MINIMAL LENGTH ELEMENTS OF EXTENDED AFFINE WEYL GROUPS

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Abstract. Let W be an extended affine Weyl group. We prove that the minimal length elements $w_{\mathcal{O}}$ of any conjugacy class $\mathcal{O}$ of W satisfy some nice properties, generalizing results of Geck and Pfeiffer [7] on finite Weyl groups. We also study a special class of conjugacy classes, the straight conjugacy classes. These conjugacy classes are in a natural bijection with the Frobenius-twisted conjugacy classes of some p-adic group and satisfy additional interesting properties. Furthermore, we discuss some applications to the affine Hecke algebra H. We prove that $T_{w_{\mathcal{O}}}$, where $\mathcal{O}$ ranges over all the conjugacy classes of W, forms a basis of the cocenter $H/[H, H]$. We also introduce the class polynomials, which play a crucial role in the study of affine Deligne-Lusztig varieties [12].

Introduction

0.1. Let W be a finite Weyl group and $\mathcal{O}$ be a conjugacy class of W. In [7] and [8], Geck and Pfeiffer proved the following remarkable properties:

(1) For any $w \in \mathcal{O}$, there exists a sequence of conjugations by simple reflections that reduces w to a minimal length element in $\mathcal{O}$, with the lengths of the elements in the sequence weakly decreasing;

(2) If w and w' are both of minimal length in $\mathcal{O}$, then they are strongly conjugate.

Such properties play an important role in the study of finite Hecke algebras. They lead to the definition and determination of "character tables" for finite Hecke algebras, analogous to character tables for finite groups. They also play a role in the study of Deligne-Lusztig varieties (see, e.g. [23], [2], and [13]) and in the study of links between conjugacy classes in finite Weyl groups and unipotent conjugacy classes in reductive groups (see [21]).

0.2. The main purpose of this paper is to study minimal length elements in a conjugacy class of an (extended) affine Weyl group and to establish some remarkable properties. These properties play an important role in the study of affine Hecke algebras and p-adic groups. We

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will discuss some applications to affine Hecke algebras in §0.4. These properties also play a key role in the study of affine Deligne-Lusztig varieties, see [12] and [5].

The minimal length elements for some affine Weyl groups of classical types were first studied by the first author in [11] using a case-by-case analysis. The method we use here is quite different. We give a case-free proof that works for all cases, including the affine Weyl groups of exceptional type, which seem very difficult using the approach in [11]. The present method is based on three main ingredients:

- the "partial conjugation" method introduced in [10];
- the geometric interpretation of the length function in terms of alcoves and Weyl chambers introduced in [14];
- straight conjugacy classes and Newton points.

The first two ingredients were also used by the authors in [14] to provide a case-free proof of the remarkable properties mentioned in §0.1 for finite Weyl groups.

The third ingredient is a new feature for affine Weyl groups. Not only is it a crucial ingredient that allows us to pass from the finite Weyl groups to affine Weyl groups; it also has independent interest. We will discuss this in more detail below.

0.3. For simplicity, we only consider (unextended) affine Weyl groups and (untwisted) conjugacy classes in the introduction. However, we will also cover the general case in this paper.

Let $P_{\mathbb{Q}}^+$ be the set of dominant rational coweights. To each element x in the affine Weyl group W, we may associate the dominant Newton point $\bar{\nu}_x \in P_{\mathbb{Q}}^+$ (see §3.4). We call an element x *straight* if $\ell(x) = \langle \bar{\nu}_x, 2\rho \rangle$, where ρ is the sum of fundamental weights. This is equivalent to saying that $\ell(x^n) = n\ell(x)$ for all $n \geqslant 0$. A conjugacy class is called straight if it contains a straight element. The minimal length elements in a straight conjugacy class are just the straight elements it contains. The notion of straight element/conjugacy class was first introduced by Krammer in [18] to study the conjugacy problem.

The first author observed in [11] that the straight conjugacy classes have a geometric meaning: there is a natural bijection between the set of Frobenius-twisted conjugacy classes of a p-adic group and the set of straight conjugacy classes of the corresponding affine Weyl group W. There is no known counterpart for finite Weyl groups.

We prove that

Theorem A. (=Theorem 2.9 and Theorem 3.8)

Let W be an affine Weyl group and $\mathcal{O}$ be a conjugacy class of W. Then

(1) For any $w \in \mathcal{O}$, there exists a sequence of conjugations by simple reflections that reduces w to a minimal length element in $\mathcal{O}$, with the lengths of the elements in the sequence weakly decreasing.

