${}^\lambda V' \neq 0$, $\lambda \in P^+$ implies $\lambda \in \Lambda$. Put $A(\Lambda) := O_\Lambda A$, we know that $(A(\Lambda), P^+, A)$ is a triple which satisfies Woodcock's condition.

Remark 4.3. [28] Let Λ be a finite ideal of P^+ . Denote $A_R(\Lambda) = R \otimes_{\mathcal{A}} A(\Lambda)$ and $A_R = R \otimes_{\mathcal{A}} A$, where $A(\Lambda)$ is defined as above. Moreover, let λ be the maximal one in Λ and $\Gamma = \Lambda \setminus \{\lambda\}$, then we have a short exact sequence in U -bimodule

$$(4.1) \quad 0 \rightarrow A_R(\Gamma) \rightarrow A_R(\Lambda) \rightarrow \nabla_R(\lambda) \otimes \nabla_R(\lambda)' \rightarrow 0,$$

which maps those global basis elements in $A_R(\Lambda) \setminus A_R(\Gamma)$ bijectively onto the standard global basis of $\nabla(\lambda) \otimes \nabla(\lambda)'$.

Therefore, the triple $(A_R(\lambda), P^+, A_R)$ satisfies the Woodcock condition defined above, where the ground ring R has a ring morphism $\mathbb{Z}[q, q^{-1}] \rightarrow R$.

Definition 4.4. [17] For $\mu \in W \cdot \lambda$ with $\lambda \in P^+$, W is the weyl group associated with root datum of U . Let ν_μ be the element of global basis of $\Delta(\lambda)$ with weight μ . If $w \in W$ with $m = \langle \alpha^\vee, w\lambda \rangle \geq 0$, we have

$$\nu_{s_i w \lambda} = f_i^{(m)} \nu_{w \lambda}, \quad \nu_{w \lambda} = e_i^{(m)} \nu_{s_i w \lambda}.$$

Put $\Delta^\#(\mu) := U^\# \cdot \nu_\mu$. They are the Demazure modules associated to μ .

Example 4.5. [28] Dualizing short exact sequence (4.1) and induction on $|\Lambda|$, we give an algebra epimorphism $U_R \rightarrow S_R$, $u \mapsto u \cdot 1$. Write S_R^b for the image of U_R^b under this map.

The Borel Schur algebras still fit the framework of Woodcock's condition:

Let $\Xi \leq P$ be a finite ideal for the antipodal excellent order, then $W\Xi \cap P^+$ is a finite ideal in P^+ . Thus we may choose $W\Xi \cap P^+$ to be the ideal Λ . Let $F_\Xi : S_R^b \mathbf{mod} \rightarrow S_R^b \mathbf{mod}$ take V to its largest quotient with weight in Ξ , a right exact functor.

Put $S_R^b(\Xi) = F_\Xi S_R^b$. By the alternative version in [25], if μ is maximal in Ξ for the antipodal excellent order, then there is a short exact sequence of bimodules

$$(4.2) \quad 0 \rightarrow \Delta_R^b(\mu) \otimes \bar{\Delta}_R^\#(\mu)^\tau \rightarrow S_R^b(\Xi) \rightarrow S_R^b(\Xi \setminus \{\mu\}) \rightarrow 0.$$

By induction on $|\Xi|$ it now can be shown that S_R^b is finitely generated free over R . Therefore, the $A_R^b(\Xi) := (S_R^b(\Xi))^*$ forms a filtered system of R -coalgebras. Write C_R^b as the colimit of this system, which is a free R -coalgebra.

Suppose one takes P for both P and P^+ , \preceq for order \leq , and defines the maps $A_R^b \rightarrow \coprod_{\lambda \in P} R(\lambda)$ by using the idempotent ε_μ . Then the triple (A_R^b, P, C_R^b) satisfies the Woodcock's condition. It implies that the roles of the $\Delta(\mu)$ and $\Delta'(\mu)$ are respectively played by $\Delta_R^b(\mu)$ and $\bar{\Delta}_R^\#(\mu)'$.

The triple $(A(\Lambda), P^+, C)$ with Woodcock condition have some very interesting homological consequences [28]. One of them is called *Ext-reciprocity*. For $V \in S(\Lambda)\text{-Mod}$, $X \in R\text{-Mod}$, and $\mu \in P$, we have

$$(4.3) \quad \text{Ext}_{S(\Lambda)}^i(S(\Lambda)^\mu \otimes X, V) \cong \text{Ext}_R^i(X, {}^\mu V) \quad \forall i \geq 0,$$

$$(4.4) \quad \text{Ext}_{S(\Lambda)}^i(V, A(\Lambda)^\mu \otimes X) \cong \text{Ext}_R^i({}^\mu V, X) \quad \forall i \geq 0.$$

A q -analogue Kempf's vanishing theorem was established by some properties of the crystal basis proved by Kashiwara in order to obtain the refined Demazure character formula in [17][22].

