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An Extended Multistep Shanks Transformation and Convergence Acceleration Algorithm with Their Convergence and Stability Analysis

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Abstract The molecule solution of an extended discrete Lotka–Volterra equation is constructed, from which a new sequence transformation is proposed. A convergence acceleration algorithm for implementing this sequence transformation is found. It is shown that our new sequence transformation accelerates some kinds of linearly convergent sequences and factorially convergent sequences with good numerical stability. Some numerical examples are also presented.

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1 Introduction

Some intimate relations between certain numerical algorithms and integrable systems have been revealed in recent years, which leads to a reinvestigation of both objects.

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On the one hand, many algorithms in numerical analysis when considered as dynamical systems, have a variety of interesting dynamical behavior (see [17]). One of the intriguing properties is *integrability*, which distinguishes these numerical schemes from others, and attracts workers in the field of integrable systems to study integrable properties of numerical algorithms. In the literature, integrability of numerical algorithms always appears in various guises such as properties of *invariance*, *compatibility* and *identity*. To be precise, the notion of invariance refers to the existence of a sufficient number of conserved quantities, which is a key feature of the Liouville integrability [4] in some sense. And the compatibility means that the equation can be seen as the compatibility condition of some linear problems. Finally, the identity property indicates that, essentially, integrable equations are some kind of determinantal (or Pfaffian) identities on so-called τ-function level [23,39]. For example, Gauss arithmetic-geometric mean algorithm [3,6,31] for computing the first elliptic integral can be viewed as an illustration of invariance. In fact, this algorithm corresponds to a discrete-time integrable equation, with the corresponding elliptic integral as its conserved quantity. In addition, the qd-algorithm [21,38], which plays a significant role in the theory of formal orthogonal polynomials and Padé approximants [5,9,35], is nothing but the compatibility condition of the spectral problem related to the discrete-time Toda equation [22]. For more examples, please consult [28–30,36,48] and references therein.

On the other hand, some integrable equations can lead to new algorithms. For instance, the discrete Lotka–Volterra equation can be used as an efficient algorithm to compute singular values [25,26,50]. Moreover, from the identity property of integrable systems, two new convergence acceleration algorithms have been constructed in [13,19]. Based on the observation that the integrable equation provided by the new algorithm given in [13] is only a special case of the extended Lotka–Volterra equation which was first proposed in [32] (more results can be found in [24]), it is a natural question whether new convergence acceleration algorithms may be obtained from other cases. This is what we want to do in this article.

Our main tools are determinant techniques and Hirota's bilinear method [23], which was invented by Hirota to solve integrable nonlinear differential or difference evolution equations having soliton solutions. The essence of Hirota's bilinear method is to change nonlinear differential or difference equations into a type of bilinear ones (often called *bilinear form*) through dependent variables transformations. It should be pointed out that there are various kinds of solutions to integrable equations, among which *molecule solutions* are closely related to sequence transformation, and one should notice that the terminology molecule solution specifically stands for the solutions of the semi-infinite and the finite nonperiodic equation with boundary values. For example, the famous Toda equation

$$\frac{d^2x_k}{dt^2} = e^{x_{k-1}-x_k} - e^{x_k-x_{k+1}}. \tag{1}$$

If $k = 0, 1, 2, \ldots$ with the boundary condition $x_0(t) = -\infty$, we call (1) the *semi-infinite Toda equation* or *infinite Toda molecule equation*; if $x_0(t) = -\infty$ and $x_{N+1} = +\infty$, we call (1) the *finite nonperiodic Toda equation* or *finite molecule Toda equation*.

Solutions corresponding to the above two equations are called Toda molecule solutions. Before presenting the extended discrete Lotka–Volterra equation, let us give a sketch of convergence acceleration algorithms and sequence transformations.

Convergence acceleration algorithms are an important class of numerical algorithms, which are used to accelerate the convergence of a given sequence. In numerical analysis, many methods produce sequences, for example iterative methods, perturbation methods, discretization methods and so on. Sometimes, the convergence of these sequences is so slow that the corresponding numerical methods are ineffective in practice. This is why we study *sequence transformations*, which are based on the idea of *extrapolation* [14,46]. Let (S_n) be a sequence converging to a limit S, satisfying

$$\lim_{n\to} \frac{S_{n+1} - S}{S_n - S} = \lambda.$$

When $0 < |\lambda| < 1$, we say that the sequence (S_n) converges *linearly*; when $\lambda = 1$, we say that this sequence converges *logarithmically*; and when $\lambda = 0$, it converges *superlinearly*. A sequence transformation $T : (S_n) \to (T_n)$, transforms this sequence to a new sequence (T_n), which converges faster to the same limit S under some assumptions, that is,

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

Sequence transformations are most useful in the case of linear and logarithmic convergence and also for certain types of divergence. However, sequence transformations normally accomplish little in the case of superlinear convergence. Fortunately, superlinearly convergent sequences usually converge so well that there is no compelling need to speed up their convergence by applying convergence acceleration techniques. There are many sequence transformations (see e.g. [9–12,14,46,51,52] and the reference therein), among which the most well known is Aitken's Δ^2 process due to Aitken [1], who used it to accelerate the convergence of Bernoulli's method for computing the dominant zero of a polynomial. Furthermore, Pennacchi [37] considered transformations of the form

$$C_n(p, m) = S_n + \frac{P_m(\Delta S_n, \ldots, \Delta S_{n+p-1})}{Q_{m-1}(\Delta S_n, \ldots, \Delta S_{n+p-1})}, \tag{2}$$

where P_m and Q_{m-1} are homogeneous polynomials of degree m and $m-1$, respectively, in p variables $\Delta S_n, \Delta S_{n+1}, \ldots, \Delta S_{n+p-1}$, and p, m are positive integers. Such a transformation is called *rational transformation of type* (p, m), denoted by $C_n(p, m)$. In this sense, Aitken's Δ^2 process is a rational transformation of type $(2, 2)$, and Pennacchi proved that any rational transformation of type $(2, m)$ with $m \geq 2$ which accelerates the set of linearly convergent sequences is equivalent to Aitken's process. He also gave a rational transformation of type $(3, 2)$

$$C_n(3, 2) = S_n + \frac{\Delta S_n[\Delta S_n - \Delta S_{n+1}] + [\Delta S_n \Delta S_{n+2} - (\Delta S_{n+1})^2]}{\Delta S_n - 2\Delta S_{n+1} + \Delta S_{n+2}}, \tag{3}$$

which accelerates the set of linearly convergent sequences.

For many sequence transformations obtained by extrapolation methods, new sequences can be expressed as ratios of two determinants. By using some determinantal identities, we can obtain a *recursive algorithm* for implementing the corresponding sequence transformation, such an algorithm is called *extrapolation algorithm*, or *convergence acceleration algorithm*. So far, many convergence acceleration algorithms have been found and investigated, such as the famous ε-algorithm proposed by Wynn [53], and some of its generalizations [8,16]. It is worth mentioning that sequence transformations are now also considered in the recently published NIST Handbook of Mathematical Functions [33, Chapter 3.9]. For more results, please refer to [14,46,51,52].

Then we return to the extended Lotka–Volterra equation, which is expressed as

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k-\frac{q-1}{2}+i}\right) = \prod_{i=0}^{N-1} a_{k-\frac{q-1}{2}+i} - \prod_{i=0}^{N-1} a_{k+\frac{q-1}{2}-i}, \\ q, N = 1, 2, \ldots, \text{ and } q \neq N, \tag{4}$$

or

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k-\frac{q-1}{2}+i}\right) = \prod_{i=0}^{-N-1} a^{-1}_{k-\frac{q+1}{2}-i} - \prod_{i=0}^{-N-1} a^{-1}_{k+\frac{q+1}{2}+i}, \quad q, -N = 1, 2, \ldots. \tag{5}$$

In [13], a new convergence acceleration algorithm was obtained from the discretization of (5) when $N = -1$. Now we consider equation (4), with $N = q + 1$. In this case, it can be written as

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k+i}\right) = \left(\prod_{i=0}^{q-1} a_{k+i}\right)(a_{k+q} - a_{k-1}),$$

with the following difference equation as its time discretization:

$$\left(\prod_{m=0}^{M_{k-1}} \frac{1 + a^{(n+mp+p+1)}_{k-mq-1}}{1 + a^{(n+mp+1)}_{k-mq-1}}\right)\left(1 + a^{(n+1)}_{k-1}\right)\prod_{i=0}^{q-1} a^{(n+1)}_{k+i} \\ = \left(\prod_{m=0}^{M_k} \frac{1 + a^{(n+mp+p)}_{k-mq}}{1 + a^{(n+mp)}_{k-mq}}\right)\left(1 + a^{(n)}_{k+q}\right)\prod_{i=0}^{q-1} a^{(n)}_{k+i}, \tag{6}$$

while $p = 0, 1, \ldots,$ and the nonnegative integer M_k is defined as $M_k = \lfloor k/q \rfloor + 1$, where $\lfloor x \rfloor$ stands for the greatest integer not exceeding x.