(2) If w and w' are both of minimal length in $\mathcal{O}$, then they are strongly conjugate.

(3) If, moreover, $\mathcal{O}$ is straight, then any two minimal length elements are conjugate by "cyclic shifts".

Theorem B. (=Theorem 3.3 and Theorem 3.4)

Let W be an affine Weyl group. Then

(1) The map $f : W \to P_{\mathbb{Q}}^+$, $x \mapsto \bar{\nu}_x$ is constant on each conjugacy class of W.

(2) The map f induces a bijection from the set of straight conjugacy classes of W to $f(W)$.

(3) Any conjugacy class $\mathcal{O}$ of W can be "reduced" to the unique straight conjugacy class in the fiber of $f(\mathcal{O})$ in the sense of Theorem 3.4.

The statement of Theorem 3.4 is technical and we don't include it here. We would like to point out that in fact Theorem B implies Theorem A. Moreover, Theorem B is a crucial ingredient in the study of affine Deligne-Lusztig varieties in [12]. Theorem A is not enough for this purpose.

0.4. Now we discuss some applications to affine Hecke algebras.

The affine Hecke algebra H is a free $\mathbb{Z}[q, q^{-1}]$-module with basis T_w for w ranges over elements in W. By the density theorem and the trace Paley-Wiener theorem [17], if q is a power of a prime, then the trace function gives a natural bijection from the dual space of the cocenter $H/[H, H]$ to the space of the Grothendieck group of representations of H.

We prove that

Theorem C. (=Corollary 5.2 and Theorem 6.7)

(1) Let $\mathcal{O}$ be a conjugacy class of W and $w_{\mathcal{O}}$ be a minimal length representative. Then the image of $T_{w_{\mathcal{O}}}$ in the cocenter H does not depend on the choice of a minimal length representative $w_{\mathcal{O}}$. We denote the image by $T_{\mathcal{O}}$.

(2) The set $\{T_{\mathcal{O}}\}$, where $\mathcal{O}$ ranges over all the conjugacy classes of W, is a basis of the cocenter $H/[H, H]$.

Here part (1) and the fact that $\{T_{\mathcal{O}}\}$ spans the cocenter follow from the special properties for W discussed above, and the fact that $\{T_{\mathcal{O}}\}$ is a linearly independent set follows from the density theorem for affine Weyl groups, which will be proved in section 6.

As a consequence,

Theorem D. (=Part of Theorem 5.3)

For any $w \in W$, the image of T_w in the cocenter of H is a linear combination of $T_{\mathcal{O}}$ and the coefficients are the "class polynomials" $f_{w,\mathcal{O}}$.

It is worth mentioning that the class polynomials are closely related to the affine Deligne-Lusztig varieties [12, Theorem 6.1 & Proposition 8.3].

1. Preliminary

1.1. Let S be a finite set and $(m_{st})_{s,t\in S}$ be a matrix with entries in $\mathbb{N}\cup\{\infty\}$ such that $m_{ss}=1$ and $m_{st}=m_{ts}\geqslant 2$ for all $s\neq t$. Let W be a group generated by S with relations $(st)^{m_{st}}=1$ for $s,t\in S$ with $m_{s,t}<\infty$. We say that (W,S) is a *Coxeter group*. Sometimes we just call W itself a Coxeter group.

Let $\mathrm{Aut}(W,S)$ be the group of automorphisms of the group W that preserve S. Let Ω be a group with a group homomorphism to $\mathrm{Aut}(W,S)$. Set $\tilde W=W\rtimes\Omega$. Then an element in $\tilde W$ is of the form $w\delta$ for some $w\in W$ and $\delta\in\Omega$. We have that $(w\delta)(w'\delta')=w\delta(w')\delta\delta'\in\tilde W$ with $\delta,\delta'\in\Omega$. Since we are mainly interested in the action of Ω on W, we may assume without loss of generality that Ω is finite.

For $w\in W$ and $\delta\in\Omega$, we set $\ell(w\delta)=\ell(w)$, where $\ell(w)$ is the length of w in the Coxeter group (W,S). Thus Ω consists of length 0 elements in $\tilde W$. We sometimes call the elements in Ω *basic elements* in $\tilde W$.

We are mainly interested in the W-conjugacy classes in $\tilde W$. By [6, Remark 2.1], for any $\delta\in\Omega$, the map $W\to\tilde W$, $w\mapsto w\delta$ gives a bijection between the δ-conjugacy classes in W and the W-conjugacy classes in $\tilde W$ that are contained in $W\delta$.

