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A direct method for evaluating some nice Hankel determinants and proofs of several conjectures

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ABSTRACT

In this paper, we propose a direct method to evaluate Hankel determinants for some generating functions satisfying a certain type of quadratic equations, which cover generating functions of Catalan numbers, Motzkin numbers and Schröder numbers. Additionally, four recent conjectures proposed by Cigler (2011) [3] are proved.

1. Introduction

Hankel determinants of path counting numbers have appeared frequently in the literature. For example, the first three Hankel determinants of the well known Catalan numbers (cf. e.g. [1]) are

$$\begin{aligned}
&det(C_{i+j})_{i,j=0}^{n-1} = 1,\\
&det(C_{i+j+1})_{i,j=0}^{n-1} = 1,\\
&det(C_{i+j+2})_{i,j=0}^{n-1} = n+1,
\end{aligned}$$

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where the nth Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$. As for Motzkin numbers (cf. e.g. [2,3]) generated by $M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$ with $M_0 = 1$, we have

$$\begin{aligned}
&det(M_{i+j})_{i,j=0}^{n-1} = 1,\\
&det(M_{i+j+1})_{i,j=0}^{n-1} = Fib_{n+1}(1, -1),\\
&det(M_{i+j+2})_{i,j=0}^{n-1} = \textstyle\sum_{j=0}^{n} \left(Fib_{j+1}(1, -1)\right)^2,
\end{aligned}$$

where $Fib_n(x, s)$ is the Fibonacci polynomial defined by the recurrence $Fib_n(x, s) = xFib_{n-1}(x, s) + sFib_{n-2}(x, s)$ with values $Fib_0(x, s) = 0$ and $Fib_1(x, s) = 1$. Besides, the Hankel determinants of the Schröder numbers (cf. e.g. [4]) are

$$\begin{aligned}
&det(S_{i+j})_{i,j=0}^{n-1} = 2^{\binom{n}{2}},\\
&det(S_{i+j+1})_{i,j=0}^{n-1} = 2^{\binom{n+1}{2}},\\
&det(S_{i+j+2})_{i,j=0}^{n-1} = (2^{n+1} - 1)2^{\binom{n+1}{2}},
\end{aligned}$$

where $S_n = S_{n-1} + \sum_{i=0}^{n-1} S_i S_{n-1-i}$ with $S_0 = 1$ and $\binom{\cdot}{\cdot}$ is a binomial symbol. It is remarked that Krattenthaler has described several methods to evaluate determinants and listed many known determinant evaluations in [5,6]. For more information, please consult [7–14] and so on.

Recently, Cigler [3] considered Hankel determinants of three sequences $\{c(n, m, a, b)\}_{n=0}^{\infty}$, $\{C(n, m, a, b, t)\}_{n=0}^{\infty}$ and $\{g(n, m, a, b)\}_{n=0}^{\infty}$, whose generating functions respectively satisfy equations

$$\begin{aligned}
f_m(z, a, b) &= \sum_{n\geqslant 0} c(n, m, a, b)z^n\\
&= 1 + azf_m(z, a, b) + bz^m f_m(z, a, b)^2, &\quad (1)\\
F_m(z, a, b, t) &= \sum_{n\geqslant 0} C(n, m, a, b, t)z^n\\
&= 1 + (a + t)zF_m(z, a, b, t) + bz^m F_m(z, a, b, t)f_m(z, a, b), &\quad (2)\\
G_m(z, a, b) &= \sum_{n\geqslant 0} g(n, m, a, b)z^n\\
&= 1 + azG_m(z, a, b) + 2bz^m G_m(z, a, b)f_m(z, a, b). &\quad (3)
\end{aligned}$$

Here a, b, t are arbitrary given complex numbers and m is a fixed positive integer. Obviously, $\{c(n, 1, 0, 1)\}_{n=0}^{\infty}$, $\{c(n, 2, 1, 1)\}_{n=0}^{\infty}$ and $\{c(n, 1, 1, 1)\}_{n=0}^{\infty}$ give Catalan numbers, Motzkin numbers and Schröder numbers, respectively.

It is noted that the coefficients $c(n, m, a, b)$, $C(n, m, a, b)$ and $g(n, m, a, b)$ can be regarded as weights of some lattice paths set. As is known, we may define the weight w of a path as the product of all steps of the path and the weight of a set of paths as the sum of their weights. For given m, consider lattice paths from $(0, 0)$ to $(n, 0)$ with horizontal steps $H = (1, 0)$, up-steps $U = (1, 1)$ and down-steps $D = (m - 1, -1)$ of width $m - 1$. Then $c(n, m, a, b)$ is the weight of the set of all non-negative lattice paths from $(0, 0)$ to $(n, 0)$ and $g(n, m, a, b)$ is the weight of the set of all lattice paths, where $w(U) = 1$, $w(H) = a$ and $w(D) = b$. If $w(H) = a + t$ when H lies on height 0 and $w(H) = a$ in other cases, then $C(n, m, a, b)$ is the weight of the set of all non-negative lattice paths from $(0, 0)$ to $(n, 0)$.

Cigler considered Hankel determinants

$$\begin{aligned}
&d_r^{(m)}(n, a, b) = det(c(i + j + r, m, a, b))_{i,j=0}^{n-1},\\
&D_r^{(m)}(n, a, b, t) = det(C(i + j + r, m, a, b, t))_{i,j=0}^{n-1},\\
&dd_r^{(m)}(n, a, b) = det(g(i + j + r, m, a, b))_{i,j=0}^{n-1}
\end{aligned}$$

for $r = 0, 1, 2$. He used well-known orthogonal polynomials approach (cf. e.g. [3,6,15]), Gessel–Viennot–Lindström theorem (cf. e.g. [16–18]) and the continued fractions method [18,19] to compute the Hankel determinants and successfully obtained some results for special cases. However, for general cases, he only listed several conjectures (i.e. Conjecture 6.8, 7.5, 7.6, 7.7 in [3]). To the best of our knowledge, it still remains to be open as to how to prove these four conjectures [20].

The purpose of this paper is to give a direct method to compute Hankel determinants. This method works for generating functions satisfying a certain type of quadratic function. Then we apply this method to give rigorous proofs to Cigler's four conjectures.

2. Hankel determinants for some quadratic generating functions

Consider the sequences $\{c(n, m, a, b)\}_{n=0}^{\infty}$, whose generating function satisfies (1). In this case, it is easy to see that $c(n, m, a, b)$ satisfy the recurrence relation

$$c_n = ac_{n-1} + b \sum_{i=0}^{n-m} c_i c_{n-m-i} \tag{4}$$

with $c_0 = 1$. Here we use the abbreviations without a, b, m. If we let $c_n = 0$ for $n < 0$, then (4) hold for any integer n.

In this section, we will evaluate the Hankel determinants $d_0^{(m)}(n)$, $d_1^{(m)}(n)$, $d_2^{(m)}(n)$. It is also noted that some results in this section have been proved in [3], but here we will prove them again by our method.

To begin with, we show several lemmas, which give some relations on Hankel determinants. All the proofs can be completed by three steps:

- Step 1: Perform column operations recursively.
- Step 2: If necessary, decompose the determinant obtained by Step 1 to several parts.
- Step 3: Perform row operations recursively in order to get a determinant in a simple expression.

