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A conjecture based on Somos-4 sequence and its extension

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ABSTRACT

In two recent papers by Barry (2010) [29] and (2011) [30], it is conjectured that Somos-4 admits a solution expressed in terms of Hankel determinant with its elements satisfying a convolution recursion relation. In this paper, Barry's conjecture on Somos-4 is firstly confirmed. Actually, we present a more generalized result. The proof is mainly based on new findings on properties for so-called Block–Hankel determinants. The method can also be used to prove another conjecture proposed by Michael Somos, which has been solved by Guoce Xin.

1. Introduction

A Somos-k ($k \geqslant 4$) sequence originally introduced by Michael Somos is a sequence of numbers defined by a recurrence as follows:

$$S_n S_{n-k} = \sum_{i=1}^{[k/2]} x_i S_{n-i} S_{n-k+i}, \tag{1}$$

where the x_i are given integers. In recent years, such sequences have attracted a great deal of interest of researchers in number theory, algebraic combinatorics, statistical mechanics, as well as discrete integrable systems.

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For example, combinatorial interpretations for Somos-4 (A006720) and Somos-5 (A006721) have been found by Speyer [1] and for Somos-6 (A006722) and Somos-7 (A006723) by Carroll and Speyer [2]. For the case of $k = 4, 5, 6, 7$, if the coefficients are $x_i = 1\,(1 \leqslant i \leqslant [k/2])$ and the initial values are given by $S_i = 1\,(0 \leqslant i \leqslant k-1)$, then quadratic relation (1) can produce a sequence of integers [3–5], which is a property of what we call integrality. Such a property was first found by Michael Somos. It was realized that the deeper reason behind this lay in the fact that the recurrence (1) satisfies Laurent phenomenon, meaning that the iterates are polynomials in the four initial data, and their inverses, and these Laurent polynomials have integer coefficients. The Gale–Robinson conjecture[6,7] confirmed by Fomin and Zelevinsky [8] indicates that Somos-k ($k = 4, 5, 6, 7$) sequences exhibit the Laurent property, which is an essential feature of Fomin and Zelevinsky's theory of cluster algebras [9,10]. For the case $k = 4$ and $k = 5$, Hone constructed explicit solutions to the bilinear recurrence (1) with complex numbers x_i in terms of the Weierstrass sigma function in [11,12], respectively. He also indicated that Somos-4 can be thought of as an integrable symplectic map [13,14] by making a change of variables. But, unfortunately, it is not obvious from the form of the Weierstrass sigma function that sequences have the Laurent phenomenon. For more details, please consult [15–24].

Furthermore, there also exist several conjectures about Somos-4 or Somos-5. For instance, Michael Somos [25] observed that $y(z)$ given by $y - y^2 = z - z^3$ yields the Somos-4 sequence, which has been proved by Xin [26]. Gosper and Schroeppel [27] made two *near-addition formulas* for Somos-4 and Somos-5, which has been solved by Ma [28]. More recently, Barry proposed another conjecture about Somos-4 in the paper [29,30], which is still open. We restate it as the following:

Conjecture 1.1. *Assume that a_n satisfies a convolution recursion relation:*

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ \alpha, & \text{if } n = 1; \\ \alpha a_{n-1} + \beta a_{n-2} + \gamma \sum_{i=0}^{n-2} a_i a_{n-2-i}, & \text{if } n > 1. \end{cases} \tag{2}$$

Then the Hankel determinant $H_n = det(a_{i+j})_{0 \leqslant i,j \leqslant n-1}$ is a $(\alpha^2\gamma^2, \gamma^2(\beta+\gamma)^2 - \alpha^2\gamma^3)$ Somos-4 sequence.

Here we use the notation $(x_1, x_2, \ldots, x_{[\frac{k}{2}]})$ Somos-k sequence, which represents that the sequence satisfies recursion relation (1).

The main purpose of this paper is to show the following result:

Theorem 1.2. *For a fixed positive integer p, let $a_n^{(p)}$ be computed via the following convolution recurrence:*

$$a_n^{(p)} = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \alpha a_{n-1}^{(p)} + \beta a_{n-2}^{(p)} + \gamma \sum_{i=0}^{[\frac{n-2}{p}]} a_{pi}^{(p)} a_{n-2-pi}^{(p)}, & \text{if } n = kp+1, k \geqslant 0 \\ \alpha a_{n-1}^{(p)} + \gamma \sum_{i=1}^{[\frac{n-2}{p}]} a_{pi}^{(p)} a_{n-2-pi}^{(p)}, & \text{if } n = kp+j, k \geqslant 0, 2 \leqslant j \leqslant p. \end{cases} \tag{3}$$

Then we have

(i) $h_n^{(1)}$ is a $(\alpha^2\gamma^2, \gamma^2(\beta+\gamma)^2 - \alpha^2\gamma^3)$ Somos-4 sequence.
(ii) $h_n^{(2)}$ is a $(\alpha^6\gamma^3 - \alpha^4\gamma^3(\beta+\gamma), \alpha^6\gamma^4(\beta+\gamma)(\beta+2\gamma) - \alpha^8\gamma^5)$ Somos-5 sequence.

