The dynamics of conservative peakons in the NLS hierarchy

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Abstract

Using the tri-hamiltonian splitting method, the authors of [1] derived two $U(1)$-invariant nonlinear PDEs that arise from the hierarchy of the nonlinear Schrödinger equation and admit peakons (*non-smooth solitons*). In the present paper, these two peakon PDEs are generalized to a family of $U(1)$-invariant peakon PDEs parametrized by the real projective line $\mathbf{RP}^1$. All equations in this family are shown to posses *conservative peakon solutions* (whose Sobolev $H^1(\mathbf{R})$ norm is time invariant). The Hamiltonian structure for the sector of conservative peakons is identified and the peakon ODEs are shown to be Hamiltonian with respect to several Poisson structures. It is shown that the resulting Hamilonian peakon flows in the case of the two peakon equations derived in [1] form orthogonal families, while in general the Hamiltonian peakon flows for two different equations in the general family intersect at a fixed angle equal to the angle between two lines in $\mathbf{RP}^1$ parametrizing those two equations. Moreover, it is shown that inverse spectral methods allow one to solve explicitly the dynamics of conservative peakons using explicit solutions to a certain interpolation problem. The graphs of multipeakon solutions confirm the existence of multipeakon breathers as well as asymptotic formation of pairs of two peakon bound states in the non-periodic time domain.

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Contents

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1 Introduction

The origin of the present paper goes back at least to the fundamental paper by R.Camassa and D.Holm [4] in which they proposed the nonlinear partial differential equation

$$m_t + um_x + 2u_x m + 2\kappa u_x = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

as a model equation for the shallow water waves. The coefficient κ appearing in (1.1) is proportional to a critical shallow water wave speed. One of the results of [4] was that (1.1) has soliton solutions which are no longer smooth in the limit of $\kappa \to 0^+$. These non-smooth solitons have the form of *peakons*

$$u = \sum_{j=1}^{n} m_j(t) e^{-|x - x_j(t)|}, \tag{1.2}$$

where all coefficients $m_j(t)$ and the positions $x_j(t)$ are assumed to be smooth as functions of t. It is elementary to see that $m = u - u_{xx}$ becomes a discrete measure $m = 2\sum_j m_j \delta_{x_j}$ and that when the CH equation (1.1) for $\kappa = 0$ is interpreted in an appropriate weak sense, it turns into a system of Hamilton's equations of motion

$$\dot{x}_j = \{x_j, H\}, \qquad \dot{m}_j = \{m_j, H\}, \tag{1.3}$$

with the Hamiltonian $H = \frac{1}{2}\sum_{i,j} m_i m_j e^{-|x_i - x_j|}$ and with x_j, m_j being canonical conjugate variables. This system is Lax integrable in the sense that it can be written as a matrix Lax equation [4]. Moreover, there exists an explicit solution in terms of Stieltjes continued fractions [2]. The CH equation has generated over the years a remarkably strong response from the scientific community, attracted to its unique features such as the breakdown of regularity of its solutions [7, 20, 21, 3] and the stability of its solutions, including the peakon solutions [8, 10, 9]. One of the outstanding issues that has emerged over the last two decades has been the question of understanding, or perhaps even classifying, equations sharing these distinct features of the CH equation. One of the earliest results in this direction was obtained by P. Olver and P. Rosenau in [22] who put forward an elegant method, called by them a tri-Hamiltionian duality, which provided an intriguing way of deriving not only the CH equation but also other equations possessing peakon solutions. The main idea of that paper can be illustrated, as in their paper, on the example of the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x. \tag{1.4}$$

This equation is known to have a bi-Hamiltonian structure. Indeed, using the two compatible Hamiltonian operators

$$J_1 = D_x, \qquad J_2 = D_x^3 + uD_x + D_x u, \tag{1.5}$$

and the Hamiltonians

$$H_1 = \tfrac{1}{2}\int u^2 dx, \qquad H_2 = \tfrac{1}{2}\int (-u_x^2 + u^3) dx, \tag{1.6}$$

the KdV equation (1.4) can be written as

$$u_t = \{u, H_1\}_{J_2} = \{u, H_2\}_{J_1}, \tag{1.7}$$

where the Poisson bracket associated with the Hamiltonian operator J is given by $\{f, g\}_J = \int \frac{\delta f}{\delta u} J \frac{\delta g}{\delta u} dx$ for smooth in u functionals f, g. The decisive step is now to redistribute the Hamiltonian operators by splitting and reparametrizing differently the compatible triple $D_x, D_x^3, uD_x + D_x u$ of Hamiltonian operators to create a new compatible pair

$$\hat{J}_1 = D_x - D_x^3 = D_x \Delta, \qquad \hat{J}_2 = mD + Dm, \tag{1.8}$$

with $m = (1 - D_x^2)u = \Delta u$. The appearance of the factorization of one of the Hamiltonian operators in terms of the Helmholtz operator Δ is of paramount importance for creating "peakon" equations. In particular, the CH equation (1.1) can be written now in bi-Hamiltonian form

$$m_t = \{m, \hat{H}_1\}_{\hat{J}_2} = \{m, \hat{H}_2\}_{\hat{J}_1}, \tag{1.9}$$

with the Hamiltonians

$$\hat{H}_1 = \tfrac{1}{2}\int u m dx, \qquad \hat{H}_2 = \tfrac{1}{2}\int (-u u_x^2 + u^3) dx. \tag{1.10}$$

It is shown in [22] that applying this methodology to the modified KdV equation one obtains the nonlinear partial differential equation

$$m_t + \big((u^2 - u_x^2)m\big)_x = 0, \qquad m = u - u_{xx}. \tag{1.11}$$

Equation 1.11 appeared also in the papers of T. Fokas [11] and B. Fuchssteiner [12], and was, later, rediscovered by Z. Qiao [23, 24]. Some early work on the Lax formulation of this equation was done by J. Schiff [25] . Recently this equation has attracted a considerable attention from many authors [17, 13, 18, 15, 14, 6].

Interestingly enough, the same philosophy applied to the non-linear Schrödinger equation

$$u_t = i(u_{xx} + |u|^2 u) \tag{1.12}$$

appeared to produce no peakon equations. We recall that the bi-Hamiltonian formulation of (1.12) which was used in [22] is based on the standard NLS Hamiltonian operators

$$J_1(F) = \mathrm{i}F, \qquad J_2(F) = D_x F + u D_x^{-1}(\bar{u}F - u\bar{F}), \tag{1.13}$$

written in action on densities F. The two redistributed Hamiltonian operators

$$\hat{J}_1(F) = (D_x + i)F, \qquad \hat{J}_2(F) = m D_x^{-1}(\bar{m}F - m\bar{F})$$

do not contain the Helmholtz operator Δ in their factorizations. This puzzling situation was resolved in the paper [1] by S. Anco and F. Mobasheramini in which the authors proposed a different choice of Hamiltonian operators resulting in the bi-Hamiltonian formulation of **two** new peakon equations

$$m_t + ((|u|^2 - |u_x|^2)m)_x + 2\mathrm{i}\,\mathrm{Im}(\bar{u}u_x)m = 0, \tag{1.14}$$

called the Hirota-type (HP) peakon equation, and

$$\mathrm{i}m_t + 2\mathrm{i}(\mathrm{Im}(\bar{u}u_x)m)_x + (|u|^2 - |u_x|^2)m = 0, \tag{1.15}$$

called the NLS-type (NLSP) peakon equation. Unlike other peakon equations, both the HP and NLSP equations display the same $U(1)$-invariance as the NLS equation.

In the remainder of this introduction, we outline the content of individual sections, highlighting the main results.

To begin, in Section 2 we review the Hamiltonian setup for both HP and NLSP following [1].

In Section 3 we give a unifying perspective on the Lax pairs for (1.14) and (1.15), showing that these equations are just two members of a family of peakon equations parametrized by the real projective line $\mathbf{RP}^1$.

In Section 4 we introduce and study *conservative* peakons obtained from the distributional formulation of the Lax pairs discussed in 3. This type of peakon solutions not only preserves the Sobolev H^1 norm but also admits multiple Hamiltonian formulations which we study in detail, giving in particular a unifying Hamiltonian formulation for the whole family of conservative peakon equations.

In Section 5 we concentrate on the isospectral boundary value problem relevant for the peakon equation (4.3).

In Section 6 we develop a two-step procedure for solving the inverse problem for the HP equation which, effectively, recovers the peakon measure (4.1) in terms of solutions to an interpolation problem stated in Theorem 6.3. We provide graphs of solutions obtained from explicit formulas and give preliminary comments about the dynamics of peakon solutions.

2 Hamiltonian structure of NLS and Hirota peakon equations

This section is a condensed summary of the part of [1] relevant to the present paper.

Hamiltonian structures go hand in hand with Poisson brackets. In particular, a linear operator $\mathcal{E}$ is a Hamiltonian operator iff its associated bracket

$$\{F, G\}_{\mathcal{E}} = \langle \delta F/\delta\bar{m}, \mathcal{E}(\delta G/\delta\bar{m})\rangle \tag{2.1}$$

on the space of functionals of $(m, \bar{m})$ is a Poisson bracket, namely, this bracket is skew-symmetric and obeys the Jacobi identity, turning the space of functionals into a Lie algebra. Here $\langle f, g\rangle$ is a real, symmetric, bilinear form defined as

$$\langle f, g\rangle = \int_{\mathbf{R}} (\bar{f}(x)g(x) + f(x)\bar{g}(x))\, dx, \tag{2.2}$$

thus equipping the space of L^2 complex functions with a real, positive-definite, inner product. Note that if F, G are real functionals, then the bracket (2.1) takes the form

$$\{F, G\} = \int_{\mathbf{R}} \big((\delta F/\delta m)\, \mathcal{E}(\delta G/\delta\bar{m}) + (\delta F/\delta\bar{m})\, \bar{\mathcal{E}}(\delta G/\delta m)\big)\, dx, \tag{2.3}$$

where the variational derivative of each functional with respect to $(m, \bar{m})$ is defined relative to the inner product by

$$\delta H = \langle \delta H/\delta m, \delta\bar{m}\rangle = \langle \delta H/\delta\bar{m}, \delta m\rangle \tag{2.4}$$

for each real functional H. In explicit form, the variational derivative of real functional $H = \int_{\mathbf{R}} h\, dx$ is given in terms of the density h by the relation

$$\delta H/\delta m = E_m(h), \quad \delta H/\delta\bar{m} = E_{\bar{m}}(h), \tag{2.5}$$

where E_v denotes the Euler operator with respect to a variable v.

Another key relationship is that a Poisson bracket and a Hamiltonian operator satisfy $\{m, H\}_{\mathcal{E}} = \mathcal{E}(\delta H/\delta\bar{m})$ and, symmetrically, $\{\bar{m}, H\}_{\mathcal{E}} = \bar{\mathcal{E}}(\delta H/\delta m)$.

To proceed, we first introduce the 1-D Helmholtz operator

$$\Delta = 1 - D_x^2 \tag{2.6}$$

which connects m and u through

$$m = u - u_{xx} = \Delta u. \tag{2.7}$$

Both the HP and NLSP equations share two compatible Hamiltonian operators, stated in [1], namely

$$\mathcal{H} = 2\mathrm{i}\Delta, \quad \mathcal{D} = 2\mathrm{i}(mD_x^{-1}\,\mathrm{Re}\,\bar{m}D_x + \mathrm{i}D_x m D_x^{-1}\,\mathrm{Im}\,\bar{m}). \tag{2.8}$$

Compatibility means that every linear combination $c_1\mathcal{H} + c_2\mathcal{D}$ of these two Hamiltonian operators is a Hamiltonian operator. Note, compared to the operators presented in [1], here $\mathcal{H}$ and $\mathcal{D}$ have been normalized by a factor of 2 that corresponds to our choice of normalization for the nonlinear terms in the equations (1.14)–(1.15).