1.2. For $w,w'\in\tilde W$ and $s\in S$, we write $w\xrightarrow{s}w'$ if $w'=sws$ and $\ell(w')\leqslant\ell(w)$. We write $w\to w'$ if there is a sequence $w=w_0,w_1,\cdots,w_n=w'$ of elements in $\tilde W$ such that for any k, $w_{k-1}\xrightarrow{s}w_k$ for some $s\in S$.

We write $w\approx w'$ if $w\to w'$ and $w'\to w$. In this case, we say that w and w' are conjugate by "cyclic shifts". It is easy to see that $w\approx w'$ if $w\to w'$ and $\ell(w)=\ell(w')$.

We call $\tilde w,\tilde w'\in\tilde W$ *elementarily strongly conjugate* if $\ell(\tilde w)=\ell(\tilde w')$ and there exists $x\in W$ such that $\tilde w'=x\tilde w x^{-1}$ and $\ell(x\tilde w)=\ell(x)+\ell(\tilde w)$ or $\ell(\tilde w x^{-1})=\ell(x)+\ell(\tilde w)$. We call $\tilde w,\tilde w'$ *strongly conjugate* if there is a sequence $\tilde w=\tilde w_0,\tilde w_1,\cdots,\tilde w_n=\tilde w'$ such that for each i, $\tilde w_{i-1}$ is elementarily strongly conjugate to $\tilde w_i$. We write $\tilde w\sim\tilde w'$ if $\tilde w$ and $\tilde w'$ are strongly conjugate. We write $\tilde w\tilde\sim\tilde w'$ if $\tilde w\sim\delta\tilde w'\delta^{-1}$ for some $\delta\in\Omega$.

The following result is proved in [7], [6] and [10] via a case-by-case analysis with the aid of computer for exceptional type. A case-free proof which does not rely on computer calculation was recently obtained in [14].

Theorem 1.1. *Assume that W is a finite Coxeter group. Let $\mathcal{O}$ be a conjugacy class in $\tilde W$ and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to w'$.
(2) Let $w, w' \in \mathcal{O}_{\min}$, then $w \sim w'$.

The main purpose of this paper is to extend the above theorem to the cases of affine Weyl groups and to discuss its application to affine Hecke algebras. To do this, we first recall some basic facts on affine Weyl groups and Bruhat-Tits building.

1.3. Let Φ be a reduced root system and W_0 the corresponding finite Weyl group. Then (W_0, S_0) is a Coxeter group, where S_0 is the set of simple reflections in W_0.

Let Q be the coroot lattice spanned by $\Phi^\vee$ and

$$W = Q \rtimes W_0 = \{t^\chi w; \chi \in Q, w \in W_0\}$$

be the affine Weyl group. The multiplication is given by the formula $(t^\chi w)(t^{\chi'} w') = t^{\chi + w\chi'} ww'$. Moreover, (W, S) is a Coxeter group, where $S \supset S_0$ is the set of simple reflections in W.

The length function on W is given by the following formula (see [16])

$$\ell(t^\chi w) = \sum_{\alpha, w^{-1}(\alpha) \in \Phi^+} |\langle \chi, \alpha \rangle| + \sum_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} |\langle \chi, \alpha \rangle - 1|.$$

1.4. Let $V = Q \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have a natural action of $\tilde{W}$ on V. For $x, y \in V$, define $(x, y) = \sum_{\alpha \in \Phi} \langle x, \alpha \rangle \langle y, \alpha \rangle$. Then by [1, Ch. VI, §1, no.1, Proposition 3], $(\,,\,)$ is a positive-definite symmetric bilinear form on V invariant under W. We define the norm $|| \cdot || : V \to \mathbb{R}$ by $||x|| = \sqrt{(x, x)}$ for $x \in V$.

For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, define $H_{\alpha,k} = \{x \in V; \langle x, \alpha \rangle = k\}$. Let $\mathfrak{H} = \{H_{\alpha,k};\ \alpha \in \Phi, k \in \mathbb{Z}\}$. For each hyperplane $H \in \mathfrak{H}$, let $s_H \in W$ be the orthogonal reflection with respect to H. Connected components of $V - \cup_{H \in \mathfrak{H}} H$ are called *alcoves*. We denote by $\bar{A}$ the closure of an alcove A. We denote by Δ the fundamental alcove, i.e. the alcove in the dominant chamber such that $0 \in \bar{\Delta}$.

