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Moment modification, multipeakons, and nonisospectral generalizations

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Abstract

Firstly, a formal correspondence is established between the Camassa–Holm (CH) equation and a two-component modified CH (or called SQQ) equation according to the method of *moment modification* for multipeakon formulae. Secondly, based on the generalized nonisospectral CH equation in Chang et al. (2014) [14] and the interlacing multipeakons of the two-component modified CH equation in Chang et al. (2016) [15], we propose a new generalized two-component modified CH equation with two parameters, which possesses a nonisospectral Lax pair. The proposed equation still admits multipeakon solutions of explicit and closed form. Sufficient conditions for global existence of solutions are given and two concrete examples with certain interesting phenomenon are presented. Last of all, as a by-product, a generalized nonisospectral modified CH equation is deduced, together with its Lax pair.

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Contents

1. Introduction

The Camassa–Holm (CH) equation [12,13]

$$m_t + (mu)_x + mu_x = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

is an integrable shallow water wave equation well studied in the last two decades. It firstly appeared in the work of Fuchssteiner and Fokas [28] as an abstract bi-Hamiltonian equation. But it didn't attract much more attention until it was rediscovered, by Camassa and Holm [12], and shown to admit peaked solitons (called peakons) as solutions. From then on, the CH equation has been investigated deeply and found to possess many interesting properties, such as describing breaking waves [21,35] and admitting explicitly solvable N-peakons [2,3,23], which are orbitally stable in some sense [22,38]. The other features that have attracted a lot of attention are its geometric interpretation on the Bott–Virasoro group [39,41] and its connection with the Korteweg–de Vries (KdV) equation according to tri-Hamiltonian duality [42].

The N-peakon dynamics of the CH equation is described by certain ODE system. Due to the Lax integrability, the ODE system can be solved by using inverse spectral method so that the explicit formulae of N-peakon solution to the CH equation were obtained [3]. The closed form of the solution is expressed in terms of the orthogonal polynomials with a discrete measure. Besides, it is also noted that the CH peakon flow may be projected to the finite Toda flow [4].

The discovery of CH peakons resulted in an increasing interest in search of new integrable equations. The modified Camassa–Holm (mCH) equation:

$$m_t + [m(u^2 - u_x^2)]_x = 0, \qquad m = u - u_{xx} \tag{1.2}$$

is just one of intriguing modifications. It originally appeared as a new integrable system in the works of Fokas [26] and Fuchssteiner [27] as well as Olver and Rosenau [42]. As mentioned by Fokas, this equation arises in the theory of nonlinear water waves. In the latter, this equation followed from the general method of tri-Hamiltonian duality applied to the bi-Hamiltonian representation of the modified KdV equation. It was rederived by Qiao [45,46] and proved to possess a Lax pair, which also appeared in an early work of Schiff [50]. Recently, some of its interesting properties were investigated, such as [16–18,30–32,37,40]. It is noted that (1.2) is also called the FORQ equation in the references.

Moreover, a two-component integrable extension of the mCH equation (1.2) was proposed by Song, Qu and Qiao [51]:

$$m_t + [(u-u_x)(v+v_x)m]_x = 0, \tag{1.3a}$$

$$n_t + [(u-u_x)(v+v_x)n]_x = 0, \tag{1.3b}$$

$$m = u - u_{xx}, \quad n = v - v_{xx}, \tag{1.3c}$$

which we call the 2-mCH equation for simplicity (sometimes it is also called the SQQ system in the literature). This system is proven to possess infinitely many conservation laws and a Lax formulation. And, it is geometrically integrable since it describes pseudospherical surfaces. Subsequently, its bi-Hamiltonian structure was derived by Tian and Liu [53] and its interlacing peakons were studied in [15] very recently.

In the present paper, we are interested in the moment problem involved in multipeakons, and extending the equations to nonisospectral case.

On one hand, we plan to investigate what we can get from the CH peakon flow by *moment modification* (See Definition 2.1).

As is known in [4], there exists a correspondence between the CH peakon and finite Toda flow. Besides, the Toda flow is related to Kac–van Moerbeke flow (or the Lotka–Volterra lattice or the Langmuir lattice [19,36,54]) from different attitudes, such as the theory of orthogonal polynomials (OPs) [1,5,43] or Jacobi operator [52], Stieltjes function [44], Bäcklund transformation [33]. In Section 2, more details on their correspondence using *moment modification* are emphasized from the view of the theory of OPs.

According to *moment modification* of multipeakon formulae of the CH equation, we get an ODE system, which turns out to be an interlacing peakon flow of the 2-mCH equation. In other words, the 2-mCH equation may be reproduced from the CH equation according to *moment modification* of multipeakon formulae.

On the other hand, nonisospectral equations are of interests in some sense. There exist many nonisospectral deformations of classic integrable systems in the literature (see, for example, [6–11,20,29,48,49]). Some nonisospectral soliton equations are demonstrated to describe solitary waves in a certain type of nonuniform media [10,11,34]. Also, it is noted that, some important soliton systems, such as the Bianchi system and the Ernst equation, admit nonisospectral linear representations, which are helpful for finding some geometric properties of their own (see [47, Chapter 8]). Recently, some nonisospectral generalizations of the CH equation have been proposed [14,24,25]. The proposed equation in [14] admits multipeakon solutions, while those in [24,25] do not.

Inspired by the work on the generalized nonisospectral CH equation [14], we shall generalize the 2-mCH equation into nonisospectral case. Starting from the explicit formulae of interlacing multipeakons of the 2-mCH equation [15], we operate a series of operations:

(1) **Alter the moment evolution**;
(2) **Generate an ODE system**;
(3) **Construct the corresponding PDE**;

so that a nonisospectral extension with two parameters is derived. It is noted that, although the method is algebraic, it turns out to be efficient to obtain these new results, which may not be easily derived in other ways.

In summary, the main new results in the present paper consist of two parts:

(1) **A formal correspondence between the CH equation** (1.1) **and the 2-mCH equation** (1.3) **is revealed from a *moment modification* perspective**.
(2) **A new 2-parameter integrable system with some multipeakon solution is proposed**. That is,

$$m_t + (\rho m)_x = 0, \quad n_t + (\rho n)_x = 0, \tag{1.4a}$$

$$\rho_x = (s + r)m(v + v_x) - sn(u - u_x), \tag{1.4b}$$

$$m = u - u_{xx}, \ \ n = v - v_{xx}, \tag{1.4c}$$

where $s(t)$ and $r(t)$ are arbitrary functions in t. This equation covers the 2-mCH equation (1.3) when $s = 1$, $r = 0$ by noting that, in this case, there holds

$$\rho_x = [(u - u_x)(v + v_x)]_x.$$

Besides, it is also shown to be integrable in sense of Lax pair. More exactly, it possesses a nonisospectral Lax pair, in other words, the spectrum in the Lax pair is dependent on time t instead of a constant. Therefore, we shall call it generalized nonisospectral two-component mCH (GN2-mCH) equation. The reduction at $v = u$ gives a generalized nonisospectral mCH (GNmCH) equation in sense of Lax integrability.

The paper is organized as follows: In Section 2, the correspondence of the Toda and Kac–van Moerbeke lattice is reviewed from the view of *moment modification*. We rederive the 2-mCH equation from by applying the idea of *moment modification* to the CH peakon solutions in Section 3. Nonisospectral extension of the 2-mCH equation is presented, in Section 4, together with its global interlacing multipeakons and the Lax pair. And, some special cases are studied there. Section 5 is devoted to conclusion and discussion.

2. From Toda to Kac–van Moerbeke lattices

In this section, we shall see how to derive the Kac–van Moerbeke lattice from the Toda lattice by *moment modification* from the view of orthogonal polynomials. Related materials can be found in the papers e.g. [19,44,52], but the description here is a bit different.

2.1. OPs and Toda lattices

To begin with, let's review how to connect the orthogonal polynomials (OPs) and the symmetric orthogonal polynomials (SOPs) by *moment modification*.