In this article, we first derive the bilinear form of the discrete equation (6), and then construct its molecule solution, from which we obtain a new sequence transformation.

We also show that there exists a two-dimensional difference equation, which shares the same bilinear form with Eq. (6) and can be used as a recursive algorithm for the implementation of the new sequence transformation.

Our article is organized as follows: In Sect. 2, we will derive the molecule solution of Eq. (6) with the help of bilinear method and determinantal identities. In Sect. 3, a new sequence transformation is constructed, and also its corresponding recursive algorithm. In Sect. 4, we will give the convergence and stability analysis of the new sequence transformation. In Sect. 5, some numerical examples are proposed. Section 6 is devoted to conclusion and discussions.

2 Molecule solution of Eq. (6)

In this section, we construct the molecule solution of the extended discrete Lotka–Volterra equation by using Hirota's bilinear method and determinantal identities [2,15].

It can be proved that under the dependent variable transformation

$$a_k^{(n)} = \frac{f_{k-1}^{(n+p+1)} f_{k+q+1}^{(n)}}{f_k^{(n+p+1)} f_{k+q}^{(n)}}, \tag{7}$$

with $f_k^{(n)}$ satisfying initial conditions $f_{-q}^{(n)} = \cdots = f_0^{(n)} \equiv 1$, the extended discrete Lotka–Volterra equation (6) could be transformed into the following bilinear form

$$f_{k+q}^{(n+1)} f_k^{(n+p)} - f_k^{(n+p+1)} f_{k+q}^{(n)} = f_{k+q+1}^{(n)} f_{k-1}^{(n+p+1)}, \quad k = -q+1, -q+2, \ldots. \tag{8}$$

We now introduce an intermediate auxiliary variable $g_k^{(n)}$, and give a class of bilinear equations

$$f_{m(q+1)+i}^{(n)} g_{m(q+1)+i}^{(n+1)} - f_{m(q+1)+i}^{(n+1)} g_{m(q+1)+i}^{(n)} = f_{m(q+1)+i-1}^{(n+1)} f_{m(q+1)+i+1}^{(n)}, \tag{9}$$

$$f_{m(q+1)+1}^{(n)} g_{m(q+1)+1}^{(n+1)} - f_{m(q+1)+1}^{(n+1)} g_{m(q+1)+1}^{(n)} = -f_{m(q+1)}^{(n+1)} f_{m(q+1)+2}^{(n)}, \tag{10}$$

$$f_{(m+1)(q+1)+i}^{(n)} g_{m(q+1)+i}^{(n+p+1)} - f_{m(q+1)+i}^{(n+p+1)} g_{(m+1)(q+1)+i}^{(n)} = f_{m(q+1)+i+1}^{(n+p)} f_{m(q+1)+i+q}^{(n+1)}, \tag{11}$$

$$f_{(m+1)(q+1)+1}^{(n)} g_{m(q+1)+1}^{(n+p+1)} - f_{m(q+1)+1}^{(n+p+1)} g_{(m+1)(q+1)+1}^{(n)} = -f_{m(q+1)+2}^{(n+p)} f_{(m+1)(q+1)}^{(n+1)}, \tag{12}$$

which can yield (8) by eliminating $g_k^{(n)}$, where m is an arbitrary integer, $i = 2, \ldots, q+1$. In Sect. 3, we will see that the technique of introducing the auxiliary variable $g_k^{(n)}$ play an important role in setting up a connection between Eq. (5) and sequence transformation. In order to get the molecule solution of Eq. (6), we only need to study the bilinear equations (9)–(12) instead, whose initial conditions are given by

$$f_{-q}^{(n)} = \cdots = f_0^{(n)} = 1, \tag{13}$$

$$g_{-q}^{(n)} = 0, g_{-q+1}^{(n)} = \cdots = g_{-1}^{(n)} = n, g_0^{(n)} = S_n, \tag{14}$$

where $n = 1, 2, \ldots$, and (S_n) is a given sequence.

In fact, if we set

$$\Psi_m^{(p,q)}(v_n) = \begin{vmatrix} v_{n+(m-1)p} & v_{n+(m-1)p+1} & \cdots & v_{n+(m-1)(p+1)} \\ \Delta^q v_{n+(m-2)p} & \Delta^q v_{n+(m-2)p+1} & \cdots & \Delta^q v_{n+(m-2)p+m-1} \\ \vdots & \vdots & & \vdots \\ \Delta^{(m-1)q} v_n & \Delta^{(m-1)q} v_{n+1} & \cdots & \Delta^{(m-1)q} v_{n+m-1} \end{vmatrix},$$

$$\Psi_{-1}^{(p,q)}(v_n) = 0, \ \Psi_0^{(p,q)}(v_n) = 1,$$

$$\Phi_m^{(p,q)}(v_n) = \begin{vmatrix} n+(m-1)p & n+(m-1)p+1 & \cdots & n+(m-1)(p+1) \\ v_{n+(m-2)p} & v_{n+(m-2)p+1} & \cdots & v_{n+(m-2)p+m+1} \\ \vdots & \vdots & & \vdots \\ \Delta^{(m-2)q} v_n & \Delta^{(m-2)q} v_{n+1} & \cdots & \Delta^{(m-2)q} v_{n+m-1} \end{vmatrix},$$

$$\Phi_{-1}^{(p,q)}(v_n) = 0, \ \Phi_0^{(p,q)}(v_n) = 1,$$

then we have the following theorem.

Theorem 1 *The molecule solutions to bilinear equations* (9)–(12) *with initial conditions* (13)–(14) *can be expressed as*

$$f_{m(q+1)+i}^{(n)} = \Psi_{m+1}(\Delta^i S_n), \quad i = 1, \ldots, q+1,$$
$$g_{m(q+1)}^{(n)} = \Psi_{m+1}(S_n), \ g_{m(q+1)+1}^{(n)} = \Psi_m(\Delta^{q+2} S_n),$$
$$g_{m(q+1)+i}^{(n)} = \Phi_{m+2}(\Delta^{i-1} S_n), \quad i = 2, \ldots, q.$$

where $q, n = 1, 2, \ldots, \ m = 0, 1, \ldots$, *and the common upper index* (p, q) *of the* Ψ*s and* Φ*s has been dropped for the sake of simplicity.*

Proof Firstly, we prove Eqs. (9) and (10), which are equivalent to the following identities:

$$\begin{aligned} &\Phi_{m+2}(\Delta^{i-1} S_{n+1})\Psi_{m+1}(\Delta^i S_n) - \Phi_{m+2}(\Delta^{i-1} S_n)\Psi_{m+1}(\Delta^i S_{n+1}) \\ &\quad = \Psi_{m+1}(\Delta^{i-1} S_{n+1})\Psi_{m+1}(\Delta^{i+1} S_n), \quad i = 2, \ldots, q, \end{aligned} \tag{15}$$

$$\begin{aligned} &\Psi_{m+1}(S_{n+1})\Psi_m(\Delta^{q+1} S_n) - \Psi_{m+1}(S_n)\Psi_m(\Delta^{q+1} S_{n+1}) \\ &\quad = \Psi_m(\Delta^q S_{n+1})\Psi_{m+1}(\Delta S_n), \end{aligned} \tag{16}$$

$$\begin{aligned} &\Psi_{m+1}(\Delta S_n)\Psi_m(\Delta^{q+2} S_{n+1}) - \Psi_{m+1}(\Delta S_{n+1})\Psi_m(\Delta^{q+2} S_n) \\ &\quad = -\Psi_m(\Delta^{q+1} S_{n+1})\Psi_{m+1}(\Delta^2 S_n). \end{aligned} \tag{17}$$

Since (17) can be obtained in a similar way to (16), here we only prove (15) and (16).