Let $H \in \mathfrak{H}$. If the interior $H_A = (H \cap \bar{A})^\circ \subset H \cap \bar{A}$ spans H, then we call H a *wall* of A and H_A a *face* of A.

For $p \neq q \in V$, we denote by $L(p, q) \subset V$ the affine subspace spanned by p and q.

Let $K \subset V$ be a convex subset. We call $x \in K$ a *regular point* of K if for any $H \in \mathfrak{H}$, $x \in H$ implies that $K \subset H$. It is clear that all the regular points of K form an open dense subset of K.

1.5. The action of $\tilde{W}$ on V sends hyperplanes in $\mathfrak{H}$ to hyperplanes in $\mathfrak{H}$ and thus induces an action on the set of alcoves. It is known that the affine Weyl group W acts simply transitively on the set of alcoves. For any alcove A, we denote by x_A the unique element in W such that $x_A \Delta = A$.

For any $\tilde{w} \in \tilde{W}$ and alcove A, set $\tilde{w}_A = x_A^{-1} \tilde{w} x_A$. Then any element in the W-conjugacy class of $\tilde{w}$ is of the form $\tilde{w}_A$ for some alcove A.

For any two alcoves A, A', let $\mathfrak{H}(A, A')$ denote the set of hyperplanes in $\mathfrak{H}$ separating them. Then $H \in \mathfrak{H}(\Delta, \tilde{w}\Delta)$ if and only if $\tilde{w}^{-1}\alpha$ is a negative affine root, where α is the positive affine root corresponding to H. In this case, $\ell(s_H \tilde{w}) < \ell(\tilde{w})$. We also have that $\ell(\tilde{w}) = \sharp\mathfrak{H}(\Delta, \tilde{w}\Delta)$.

2. Minimal length elements in affine Weyl groups

Unless otherwise stated, we write W_0 for finite Weyl group and W for affine Weyl group in the rest of this paper.

2.1. Similar to [14], we may view conjugation by a simple reflection in the following way.

Let A, A' be two alcoves with a common face $H_A = H_{A'}$, here $H \in \mathfrak{H}$. Let s_H be the reflection along H and set $s = x_A^{-1} s_H x_A$. Then $s \in S$. Now

$$\tilde{w}_{A'} = (s_H x_A)^{-1} \tilde{w} (s_H x_A) = s x_A^{-1} \tilde{w} x_A s = s \tilde{w}_A s$$

is obtained from $\tilde{w}_A$ by conjugating the simple reflection s. Similar to [14, Lemma 1.1], we have the following criterion to check if $\ell(\tilde{w}_{A'}) > \ell(\tilde{w}_A)$.

Lemma 2.1. *We keep the notations as above. Define $f_{\tilde{w}} : V \to \mathbb{R}$ by $v \mapsto ||\tilde{w}(v) - v||^2$. Let h be a regular point in H_A and $v \in V$ such that $(v, h - h') = 0$ for all $h' \in H_A$ and $h - \epsilon v \in A$ for sufficient small $\epsilon > 0$. Set*

$$D_v f_{\tilde{w}}(h) = \lim_{t \to 0} \frac{f_{\tilde{w}}(h + tv) - f_{\tilde{w}}(h)}{t} = 2(\tilde{w}(h) - h, \tilde{w}(v) - \tilde{w}(0) - v).$$

If $\ell(\tilde{w}_{A'}) = \ell(s\tilde{w}_A s) = \ell(\tilde{w}_A) + 2$, then $D_v f_{\tilde{w}}(h) > 0$.

2.2. Let $\text{grad} f_{\tilde{w}}$ denote the gradient of the function $f_{\tilde{w}}$ on V, that is, for any other vector field X on V, we have $X f_{\tilde{w}} = (X, \text{grad} f_{\tilde{w}})$. Here we naturally identify V with the tangent space of any point in V.

We'll describe where the gradient vanishes. To do this, we introduce an affine subspace $V_{\tilde{w}}$.