Given a measure $d\mu(x)$ defined on $(0, \infty)$ with the associated moments

$$c_i = \int x^i d\mu(x),$$

consider the monic polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to $d\mu(x)$. Assuming that the corresponding Hankel determinants

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} \neq 0, \qquad n \in \mathbb{Z}_+,$$

with the convention $\Delta_0 = 1$ and $\Delta_k = 0$ for $k < 0$, the OP sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ are uniquely determined and can be expressed in terms of

$$\phi_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

In this situation, the OP sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ satisfy

$$\int \phi_n \phi_m d\mu = h_n \delta_{nm} \qquad \text{with} \quad h_n = \frac{\Delta_{n+1}}{\Delta_n}.$$

As is known, $\{\phi_n(x)\}_{n=0}^{\infty}$ satisfy the three term recurrence

$$\phi_{n+1}(x) = (x - w_n)\phi_n(x) - u_n \phi_{n-1}(x), \tag{2.1}$$

with $u_0 = 0$, where

$$u_n = \frac{\int x\phi_n \phi_{n-1} d\mu}{\int \phi_{n-1}^2 d\mu}, \qquad w_n = \frac{\int x\phi_n^2 d\mu}{\int \phi_n^2 d\mu}.$$

In our setup, u_n and w_n will be given in terms of the moments c_i, i.e.

$$u_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad w_n = \frac{\Gamma_{n+1}}{\Delta_{n+1}} - \frac{\Gamma_n}{\Delta_n}, \tag{2.2}$$

where

$$\Gamma_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_{2n-3} \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix}$$

with the convention $\Gamma_1 = c_1$ and $\Gamma_k = 0$ for $k \leq 0$.

2.2. *Moment modification*

Let us see some modification on the moments $\{c_n\}_{n=0}^{\infty}$. We shall use the following definition for *moment modification*.

Definition 2.1. The change of the associated moment based on modifying a measure is called **moment modification**.

For the measure $d\mu(x)$ in the above subsection, we consider the modified measure $d\tilde{\mu}(x) = \frac{1}{2}d\mu(|x|)$ defined on $(-\infty, \infty)$. The resulted moment sequence $\{\tilde{c}_n\}_{n=0}^{\infty}$ based on the modified measure $d\tilde{\mu}(x)$ become

$$\tilde{c}_{2n} = c_{2n} \neq 0, \qquad \tilde{c}_{2n+1} = 0. \tag{2.3}$$

In this case, if we denote $\tilde{c}_{2n}$ by d_n for simplicity, then we can obtain

$$\tilde{\Delta}_{2n} = H_n^0 H_n^1, \qquad \tilde{\Delta}_{2n+1} = H_{n+1}^0 H_n^1, \qquad \tilde{\Gamma}_n = 0, \tag{2.4}$$

according to Laplace's expansion of a determinant by complementary minors, where $\tilde{\Delta}_n$ are the Hankel determinants generated by the moments $\{\tilde{c}_n\}_{n=0}^{\infty}$ and

$$H_n^l = \begin{vmatrix} d_l & d_{l+1} & \cdots & d_{l+n-1} \\ d_{l+1} & d_{l+2} & \cdots & d_{l+n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{l+n-1} & d_{l+n} & \cdots & d_{l+2n-2} \end{vmatrix} \neq 0$$

for $n, l \geq 0$ with the convention $H_0^l = 1$, $H_n^l = 0$ for $l \geq 0, n < 0$. And the OPs $\{\phi_n(x)\}_{n=0}^{\infty}$ reduce to a class of so-called symmetric orthogonal polynomials (SOPs) $\{\psi_n(x)\}_{n=0}^{\infty}$, which are given by

$$\psi_{2n}(x) = \frac{1}{H_n^0} \begin{vmatrix} d_0 & d_1 & \cdots & d_n \\ d_1 & d_2 & \cdots & d_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1} & d_n & \cdots & d_{2n-1} \\ 1 & x^2 & \cdots & x^{2n} \end{vmatrix}, \qquad \psi_{2n+1}(x) = \frac{1}{H_n^1} \begin{vmatrix} d_1 & d_2 & \cdots & d_{n+1} \\ d_2 & d_3 & \cdots & d_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n+1} & \cdots & d_{2n} \\ x & x^3 & \cdots & x^{2n+1} \end{vmatrix}.$$

(Note that, for convenience, the notation ψ_n is introduced to distinguish SOPs from OPs, and likewise v_n later takes the place of u_n in the case of SOPs.) The orthogonal relation becomes

$$\int \psi_n \psi_m d\mu = g_n \delta_{nm}$$

with

$$g_{2n} = \frac{H_{n+1}^0}{H_n^0}, \qquad g_{2n+1} = \frac{H_{n+1}^1}{H_n^1}.$$

We note that the SOPs consist of even functions $\{\psi_{2n}\}_{n=0}^{\infty}$ and odd functions $\{\psi_{2n+1}\}_{n=0}^{\infty}$ in x.

It is also noted that $\{\psi_n(x)\}_{n=0}^{\infty}$ satisfy a simpler three term recurrence

$$\psi_{n+1}(x) = x\psi_n(x) - v_n \psi_{n-1}(x) \tag{2.5}$$

with

$$v_n = \frac{\int x\psi_n \psi_{n-1} d\mu}{\int \psi_{n-1}^2 d\mu}.$$

In terms of d_i, we obtain

$$v_{2n} = \frac{H_{n+1}^0 H_{n-1}^1}{H_n^0 H_n^1}, \qquad v_{2n+1} = \frac{H_n^0 H_{n+1}^1}{H_{n+1}^0 H_n^1}.$$

Note that $v_0 = 0$. In fact, this formula is also obtained from the expression (2.2) under the moment modification (2.3) by use of (2.4). Obviously, in this case, w_n becomes zero, which also implies the special three term recurrence in (2.5).

2.3. Time evolution

So far, the objects we discuss are independent of time t. Now we will make them depend on time t.

Suppose that the moments c_i admits time evolution

$$\dot{c}_i = c_{i+1}.$$

Then $\phi_n(x,t)$ evolve according to (see e.g. [44, Th. 1])

$$\dot{\phi}_n(x,t) = -u_n(t)\phi_{n-1}(x,t). \tag{2.6}$$

In fact, the compatibility condition of (2.1) and (2.6) leads to

$$\dot{w}_n = u_{n+1} - u_n, \qquad \dot{u}_n = u_n(w_n - w_{n-1}), \qquad n = 0, 1, 2... \tag{2.7}$$

with $u_0 = 0$, which is no other than the semi-infinite Toda lattice with $u_0 = 0$. From the derivation, it is obvious that

$$u_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad w_n = \frac{\Gamma_{n+1}}{\Delta_{n+1}} - \frac{\Gamma_n}{\Delta_n}$$

with

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} \neq 0, \qquad \Gamma_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_{2n-3} \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix}, \qquad \dot{c}_i = c_{i+1}$$

is a solution to the semi-infinite Toda lattice (2.7).

When we impose the time evolution on d_i by

$$\dot{d}_i = d_{i+1},$$

the corresponding SOPs flow satisfy (see e.g. [44, p. 510])

$$\dot{\psi}_n(x,t) = -v_n(t)v_{n-1}(t)\psi_{n-2}(x,t). \tag{2.8}$$

By the compatibility condition of (2.5) and (2.8), one can get

$$\dot{v}_n = v_n(v_{n+1} - v_{n-1}), \qquad n = 0, 1, 2, ... \tag{2.9}$$

with $v_0 = 0$, which is nothing but the semi-infinite Kac–van Moerbeke lattice (or the Lotka–Volterra lattice or the Langmuir lattice [19,54]) with $v_0 = 0$. We also obviously obtain that

$$v_{2n} = \frac{H^0_{n+1} H^1_{n-1}}{H^0_n H^1_n}, \qquad v_{2n+1} = \frac{H^0_n H^1_{n+1}}{H^0_{n+1} H^1_n}$$

with

$$H^l_k = \begin{vmatrix} d_l & d_{l+1} & \cdots & d_{l+k-1} \\ d_{l+1} & d_{l+2} & \cdots & d_{l+k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{l+k-1} & d_{l+k} & \cdots & d_{l+2k-2} \end{vmatrix} \neq 0, \qquad \dot{d}_i = d_{i+1}$$

is a solution to the semi-infinite Kac–van Moerbeke lattice (2.9).

3. From CH to 2-mCH equations

We reviewed how to connect the Toda and Kac–van Moerbeke lattices from the point of *moment modification* in the previous section. As is indicated in [4], Beals, Sattinger and Szmigielski gave a correspondence of the Toda, Jacobi, and CH multipeakon flows. In a recent paper [15], the authors established the picture among the Kac–van Moerbeke, Jacobi and the 2-mCH peakon flow. In this section, we shall show how to derive the 2-mCH equation (1.3) from the CH equation (1.1) in the sense of *moment modification*.