Set

$$D_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ S_{n+mp} & S_{n+mp+1} & \cdots & S_{n+mp+m+1} \\ \vdots & \vdots & & \vdots \\ \Delta^{mq} S_n & \Delta^{mq} S_{n+1} & \cdots & \Delta^{mq} S_{n+m+1} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ n+(m+1)p & n+(m+1)p+1 & \cdots & n+(m+1)p+m+2 \\ \Delta^{i-1} S_{n+mp} & \Delta^{i-1} S_{n+mp+1} & \cdots & \Delta^{i-1} S_{n+mp+m+2} \\ \vdots & \vdots & & \vdots \\ \Delta^{i-1+mq} S_n & \Delta^{i-1+mq} S_{n+1} & \cdots & \Delta^{i-1+mq} S_{n+m+2} \end{vmatrix}.$$

If we use $D\begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix}$ to denote the determinant with the $i_1 < \cdots < i_n$ *-th* rows and the $j_1 < \cdots < j_n$ *-th* columns removed from the original determinant D, then the Jacobi identity [2,15] can be written as

$$D \cdot D\begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D\begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdot D\begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D\begin{bmatrix} i_1 \\ j_2 \end{bmatrix} \cdot D\begin{bmatrix} i_2 \\ j_1 \end{bmatrix}. \tag{18}$$

Applying (18) to D_1 and D_2, and noticing that

$$\begin{aligned} &D_1 = \Psi_{m+1}(\Delta S_n),\ \ D_1\begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^q S_{n+1}), \\ &D_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Psi_{m+1}(S_{n+1}),\ \ D_1\begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{q+1} S_n), \\ &D_1\begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Psi_{m+1}(S_n),\ \ D_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_m(\Delta^{q+1} S_{n+1}), \\ &D_2 = \Psi_{m+1}(\Delta^{i+1} S_n),\ \ D_2\begin{bmatrix} 1 & 2 \\ 1 & m+3 \end{bmatrix} = \Psi_{m+1}(\Delta^{i-1} S_{n+1}), \\ &D_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Phi_{m+2}(\Delta^{i-1} S_{n+1}),\ \ D_2\begin{bmatrix} 2 \\ m+3 \end{bmatrix} = \Psi_{m+1}(\Delta^i S_n), \\ &D_2\begin{bmatrix} 1 \\ m+3 \end{bmatrix} = \Phi_{m+2}(\Delta^{i-1} S_n),\ \ D_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m+1}(\Delta^i S_{n+1}), \end{aligned}$$

then we get (15) and (16) immediately.

Next, we consider Eqs. (11) and (12), which are equivalent to

$$\begin{aligned} &\Psi_{m+1}(\Delta^i S_n)\Phi_{m+1}(\Delta^{i-1} S_{n+p+1}) - \Psi_m(\Delta^i S_{n+p+1})\Phi_{m+2}(\Delta^{i-1} S_n) \\ &\quad = \Psi_m(\Delta^{i+1} S_{n+p})\Psi_{m+1}(\Delta^{i-1} S_{n+1}),\ \ i = 2, \ldots, q, \end{aligned} \tag{19}$$

$$\Psi_m(S_{n+p+1})\Psi_m(\Delta^{q+1}S_n) - \Psi_{m+1}(S_n)\Psi_{m-1}(\Delta^{q+1}S_{n+p+1})$$
$$= \Psi_m(\Delta S_{n+p})\Psi_m(\Delta^q S_{n+1}), \tag{20}$$
$$\Psi_m(\Delta S_{n+p+1})\Psi_m(\Delta^{q+2}S_n) - \Psi_{m+1}(\Delta S_n)\Psi_{m-1}(\Delta^{q+2}S_{n+p+1})$$
$$= \Psi_m(\Delta^2 S_{n+p})\Psi_m(\Delta^{q+1}S_{n+1}). \tag{21}$$

We use the Jacobi identity (18) to show the validity of (19) and (20). Set

$$D_3 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ n+mp & n+mp+1 & \cdots & n+mp+m & 0 \\ \Delta^i S_{n+(m-1)p} & \Delta^i S_{n+(m-1)p+1} & \cdots & \Delta^i S_{n+(m-1)p+m} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{i+(m-2)q} S_{n+p} & \Delta^{i+(m-2)q} S_{n+p+1} & \cdots & \Delta^{i+(m-2)q} S_{n+p+m} & 0 \\ \Delta^{i+(m-1)q} S_n & \Delta^{i+(m-1)q} S_{n+1} & \cdots & \Delta^{i+(m-1)q} S_{n+m} & 1 \end{vmatrix},$$

$$D_4 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_{n+mp} & S_{n+mp+1} & \cdots & S_{n+mp+m} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{(m-1)q} S_{n+p} & \Delta^{(m-1)q} S_{n+p+1} & \cdots & \Delta^{(m-1)q} S_{n+p+m} & 0 \\ \Delta^{mq} S_n & \Delta^{mq} S_{n+1} & \cdots & \Delta^{mq} S_{n+m} & 1 \end{vmatrix},$$

we have the following relations

$$D_3 = \Psi_{m-1}(\Delta^{i+2}S_{n+p}), \ \ D_3\begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^i S_{n+1}),$$
$$D_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Phi_m(\Delta^i S_{n+p+1}), \ \ D_3\begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{i+1}S_n),$$
$$D_3\begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Phi_{m+1}(\Delta^i S_n), \ \ D_3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m-1}(\Delta^{i+1}S_{n+p+1}),$$
$$D_4 = \Psi_m(\Delta S_{n+p}), \ \ D_4\begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^q S_{n+1}),$$
$$D_4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Psi_m(S_{n+p+1}), \ \ D_4\begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{q+1}S_n),$$
$$D_4\begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Psi_{m+1}(S_n), \ \ D_4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m-1}(\Delta^{q+1}S_{n+p+1}).$$

Then Eqs. (19) and (20) are obtained by applying Jacobi identity (18) to D_3 and D_4 (with $i_1 = 1, i_2 = 2\ ; j_1 = 1,\ j_2 = m+2$), respectively.

The proof of (21) is nearly the same as that of (20), thus we omit it.

Consequently, Eqs. (9)–(12) hold, which complete the proof.

From Theorem 1 and the dependent variable transformation (7), we obtain the molecule solution of (6) immediately.

3 A new sequence transformation and the corresponding recursive algorithm

In this section, we construct a new sequence transformation related to the molecule solution given by Theorem 1, and derive a convergence acceleration algorithm for its implementation.

Let us consider a new sequence transformation defined by

$$T_k^{(p,q)}(S_n) = \frac{\begin{vmatrix} S_{n+kp} & S_{n+kp+1} & \cdots & S_{n+k(p+1)} \\ \Delta^q S_{n+(k-1)p} & \Delta^q S_{n+(k-1)p+1} & \cdots & \Delta^q S_{n+(k-1)p+k} \\ \vdots & \vdots & & \vdots \\ \Delta^{kq} S_n & \Delta^{kq} S_{n+1} & \cdots & \Delta^{kq} S_{n+k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^q S_{n+(k-1)p} & \Delta^q S_{n+(k-1)p+1} & \cdots & \Delta^q S_{n+(k-1)p+k} \\ \vdots & \vdots & & \vdots \\ \Delta^{kq} S_n & \Delta^{kq} S_{n+1} & \cdots & \Delta^{kq} S_{n+k} \end{vmatrix}}, \tag{22}$$

where $k, p = 0, 1, \ldots, q = 1, 2, \ldots$, and $p \leq q$. Hereinafter, the values of p and q will always be taken like this. Obviously, when $k = 1$, $p = 1$ and $q = 2$, (22) is nothing but the rational transformation $C_n(3, 2)$ given by (3), and when $p = 0$, it is equivalent to the multistep Shanks' transformation proposed in [13]. Thus, (22) is an extension of the both. We mention that $T_k^{(p,q)}(S_n)$ can also be expressed as

$$T_k^{(p,q)}(S_n) = \frac{\Psi_{k+1}^{(p,q)}(S_n)}{\Psi_k^{(p,q)}(\Delta^{q+1} S_n)} = \frac{g_{k(q+1)}^{(n)}}{f_{k(q+1)}^{(n)}},$$

which motivates us to implement the dependent variable transformation

$$u_k^{(n)} = \frac{g_k^{(n)}}{f_k^{(n)}} \tag{23}$$

to the bilinear equations (9)–(12) to see whether there exists a recursive relation satisfied by $u_k^{(n)}$. In fact, we have the following theorem.

Theorem 2 *If $g_k^{(n)}$ and $f_k^{(n)}$ satisfy the bilinear equations (9)–(12), then $u_k^{(n)}$ defined by (23) can be computed recursively:*

$$u_{k+1}^{(n)} = u_{k-q}^{(n+p+1)} - \frac{(u_{k-q-1}^{(n+p+1)} - u_k^{(n)})(u_{k-q}^{(n+p+1)} - u_{k-q}^{(n+p)})}{u_k^{(n+1)} - u_k^{(n)}}, \quad n, k = 1, 2, \ldots, \tag{24}$$

with the initial values

$$u_{-q}^{(n)} = 0,\ u_{-q+1}^{(n)} = \cdots = u_{-1}^{(n)} = n,\ u_0^{(n)} = S_n,\ u_1^{(n)} = \frac{1}{\Delta S_n}, \quad n = 1, 2, \ldots. \tag{25}$$

Proof It is obvious that the initial conditions (25) can be obtained directly from the dependent variable transformation (23) and Theorem 1. Thus, we only need to prove equation (24), which is equivalent to the following identity

$$\left(u_{k+q}^{(n+1)}-u_{k+q}^{(n)}\right)\left(u_{k}^{(n+p+1)}-u_{k+q+1}^{(n+1)}\right)=\left(u_{k-1}^{(n+p+1)}-u_{k+q}^{(n)}\right)\left(u_{k}^{(n+p+1)}-u_{k}^{(n+p)}\right). \tag{26}$$

From the dependent variable transformation (23) and Eqs. (9)–(12), we obtain

$$u_k^{(n+1)}-u_k^{(n)}=\frac{f_{k-1}^{(n+1)}f_{k+1}^{(n)}}{f_k^{(n+1)}f_k^{(n)}}, \tag{27}$$

$$u_k^{(n+p+1)}-u_{k+q+1}^{(n)}=\frac{f_{k+1}^{(n+p)}f_{k+q}^{(n+1)}}{f_k^{(n+p+1)}f_{k+q+1}^{(n)}}, \tag{28}$$

where $k\neq m(q+1)+1$, and when $k=m(q+1)+1$, it only needs to change the sign of the right hand side of (27) and (28).