Each Hamiltonian operator (2.8) defines a respective Poisson bracket $\{F, G\}_{\mathcal{H}}$ and $\{F, G\}_{\mathcal{D}}$. The bi-Hamiltonian structure of the NLSP equation is given by

$$m_t = \mathrm{i}(|u|^2 - |u_x|^2)m - 2(\mathrm{Im}(\bar{u}u_x)m)_x = \mathcal{D}(\delta H^{(0)}/\delta\bar{m}) = \mathcal{H}(\delta H^{(1)}/\delta\bar{m}), \tag{2.9}$$

or equivalently,

$$m_t = \{m, H^{(0)}\}_{\mathcal{D}} = \{m, H^{(1)}\}_{\mathcal{H}}, \tag{2.10}$$

where

$$H^{(0)} = \int_{\mathbf{R}} \mathrm{Re}(\bar{u}m)\, dx \ = \int_{\mathbf{R}} (|u|^2 + |u_x|^2)\, dx \tag{2.11}$$

and

$$H^{(1)} = \int_{\mathbf{R}} \big(\tfrac{1}{4}(|u|^2 - |u_x|^2)\,\mathrm{Re}(\bar{u}m) + \tfrac{1}{2}\,\mathrm{Im}(\bar{u}u_x)\,\mathrm{Im}(\bar{u}_x m)\big)\, dx \tag{2.12}$$

are the Hamiltonian functionals. Both functionals $H^{(0)}$ and $H^{(1)}$ are conserved for smooth solutions $u(t,x)$ with appropriate decay conditions at $|x| \to \infty$.

Likewise, the bi-Hamiltonian structure of the HP equation is given by

$$m_t = -((|u|^2 - |u_x|^2)m)_x - 2\mathrm{i}\,\mathrm{Im}(\bar{u}u_x)m = -\mathcal{D}(\delta E^{(0)}/\delta \bar{m}) = \mathcal{H}(\delta E^{(1)}/\delta \bar{m}) \tag{2.13}$$

where

$$E^{(0)} = \int_{\mathbf{R}} \mathrm{Im}(\bar{u}m_x)\,dx = \int_{\mathbf{R}} \mathrm{Im}(u_x\bar{m})\,dx \tag{2.14}$$

and

$$E^{(1)} = \int_{\mathbf{R}} \big(\tfrac{1}{4}(|u|^2 - |u_x|^2)\,\mathrm{Im}(\bar{u}_x m) - \tfrac{1}{2}\,\mathrm{Im}(\bar{u}u_x)\,\mathrm{Re}(\bar{u}m)\big)\,dx \tag{2.15}$$

are the Hamiltonian functionals. These two functionals $E^{(0)}$ and $E^{(1)}$ are conserved for smooth solutions $u(t,x)$ with appropriate decay conditions. In terms of Poisson brackets, the corresponding structure is

$$m_t = \{m, -E^{(0)}\}_{\mathcal{D}} = \{m, E^{(1)}\}_{\mathcal{H}}. \tag{2.16}$$

We now make some brief comments about the conserved Hamiltonians $H^{(0)}$ and $E^{(0)}$, going beyond the presentation in [1]. Recall that once a Hamiltonian operator $\mathcal{E}$ (or the corresponding Poisson bracket) is given, any real functional H gives rise to a Hamiltonian vector field

$$\mathrm{X}_H = \eta\partial_m + \bar{\eta}\partial_{\bar{m}}, \qquad \text{where } \eta = \mathcal{E}(\delta H/\delta\bar{m}) = \{m, H\}_{\mathcal{E}},$$

acting on the space of densities of real functionals. First, using the Hamiltonian operator $\mathcal{H}$, we see that

$$\mathcal{H}(\delta H^{(0)}/\delta\bar{m}) = 2im, \quad \mathcal{H}(\delta E^{(0)}/\delta\bar{m}) = 2m_x$$

respectively, produce, after a simple rescaling, the Hamiltonian vector fields $\mathrm{X}_{\mathrm{phas.}} = \mathrm{i}m\partial_m - \mathrm{i}\bar{m}\partial_{\bar{m}}$ and $\mathrm{X}_{\mathrm{trans.}} = m_x\partial_m + \bar{m}_x\partial_{\bar{m}}$. These two vector fields are the respective generators of phase rotations $(m,\bar{m}) \to (e^{\mathrm{i}\phi}m, e^{-\mathrm{i}\phi}\bar{m})$ and x-translations $x \to x+\varepsilon$, where ϕ, ε are arbitrary (real) constants. Next, by direct computation, we find that

$$\{H^{(0)}, E^{(0)}\}_{\mathcal{H}} = 0, \quad \{H^{(0)}, E^{(0)}\}_{\mathcal{D}} = 0. \tag{2.17}$$

These brackets show that $\mathrm{X}_{\mathrm{phas.}}$ and $\mathrm{X}_{\mathrm{trans.}}$ are commuting symmetry vector fields for both the NLSP equation and the HP equation. As a consequence, we note the following useful features of these Hamiltonians.

Remark 2.1. Each Hamiltonian $H^{(0)}$ and $E^{(0)}$ is conserved for both the NLSP equation and the HP equation, and these Hamiltonians are invariant under the symmetries generated by the commuting Hamiltonian vector fields $\mathrm{X}_{\mathrm{phas.}}$ and $\mathrm{X}_{\mathrm{trans.}}$. In addition, $H^{(0)} = ||u||^2_{H^1}$ is the square of the Sobolev norm of $u(t,x)$.

Some insight into the meaning of $E^{(0)}$ comes from expressing these Hamiltonians in terms of a Fourier representation $u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ikx}a(k,t)\,dk$. A simple computation gives $H^{(0)} = \int_{\mathbf{R}}(1+k^2)|a(k,t)|^2\,dk$ and $E^{(0)} = \int_{\mathbf{R}} k(1+k^2)|a(k,t)|^2\,dk$, which shows that the conserved density arising from $E^{(0)}$ is k times the conserved positive-definite density given by $H^{(0)}$. This is analogous to the relationship between the well-known conserved energy and momentum quantities for the NLS equation [16]. Since $H^{(0)}$ plays the role of a conserved positive-definite energy for both the HP and NLSP equations, we can thereby view $E^{(0)}$ as being a conserved indefinite-sign momentum for these equations.

In the the next section Section 3 we will revisit the derivation of the HP and NLSP equations (1.15) and (1.14) starting from a unified perspective provided by their Lax pair formulation.

3 A unified Lax pair

We begin by showing how the Lax pairs in [1] for the NLSP and HP equations can be unified.

For $\lambda \in \mathbf{C}$, consider the family of $\mathfrak{sl}(2,\mathbf{C})$ matrices

$$U = \frac{1}{2}\begin{bmatrix} -1 & \lambda m \\ -\lambda\bar{m} & 1 \end{bmatrix}, \quad V = \frac{1}{2}\begin{bmatrix} \sigma(2\lambda^{-2}+Q) & -2\sigma\lambda^{-1}(u-u_x)-\lambda mJ \\ 2\sigma\lambda^{-1}(\bar{u}+\bar{u}_x)+\lambda\bar{m}J & -\sigma(2\lambda^{-2}+Q) \end{bmatrix} \tag{3.1}$$

parametrized by two complex valued functions m and u, and a complex constant σ, where

$$Q = (u-u_x)(\bar{u}+\bar{u}_x) = |u|^2 - |u_x|^2 - 2i\,\mathrm{Im}(\bar{u}u_x), \tag{3.2}$$

and J is a complex function that we will now determine.

Remark 3.1. We point out that V is not uniquely determined; in particular we can add to it a λ dependent multiple of the identity. This becomes a necessity when boundary conditions are imposed. We will return to this point further into the paper.

We impose on the pair (U,V) the zero-curvature equation

$$U_t - V_x + [U,V] = 0, \tag{3.3}$$

which gives

$$m_t + (mJ)_x + (J-\sigma Q)m = 0, \tag{3.4a}$$
$$\bar{m}_t + (\bar{m}J)_x - (J-\sigma Q)\bar{m} = 0, \tag{3.4b}$$
$$m = u - u_{xx}. \tag{3.4c}$$

For these equations to be compatible, J must be real and $J-\sigma Q$ must be purely imaginary, which implies

$$J = \mathrm{Re}(\sigma Q) = \mathrm{Re}(\sigma)(|u|^2-|u_x|^2) + 2\,\mathrm{Im}(\sigma)\,\mathrm{Im}(\bar{u}u_x) \tag{3.5}$$

and

$$J - \sigma Q = -\mathrm{i}\,\mathrm{Im}(\sigma Q) = 2\mathrm{i}\,\mathrm{Re}(\sigma)\,\mathrm{Im}(\bar{u}u_x) - \mathrm{i}\,\mathrm{Im}(\sigma)(|u|^2-|u_x|^2).$$

Consequently, the zero-curvature equation (3.3) becomes

$$m_t + \mathrm{Re}(\sigma)\big(((|u|^2-|u_x|^2)m)_x + 2i\,\mathrm{Im}(\bar{u}u_x)m\big) + \mathrm{Im}(\sigma)\big(2(\mathrm{Im}(\bar{u}u_x)m)_x - i(|u|^2-|u_x|^2)m\big) = 0 \tag{3.6}$$

which is, loosely speaking, a linear combination of the HP and NLSP equations. In particular, the HP equation is obtained for $\sigma = 1$, and the NLSP equation is obtained for $\sigma = i$. We note, however, that by rescaling the t variable we can put $|\sigma| = 1, \mathrm{Im}(\sigma) \geq 0$. In this sense Equation 3.4a simplifies to

$$m_t + (\mathrm{Re}(e^{\mathrm{i}\theta}Q)m)_x - \mathrm{i}\,\mathrm{Im}(e^{\mathrm{i}\theta}Q)m = 0, \tag{3.7}$$

where the angle θ can be restricted to $[0,\pi)$. This angle has a simple geometric interpretation as being a local parameter in the real projective line $\mathbf{RP}^1$. Thus different equations in this family of PDEs correspond to different points in $\mathbf{RP}^1$; in practice though different equations are obtained by rigid rotations of Q by the angle θ (see more on this item below).

We conclude that the unified equation (3.7) is, at least formally, a Lax integrable system, as it arises from a Lax pair (3.1), (3.2), with J given by (3.5) and $\sigma = e^{\mathrm{i}\theta}$. The unified equation also possesses a bi-Hamiltonian structure given by the Hamiltonian operators (2.8) shared by the HP and NLSP equations, using Hamiltonians that are given by a linear combination of the HP and NLSP Hamiltonians:

$$m_t = \mathcal{D}(\delta K^{(0)}/\delta\bar{m}) = \mathcal{H}(\delta K^{(1)}/\delta\bar{m}) \tag{3.8}$$

where

$$K^{(0)} = \cos\theta(-E^{(0)}) + \sin\theta H^{(0)}, \quad K^{(1)} = \cos\theta E^{(1)} + \sin\theta\, H^{(1)}. \tag{3.9}$$

The unified equation (3.7) reveals an interesting symmetry between the HP and NLSP equations. Indeed, the HP and NLSP equations are respectively given by

$$m_t + (\mathrm{Re}(Q)m)_x - i\,\mathrm{Im}(Q)m = 0, \quad \theta = 0, \tag{3.10}$$
$$m_t - (\mathrm{Im}(Q)m)_x - i\,\mathrm{Re}(Q)m = 0, \quad \theta = \tfrac{\pi}{2}. \tag{3.11}$$

Thus, these two equations are related by a phase rotation of Q by the angle $\theta = \frac{\pi}{2}$.