3.1. Multipeakons of CH equation

First of all, let's review the explicit formula for the N-peakon solution of the CH equation, which was obtained by Beals, Sattinger and Szmigielski [2,3]. We remark that the formulae below are almost the same as those in [3] except a scaling transformation

$$x \to 2\tilde{x}, \quad t \to 2\tilde{t}, \quad u \to \tilde{u}, \quad m \to \frac{1}{2}\tilde{m}.$$

That's because we focus on the transformed form (1.1) instead of

$$\tilde{m}_{\tilde{t}} + (\tilde{m}\tilde{u})_x + \tilde{m}\tilde{u}_{\tilde{x}} = 0, \qquad 2\tilde{m} = 4\tilde{u} - \tilde{u}_{\tilde{x}\tilde{x}}.$$

The N-peakon problem is equivalent to solving a finite dimension ODE system obtained from original PDE (1.1). They worked out the ODE system by use of inverse spectral method. The procedure is as follows:

(1) From PDE to ODEs.
When u takes the form of

$$u = \sum_{j=1}^{N} m_j(t) e^{-|x - x_j(t)|}$$

m can be viewed as a discrete measure with weights m_j at locations x_j:

$$m = 2\sum_{j=1}^{N} m_j \delta(x - x_j),$$

which ensures that equation (1.1) is a well posed distributional problem.
Using elementary distribution calculus, one can find that the PDEs (1.1) hold if x_k, m_k satisfy the ODEs:

$$\dot{x}_j = u(x_j), \quad \dot{m}_j = -\langle u_x \rangle (x_j) m_j, \tag{3.1}$$

where the notation $\langle f \rangle$ denotes the average of its left and right-hand limits, that is,

$$\langle f \rangle (a) = \frac{f(a+) + f(a-)}{2}.$$

Note that u_x has a jump at every point x_j.

(2) To solve the ODEs by inverse spectral method.
The ODEs (3.1) may be regarded as a system undergoing an isospectral deformation, which suggests that it can be solved by inverse spectral method. Without loss of generality, the locations x_j at initial time can be ordered as

$$x_1(0) < x_2(0) < \cdots < x_N(0).$$

For simplicity, here we consider positive amplitudes $m_j(0)$, i.e. $m_j(0) > 0$, $j = 1, 2..., N$. By employing Liouville transformation, the corresponding spectral problem to solve can be simplified to the following string problem

$$\tilde{f}_{yy}(y) = -\lambda g(y)\tilde{f}(y), \quad -1 < y < 1; \qquad \tilde{f}(-1) = \tilde{f}(1) = 0,$$
$$g = \sum_{j=1}^{N} g_j \delta(y - y_j), \quad -1 = y_0 < y_1 < y_2 < \cdots < y_N < y_{N+1} = 1,$$

where

$$y_j = \tanh(x_j/2), \quad g_j = \frac{2m_j}{1 - y_j^2}.$$

Since all the g_j are positive, the eigenvalues λ_j for this problem are simple and positive. In order to study the inverse spectral problem, the corresponding Weyl function was introduced. It is shown that the string data $\{g_j, y_j\}_{j=1}^N$ can be one-to-one mapped into the scattering data $\{\lambda_j, a_j\}_{j=1}^N$. Stieltjes' theorem on continued fraction was employed to explicitly recover $\{g_j, y_j\}_{j=1}^N$ from $\{\lambda_j, a_j\}_{j=1}^N$. The data $\{\lambda_j, a_j\}_{j=1}^N$ evolved according to

$$\dot{\lambda}_j = 0, \qquad \dot{a}_j = \frac{a_j}{\lambda_j}.$$

In summary, the following theorem holds.

Theorem 3.1 *(Beals, Sattinger and Szmigielski [3]). The Camassa–Holm equation* (1.1) *admits the N-peakon solution*

$$u(x,t) = \sum_{j=1}^{N} m_j(t) \exp(-|x - x_j(t)|)$$

with

$$x_j = \log\left(\frac{1 + y_j}{1 - y_j}\right), \quad m_j = \frac{1}{2} g_j (1 - y_j^2)$$

and

$$y_j = \frac{\Delta_{N-j+1}^0 - \frac{1}{2}\Delta_{N-j}^2}{\Delta_{N-j+1}^0 + \frac{1}{2}\Delta_{N-j}^2}, \quad g_j = \frac{(\Delta_{N-j+1}^0 + \frac{1}{2}\Delta_{N-j}^2)^2}{\Delta_{N-j+1}^1 \Delta_{N-j}^1}. \tag{3.2}$$

Here $\Delta_k^l = \det(A_{i+j+l}(t))_{i,j=0}^{k-1}$ *and the moments* $A_k(t)$ *are restricted by*

$$A_k(t) = \int x^k e^{\frac{t}{x}} d\nu(x) = \sum_{j=1}^{N} (\lambda_j)^k a_j(t), \qquad d\nu(x) = \sum_{j=1}^{N} a_j(0)\delta(x-\lambda_j)dx \tag{3.3}$$

with $a_j(t) = a_j(0)e^{\frac{t}{\lambda_j}}$, $j \geq 1$ *and positive constants* $\lambda_j, a_j(0)$, $j \geq 1$.

Remark 3.2. The solution formulae above are not quoted exactly as they are given in the original paper [3]. And the notation Δ_j^0 corresponds to $\tilde{\Delta}_j^0$ in [3].

Remark 3.3. For $l \geq 0$, minors $\Delta_k^l, 0 \leq k \leq N$ are strictly positive so that the order $x_1(t) < x_2(t) < \cdots < x_N(t)$ holds in all the time t. Thus, the theorem above are valid globally. Besides, the $m_j(t)$ are always positive when positive $m_j(0)$ are given. Therefore, this corresponds to the pure N-peakon solution. Please refer [3] for more information.

Remark 3.4. When the initial momenta $m_j(0)$, $1 \leq j \leq N$ are of both signs, collisions will occur. This corresponds to the peakon–antipeakon case [3], for which the spectral data $\{\lambda_j\}_{j=1}^N$ are real, distinct and could be negative but never be zero, and $\{a_j\}_{j=1}^N$ are still positive. It deserves to note that the formulae given in Th. 3.1 are at least valid for local time.

3.2. Moment modification

Our motivation comes from the correspondence of the CH peakon solution and the Toda flow, together with the relation between the Toda and Kac–van Moerbeke lattice. We would like to see what will happen if *moment modification* is applied to the CH peakon solution.

As is known, the bridge between the Toda and Kac–van Moerbeke lattices is the moment modification (2.3), which leads to the restriction of Hankel determinants $\Delta_n \to H_n^0, H_n^1$, and subsequently the variable modification $u_n \to v_n$. (Here the symbols correspond to those in Section 2.) Thus, let's start from the moments (3.3) and their Hankel determinants Δ_k^l, and then variables g_j, y_j in (3.2).

We shall employ the moment modification formally based on the modified measure defined on $(-\infty, \infty)$

$$d\tilde{\nu}(x) = \frac{1}{2x} d\hat{\nu}(x),$$

where $d\hat{\nu}(x)$ is a symmetric constraint of $d\mu(x)$, i.e.

$$d\hat{\nu}(x) = \sum_{j=-K}^{K} a_{|j|}\delta(|x| - \lambda_{|j|})dx.$$

This results in

$$\tilde{A}_{2k} = \int x^{2k} d\tilde{\nu}(x) = 0, \qquad A_{2k+1} = \int x^{2k+1} d\tilde{\nu}(x) = \sum_{j=1}^{K} (\zeta_j)^k b_j \triangleq B_k \neq 0$$

with

$$b_j = \frac{a_j}{\lambda_j}, \qquad \zeta_j = \lambda_j^2.$$

In this case, for the Hankel determinants $\tilde{\Delta}_k^l = \det(\tilde{A}_{i+j+l})_{i,j=0}^{k-1}$, it is not difficult to see, by Laplace's expansion along multiple rows and columns, that

$$\tilde{\Delta}_{2k}^{2l} = (-1)^k (H_k^l)^2, \qquad \tilde{\Delta}_{2k}^{2l+1} = H_k^l H_k^{l+1},$$
$$\tilde{\Delta}_{2k+1}^{2l} = 0, \qquad \tilde{\Delta}_{2k+1}^{2l+1} = H_{k+1}^l H_k^{l+1},$$

for $k \in N$, where $H_k^l = \det(B_{i+j+l})_{i,j=0}^{k-1}$. And then, for (3.2), it is easy to get

$$g_{2K-2j+1} = \frac{(H_j^0)^2}{H_j^1 H_{j-1}^1}, \qquad g_{2K-2j+2} = \frac{1}{4}\frac{(H_{j-1}^1)^2}{H_j^0 H_{j-1}^0},$$
$$y_{2K-2j+1} = 1, \qquad y_{2K-2j+2} = -1$$

for $j = 1, 2, \ldots, K$. The above formulae are valid when b_j and ζ_j are positive, and ζ_j are distinct. The reason is that $H_k^l, k = 0, 1, \ldots, K$ are strictly positive in this case, which may be shown in the same way as the positivity of Δ_k^l in [3,14].

Remark 3.5. Note that we choose the moment modification so that $\tilde{A}_{2k} = 0,\ \tilde{A}_{2k+1} \neq 0$ rather than $\tilde{A}_{2k+1} = 0,\ \tilde{A}_{2k} \neq 0$. That's because the latter case will result in the singularity for g_j, which makes the modification meaningless. Similar phenomenon will occur when the moment modification $\tilde{c}_{2n} = 0,\ \tilde{c}_{2n+1} \neq 0$ is applied to the Toda lattice in Section 2.