(iii) If $\beta+\gamma=0$, then $h_n^{(p)}$ satisfies the relation:

$$h_{n+p+3}^{(p)}h_n^{(p)} = (\alpha^p\gamma)^{p+1}h_{n+p+2}^{(p)}h_{n+1}^{(p)} - \frac{(\alpha^p\gamma)^{2p+1}}{\alpha^p}h_{n+p+1}^{(p)}h_{n+2}^{(p)},$$

where $h_n^{(p)} = det\left(a_{ip+j}^{(p)}\right)_{0\leqslant i,j\leqslant n-1}$.

Obviously, conclusion (i) confirms the conjecture due to Barry. Besides, conclusion (ii) indicates that a Block–Hankel determinant solution to Somos-5 recurrences is obtained while conclusion (iii) yields a solution for a special case of the Gale–Robinson recurrence.

This paper is organized as follows. In Section 2, the so-called Block–Hankel determinants are investigated. As a result, several new results on Block–Hankel determinants are achieved. We present the proof of Theorem 1.2 in Section 3. Finally conclusions and discussions are given in Section 4.

2. Block–Hankel determinants and their properties

In order to complete the proof of Theorem 1.2, it is necessary to investigate the properties of determinant $h_n^{(p)}$.

As we know, Hankel determinants [31] have widely appeared in the theory of orthogonal polynomials, Padé approximation, continued fractions and combinatorial mathematics. (See [32–36] for more details.) In order to interpret a graph, Shingu and Kamioka [37] introduce the notion of Block–Hankel determinant, which is an extension of Hankel determinant. Now we give the definition of *Block–Hankel determinant*.

An $n\times n$ *Block–Hankel determinant* $(p\geqslant 1, q\geqslant 1)$ is defined by $H_n(a_r^{(p,q)}) \equiv det(a_{r+ip+jq}^{(p,q)})_{0\leqslant i,j\leqslant n-1}$. When $p=1$ and $q=1$,

$$H_n(a_r^{(1,1)}) \equiv \begin{vmatrix} a_r^{(1,1)} & a_{r+1}^{(1,1)} & \cdots & a_{r+n-1}^{(1,1)} \\ a_{r+1}^{(1,1)} & a_{r+2}^{(1,1)} & \cdots & a_{r+n}^{(1,1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+n-1}^{(1,1)} & a_{r+n}^{(1,1)} & \cdots & a_{r+2n-1}^{(1,1)} \end{vmatrix}, \tag{4}$$

which is a conventional *Hankel determinant*. The following are two other examples:

$$H_n(a_r^{(2,1)}) \equiv \begin{vmatrix} a_r^{(2,1)} & a_{r+1}^{(2,1)} & \cdots & a_{r+n-1}^{(2,1)} \\ a_{r+2}^{(2,1)} & a_{r+3}^{(2,1)} & \cdots & a_{r+n+1}^{(2,1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+2(n-1)}^{(2,1)} & a_{r+2n-1}^{(2,1)} & \cdots & a_{r+3(n-1)}^{(2,1)} \end{vmatrix} \tag{5}$$

and

$$H_n(a_r^{(4,2)}) \equiv \begin{vmatrix} a_r^{(4,2)} & a_{r+2}^{(4,2)} & \cdots & a_{r+2(n-1)}^{(4,2)} \\ a_{r+4}^{(4,2)} & a_{r+6}^{(4,2)} & \cdots & a_{r+2n+2}^{(4,2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+4(n-1)}^{(4,2)} & a_{r+4n-2}^{(4,2)} & \cdots & a_{r+6(n-1)}^{(4,2)} \end{vmatrix}. \tag{6}$$

Obviously, $h_n^{(p)}$ in Theorem 1.2 have the same structure as $H_n(a_0^{(p,1)})$. Thus, we concentrate on the study of *Block–Hankel determinants* with $q = 1$, namely, $H_n(a_r^{(p,1)})$. For simplicity, we abbreviate this as $H_n(a_r^{(p)})$.

In the sequel, we investigate some properties of *Block–Hankel determinants* $H_n(a_r^{(p)})$ with the elements in (3). The proofs of the following lemmas can be completed by the same steps:

- Step 1: Perform column operations recursively.
- Step 2: If necessary, decompose the determinant obtained by Step 1 to two parts.
- Step 3: Perform row operations recursively in order to get a determinant in a simple expression.