Notice that Ω is a finite subgroup of $\tilde{W}$. For any $\tilde{w} \in \tilde{W}$, there exists $n \in \mathbb{N}$ such that $\tilde{w}^n \in W$. Hence there exists $m \in \mathbb{N}$ such that $\tilde{w}^{mn} = t^\lambda$ for some $\lambda \in Q$. Set $\nu_{\tilde{w}} = \lambda / mn \in V$ and call it the *Newton point of* $\tilde{w}$. Then it is easy to see that $\nu_{\tilde{w}}$ doesn't depend on the choice of m and n. We set

$$V_{\tilde{w}} = \{v \in V; \tilde{w}(v) = v + \nu_{\tilde{w}}\}.$$

Lemma 2.2. *Let $\tilde{w} \in \tilde{W}$. Then $V_{\tilde{w}} \subset V$ is a nonempty affine subspace such that $V_{\tilde{w}} = \tilde{w} V_{\tilde{w}} = V_{\tilde{w}} + \nu_{\tilde{w}}$.*

Proof. Since $\tilde{w}$ is an affine transformation, for any $p \neq q \in V_{\tilde{w}}$, the affine line $L(p,q)$ is also contained in $V_{\tilde{w}}$. Thus $V_{\tilde{w}}$ is an affine subspace of V.

Now we prove that $V_{\tilde{w}}$ is nonempty. Assume $\tilde{w}^n = t^{n\nu_{\tilde{w}}}$ for some $n > 0$. Let $q \in V$. Set $p = \frac{1}{n}\sum_{i=0}^{n-1} \tilde{w}^i(q)$. Then $\tilde{w}(p) - p = \frac{1}{n}(\tilde{w}^n(p) - p) = \nu_{\tilde{w}}$. In particular, $V_{\tilde{w}} \neq \emptyset$.

For any $x \in V_{\tilde{w}}$, $||\tilde{w}^k(x) - \tilde{w}^{k-1}(x)|| = ||\tilde{w}(x) - x|| = ||\nu_{\tilde{w}}||$ and $\tilde{w}^n(x) - x = n\nu_{\tilde{w}}$. Hence $\tilde{w}^k(x) = \tilde{w}^{k-1}(x) + \nu_{\tilde{w}}$ for all $k \in \mathbb{Z}$. In particular, $\tilde{w}(x) = x + \nu_{\tilde{w}} \in V_{\tilde{w}}$. $\square$

Lemma 2.3. *Let $\tilde{w} \in \tilde{W}$. Then*

(1) For $v \in V$, grad$f_{\tilde{w}}(v) = 0$ if and only if $v \in V_{\tilde{w}}$.

(2) Let $C_{\tilde{w}} : V \times \mathbb{R} \to V$ denote the integral curve of the vector field grad$f_{\tilde{w}}$ with $C_{\tilde{w}}(v, 0) = v$ for all $v \in V$. Define

$$Lim : V \to V_{\tilde{w}}, \qquad v \mapsto \lim_{t \to -\infty} C_{\tilde{w}}(v,t) \in V_{\tilde{w}}.$$

Then $Lim : V \to V_{\tilde{w}}$ is a trivial vector bundle over $V_{\tilde{w}}$.

Proof. (1) Let $p \in V_{\tilde{w}}$. Set $T_p : V \to V$ by $v \mapsto v + p$. Define $\tilde{w}_p = T_p^{-1} \circ \tilde{w} \circ T_p$ and $V_{\tilde{w}_p} = T_p^{-1} V_{\tilde{w}} = \{v \in V;\ \tilde{w}_p(v) = v + \nu_{\tilde{w}}\}$. Then $V_{\tilde{w}_p} \subset V$ is a linear subspace. Let $V_{\tilde{w}_p}^{\perp} = \{v \in V; (v, V_{\tilde{w}_p}) = 0\}$ be its orthogonal complement.

Then $\tilde{w}_p(x+y) = \tilde{w}_p(x) + \tilde{w}_p(y) - \tilde{w}_p(0) = x + \tilde{w}_p(y)$ for any $x \in V_{\tilde{w}_p}$ and $y \in V_{\tilde{w}_p}^{\perp}$. Since $\tilde{w}_p$ is an isometry on V, we have that

$$\begin{aligned} ||x||^2 + ||y||^2 = ||x+y||^2 &= ||\tilde{w}_p(x+y) - \tilde{w}_p(0)||^2 \\ &= ||x||^2 + ||\tilde{w}_p(y) - \tilde{w}_p(0)||^2 + 2(x, \tilde{w}_p(y) - \tilde{w}_p(0)). \end{aligned}$$

In particular, $(x, \tilde{w}_p(y) - \tilde{w}_p(0)) = 0$ for all $x \in V_{\tilde{w}_p}$ and $y \in V_{\tilde{w}_p}^{\perp}$. Hence $\tilde{w}_p(y) - \tilde{w}_p(0) \in V_{\tilde{w}_p}^{\perp}$ for all $y \in V_{\tilde{w}_p}^{\perp}$. Let $M : V_{\tilde{w}_p}^{\perp} \to V_{\tilde{w}_p}^{\perp}$ be the linear transformation defined by $y \mapsto \tilde{w}_p(y) - \tilde{w}_p(0) - y$. By definition, $\ker M \subset V_{\tilde{w}_p} \cap V_{\tilde{w}_p}^{\perp} = \{0\}$ for $\tilde{w}_p(0) = \nu_{\tilde{w}}$. Hence M is invertible.