It is noted that we will use the conventions

$$\begin{aligned} &d_{-r}^{(m)}(0) = 1, \\ &d_{-r}^{(m)}(n) = 0 \ \text{ for } n < 0, \end{aligned} \tag{5}$$

in the following lemmas.

Lemma 2.1. *For all $n \in \mathbb{Z}$, $m \geqslant 1$, and $r \geqslant 0$, there holds*

$$d_{-r}^{(m)}(n) - (-1)^{\binom{r+1}{2}} b^{n-r-1} d_{r+2-m}^{(m)}(n-r-1) = [n=0]. \tag{6}$$

Here we use the Iverson bracket [ ] defined as

$$[P] = \begin{cases} 1 \text{ if } P \text{ is true;} \\ 0 \text{ otherwise.} \end{cases} \tag{7}$$

Proof. Here we only give the detailed proof for the case of $n > r$, as the result for the case of $n \leqslant r$ obviously hold by the convention (5).

Step 1: Perform column operations. Let $C_0, C_1, \dots, C_{n-1}$ denote the columns of the corresponding matrix. Change C_N to $C_N - aC_{N-1} - b\sum_{i=0}^{N-m} c_i C_{N-m-i}$ for $N = n-1, n-2, \dots, r+1$, then, by using the recursion relation (4), we have

$$d_{-r}^{(m)}(n)=\begin{vmatrix} c_{-r} & \dots & c_0 & 0 & \dots & 0 \\ c_{-r+1} & \dots & c_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_0 & \dots & c_r & 0 & \dots & 0 \\ c_1 & \dots & c_{r+1} & bc_0c_{r-m+2} & \dots & bc_0c_{n-m} \\ c_2 & \dots & c_{r+2} & b\sum_{i=0}^{1} c_ic_{r-m+3-i} & \dots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{-r+n-1} & \dots & c_{n-1} & b\sum_{i=0}^{-r+n-2} c_ic_{-m+n-i} & \dots & b\sum_{i=0}^{-r+n-2} c_ic_{2n-r-m-2-i} \end{vmatrix}$$

$$=(-1)^{\binom{r+1}{2}}\begin{vmatrix} bc_0c_{r-m+2} & \dots & bc_0c_{n-m} \\ b\sum_{i=0}^{1} c_ic_{r-m+3-i} & \dots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \ddots & \vdots \\ b\sum_{i=0}^{-r+n-2} c_ic_{-m+n-i} & \dots & b\sum_{i=0}^{-r+n-2} c_ic_{2n-r-m-2-i} \end{vmatrix}.$$

Here we use the equivalent recurrence relation $c_n = ac_{n-1} + b\sum_{i=0}^{n-1} c_ic_{n-m-i}$, as $c_k = 0$ for $k < 0$ hold .

Step 3: Perform row operations for the above determinant. For $k = 2, 3, \dots, n-r-1$, we subtract the ith row multiplied by c_{k-i}/c_0 for $i = 1, \dots, k-1$. Then it follows that (6) holds. □

By this lemma, we list some obvious formulae in the following, which will be useful later.

Corollary 2.2. *Let $n > 0$. Then, for $m \geqslant 1$,*

$$d_0^{(m)}(n) = b^{n-1}d_{2-m}^{(m)}(n-1), \tag{8}$$
$$d_{-1}^{(m)}(n) = -b^{n-2}d_{3-m}^{(m)}(n-2), \tag{9}$$
$$d_{-2}^{(m)}(n) = -b^{n-3}d_{4-m}^{(m)}(n-3), \tag{10}$$
$$d_{-m}^{(m)}(n) = (-1)^{\binom{m+1}{2}}b^{n-m-1}d_2^{(m)}(n-m-1), \tag{11}$$
$$d_{1-m}^{(m)}(n) = (-1)^{\binom{m}{2}}b^{n-m}d_1^{(m)}(n-m), \tag{12}$$

For $m \geqslant 2$,

$$d_{2-m}^{(m)}(n) = (-1)^{\binom{m-1}{2}}b^{n-m+1}d_0^{(m)}(n-m+1). \tag{13}$$

For $m \geqslant 3$,

$$d_{3-m}^{(m)}(n) = (-1)^{\binom{m-2}{2}}b^{n-m+2}d_{-1}^{(m)}(n-m+2), \tag{14}$$

For $m \geqslant 4$,

$$d_{4-m}^{(m)}(n) = (-1)^{\binom{m+1}{2}}b^{n-m+3}d_{-2}^{(m)}(n-m+3). \tag{15}$$

Lemma 2.3.

$$d_1^{(m)}(n) = ab^{n-1} d_{3-m}^{(m)}(n-1) + b^n d_{1-m}^{(m)}(n) \tag{16}$$

holds for $m \geqslant 1$, $n \in \mathbb{Z}$.

Proof. Obviously, the result holds for $n \leqslant 0$. We will prove the result for the case $n > 0$ in the following.

Step 1: Perform column operations. Let $C_0, C_1, \ldots, C_{n-1}$ denote the columns of the corresponding matrix. Change C_N to $C_N - aC_{N-1} - b\sum_{i=0}^{N-m} c_i C_{N-m-i}$ for $N = n-1, n-2, \ldots, 1$, then, by using the recursion relation (4), we have

$$d_1^{(m)}(n) = \begin{vmatrix} c_1 & bc_0c_{2-m} & \cdots & bc_0c_{n-m} \\ c_2 & b\sum_{i=0}^{1} c_ic_{3-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & b\sum_{i=0}^{n-1} c_ic_{n+1-m-i} & \cdots & b\sum_{i=0}^{-m+n-2} c_ic_{2n-m-1-i} \end{vmatrix}$$

$$= \begin{vmatrix} ac_0 + bc_0c_{1-m} & bc_0c_{2-m} & \cdots & bc_0c_{n-m} \\ ac_1 + b\sum_{i=0}^{1} c_ic_{2-m-i} & b\sum_{i=0}^{1} c_ic_{3-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \ddots & \vdots \\ ac_{n-1} + b\sum_{i=0}^{n-1} c_ic_{n-m-i} & b\sum_{i=0}^{n-1} c_ic_{n+1-m-i} & \cdots & b\sum_{i=0}^{n-1} c_ic_{2n-m-1-i} \end{vmatrix}.$$

Step 2: Decompose $d_1^{(m)}(n)$ into two parts along the first column, namely,

$$\begin{vmatrix} ac_0 & bc_0c_{2-m} & \cdots & bc_0c_{n-m} \\ ac_1 & b\sum_{i=0}^{1} c_ic_{3-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \ddots & \vdots \\ ac_{n-1} & b\sum_{i=0}^{n-1} c_ic_{n+1-m-i} & \cdots & b\sum_{i=0}^{n-1} c_ic_{2n-m-1-i} \end{vmatrix}$$

and

$$\begin{vmatrix} bc_0c_{1-m} & bc_0c_{2-m} & \cdots & bc_0c_{n-m} \\ b\sum_{i=0}^{1} c_ic_{2-m-i} & b\sum_{i=0}^{1} c_ic_{3-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \ddots & \vdots \\ b\sum_{i=0}^{n-1} c_ic_{n-m-i} & b\sum_{i=0}^{n-1} c_ic_{n+1-m-i} & \cdots & b\sum_{i=0}^{n-1} c_ic_{2n-m-1-i} \end{vmatrix}.$$