Lemma 2.1. *For $n \geqslant 1$, we have*

$$H_n(a_0^{(p)}) = (\alpha^p\gamma)^{n-1}H_{n-1}(a_{-1}^{(p)}) + (\beta+\gamma)(\alpha^p\gamma)^{n-2}J_{n-1}(a_0^{(p)}), \tag{7}$$

where

$$J_n(a_0^{(p)}) \equiv \begin{vmatrix} A_{p-1}^{(p)} & a_0^{(p)} & a_1^{(p)} & \cdots & a_{n-2}^{(p)} \\ A_{2p-1}^{(p)} & a_p^{(p)} & a_{p+1}^{(p)} & \cdots & a_{n+p-2}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{np-1}^{(p)} & a_{(n-1)p}^{(p)} & a_{(n-1)p+1}^{(p)} & \cdots & a_{n+(n-1)p-2}^{(p)} \end{vmatrix} \tag{8}$$

and

$$H_n(a_{-1}^{(p)}) = -(\alpha^p\gamma)^{n-2}K_{n-1}(a_{-1}^{(p)}), \tag{9}$$

where

$$K_n(a_{-1}^{(p)}) \equiv \begin{vmatrix} A_{p-1}^{(p)} & a_{-1}^{(p)} & a_0^{(p)} & \cdots & a_{n-3}^{(p)} \\ A_{2p-1}^{(p)} & a_{p-1}^{(p)} & a_p^{(p)} & \cdots & a_{n+p-3}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{np-1}^{(p)} & a_{(n-1)p-1}^{(p)} & a_{(n-1)p}^{(p)} & \cdots & a_{n+(n-1)p-3}^{(p)} \end{vmatrix}. \tag{10}$$

Here $A_{kp-1}^{(p)}$ is defined by

$$A_{kp-1}^{(p)} = a_{kp-1}^{(p)} - \sum_{i=1}^{k-1} A_{ip-1}^{(p)} \frac{a_{(k+1-i)p}^{(p)}}{a_p^{(p)}}, \quad A_{p-1}^{(p)} = a_{p-1}^{(p)}. \tag{11}$$

Proof. Here we only give the proof of Eq. (7). By making the appropriate substitutions, Eq. (9) can be obtained in a similar manner without *Step 2*.

Step 1: We rewrite

$$H_n(a_0^{(p)}) = \begin{vmatrix} a_0^{(p)} & a_1^{(p)} & \cdots & a_{n-1}^{(p)} \\ a_p^{(p)} & a_{p+1}^{(p)} & \cdots & a_{n+p-1}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & a_{(n-1)p+1}^{(p)} & \cdots & a_{n+(n-1)p-1}^{(p)} \end{vmatrix} \tag{12}$$

by using the recursion relation (3) to obtain Eq. (7). We first subtract the $(ip+1)$th column multiplied by $\gamma a_{n-3-ip}^{(p)}$ from the nth column for $i = 1, 2, \ldots, [\frac{n-3}{p}]$. Next, if $n = kp+2$, $k = 1, 2, \ldots$, subtract the $(n-2)$th column multiplied by $(\beta+\gamma)$ from the nth column. And then, subtracting the $(n-1)$th column multiplied by α, we have

$$H_n\left(a_0^{(p)}\right) = \begin{vmatrix} a_0^{(p)} & \cdots & a_{n-2}^{(p)} & 0 \\ a_p^{(p)} & \cdots & a_{n+p-2}^{(p)} & \gamma a_p^{(p)} a_{n-3}^{(p)} \\ a_{2p}^{(p)} & \cdots & a_{n+2p-2}^{(p)} & \gamma \sum_{i=1}^{2} a_{pi}^{(p)} a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & \cdots & a_{n+(n-1)p-2}^{(p)} & \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}. \tag{13}$$

Applying a similar procedure to the $(n-1)$th, ..., 2nd columns, we obtain

$$H_n\left(a_0^{(p)}\right) = \begin{vmatrix} a_0^{(p)} & 0 & \cdots & 0 \\ a_p^{(p)} & (\beta+\gamma)a_{p-1}^{(p)} + \gamma a_p^{(p)} a_{-1}^{(p)} & \cdots & \gamma a_p^{(p)} a_{n-3}^{(p)} \\ a_{2p}^{(p)} & (\beta+\gamma)a_{2p-1}^{(p)} + \gamma \sum_{i=1}^{2} \gamma a_{pi}^{(p)} a_{2p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=1}^{2} a_{pi}^{(p)} a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & (\beta+\gamma)a_{(n-1)p-1}^{(p)} + \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{(n-1)p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}. \tag{14}$$