Consider the case when $k=m(q+1)+i$, $i=3,\ldots,q+1$, in (26). We have

$$\begin{aligned}
&\left(u_{k+q}^{(n+1)}-u_{k+q}^{(n)}\right)\left(u_{k}^{(n+p+1)}-u_{k+q+1}^{(n+1)}\right)\\
&\quad=\frac{f_{k+q-1}^{(n+1)}f_{k+q+1}^{(n)}}{f_{k+q}^{(n+1)}f_{k+q}^{(n)}}\cdot\frac{f_{k+1}^{(n+p)}f_{k+q}^{(n+1)}}{f_{k}^{(n+p+1)}f_{k+q+1}^{(n)}}=\frac{f_{k+q-1}^{(n+1)}f_{k+1}^{(n+p)}}{f_{k}^{(n+p+1)}f_{k+q}^{(n)}},\\
&\left(u_{k-1}^{(n+p+1)}-u_{k+q}^{(n)}\right)\left(u_{k}^{(n+p+1)}-u_{k}^{(n+p)}\right)\\
&\quad=\frac{f_{k}^{(n+p)}f_{k+q-1}^{(n+1)}}{f_{k-1}^{(n+p+1)}f_{k+q}^{(n)}}\cdot\frac{f_{k-1}^{(n+p+1)}f_{k+1}^{(n+p)}}{f_{k}^{(n+p+1)}f_{k}^{(n+p)}}=\frac{f_{k+q-1}^{(n+1)}f_{k+1}^{(n+p)}}{f_{k}^{(n+p+1)}f_{k+q}^{(n)}},
\end{aligned}$$

which shows that (26) holds when $k=m(q+1)+i$, $i=3,\ldots,q+1$. The proofs of this identity when $k=m(q+1)+1, m(q+1)+2$ are nearly the same, and thus be omitted.

Consequently, (26) holds for all $k\in\mathbb{N}$, which implies the validity of (24). This completes the proof. □

It is obvious that $u_{kT}^{(n)}$ is nothing but the new sequence transformation (22). Thus, according to Theorem 2, $T_k^{(p,q)}:(S_n)\to(u_{kT}^{(n)})$ can be implemented via (24) with initial values (25). In other words, (24) together with (25) can be viewed as a convergence acceleration algorithm corresponding to sequence transformation $T_k^{(p,q)}$.

Since transformation (22) can be regarded as an extension of the multistep Shanks' transformation, it is natural to investigate the relationship between their corresponding recursive algorithms. In fact, the following corollary shows that the multistep ε-algorithm given in [13] is just a special case of our new algorithm.

Corollary 1 *If we set $p = 0$ in the new algorithm* (24)*, then it can be reduced to the multistep ε-algorithm* [13, p.5].

Proof In this case, (24) is written as

$$\left(u_{k+1}^{(n)} - u_{k-q}^{(n+1)}\right)\left(u_{k}^{(n+1)} - u_{k}^{(n)}\right) = \left(u_{k}^{(n)} - u_{k-q-1}^{(n+1)}\right)\left(u_{k-q}^{(n+1)} - u_{k-q}^{(n)}\right).$$

Multiplying both sides of the above equation by $\prod_{i=1}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right)$, we obtain

$$\left(u_{k+1}^{(n)} - u_{k-q}^{(n+1)}\right)\prod_{i=0}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right) = \left(u_{k}^{(n)} - u_{k-q-1}^{(n+1)}\right)\prod_{i=1}^{q}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right),$$

which can be simplified further yielding

$$u_{k+1}^{(n)} = u_{k-q}^{(n+1)} + \frac{1}{\prod_{i=0}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right)}.$$

This formula is nothing but the multistep ε-algorithm, corresponding to $m = q$. □

As the end of this section, we give the kernel of the new sequence transformation, that is the set of sequences which would be transformed into a constant.

Theorem 3 *A necessary and sufficient condition for all $n > N$, $T_k^{(p,q)}(S_n) = S$ is that for $\forall n > N$,*

$$S_{n+kp} = S + a_1\Delta^q S_{n+(k-1)p} + \cdots + a_k\Delta^{kq} S_n,$$

where N is a given positive integer, $S = \lim_{n\to\infty} S_n$, $a_1, \ldots, a_k$ are constants independent of n and $a_k \neq 0$, $k = 1, 2, \ldots$.

4 Convergence and stability analysis

The analysis of convergence and stability is an important topic in the theory of convergence acceleration. In this section, we consider the convergence and stability of the sequence transformation (22) as it is applied to the following three different sequences.

I. Logarithmically convergent model sequences where

$$S_n \sim S + \sum_{i=0}^{\infty}\alpha_i n^{\gamma-i} \;\; as \;\; n \to \infty; \; \alpha_0 \neq 0, \gamma < 0. \tag{29}$$

II. Linearly convergent model sequences where

$$S_n \sim S + \xi^n \sum_{i=0}^{\infty}\alpha_i n^{\gamma-i} \;\; as \;\; n \to \infty; \; \alpha_0 \neq 0, |\xi| < 1 \text{ or } \xi = -1, \gamma < 0. \tag{30}$$

III. Factorially convergent model sequences where

$$S_n \sim S + (n!)^{-r}\xi^n \sum_{i=0}^{\infty} \alpha_i n^{\gamma-i} \;\; as \;\; n \to \infty; \alpha_0 \neq 0, r = 1, 2 \dots . \tag{31}$$

4.1 Convergence analysis

We give the following two lemmas, which are useful in the subsequent proofs. In fact, the first lemma is an obvious result of asymptotic expansions.

Lemma 1 *Given* (A_n) *a sequence, assume* $A_n \sim \sum_{i=0}^{\infty} a_i n^{\gamma-i}$, *as* $n \to \infty$; $a_0 \neq 0$, *then we have*

(i) *if* $\gamma \neq 0$, $\Delta A_n \sim \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i-1}$, *as* $n \to \infty; \widehat{a}_0 = \gamma a_0 \neq 0$;
(ii) *if* $\gamma = 0$, $\Delta A_n \sim \sum_{i=\mu}^{\infty} \widehat{a}_i n^{-i-1}$, *as* $n \to \infty; \widehat{a}_\mu = -\mu a_\mu \neq 0$ *(*a_μ *is the first nonzero* a_i *with* $i \geq 1$*);*
(iii) *if* $\xi \neq 1$, $\Delta^k(\xi^n A_n) \sim \xi^n \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i}$, *as* $n \to \infty; \widehat{a}_0 = (\xi - 1)^k a_0 \neq 0$;
(iv) *if* $r = 1, 2 \dots$, $\Delta(\frac{\xi^n}{(n!)^r} A_n) \sim \frac{\xi^n}{(n!)^r} \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i}$, *as* $n \to \infty; \widehat{a}_0 = -a_0 \neq 0$;
(v) *if* $r = 1, 2 \dots$, $\Delta(\frac{(n!)^r}{\xi^n} A_n) \sim \frac{(n!)^r}{\xi^n} \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma+r-i}$, *as* $n \to \infty; \widehat{a}_0 = \frac{1}{\xi} a_0 \neq 0$.