Step 3: Perform row operations for the above two determinants. For $k = 2, 3, \ldots, n$, we subtract the ith row multiplied by c_{k-i}/c_0 for $i = 1, \ldots, k-1$. Then it follows that (16) holds. □

Lemma 2.4.

$$d_0^{(m)}(n) = d_2^{(m)}(n-1) + b^n d_{-m}^{(m)}(n) - a^2 b^{n-2} d_{4-m}^{(m)}(n-2) - 2ab^{n-1}\tilde{d}_{1-m}^{(m)}(n-1) \tag{17}$$

holds for $n \geqslant 2$, $m \geqslant 1$, *where*

$$\tilde{d}_{1-m}^{(m)}(n) = \begin{vmatrix} c_{1-m} & c_{2-m} & \cdots & c_{n-m} \\ c_{3-m} & c_{4-m} & \cdots & c_{n+2-m} \\ c_{4-m} & c_{5-m} & \cdots & c_{n+3-m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n+1-m} & c_{n+2-m} & \cdots & c_{2n-m} \end{vmatrix}$$

with $\tilde{d}_{1-m}^{(m)}(0) = 1$ *and* $\tilde{d}_{1-m}^{(m)}(n) = 0$ *for* $n < 0$.

Proof. Obviously we have $d_0^{(m)}(n) = d_2^{(m)}(n-1) + H$, where

$$H = \begin{vmatrix} 0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_n \\ c_2 & c_3 & c_4 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix}.$$

Based on this observation, in order to show that (17) holds, it suffices to prove that $H = b^n d_{-m}^{(m)}(n) - a^2 b^{n-2} d_{4-m}^{(m)}(n-2) - 2ab^{n-1}\tilde{d}_{1-m}^{(m)}(n-1)$.

Step 1: Perform column operations. Let $C_0, C_1, \ldots, C_{n-1}$ denote the columns of the corresponding matrix. Change C_N to $C_N - aC_{N-1} - b\sum_{i=0}^{N-m} c_i C_{N-m-i}$ for $N = n-1, n-2, \ldots, 1$, then, by using the recursion relation (4), we have

$$H = \begin{vmatrix} 0 & ac_0 + bc_0 c_{1-m} & bc_0 c_{2-m} & \cdots & bc_0 c_{n-1-m} \\ c_1 & bc_0 c_{2-m} & bc_0 c_{3-m} & \cdots & bc_0 c_{n-m} \\ c_2 & b\sum_{i=0}^{1} c_i c_{3-m-i} & b\sum_{i=0}^{1} c_i c_{4-m-i} & \cdots & b\sum_{i=0}^{1} c_i c_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & b\sum_{i=0}^{n-2} c_i c_{n-m-i} & b\sum_{i=0}^{n-2} c_i c_{n-m+1-i} & \cdots & b\sum_{i=0}^{n-2} c_i c_{2n-m-2-i} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & ac_0 + bc_0c_{1-m} & bc_0c_{2-m} & \cdots & bc_0c_{n-1-m} \\ ac_0 + bc_0c_{1-m} & bc_0c_{2-m} & bc_0c_{3-m} & \cdots & bc_0c_{n-m} \\ ac_1 + b\sum_{i=0}^{1} c_ic_{2-m-i} & b\sum_{i=0}^{1} c_ic_{3-m-i} & b\sum_{i=0}^{1} c_ic_{4-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ac_{n-2} + b\sum_{i=0}^{n-2} c_ic_{n-m-1-i} & b\sum_{i=0}^{n-2} c_ic_{n-m-i} & b\sum_{i=0}^{n-2} c_ic_{n-m+1-i} & \cdots & b\sum_{i=0}^{n-2} c_ic_{2n-m-2-i} \end{vmatrix}.$$

Step 2: Let $A + B$ be the first column of H and $C + D$ be the first row with $A = \begin{pmatrix} 0 \\ ac_0 \\ ac_1 \\ \vdots \\ ac_{n-2} \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ ac_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T$. Then by multilinearity, we have

$$H = \Phi(A + B, C + D) = \Phi(A, C) + \Phi(B, C) + \Phi(A, D) + \Phi(B, D) = H_1 + H_2 + H_3 + H_4,$$

where

$$H_1 = \begin{vmatrix} 0 & ac_0 & 0 & \cdots & 0 \\ ac_0 & bc_0c_{2-m} & bc_0c_{3-m} & \cdots & bc_0c_{n-m} \\ ac_1 & b\sum_{i=0}^{1} c_ic_{3-m-i} & b\sum_{i=0}^{1} c_ic_{4-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ac_{n-2} & b\sum_{i=0}^{n-2} c_ic_{n-m-i} & b\sum_{i=0}^{n-2} c_ic_{n-m+1-i} & \cdots & b\sum_{i=0}^{n-2} c_ic_{2n-m-2-i} \end{vmatrix},$$

$$H_2 = \begin{vmatrix} 0 & ac_0 & 0 & \cdots & 0 \\ bc_0c_{1-m} & bc_0c_{2-m} & bc_0c_{3-m} & \cdots & bc_0c_{n-m} \\ b\sum_{i=0}^{1} c_ic_{2-m-i} & b\sum_{i=0}^{1} c_ic_{3-m-i} & b\sum_{i=0}^{1} c_ic_{4-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b\sum_{i=0}^{n-2} c_ic_{n-m-1-i} & b\sum_{i=0}^{n-2} c_ic_{n-m-i} & b\sum_{i=0}^{n-2} c_ic_{n-m+1-i} & \cdots & b\sum_{i=0}^{n-2} c_ic_{2n-m-2-i} \end{vmatrix},$$

$$H_3 = \begin{vmatrix} 0 & bc_0c_{1-m} & bc_0c_{2-m} & \cdots & bc_0c_{n-1-m} \\ ac_0 & bc_0c_{2-m} & bc_0c_{3-m} & \cdots & bc_0c_{n-m} \\ ac_1 & b\sum\limits_{i=0}^{1} c_ic_{3-m-i} & b\sum\limits_{i=0}^{1} c_ic_{4-m-i} & \cdots & b\sum\limits_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ac_{n-2} & b\sum\limits_{i=0}^{n-2} c_ic_{n-m-i} & b\sum\limits_{i=0}^{n-2} c_ic_{n-m+1-i} & \cdots & b\sum\limits_{i=0}^{n-2} c_ic_{2n-m-2-i} \end{vmatrix}$$

and

$$H_4 = \begin{vmatrix} 0 & bc_0c_{1-m} & bc_0c_{2-m} & \cdots & bc_0c_{n-1-m} \\ bc_0c_{1-m} & bc_0c_{2-m} & bc_0c_{3-m} & \cdots & bc_0c_{n-m} \\ b\sum\limits_{i=0}^{1} c_ic_{2-m-i} & b\sum\limits_{i=0}^{1} c_ic_{3-m-i} & b\sum\limits_{i=0}^{1} c_ic_{4-m-i} & \cdots & b\sum\limits_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b\sum\limits_{i=0}^{n-2} c_ic_{n-m-1-i} & b\sum\limits_{i=0}^{n-2} c_ic_{n-m-i} & b\sum\limits_{i=0}^{n-2} c_ic_{n-m+1-i} & \cdots & b\sum\limits_{i=0}^{n-2} c_ic_{2n-m-2-i} \end{vmatrix}.$$