The bi-Hamiltonian formulations of the HP equation, NLSP equation, and the generalized equation (3.7) exhibit the same symmetry. If we write

$$P = (u - u_x)\bar{m} \tag{3.12}$$

then we have the relations

$$\mathrm{Re}(P) = \mathrm{Re}(u\bar{m}) - \tfrac{1}{2}(\mathrm{Re}(Q))_x, \quad \mathrm{Im}(P) = -\,\mathrm{Im}(u_x\bar{m}) - \tfrac{1}{2}(\mathrm{Im}(Q))_x,$$

where $\mathrm{Re}(u\bar{m})$ is the density for $H^{(0)}$, and $\mathrm{Im}(u_x\bar{m})$ is the density for $E^{(0)}$. Likewise, for the quantity QP we obtain

$$\begin{aligned}\mathrm{Re}(QP) &= \mathrm{Re}(Q)\,\mathrm{Re}(u\bar{m}) - \mathrm{Im}(Q)\,\mathrm{Im}(\bar{u}_x m) + \tfrac{1}{4}(\mathrm{Im}^2(Q) - \mathrm{Re}^2(Q))_x\\ \mathrm{Im}(QP) &= \mathrm{Re}(Q)\,\mathrm{Im}(\bar{u}_x m) + \mathrm{Im}(Q)\,\mathrm{Re}(\bar{u}m) - \tfrac{1}{2}(\mathrm{Im}(Q)\,\mathrm{Re}(Q))_x,\end{aligned}$$

where $\frac{1}{4}(\mathrm{Re}(Q)\,\mathrm{Re}(u\bar{m}) - \mathrm{Im}(Q)\,\mathrm{Im}(\bar{u}_x m))$ is the density for $H^{(1)}$ (see (2.12)) and $\frac{1}{4}(\mathrm{Re}(Q)\,\mathrm{Im}(\bar{u}_x m) + \mathrm{Im}(Q)\,\mathrm{Re}(\bar{u}m))$ for $E^{(1)}$ (see (2.15)). These relations show that the densities (modulo irrelevant boundary terms) for $K^{(0)}$ and $K^{(1)}$ are given by $\mathrm{Im}(e^{\mathrm{i}\theta}P)$ and $\frac{1}{4}\,\mathrm{Im}(e^{\mathrm{i}\theta}QP)$, respectively. Hence, we obtain

$$K^{(0)} = \int_{\mathbf{R}} \mathrm{Im}(e^{\mathrm{i}\theta}P)\,dx, \quad K^{(1)} = \int_{\mathbf{R}} \tfrac{1}{4}\,\mathrm{Im}(e^{\mathrm{i}\theta}QP)\,dx. \tag{3.13}$$

In particular, from these expressions for the unified Hamiltonians, we see that the bi-Hamiltonian structures of the HP and NLSP equations are related by the phase rotation by $\pi/2$.

In section 4 we introduce a *sector of conservative peakons* for Equation 3.7, concentrating mostly on the cases of $\theta = 0$ (HP) and $\theta = \frac{\pi}{2}$ (NLSP).

4 Conservative peakons

The peakon Ansatz [4]

$$u = \sum_{j=1}^{N} m_j e^{-|x-x_j|}$$

was originally designed for real m_j, x_j. For the HP and NLSP equations (3.10) and (3.11), the coefficients m_j are complex and x_j are real, resulting in $m = u - u_{xx}$ being a complex discrete measure

$$m = 2\sum_{j=1}^{N} m_j \delta_{x_j}. \tag{4.1}$$

Thus both equations (3.10) and (3.11), and more generally (3.7), must be viewed as distribution equations. To this end the products $\mathrm{Im}(Q)m$ and $\mathrm{Re}(Q)m$ need to be defined, and accordingly Qm needs to be defined. By analyzing the distributional Lax pair in a similar way to [6], we can show that the choice consistent with Lax integrability is to take

$$Qm = \langle Q\rangle m \overset{\text{def}}{=} 2\sum_{j=1}^{N} \langle Q\rangle(x_j) m_j \delta_{x_j}, \tag{4.2}$$

where $\langle Q\rangle(x_j)$ denotes the arithmetic average of the right and left hand limits at x_j.

Remark 4.1. Many previous investigations of peakon equations, particularly on global existence and wave breaking for the mCH equation [13], have defined distribution products differently by using a weak (integral) formulation of the peakon equation. The same approach was taken in [1] to derive single peakon weak solutions and peakon breather weak solutions of the HP and NLSP equations, but as pointed out in that paper, the HP and NLSP equations do not appear to have a weak formulation that allows multi-peakon solutions to be derived. Indeed, the choice of defining distribution products used here (4.2) appears to be the only way to obtain multi-peakon solutions for these two equations, as well as for the general family (3.7). As a consequence, the conservative single peakon and peakon breather solutions that will be obtained later in this paper differ from the single peakon weak solutions and peakon breather weak solutions presented in [1]. Most importantly, conservative N-peakon solutions will be derived for all $N \geq 1$.

Since m_js are complex, we will use polar co-ordinates:

$$m_j = |m_j|\, e^{i\omega_j}$$

Using these definitions, we obtain the following systems of ODEs from the peakon equations (3.10) and (3.11).

Proposition 4.2. *For the peakon Ansatz (1.2), suppose the ill-defined product Qm is regularized according to (4.2). Then the HP equation (3.10) reduces to*

$$\dot{x}_j = \big\langle \mathrm{Re}\, Q \big\rangle(x_j), \quad \dot{\omega}_j = \big\langle \mathrm{Im}\, Q \big\rangle(x_j), \quad \frac{d|m_j|}{dt} = 0, \quad j = 1, \dots, N. \tag{4.3}$$

Likewise, the NLSP equation (3.11) reduces to

$$\dot{x}_j = -\big\langle \mathrm{Im}\, Q \big\rangle(x_j), \quad \dot{\omega}_j = \big\langle \mathrm{Re}\, Q \big\rangle(x_j), \quad \frac{d|m_j|}{dt} = 0, \quad j = 1, \dots, N, \tag{4.4}$$

while in the general case of Equation 3.7 the peakon ODEs read:

$$\dot{x}_j = \big\langle \mathrm{Re}(e^{\mathrm{i}\theta} Q) \big\rangle(x_j), \quad \dot{\omega}_j = \big\langle \mathrm{Im}(e^{\mathrm{i}\theta} Q) \big\rangle(x_j), \quad \frac{d|m_j|}{dt} = 0, \quad j = 1, \dots, N, \quad 0 \leq \theta < \pi. \tag{4.5}$$

It is easy to see that the vector fields on the right hand sides of equations (4.3) and (4.4) are orthogonal. The following conclusion about the geometry of solution curves of peakon ODEs is straightforward.

Corollary 4.3. *The family of solution curves to the ODE system (4.3) is orthogonal to the family of solution curves to the ODE system (4.4). In general, the family of solution curves to the ODE system (4.3) is at the angle θ to the family of solution curves to the ODE system (4.5).*

We can write these ODE systems in a simpler form in terms of a complex variable

$$X_j = x_j + \mathrm{i}\omega_j \tag{4.6}$$

which combines the positions and phases.

Lemma 4.4. *The ODE systems (4.3) and (4.4) can be expressed in the complex-variable form*

$$\dot{X}_j = \big\langle Q \big\rangle(x_j), \tag{4.7}$$

$$\dot{X}_j = i\big\langle Q \big\rangle(x_j), \tag{4.8}$$

and similarly for system (4.5),

$$\dot{X}_j = e^{\mathrm{i}\theta}\big\langle Q \big\rangle(x_j) \tag{4.9}$$

holds.

Remark 4.5. We recall [5] that the peakons for the two-component modified Camassa-Holm (2mCH) equation satisfy an identically looking ODE system $\dot{x}_j = \langle Q \rangle(x_j)$, but with an important difference that $Q(x) = (u - u_x)(v + v_x)$, where (u, v) are the two (real) components. One natural reduction of the 2mCH equation is the modified Camassa-Holm (1mCH) equation obtained by putting $v = u$ [6]. In a way the present paper is about the reduction $v = \bar{u}$. However, one needs to keep in mind that the work in [5] is restricted to the real case, so the results of that paper do not apply in any direct way to the present situation. Nevertheless, for reasons that are not fully understood at this moment, the solution to the inverse problem associated with (3.10) or (3.11) turns out to have more similarities with the inverse problem for 1mCH peakons studied in [6] rather than with the one for the 2mCH peakons in [5].

4.1 Poisson bracket

We will now introduce a Poisson structure that will allow systems Equation 4.7 and Equation 4.8 to arise as Hamilton's equations. We observe that the vector field in equations (4.3) and (4.4) is not Lipschitz in the whole space $\mathbf{R}^{2N}$ of (x_j, ω_j)s. To remedy this, we will have to avoid the hyperplanes $x_i = x_j, i \neq j$, for example, by restricting our attention to the region of positions where the ordering $x_1 < x_2 < \cdots < x_N$ holds. Let us then denote that region

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^N : x_1 < x_2 < \cdots < x_N\}$$

and, subsequently, define the pertinent phase space as follows.

Definition 4.6.

$$\mathcal{M} = \mathcal{P} \times T^N \tag{4.10}$$

where T^N is the N-dimensional torus of angles $\omega_1, \omega_2, \cdots, \omega_N$.

Locally, it is convenient to think of a point $\xi = (x_1, x_2, \cdots, x_N, \omega_1, \omega_2, \cdots, \omega_N) \in \mathcal{M}$ as a complex vector $X = (X_1, X_2, \cdots, X_N)$ where X_j was introduced in Equation 4.6. Consequently, any function $f(\xi) \in \mathcal{C}^\infty(\mathcal{M})$ can be viewed as a smooth function f of X and its complex conjugate $\bar{X}$, namely $f = f(\xi) = f(X, \bar{X})$.

Proposition 4.7. *The bracket*

$$\{X_j, X_k\} = \operatorname{sgn}(j-k), \quad \{X_j, \bar{X}_k\} = 0, \quad \{\bar{X}_j, \bar{X}_k\} = \operatorname{sgn}(j-k), \tag{4.11}$$

defines a Poisson structure on $\mathcal{C}^\infty(\mathcal{M})$.

Proof. It suffices to observe that Equation 4.11 is equivalent to

$$\{x_j, x_k\} = \tfrac{1}{2}\operatorname{sgn}(j-k), \quad \{\omega_j, \omega_k\} = -\tfrac{1}{2}\operatorname{sgn}(j-k), \quad \{\omega_j, x_k\} = 0. \tag{4.12}$$

This set of brackets defines a skew symmetric matrix Ω_{ab} on $\mathbf{R}^{2N}$ with block form

$$\Omega = \begin{pmatrix} [\frac{1}{2}\operatorname{sgn}(j-k)] & 0 \\ 0 & -[\frac{1}{2}\operatorname{sgn}(j-k)] \end{pmatrix},$$

each block having dimension $N \times N$. Then upon setting

$$\{f, g\}(\xi) = \sum_{a,b=1}^{2N} \Omega_{ab} \frac{\partial f}{\partial \xi_a} \frac{\partial g}{\partial \xi_b} \tag{4.13}$$

we obtain the desired Poisson structure on $\mathcal{C}^\infty(\mathcal{M})$, since the skew symmetric matrix Ω is ξ-independent and thus the bracket (4.13) automatically satisfies the Jacobi identity. □

Remark 4.8. Since Ω is full rank the Poisson bracket given by Equation 4.11 equips $\mathcal{M}$ with a symplectic structure.

Before we prove the main statement of this section we need to express $H^{(0)}$ and $E^{(0)}$ (see Equation 2.11 and Equation 2.14) in terms of coordinates on $\mathcal{M}$. The detailed computations are provided in A (see also Lemma 5.13 for a spectral interpretation of both quantities).

Lemma 4.9. *Let u be given by the peakon Ansatz (1.2) and let the multiplication of the singular term Qm be defined by (4.2). Then*

$$H^{(0)}\big|_{\mathcal{M}} = 4\operatorname{Re}\Big(\sum_{k<l} |m_k|\,|m_l|\, e^{X_k - X_l}\Big) + 2\sum_l |m_l|^2, \tag{4.14}$$

$$E^{(0)}\big|_{\mathcal{M}} = -4\operatorname{Im}\Big(\sum_{k<l} |m_k|\,|m_l|\, e^{X_k - X_l}\Big). \tag{4.15}$$

Theorem 4.10. *Equation 4.3 and Equation 4.4 are Hamilton's equations of motion with respect to the Poisson structure given by Equation 4.12 and Hamiltonians $H^{(0)}$ and $E^{(0)}$ respectively. In terms of the complex variable X we have*

$$\dot{X}_j = \{X_j, H^{(0)}\} \tag{4.16}$$

for the HP peakon flow (4.3),

$$\dot{X}_j = \{X_j, E^{(0)}\} \tag{4.17}$$

for the NLSP peakon flow (4.4), and

$$\dot{X}_j = \{X_j, K^{(0)}\} \tag{4.18}$$

for the general peakon flow (4.5).