Lemma 2 *If* $u_k^{(n)}$ *is computed by algorithm* (24)*, then* $u_{(k+1)(q+1)}^{(n)}$ *can be expressed as*

$$\begin{aligned} u_{(k+1)(q+1)}^{(n)} = \frac{1}{\prod_{i=1}^{q} \Delta u_{k(q+1)+i}^{(n)}} \Bigg\{ & u_{k(q+1)}^{(n+p+1)} \Delta u_{k(q+1)+1}^{(n)} \prod_{i=2}^{q} \Delta u_{k(q+1)+i}^{(n)} \\ & + u_{k(q+1)+1}^{(n)} \Delta u_{k(q+1)}^{(n+p)} \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)} \\ & - \Delta u_{k(q+1)}^{(n+p)} u_{k(q+1)-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)} \Bigg\} \\ & n = 1, 2, \dots, k = 0, 1 \dots , \end{aligned} \tag{32}$$

where the forward difference operator Δ *is applied to superscripts.*

Proof Equation (24) can be rewritten as

$$u_{m+1}^{(n)} - u_{m-q}^{(n+p+1)} = \frac{\Delta u_{m-q}^{(n+p)}}{\Delta u_m^{(n)}} \left(u_m^{(n)} - u_{m-q-1}^{(n+p+1)} \right).$$

Multiplying together these equations for $m = (k + 1)(q + 1) - 1, (k + 1)(q + 1) - 2, \dots k(q + 1) + 1$, we will obtain

$$u_{(k+1)(q+1)}^{(n)} - u_{k(q+1)}^{(n+p+1)} = \frac{\prod_{i=2}^{q+1} \Delta u_{(k-1)(q+1)+i}^{(n+p)}}{\prod_{i=1}^{q} \Delta u_{k(q+1)+i}^{(n)}} \left(u_{k(q+1)+1}^{(n)} - u_{k(q+1)-q}^{(n+p+1)} \right),$$

which is equivalent to the expression (32).

Then we have the following convergence theorem.

Theorem 4 *If we apply the new algorithm* (24) *together with the initial conditions* (25) *to sequence* (S_n)*, then for any nonnegative integer* k*, we have:*

(i) *If* (S_n) *behaves like* (29)*, then*

$$u_{k(q+1)}^{(n)} - S \sim (-1)^k \alpha_0 \frac{q^k \cdot k!}{(\gamma - q)(\gamma - 2q)\cdots(\gamma - kq)} n^{\gamma} \quad as \quad n \to \infty. \tag{33}$$

(ii) *If* (S_n) *behaves like* (30)*, then*

$$u_{k(q+1)}^{(n)} - S \sim \xi^n \sum_{i=0}^{\infty} \alpha_{k,i}^{(0)} n^{\gamma_k - i} \quad as \quad n \to \infty, \alpha_{k,0}^{(0)} \neq 0, \tag{34}$$

$$u_{k(q+1)+1}^{(n)} \sim \xi^{-n} \sum_{i=0}^{\infty} \alpha_{k,i}^{(1)} n^{-\gamma_k - i} \quad as \quad n \to \infty, \alpha_{k,0}^{(1)} = \frac{1}{\alpha_{k,0}^{(0)}(\xi - 1)} \neq 0, \tag{35}$$

$$u_{k(q+1)+j}^{(n)} \sim n + \sum_{i=0}^{\infty} \alpha_{k,i}^{(j)} n^{-i} \quad as \quad n \to \infty,$$
$$\alpha_{k,0}^{(j)} = (k+1) \left(p + 1 + \frac{\xi}{1 - \xi} \right), \ j = 2, 3 \ldots q, \tag{36}$$

with $\gamma_0 = \gamma$, $\gamma_k = \gamma_{k-1} - 2 - \mu_k$, $k = 1, 2 \ldots$, *where* μ_k *are some nonnegative integers.*

(iii) *If* (S_n) *behaves like* (31)*, then*

$$u_{k(q+1)}^{(n)} - S \sim \frac{\xi^n}{(n!)^r} \sum_{i=0}^{\infty} \alpha_{k,i}^{(0)} n^{\gamma_k - i} \quad as \quad n \to \infty, \alpha_{k,0}^{(0)} \neq 0, \tag{37}$$

$$u_{k(q+1)+1}^{(n)} \sim \frac{(n!)^r}{\xi^n} \sum_{i=0}^{\infty} \alpha_{k,i}^{(1)} n^{-\gamma_k - i} \quad as \quad n \to \infty, \alpha_{k,0}^{(1)} = -\frac{1}{\alpha_{k,0}^{(0)}} \neq 0, \tag{38}$$

$$u_{k(q+1)+j}^{(n)} \sim n + p + 1 + \sum_{i=0}^{\infty} \alpha_{k,i}^{(j)} n^{-r-i} \quad as \quad n \to \infty,$$
$$\alpha_{k,0}^{(j)} = (k+1)\xi, \ j = 2, 3 \ldots q, \tag{39}$$

with $\gamma_0 = \gamma$, $\gamma_k = \gamma_{k-1} - r - 2 - \mu_k$ *when* $p = 0$ *and* $\gamma_k = \gamma_{k-1} - (p+1)r - 1 - \mu_k$ *when* $p \neq 0$*, where* μ_k *are some nonnegative integers.*

Proof (i) For logarithmically convergent sequences in (29), the convergence results can be obtained by following the similar steps given by Garibotti–Grinstein [18] in the proof of ε-algorithm. Here we omit the details.

(ii) Now we prove the convergence results for linearly convergent sequences in (30). We proceed by induction on k.

Base step. Consider the case when $k = 0$. On the one hand, since $u_0^{(n)} = S_n$ and $u_1^{(n)} = 1/\Delta S_n$, it is obvious that (34) and (35) hold for $k = 0$ with $\gamma_0 = \gamma$, $\alpha_{k,i}^{(0)} = \alpha_0$. On the other hand, according to the recursive relation (24) and initial values (25), expression (36) for $k = 0$ can be easily obtained with the help of Lemma 2.

Inductive step. Assume that expressions (34)–(36) hold for $k = 1, 2, \ldots, m$, where m is a positive integer. Next, we will prove that they also hold for $k = m + 1$.

Firstly, consider the proof of (34) when $k = m + 1$. Subtracting S form both sides of Eq. (32) in Lemma 2, we get

$$
\begin{aligned}
u_{(k+1)(q+1)}^{(n)} - S = \frac{1}{\prod_{i=1}^{q} \Delta u_{k(q+1)+i}^{(n)}} \Bigg\{ & \left(u_{k(q+1)}^{(n+p+1)} - S\right) \Delta u_{k(q+1)+1}^{(n)} \prod_{i=2}^{q} \Delta u_{k(q+1)+i}^{(n)} \\
& + u_{k(q+1)+1}^{(n)} \Delta \left(u_{k(q+1)}^{(n+p)} - S\right) \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)} \\
& - \Delta \left(u_{k(q+1)}^{(n+p)} - S\right) u_{k(q+1)-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)} \Bigg\}.
\end{aligned} \tag{40}
$$

For simplicity, set

$$
\begin{aligned}
A_k^{(n)} &= \left(u_{k(q+1)}^{(n+p+1)} - S\right) \Delta u_{k(q+1)+1}^{(n)} \prod_{i=2}^{q} \Delta u_{k(q+1)+i}^{(n)} \\
&\quad + u_{k(q+1)+1}^{(n)} \Delta \left(u_{k(q+1)}^{(n+p)} - S\right) \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)}, \\
B_k^{(n)} &= \Delta \left(u_{k(q+1)}^{(n+p)} - S\right) u_{k(q+1)-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)(q+1)+i}^{(n+p)}, \\
C_k^{(n)} &= \prod_{i=1}^{q} \Delta u_{k(q+1)+i}^{(n)},
\end{aligned}
$$

then (40) can be written as

$$
u_{(k+1)(q+1)}^{(n)} - S = \frac{A_k^{(n)} - B_k^{(n)}}{C_k^{(n)}}. \tag{41}
$$

Thus, in order to prove (34) for $k = m + 1$, we only need to analyze the asymptotic behaviours of $A_m^{(n)}$, $B_m^{(n)}$ and $C_m^{(n)}$ as $n \to \infty$, respectively.

In fact, according to the inductive hypothesis and Lemma 1, we have

$$\Delta\left(u_{k(q+1)}^{(n)}-S\right)\sim\xi^{n}\sum_{i=0}^{\infty}\hat{\alpha}_{k,i}^{(0)}n^{\gamma_k-i},\ as\ n\to\infty,\quad \hat{\alpha}_{k,0}^{(0)}=(\xi-1)\alpha_{k,0}^{(0)},\tag{42}$$

$$\Delta u_{k(q+1)+1}^{(n)}\sim\xi^{-n}\sum_{i=0}^{\infty}\hat{\alpha}_{k,i}^{(1)}n^{-\gamma_k-i},\ as\ n\to\infty,\ \hat{\alpha}_{k,0}^{(1)}=\left(\frac{1}{\xi}-1\right)\alpha_{k,0}^{(1)},\tag{43}$$

$$\Delta u_{k(q+1)+j}^{(n)}\sim 1+\sum_{i=2}^{\infty}\hat{\alpha}_{k,i}^{(j)}n^{-i},\ as\quad n\to\infty,\quad j=2,3\ldots q.\tag{44}$$

Furthermore,

$$\left(u_{k(q+1)}^{(n+p)}-S\right)u_{k(q+1)+1}^{(n)}\sim\sum_{i=0}^{\infty}b_{k,i}n^{-i},\quad as\ \ n\to\infty,$$

which leads to

$$\Delta\left[\left(u_{k(q+1)}^{(n+p)}-S\right)u_{k(q+1)+1}^{(n)}\right]\sim\sum_{i=2}^{\infty}\hat{b}_{k,i}n^{-i},\ as\ n\to\infty,$$

where $k=1,\ldots,m$.