Step 3: Perform row operations for the determinants H_1, H_2, H_3, H_4. For $k = 3, 4, \dots, n$, we subtract the ith row multiplied by c_{k-i}/c_0 for $i = 2, 3, \dots, k-1$. we can see that $H_1 = -a^2b^{n-2}d_{4-m}^{(m)}(n-2)$, $H_4 = b^nd_{-m}^{(m)}(n)$ and

$$H_2 = H_3 = -ab^{n-1} \begin{vmatrix} c_{1-m} & c_{2-m} & \cdots & c_{n-1-m} \\ c_{3-m} & c_{4-m} & \cdots & c_{n+1-m} \\ c_{4-m} & c_{5-m} & \cdots & c_{n+2-m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-m} & c_{n+1-m} & \cdots & c_{2n-2-m} \end{vmatrix}.$$

Thus, (17) is proved. □

Lemma 2.5.

$$\begin{aligned} &\tilde{d}_{1-m}^{(m)}(n) - (-1)^{\binom{m-1}{2}} b^{2n-2m}(b\tilde{d}_{1-m}^{(m)}(n-m) + ad_{4-m}^{(m)}(n-m-1)) \\ &= [n=0] - (-1)^{\binom{m-1}{2}} b[n=m] \end{aligned} \tag{18}$$

holds for all $n \in \mathbb{Z}$ and $m \geqslant 3$.

Proof. Obviously, the result for $n \leqslant m$ can be obtained by noting that $\tilde{d}_{1-m}^{(m)}(i) = 0$ for $i = 1, 2, \dots, m-1$ because the elements of the first row are zeros and $\tilde{d}_{1-m}^{(m)}(m) = 0$ because the $(m-1)$th column is a multiple of the $(m-2)$th column. In the following, we give a detailed proof for the case $n > m$.

First, let's consider $\tilde{d}_{1-m}^{(m)}(n)$. Let $C_0, C_1, \ldots, C_{n-1}$ denote the columns of the corresponding matrix. Change C_N to $C_N - aC_{N-1} - b\sum_{i=0}^{N-m} c_i C_{N-m-i}$ for $N = n-1, n-2, \ldots, m$, then, by using the recursion relation (4), we have, for $m \geqslant 3$ and $n > m$,

$$\tilde{d}_{1-m}^{(m)}(n) = \begin{vmatrix} c_{1-m} & \ldots & c_{-2} & c_{-1} & c_0 & 0 & \ldots & 0 \\ c_{3-m} & \ldots & c_0 & c_1 & c_2 & 0 & \ldots & 0 \\ c_{4-m} & \ldots & c_1 & c_2 & c_3 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \\ c_0 & \ldots & c_{m-3} & c_{m-2} & c_{m-1} & 0 & \ldots & 0 \\ c_1 & \ldots & c_{m-2} & c_{m-1} & c_m & bc_0c_1 & \ldots & bc_0c_{n-m} \\ c_2 & \ldots & c_{m-1} & c_m & c_{m+1} & b\sum_{i=0}^{1} c_ic_{2-i} & \ldots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \\ c_{n-m+1} & \ldots & c_{n-2} & c_{n-1} & c_n & b\sum_{i=0}^{n-m} c_ic_{n-m+1-i} & \ldots & b\sum_{i=0}^{n-m} c_ic_{2n-2m-i} \end{vmatrix}.$$

Subtracting the $(m-2)$th column multiplied by a from the $(m-1)$th column, we see

$$\tilde{d}_{1-m}^{(m)}(n)$$
$$= \begin{vmatrix} c_{1-m} & \ldots & c_{-2} & 0 & c_0 & 0 & \ldots & 0 \\ c_{3-m} & \ldots & c_0 & 0 & c_2 & 0 & \ldots & 0 \\ c_{4-m} & \ldots & c_1 & 0 & c_3 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \\ c_0 & \ldots & c_{m-3} & 0 & c_{m-1} & 0 & \ldots & 0 \\ c_1 & \ldots & c_{m-2} & bc_0c_{-1} & c_m & bc_0c_1 & \ldots & bc_0c_{n-m} \\ c_2 & \ldots & c_{m-1} & b\sum_{i=0}^{1} c_ic_{-i} & c_{m+1} & b\sum_{i=0}^{1} c_ic_{2-i} & \ldots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \\ c_{n-m+1} & \ldots & c_{n-2} & b\sum_{i=0}^{n-m} c_ic_{n-m-1-i} & c_n & b\sum_{i=0}^{n-m} c_ic_{n-m+1-i} & \ldots & b\sum_{i=0}^{n-m} c_ic_{2n-2m-i} \end{vmatrix}.$$

Expanding the above determinant along the first $m-1$ rows and the $1, 2, \ldots, m-2, m$th columns, we get

$$\tilde{d}_{1-m}^{(m)}(n) = -\begin{vmatrix} 0 & 0 & \ldots & 0 & c_0 \\ 0 & 0 & \ldots & c_0 & c_2 \\ 0 & 0 & \ldots & c_1 & c_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_0 & c_1 & \ldots & c_{m-3} & c_{m-1} \end{vmatrix}$$

$$
\times \begin{vmatrix} bc_0c_{-1} & bc_0c_1 & \cdots & bc_0c_{n-m} \\ b\sum_{i=0}^{1} c_ic_{-i} & b\sum_{i=0}^{1} c_ic_{2-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-m+1-i} \\ \vdots & \vdots & \ddots & \vdots \\ b\sum_{i=0}^{n-m} c_ic_{n-m-1-i} & b\sum_{i=0}^{n-m} c_ic_{n-m+1-i} & \cdots & b\sum_{i=0}^{n-m} c_ic_{2n-2m-i} \end{vmatrix}
$$

$$
= -(-1)^{\binom{m-1}{2}} b^{n-m+1} \begin{vmatrix} c_{-1} & c_0 & \cdots & c_{n-m-1} \\ c_1 & c_2 & \cdots & c_{n-m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-m} & c_{n+1-m} & \cdots & c_{2n-2m} \end{vmatrix}
$$

$$
= -(-1)^{\binom{m-1}{2}} b^{n-m+1} \tilde{d}_{-1}^{(m)}(n-m+1). \tag{19}
$$

Here the second identity is a consequence of performing row operations and taking the transpose. That is, for $k = 2, 3, \ldots, n-m+1$, we subtract the ith row multiplied by c_{k-i}/c_0 for $i = 1, \ldots, k-1$, then, transpose the result.