Proof. We will first compute $\{X_j, H^{(0)}\}$ using Equation 4.14; for convenience we abbreviate *c.c.* to mean the complex conjugate. We have

$$\begin{aligned}\{X_j, H^{(0)}\} =&2\{X_j, \sum_{k<l} |m_k|\,|m_l|\,e^{X_k - X_l} + \text{c.c.}\} \overset{\textit{Equation 4.11}}{=} 2\sum_{k<l} |m_k|\,|m_l|\,e^{X_k-X_l}\{X_j, X_k - X_l\} = \\ &2\sum_{k<l} |m_k|\,|m_l|\,e^{X_k-X_l}\big(\operatorname{sgn}(j-k) - \operatorname{sgn}(j-l)\big) = 4\sum_{k<j<l} |m_k|\,|m_l|\,e^{X_k-X_l} + \\ &2\,|m_j|\,\Big(\sum_{k<j} |m_k|\,e^{X_k-X_j} + \sum_{j<k} |m_k|\,e^{X_j-X_k}\Big) \overset{\textit{Lemma A.4}}{=} \langle Q\rangle(x_j).\end{aligned}$$

Likewise,

$$\begin{aligned}\{X_j, E^{(0)}\} =&2\mathrm{i}\{X_j, \sum_{k<l} |m_k|\,|m_l|\,e^{X_k - X_l} - \text{c.c.}\} \overset{\textit{Equation 4.11}}{=} 2\mathrm{i}\sum_{k<l} |m_k|\,|m_l|\,e^{X_k-X_l}\{X_j, X_k - X_l\} = \\ &2\mathrm{i}\sum_{k<l} |m_k|\,|m_l|\,e^{X_k-X_l}\big(\operatorname{sgn}(j-k) - \operatorname{sgn}(j-l)\big) = 4\mathrm{i}\sum_{k<j<l} |m_k|\,|m_l|\,e^{X_k-X_l} + \\ &2\mathrm{i}\,|m_j|\,\Big(\sum_{k<j} |m_k|\,e^{X_k-X_j} + \sum_{j<k} |m_k|\,e^{X_j-X_k}\Big) \overset{\textit{Lemma A.4}}{=} \mathrm{i}\langle Q\rangle(x_j).\end{aligned}$$

Finally, the general case can be verified by using the above results and (3.9). □

Remark 4.11. In addition to the Poisson bracket (4.11), there is a second Poisson structure on $\mathcal{C}^\infty(\mathcal{M})$ defined by another bracket

$$\{X_j, X_k\}_{\frac{\pi}{2}} = \mathrm{i}\operatorname{sgn}(j-k), \quad \{X_j, \bar{X}_k\}_{\frac{\pi}{2}} = 0, \quad \{\bar{X}_j, \bar{X}_k\}_{\frac{\pi}{2}} = -\mathrm{i}\operatorname{sgn}(j-k), \tag{4.19}$$

or, equivalently,

$$\{x_j, x_k\}_{\frac{\pi}{2}} = 0, \quad \{\omega_j, \omega_k\}_{\frac{\pi}{2}} = 0, \quad \{\omega_j, x_k\}_{\frac{\pi}{2}} = \tfrac{1}{2}\operatorname{sgn}(j-k). \tag{4.20}$$

The rationale for the subscript $\frac{\pi}{2}$ will be explained below, but for now we note that the skew symmetric matrix Ω takes the form:

$$\Omega_{\frac{\pi}{2}} = \begin{pmatrix} 0 & [\frac{1}{2}\operatorname{sgn}(j-k)] \\ [\frac{1}{2}\operatorname{sgn}(j-k)] & 0\end{pmatrix},$$

and both of the peakon equations (4.3) and (4.4) remain Hamiltonian, although with swapped Hamiltonians

$$\dot{X}_j = \{X_j, H^{(0)}\}_{\frac{\pi}{2}} \tag{4.21}$$

for the peakon NLSP equation (4.8) and

$$\dot{X}_j = \{X_j, -E^{(0)}\}_{\frac{\pi}{2}} \tag{4.22}$$

for the peakon HP equation (4.7).

The second bracket appears to be more natural one, since $-E^{(0)}$ is the Hamiltonian for HP and since this bracket arises from reduction of the Hamiltonian structure given by $\mathcal{D}$, though the reduction is slightly singular. This point will be elaborated on elsewhere. However, there is another, perhaps more unifying, point of view that we would like to mention here. To this end we define a θ-dependent Poisson structure

Definition 4.12.

$$\{X_j, X_k\}_\theta = e^{\mathrm{i}\theta}\,\mathrm{sgn}(j-k), \quad \{X_j, \bar{X}_k\}_\theta = 0, \quad \{\bar{X}_j, \bar{X}_k\}_\theta = e^{-\mathrm{i}\theta}\,\mathrm{sgn}(j-k), \tag{4.23}$$

or, equivalently,

$$\{x_j, x_k\}_\theta = \tfrac{\cos\theta}{2}\,\mathrm{sgn}(j-k), \quad \{\omega_j, \omega_k\}_\theta = -\tfrac{\cos\theta}{2}\,\mathrm{sgn}(j-k), \quad \{\omega_j, x_k\}_\theta = \tfrac{\sin\theta}{2}\,\mathrm{sgn}(j-k). \tag{4.24}$$

The skew symmetric matrix Ω now takes the form

$$\Omega_\theta = \begin{pmatrix} [\frac{\cos\theta}{2}\,\mathrm{sgn}(j-k)] & [\frac{\sin\theta}{2}\,\mathrm{sgn}(j-k)] \\ [\frac{\sin\theta}{2}\,\mathrm{sgn}(j-k)] & -[\frac{\cos\theta}{2}\,\mathrm{sgn}(j-k)] \end{pmatrix},$$

which clearly combines both previous cases. More importantly, the following lemma holds, the proof of which is just a simple modification of the proof of Theorem 4.10.

Lemma 4.13. *The θ family of equations (4.9) is Hamiltonian with respect to the Poisson bracket (4.23) with a fixed Hamiltonian $H^{(0)}$, that is Equation 4.9 can be written*

$$\dot{X}_j = \{X_j, H^{(0)}\}_\theta, \qquad 0 \le \theta < \pi. \tag{4.25}$$

We will conclude this subsection by stating an easy corollary focusing again on special cases of HP and NLSP equations, followed by a theorem about the norm preservation for the θ family.

Corollary 4.14. *In the original variables (m_j, x_j), and written in the notation consistent with (4.23), the Poisson brackets (4.11) and (4.19) are given by, respectively,*

$$\begin{aligned} &\{m_j, m_k\}_0 = \tfrac{1}{2}\,\mathrm{sgn}(j-k) m_j m_k, \quad \{m_j, \bar{m}_k\}_0 = -\tfrac{1}{2}\,\mathrm{sgn}(j-k) m_j \bar{m}_k, \\ &\{x_j, x_k\}_0 = \tfrac{1}{2}\,\mathrm{sgn}(j-k), \\ &\{x_j, m_k\}_0 = 0, \end{aligned}$$

and

$$\begin{aligned} &\{m_j, m_k\}_{\frac{\pi}{2}} = 0, \qquad \{m_j, \bar{m}_k\}_{\frac{\pi}{2}} = 0, \\ &\{x_j, x_k\}_{\frac{\pi}{2}} = 0, \\ &\{x_j, m_k\}_{\frac{\pi}{2}} = \tfrac{\mathrm{i}}{2}\,\mathrm{sgn}(j-k) m_j. \end{aligned}$$

Both Poisson structures can be derived from the first Hamiltonian structure of the NLSP and HP equations by a (singular) reduction process. This topic will be taken up in another publication.

We recall, as shown in section 2, that $H^{(0)}$ is conserved for both the HP and NLSP equations; this was then further amplified in the peakon sector for all equations in the θ family of equations (Lemma 4.13). We stress that, at least in the peakon sector, all equations in the θ family share the same Hamiltonian, but their Hamiltonian structure deforms. Since $||u||^2_{H^1} = H^{(0)}$ is the square of the Sobolev norm, we have the following theorem which justifies the name " conservative peakons". We emphasize that this theorem is valid not only for the HP and NLSP peakons but, thanks to Lemma 4.13, for the whole peakon θ family (4.9).

Theorem 4.15. *Let u be given by the peakon Ansatz (1.2) and let the singular term Qm be regularized by (4.2). Then for any $0 \le \theta < \pi$:*

$$\frac{d}{dt}||u||_{H^1} = 0.$$

5 HP equation; the spectral theory

For the reminder of this work we will concentrate mostly on the HP case, and to some extent on the NLSP case, leaving more in-depth analysis of the θ family for future investigations.

As was indicated in Remark 3.1 the Lax pair can be modified by a multiple of identity. This is effectively changing what appeared to be an $\mathfrak{sl}(2,\mathbf{C})$ theory to a $\mathfrak{gl}(2,C)$ theory. We take the Lax pair for the HP equation Equation 3.10 to be (compare with (3.1), (3.2), (3.5))

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \Psi = \begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}, \tag{5.1}$$

where

$$U = \tfrac{1}{2}\begin{bmatrix}-1 & \lambda m\\ -\lambda\bar{m} & 1\end{bmatrix},$$

$$V = \tfrac{1}{2}\begin{bmatrix}4\lambda^{-2}+Q & -2\lambda^{-1}(u-u_x)-\lambda m\,\mathrm{Re}(Q)\\ 2\lambda^{-1}(\bar{u}+\bar{u}_x)+\lambda\bar{m}\,\mathrm{Re}(Q) & -Q\end{bmatrix},$$

with Q given by expression (3.2). This choice of V is compatible, as opposed to V in Equation 3.1, with the asymptotic behaviour $\Psi = \begin{bmatrix}0\\ e^x\end{bmatrix}$ as $x \to -\infty$. This type of asymptotic adjustment is present in all peakon equations known to us (e.g. [3], [19]). Performing on (5.1) a $GL(2,\mathbf{C})$ gauge transformation

$$\Phi = \mathrm{diag}(\lambda^{-1}e^{\frac{x}{2}}, e^{-\frac{x}{2}})\Psi$$

yields a simpler x-equation

$$\Phi_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\Phi, \qquad g = \sum_{j=1}^{N} g_j\delta_{x_j}, \qquad h = \sum_{j=1}^{N} h_j\delta_{x_j}, \tag{5.2}$$

where $g_j = \bar{m}_j e^{-x_j}$, $h_j = m_j e^{x_j}$, $z = \lambda^2$. For future use we note, using the complex-variable notation (4.6), that

$$g_j = |m_j|\, e^{-X_j}, \qquad h_j = |m_j|\, e^{X_j}, \tag{5.3}$$

hence $g_jh_j = |m_j|^2$.

We can impose the boundary conditions $\Phi_1(-\infty) = 0$ and $\Phi_2(+\infty) = 0$ without violating the compatibility of the Lax pair (5.1). The argument in support of that is similar to other peakon cases, most notably to the modified CH equation [6], so we skip it in this paper. However, to make the boundary value problem

$$\Phi_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\Phi, \qquad \Phi_1(-\infty) = \Phi_2(+\infty) = 0, \tag{5.4}$$

well posed, we need to define the multiplication of the measures h and g by Φ on their singular support, namely at the points x_j. It can be shown in a way similar to what was done in [6] that if we require that Φ be left continuous and define $\Phi_a\delta_{x_j} = \Phi_a(x_j)\delta_{x_j}, a = 1, 2$, then this choice makes the Lax pair (5.1) well defined as a distributional Lax pair, and the compatibility condition of the x and t components of the Lax pair indeed implies the peakon HP equation (4.3).

The solution Φ is a piecewise constant function in x which, for convenience, we can normalize by setting $\Phi_2(-\infty) = 1$. The distributional boundary value problem (5.4), whenever m is a discrete measure, is equivalent to a finite difference equation.

Lemma 5.1. *Let $q_k = \Phi_1(x_k+)$, $p_k = \Phi_2(x_k+)$, then the finite-difference form of the boundary value problem is given by*

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le N,\\ p_k - p_{k-1} &= -zg_k q_{k-1}, \qquad 1 \le k \le N,\\ q_0 = 0, \quad p_0 &= 1, \quad p_N(z) = 0. \end{aligned} \tag{5.5}$$

An easy proof by induction leads to the following result for the associated initial value problem.