Woodcock used that the ideal of using the properties of the crystal basis to obtain a quantized Kempf's vanishing theorem also works for Schur algebra's version. Let us recall the *Kempf's vanishing theorem* in Woodcock [28]:

For any R -module X and $\mu \in P$, note that μ^+ as the unique dominant weight in orbit $W\mu$. We have

$$\begin{aligned} \text{Ext}_{S_R^b}^i(S_R, \nabla_R^b(\mu) \otimes X) &\cong \text{Ext}_R^i({}^\mu S_R, X) \cong \text{Ext}_R^i(\Delta_R(\mu^+)^\tau, X) \\ &\cong \begin{cases} \Delta_R(\mu^+) \otimes X & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \end{aligned}$$

Remark 4.6. *The case $\mu = \mu^+$ of 4.5 is Kempf's vanishing theorem in [22], since $H^i(U_R/U_R^b) \cong \text{Ext}_{S_R^b(W\Lambda)}^i(S_R(W\Lambda), \Delta(\lambda))$ for all $i \geq 0$. If Λ is a finite ideal in P^+ , and $V \in S_R^b(W\Lambda)\text{-mod}$.*

Moreover, in this situation, we have $\nabla_R^b(\mu) = L_\mu$, and $S_R^b = S_R^-$. Thanks to the definition of Demazure modules in 4.4, which means the module L_λ is $\text{Hom}_{S_R^-}(S_R, -)$ -acyclic.

Theorem 4.7. *For $\lambda \in \Lambda^+(n, r)$, the complex $S_R(n, r) \otimes_{S_R^+(n, r)} \mathbb{C}_*^+(L_\lambda)$ is a projective resolution of $W_\lambda^L := S_R(n, r) \otimes_{S_R^+(n, r)} L_\lambda$ over $S_R(n, r)$.*

Proof. Given $\lambda \in \Lambda^+(n, r)$, denote the complex $\mathbb{B}(R) := S_R \otimes_{S_R^+} \mathbb{C}_*^+(L_\lambda)$. Since the S_R -module isomorphism holds:

$$\mathbb{B}(R) \cong \bigoplus_{\mu^{(1)} \triangleright \dots \triangleright \mu^{(k)} \triangleright \lambda} S_R \psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \dots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda,$$

and $S_R \psi_{\mu^{(i)}} \otimes_R \psi_{\mu^{(i)}} J_1 \psi_{\mu^{(i+1)}}$ with $1 \leq i \leq k$ and $\psi_{\mu^{(k)}} J_1 \psi_\lambda$ are all free R -module by Proposition 3.11, it follows that all factors in $\mathbb{B}(R)$ are free R -modules.

The short exact sequence

$$0 \rightarrow H_i(\mathbb{B}(\mathbb{Z})) \otimes R \rightarrow H_i(\mathbb{B}(R)) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{i-1}(\mathbb{B}(\mathbb{Z})), R) \rightarrow 0$$

follows from the Universal Coefficient Theorem over the complex $\mathbb{B}(\mathbb{Z}) \otimes_{\mathbb{Z}} R \cong \mathbb{B}(R)$.

Hence, in order to show that the complex $\mathbb{B}(R)$ are acyclic, it is enough to prove the claim that the complex $\mathbb{B}(\mathbb{Z})$ is acyclic.

We already know that $H_i(\mathbb{B}(\mathbb{Z}))$ is a finitely generated abelian group. Therefore we can write $H_i(\mathbb{B}(\mathbb{Z}))$ as $\mathbb{Z}^\alpha \bigoplus_{p \text{ is a prime } s \geq 1} \bigoplus (\mathbb{Z}/p^s \mathbb{Z})^{\alpha_{ps}}$, where only finitely many of the integer α, α_{ps} are different from zero. For every prime p denote by $\bar{\mathbb{F}}_p$ the algebraic closure \mathbb{F}_p . We get $H_i(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p \cong \bar{\mathbb{F}}_p^{\sum_{s \geq 1} \alpha_{ps}}$, and also $H_k(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^\alpha$.

Thus, in order to prove the above claim, we only need to show that $H_i(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $H_i(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p = 0$ for any prime number p .