For $x \in V_{\tilde{w}_p}$ and $y \in V_{\tilde{w}_p}^{\perp}$,

$$\begin{aligned} f_{\tilde{w}_p}(x+y) = ||\tilde{w}_p(x+y) - (x+y)||^2 &= ||M(y)||^2 + ||\tilde{w}_p(0)||^2 \\ &= (y, {}^tMM(y)) + ||\nu_{\tilde{w}}||^2, \end{aligned}$$

where ${}^tM$ is the transpose of M with respect to the inner product $(,)$ on V. Thus grad$f_{\tilde{w}_p}(x+y) = 2{}^tMM(y) \in V_{\tilde{w}_p}^{\perp}$. Hence grad$f_{\tilde{w}_p}$ which vanishes exactly on $V_{\tilde{w}_p}$.

Notice that $f_{\tilde{w}_p} = f_{\tilde{w}} \circ T_p$ and T_p is an isometry. We have that grad$f_{\tilde{w}}(v) = $ grad$f_{\tilde{w}_p}(v - p)$ for any $v \in V$. Hence grad$f_{\tilde{w}}$ vanishes exactly on $V_{\tilde{w}}$.

(2) The integral curve of grad$f_{\tilde{w}_p}$ can be written explicitly as $C_{\tilde{w}_p}(x + y, t) = x + \exp(2t{}^tMM)(y)$ for any $x \in V_{\tilde{w}_p}$ and $y \in V_{\tilde{w}_p}^{\perp}$. Hence the integral curve $C_{\tilde{w}}(t, v)$ of grad$f_{\tilde{w}}$ is given by $C_{\tilde{w}}(x + y, t) = x +$

$\exp(2t{}^tMM)(y)$ for $x \in V_{\tilde{w}}$ and $y \in V_{\tilde{w}_p}^{\perp}$. Since ${}^tMM$ is self adjoint with positive eigenvalues, $\lim_{t\to-\infty} \exp(2t{}^tMM) = 0$. Hence $\mathrm{Lim}(x+y) = x$ for any $x \in V_{\tilde{w}}$ and $y \in V_{\tilde{w}_p}^{\perp}$. Thus Lim is a trivial vector bundle over $V_{\tilde{w}}$. □

Proposition 2.4. *Let $\tilde{w} \in \tilde{W}$ and A be an alcove. Then there exists an alcove A' such that $\bar{A}'$ contains a regular point of $V_{\tilde{w}}$ and $\tilde{w}_A \to \tilde{w}_{A'}$.*

Remark. The proof is similar to [14, Proposition 1.2]. The difference is that we consider here $D_{\tilde{w}} = \{v \in V; v \notin C_{\tilde{w}}(V^{\geqslant 2}, \mathbb{R}) \cup \mathrm{Lim}^{-1}(V_{\tilde{w}}^{\geqslant 1})\}$, where $V_{\tilde{w}}^{\geqslant 1} \subset V_{\tilde{w}}$ is the complement of the set of regular points of $V_{\tilde{w}}$ and $V^{\geqslant 2}$ be the complement of all alcoves and faces in V. We omit the details.

2.3. As a consequence, there exists a minimal length element in the conjugacy class of $\tilde{w}$ which is of the form $\tilde{w}_A$ for some alcove A with $V_{\tilde{w}} \cap \bar{A} \neq \emptyset$. However, not every minimal length element is of this form. Now we give an example.

Let W be the affine group of type $\tilde{A}_2$ with a set of simple reflections $\{s_1 = s_{\alpha_1}, s_2 = s_{\alpha_2}, s_0 = t^{\alpha_1^\vee + \alpha_2^\vee} s_1 s_2 s_1\}$, where α_1, α_2 are the simple roots. The corresponding fundamental coweight is denoted by $\omega_1^\vee, \omega_2^\vee$ respectively. Let $\delta \in \mathrm{Aut}W$ such that $\delta : s_1 \mapsto s_2, s_2 \mapsto s_1, s_0 \mapsto s_0$. Let $\tilde{w} = t^{2\alpha_1^\vee + 2\alpha_2^\vee}\delta$ and $A = t^{\alpha_1^\vee} s_1 s_2 \Delta$. Then $\ell(\tilde{w}) = \ell(\tilde{w}_A) = 8$ and $\tilde{w}_A$ is of minimal length in $W \cdot \tilde{w}$. Note that $V_{\tilde{w}} = \{v \in V; (v, \alpha_1^\vee - \alpha_2^\vee) = 0\}$. The vertices (extremal points) of $\bar{A}$ are $\omega_1^\vee$, $\omega_1^\vee - \omega_2^\vee$ and $2\omega_1^\vee - \omega_2^\vee$ which all lie in the same connected component of $V - V_{\tilde{w}}$. Hence $V_{\tilde{w}} \cap \bar{A} = \emptyset$.