Lemma 5.2. *Consider the initial value problem*

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le N, \\ p_k - p_{k-1} &= -z g_k q_{k-1}, \quad 1 \le k \le N, \\ q_0 = 0, &\quad p_0 = 1. \end{aligned} \tag{5.6}$$

Then $q_k(z)$ is a polynomial of degree $\lfloor \frac{k-1}{2} \rfloor$ in z, and $p_k(z)$ is a polynomial of degree $\lfloor \frac{k}{2} \rfloor$, respectively.

We remark that the finite-difference form of the boundary value problem (5.6) admits a simple matrix representation

$$\begin{bmatrix} q_k \\ p_k \end{bmatrix} = T_k \begin{bmatrix} q_{k-1} \\ p_{k-1} \end{bmatrix}, \qquad\qquad T_k = \begin{bmatrix} 1 & h_k \\ -z g_k & 1 \end{bmatrix}, \tag{5.7}$$

and observe that in view of (5.3)

$$\det T_k = 1 + |m_k|^2 z. \tag{5.8}$$

Definition 5.3. A complex number z is an *eigenvalue* of the boundary value problem (5.5) if there exists a solution $\{q_k, p_k\}$ to (5.6) for which $p_N(z) = 0$. The set of all eigenvalues is the *spectrum* of the boundary value problem (5.5).

Remark 5.4. Clearly, $z = 0$ is not an eigenvalue.

To encode the spectral data we introduce the *Weyl function*

$$W(z) = \frac{q_N(z)}{p_N(z)}. \tag{5.9}$$

If the spectrum of the boundary problem (5.5) is simple, $W(z)$ can be written as

$$W(z) = c + \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \frac{b_j}{\zeta_j - z}. \tag{5.10}$$

Remark 5.5. In contrast to the situation for the 1mCH equation in [6] we no longer expect in general the spectrum to be either simple or real.

Regardless of the nature of the spectrum we easily obtain the following result by examining the t part of the Lax pair (5.1) in the region $x > x_N$.

Lemma 5.6. *Let $\{q_k, p_k\}$ satisfy the system of difference equations (5.6). Then*

$$\dot{q}_N = \frac{2}{z} q_N - \frac{2L}{z} p_N, \qquad \dot{p}_N = 0, \tag{5.11}$$

where $L = \sum_{j=1}^N h_j$. Thus $p_N(z)$ is independent of time and, in particular, its zeros, i.e. the spectrum, are time invariant. Moreover,

$$\dot{W} = \frac{2}{z} W - \frac{2L}{z}. \tag{5.12}$$

If the spectrum is simple we have further simplification of the time evolution.

Corollary 5.7. *Suppose $p_N(z)$ has simple roots. Then the data in the Weyl function Equation 5.10 has the time evolution*

$$\dot{c} = 0, \quad \dot{\zeta}_j = 0, \quad \dot{b}_j = \frac{2}{\zeta_j} b_j. \tag{5.13}$$

Let us recall a notation introduced in [6] to present in a compact form expressions appearing in the solution to the inverse problem; these expressions call for choices of j-element index sets I and J from the set $[k] = \{1, 2, \dots, k\}$. Henceforth we will use the notation $\binom{[k]}{j}$ for the set of all j-element subsets of $[k]$, listed in increasing order; for example $I \in \binom{[k]}{j}$ means that $I = \{i_1, i_2, \dots, i_j\}$ for some increasing sequence $i_1 < i_2 < \cdots < i_j \le k$. Furthermore, given the multi-index I and a vector $\mathbf{g} = (g_1, g_2, \cdots, g_k)$ we will abbreviate $\mathbf{g}_I = g_{i_1} g_{i_2} \dots g_{i_j}$ etc.

Definition 5.8. Let $I, J \in \binom{[k]}{j}$, or $I \in \binom{[k]}{j+1}, J \in \binom{[k]}{j}$. Then I, J are said to be *interlacing* if

$$i_1 < j_1 < i_2 < j_2 < \cdots < i_j < j_j$$

or,

$$i_1 < j_1 < i_2 < j_2 < \cdots < i_j < j_j < i_{j+1},$$

in the latter case. We abbreviate this condition as $I < J$ in either case, and, furthermore, use this same notation for $I \in \binom{[k]}{1}, J \in \binom{[k]}{0}$.

By a straightforward computation of the coefficients of $p_N = 1 - M_1 z + \cdots + \cdots M_j(-z)^j + \cdots$ (see Corollary 2.7 in [6]) we obtain the following description of constants of motion.

Lemma 5.9. *The quantities*

$$M_j = \sum_{\substack{I,J \in \binom{[N]}{j} \\ I<J}} h_I g_J, \qquad 1 \le j \le \lfloor \tfrac{N}{2} \rfloor$$

comprise a set of $\lfloor \frac{N}{2} \rfloor$ *constants of motion for the system* (4.3).

Example 5.10. Let us consider the case $N = 4$. Then the constants of motion, written in terms of the complex variables X_j (see (4.6)), with positions x_j satisfying $x_1 < x_2 < x_3 < x_4$, are

$$\begin{aligned} M_1 &= |m_1 m_2|\, e^{X_1 - X_2} + |m_1 m_3|\, e^{X_1 - X_3} + |m_1 m_4|\, e^{X_1 - X_4} + |m_2 m_3|\, e^{X_2 - X_3} + |m_2 m_4|\, e^{X_2 - X_4} + \\ &\quad |m_3 m_4|\, e^{X_3 - X_4}, \\ M_2 &= |m_1 m_2 m_3 m_4|\, e^{X_1 - X_2 + X_3 - X_4}. \end{aligned}$$

We have the following, very preliminary, characterization of the spectrum.

Lemma 5.11. *If all the angles* ω_j *in the parametrization given by Equation 4.6 satisfy*

$$-\frac{\pi}{2N} \le \omega_j \le \frac{\pi}{2N}, \tag{5.14}$$

then the spectrum of the boundary value problem (5.5) *is a finite subset of*

$$\{z | \ -\pi < \arg(z) < \pi\},$$

namely, there are no eigenvalues on the negative real axis.

Proof. Suppose there exists a positive number $\zeta_0 > 0$ for which $-\zeta_0$ is an eigenvalue, hence

$$p_N(-\zeta_0) = 0.$$

By Corollary (2.7) in [6] we have

$$p_N(z) = 1 + \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \Big(\sum_{\substack{I,J \in \binom{[N]}{j} \\ I<J}} h_I g_J \Big) (-z)^j.$$

Under condition (5.14), and recalling the parametrization of g_is and h_js (see (5.3)), it is straightforward to see that the coefficients of $(-z)^j$ satisfy

$$-\frac{\pi}{2} \le \arg \Big(\sum_{\substack{I,J \in \binom{[k]}{j} \\ I<J}} h_I g_J \Big) \le \frac{\pi}{2}$$

leading to

$$\mathrm{Re} \Big(\sum_{\substack{I,J \in \binom{[k]}{j} \\ I<J}} h_I g_J \Big) \ge 0.$$

Therefore, we have

$$\operatorname{Re}\left(p_N(-\zeta_0)\right) = 1 + \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \operatorname{Re}\Big(\sum_{\substack{I,J \in \binom{[k]}{j} \\ I<J}} h_I g_J \Big) (\zeta_0)^j > 0,$$

contradicting $p_N(-\zeta_0) = 0$. □

With the additional assumptions on the angles ω_j in place, we can improve upon (5.10).

Lemma 5.12. *Let W be the Weyl function (5.9). Suppose the spectrum of the boundary problem (5.5) is simple, and $\omega_j s$ satisfy condition (5.14). Then $W(z)$ can be expressed as*

$$W(z) = c + \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \frac{b_j}{\zeta_j - z}, \qquad b_j \neq 0, \tag{5.15}$$

where $c \neq 0$ when N is odd, and $c = 0$ when N is even.

Proof. The claim about c can be verified by examining the degree of p_N and q_N.

To prove the nonzero property of b_j we suppose that there exists some $b_j = 0$, which means that for some ζ_0

$$p_N(\zeta_0) = q_N(\zeta_0) = 0.$$

By the recursive relation (5.6) we have

$$-q_{N-1}(\zeta_0) = h_N p_{N-1}(\zeta_0), \qquad p_{N-1}(\zeta_0) = \zeta_0 g_N q_{N-1}(\zeta_0),$$

leading to

$$\left(1 + \zeta_0 |m_N|^2\right) p_{N-1}(\zeta_0) = 0.$$

Furthermore, since ζ_0 can not be negative by Lemma 5.11, we obtain $p_{N-1}(\zeta_0) = 0$. Then the second relation above, taking into account that 0 is not an eigenvalue, implies $q_{N-1}(\zeta_0) = 0$. By implementing the above argument and using (5.6) recursively we eventually get

$$p_1(\zeta_0) = q_1(\zeta_0) = 0,$$

thus contradicting $p_1 = 1$ obtained from the first iteration of (5.6). Therefore the proof is completed. □

We finish this section by commenting about the connection between the constant of motion M_1 and the two conserved Hamiltonians $H^{(0)}$ and $E^{(0)}$ (see (2.11) and (2.14)).

Lemma 5.13. *The Hamiltonians $H^{(0)}$ and $E^{(0)}$ are related to the constant of motion M_1 by*

$$H^{(0)} = 2 \sum_{j=1}^{N} |m_j|^2 + 4 \operatorname{Re}(M_1), \qquad E^{(0)} = -4 \operatorname{Im}(M_1), \tag{5.16}$$

where each $|m_j|$ is itself a constant of motion.

Proof. From Lemma 5.9,

$$M_1 = \sum_{1 \le j < k \le N} |m_j||m_k| e^{X_j - X_k}.$$

Then the result follows immediately from Lemma A.1 and Lemma A.2. □

6 Inverse Problems

6.1 First inverse problem: spectral data $\Rightarrow (g_j, h_j)$

We will formulate the inverse problem in the case where the spectrum is given by a collection of distinct complex numbers ζ_j, $1 \le j \le \lfloor \frac{N}{2} \rfloor$, none of which lies on the negative real axis (see Lemma 5.11 for one scenario ensuring the validity of the latter condition).

Definition 6.1. Given a rational function

$$W(z) = c + \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} \frac{b_j}{\zeta_j - z}, \tag{6.1}$$

and a collection of distinct positive numbers $|m_j|$, find complex constants g_j, h_j, for $1 \le j \le N$, such that $g_j h_j = |m_j|^2$ and also such that the solution of the initial value problem

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le N, \\ p_k - p_{k-1} &= -z g_k q_{k-1}, \quad 1 \le k \le N, \\ q_0 = 0, &\quad p_0 = 1, \end{aligned}$$

satisfies

$$W(z) = \frac{q_N(z)}{p_N(z)}.$$

Remark 6.2. The non degeneracy condition that the positive numbers $|m_j|$ be distinct will be eventually relaxed; the condition simplifies the derivation of the inverse formulas.

First we give a brief summary of main ideas behind the solution of the inverse problem stated in Definition 6.1. The main tool is a certain **interpolation problem** (see [6] for details). In short, let us rewrite (5.7) in terms of the Weyl function W, iterating down (5.7) k times starting with the highest index N:

$$\begin{bmatrix} W(z) \\ 1 \end{bmatrix} = T_N(z) T_{N-1}(z) \dots T_{N-k+1}(z) \begin{bmatrix} \frac{q_{N-k}(z)}{p_N(z)} \\ \frac{p_{N-k}(z)}{p_N(z)} \end{bmatrix}. \tag{6.2}$$

Then by using the transpose of the matrix of cofactors (*adjugate*) of each $T_j(z)$, and denoting $\begin{bmatrix} 1 & -h_j \\ z g_j & 1 \end{bmatrix} \stackrel{\text{def}}{=} C_{N-j+1}(z)$, one can express equation (6.2) as

$$C_k(z) \dots C_1(z) \begin{bmatrix} W(z) \\ 1 \end{bmatrix} = \det(T_N(z)) \det(T_{N-1}(z)) \dots \det(T_{N-k+1}(z)) \begin{bmatrix} \frac{q_{N-k}(z)}{p_N(z)} \\ \frac{p_{N-j}(z)}{p_N(z)} \end{bmatrix}.$$

Recalling that $\det T_j(z) = 1 + z\,|m_j|^2$ and using our assumption that none of the roots of $p_N(z)$ lies on the negative real axis, we conclude

$$\left(C_k(z) \dots C_1(z) \begin{bmatrix} W(z) \\ 1 \end{bmatrix} \right) \Big|_{z = -\frac{1}{|m_{N-i+1}|^2}} = 0, \qquad \text{for any } 1 \le i \le k. \tag{6.3}$$

Equation 6.3 can be interpreted as an interpolation problem.