Let \mathbb{L} denote either \mathbb{Q} or the field $\bar{\mathbb{F}}_p$ for a prime p . Then, $H_i(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{L}$ is a submodule of $H_i(\mathbb{B}(\mathbb{L}))$ due to the Universal Coefficient Theorem. Hence, in order to prove $H_i(\mathbb{B}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{L} = 0$, it is enough to show that $H_i(\mathbb{B}(\mathbb{L})) = 0$.

Now we prove that $H_i(\mathbb{B})(\mathbb{L}) = 0$.

First, the algebra $S_{\mathbb{L}}(n, r)$ has an anti-involution $\xi : S_{\mathbb{L}}(n, r) \rightarrow S_{\mathbb{L}}(n, r)$ defined on the basis elements by $\varphi_{\lambda\mu}^d \mapsto \varphi_{\mu\lambda}^{d^{-1}}$. The image of $S_{\mathbb{L}}^+(n, r)$ under ξ is the subalgebra $S_{\mathbb{L}}^-(n, r)$ of $S_{\mathbb{L}}(n, r)$.

As in Definition 3.9, this homomorphism ξ induces a contravariant equivariance of categories $\mathcal{J} : S_{\mathbb{L}}^+(n, r)\mathbf{mod} \rightarrow S_{\mathbb{L}}^-(n, r)\mathbf{mod}$.

By Lemma 3.10, the following exact functors are isomorphic:

$$\begin{aligned} \mathcal{J} \circ \mathrm{Hom}_{S_{\mathbb{L}}^-(n, r)}(S_{\mathbb{L}}, -) \circ \mathcal{J} : S_{\mathbb{L}}^+(n, r)\mathbf{mod} &\rightarrow S_{\mathbb{L}}(n, r)\mathbf{mod} \\ S_{\mathbb{L}}(n, r) \otimes_{S_{\mathbb{L}}^+(n, r)} - : S_{\mathbb{L}}^+(n, r)\mathbf{mod} &\rightarrow S_{\mathbb{L}}(n, r)\mathbf{mod}. \end{aligned}$$

Second, we recall that $\mathbb{B}(\mathbb{L}) = S_{\mathbb{L}} \otimes_{S_{\mathbb{L}}^+} \mathbb{C}_*^+(L_\lambda) \cong \mathcal{J}(\mathrm{Hom}_{S_{\mathbb{L}}^-}(S_{\mathbb{L}}, \mathcal{J}(B_*(L_\lambda))))$. Furthermore, by *Kempf's vanishing theorem* we already know that the module L_λ is $\mathrm{Hom}_{S_{\mathbb{L}}^-}(S_{\mathbb{L}}, -)$ -acyclic. Thus, after applying \mathcal{J} , we get that the complex $\mathbb{B}(\mathbb{L})$ is exact. \square

5. The Boltje-Maisch complex

Suppose that $n \geq r$, then we know that there is a partition $\delta := (1, \dots, 1, 0, \dots, 0) \in \Lambda(n, r)$. Then there is a obvious isomorphism of algebras $\phi : \mathcal{H}_r \cong \psi_\delta S_R(n, r) \psi_\delta$. Therefore we can tell that if M is an $S_R(n, r)$ -module, the $\psi_\delta S_R(n, r) \psi_\delta$ -module $\psi_\delta M$ induced by ϕ which is a $R\mathcal{H}_r$ -module too. In fact the map $M \mapsto \psi_\delta M$ is functorial, and it defines a functor $\mathcal{S}_r : S_R(n, r)\mathbf{mod} \rightarrow R\mathcal{H}_r\mathbf{mod}$ which was named *Schur functor* in [16].

In this section we show that for any $\lambda \in \Lambda^+(n, r)$ the complex $\mathcal{S}_r(S_R \otimes_{S_R^+} \mathbb{C}_*^+(L_\lambda))$ is isomorphic to the complex which has been constructed in [22]. Here, we call it *Boltje-Maisch complex*.

First of all, we start with some notations and conventions.

Definition 5.1. [4] *For any $\lambda, \mu \in \Lambda(n, r)$, there is a R -submodule of $\mathrm{Hom}_{\mathcal{H}_r}(M^\mu, M^\lambda)$. One can denote it by $\mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^\mu, M^\lambda)$, which is a free R -module in the following form:*

$$(5.1) \quad \mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^\mu, M^\lambda) := \bigoplus_{d \in \Omega_{\lambda\mu}^>} R\psi_{\lambda\mu}^d.$$

Since $\psi_\lambda S_R^+(n, r) \psi_\mu = \psi_{\lambda\lambda}^1 \bigoplus_{\substack{d \in \Omega_{\theta\eta}^> \\ \theta, \eta \in \Lambda(n, r)}} R\psi_{\theta\eta}^d \psi_{\mu\mu}^1 = \bigoplus_{d \in \Omega_{\lambda\mu}^>} R\psi_{\lambda\mu}^d$, we know that $\mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^\mu, M^\lambda)$ equals $\psi_\lambda S_R^+(n, r) \psi_\mu$. Moreover, for $S_R^+(n, r) = J_1 \oplus R_\Lambda$, we have $\mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^\mu, M^\lambda) = \psi_\lambda J_1 \psi_\mu$ if $\lambda \triangleright \mu$.