2.4. Now we recall the "partial conjugation action" introduced in [10].

For $J \subset S$, we denote by W_J the standard parabolic subgroup of W generated by J and by ${}^J\tilde{W}$ the set of minimal coset representatives in $W_J \backslash \tilde{W}$.

For $\tilde{w} \in {}^J\tilde{W}$, set

$$I(J, \tilde{w}) = \max\{K \subset J; \tilde{w}(K) = K\}.$$

The following result is proved in [10, Section 2 & 3]. See also [11, Theorem 2.1].

Theorem 2.5. *Let $J \subset S$ such that W_J is finite. We consider the (partial) conjugation action of W_J on $\tilde{W}$. Let $\mathcal{O}$ be an orbit. Then*

(1) There exists $\tilde{w} \in {}^J\tilde{W}$, such that for any $\tilde{w}' \in \mathcal{O}$, there exists $x \in W_{I(J,\tilde{w})}$ such that $\tilde{w}' \to x\tilde{w}$.

(2) If $\tilde{w}', \tilde{w}'' \in \mathcal{O}_{\min}$, then $\tilde{w}' \sim \tilde{w}''$.

2.5. We'll show that $\tilde{w}_{A'}$ appeared in Proposition 2.4 is of the form xy for some $y \in {}^J\tilde{W}$ and $x \in W_{I(J,\tilde{w})}$. To do this, we introduce some more notations.

Let $K \subset V$ be a convex subset. Let $\mathfrak{H}_K = \{H \in \mathfrak{H}; K \subset H\}$ and $W_K \subset W$ be the subgroup generated by s_H with $H \in \mathfrak{H}_K$. For any two alcoves A and A', define $\mathfrak{H}_K(A, A') = \mathfrak{H}(A, A') \cap \mathfrak{H}_K$.

Let A be an alcove. We set $W_{K,A} = x_A^{-1} W_K x_A$ and $I(K,A) = \{s_H \in S; K \subset x_A H\}$. If $\bar{A}$ contains a regular point of K, then $W_{K,A} = W_{I(K,A)}$.

Lemma 2.6. *Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}$ be an affine subspace with $\tilde{w}(K) = K$. Let A be an alcove such that A and $\tilde{w}A$ are in the same connected component C of $V - \cup_{H \in \mathfrak{H}_K} H$. Assume furthermore that $\bar{A}$ contains an element $v \in K$ such that for each $H \in \mathfrak{H}$, $v, \tilde{w}(v) \in H$ implies that $K \subset H$. Then*

$$\ell(\tilde{w}_A) = \sharp \mathfrak{H}(A, \tilde{w}A) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle.$$

Here ρ is the half sum of the positive roots in Φ and $\bar{\nu}_{\tilde{w}}$ is the unique dominant element in the W_0-orbit of $\nu_{\tilde{w}}$.

Proof. By our assumption, $\tilde{w}$ fixes C. Hence $\mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A) \subset \mathfrak{H} - \mathfrak{H}_K$ for any $i, j \in \mathbb{Z}$. Since $v \in \bar{A}$ and $\tilde{w}(v) \in \tilde{w}\bar{A}$, any $H \in \mathfrak{H}(A, \tilde{w}A)$ intersects with the closed interval $[v, \tilde{w}(v)]$ at a single point.

If $\nu_{\tilde{w}} = 0$, then $\tilde{w}(v) = v$. For any $H \in \mathfrak{H}(A, \tilde{w}A)$, we have $v \in H$, hence $H \in \mathfrak{H}_K$. That is a contradiction. Hence $\mathfrak{H}(A, \tilde{w}A) = \emptyset$ and $\ell(\tilde{w}_A) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle = 0$.