Step 2: Decompose $H_n(a_0^{(p)})$ into two parts, namely,

$$\tilde{H}_n(a_0^{(p)}) = \begin{vmatrix} a_0^{(p)} & 0 & \cdots & 0 \\ a_p^{(p)} & (\beta+\gamma)a_{p-1}^{(p)} & \cdots & \gamma a_p^{(p)} a_{n-3}^{(p)} \\ a_{2p}^{(p)} & (\beta+\gamma)a_{2p-1}^{(p)} & \cdots & \gamma \sum_{i=1}^{2} a_{pi}^{(p)} a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & (\beta+\gamma)a_{(n-1)p-1}^{(p)} & \cdots & \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}$$

and

$$\hat{H}_n(a_0^{(p)}) = \begin{vmatrix} a_0^{(p)} & 0 & \cdots & 0 \\ a_p^{(p)} & \gamma a_p^{(p)} a_{-1}^{(p)} & \cdots & \gamma a_p^{(p)} a_{n-3}^{(p)} \\ a_{2p}^{(p)} & \gamma \sum_{i=1}^{2} a_{pi}^{(p)} a_{2p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=1}^{2} a_{pi}^{(p)} a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{(n-1)p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=1}^{n-1} a_{pi}^{(p)} a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}. \tag{15}$$

Step 3: Lastly, we perform the same row operations to both the determinants $\tilde{H}_n(a_0^{(p)})$ and $\hat{H}_n(a_0^{(p)})$. For $k = 3, 4, \ldots,$ we subtract the ith multiplied by $a^{(p)}_{(k-i+1)p}/a_p^{(p)}$ for $i = 2, \ldots, k-1$. Then the result in (7) obviously holds. $\square$

In addition, $H_n(a_{-1}^{(p)})$ and $H_n(a_0^{(p)})$ respectively have another expression.

Lemma 2.2. *For $n \geqslant 1$, we have*

$$H_n(a_0^{(p)}) = \gamma^{n-1}H_{n-1}(a_{p-1}^{(p)}) - \beta\gamma^{n-2}L_{n-1}(a_p^{(p)}), \tag{16}$$

where

$$L_n(a_p^{(p)}) \equiv \begin{vmatrix} B_{p-1}^{(p)} & a_p^{(p)} & a_{p+1}^{(p)} & \cdots & a_{n+p-2}^{(p)} \\ B_{2p-1}^{(p)} & a_{2p}^{(p)} & a_{2p+1}^{(p)} & \cdots & a_{n+2p-2}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{np-1}^{(p)} & a_{np}^{(p)} & a_{np+1}^{(p)} & \cdots & a_{n+np-2}^{(p)} \end{vmatrix} \tag{17}$$

and

$$H_n(a_{-1}^{(p)}) = -\gamma^{n-2}L_{n-1}(a_{p-1}^{(p)}), \tag{18}$$

where

$$L_n(a_{p-1}^{(p)}) \equiv \begin{vmatrix} B_{p-1}^{(p)} & a_{p-1}^{(p)} & a_p^{(p)} & \cdots & a_{n+p-3}^{(p)} \\ B_{2p-1}^{(p)} & a_{2p-1}^{(p)} & a_{2p}^{(p)} & \cdots & a_{n+2p-3}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{np-1}^{(p)} & a_{np-1}^{(p)} & a_{np}^{(p)} & \cdots & a_{n+np-3}^{(p)} \end{vmatrix}. \tag{19}$$

Here $B_{kp-1}^{(p)} (k \geqslant 1)$ is defined by

$$B_{kp-1}^{(p)} = a_{kp-1}^{(p)} - \sum_{i=1}^{k-1} B_{ip-1}^{(p)} \frac{a_{(k-i)p}^{(p)}}{a_0^{(p)}}, \quad B_{p-1}^{(p)} = a_{p-1}^{(p)}. \tag{20}$$

Proof. *Step 1*: We start by performing column operations. Firstly, subtract the $(ip+1)$th column multiplied by $a_{n-3-ip}^{(p)}$ from the nth column for $i = 0, 1, \ldots, [\frac{n-3}{p}]$. Next, if $n = kp+2$, $k = 1, 2, \ldots$, subtract the $(n-2)$th column multiplied by β from the nth column. And then, subtracting the $(n-1)$th column multiplied by α and adding the $(n-2)$th column multiplied by $\gamma a_0^{(p)}$, we obtain

$$H_n(a_0^{(p)}) = \begin{vmatrix} a_0^{(p)} & 0 & \cdots & 0 \\ a_p^{(p)} & \beta a_{p-1}^{(p)} + \gamma a_0^{(p)} a_{p-1}^{(p)} & \cdots & \gamma a_0^{(p)} a_{n+p-3}^{(p)} \\ a_{2p}^{(p)} & \beta a_{2p-1}^{(p)} + \gamma \sum_{i=0}^{1} a_{pi}^{(p)} a_{2p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=0}^{1} a_{pi}^{(p)} a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & \beta a_{(n-1)p-1}^{(p)} + \gamma \sum_{i=0}^{n-2} a_{pi}^{(p)} a_{(n-1)p-1-pi}^{(p)} & \cdots & \gamma \sum_{i=0}^{n-2} a_{pi}^{(p)} a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}.$$

Next, following *Step 2 and 3*, we can easily obtain the result in (16).