Theorem 6.3 ([6])**.** *Let the matrix of products of Cs in equation (6.3) be denoted by* $\begin{bmatrix} a_k(z) & b_k(z) \\ c_k(z) & d_k(z) \end{bmatrix} \stackrel{\text{def}}{=} \hat{S}_k(z)$. *Then the polynomials $a_k(z), b_k(z), c_k(z), d_k(z)$ solve the following interpolation* *problem:*

$$a_k(-\frac{1}{|m_{N-i+1}|^2})W(-\frac{1}{|m_{N-i+1}|^2})+b_k(-\frac{1}{|m_{N-i+1}|^2})=0, \qquad 1\le i\le k, \tag{6.4a}$$

$$\deg a_k=\lfloor\frac{k}{2}\rfloor, \qquad \deg b_k=\lfloor\frac{k-1}{2}\rfloor, \qquad a_k(0)=1, \tag{6.4b}$$

$$c_k(-\frac{1}{|m_{N-i+1}|^2})W(-\frac{1}{|m_{N-i+1}|^2})+d_k(-\frac{1}{|m_{N-i+1}|^2})=0, \qquad 1\le i\le k, \tag{6.4c}$$

$$\deg c_k=\lfloor\frac{k+1}{2}\rfloor, \qquad \deg d_k=\lfloor\frac{k}{2}\rfloor, \qquad c_k(0)=0, \quad d_k(0)=1. \tag{6.4d}$$

Remark 6.4. The appearance of the label $N-j+1$ in the above formulation is fully explained in [6]; roughly, this way of counting is typical of the right, rather than left, initial value problem. This effectively results in the *counting* of the masses from right to left rather than from left to right.

Now we outline our strategy for solving the inverse problem given by Definition 6.1:

1. given W and $\{|m_j|\}$ we solve the interpolation problem of Theorem 6.3 for polynomials a_k, b_k, c_k, d_k, or equivalently the matrix $\hat{S}_k(z)$ (see the theorem above for the definition);
2. using the relation between $\hat{S}_k(z)$ and the transition matrices T_js which depend on h_js and g_js, we establish how the coefficients in the polynomials a_k, b_k, c_k, d_k are built out of h_js and g_js, leading to formulas expressing h_js and g_js as ratios of certain coefficients of a_k, b_k, c_k, d_k.

The algebraic solution to the interpolation problem stated in Theorem 6.3 was essentially given in [6] with one important *caveat*: the problem is now complex since both the spectrum and the residues b_j in (6.1) are complex. Luckily, even though this affects the global existence when time is switched on, it nevertheless has no bearing on the algebraic formulation.

We begin the presentation of formulas by introducing some additional notation. Thus we denote: $[i,j]=\{i,i+1,\cdots,j\}$, $\binom{[1,K]}{k}=\{J=\{j_1,j_2,\cdots,j_k\}|j_1<\cdots<j_k, j_i\in[1,K]\}$ and $i'=N-i+1$. Moreover, given two sets of vectors $\mathbf{x}=(x_1,x_2,\ldots,x_N)$, $\mathbf{y}=(y_1,y_2,\cdots,y_N)$ and two ordered multi-index sets I, J we define

$$\mathbf{x}_J=\prod_{j\in J}x_j, \qquad \Delta_J(\mathbf{x})=\prod_{i<j\in J}(x_j-x_i),$$

$$\Delta_{I,J}(\mathbf{x};\mathbf{y})=\prod_{i\in I}\prod_{j\in J}(x_i-y_j), \qquad \Gamma_{I,J}(\mathbf{x};\mathbf{y})=\prod_{i\in I}\prod_{j\in J}(x_i+y_j),$$

along with the convention

$$\Delta_\emptyset(\mathbf{x})=\Delta_{\{i\}}(\mathbf{x})=\Delta_{\emptyset,J}(\mathbf{x};\mathbf{y})=\Delta_{I,\emptyset}(\mathbf{x};\mathbf{y})=\Gamma_{\emptyset,J}(\mathbf{x};\mathbf{y})=\Gamma_{I,\emptyset}(\mathbf{x};\mathbf{y})=1,$$
$$\binom{[1,K]}{0}=1; \qquad \binom{[1,K]}{k}=0, \quad k>K.$$

Building in an essential way on work [6] we can now give an algebraic solution to the inverse problem stated in Definition 6.1, postponing more delicate issues like global existence to future studies.

Theorem 6.5. *Given the vector* $\mathbf{e}=(e_1,e_2,\cdots,e_N), e_j=\frac{1}{|m_{j'}|^2}$ *as well as the vector of residues of the Weyl function* $\mathbf{b}=(b_1,b_2,\cdots,b_{\lfloor\frac{N}{2}\rfloor})$*, let*

$$\mathcal{D}_k^{(l,p)}=\begin{cases}\Delta_{[1,k]}(\mathbf{e})\displaystyle\sum_{J\in\binom{[\lfloor\frac{N}{2}\rfloor]}{j}}\frac{\Delta_J(\zeta)^2(\zeta_J)^p\mathbf{b}_J}{\Gamma_{[1,k],J}(\mathbf{e};\zeta)}, & \text{if either } c=0 \text{ or } p+l-1<k-l;\\ \Delta_{[1,k]}(\mathbf{e})\cdot\Big(\displaystyle\sum_{J\in\binom{[\lfloor\frac{N}{2}\rfloor]}{j}}\frac{\Delta_J(\zeta)^2(\zeta_J)^p\mathbf{b}_J}{\Gamma_{[1,k],J}(\mathbf{e};\zeta)}+c\sum_{J\in\binom{[\lfloor\frac{N}{2}\rfloor]}{j-1}}\frac{\Delta_J(\zeta)^2(\zeta_J)^p\mathbf{b}_J}{\Gamma_{[1,k],J}(\mathbf{e};\zeta)}\Big), & \text{if } c\neq 0 \text{ and } p+l-1=k-l.\end{cases} \tag{6.5}$$

Then, provided $\mathcal{D}_k^{(l,p)} \neq 0$ for $1 \le k \le N$, there exists a unique solution to the inverse problem specified in Definition 6.1:

$$g_{k'} = \frac{\mathcal{D}_k^{(\frac{k-1}{2},1)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}}{\mathbf{e}_{[1,k]}\mathcal{D}_k^{(\frac{k+1}{2},0)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}, \qquad \text{if } k \text{ is odd}, \tag{6.6a}$$

$$g_{k'} = \frac{\mathcal{D}_k^{(\frac{k}{2},1)}\mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}}{\mathbf{e}_{[1,k]}\mathcal{D}_k^{(\frac{k}{2},0)}\mathcal{D}_{k-1}^{(\frac{k}{2},0)}}, \qquad \text{if } k \text{ is even}. \tag{6.6b}$$

Likewise,

$$h_{k'} = \frac{\mathbf{e}_{[1,k-1]}\mathcal{D}_k^{(\frac{k+1}{2},0)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}{\mathcal{D}_k^{(\frac{k-1}{2},1)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}}, \qquad \text{if } k \text{ is odd}, \tag{6.7a}$$

$$h_{k'} = \frac{\mathbf{e}_{[1,k-1]}\mathcal{D}_k^{(\frac{k}{2},0)}\mathcal{D}_{k-1}^{(\frac{k}{2},0)}}{\mathcal{D}_k^{(\frac{k}{2},1)}\mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}}, \qquad \text{if } k \text{ is even}. \tag{6.7b}$$

6.2 Second inverse problem: $(g_j, h_j) \Rightarrow X_j$

Finally, the relations (see equation (5.3))

$$h_j = |m_j|\, e^{X_j}, \quad g_j = |m_j|\, e^{-X_j}, \quad X_j = x_j + i\omega_j$$

imply

$$X_j = \ln \frac{h_j}{|m_j|} = \ln \frac{|m_j|}{g_j}.$$

Hence we arrive at the inverse formulae relating the spectral data and the positions and the momenta of the peakons.

Theorem 6.6. *Let W, given by Definition 6.1, be the Weyl function for the boundary value problem Equation 5.5 with the associated spectral data $\{\zeta_j, b_j, c\}$. Then the positions x_j and the phases ω_j (of peakons) in the discrete measure $m = 2\sum_{j=1}^N |m_j|\, e^{i\omega_j}\delta_{x_j}$ can be expressed in terms of the spectral data as:*

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]}|\mathcal{D}_k^{(\frac{k+1}{2},0)}||\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}|}{|m_{k'}||\mathcal{D}_k^{(\frac{k-1}{2},1)}||\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}|}, \qquad \text{if } k \text{ is odd}, \tag{6.8a}$$

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]}|\mathcal{D}_k^{(\frac{k}{2},0)}||\mathcal{D}_{k-1}^{(\frac{k}{2},0)}|}{|m_{k'}||\mathcal{D}_k^{(\frac{k}{2},1)}||\mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}|}, \qquad \text{if } k \text{ is even}, \tag{6.8b}$$

$$e^{i\omega_{k'}} = \frac{\mathcal{D}_k^{(\frac{k+1}{2},0)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}{\mathcal{D}_k^{(\frac{k-1}{2},1)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}} \frac{|\mathcal{D}_k^{(\frac{k-1}{2},1)}||\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}|}{|\mathcal{D}_k^{(\frac{k+1}{2},0)}||\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}|}, \qquad \text{if } k \text{ is odd}, \tag{6.8c}$$

$$e^{i\omega_{k'}} = \frac{\mathcal{D}_k^{(\frac{k}{2},0)}\mathcal{D}_{k-1}^{(\frac{k}{2},0)}}{\mathcal{D}_k^{(\frac{k}{2},1)}\mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}} \frac{|\mathcal{D}_k^{(\frac{k}{2},1)}||\mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}|}{|\mathcal{D}_k^{(\frac{k}{2},0)}||\mathcal{D}_{k-1}^{(\frac{k}{2},0)}|}|m_{k'}|, \qquad \text{if } k \text{ is even}, \tag{6.8d}$$

with $\mathcal{D}_k^{(l,p)}$ defined in (6.5), $k' = N - k + 1$, $1 \le k \le N$ and the convention that $\mathcal{D}_0^{l,p} = 1$.

Example 6.7 (1-peakon solution)**.**

$$X_1 = \ln \frac{h_1}{|m_1|},$$

where

$$h_1 = c.$$

This case does not require the inverse spectral machinery.

Example 6.8 (2-peakon solution)**.**

$$X_j = \ln \frac{h_j}{|m_j|}, \qquad j = 1, 2,$$

where

$$h_1 = \frac{b_1}{\zeta_1(1+\zeta_1|m_2|^2)}, \quad h_2 = \frac{b_1|m_2|^2}{1+\zeta_1|m_2|^2}, \quad b_1(t) = b_1(0)e^{\frac{2t}{\zeta_1}}. \tag{6.9}$$

Observe that $X_2 - X_1 = \ln |m_1|\,|m_2|\,\zeta_1$ so both the distance between the peakons and their relative phases are constant in time (see Figure 1). Note, however, that to respect the ordering $x_1 < x_2$,

$$\frac{1}{|\zeta_1|} < |m_1|\,|m_2|$$

must hold.

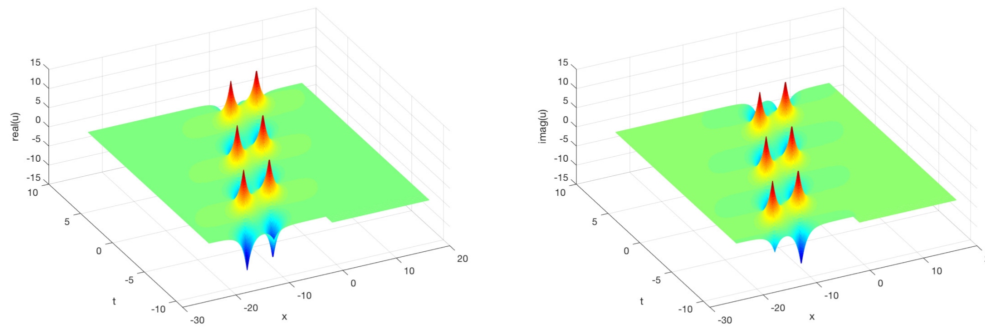

Figure 1: 2-peakon solution; $b_1(0) = 2+i,\ \zeta_1 = 1+i,\ |m_1| = 10,\ |m_2| = 10$. Peakons form a bound state.