Our moment modification formally implies that the odd sites y_{2j-1} are located at $+1$ while the even sites y_{2j} are all at -1. Even though the interpretation of the problem in terms of a string is no longer valid, we can nevertheless investigate how the modification of the spectral data results in a new difference spectral problem.

In the CH theory [3, eq. (4.3)], the related spectral problem reads

$$q_j - q_{j-1} = l_{j-1} p_j, \quad p_j - p_{j-1} = z g_{j-1} q_{j-1}, \qquad q_0 = q_{N+1} = 0,$$

where $l_j = y_{j+1} - y_j$.

In the special case of $y_{2j-1} = 1, y_{2j} = -1$ and $N = 2K$, one is led to a formal spectral problem

$$q_{2j} - q_{2j-1} = -2p_{2j}, \qquad q_{2j+1} - q_{2j} = 2p_{2j+1},$$
$$p_{2j} - p_{2j-1} = z g_{2j-1} q_{2j-1}, \quad p_{2j+1} - p_{2j} = z g_{2j} q_{2j}$$
$$q_0 = q_{2K+1} = 0.$$

By eliminating the variables p_j, we get

$$q_{2j} - q_{2j-2} = -2z g_{2j-1} q_{2j-1}, \quad q_{2j+1} - q_{2j-1} = 2z g_{2j} q_{2j}, \quad q_0 = q_{2K+1} = 0,$$

which are nothing but the related spectral problems for multipeakons of the 2-mCH equation studied in [15, eq. (3.7)] by setting $q_{2j} = \Theta_j, q_{2j-1} = \Pi_j$. This implies a formal connection between the CH peakon solutions and the 2-mCH multipeakons. There is a comparison between the related spectral problems for the CH peakons and the 2-mCH multipeakons in [15], where string problems with different boundary conditions are involved, however, the interpretation here is somewhat different. In the next subsection, we shall impose time evolution on the variables g_j to construct the 2-mCH equation.

3.3. Rederivation of 2-mCH equation

As is shown in [14], the CH peakon flow is equivalent to an ODE system on g_j, y_j. In this subsection, we would like to impose certain time evolution on the restricted variables g_j so as to obtain a new ODE system, from which a peakon flow can be constructed. Accordingly, we will get a PDE, which admits certain peakon solution.

For our convenience, we start from the moments

$$B_k(t) = \sum_{j=1}^{K} (\zeta_j)^k b_j(t), \qquad k \in \mathbb{Z}$$

with $b_j(t) = b_j(0)e^{\frac{2t}{\zeta_j}}$, which result in

$$\dot{B}_k = 2B_{k-1}.$$

Here $b_j(0)$ and ζ_j are some positive constants, and ζ_j are distinct.

Define the Hankel determinants $H_k^l(t) = \det(B_{i+j+l}(t))_{i,j=0}^{k-1}$. Generally, we have the convention

$$H_0^l = 1, \quad H_k^l = 0, \qquad k < 0, \ l \in \mathbb{Z}.$$

In fact, there are some underlying properties behind them, some of which are listed below for reader's convenience. The proofs can be found in, such as, [3,14,15] (especially Section 3 in [14]). But, for completeness, we shall present some detailed proofs in Appendix A.

Lemma 3.6. *For $l \in \mathbb{Z}$, there hold*

(1) Linear identities.

$$H_k^l = 0, \qquad k > K,$$

$$H_K^l = \prod_{i=1}^{K} [b_i(\zeta_i)^l] \prod_{1 \le i < j \le K} (\zeta_j - \zeta_i)^2 > 0.$$

Actually, we also have

$$H_k^l > 0, \qquad k = 0, 1, \ldots, K.$$

(2) Bilinear identities

$$
\begin{aligned}
H_{k+2}^{l-1}H_k^{l+1} &= H_{k+1}^{l-1}H_{k+1}^{l+1} - (H_{k+1}^{l})^2,\\
H_{k+1}^{l-1}H_k^{l+1} &= G_{k+1}^{l-1}H_k^{l} - G_k^{l-1}H_{k+1}^{l},\\
H_{k+1}^{l}G_k^{l} &= H_k^{l+1}G_{k+1}^{l-1} - H_{k+1}^{l-1}H_k^{l+2},\\
H_{k+2}^{l-1}H_k^{l+2} &= H_{k+1}^{l+1}G_{k+1}^{l-1} - H_{k+1}^{l}G_{k+1}^{l},
\end{aligned}
$$

for any $k \geq -1$.

(3) Combinations of identities

$$
\begin{aligned}
\sum_{l=k}^{K} \frac{(H_l^0)^2}{H_l^1 H_{l-1}^1} &= \frac{H_k^{-1}}{H_{k-1}^1},\\
\sum_{l=1}^{k} \frac{(H_{l-1}^1)^2}{H_l^0 H_{l-1}^0} &= \frac{H_{k-1}^2}{H_k^0},\\
\sum_{l=1}^{k} \frac{H_l^0 H_{l-1}^2}{H_l^1 H_{l-1}^1} &= \frac{G_k^0}{H_k^1},
\end{aligned}
$$

for $1 \leq k \leq K$.

Here G_k^l *is defined by*

$$
G_k^l = \begin{vmatrix}
B_l & B_{l+2} & B_{l+3} & \cdots & B_{l+k}\\
B_{l+1} & B_{l+3} & B_{l+4} & \cdots & B_{l+k+1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
B_{l+k-1} & B_{l+k+1} & B_{l+k+2} & \cdots & B_{l+2k-1}
\end{vmatrix}
$$

with the convention $G_1^l = B_l,\ G_k^l = 0,\ k \leq 0$.

Lemma 3.7. *If* $\dot{B}_k = 2B_{k-1}$, *then* H_k^l *evolve according to*

$$
\dot{H}_k^l = 2\, G_k^{l-1}, \qquad 1 \leq k \leq K.
$$

Proof. This is a consequence of employing differential rule for determinants. □

Now, let's turn to investigate the variables g_j. Recall that

$$
g_{2K-2j+1} = \frac{(H_j^0)^2}{H_j^1 H_{j-1}^1}, \qquad g_{2K-2j+2} = \frac{1}{4}\frac{(H_{j-1}^1)^2}{H_j^0 H_{j-1}^0}, \qquad j = 1, \ldots, K. \tag{3.4}
$$

After trial and error, we find g_j satisfy some ODEs. More exactly, we have the following claim.

Theorem 3.8. *For $j = 1, \ldots, K$, there hold*

$$\dot{g}_{2j-1} = 8\, g_{2j-1} \sum_{i=j}^{K} g_{2i} \left(2 \sum_{i=1}^{j} g_{2i-1} - g_{2j-1} \right), \tag{3.5}$$

$$\dot{g}_{2j} = -8\, g_{2j} \sum_{i=1}^{j} g_{2i-1} \left(2 \sum_{i=j}^{K} g_{2i} - g_{2j} \right). \tag{3.6}$$

Proof. See Appendix B for the detailed proof. □

A series of simple calculation will result in

Corollary 3.9. *For $j = 1, 2, \ldots, K$, if we let*

$$p_j = \ln \frac{g_{2j-1}}{m_{2j-1}}, \qquad q_j = \ln \frac{n_{2j}}{4g_{2j}},$$

where m_j, n_j are some positive constants, then

$$\dot{p}_j = 2 \sum_{i=j}^{K} n_{2i} e^{p_j - q_i} \left(2 \sum_{i=1}^{j} m_{2i-1} e^{p_i - p_j} - m_{2j-1} \right),$$

$$\dot{q}_j = -2 \sum_{i=1}^{j} m_{2i-1} e^{p_i - q_j} \left(2 \sum_{i=j}^{K} n_{2i} e^{q_j - q_i} - n_{2j} \right).$$

Moreover, if we let

$$u(x,t) = \sum_{j=1}^{K} m_{2i-1} e^{-|x - p_j(t)|}, \qquad v(x,t) = \sum_{j=1}^{K} n_{2i} e^{-|x - q_j(t)|},$$

with

$$p_1 < q_1 < p_2 < \cdots < p_K < q_K,$$

then

$$\dot{p}_j = \big(u(p_j) - \langle u_x \rangle (p_j)\big) \big(v(p_j) + v_x(p_j)\big),$$
$$\dot{q}_j = \big(u(q_j) - u_x(q_j)\big) \big(v(q_j) + \langle v_x \rangle (q_j)\big).$$

Here we recall that the notation $\langle f \rangle$ means the average of f. Also note that u_x, v_x have jumps at the points p_j, q_j, respectively.