Next, we consider the determinant $\tilde{d}_{-1}^{(m)}(n-m+1)$. Let $C_0, C_1, \ldots, C_{n-m}$ denote the columns of the corresponding matrix. Change C_N to $C_N - aC_{N-1} - b\sum_{i=0}^{N-m} c_iC_{N-m-i}$ for $N = n-m, n-m-1, \ldots, 2$, then, by using the recursion relation (4), we have

$$
\tilde{d}_{-1}^{(m)}(n-m+1)
$$

$$
= \begin{vmatrix} c_{-1} & c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_2 & bc_0c_{3-m} & bc_0c_{4-m} & \cdots & bc_0c_{n-2m+1} \\ c_2 & c_3 & b\sum_{i=0}^{1} c_ic_{4-m-i} & b\sum_{i=0}^{1} c_ic_{5-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-2m+2-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ c_{n-m} & c_{n-m+1} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+2-i} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+3-i} & \cdots & b\sum_{i=0}^{n-m-1} c_ic_{2n-3m-i} \end{vmatrix}
$$

$$
= -\begin{vmatrix} c_1 & bc_0c_{3-m} & bc_0c_{4-m} & \cdots & bc_0c_{n-2m+1} \\ c_2 & b\sum_{i=0}^{1} c_ic_{4-m-i} & b\sum_{i=0}^{1} c_ic_{5-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-2m+2-i} \\ \vdots & \vdots & \ddots & \vdots & \\ c_{n-m} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+2-i} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+3-i} & \cdots & b\sum_{i=0}^{n-m-1} c_ic_{2n-3m-i} \end{vmatrix}.
$$

Noting the first column can be written $A+B$ with $A=\begin{pmatrix} ac_0 \\ ac_1 \\ \vdots \\ ac_{n-m-1} \end{pmatrix}$ and $B=\begin{pmatrix} bc_0c_{1-m} \\ b\sum_{i=0}^{1} c_ic_{2-m-i} \\ \vdots \\ b\sum_{i=0}^{n-m-1} c_ic_{n-2m-i} \end{pmatrix}$,

we can decompose the determinant into two parts by linearity:

$$-\begin{vmatrix} ac_0 & bc_0c_{3-m} & bc_0c_{4-m} & \cdots & bc_0c_{n-2m+1} \\ ac_1 & b\sum_{i=0}^{1} c_ic_{4-m-i} & b\sum_{i=0}^{1} c_ic_{5-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-2m+2-i} \\ \vdots & \vdots & \ddots & \vdots & \\ ac_{n-m-1} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+2-i} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+3-i} & \cdots & b\sum_{i=0}^{n-m-1} c_ic_{2n-3m-i} \end{vmatrix}$$

and

$$-\begin{vmatrix} bc_0c_{1-m} & bc_0c_{3-m} & bc_0c_{4-m} & \cdots & bc_0c_{n-2m+1} \\ b\sum_{i=0}^{1} c_ic_{2-m-i} & b\sum_{i=0}^{1} c_ic_{4-m-i} & b\sum_{i=0}^{1} c_ic_{5-m-i} & \cdots & b\sum_{i=0}^{1} c_ic_{n-2m+2-i} \\ \vdots & \vdots & \ddots & \vdots & \\ b\sum_{i=0}^{n-m-1} c_ic_{n-2m-i} & b\sum_{i=0}^{n-m-1} c_ic_{n-2m+2-i} & b\sum_{i=0}^{n-m-1} c_ic_{n-m+3-i} & \cdots & b\sum_{i=0}^{n-m-1} c_ic_{2n-3m-i} \end{vmatrix}.$$

Then, perform row operations for the above two determinants. For $k=2,3,\ldots,n-m$, we subtract the ith row multiplied by c_{k-i}/c_0 for $i=1,\ldots,k-1$, we can get

$$\tilde{d}_{-1}^{(m)}(n-m+1)=-b^{n-m}\tilde{d}_{1-m}^{(m)}(n-m)-ab^{n-m-1}d_{4-m}^{(m)}(n-m-1). \tag{20}$$

With the help of (19) and (20), the result (18) follows. □

Applying these lemmas, we can obtain the following result. It is noted that we care about the determinants of nonnegative order in the following theorems.

Theorem 2.6. *For $m\geqslant 2$, we have*

$$\begin{aligned} d_0^{(m)}(mn)&=(-1)^{\binom{m-1}{2}n}b^{n(mn-1)}, \\ d_0^{(m)}(mn+1)&=(-1)^{\binom{m-1}{2}n}b^{n(mn+1)} \end{aligned} \tag{21}$$

and $d_0^{(m)}(n)=0$ else.
For $m\geqslant 3$, we have

$$\begin{aligned} d_1^{(m)}(mn)&=(-1)^{\binom{m}{2}n}b^{mn^2}, \\ d_1^{(m)}(mn+1)&=(-1)^{\binom{m}{2}n}(n+1)ab^{mn^2+2n}, \\ d_1^{(m)}(mn-1)&=-(-1)^{\binom{m}{2}n}nab^{mn^2-2n} \end{aligned} \tag{22}$$

and $d_1^{(m)}(n) = 0$ *else.*
For $m \geqslant 4$, *we have*

$$\begin{aligned} &d_2^{(m)}(mn) = (-1)^{\binom{m-1}{2}n}(n+1)b^{mn^2+n}, \\ &d_2^{(m)}(mn+1) = (-1)^{\binom{m-1}{2}n}(n+1)^2 a^2 b^{mn^2+3n}, \\ &d_2^{(m)}(mn-1) = (-1)^{\binom{m-1}{2}n} n b^{mn^2-n}, \\ &d_2^{(m)}(mn-2) = -(-1)^{\binom{m-1}{2}n} n^2 a^2 b^{mn^2-3n} \end{aligned} \tag{23}$$

and $d_2^{(m)}(n) = 0$ *else.*

Remark: Since (23) includes the content of Conjecture 7.5 in [3], this conjecture can be solved as a corollary of Theorem 2.6.

Proof. First, let us compute $d_0^{(m)}(n)$. From (8) and (13), we obtain

$$d_0^{(m)}(n) = (-1)^{\binom{m-1}{2}} b^{2n-m-1} d_0^{(m)}(n-m) \tag{24}$$

for $m \geqslant 2$. Noting that $d_0^{(m)}(0) = d_0^{(m)}(1) = 1$, we can derive $d_0^{(m)}(mn)$ and $d_0^{(m)}(mn+1)$ by induction. Additionally, from the recurrence relation of c_n, we see that $c_i = a^i$ for $i = 0, 1, \dots, m-1$. This lead to $d_0^{(m)}(n) = 0$, $2 \leqslant n \leqslant m-1$ because the second column is a multiple of the first. From (24), we can confirm that $d_0^{(m)}(mn+i) = 0$ for $2 \leqslant i \leqslant m-1$ by induction. Thus, (21) is proved.

Next, let us consider $d_1^{(m)}(n)$. From (12), (14) and (16), we obtain

$$d_1^{(m)}(n) = (-1)^{\binom{m-2}{2}} ab^{2n-m} d_{-1}^{(m)}(n-m+1) + (-1)^{\binom{m}{2}} b^{2n-m} d_1^{(m)}(n-m) \tag{25}$$

for $m \geqslant 3$. Thus, it is possible to compute $d_1^{(m)}(n)$ if $d_{-1}^{(m)}(n)$ is known.