This inequality can also be arranged to hold in particular if ζ_1 is purely imaginary. Thus there exist two-peakon breather solutions (see Figure 2).

Example 6.9 (3-peakon solution)**.**

$$X_j = \ln \frac{h_j}{|m_j|}, \qquad b_1(t) = b_1(0)e^{\frac{2t}{\zeta_1}}, \qquad j = 1, 2, 3,$$

where

$$h_1 = \frac{b_1 c}{\zeta_1\left(b_1\zeta_1|m_2|^2|m_3|^2 + c(1+\zeta_1|m_2|^2)(1+\zeta_1|m_3|^2)\right)}, \tag{6.10a}$$

$$h_2 = \frac{b_1|m_2|^2}{b_1\zeta_1|m_2|^2|m_3|^2 + c(1+\zeta_1|m_2|^2)(1+\zeta_1|m_3|^2)}\left(\frac{b_1|m_3|^2}{1+\zeta_1|m_3|^2} + c\right), \tag{6.10b}$$

$$h_3 = \frac{b_1|m_3|^2}{1+\zeta_1|m_3|^2} + c. \tag{6.10c}$$

The formulas (6.10a) are local formulas, nevertheless we can prove, under appropriate conditions on the initial data, that there always exists a global (i.e. valid for arbitrary time $t \in (t_0, +\infty)$) 3-peakon solution, originating at some initial time t_0.

Theorem 6.10. *Suppose that*

$$\mathrm{Re}(\zeta_1) > 0, \qquad \frac{1}{|\zeta_1|} < |m_2|\,|m_3|\,.$$

Then there always exists a choice of $b_1(0)$ for which the solutions $X_j, j = 1, 2, 3$ are global.

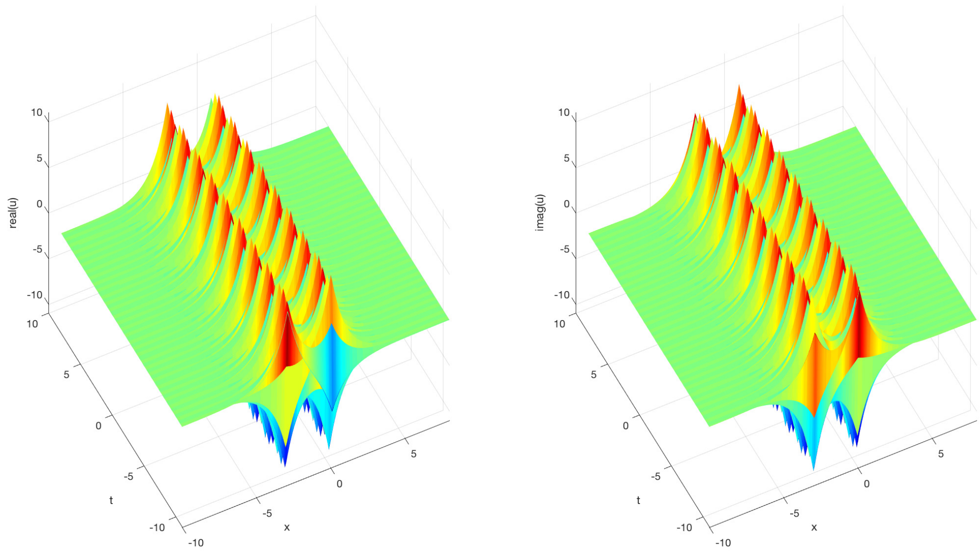

Figure 2: Periodic 2-peakon breather; $b_1(0) = 2 + i,\ \zeta_1 = 0.2i,\ |m_1| = 10,\ |m_2| = 10.$

Proof. By construction if all $h_j, j = 1, 2, 3$ are well defined and the ordering condition $x_1 < x_2 < x_3$ holds then X_1, X_2, X_3 satisfy (4.7). Let us consider $b_1(t) = b_1(0)e^{\frac{2t}{\zeta_1}}$ with fixed, but otherwise arbitrary, $b_1(0)$. Since $\mathrm{Re}(\zeta_1) > 0$ the modulus of $b_1(t)$ can be made arbitrary large by choosing t large enough. Thus for t large enough none of the denominators in h_1, h_2 can become 0 while the denominator of h_3 is never 0 in view of the assumption on ζ_1. We observe that the ordering condition $x_1 < x_2 < x_3$ can be stated

$$\left|\frac{h_1}{m_1}\right| < \left|\frac{h_2}{m_2}\right| < \left|\frac{h_3}{m_3}\right|. \tag{6.11}$$

Let us now analyze the ordering condition in the region $t \to \infty$. We have

$$\left|\frac{h_1}{m_1}\right| \approx \frac{|c|}{|\zeta_1|^2 |m_1| |m_2|^2 |m_3|^2}, \tag{6.12}$$

$$\left|\frac{h_2}{m_2}\right| \approx \frac{|b_1|}{|\zeta_1| |m_2| \left|(1 + \zeta_1 |m_3|^2)\right|}, \tag{6.13}$$

$$\left|\frac{h_3}{m_3}\right| \approx \frac{|b_1| |m_3|}{\left|(1 + \zeta_1 |m_3|^2)\right|}, \tag{6.14}$$

and in that region, enforcing (6.12), one is led to

$$\frac{\left|c(1 + \zeta_1 |m_2|^2)\right|}{|\zeta_1| |m_1| |m_2| |m_3|^2} < |b_1|\,, \tag{6.15}$$

$$\frac{1}{|\zeta_1|} < |m_2| |m_3|\,, \tag{6.16}$$

the first holding in the asymptotic region without any further assumptions, the second holding in view of the assumption on ζ_1. However, in view of continuity in t we can extend the asymptotic inequalities to a region $[t_0, \infty)$, for some $t_0 \geq 0$, without violating the inequalities (6.11). Now it suffices to choose a new $\tilde{b}_1(0) = b_1(0)e^{\frac{2t_0}{\zeta_1}}$ and construct h_1, h_2, h_3 using the inverse formulas (6.10a). The resulting x_1, x_2, x_3 will by construction satisfy the ordering condition for arbitrary $t > 0$. $\square$

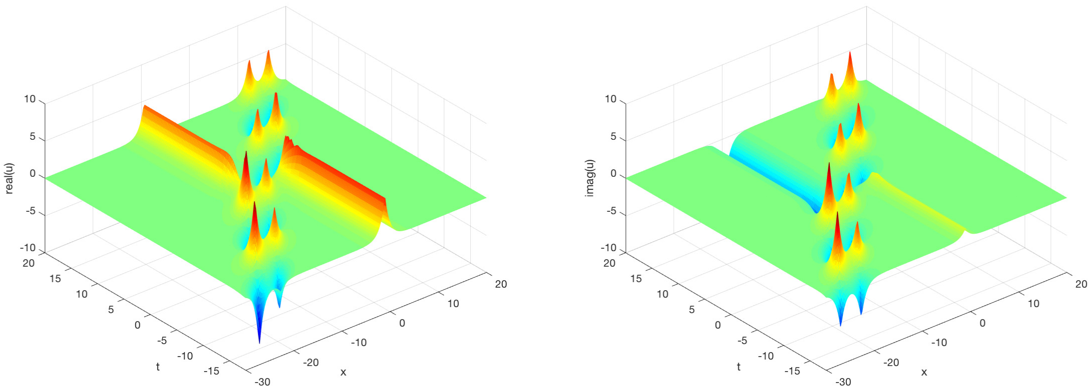

Figure 3: 3-peakon solution; $b_1(0) = 1 + i,\ c = 2 + 0.5i,\ \zeta_1 = 1.5 + i,\ |m_1| = 8,\ |m_2| = 5,\ |m_3| = 6$

Once the existence of global solutions is established the asymptotic behaviour follows from explicit formulas (6.10a). In particular we see that, asymptotically, peakons pair up as illustrated by Figure 3 which shows the 3 dimensional evolution of profiles $\mathrm{Re}(u), \mathrm{Im}(u)$ while Figure 4 illustrates the evolution of $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ relative to the graph of $|u|$. The interaction between peakons is best captured through the graphs of their trajectories (see Figure 5).

Corollary 6.11. *Suppose that $b_1(0)$ and ζ_1 satisfy conditions of Theorem 6.10. Then the first peakon stops in the asymptotic region, while the second and third form a bound pair moving with speed $\frac{2\,\mathrm{Re}\,\zeta_1}{|\zeta_1|^2}$.*

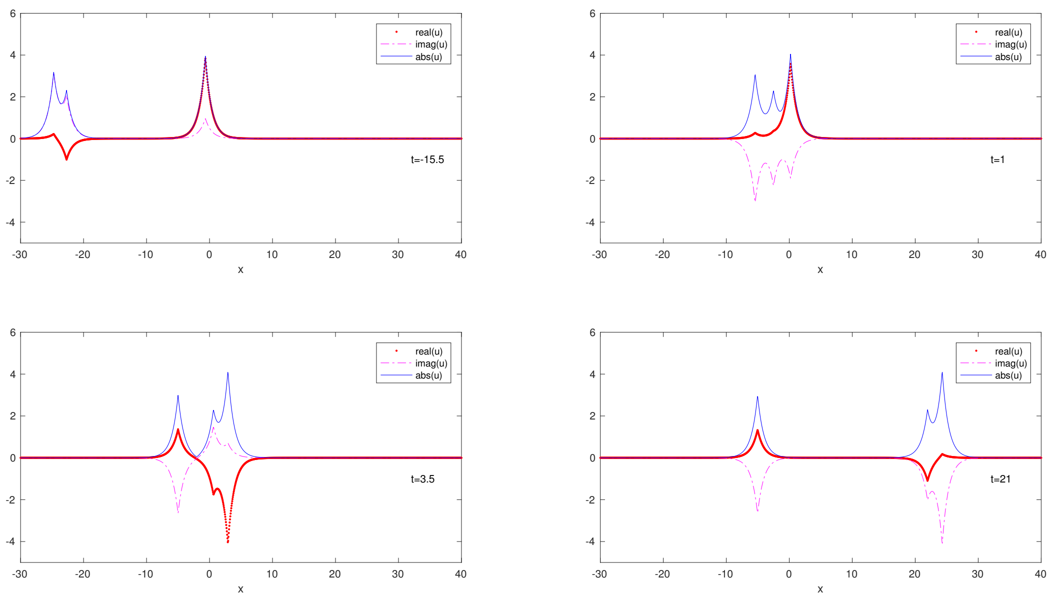

Figure 4: A superimposed view of the amplitude $|u|$ as well as $\mathrm{Re}\,u$ and $\mathrm{Im}\,u$. Asymptotic pairing is visible in graphs of all these three quantities.

We also point out that this analysis can be carried out for purely imaginary ζ_1 by forcing $b_1(0)$ to be large enough to satisfy (6.15) while at the same time imposing (6.16); the resulting 3-peakon breather is graphed in Figure 6.

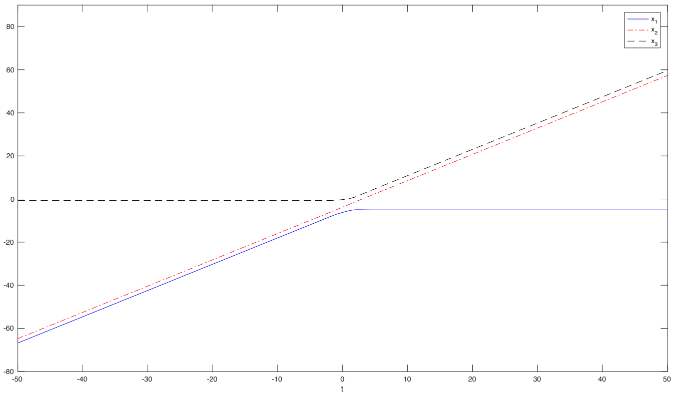

Figure 5: Asymptotic pairing of positions x_2, x_3 for large positive times. Observe that initially the first and the second peakons form a bound state; the interaction at $t = 0$ with the third peakon breaks the bond and a new bond emerges, while the first peakon slows to a halt.