With the help of the above relations, we obtain

$$\begin{aligned}
A_m^{(n)} &\sim \left(u_{m(q+1)}^{(n+p+1)}-S\right)\Delta u_{m(q+1)+1}^{(n)}\prod_{j=2}^{q}\left(1+\sum_{i=2}^{\infty}\hat{\alpha}_{m,i}^{(j)}n^{-i}\right)\\
&\quad+u_{m(q+1)+1}^{(n)}\Delta\left(u_{m(q+1)}^{(n+p)}-S\right)\prod_{j=2}^{q}\left(1+\sum_{i=2}^{\infty}\hat{\alpha}_{m-1,i}^{(j)}(n+p)^{-i}\right)\\
&\sim\left(u_{m(q+1)}^{(n+p+1)}-S\right)\Delta u_{m(q+1)+1}^{(n)}\left(1+\mathcal{O}(n^{-2})\right)\\
&\quad+u_{m(q+1)+1}^{(n)}\Delta\left(u_{m(q+1)}^{(n+p)}-S\right)\left(1+\mathcal{O}(n^{-2})\right)\\
&\sim\Delta\left[\left(u_{m(q+1)}^{(n+p)}-S\right)u_{m(q+1)+1}^{(n)}\right]\left(1+\mathcal{O}(n^{-2})\right)\\
&\sim\sum_{i=2}^{\infty}\beta_{m,i}n^{-i},\ as\ \ n\to\infty,
\end{aligned}\tag{45}$$

$$\begin{aligned}
B_m^{(n)} &\sim\left[\xi^{-n-p-1}\sum_{i=0}^{\infty}\alpha_{m-1,i}^{(1)}(n+p+1)^{-\gamma_{m-1}-i}\right]\\
&\quad\cdot\left[\xi^{n+p}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(0)}(n+p)^{\gamma_m-i}\right]\prod_{j=2}^{q}\left(1+\sum_{i=2}^{\infty}\hat{\alpha}_{m-1,i}^{(j)}(n+p)^{-i}\right)
\end{aligned}$$

$$
\begin{aligned}
&\sim \left[\xi^{-n-p-1}\sum_{i=0}^{\infty}\alpha_{m-1,i}^{(1)}(n+p+1)^{-\gamma_{m-1}-i}\right]\cdot \\
&\quad\times\left[\xi^{n+p}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(0)}(n+p)^{\gamma_m-i}\right]\Big(1+\mathcal{O}(n^{-2})\Big) \\
&\sim \sum_{i=0}^{\infty}\theta_{m,i}n^{\gamma_m-\gamma_{m-1}-i},\ \ as\ \ n\to\infty,
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
C_m^{(n)} &\sim \left[\xi^{-n}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(1)}n^{-\gamma_m-i}\right]\prod_{j=2}^{q}\left(1+\sum_{i=2}^{\infty}\hat{\alpha}_{m,i}^{(j)}n^{-i}\right) \\
&\sim \left[\xi^{-n}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(1)}n^{-\gamma_m-i}\right]\Big(1+\mathcal{O}(n^{-2})\Big) \\
&\sim \xi^{-n}\sum_{i=0}^{\infty}\tau_{m,i}n^{-\gamma_m-i},\ \ as\ \ n\to\infty.
\end{aligned}
\tag{47}
$$

If we substitute the expressions (45)–(47) into (41), and use $\gamma_m-\gamma_{m-1}\le -2$, we finally get the following result

$$
\begin{aligned}
u_{(m+1)(q+1)}^{(n)}-S &\sim \xi^{n}\sum_{i=0}^{\infty}\rho_{m,i}^{(0)}n^{\gamma_m-2-i} \\
&= \xi^{n}\sum_{i=0}^{\infty}\alpha_{m+1,i}^{(0)}n^{\gamma_{m+1}-i},\ \ as\ n\to\infty,\ \ \alpha_{m+1,0}^{(0)}\neq 0,
\end{aligned}
\tag{48}
$$

with $\gamma_{m+1}=\gamma_m-2-\mu_{m+1}$ for some nonnegative integer μ_{m+1}, which implies that (34) holds for $k=m+1$.

Secondly, we prove (35) for $k=m+1$. Replacing k by $(m+1)(q+1)$, equation (24) can be written as

$$
u_{(m+1)(q+1)+1}^{(n)}=u_{m(q+1)+1}^{(n+p+1)}+\frac{\Delta u_{m(q+1)+1}^{(n+p)}}{\Delta u_{(m+1)(q+1)}^{(n)}}\left(u_{(m+1)(q+1)}^{(n)}-u_{m(q+1)}^{(n+p+1)}\right).
$$

Using the hypothesis and expression (48) we have just proved, we obtain

$$
u_{(m+1)(q+1)+1}^{(n)}\sim \xi^{-n}\sum_{i=0}^{\infty}\alpha_{m+1,i}^{(1)}n^{-\gamma_{m+1}-i}\quad as\ \ n\to\infty,
$$

where $\alpha_{m+1,0}^{(1)}=\frac{1}{\alpha_{m,0}^{(0)}(\xi-1)}$, which can be derived by the relation $\alpha_{m+1,0}^{(1)}\alpha_{m+1,0}^{(0)}=\alpha_{m,0}^{(1)}\alpha_{m,0}^{(0)}$.

Finally, we investigate the asymptotic behaviours of $u^{(n)}_{(m+1)(q+1)+j}$ with $j = 2, \ldots, q$, as $n \to \infty$. In fact, similar to the analysis of $u^{(n)}_{(m+1)(q+1)+1}$ given above, the asymptotic behaviour of $u^{(n)}_{(m+1)(q+1)+j}$ can be easily derived from that of $u^{(n)}_{(m+1)(q+1)+j-1}$ and the inductive hypothesis, that is

$$u^{(n)}_{(m+1)(q+1)+j} \sim n + \sum_{i=0}^{\infty} \alpha^{(j)}_{m+1,i} n^{-i} \quad as\ n \to \infty, \quad j = 2, 3 \ldots q,$$

with $\alpha^{(j)}_{m+1,0} = (m+2)(p+1+\frac{\xi}{1-\xi})$, which can be obtained from $\alpha^{(j+1)}_{m+1,0} - \alpha^{(j+1)}_{m,0} = \alpha^{(j)}_{m+1,0} - \alpha^{(j)}_{m,0} = p + 1 + \frac{\xi}{1-\xi}$, $j = 2, 3 \ldots q - 1$.

Consequently, expressions (34)–(36) hold for $k = m+1$, which complete the proof of (ii) by inductive principle.

(iii) The proof of (37)–(39) to factorially convergent sequences in (31) can be achieved in a similar way as we did in the case of linearly convergent sequences in part (ii).

Thus proving the theorem.

From Theorem 4, we can easily see that

$$\lim_{n\to\infty} \frac{u^{(n)}_{k(q+1)} - S}{S_n - S} \neq 0, \quad k = 1, 2, \ldots$$

for logarithmically convergent sequences (29), and

$$\lim_{n\to\infty} \frac{u^{(n)}_{k(q+1)} - S}{S_n - S} = 0, \quad k = 1, 2, \ldots$$

for both linearly convergent sequences (30) and factorially convergent sequences (31). In other words, Theorem 4 indicates that our new method accelerate the convergence of both linearly convergent sequences (30) and factorially convergent sequences (31), but fails in logarithmically convergent sequences (29). This is the same as ε-algorithm.

Let us mention that there is a classical article on the convergence of ε-algorithm by Wynn [54], whose results were later extended by Sidi [42]. They analyzed the behavior of Shanks transformation on sequences (S_n) behaving as

$$S_n \sim S + \sum_{i=1}^{\infty} \alpha_i \lambda_i^n, \quad as\ \ n \to \infty \tag{49}$$

with some conditions. And it was showed to be effective, which was expected in view of the fact that the derivation of Shanks transformation was actually based on a model sequence obtained by a truncation of the infinite series in (49). In this section, we presented some results pertaining to more general sequences for (29)–(31). These results can exactly or similarly be reduced to those on ε-algorithm [18,46].