From (9) and (14), we have

$$d_{-1}^{(m)}(n) = -(-1)^{\binom{m-2}{2}} b^{2n-m-2} d_{-1}^{(m)}(n-m) \tag{26}$$

for $m \geqslant 3$. It is easy to see that $d_{-1}^{(m)}(0) = 1$, $d_{-1}^{(m)}(1) = 0$, $d_{-1}^{(m)}(2) = -1$ and it is also noted that $d_{-1}^{(m)}(n) = 0$ for $3 \leqslant n \leqslant m-1$ because the third column is a multiple of the second. By induction, we get

$$\begin{aligned} &d_{-1}^{(m)}(mn) = -(-1)^{\binom{m-2}{2}} b^{mn^2-2n}, \\ &d_{-1}^{(m)}(mn+2) = (-1)^{\binom{m-2}{2}} b^{mn^2-2n}, \\ &d_{-1}^{(m)}(mn+i) = 0,\ 1 \leqslant i \leqslant m-1,\ i \neq 2. \end{aligned} \tag{27}$$

Obviously, $d_1^{(m)}(0) = 1$, $d_1^{(m)}(1) = a$ hold. It is also noted that $d_1^{(m)}(n) = 0$ for $2 \leqslant n \leqslant m-2$ because the second column is a multiple of the first, and $d_1^{(m)}(m-1) = -(-1)^{\binom{m}{2}} ab^{m-2}$ after simple column operations. Then, using (25), (27) and the initial values, we can prove (22) by induction.

Finally, we turn to $d_2^{(m)}(n)$. From the condensation formula for determinants (cf. [6,21,22]) we get

$$d_0^{(m)}(n) d_2^{(m)}(n-2) = d_0^{(m)}(n-1) d_2^{(m)}(n-1) - (d_1^{(m)}(n-1))^2. \tag{28}$$

Replacing n by $mn+2$ and mn respectively, we have

$$d_0^{(m)}(mn+1)d_2^{(m)}(mn+1)-(d_1^{(m)}(mn+1))^2=0, \tag{29}$$
$$d_0^{(m)}(mn)d_2^{(m)}(mn-2)+(d_1^{(m)}(mn-1))^2=0, \tag{30}$$

from which we can calculate $d_2^{(m)}(mn+1)$ and $d_2^{(m)}(mn-2)$.

From (10) and (15), we see

$$d_{4-m}^{(m)}(n)=(-1)^{\binom{m-1}{2}}b^{2n-2m+3}d_{4-m}^{(m)}(n-m), \tag{31}$$

for $m\geqslant 4$. Thus, by noting that the initial values $d_{4-m}^{(m)}(i)=0$ for $1\leqslant i\leqslant m-4$ because the elements of the first row are zeros and $d_{4-m}^{(m)}(i)=0, i=m-2, m-1$ because the $(m-1)$th column is a multiple of the $(m-2)$th column, it is easy to get

$$d_{4-m}^{(m)}(mn+i)=0,\quad 1\leqslant i\leqslant m-1,\quad i\neq m-3. \tag{32}$$

Besides, noting that $\tilde{d}_{1-m}^{(m)}(i)=0$ for $i=1,2,\ldots,m-1$ because the elements of the first row are zeros and $\tilde{d}_{1-m}^{(m)}(m)=0$ because the $(m-1)$th column is a multiple of the $(m-2)$th column, we also have $\tilde{d}_{1-m}^{(m)}(mn+i)=0$ for $2\leqslant i\leqslant m, i\neq m-2$ by use of (18) and (32). Then, with the help of (11) and (17), we have

$$d_0^{(m)}(mn+i)=d_2^{(m)}(mn+i-1)+(-1)^{\binom{m+1}{2}}b^{2nm+2i-m-1}d_2^{(m)}(nm+i-m-1)$$

for $m\geqslant 4$ and $0\leqslant i\leqslant m-2$ and $i\neq 2$. Noting that $d_2^{(m)}(0)=1, d_2^{(m)}(n)=0, 2\leqslant n\leqslant m-3$ because the second column is a multiple of the first and noting that $d_0^{(m)}(n)$ is already known, we can derive $d_2^{(m)}(mn+i)$ for $0\leqslant i\leqslant m-3$ and $i\neq 1$ by induction.

As for $d_2^{(m)}(mn-1)$, because $d_0^{(m)}(n)$, $d_1^{(m)}(mn)$ and $d_2^{(m)}(mn)$ are known and $d_0^{(m)}(mn+1)\neq 0$, we can compute them by

$$d_0^{(m)}(mn+1)d_2^{(m)}(mn-1)=d_0^{(m)}(mn)d_2^{(m)}(mn)-(d_1^{(m)}(mn))^2,$$

which is obtained by replacing n by $mn+1$ in (28). Therefore, (23) is proved and we complete the proof. □

The above theorem gives the results for most of the cases. Now we consider other cases. As for the case $m=1$, we begin with the following lemma:

Lemma 2.7.

$$d_0^{(1)}(n)=b^{n-1}(a+b)^{n-1}d_0^{(1)}(n-1)+[n=0] \tag{33}$$

holds for all $n\in\mathbb{Z}$.

Proof. We consider the case of $n\geqslant 1$. First, we rewrite the recurrence relation (4) for $\{c_n\}$ of case $m=1$ as

$$c_n=(a+b)c_{n-1}+b\sum_{i=0}^{n-2}c_ic_{n-1-i} \tag{34}$$

with $c_0=1$.

Step 1: Let $C_0, C_1, \ldots, C_{n-1}$ denote the columns of the corresponding matrix. Change C_N to $C_N-(a+b)C_{N-1}-b\sum_{i=0}^{N-2}c_iC_{N-1-i}$ for $N=n-1, n-2, \ldots, 1$, then, by using the recursion relation (4), we have

$$d_0^{(1)}(n) = \begin{vmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & bc_1c_0 & \cdots & bc_1c_{n-2} \\ c_2 & b\sum_{i=1}^{2} c_ic_{2-i} & \cdots & b\sum_{i=1}^{2} c_ic_{n-i} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & b\sum_{i=1}^{n-1} c_ic_{n-1-i} & \cdots & b\sum_{i=1}^{n-1} c_ic_{2n-3-i} \end{vmatrix}$$

$$= c_0 \begin{vmatrix} bc_1c_0 & \cdots & bc_1c_{n-2} \\ b\sum_{i=1}^{2} c_ic_{2-i} & \cdots & b\sum_{i=1}^{2} c_ic_{n-i} \\ \vdots & \ddots & \vdots \\ b\sum_{i=1}^{n-1} c_ic_{n-1-i} & \cdots & b\sum_{i=1}^{n-1} c_ic_{2n-3-i} \end{vmatrix}.$$

Step 3: Perform row operations for the above determinant. For $k = 2, 3, \ldots, n-1$, we subtract the ith row multiplied by c_{k+1-i}/c_1 for $i = 1, \ldots, k-1$. Then it follows that (33) holds. □

With the help of this lemma, we have

Theorem 2.8.

$$d_0^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n}{2}}, \tag{35}$$

$$d_1^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n+1}{2}}, \tag{36}$$

$$d_2^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n+1}{2}}\frac{(a+b)^{n+1} - b^{n+1}}{a} \tag{37}$$

Proof. Applying the above lemma and noting that $d_0^{(1)}(0) = 1$, we can derive $d_0^{(1)}(n) = a^{\binom{n}{2}}(a+b)^{\binom{n}{2}}$ by induction.

Replacing m with 1 in (12), namely,

$$d_0^{(1)}(n) = b^{n-1}d_1^{(1)}(n-1),$$

we can see (36) obviously follows.