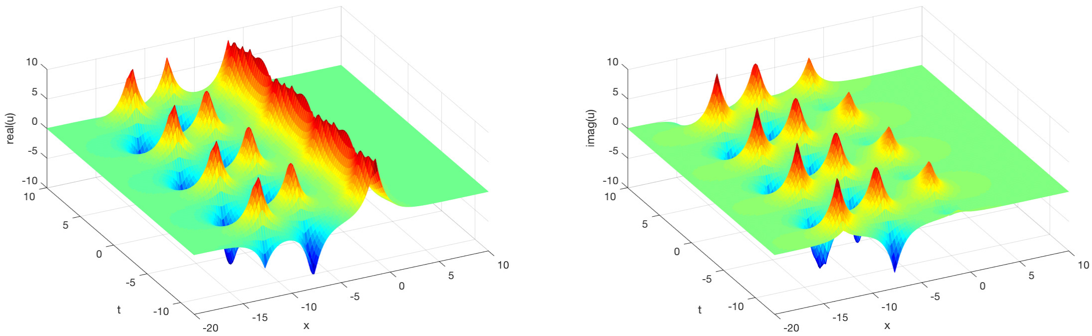

Figure 6: A 3-peakon breather; Re(u) and Im(u) graphed for $b_1(0) = 1+i,\quad c = \ 2+0.5i,\ \zeta_1 = 2i,\ |m_1| = 9,\ |m_2| = 8,\ |m_3| = 10$

The generalization of the analysis of the 3-peakon solutions done above to multipeakons will be carried out elsewhere. For now, we confine ourselves to stating the explicit formula for 4-peakons.

Example 6.12 (4-peakon solution)**.**

$$X_j = \ln \frac{h_j}{|m_j|} \qquad b_1(t) = b_1(0) e^{\frac{2t}{\zeta_1}}, \qquad b_2(t) = b_2(0) e^{\frac{2t}{\zeta_2}}, \qquad j = 1,2,3,4,$$

where

$$h_1 = \frac{b_1 b_2 (\zeta_2 - \zeta_1)^2}{\zeta_1 \zeta_2 \left(b_1 \zeta_1 (1+\zeta_2|m_2|^2)(1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 \zeta_2 (1+\zeta_1|m_2|^2)(1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2)\right)},$$

$$h_2 = |m_2|^2 \cdot \frac{b_1 b_2 (\zeta_2 - \zeta_1)^2 \left(b_1 (1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 (1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2)\right)}{(b_1 \zeta_1 (1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 \zeta_2 (1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2))}$$

$$\cdot \frac{1}{(b_1 \zeta_1 (1+\zeta_2|m_2|^2)(1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 \zeta_2 (1+\zeta_1|m_2|^2)(1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2))} \Bigg),$$

$$h_3 = \frac{\left(b_1 (1+\zeta_2|m_4|^2) + b_2 (1+\zeta_1|m_4|^2)\right) \left(b_1 (1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 (1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2)\right)}{(1+\zeta_1|m_4|^2)(1+\zeta_2|m_4|^2) \left(b_1 \zeta_1 (1+\zeta_2|m_3|^2)(1+\zeta_2|m_4|^2) + b_2 \zeta_2 (1+\zeta_1|m_3|^2)(1+\zeta_1|m_4|^2)\right)},$$

$$h_4 = |m_4|^2 \cdot \frac{b_1 (1+\zeta_2|m_4|^2) + b_2 (1+\zeta_1|m_4|^2)}{(1+\zeta_1|m_4|^2)(1+\zeta_2|m_4|^2)}.$$

We finish this section by briefly commenting about the NLSP equation in the conservative peakon sector. As we pointed out in Corollary 4.3 the peakon flows for HP and NLSP form an orthogonal family and our analysis using the inverse spectral problem carries over to the NLSP case.

We recall that Equation 3.11 is the compatibility condition of

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{6.17}$$

where

$$U = \frac{1}{2}\begin{bmatrix} -1 & \lambda m \\ -\lambda \bar{m} & 1 \end{bmatrix},$$

$$V = \frac{i}{2}\begin{bmatrix} 4\lambda^{-2} + Q & -2\lambda^{-1}(u - u_x) - \lambda m i \,\mathrm{Im}(Q) \\ 2\lambda^{-1}(\bar{u} + \bar{u}_x) + \lambda \bar{m} i \,\mathrm{Im}(Q) & -Q \end{bmatrix},$$

with Q given by (3.2).

The spectral problem is the same, namely given by Equation 5.4. The only point of departure from the HP case is the time evolution of the spectral date which can be obtained by using V from Equation 6.17 in the asymptotic region, resulting in a modest variation of Lemma 5.6. In summary, the NLSP time evolution of W is:

$$\dot{W} = i\left(\frac{2}{z}W - \frac{2L}{z}\right), \tag{6.18}$$

which in turn leads to the NLSP time evolution of the spectral data given by the following theorem (see Equation 5.13 for comparison).

Lemma 6.13. *Suppose $p_N(z)$ has simple roots. Then in the notation of Equation 5.10 the spectral data $\{\zeta_j, b_j, c\}$ evolve according to*

$$\dot{\zeta}_j = 0, \qquad \dot{b}_j = \frac{2i}{\zeta_j} b_j, \qquad \dot{c} = 0. \tag{6.19}$$

Finally, the generalization to the θ family given by (4.5) is straightforward and we only mention that the Weyl function evolves according to:

$$\dot{W} = e^{\mathrm{i}\theta}\left(\frac{2}{z}W - \frac{2L}{z}\right), \tag{6.20}$$

implying that the spectral data $\{\zeta_j, b_j, c\}$ evolves

$$\dot{\zeta}_j = 0, \qquad \dot{b}_j = \frac{2e^{\mathrm{i}\theta}}{\zeta_j} b_j, \qquad \dot{c} = 0. \tag{6.21}$$

7 Conclusions

We showed that the NLS-type peakon equations introduced in [1] can be generalized to a family of peakon equations parametrized by the real projective line. We studied the sector of conservative peakon solutions for which we formulated and solved the inverse problem resulting in explicit peakon solutions of various types such as peakons and periodic peakon breathers. This work opens multiple paths to further studies. Some of the outstanding issues are:

1. the Poisson structures for the conservative peakon sector were obtained by analyzing the (singular) limit of the Hamiltonian structures valid in the smooth case, followed by, what amounts to, guessing the right structure and it is of general interest to understand more deeply which Hamiltonian structures survive that singular limit;

2. the inverse problem was solved under the assumption of simple eigenvalues and it is not known to us what dynamical consequences will result from lifting of that assumption;

3. the analysis of the global (in t) existence of solutions was only done for a small number of peakons and a generalization is called for;

4. we have done no stability analysis of conservative peakons and this remains an important open problem.

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Appendix A Some results for the conservative peakon sector

In this Appendix we collect some formulas used in Section 4.1 for the discussion of Poisson structures and for the Hamiltonian form of the peakon flows Equation 4.7 and Equation 4.8.

Our main goal is to express $H^{(0)}$ and $E^{(0)}$ in terms of the complex variable X_j introduced in (4.6). Throughout we work with the ordering $x_1 < x_2 < \cdots < x_N$.

Lemma A.1.

$$H^{(0)} = ||u||^2_{H^1} = 4\,\mathrm{Re}\,\big(\sum_{k<l} |m_k|\,|m_l|\,e^{X_k - X_l}\big) + 2\sum_l |m_l|^2\,. \tag{A.1}$$

Proof. We recall (see Equation 2.11)

$$\begin{aligned} H^{(0)} =& \mathrm{Re}\int \bar{u} m dx = 2\,\mathrm{Re}\,\big(\sum_{k,l} m_k \bar{m}_l e^{-|x_k - x_l|}\big) = 2\,\mathrm{Re}\,\big(\sum_{k<l} m_k\bar{m}_l e^{x_k - x_l} + \sum_k |m_k|^2 + \\ & \sum_{l<k} m_k \bar{m}_l e^{x_l - x_k}\big) = 2\,\mathrm{Re}\,\big(\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l} + \sum_k |m_k|^2 + \sum_{l<k}|m_k|\,|m_l|\,e^{\bar{X}_l - \bar{X}_k}\big) = \\ & 4\,\mathrm{Re}\,\big(\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l}\big) + 2\sum_k |m_k|^2\,. \end{aligned}$$

□

Lemma A.2.

$$E^{(0)} = \mathrm{Im}\int u_x \bar{m} dx = -4\,\mathrm{Im}(\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l}) \tag{A.2}$$

Proof. By definition

$$\begin{aligned} &\mathrm{Im}\int u_x \bar{m} dx = 2\,\mathrm{Im}\sum_l \langle u_x\rangle(x_l)\bar{m}_l = 2\,\mathrm{Im}\sum_{k,l} m_k\bar{m}_l\,\mathrm{sgn}(x_k - x_l)e^{-|x_l - x_k|} = \\ &2\,\mathrm{Im}\,\big(-\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l} + \mathrm{Im}\sum_{l<k}|m_k|\,|m_l|\,e^{\bar{X}_l - \bar{X}_k}\big) = \\ &2\,\mathrm{Im}\,\big(-\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l} + \sum_{k<l}|m_k|\,|m_l|\,e^{\bar{X}_k - \bar{X}_l}\big) = -4\,\mathrm{Im}\,\big(\sum_{k<l}|m_k|\,|m_l|\,e^{X_k - X_l}\big). \end{aligned}$$

□

Lemma A.3. *Suppose $x \notin supp(m)$ then*

$$Q(x) = 4\sum_{x_k < x < x_l} |m_k|\,|m_l|\,e^{X_k - X_l}, \tag{A.3}$$

Proof. We recall that by Equation 3.2

$$Q(x) = (u - u_x)(\bar{u} + \bar{u}_x).$$

For the peakon Ansatz (1.2)

$$\begin{aligned} u - u_x =& \sum_k m_k(1 - \mathrm{sgn}(x_k - x))e^{-|x - x_k|} = 2\sum_{x_k < x} m_k e^{-|x - x_k|} = \\ & (2\sum_{x_k < x}|m_k|\,e^{X_k})e^{-x}, \end{aligned}$$

and

$$\bar{u}+\bar{u}_x=\sum_l \bar{m}_l(1+\operatorname{sgn}(x_l-x))e^{-|x-x_l|}=2\sum_{x<x_l}\bar{m}_l e^{-|x-x_l|}=$$
$$\big(2\sum_{x<x_l}|m_l|\,e^{-X_l}\big)e^x,$$

thus proving the claim. ∎

Lemma A.4. *Let $x_j \in supp(m)$. Then*

$$\langle Q\rangle(x_j)=4\sum_{k<j<l}|m_k|\,|m_l|\,e^{X_k-X_l}+2\,|m_j|\,\big(\sum_{k<j}|m_k|\,e^{X_k-X_j}+\sum_{j<k}|m_k|\,e^{X_j-X_k}\big).$$

Proof. A straightforward computation yields

$$\langle Q\rangle(x_j)=\frac{1}{2}\big(Q(x_j+)+Q(x_j-)\big)\overset{Lemma\ A.3}{=}2\big(\sum_{x_k<x_j+<x_l}|m_k|\,|m_l|\,e^{X_k-X_l}+\sum_{x_k<x_j-<x_l}|m_k|\,|m_l|\,e^{X_k-X_l}\big)=$$
$$2\big(\sum_{k\le j<l}|m_k|\,|m_l|\,e^{X_k-X_l}+\sum_{k<j\le l}|m_k|\,|m_l|\,e^{X_k-X_l}\big)=$$
$$4\sum_{k<j<l}|m_k|\,|m_l|\,e^{X_k-X_l}+2\,|m_j|\,\big(\sum_{k<j}|m_k|\,e^{X_k-X_j}+\sum_{j<k}|m_k|\,e^{X_j-X_k}\big).$$

∎

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