The proof of Eq. (18) is similar, the details of which is omitted here. □

Employing the similar procedure in the proof of Lemma 2.1 to $H_n(a_r^{(p)})(1 \leqslant r \leqslant p-1)$ without *Step 2*, we also have the following lemma.

Lemma 2.3. *For $n \geqslant 1$, we have*

$$H_n(a_r^{(p)}) = \alpha^r(\alpha^p\gamma)^{n-1}H_{n-1}(a_{r-1}^{(p)}), \quad 1 \leqslant r \leqslant p-1. \tag{21}$$

At last, we analyze the term $H_n(a_p^{(p)})$ and another expression of $H_n(a_{p-1}^{(p)})$.

Lemma 2.4. *For $n \geqslant 1$, we have*

$$H_n(a_p^{(p)}) = \alpha^p(\alpha^p\gamma)^{n-1}H_{n-1}(a_{p-1}^{(p)}) + \alpha^p(\beta+\gamma)(\alpha^p\gamma)^{n-2}J_{n-1}(a_p^{(p)}), \tag{22}$$

where

$$J_n(a_p^{(p)}) \equiv \begin{vmatrix} A_{2p-1}^{(p)} & a_p^{(p)} & a_{p+1}^{(p)} & \cdots & a_{n+p-2}^{(p)} \\ A_{3p-1}^{(p)} & a_{2p}^{(p)} & a_{2p+1}^{(p)} & \cdots & a_{n+2p-2}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{(n+1)p-1}^{(p)} & a_{np}^{(p)} & a_{np+1}^{(p)} & \cdots & a_{n+np-2}^{(p)} \end{vmatrix} \tag{23}$$

and

$$H_n(a_{p-1}^{(p)}) = -\alpha^p(\alpha^p\gamma)^{n-2}K_{n-1}(a_{p-1}^{(p)}), \tag{24}$$

where

$$K_n(a_{p-1}^{(p)}) \equiv \begin{vmatrix} A_{2p-1}^{(p)} & a_{p-1}^{(p)} & a_p^{(p)} & \cdots & a_{n+p-3}^{(p)} \\ A_{3p-1}^{(p)} & a_{2p-1}^{(p)} & a_{2p}^{(p)} & \cdots & a_{n+2p-3}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{(n+1)p-1}^{(p)} & a_{np-1}^{(p)} & a_{np}^{(p)} & \cdots & a_{n+np-3}^{(p)} \end{vmatrix}. \tag{25}$$

Proof. Likewise, we just give the detailed proof of Eq. (22), as Eq. (24) can be proved in a similar way without *Step 2*.

Step 1: We start by performing column operations. Firstly, subtract the $(ip+1)$th column multiplied by $a_{n-3-ip}^{(p)}$ from the nth column for $i = 1, 2, \ldots, [\frac{n-3}{p}]$. Next, if $n = kp+2$, $k = 1, 2, \ldots$, subtract the $(n-2)$th column multiplied by $(\beta+\gamma)$. And then, subtracting the $(n-1)$th column multiplied by α , we get

$$H_n(a_p^{(p)}) = \begin{vmatrix} a_p^{(p)} & (\beta+\gamma)a_{p-1}^{(p)} + \gamma a_p^{(p)}a_{-1}^{(p)} & \cdots & \gamma a_p^{(p)}a_{n-3}^{(p)} \\ a_{2p}^{(p)} & (\beta+\gamma)a_{2p-1}^{(p)} + \gamma\sum_{i=1}^{2}\gamma a_{pi}^{(p)}a_{2p-1-pi}^{(p)} & \cdots & \gamma\sum_{i=1}^{2}a_{pi}^{(p)}a_{n+2p-3-pi}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)p}^{(p)} & (\beta+\gamma)a_{(n-1)p-1}^{(p)} + \gamma\sum_{i=1}^{n}a_{pi}^{(p)}a_{(n-1)p-1-pi}^{(p)} & \cdots & \gamma\sum_{i=1}^{n}a_{pi}^{(p)}a_{n+(n-1)p-3-pi}^{(p)} \end{vmatrix}.$$

Then the result in (22) can be obtained by following *Step 2* and *Step 3*. □

3. The proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. The following derivations hold for any positive integer p as long as the expressions make sense.