Using the condensation formula

$$d_0^{(1)}(n)d_2^{(1)}(n-2) = d_0^{(1)}(n-1)d_2^{(1)}(n-1) - (d_1^{(1)}(n-1))^2$$

and the initial value $d_2^{(1)}(0) = 1$, (37) can be confirmed by induction. Therefore, the proof is completed. □

For the case $m = 2$, here we only need to calculate $d_1^{(2)}(n)$ and $d_2^{(2)}(n)$.

Noting that (9) and (16) give

$$d_1^{(2)}(n) = ab^{n-1}d_1^{(2)}(n-1) - b^{2n-2}d_1^{(2)}(n-2),$$

and the initial values $d_1^{(2)}(0) = 1$ and $d_1^{(2)}(1) = a$, we can express $d_1^{(2)}(n)$ by the Fibonacci polynomial, where the Fibonacci polynomial $Fib_n(x, s)$ is defined by the recurrence $Fib_n(x, s) = xFib_{n-1}(x, s) + sFib_{n-2}(x, s)$ with values $Fib_0(x, s) = 0$ and $Fib_1(x, s) = 1$.

Using the condensation formula

$$d_0^{(2)}(n)d_2^{(2)}(n-2) = d_0^{(2)}(n-1)d_2^{(2)}(n-1) - (d_1^{(2)}(n-1))^2$$

and the initial value $d_2^{(2)}(0) = 1$, $d_2^{(2)}(n)$ can be obtained by induction. Thus, we have

Theorem 2.9.

$$\begin{aligned}
&d_0^{(2)}(n) = b^{\binom{n}{2}},\\
&d_1^{(2)}(n) = b^{\binom{n}{2}} Fib_{n+1}(a, -b),\\
&d_2^{(2)}(n) = b^{\binom{n}{2}} \textstyle\sum_{j=0}^{n} b^{n-j} \left(Fib_{j+1}(a, -b)\right)^2 .
\end{aligned}$$

For the case $m = 3$, Theorem 2.6 states the result for $d_0^{(3)}(n)$ and $d_1^{(3)}(n)$. Using (18) with $m = 3$, we have

$$\tilde{d}_{-2}^{(3)}(n) = -b^{2n-5}\tilde{d}_{-2}^{(3)}(n-3) - ab^{2n-6}d_1^{(3)}(n-4) + [n=0] + b[n=3],$$

from which we can firstly obtain

$$\begin{aligned}
&\tilde{d}_{-2}^{(3)}(3n) = (-1)^{n+1}\tfrac{n(n-1)}{2}a^2b^{3n^2-2n-1} + [n=0],\\
&\tilde{d}_{-2}^{(3)}(3n+1) = (-1)^n nab^{3n^2-1},\\
&\tilde{d}_{-2}^{(3)}(3n+2) = (-1)^n\tfrac{n(n+1)}{2}a^2b^{3n^2+2n-1}.
\end{aligned}$$

by induction and the initial values $\tilde{d}_{-2}^{(3)}(0) = 1$, $\tilde{d}_{-2}^{(3)}(1) = 0$, $\tilde{d}_{-2}^{(3)}(2) = 0$.

With the help of (11) and (17), we have

$$d_0^{(3)}(n) = d_2^{(3)}(n-1) + b^{2n-4}d_2^{(3)}(n-4) - a^2b^{n-2}d_1^{(3)}(n-2) - 2ab^{n-1}\tilde{d}_{-2}^{(3)}(n-1),$$

using which and the initial value $d_2^{(3)}(0) = 1$, $d_2^{(3)}(1) = a^2$ and $d_2^{(3)}(0) = a^3b - b^2$, then, we can derive the result for $d_2^{(3)}(n)$ by induction.

Theorem 2.10.

$$\begin{aligned}
&d_2^{(3)}(3n) = (-1)^{n+1}b^{3n^2+n-1}\left(a^3\textstyle\sum_{i=0}^{n} i^2 - (n+1)b\right),\\
&d_2^{(3)}(3n+1) = (-1)^n(n+1)^2a^2b^{3n^2+3n},\\
&d_2^{(3)}(3n+2) = (-1)^n b^{3n^2+5n+1}\left(a^3\textstyle\sum_{i=0}^{n+1} i^2 - (n+1)b\right).
\end{aligned}$$

3. Determinants $D_r^{(m)}(n, a, b, t)$ and $dd_r^{(m)}(n, a, b)$ for $r = 0, 1, 2$

In this section, we consider Hankel determinants $D_r^{(m)}(n, a, b, t)$ and $dd_r^{(m)}(n, a, b)$ for $r = 0, 1, 2$, which we will abbreviate as $D_r^{(m)}(n)$ and $dd_r^{(m)}(n)$, respectively. The steps are similar to those in Section 2.

First, from (2) and (3), it is easy to see that the following recurrence relations hold:

$$C_n = (a+t)C_{n-1} + b\sum_{i=0}^{n-m} C_i c_{n-m-i}, \tag{38}$$

$$g_n = ag_{n-1} + 2b\sum_{i=0}^{n-m} g_i c_{n-m-i} \tag{39}$$

with $C_0 = 1$ and $g_0 = 1$. Here we omit parameters a, b, t, m for simplicity.

Similar to the proofs of lemmas and corollary in Section 2, we can also prove the following results. Here we omit the details.

Lemma 3.1. *Let n be large enough. Then,*

$$\begin{aligned}
&D_0^{(m)}(n) = b^{n-1}d_{2-m}^{(m)}(n-1),\\
&dd_0^{(m)}(n) = (2b)^{n-1}d_{2-m}^{(m)}(n-1),\\
&D_1^{(m)}(n) = (a+t)b^{n-1}d_{3-m}^{(m)}(n-1) + b^n d_{1-m}^{(m)}(n),\\
&dd_1^{(m)}(n) = a(2b)^{n-1}d_{3-m}^{(m)}(n-1) + (2b)^n d_{1-m}^{(m)}(n),\\
&D_0^{(m)}(n) = D_2^{(m)}(n-1)+b^n d_{-m}^{(m)}(n)-(a+t)^2 b^{n-2} d_{4-m}^{(m)}(n-2)-2(a+t)b^{n-1}\tilde{d}_{1-m}^{(m)}(n-1),\\
&dd_0^{(m)}(n) = dd_2^{(m)}(n-1)+(2b)^n d_{-m}^{(m)}(n)-a^2(2b)^{n-2} d_{4-m}^{(m)}(n-2)-2a(2b)^{n-1}\tilde{d}_{1-m}^{(m)}(n-1),\\
&D_0^{(1)}(n) = b^{n-1}(a+b+t)^{n-1}d_0^{(1)}(n-1),\\
&dd_0^{(1)}(n) = (2b)^{n-1}(a+2b)^{n-1}d_0^{(1)}(n-1).
\end{aligned}$$

Employing Corollary 2.2 and Lemma 3.1, we can evaluate $D_r^{(m)}(n, a, b, t)$ and $dd_r^{(m)}(n, a, b)$ for $r = 0, 1, 2$. The proofs are similar to those of theorems in Section 2. Some results can be achieved more easily when using the known results in Section 2.