As is known, if u, v take the forms

$$u(x,t)=\sum_{j=1}^{K} m_{2i-1} e^{-|x-p_j(t)|}, \qquad v(x,t)=\sum_{j=1}^{K} n_{2i} e^{-|x-q_j(t)|},$$

then $m=u-u_{xx},\ n=v-v_{xx}$ give discrete measures, namely,

$$m=2\sum_{j=1}^{K} m_{2i-1}\delta(x-p_j), \qquad n=2\sum_{j=1}^{K} n_{2i}\delta(x-q_j).$$

With the help of distribution calculations, we can conclude that u, v, m, n satisfy the PDE

$$\begin{aligned} m_t+[(u-u_x)(v+v_x)m]_x&=0,\\ n_t+[(u-u_x)(v+v_x)n]_x&=0,\\ m=u-u_{xx},\ \ n&=v-v_{xx}, \end{aligned}$$

in the sense of distributions, where the singular products fm for any piecewise smooth function f are defined as $\langle f\rangle m$. This PDE is nothing but the 2-mCH equation proposed in [51]. So far, we have explained how to construct the 2-mCH equation from the CH equation.

The procedure above also implies some special interlacing peakon solution to the 2-mCH equation (1.3), which has been studied in [15]. For completeness, we restate the interlacing peakon solution to the 2-mCH equation shown in [15], which simultaneously summarize the result above.

Theorem 3.10. *Given*

$$p_1(0)<q_1(0)<p_2(0)<\cdots<p_K(0)<q_K(0),$$

the 2-mCH equation (1.3) *admits the multipeakon solution*

$$u(x,t)=\sum_{j=1}^{K} m_j \exp(-|x-p_j(t)|), \quad v(x,t)=\sum_{j=1}^{K} n_j \exp(-|x-q_j(t)|) \tag{3.7}$$

with

$$p_{K+1-j}=\ln\left(\frac{1}{m_{K+1-j}}\cdot\frac{(H_j^0)^2}{H_j^1 H_{j-1}^1}\right), \qquad q_{K+1-j}=\ln\left(n_{K+1-j}\cdot\frac{H_j^0 H_{j-1}^0}{(H_{j-1}^1)^2}\right) \tag{3.8}$$

and the positive constants m_j and n_j. Here, $H_k^l(t)=\det(B_{i+j+l}(t))_{i,j=0}^{k-1}$ and the moments $B_k(t)$ are given by

$$B_k(t)=\int \zeta^k d\mu_t(\zeta),$$

where $d\mu_t = \sum_{j=1}^{K} b_j(0)e^{\frac{2t}{\zeta_j}}\delta_{\zeta_j} = e^{\frac{2t}{\zeta}}d\mu_0$ *with some positive* $b_j(0)$, *and some distinct and positive constants* ζ_j.

Theorem 3.10 is at least true for all t in some open interval containing $t = 0$. Sufficient conditions for the global existence in t have been given in [15]. We end this section by stating the conditions ensuring the global existence.

Theorem 3.11. *Given*

$$\{b_j > 0, \zeta_j > 0 : 1 \le j \le K,\ \zeta_j < \zeta_{j+1}\},$$

suppose the masses m_{2k-1}, n_{2k} *satisfy*

$$\begin{aligned} &M_j < m_j n_j, && 1 \le j \le K, \\ &N_j > n_j m_{j+1}, && 1 \le j \le K-1, \end{aligned}$$

where

$$M_j = \frac{\zeta_K^{K-j}}{\zeta_1^{K+1-j}}, \qquad N_k = \frac{\zeta_1^{K-j}}{(K-j)\zeta_K^{K-1-j}} \frac{\left(\min_i(\zeta_{i+1}-\zeta_i)\right)^{2(K-1-j)}}{\left(\zeta_K-\zeta_1\right)^{2(K-j)}}.$$

Then the multipeakon solutions (3.7) *are valid for all* $t \in \mathbb{R}$.

4. The GN2-mCH equation

The 2-mCH equation can be extended to nonisospectral case by applying the idea for the nonisospectral generalizations of CH equation proposed in [14]. More precisely, we have implemented an inverse calculation by employing determinant technique. From the explicit formulae of interlacing multipeakons of the 2-mCH equation in Theorem 3.10, we firstly alter the evolution with respect to time t for the moments. Then we deduce the dynamical system by using the determinant identities. At last, the corresponding partial differential equation is obtained. To the best of our knowledge, this equation is novel.

4.1. Derivation

In this subsection, we shall present the detailed derivation of the generalized nonisospectral two-component mCH equation.

(1). Alter the moment evolution.

Introduce two arbitrary functions $r(t)$ and $s(t)$. Suppose that $\zeta_j(t)$ depend on time t according to

$$\dot{\zeta}_j = 2r$$

with positive and distinct $\zeta_j(0)$, and $b_j(t)$ satisfy

$$\dot{b}_j = \frac{2s}{\zeta_j} b_j,$$

with positive $b_j(0)$. As before, the moments $B_k(t)$ take the form of

$$B_k(t) = \sum_{j=1}^{K} (\zeta_j(t))^k b_j(t), \qquad k \in \mathbb{Z}.$$

In this case, we have

$$\dot{B}_k = (2rk + 2s) B_{k-1},$$

which is our modification on moment evolution.

(2). Generate an ODE system.

Since Lemma 3.6 does not depend on time t, it still holds in this case. However, the Hankel determinants $H_k^l = \det(B_{i+j+l})_{i,j=0}^{k-1}$ acquire a different time evolution.

Lemma 4.1. *If $\dot{B}_k = (2rk + 2s)B_{k-1}$, then H_k^l evolve according to*

$$\dot{H}_k^l = (2rl + 2s)\; G_k^{l-1}, \qquad 1 \le k \le K.$$

Proof. The result follows by that in Lemma 4.2 of [14]. □

Introduce dependent variables

$$g_{2K-2j+1} = \frac{1}{m_{K-j+1}} \frac{(H_j^0)^2}{H_j^1 H_{j-1}^1}, \qquad g_{2K-2j+2} = \frac{1}{n_{K-j+1}} \frac{(H_{j-1}^1)^2}{H_j^0 H_{j-1}^0}, \qquad j = 1, \ldots, K, \tag{4.1}$$

where m_j, n_j are positive constants. After many attempts, we obtain the following result, whose proof will be given in Appendix C.

Theorem 4.2. *For $j = 1, \ldots, K$, there hold*

$$\begin{aligned} \dot{g}_{2j-1} =& 2s g_{2j-1} \sum_{i=j}^{K} n_i g_{2i} \left(2 \sum_{i=1}^{j} m_i g_{2i-1} - m_j g_{2j-1} \right) - 2r g_{2j-1} \sum_{i=j}^{K} \sum_{l=i}^{K} m_i g_{2i-1} n_l g_{2l} \\ & - 2r\; g_{2j-1} \sum_{i=j+1}^{K} \sum_{l=i}^{K} m_i g_{2i-1} n_l g_{2l}, \end{aligned} \tag{4.2}$$

$$\dot{g}_{2j} = - 2s g_{2j} \sum_{i=1}^{j} m_i g_{2i-1} \left(2 \sum_{i=j}^{K} n_i g_{2i} - n_j g_{2j} \right) + 4r g_{2j} \sum_{i=j+1}^{K} \sum_{l=i}^{K} m_i g_{2i-1} n_l g_{2l}. \tag{4.3}$$

In summary, we created an ODE system, which is satisfied by g_j.

(3). Construct the corresponding PDE.