Boltje and Maisch defined a complex \widetilde{B}_*^λ in Section 3.1 of [5], as following:

For some $\lambda \in \Lambda^+(n, r)$, $\widetilde{B}_{-1}^\lambda$ is the dual Specht module that relates to λ and \widetilde{B}_0^λ defined as $\mathrm{Hom}_R(M^\lambda, R)$. When $k \geq 1$, \widetilde{B}_k^λ is defined as the direct sum over all sequence $(\mu^{(1)} \triangleright \dots \triangleright \mu^{(k)} \triangleright \lambda)$ as

$$\bigoplus_{\substack{\mu^{(1)} \triangleright \dots \triangleright \mu^{(k)} \triangleright \lambda \\ \mu^{(1)}, \dots, \mu^{(k)} \in \Lambda^+(n, r)}} \mathrm{Hom}_R(M^{\mu^{(1)}}, R) \otimes_R \mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^{\mu^{(2)}}, M^{\mu^{(1)}}) \otimes_R \dots \otimes_R \mathrm{Hom}_{\mathcal{H}_r}^\wedge(M^\lambda, M^{\mu^{(k)}}).$$

The differential d_k , $k \geq 1$, in \widetilde{B}_*^λ is given by the formula

$$(5.2) \quad d_k(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = \sum_{t=0}^{k-1} (-1)^t f_0 \otimes \cdots \otimes f_t \circ f_{t+1} \otimes \cdots \otimes f_k.$$

and when $k = 0$, we put

$$(5.3) \quad d_0^\lambda : \widetilde{B}_0^\lambda = \text{Hom}_R(M^\lambda, R) \rightarrow \text{Hom}_R(S^\lambda, R) = \widetilde{B}_{-1}^\lambda, \quad \varepsilon \mapsto \varepsilon|_{S^\lambda}.$$

and finally obtain a chain complex with only finite no trivial terms:

$$(5.4) \quad \widetilde{B}_*^\lambda : \quad 0 \rightarrow \widetilde{B}_{f(\lambda)}^\lambda \xrightarrow{d_{f(\lambda)}^\lambda} \widetilde{B}_{f(\lambda)-1}^\lambda \xrightarrow{d_{f(\lambda)-1}^\lambda} \cdots \xrightarrow{d_1^\lambda} \widetilde{B}_0^\lambda \xrightarrow{d_0^\lambda} \widetilde{B}_{-1}^\lambda \rightarrow 0,$$

where $f(\lambda)$ is a positive integer by Proposition 3.13.

According to Theorem 4.2 and 4.4 in [4], we have:

Lemma 5.2. [5] \widetilde{B}_*^λ are exact in degree 0 and -1 .

Theorem 5.3. For $\lambda \in \Lambda^+(n, r)$, the complex \widetilde{B}_*^λ is isomorphic to the complex

$$\mathcal{S}_r(S_R(n, r) \otimes_{S_R^+(n, r)} \mathbb{C}_*^+(L_\lambda)).$$

Proof. For convenience, we denote the complex $\mathcal{S}_r(S_R(n, r) \otimes_{S_R^+(n, r)} \mathbb{C}_*^+(L_\lambda))$ by \widehat{B}_*^λ .

Since the complex \widehat{B}_*^λ is exact, and the exactness of \widetilde{B}_*^λ in degree 0 and -1 has been treated in Lemma 5.2, we only need to establish the isomorphism in the non-negative degrees. The isomorphism in the degree -1 will follow.