4.2 Stability

We now turn to the investigation of stability. From equation (24), we obtain

$$u_{(k+1)(q+1)}^{(n)} = \lambda_k^{(n)} u_{k(q+1)}^{(n+p+1)} + \mu_k^{(n)} u_{k(q+1)}^{(n+p)}, \quad \lambda_k^{(n)} + \mu_k^{(n)} = 1, \tag{50}$$

where

$$\lambda_k^{(n)} = \frac{u_{k(q+1)+q}^{(n+1)} - u_{(k-1)(q+1)+q}^{(n+p+1)}}{\Delta u_{k(q+1)+q}^{(n)}}, \quad \mu_k^{(n)} = -\frac{u_{k(q+1)+q}^{(n)} - u_{(k-1)(q+1)+q}^{(n+p+1)}}{\Delta u_{k(q+1)+q}^{(n)}}. \tag{51}$$

Using mathematical induction on k and noticing that $u_0^{(n)} = S_n$, we have

$$u_{k(q+1)}^{(n)} = \sum_{i=0}^{k} \gamma_{k,i}^{(n)} S_{n+kp+i}, \quad \sum_{i=0}^{k} \gamma_{k,i}^{(n)} = 1. \tag{52}$$

From the context of other extrapolation methods [40,43,44,47], the quantities of relevance to stability are

$$\Gamma_k^{(n)} = \sum_{i=0}^{k} |\gamma_{k,i}^{(n)}|.$$

In fact, if $\tilde{S}_n = S_n + \epsilon_n$, $\tilde{\gamma}_{k,i}^{(n)} = \gamma_{k,i}^{(n)} + \delta_{k,i}^{(n)}$ are the computed quantities with small perturbations, then the calculated values $\tilde{u}_{k(q+1)}^{(n)}$ are given by

$$\tilde{u}_{k(q+1)}^{(n)} = \sum_{i=0}^{k} \tilde{\gamma}_{k,i}^{(n)} \tilde{S}_{n+kp+i} = u_{k(q+1)}^{(n)} + \sum_{i=0}^{k} \gamma_{k,i}^{(n)} \epsilon_{n+kp+i} + \sum_{i=0}^{k} \delta_{k,i}^{(n)} \tilde{S}_{n+kp+i}.$$

Assume the $\tilde{S}_n$ and the $\tilde{\gamma}_{k,i}$ have been computed with machine precision, that is, $|\epsilon_i| = |S_i||\rho_i|$, $|\delta_{k,i}^{(n)}| = |\gamma_{k,i}^{(n)}||\eta_{k,i}^{(n)}|$, where $|\rho_i|$, $|\eta_{k,i}^{(n)}| \le \omega$, ω being the roundoff unit of the arithmetic being used, then

$$|\tilde{u}_{k(q+1)}^{(n)} - S| \le |u_{k(q+1)}^{(n)} - S| + \omega \left[\sum_{i=0}^{k} |\gamma_{k,i}^{(n)}||S_{n+kp+i}| + \sum_{i=0}^{k} |\gamma_{k,i}^{(n)}||\tilde{S}_{n+kp+i}| \right].$$

Noticing that $\omega|\tilde{S}_i| = \omega|S_i| + \mathcal{O}(\omega^2)$, we have

$$|\tilde{u}_{k(q+1)}^{(n)} - S| \lessapprox |u_{k(q+1)}^{(n)} - S| + 2\omega \sum_{i=0}^{k} |\gamma_{k,i}^{(n)}||S_{n+kp+i}|.$$

In case S_n converges, then $S_i \approx S$ for all large i. Therefore, when $S \neq 0$,

$$\frac{|\tilde{u}_{k(q+1)}^{(n)} - S|}{|S|} \leq\approx \frac{|u_{k(q+1)}^{(n)} - S|}{|S|} + 2\omega \sum_{i=0}^{k} |\gamma_{k,i}^{(n)}| = \frac{|u_{k(q+1)}^{(n)} - S|}{|S|} + 2\omega \Gamma_k^{(n)},$$

which implies that $\Gamma_k^{(n)}$ control the propagation of errors in computing process. When $sup_n \Gamma_k^{(n)} = \infty$, the sequence $\left(u_{k(q+1)}^{(n)}\right)_{n=0}^{\infty}$ is unstable, and when $sup_n \Gamma_k^{(n)} < \infty$, it is stable. Since $\sum_{i=0}^{k} \gamma_{k,i}^{(n)} = 1$, we hope these $\Gamma_k^{(n)}$ are as close to 1 as possible to get good numerical stability. Next, we will consider the asymptotic behaviour of $\Gamma_k^{(n)}$, as $n \to \infty$.

As the following lemma can be proved in a way similar to [45], we simply list it without proof.

Lemma 3 *Let* $P_k^{(n)}(z) = \sum_{i=0}^{k} \gamma_{k,i}^{(n)} z^i$, *then*

$$P_{k+1}^{(n)}(z) = \lambda_k^{(n)} z P_k^{(n+p+1)}(z) + \mu_k^{(n)} P_k^{(n+p)}(z),$$

where $\gamma_{k,i}^{(n)}, \lambda_k^{(n)}, \mu_k^{(n)}$ *are the same as* (51)–(52), *and* $k = 0, 1, \ldots, n = 1, 2, \ldots$.

Lemma 4 *For any nonnegative integer k,*

(i) *if* (S_n) *behaves like* (30), *then*

$$\lambda_k^{(n)} \sim \frac{1}{1-\xi} \quad and \quad \mu_k^{(n)} \sim \frac{-\xi}{1-\xi}, \quad as \quad n \to \infty.$$

(ii) *if* (S_n) *behaves like* (31), *then*

$$\lambda_k^{(n)} \sim 1 \quad and \quad \mu_k^{(n)} \sim 0, \quad as \quad n \to \infty, \quad if\ q = 1,$$
$$\lambda_k^{(n)} \sim -p \quad and \quad \mu_k^{(n)} \sim p+1, \quad as \quad n \to \infty, \quad if q > 1.$$

where the quantities $\lambda_k^{(n)}$ *and* $\mu_k^{(n)}$ *are defined by* (51).

Proof The proof can be easily obtained by using the expressions (51) and the results of Theorem 4.

Theorem 5 *For any nonnegative integer k,*

(i) *If* (S_n) *behaves like* (30), *then*

$$P_k^{(n)}(z) \sim \left(\frac{\xi - z}{\xi - 1}\right)^k \quad and \quad \Gamma_k^{(n)} \sim \left(\frac{|\xi| + 1}{|\xi - 1|}\right)^k, \quad as \quad n \to \infty. \tag{53}$$

(ii) If (S_n) *behaves like* (31)*, then*

$$P_k^{(n)}(z) \sim z^k \quad and \quad \Gamma_k^{(n)} \sim 1, \quad as\ n \to \infty, \quad if\ q = 1, \tag{54}$$

$$P_k^{(n)}(z) \sim (p+1-pz)^k \quad and \quad \Gamma_k^{(n)} \sim (2p+1)^k \quad as\ n \to \infty, \quad if\ q > 1. \tag{55}$$

Proof Combining Lemmas 3 and 4, expressions (53)–(55) hold immediately by induction on k.

Remark 1 Since our new method is ineffective on logarithmically convergent sequences (29), we only consider the stability corresponding to linearly convergent sequences (30) and factorially convergent sequences (31) in the above theorem.

We close this section by concluding that our new sequence transformation is stable for both linearly convergent sequences (30) and factorially convergent sequences (31). Concretely, for linearly convergent sequences, the stability is better when ξ is a real negative number, while becomes weak when ξ approaches 1, since $\Gamma_k^{(n)} \to \infty$ as $\xi \to 1$. Noticing that ξ^l with some positive integer $l \geq 2$ is farther away from 1, we propose to apply the method to the subsequences (S_{ln}) for better numerical stability. This strategy is APS [46], which was first proposed by Sidi [41]. As for factorially convergent sequences, our new sequence transformation (22) with $q = 1$ has better stability than that with $q > 1$. In addition, for a fixed $q > 1$, the sequence transformation becomes more and more stable as p shrinks to 0.

5 Numerical examples

In this section, we give some numerical examples, which illustrate the performance of algorithm (24)–(25) numerically. We also use the ε-algorithm [53] and the multistep ε-algorithm [20, 13] to make comparisons.