Introduce the transformations

$$p_j = \ln g_{2j-1}, \qquad q_j = -\ln g_{2j}, \qquad j = 1, 2, \dots, K.$$

Then Theorem 4.2 gives

$$\begin{aligned}\dot{p}_j =& 2s \sum_{i=j}^{K} n_i e^{p_j - q_i} \left(2 \sum_{i=1}^{j} m_i e^{p_i - p_j} - 1 \right) - 2r \sum_{i=j}^{K} \sum_{l=i}^{K} m_i n_l e^{p_i - q_l} \\ & - 2r \sum_{i=j+1}^{K} \sum_{l=i}^{K} m_i n_l e^{p_i - q_l}, \end{aligned} \tag{4.4}$$

$$\dot{q}_j = 2s \sum_{i=1}^{j} m_i e^{p_i - q_j} \left(2 \sum_{i=j}^{K} n_i e^{q_j - q_i} - 1 \right) - 4r \sum_{i=j+1}^{K} \sum_{l=i}^{K} m_i n_l e^{p_i - q_l}. \tag{4.5}$$

We recall that, when u, v are taken as

$$u(x,t) = \sum_{j=1}^{K} m_j e^{-|x - p_j(t)|}, \qquad v(x,t) = \sum_{j=1}^{K} n_j e^{-|x - q_j(t)|},$$

with

$$p_1 < q_1 < p_2 < \cdots < p_K < q_K$$

m, n may be regarded as measures

$$m = 2 \sum_{j=1}^{K} m_j \delta(x - p_j), \qquad n = 2 \sum_{j=1}^{K} n_j \delta(x - q_j).$$

Note that

$$\begin{aligned} & \sum_{i=j}^{K} \sum_{l=i}^{K} m_i n_l e^{p_i - q_l} + \sum_{i=j+1}^{K} \sum_{l=i}^{K} m_i n_l e^{p_i - q_l} \\ =& \frac{1}{4} \int_{p_j-}^{+\infty} m(y)(v(y) + v_y(y)) dy + \frac{1}{4} \int_{p_j+}^{+\infty} m(y)(v(y) + v_y(y)) dy \end{aligned}$$

and

$$\sum_{i=j+1}^{K}\sum_{l=i}^{K} m_i n_l e^{p_i-q_l} = \frac{1}{4}\int_{q_j}^{+\infty} m(y)(v(y)+v_y(y))dy.$$

Therefore, (4.4) and (4.5) can be equivalently written as

$$\dot{p}_j = s\left(u(p_j) - \langle u_x(p_j)\rangle\right)\left(v(p_j)+v_x(p_j)\right) - r\left\langle \int_{x}^{+\infty} m(y)(v(y)+v_y(y))dy \right\rangle(p_j),$$

$$\dot{q}_j = s\left(u(q_j) - u_x(q_j)\right)\left(v(q_j)+\langle v_x(q_j)\rangle\right) - r\left\langle \int_{x}^{+\infty} m(y)(v(y)+v_y(y))dy \right\rangle(q_j),$$

which motivate us to get an nonlocal PDE

$$m_t + \left[\left(s(u-u_x)(v+v_x) - r\int_{x}^{+\infty} m(y)(v(y)+v_y(y))dy\right)m\right]_x = 0,$$

$$n_t + \left[\left(s(u-u_x)(v+v_x) - r\int_{x}^{+\infty} m(y)(v(y)+v_y(y))dy\right)n\right]_x = 0,$$

$$m = u - u_{xx}, \qquad n = v - v_{xx},$$

with the help of distribution calculations. Again, note that the notation $\langle f\rangle$ denotes the average of f and we suppose the singular products fm for any piecewise smooth function f are defined as $\langle f\rangle m$.

By introducing

$$\rho = s(u-u_x)(v+v_x) - r\int_{x}^{+\infty} m(y)(v(y)+v_y(y))dy$$

we finally get the local form (1.4).

Combining the content in this section and Theorem 3.10, we also actually obtain the following result on interlacing peakons of the equation (1.4).

Theorem 4.3. *Given*

$$p_1(0) < q_1(0) < p_2(0) < \cdots < p_K(0) < q_K(0),$$

the GN2-mCH equation (1.4) *admits the multipeakon solution*

$$u(x,t) = \sum_{j=1}^{K} m_j \exp(-|x-p_j(t)|), \quad v(x,t) = \sum_{j=1}^{K} n_j \exp(-|x-q_j(t)|) \tag{4.6}$$

with

$$p_{K+1-j} = \ln\left(\frac{1}{m_{K+1-j}} \cdot \frac{(H_j^0)^2}{H_j^1 H_{j-1}^1}\right), \qquad q_{K+1-j} = \ln\left(n_{K+1-j} \cdot \frac{H_j^0 H_{j-1}^0}{(H_{j-1}^1)^2}\right)$$

and the positive constants m_j and n_j. Here, $H_k^l(t) = \det(B_{i+j+l}(t))_{i,j=0}^{k-1}$ and the moments $B_k(t)$ are given by

$$B_k(t) = \int \zeta(t)^k d\mu_t(\zeta),$$

where

$$\mu_t = \sum_{j=1}^{K} b_j(0) e^{\int_0^t \frac{2s(\tau)}{\zeta_j(\tau)} d\tau} \delta_{\zeta_j(t)}, \qquad \zeta_j(t) = 2\int_0^t r(\tau)d\tau + \zeta_j(0),$$

with some positive $b_j(0)$ and some distinct and positive constants $\zeta_j(0)$.

Theorem 4.3 is at least valid for all t in some open interval containing $t = 0$. A slight modification to Theorem 3.11 will yield a sufficient condition for global existence of (4.6).

Theorem 4.4. *Given*

$$\{b_j(0) > 0, \zeta_j(0) > 0 : 1 \le j \le K,\ \zeta_j(0) < \zeta_{j+1}(0)\},$$

suppose that there exist constants $\tilde{\zeta}_1$, $\tilde{\zeta}_K$ so that

$$\zeta_1(t) \ge \tilde{\zeta}_1 > 0, \qquad 0 < \zeta_K(t) \le \tilde{\zeta}_K.$$

Then the multipeakon solutions (4.6) are valid for all $t \in \mathbb{R}$ as long as the masses m_j, n_j satisfy

$$\begin{aligned} M_j &< m_j n_j, && 1 \le j \le K, \\ N_j &> n_j m_{j+1}, && 1 \le j \le K-1, \end{aligned}$$

where

$$M_j = \frac{\tilde{\zeta}_K^{K-j}}{\tilde{\zeta}_1^{K+1-j}}, \qquad N_k = \frac{\tilde{\zeta}_1^{K-j}}{(K-j)\tilde{\zeta}_K^{K-1-j}} \frac{\left(\min_i(\zeta_{i+1} - \zeta_i)\right)^{2(K-1-j)}}{\left(\zeta_K - \zeta_1\right)^{2(K-j)}}.$$

Proof. The assumption ensures

$$b_j(t) > 0,$$

and $\{\zeta_j\}_{j=1}^K$ are bounded uniformly, that is

$$0 < \tilde{\zeta}_1 \le \zeta_1(t) < \zeta_{2j+1}(t) < \cdots < \zeta_K(t) \le \tilde{\zeta}_K,$$

for all $t \in \mathbb{R}$. Once we notice these properties, the proof can be achieved by following the way as that for Theorem 3.11. □

We end this subsection by remarking that the GN2-mCH (1.4) is integrable in the sense of having a Lax pair, where the spectral parameter is dependent on time t. More precisely, we have

Theorem 4.5. *The GN2-mCH (1.4) may be obtained by the compatibility condition of the following system*

$$\frac{\partial}{\partial x}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{2}U\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \frac{\partial}{\partial t}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{2}V\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$

where

$$U = \begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} (2s+r)\lambda^{-2} + \rho & -2s\lambda^{-1}(u-u_x) - \lambda m\rho \\ 2(s+r)\lambda^{-1}(v+v_x) + \lambda n\rho & -(2s+r)\lambda^{-2} - \rho \end{pmatrix},$$

with

$$\dot{\lambda}(t) = \frac{r(t)}{\lambda(t)}.$$

Proof. The proof can be achieved by differentiating the first matrix equation with respect to t and the second one with respect to x and setting $\frac{\partial^2}{\partial x\partial t}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{\partial^2}{\partial t\partial x}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$. □

4.2. A special case of GN2-mCH equation

Taking $r(t) = \sin t$, $s = \sin\frac{t}{2}$, we get a special equation

$$\begin{aligned} &m_t + (\rho m)_x = 0, \quad n_t + (\rho n)_x = 0, \\ &\rho_x = (\sin\frac{t}{2} + \sin t)m(v+v_x) - \sin\frac{t}{2}n(u-u_x), \\ &m = u - u_{xx}, \quad n = v - v_{xx}, \end{aligned}$$

which is a nonisospectral equation. That's because, from Theorem 4.5, it possesses a Lax pair with spectral parameter satisfying $\dot{\lambda}(t) = \frac{\sin t}{\lambda(t)}$.

We show two examples with $K = 1$ and $K = 2$ in order to illustrate global multipeakons in the interlacing case. The peakons here turns out to be periodic solutions in time.