Proof. We begin with four determinant identities. Define

$$
\begin{aligned}
D_1 &\equiv K_n(a_{-1}^{(p)}),\\
D_2 &\equiv J_n(a_0^{(p)}),\\
D_3 &\equiv H_n(a_{-1}^{(p)}),\\
D_4 &\equiv \begin{vmatrix} 0 & 1 & 0 & \cdots & 0\\ B_{p-1}^{(p)} & a_{p-1}^{(p)} & a_p^{(p)} & \cdots & a_{n+p-3}^{(p)}\\ B_{2p-1}^{(p)} & a_{2p-1}^{(p)} & a_{p+1}^{(p)} & \cdots & a_{n+2p-3}^{(p)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ B_{np-1}^{(p)} & a_{np-1}^{(p)} & a_{np}^{(p)} & \cdots & a_{n+np-3}^{(p)} \end{vmatrix}.
\end{aligned}
$$

Employing the Jacobi determinant identity [38,39], we have four relations:

$$
D_iD_i\begin{pmatrix}1 & n\\ 1 & n\end{pmatrix} = D_i\begin{pmatrix}1\\1\end{pmatrix}D_i\begin{pmatrix}n\\n\end{pmatrix} - D_i\begin{pmatrix}1\\n\end{pmatrix}D_i\begin{pmatrix}n\\1\end{pmatrix}, \quad i = 1, 2, 3, 4,
$$

which are equivalent to

$$K_n(a_{-1}^{(p)})H_{n-2}(a_{p-1}^{(p)}) = K_{n-1}(a_{-1}^{(p)})H_{n-1}(a_{p-1}^{(p)}) - K_{n-1}(a_{p-1}^{(p)})H_{n-1}(a_{-1}^{(p)}), \tag{26}$$

$$J_n(a_0^{(p)})H_{n-2}(a_p^{(p)}) = J_{n-1}(a_0^{(p)})H_{n-1}(a_p^{(p)}) - J_{n-1}(a_p^{(p)})H_{n-1}(a_0^{(p)}), \tag{27}$$

$$H_n(a_{-1}^{(p)})H_{n-2}(a_p^{(p)}) = H_{n-1}(a_{-1}^{(p)})H_{n-1}(a_p^{(p)}) - H_{n-1}(a_0^{(p)})H_{n-1}(a_{p-1}^{(p)}), \tag{28}$$

$$L_{n-1}(a_p^{(p)})H_{n-2}(a_{p-1}^{(p)}) = L_{n-2}(a_p^{(p)})H_{n-1}(a_{p-1}^{(p)}) + L_{n-1}(a_{p-1}^{(p)})H_{n-2}(a_p^{(p)}), \tag{29}$$

respectively.

Actually, from Lemmas 2.1–2.4, we can conclude that terms $K_n(a_{-1}^{(p)})$, $K_n(a_{p-1}^{(p)})$, $J_n(a_0^{(p)})$, $J_n(a_p^{(p)})$, $L_{n-1}(a_p^{(p)})$, $L_{n-1}(a_{p-1}^{(p)})$ and $H_n(a_{p-1}^{(p)})$ are all linear combinations of the variables $H_n(a_{-1}^{(p)})$, $H_n(a_0^{(p)})$ and $H_n(a_p^{(p)})$. When these substitutions are applied to relations (26)–(29), it results in four equations (for simplicity, here we shall abbreviate $H_n(a_{-1}^{(p)})$, $H_n(a_0^{(p)})$ and $H_n(a_p^{(p)})$ to $e_n^{(p)}$, $h_n^{(p)}$ and $g_n^{(p)}$, respectively):

$$e_{n+1}^{(p)}h_{n-p-1}^{(p)} = (a_p^{(p)}\gamma)^p e_n^{(p)}h_{n-p}^{(p)} - \gamma(a_p^{(p)}\gamma)^{2p-2}e_{n-1}^{(p)}h_{n-p+1}^{(p)}, \tag{30}$$

$$h_{n+1}^{(p)}g_{n-2}^{(p)} = (a_p^{(p)}\gamma)h_n^{(p)}g_{n-1}^{(p)} - \gamma h_{n-1}^{(p)}g_n^{(p)}, \tag{31}$$

$$e_n^{(p)}g_{n-2}^{(p)} = e_{n-1}^{(p)}g_{n-1}^{(p)} - \prod_{i=1}^{p-1}[a_i^{(p)}(a_p^{(p)}\gamma)^{n-1-i}]h_{n-1}^{(p)}h_{n-p}^{(p)}, \tag{32}$$

$$e_n^{(p)}g_{n-2}^{(p)} = \frac{1}{\beta}\left[\gamma\prod_{i=1}^{p-1}\left[a_i^{(p)}(a_p^{(p)}\gamma)^{n-1-i}\right]h_{n-1}^{(p)}h_{n-p}^{(p)} - \prod_{i=1}^{p-1}\left[a_i^{(p)}(a_p^{(p)}\gamma)^{n-2-i}\right]h_n^{(p)}h_{n-p-1}^{(p)}\right]. \tag{33}$$