Example 1 Consider the following alternating series

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1},$$

with $\lim_{n\to\infty} S_n = S = \pi/4$. According to the *Boole summation formula* [27,33,49], we have

$$S_n - \frac{\pi}{4} \sim \frac{(-1)^n}{2}\left(\frac{1}{2n+1} + \mathcal{O}(n^{-2})\right), \text{ as } n \to \infty.$$

This asymptotic expansion is a special case of (30), with $\xi = -1, \gamma = -1$ and $\alpha_0 = \frac{1}{4} \neq 0$. The numerical results corresponding to different choices of p and q are presented in the following table.

n	$\lvert S_n - S\rvert$	$\left\lvert \varepsilon_{2\left\lfloor\frac{n-1}{2}\right\rfloor}^{\left(n-2\left\lfloor\frac{n-1}{2}\right\rfloor\right)} - S\right\rvert$ $(p=0, q=1)$	$\left\lvert u_{3\left\lfloor\frac{n}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n}{3}\right\rfloor+1\right)} - S\right\rvert$ $(p=0, q=2)$	$\left\lvert u_{3\left\lfloor\frac{n-1}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n-1}{3}\right\rfloor\right)} - S\right\rvert$ $(p=1, q=2)$	$\left\lvert u_{3\left\lfloor\frac{n-2}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n-2}{3}\right\rfloor-1\right)} - S\right\rvert$ $(p=2, q=2)$
5	0.0495	1.8742×10^{-4}	1.7025×10^{-3}	7.9837×10^{-4}	2.2411×10^{-3}
10	0.0249	3.7074×10^{-8}	9.1074×10^{-6}	7.4099×10^{-7}	1.9801×10^{-5}
15	0.0166	4.2286×10^{-12}	1.3171×10^{-7}	4.1011×10^{-10}	2.0518×10^{-8}
20	0.0125	3.3307×10^{-16}	6.6784×10^{-9}	4.3898×10^{-13}	3.0118×10^{-11}

Example 2 Consider the linearly convergent series

$$S_n = \sum_{k=1}^{n} \frac{(0.8)^k}{k},$$

which converges to $S = \ln 5 = 1.60943791\ldots$ as $n \to \infty$. As shown in [46, p.84], S_n has the following asymptotic expansion

$$S_n - \ln 5 \sim \frac{(0.8)^n}{n}\left(-4 + \mathcal{O}(n^{-1})\right), \text{ as } n \to \infty,$$

which is a special case of (30) with $\xi = 0.8$. The corresponding numerical results are presented in the following table.

n	$\lvert S_n - S\rvert$	$\left\lvert \varepsilon_{2\left\lfloor\frac{n-1}{2}\right\rfloor}^{\left(n-2\left\lfloor\frac{n-1}{2}\right\rfloor\right)} - S\right\rvert$ $(p=0, q=1)$	$\left\lvert u_{3\left\lfloor\frac{n}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n}{3}\right\rfloor+1\right)} - S\right\rvert$ $(p=0, q=2)$	$\left\lvert u_{3\left\lfloor\frac{n-1}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n-1}{3}\right\rfloor\right)} - S\right\rvert$ $(p=1, q=2)$	$\left\lvert u_{3\left\lfloor\frac{n-2}{3}\right\rfloor}^{\left(n-3\left\lfloor\frac{n-2}{3}\right\rfloor-1\right)} - S\right\rvert$ $(p=2, q=2)$
10	3.0563×10^{-2}	1.7538×10^{-4}	7.9828×10^{-3}	7.9491×10^{-3}	6.0581×10^{-3}
20	1.8920×10^{-3}	1.3000×10^{-8}	6.8015×10^{-5}	3.3291×10^{-5}	7.8235×10^{-6}
30	1.4341×10^{-4}	1.8610×10^{-10}	8.7118×10^{-6}	3.0440×10^{-7}	1.6116×10^{-7}
40	1.1916×10^{-5}	2.9263×10^{-12}	2.6159×10^{-6}	2.1194×10^{-8}	7.4446×10^{-10}

Example 3 Consider the logarithmically convergent series

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2},$$

which converges to $S = \pi^2/6$ as $n \to \infty$. Based on the *Euler-Maclaurin summation formula* (see [7,27,34] for details), we have

$$S_n - \frac{\pi^2}{6} \sim n^{-1}\left(-1 + \frac{1}{2}n^{-1} - \sum_{j=1}^{\infty} B_{2j} n^{-2j}\right), \quad \text{as} \quad n \to \infty,$$

where B_{2j} are Bernoulli numbers. It is obvious that (S_n) is a logarithmically convergent sequence with the asymptotic expansion (29), and the corresponding numerical results are shown in the following table.

n	$\lvert S_n - S\rvert$	$\left\lvert \varepsilon_{2\lfloor \frac{n-1}{2}\rfloor}^{\left(n-2\lfloor \frac{n-1}{2}\rfloor\right)} - S\right\rvert$ $(p=0, q=1)$	$\left\lvert u_{3\lfloor \frac{n}{3}\rfloor}^{(n-3\lfloor \frac{n}{3}\rfloor+1)} - S\right\rvert$ $(p=0, q=2)$	$\left\lvert u_{3\lfloor \frac{n-1}{3}\rfloor}^{\left(n-3\lfloor \frac{n-1}{3}\rfloor\right)} - S\right\rvert$ $(p=1, q=2)$	$\left\lvert u_{3\lfloor \frac{n-2}{3}\rfloor}^{\left(n-3\lfloor \frac{n-2}{3}\rfloor-1\right)} - S\right\rvert$ $(p=2, q=2)$
10	9.5166×10^{-2}	3.0460×10^{-2}	8.2278×10^{-2}	7.8642×10^{-2}	7.0766×10^{-2}
20	4.8771×10^{-2}	8.6462×10^{-3}	3.5679×10^{-2}	3.0728×10^{-2}	2.5158×10^{-2}
30	3.2784×10^{-2}	5.3752×10^{-3}	2.5080×10^{-2}	1.8725×10^{-2}	1.3836×10^{-2}
40	2.4690×10^{-2}	4.2544×10^{-3}	2.1581×10^{-2}	1.4261×10^{-2}	1.0904×10^{-2}

The above numerical examples indicate that our new algorithm indeed accelerates the linearly convergent sequences having asymptotic expansion (30), but fails in the logarithmically convergent sequences (29). That is to say, the numerical results coincide with the theoretical results presented in Sect. 4.

We also see that the new algorithm is not as good as ε-algorithm but better than multistep ε-algorithm (with the step size equals 2), while treating the linearly convergent series considered in Examples 1 and 2. However, if we consider some sequences which are very close to the kernel given in Theorem 3, the new algorithm may be faster than the ε-algorithm. The reason is that given the same initial values, the maximal subscript k of the sequence transformation $T_k^{(p,q)}(S_n)$ computed by the new algorithm is smaller than that of the Shanks transformation $e_k(S_n)$ computed by ε-algorithm, which means less iterations in some sense.

Example 4 Consider the linearly convergent sequence

$$S_n = n^2\left(\frac{-1}{2}\right)^n + \frac{n^2}{4^n},$$

which converges to $S = 0$ as $n \to \infty$. We give the numerical results corresponding to the ε-algorithm and the new algorithm (24)–(25) (with $p = q = 2$) in the following tables.

(i) ε-algorithm.

n	$\lvert e_0 - S\rvert$	$\lvert e_1 - S\rvert$	$\lvert e_2 - S\rvert$	$\lvert e_3 - S\rvert$	$\lvert e_4 - S\rvert$	$\lvert e_5 - S\rvert$
1	0.2500	0.3525	9.8263×10^{-2}	4.4373×10^{-2}	1.0125×10^{-3}	1.9609×10^{-5}
3	0.9844	9.9295×10^{-2}	1.6620×10^{-2}	1.0205×10^{-3}	6.4315×10^{-5}	
5	0.7568	1.7071×10^{-2}	2.3124×10^{-3}	7.0607×10^{-5}		
7	0.3798	2.9025×10^{-3}	2.6781×10^{-4}			
11	0.0591					

(ii) The new algorithm (24)–(25), where $p = q = 2$.

n	$\lvert T_0^{(2,2)} - S\rvert$	$\lvert T_1^{(2,2)} - S\rvert$	$\lvert T_2^{(2,2)} - S\rvert$	$\lvert T_3^{(2,2)} - S\rvert$
1	0.2500	3.0763×10^{-2}	3.7866×10^{-4}	2.1535×10^{-17}
3	0.9844	2.4721×10^{-2}	1.0110×10^{-4}	
5	0.7568	7.4612×10^{-3}		
7	0.3798	1.9328×10^{-3}		
11	0.0591			

As indicated in the above tables, in this example, the new algorithm converges faster than the ε-algorithm with smaller k.

6 Conclusion and discussions

In this article, we construct the molecule solution of an extended discrete Lotka–Volterra equation by Hirota's bilinear method, from which a new sequence transformation is derived. From the bilinear form of this extended discrete Lotka–Volterra equation, a two dimensional difference equation which can be used as a convergence acceleration algorithm to implement the new sequence transformation is generated. In addition, our new transformation is nothing but an extension of the multistep Shanks' transformation, and the multistep ε-algorithm is just a special case of our new algorithm. Then we present a rigorous convergence and stability analysis, which implies that our new method accelerates both linearly convergent sequences (30) and factorially convergent sequences (31) with good numerical stability while fails in logarithmically convergent sequences (29). Finally, we give numerical examples to demonstrate some of the preceding theoretical results.

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