Example 4.6 *($K = 1$).* Choose $b_1(0) = 1, \zeta_1(0) = 2, m_1 = 1, n_1 = 1.2$. From Theorem 4.3, we easily get

$$u(x,t) = \exp(-|x - p_1(t)|), \qquad v(x,t) = 1.2\exp(-|x - q_1(t)|),$$

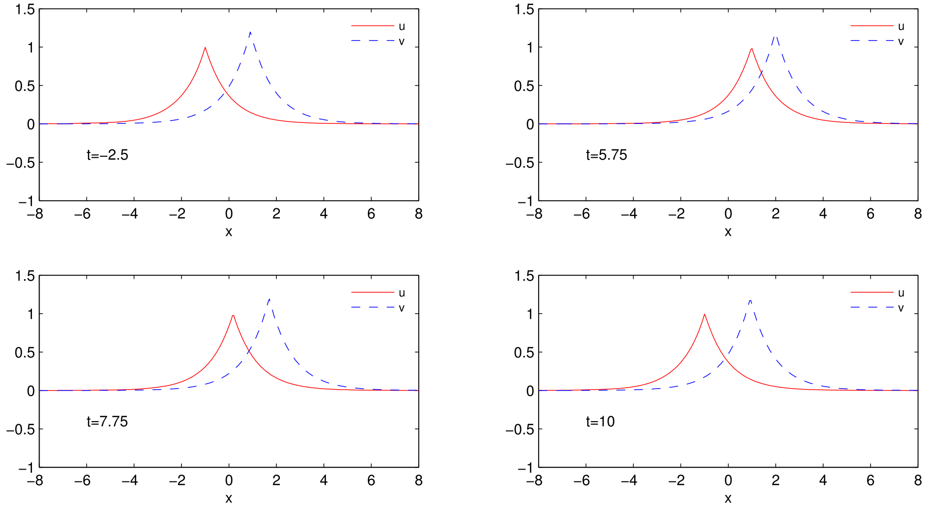

Fig. 1. A 1+1-peakon solution at time $t = -2.5,\ 5.75,\ 7.75,\ 10$ with the case of $b_1(0) = 1$, $\zeta_1(0) = 2$, $m_1 = 1$, $n_2 = 1.2$.

where

$$p_1 = \ln(b_1/\zeta_1), \qquad q_1 = \ln(2b_1)$$

with

$$\zeta_1(t) = 4 - 2\cos t, \qquad b_1(t) = \left(\frac{(\sqrt{6}+2)(\sqrt{6}-2\cos\frac{t}{2})}{(\sqrt{6}-2)(\sqrt{6}+2\cos\frac{t}{2})}\right)^{\frac{1}{\sqrt{6}}}.$$

It is a global solution since there holds,

$$q_1 - p_1 = \ln(2\zeta_1) > 0.$$

We note that the gap between the location of peak p_1 and q_1 is dependent on time t. This is different from the case for the 2-mCH equation, where the gap is invariant. A graph for interlacing peakons with $K = 1$ (see Fig. 1) using Matlab is given below by computing the explicit formula.

Example 4.7 *($K = 2$).* Choose $b_1(0) = 1$, $b_2(0) = 2$, $\zeta_1(0) = 3$, $\zeta_2(0) = 6$, $m_1 = 2.1$, $n_1 = 0.85$, $m_2 = 0.25$, $n_2 = 2.1$. From Theorem 4.3, we have

$$u(x,t) = \sum_{j=1}^{2} m_j \exp(-|x - p_j(t)|), \quad v(x,t) = \sum_{j=1}^{2} n_j \exp(-|x - q_j(t)|),$$

where

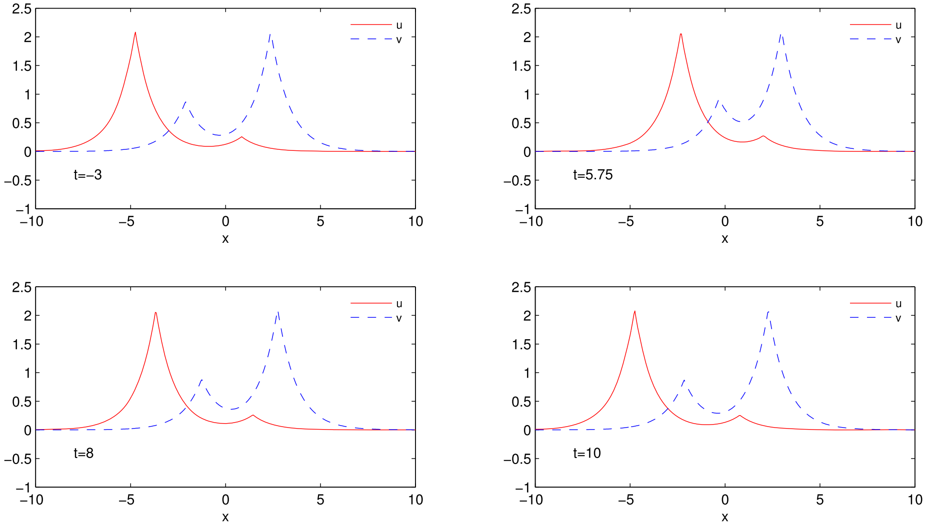

Fig. 2. A 2+2-peakon solution at time $t = -10,\ 5,\ 15,\ 25$ with the case of $b_1(0) = 1$, $b_2(0) = 2$, $\zeta_1(0) = 3$, $\zeta_2(0) = 6$, $m_1 = 2.1$, $n_2 = 0.85$, $m_3 = 0.25$, $n_4 = 2.1$.

$$p_1 = \ln\left(0.5 \cdot \frac{b_1 b_2 (\zeta_2 - \zeta_1)^2}{\zeta_1 \zeta_2 (b_1\zeta_1 + b_2\zeta_2)}\right), \qquad q_1 = \ln\left(0.85 \cdot \frac{b_1 b_2 (b_1 + b_2)(\zeta_2 - \zeta_1)^2}{(b_1\zeta_1 + b_2\zeta_2)^2}\right),$$

$$p_2 = \ln\left(4 \cdot \frac{(b_1 + b_2)^2}{b_1\zeta_1 + b_2\zeta_2}\right), \qquad q_2 = \ln(2.1 \cdot (b_1 + b_2))$$

with

$$\zeta_1(t) = 5 - 2\cos t, \qquad b_1(t) = \left(\frac{(\sqrt{7}+2)(\sqrt{7} - 2\cos\frac{t}{2})}{(\sqrt{7}-2)(\sqrt{7} + 2\cos\frac{t}{2})}\right)^{\frac{1}{\sqrt{7}}},$$

$$\zeta_2(t) = 8 - 2\cos t, \qquad b_2(t) = 2\left(\frac{(\sqrt{10}+2)(\sqrt{10} - 2\cos\frac{t}{2})}{(\sqrt{10}-2)(\sqrt{10} + 2\cos\frac{t}{2})}\right)^{\frac{1}{\sqrt{10}}}.$$

It is easy to show that the condition in Theorem 4.4 is satisfied, i.e.

$$m_1 n_1 > \frac{\tilde{\zeta}_2}{\tilde{\zeta}_1^2}, \quad m_2 n_2 > \frac{1}{\tilde{\zeta}_1},$$

$$n_1 m_2 < \frac{\tilde{\zeta}_1}{(\zeta_2 - \zeta_1)^2},$$

where $\tilde{\zeta}_1 = 3$, $\tilde{\zeta}_2 = 10$. Hence, $p_1 < q_1 < p_2 < q_2$ will be preserved at all time and one can use the explicit formulae for the 2-peakon solution at all time, resulting in the following graph (Fig. 2).

4.3. The GNmCH equation

Taking $v = u$ in the equation (1.4), we get a PDE

$$m_t + (\rho m)_x = 0, \qquad m = u - u_{xx}, \tag{4.7a}$$

$$\rho_x = (s + r)m(u + u_x) - sm(u - u_x), \tag{4.7b}$$

which we call generalized nonisospectral mCH (GNmCH) equation, since it reduces to the mCH (1.2) when $r = 0, s = 1$. The PDE (4.7) is also integrable in the sense of having a formal Lax pair. Actually, by following Theorem 4.5 in the case $u = v$, we immediately have

Theorem 4.8. *The GNmCH equation* (4.7) *may be obtained by the compatibility condition of the following system*

$$\frac{\partial}{\partial x}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{2}U\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \frac{\partial}{\partial t}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{2}V\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$

where

$$U = \begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} (2s+r)\lambda^{-2} + \rho & -2s\lambda^{-1}(u - u_x) - \lambda m\rho \\ 2(s+r)\lambda^{-1}(u + u_x) + \lambda n\rho & -(2s+r)\lambda^{-2} - \rho \end{pmatrix}$$

with

$$\dot{\lambda}(t) = \frac{r(t)}{\lambda(t)}.$$

We end this section by a remark on multipeakons for the GNmCH equation (4.7).

Remark 4.9. The multipeakons of the GNmCH equation (4.7) can not be reduced from Theorem 3.10. It turns out to have been quite complicated for the multipeakons of the mCH equation (1.2) [16,17]. We suspect the GNmCH case is much more complicated.

5. Conclusion and discussion

The method of *moment modification* is employed to find a correspondence between the CH and 2-mCH equations via the multipeakon formulae. Observing that the interlacing multipeakon formulae for the 2-mCH are expressed in terms of Hankel determinants, we perform a series of operations based on modifications of the spectral data and its time evolution to obtain a generalized nonisospectral two-component mCH equation. As a one-component reduction, a nonisospectral extension of mCH is obtained.