Now, we proceed to eliminate $e_n^{(p)}$ and $g_n^{(p)}$ from the equations above. From Eqs. (32) and (33), we immediately obtain:

$$e_{n-1}^{(p)}g_{n-1}^{(p)} = \left(1+\frac{\gamma}{\beta}\right)\prod_{i=1}^{p-1}[a_i^{(p)}(a_p^{(p)}\gamma)^{n-1-i}]h_{n-1}^{(p)}h_{n-p}^{(p)} - \frac{1}{\beta}\prod_{i=1}^{p-1}[a_i^{(p)}(a_p^{(p)}\gamma)^{n-2-i}]h_n^{(p)}h_{n-p-1}^{(p)}. \tag{34}$$

Additionally, according to relations (33) and (34), multiplying $g_{n-1}^{(p)}$ on both sides of Eq. (30), we get

$$e_n^{(p)}g_{n-1}^{(p)} = \gamma(a_p^{(p)}\gamma)^{p-2}\left(1+\frac{\gamma}{\beta}\right)\prod_{i=1}^{p-1}[a_i^{(p)}(a_p^{(p)}\gamma)^{n-1-i}]h_{n-1}^{(p)}h_{n-p+1}^{(p)} - \frac{1}{\beta}(a_p^{(p)}\gamma)^{-p}\prod_{i=1}^{p-1}[a_i^{(p)}(a_p^{(p)}\gamma)^{n-1-i}]h_{n+1}^{(p)}h_{n-p-1}^{(p)}. \tag{35}$$

In order to eliminate $e_n^{(p)}$ and $g_n^{(p)}$, we need another two equations. One is obtained by multiplying $g_n^{(p)}$ on both sides of Eq. (30), namely,

$$e_{n+1}^{(p)}g_n^{(p)}h_{n-p-1}^{(p)} = (a_p^{(p)}\gamma)^p e_n^{(p)}g_n^{(p)}h_{n-p}^{(p)} - \gamma(a_p^{(p)}\gamma)^{2p-2}e_{n-1}^{(p)}g_n^{(p)}h_{n-p+1}^{(p)}, \tag{36}$$

the other is obtained by multiplying $e_{n-1}^{(p)}$ on both sides of Eq. (31), namely,

$$h_{n+1}^{(p)}e_{n-1}^{(p)}g_{n-2}^{(p)} = (a_p^{(p)}\gamma)h_n^{(p)}e_{n-1}^{(p)}g_{n-1}^{(p)} - \gamma h_{n-1}^{(p)}e_{n-1}^{(p)}g_n^{(p)}. \tag{37}$$

Note that $e_n^{(p)}g_{n-2}^{(p)}$, $e_{n-1}^{(p)}g_{n-1}^{(p)}$ and $e_n^{(p)}g_{n-1}^{(p)}$ can be expressed in terms of $h_n^{(p)}$ in Eqs. (33), (34) and (35), therefore, according to combining Eqs. (36) and (37) and eliminating $e_{n-1}^{(p)}g_n^{(p)}$, we obtain

$$\begin{aligned}
&\gamma(\beta+\gamma)(a_p^{(p)})^{2p-2}h_n^{(p)}h_{n-1}^{(p)}h_{n-p+2}^{(p)}h_{n-p-1}^{(p)} \\
&\quad -h_{n+2}^{(p)}h_{n-1}^{(p)}h_{n-p}^{(p)}h_{n-p-1}^{(p)} + (a_p^{(p)}\gamma)^{p+1}h_{n+1}^{(p)}h_{n-1}^{(p)}{h_{n-p}^{(p)}}^2 \\
&= \gamma(\beta+\gamma)(a_p^{(p)})^{2p-2}h_{n+1}^{(p)}h_{n-2}^{(p)}h_{n-p+1}^{(p)}h_{n-p}^{(p)} \\
&\quad -h_{n+1}^{(p)}h_n^{(p)}h_{n-p+1}^{(p)}h_{n-p-2}^{(p)} + (a_p^{(p)}\gamma)^{p+1}{h_n^{(p)}}^2h_{n-p+1}^{(p)}h_{n-p-1}^{(p)}.
\end{aligned} \tag{38}$$