Theorem 3.2. *For $m \geqslant 2$, we have*

$$\begin{aligned}
&D_0^{(m)}(mn) = (-1)^{\binom{m-1}{2}n} b^{n(mn-1)},\\
&D_0^{(m)}(mn+1) = (-1)^{\binom{m-1}{2}n} b^{n(mn+1)}
\end{aligned}$$

and $D_0^{(m)}(n) = 0$ else.

For $m \geqslant 3$, we have

$$\begin{aligned}
&D_1^{(m)}(mn) = (-1)^{\binom{m}{2}n} b^{mn^2},\\
&D_1^{(m)}(mn+1) = (-1)^{\binom{m}{2}n}(t+(n+1)a) b^{mn^2+2n},\\
&D_1^{(m)}(mn-1) = -(-1)^{\binom{m}{2}n}(t+na) b^{mn^2-2n}
\end{aligned}$$

and $D_1^{(m)}(n) = 0$ else.

For $m \geqslant 4$, we have

$$\begin{aligned}
&D_2^{(m)}(mn) = (-1)^{\binom{m-1}{2}n}(n+1) b^{mn^2+n},\\
&D_2^{(m)}(mn+1) = (-1)^{\binom{m-1}{2}n}(t+(n+1)a)^2 b^{mn^2+3n},\\
&D_2^{(m)}(mn-1) = (-1)^{\binom{m-1}{2}n} n b^{mn^2-n},\\
&D_2^{(m)}(mn-2) = -(-1)^{\binom{m-1}{2}n}(t+na)^2 b^{mn^2-3n}
\end{aligned}$$

and $D_2^{(m)}(n) = 0$ else.

Theorem 3.3. *For $m \geqslant 2$, we have*

$$dd_0^{(m)}(mn) = (-1)^{\binom{m-1}{2}n}2^{mn-1}b^{n(mn-1)},$$
$$dd_0^{(m)}(mn+1) = (-1)^{\binom{m-1}{2}n}2^{mn}b^{n(mn+1)}$$

and $dd_0^{(m)}(n) = 0$ else.

For $m \geqslant 3$, we have

$$dd_1^{(m)}(mn) = (-1)^{\binom{m}{2}n}2^{mn}b^{mn^2},$$
$$dd_1^{(m)}(mn+1) = (-1)^{\binom{m}{2}n}(2n+1)ab^{mn^2+2n},$$
$$dd_1^{(m)}(mn-1) = -(-1)^{\binom{m}{2}n}(2n-1)2^{mn-2}ab^{mn^2-2n}$$

and $dd_1^{(m)}(n) = 0$ else.

For $m \geqslant 4$, we have

$$dd_2^{(m)}(mn) = (-1)^{\binom{m-1}{2}n}(2n+1)2^{mn}b^{mn^2+n},$$
$$dd_2^{(m)}(mn+1) = (-1)^{\binom{m-1}{2}n}(2n+1)^2 2^{mn}a^2b^{mn^2+3n},$$
$$dd_2^{(m)}(mn-1) = (-1)^{\binom{m-1}{2}n}(2n-1)2^{mn-1}b^{mn^2-n},$$
$$dd_2^{(m)}(mn-2) = -(-1)^{\binom{m-1}{2}n}(2n-1)^2 2^{mn-3}a^2b^{mn^2-3n}$$

and $dd_2^{(m)}(n) = 0$ else.

Theorem 3.4.

$$D_0^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n}{2}},$$
$$D_1^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n}{2}}\left((a+b)^n + t\frac{(a+b)^n-b^n}{a}\right),$$
$$D_2^{(1)}(n) = b^{\binom{n}{2}}(a+b)^{\binom{n}{2}}\sum_{j=0}^{n}(a+b)^{n-j}b^{n-j}\left((a+b)^j + t\frac{(a+b)^j-b^j}{a}\right)^2.$$

$$dd_0^{(1)}(n) = 2^{n-1}b^{\binom{n}{2}}(a+b)^{\binom{n}{2}},$$
$$dd_1^{(1)}(n) = 2^{n-1}b^{\binom{n}{2}}(a+b)^{\binom{n}{2}}\left((a+b)^n + b^n\right),$$
$$dd_2^{(1)}(n) = 2^n b^{\binom{n+1}{2}}(a+b)^{\binom{n+1}{2}}\left(1 + \sum_{j=1}^{n}\frac{\left((a+b)^j+b^j\right)^2}{2(a+b)^j b^j}\right).$$

Theorem 3.5.

$$D_0^{(2)}(n) = b^{\binom{n}{2}},$$
$$D_1^{(2)}(n) = b^{\binom{n}{2}}\left(Fib_{n+1}(a,-b) + tFib_n(a,-b)\right),$$
$$D_2^{(2)}(n) = b^{\binom{n}{2}}\sum_{j=0}^{n}b^{n-j}\left(Fib_{j+1}(a,-b) + tFib_j(a,-b)\right)^2.$$

$$dd_0^{(2)}(n) = 2^{n-1}b^{\binom{n}{2}},$$
$$dd_1^{(2)}(n) = 2^{n-1}b^{\binom{n}{2}}\left(aFib_n(a,-b) - 2^{n-1}bFib_{n-1}(a,-b)\right),$$
$$dd_2^{(2)}(n) = 2^n b^{\binom{n+1}{2}}\left(1 + \sum_{j=1}^{n}\frac{\left(Fib_j(a,-b) - 2^{j-1}bFib_{j-1}(a,-b)\right)^2}{2b^j}\right).$$

Theorem 3.6.

$$D_2^{(3)}(3n) = (-1)^n b^{3n^2+n-1}\left(a^3\sum_{i=0}^{n} i^2 - (n+1)b + nat(t+(n+1)a)\right),$$
$$D_2^{(3)}(3n+1) = (-1)^n\,(t+(n+1)a)^2\, b^{3n^2+3n},$$
$$D_2^{(3)}(3n+2) = (-1)^n b^{3n^2+5n+1}\left(a^3\sum_{i=0}^{n+1} i^2 - (n+1)b + (n+1)at(t+(n+2)a)\right).$$

$$dd_2^{(3)}(3n) = (-1)^n\left((2n+1)^2 2^{3n} b^{3n^2+n} - \binom{2n+1}{3} 2^{3n-1} a^3 b^{3n^2+n-1}\right),$$
$$dd_2^{(3)}(3n+1) = (-1)^n (2n+1)^2 2^{3n} a^2 b^{3n^2+3n},$$
$$dd_2^{(3)}(3n+2) = (-1)^{n+1}\left((2n+1)2^{3n+2} b^{3n^2+5n+2} - \binom{2n+3}{3} 2^{3n+1} a^3 b^{3n^2+5n+1}\right).$$

Remark: We solve Conjecture 7.6 and Conjecture 7.7 in [3], as the content of Conjecture 7.6 is a part of Theorem 3.2 and Conjecture 7.7 is just Theorem 3.3. We also prove Conjecture 6.8 in [3], which states the results about $dd_r^{(3)}(n)$ for $r = 0, 1, 2$.

4. Conclusion and discussions

In this paper, we evaluate the first three Hankel determinants for three sequences, whose generating functions satisfy a certain type of quadratic functions. As a result, we have confirmed four conjectures proposed by Cigler in [3].

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