Although the method in the present paper is algebraic, sometimes it is efficient to obtain new results, which may not be easily derived in other ways. The idea of *moment modification* is of potential application for producing more novel integrable systems with multipeakon solutions. We shall investigate more examples in the near future. Besides, our inverse process for nonisospectral generalizations has been successfully applied to the CH equation in [14] and the 2-mCH equation in the present paper. It is possible to generate more nonisospectral equations by using this strategy.

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Appendix A. The proof of Lemma 3.6

Lemma 3.6 describes some properties of Hankel determinants $H_k^l(t)$ with the specific moment structure. Since it leaves the time evolution out of account, it works throughout the paper.

The proof of Theorem 3.6. In fact, the Hankel determinants $H_k^l(t)$ can be evaluated explicitly. The linear relations follows from Lemma 5.2 in [15]. The bilinear identities are the consequences of applying Jacobi identity and have been shown in Lemma 3.3 in [14].

The combinations of identities can be demonstrated by following a similar way as the proof of Corollary 3.4 in [14]. As for the first relation, by using the bilinear identity, we have

$$\begin{aligned}\sum_{l=k}^{K}\frac{(H_l^0)^2}{H_l^1H_{l-1}^1}&=\sum_{l=k}^{K}\frac{H_l^1H_l^{-1}-H_{l+1}^{-1}H_{l-1}^1}{H_l^1H_{l-1}^1}\\&=\sum_{l=k}^{K}\left(\frac{H_l^{-1}}{H_{l-1}^1}-\frac{H_{l+1}^{-1}}{H_l^1}\right)=\frac{H_k^{-1}}{H_{k-1}^1},\end{aligned}$$

where we have employed the result $H_{K+1}^1=0$. Similarly,

$$\begin{aligned}\sum_{l=1}^{k}\frac{(H_{l-1}^1)^2}{H_l^0H_{l-1}^0}&=\sum_{l=1}^{k}\frac{H_{l-1}^0H_{l-1}^2-H_l^0H_{l-2}^2}{H_l^0H_{l-1}^0}\\&=\sum_{l=1}^{k}\left(\frac{H_{l-1}^2}{H_l^0}-\frac{H_{l-2}^2}{H_{l-1}^0}\right)=\frac{H_{k-1}^2}{H_k^0},\end{aligned}$$

where we have used the convention $H_{-1}^2=0$. The third relation is proved by noting

$$\begin{aligned}\sum_{l=1}^{k}\frac{H_l^0H_{l-1}^2}{H_l^1H_{l-1}^1}&=\sum_{l=1}^{k}\frac{G_l^0H_{l-1}^1-G_{l-1}^0H_l^1}{H_l^1H_{l-1}^1}\\&=\sum_{l=1}^{k}\left(\frac{G_l^0}{H_l^1}-\frac{G_{l-1}^0}{H_{l-1}^1}\right)=\frac{G_k^0}{H_k^1},\end{aligned}$$

where the convention $G_0^0=0$ is used. □

Appendix B. The proof of Theorem 3.8

Note that Theorem 3.8 works under the assumption $\dot{B}_k = 2B_{k-1}$. Therefore it only applies in Section 3.

The proof of Theorem 3.8. If we write $j' = K - j + 1$, then it is easy to see that (3.5) is equivalent to

$$\dot{g}_{2j'-1} = 8\, g_{2j'-1} \sum_{i=1}^{j} g_{2i'} \left(2 \sum_{i=j}^{K} g_{2i'-1} - g_{2j'-1} \right).$$

Substituting the expressions (3.4) into the above equation and employing the combined identities in Lemma 3.6 and the time evolution in Lemma 3.7, we get

$$2G_j^{-1} H_j^1 H_{j-1}^1 - H_j^0 (G_j^0 H_{j-1}^1 + H_j^1 G_{j-1}^0) = 2H_{j-1}^2 H_j^1 H_j^{-1} - H_{j-1}^2 (H_j^0)^2,$$

which is what we need to prove.

By using the bilinear identities in Lemma 3.6, the above equation becomes

$$\begin{aligned} &2G_j^{-1} H_j^1 H_{j-1}^1 - H_j^0 G_j^0 H_{j-1}^1 - H_j^1 (G_j^{-1} H_{j-1}^1 - H_j^{-1} H_{j-1}^2) \\ =&2H_{j-1}^2 H_j^1 H_j^{-1} - H_{j-1}^2 (H_j^1 H_j^{-1} - H_{j-1}^1 H_{j+1}^{-1}) \end{aligned}$$

which reduces to

$$G_j^{-1} H_j^1 - H_j^0 G_j^0 = H_{j-1}^2 H_{j+1}^{-1}.$$

Actually, this is also a bilinear identity in Lemma 3.6, which conclude (3.5).

Next, we turn to the proof of (3.6). Similarly, (3.6) is equivalent to

$$\dot{g}_{2j'} = -8\, g_{2j'} \sum_{i=j}^{K} g_{2i'-1} \left(2 \sum_{i=1}^{j} g_{2i'} - g_{2j'} \right),$$

which may be written as

$$2H_j^0 H_{j-1}^0 G_{j-1}^0 - H_{j-1}^1 (G_j^{-1} H_{j-1}^0 + H_j^0 G_{j-1}^{-1}) = -2H_j^{-1} H_{j-1}^0 H_{j-1}^2 + H_j^{-1} (H_{j-1}^1)^2$$

by using (3.4) and employing the combined identities in Lemma 3.6 and the time evolution in Lemma 3.7.

With the help of the bilinear identities in Lemma 3.6, the above equation results in

$$\begin{aligned} &2H_j^0 H_{j-1}^0 G_{j-1}^0 - H_{j-1}^0 (H_j^0 G_{j-1}^0 + H_j^{-1} H_{j-1}^2) - H_{j-1}^1 H_j^0 G_{j-1}^{-1} \\ =& -2H_j^{-1} H_{j-1}^0 H_{j-1}^2 + H_j^{-1} (H_{j-2}^2 H_j^0 - H_{j-1}^0 H_{j-1}^2) \end{aligned}$$

After elimination, we see it suffices to prove

$$H^0_{j-1}G^0_{j-1} - H^1_{j-1}G^{-1}_{j-1} = -H^{-1}_j H^2_{j-2},$$

which is no other than one of the bilinear identities in Lemma 3.6. Thus (3.6) is proved. □

Appendix C. The proof of Theorem 4.2

Recall that Theorem 4.2 is based on the assumption $\dot{B}_k = (2rk + 2s)B_{k-1}$.

The proof of Theorem 4.2. The proof is similar that for Theorem 3.8 and we will employ some conclusion there.

Denote $j' = K - j + 1$ for simplicity. It is not hard to see that (4.2) is equivalent to

$$\begin{aligned}\dot{g}_{2j'-1} =& 2s\ g_{2j'-1}\sum_{i=1}^{j} g_{2i'}\left(2\sum_{i=j}^{K} g_{2i'-1} - g_{2j'-1}\right) - 2r g_{2j'-1}\sum_{i=1}^{j}\sum_{l=1}^{i} m_{i'}g_{2i'-1}n_{l'}g_{2l'},\\ &- 2r g_{2j'-1}\sum_{i=1}^{j-1}\sum_{l=1}^{i} m_{i'}g_{2i'-1}n_{l'}g_{2l'}.\end{aligned}$$

Substituting the expressions (4.1) into the above equation and employing the combined identities in Lemma 3.6 and the time evolution in Lemma 4.1, we will see that it suffices to prove

$$2G^{-1}_j H^1_j H^1_{j-1} - H^0_j(G^0_j H^1_{j-1} + H^1_j G^0_{j-1}) = 2H^2_{j-1}H^1_j H^{-1}_j - H^2_{j-1}(H^0_j)^2,$$

which has been shown in Appendix B.

Next, we turn to the proof of (4.3). Similarly, (4.3) is equivalently written as

$$\dot{g}_{2j'} = -2s\ g_{2j'}\sum_{i=j}^{K} g_{2i'-1}\left(2\sum_{i=1}^{j} g_{2i'} - g_{2j'}\right) + 4r g_{2j'}\sum_{i=1}^{j-1}\sum_{l=1}^{i} m_{i'}g_{2i'-1}n_{l'}g_{2l'},$$

which results in

$$2H^0_j H^0_{j-1}G^0_{j-1} - H^1_{j-1}(G^{-1}_j H^0_{j-1} + H^0_j G^{-1}_{j-1}) = -2H^{-1}_j H^0_{j-1}H^2_{j-1} + H^{-1}_j(H^1_{j-1})^2$$

by using (4.1) and employing the combined identities in Lemma 3.6 and the time evolution in Lemma 4.1. Actually, this formula is true, which is also shown in Appendix B. So far, the proof is completed. □

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