With the consequence of Proposition 3.12, we can write the factor \widehat{B}_k^λ as a direct sum of

$$(5.5) \quad \psi_\delta S_R(n, r) \psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda,$$

where subscripts satisfy $\mu^{(1)} \triangleright \cdots \triangleright \mu^{(k)} \triangleright \lambda$. Furthermore, it is straightforward that the summand (5.5) is isomorphic to

$$\text{Hom}_{\mathcal{H}_r}(M^{\mu^{(1)}}, \mathcal{H}_r) \otimes_R \text{Hom}_{\mathcal{H}_r}^\wedge(M^{\mu^{(2)}}, M^{\mu^{(1)}}) \otimes_R \cdots \otimes_R \text{Hom}_{\mathcal{H}_r}^\wedge(M^\lambda, M^{\mu^{(k)}}),$$

First, to show the correspondence of factors between \widehat{B}_k^λ and \widetilde{B}_*^λ in non-negative degrees, it is enough an isomorphism of \mathcal{H}_r -modules $\phi_\nu : \text{Hom}_{\mathcal{H}_r}(M^\nu, \mathcal{H}_r) \rightarrow \text{Hom}_R(M^\nu, R)$, to find for every $\nu \in \Lambda(n, r)$.

With the natural anti-automorphism of $\chi : \mathcal{H}_r \rightarrow \mathcal{H}_r$ and $f \in \text{Hom}_{\mathcal{H}_r}(M^\nu, \mathcal{H}_r)$, we can define right module structure of \mathcal{H}_r on $\text{Hom}_{\mathcal{H}_r}(M^\nu, \mathcal{H}_r)$ and $\text{Hom}_R(M^\nu, R)$, which is given by the formula $(f\sigma)(m) = f(m \cdot \chi(\sigma))$, where $m \in M^\nu$, and $\sigma \in \mathcal{H}_r$.

For $f \in \text{Hom}_{\mathcal{H}_r}(M^\nu, \mathcal{H}_r)$ and $m \in M^\nu$, define $\phi_\nu(f)(m)$ to be the coefficient of T_{id} in $f(m) \in \mathcal{H}_r$. Note that $\{T_w | w \in \mathfrak{S}_r\}$ is an R -basis of \mathcal{H}_r , now we claim that ϕ_ν is a homomorphism of \mathcal{H}_r -modules.

First, we have that $\phi_\nu(f) \cdot T_w = \phi_\nu(fT_w)$ because:

- (i) $(\phi_\nu(f) \cdot T_w)(m) = \phi_\nu(f)(mT_{w^{-1}}) =$ the coefficient of T_{id} in $f(mT_{w^{-1}})$,
- (ii) $\phi_\nu(fT_w)(m) =$ the coefficient of T_{id} in $(f \cdot T_w)(m) =$ the coefficient of T_{id} in $f(mT_{w^{-1}})$.

Assume that $\phi_\nu(f) = 0$, then the coefficient of T_{id} in $f(m)$ equals to 0, for any $m \in M^\nu$. By induction on the length of elements of \mathfrak{S}_r , suppose that the coefficient of T_w in $f(m)$ equals to 0, for any $m \in M^\nu$ and $\ell(w) \leq n$.

Let $w' \in \mathfrak{S}_r$ with $\ell(w') = n + 1$. Using the Mathas's formula in [21], one can find $s = (i, i + 1) \in \mathfrak{S}_r$ for some $1 \leq i \leq r - 1$ such that

$$T_{w'}T_s = qT_{w's} + (q - 1)T_{w'}, \quad \text{and} \quad n = \ell(w's) < \ell(w') = n + 1.$$

Meantime, $f(m)T_s = f(mT_s)$, and write $f(m) = \sum_{\ell(w) > n} \alpha_w T_w$ for some $\alpha_w \in R$. Then we have

$$0 = f(mT_s) = \sum_{\ell(w) > n} \alpha_w T_w \cdot T_s = q\alpha_w T_{w's} + \sum_{\substack{\ell(w) \geq n \\ w \neq w_0 s}} \alpha'_w T_w.$$

It follows that $q\alpha_w = 0$ then $\alpha_w = 0$ since $q \neq 0$. Therefore, we find out that $f(m) = 0$ for any $m \in M^\nu$ and then $f = 0$, which implies the homomorphism ϕ_ν is injective.

As free R -modules, it is obvious that the rank of $\text{Hom}_{\mathcal{H}_r}(M^\nu, \mathcal{H}_r)$ equals to $\text{Hom}_R(M^\nu, R)$, which means the homomorphism ϕ_ν is surjective. It implies that ϕ_ν is an isomorphism.

Lastly, by Definition 3.7, it is straightforward that the isomorphism ϕ_ν indeed induces a chain map from \tilde{B}_*^λ onto \hat{B}_*^λ , which is just the isomorphism from \tilde{B}_* to \hat{B}_* . \square

Remark 5.4. *With help of the above theorem and Theorem 4.7, we have shown that the Boltje-Maisch complex \tilde{B}_*^λ is exact for $\lambda \in \Lambda^+(n, r)$.*

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