Dividing by $h_n^{(p)}h_{n-1}^{(p)}h_{n-p+1}^{(p)}h_{n-p}^{(p)}$ on both sides and rearranging the relation above, for any positive integer p, we have

$$\begin{aligned}&\gamma(\beta+\gamma)(a_p^{(p)})^{2p-2}\sum_{i=0}^{p-2}\frac{h_{n-p+2+i}^{(p)}h_{n-p-1+i}^{(p)}}{h_{n-p+1+i}^{(p)}h_{n-p+i}^{(p)}}+\frac{h_{n+1}^{(p)}h_{n-p-2}^{(p)}-(a_p^{(p)}\gamma)^{p+1}h_n^{(p)}h_{n-p-1}^{(p)}}{h_{n-1}^{(p)}h_{n-p}^{(p)}}\\ &=\gamma(\beta+\gamma)(a_p^{(p)})^{2p-2}\sum_{i=0}^{p-2}\frac{h_{n-p+3+i}^{(p)}h_{n-p+i}^{(p)}}{h_{n-p+2+i}^{(p)}h_{n-p+1+i}^{(p)}}+\frac{h_{n+2}^{(p)}h_{n-p-1}^{(p)}-(a_p^{(p)}\gamma)^{p+1}h_{n+1}^{(p)}h_{n-p}^{(p)}}{h_n^{(p)}h_{n-p+1}^{(p)}},\end{aligned} \tag{39}$$

where $\sum_m^n t_i$ means an empty sum, when $m > n$. This yields a quantity independent of n, namely,

$$\gamma(\beta+\gamma)(\alpha^p\gamma)^{2p-2}\sum_{i=0}^{p-2}\frac{h_{n-p+2+i}^{(p)}h_{n-p-1+i}^{(p)}}{h_{n-p+1+i}^{(p)}h_{n-p+i}^{(p)}}+\frac{h_{n+1}^{(p)}h_{n-p-2}^{(p)}-(\alpha^p\gamma)^{p+1}h_n^{(p)}h_{n-p-1}^{(p)}}{h_{n-1}^{(p)}h_{n-p}^{(p)}}=C(p). \tag{40}$$

As a result, (i) and (ii) in Theorem 1.2 can be obtained by putting $p = 1$ and $p = 2$, respectively, where $C(p)$ can be determined by initial values.

When $\beta+\gamma = 0$, from Lemmas 2.1, 2.3 and 2.4, we conclude that $H_n(a_{-1}^{(p)})$, $H_n(a_{p-1}^{(p)})$ and $H_n(a_p^{(p)})$ all can be linearly expressed in terms of $H_n(a_0^{(p)})$. Then, substituting $H_n(a_{-1}^{(p)})$, $H_n(a_{p-1}^{(p)})$ and $H_n(a_p^{(p)})$ into Eq. (28), we immediately obtain conclusion (iii). □

4. Conclusion and discussions

In this paper, we have presented Block–Hankel determinant solutions to a series of specific Somos recurrences. The main results are summarized as follows:

(i) Conjecture 1.1 is confirmed.
(ii) A Block–Hankel determinant solution to Somos-5 is constructed.
(iii) Block–Hankel determinant solutions to other specified Somos recurrences are also given.

Noting that in our main result the elements $a_n^{(p)}$ of the Block–Hankel determinant solutions to Somos recurrences satisfy recursion relation (3), it is natural to ask whether there exist Block–Hankel determinant solutions, whose elements satisfy other recursion relations. These problems are still needed to be considered in the future.

Actually, as for Somos-5 sequence, we have the following conjecture:

Conjecture 4.1. *Let $a_n^{(2)}$ be computed via the following convolution recurrence:*

$$a_n^{(2)}=\begin{cases}0, & \text{if } n<0;\\ 1, & \text{if } n=0;\\ \alpha a_{n-1}^{(2)}+\beta a_{n-2}^{(2)}+\gamma\sum_{i=0}^{[\frac{n-2}{2}]}a_{2i}^{(2)}a_{n-2-2i}^{(2)} & \text{if } n\geqslant 1.\end{cases} \tag{41}$$

Then $h_n^{(2)} = det(a_{2i+j}^{(2)})_{0\leqslant i,j\leqslant n-1}$ is a $(\alpha^6\gamma^3+\alpha^2\gamma^2(\beta+\gamma)[\alpha^2\gamma+(\beta+\gamma)^2], -\alpha^8\gamma^5-\alpha^4\gamma^3(\beta+\gamma)[3\beta(\beta+\gamma)^2+\gamma^2(\beta+\gamma)+\alpha^2\beta(\beta-\gamma))]$ Somos-5 sequence.

Numerical experiments indicate that the above conjecture holds, however, a rigorous proof still remains open.

Additionally, as for Somos-4 sequence, there exists another conjecture made by Michael Somos [25]. In [26], Xin solved it by use of continued fraction method. It is remarked that we can also give an alternative proof to this conjecture by similar determinant technique in this paper. Here we omit the details.

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