Now we assume $\nu_{\tilde{w}} \neq 0$. Set $v_i = \tilde{w}^i(v) = v + i\nu_{\tilde{w}} \in K$ for $i \in \mathbb{Z}$. Then all the v_i span an affine line L. We prove that

(a) If $i < j$, then $\mathfrak{H}(\tilde{w}^{i-1}A, \tilde{w}^i A) \cap \mathfrak{H}(\tilde{w}^{j-1}A, \tilde{w}^j A) = \emptyset$.

Let $H \in \mathfrak{H}(\tilde{w}^{i-1}A, \tilde{w}^i A) \cap \mathfrak{H}(\tilde{w}^{j-1}A, \tilde{w}^j A)$. Then $H \cap L = H \cap [v_{i-1}, v_i] \cap [v_{j-1}, v_j] \neq \emptyset$. Thus $i = j-1$ and $v_i \in H$. Hence $H \in \mathfrak{H}(\tilde{w}^{i-1}A, \tilde{w}^j A)$. Therefore $\tilde{w}^i A$ and $\tilde{w}^j A$ are in the same connected component of $V - H$, that is, $H \notin \mathfrak{H}(\tilde{w}^{j-1}A, \tilde{w}^j A)$.

(a) is proved.

Now we prove that

(b) For $i < j$, $\mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A) = \bigcup_{k=i+1}^{j} \mathfrak{H}(\tilde{w}^{k-1}A, \tilde{w}^k A)$.

If $H \notin \bigcup_{k=i+1}^{j} \mathfrak{H}(\tilde{w}^{k-1}A, \tilde{w}^k A)$, then $\tilde{w}^i A, \tilde{w}^{i+1}A, \cdots, \tilde{w}^j A$ are all in the same connected component of $V - H$. Thus $H \notin \mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A)$.

Let $H \in \mathfrak{H}(\tilde{w}^{r-1}A, \tilde{w}^r A)$ for some $i < r \leq j$. Then $H \cap L = H \cap [v_{r-1}, v_r] = \{e\}$. If $e \notin \{v_i, v_j\}$, then $H \in \mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A)$. If $e = v_j$, then $H \in \mathfrak{H}(\tilde{w}^{j-1}A, \tilde{w}^j A)$ and v_{j-1} and v_i are in the same connected component of $V - H$. Hence $\tilde{w}^i A$ and $\tilde{w}^{j-1}A$ are in the same connected component of $V - H$, while $\tilde{w}^{j-1}A$ and $\tilde{w}^j A$ are in different connected components of $V - H$. Hence $H \in \mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A)$. If $e = v_i$, by a similar argument we have that $H \in \mathfrak{H}(\tilde{w}^i A, \tilde{w}^j A)$.

Let $n \in \mathbb{Z}$ such that $\tilde{w}^n = t^{n\nu_{\tilde{w}}}$. Then by (a) and (b), we have that

$$n \sharp \mathfrak{H}(A, \tilde{w}A) = \sum_{i=0}^{n-1} \sharp \mathfrak{H}(\tilde{w}^i A, \tilde{w}^{i+1}A) = \sharp \mathfrak{H}(A, \tilde{w}^n A) = \sharp \mathfrak{H}(A, t^{n\nu_{\tilde{w}}} A).$$

Hence $\ell(\tilde{w}_A) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$. $\square$

Proposition 2.7. *Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}$ be an affine subspace with $\tilde{w}(K) = K$. Let A be an alcove such that $\bar{A}$ contains a regular point v of K. Then $\tilde{w}_A = u\tilde{w}_{K,A}$ for some $u \in W_{I(K,A)}$ and $\tilde{w}_{K,A} \in {}^{I(K,A)}\tilde{W}^{I(K,A)}$ with $\ell(u) = \sharp\mathfrak{H}_K(A, \tilde{w}A)$, $\tilde{w}_{K,A}(I(K,A)) = I(K,A)$ and $\ell(\tilde{w}_{K,A}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho\rangle$.*

Proof. We may assume that A is the fundamental alcove Δ by replacing $\tilde{w}$ by $\tilde{w}_A$. We simply write I for $I(K, \Delta)$.

We have that $\tilde{w} = u'\tilde{w}'u''$ for some $u', u'' \in W_I$ and $\tilde{w}' \in {}^I\tilde{W}^I$. Since $\tilde{w}K = K$, then $\tilde{w}(\mathfrak{H}_K) = \mathfrak{H}_K$ and $\tilde{w}W_I\tilde{w}^{-1} = W_I$. Hence $\tilde{w}'W_I(\tilde{w}')^{-1} = W_I$ and $\tilde{w}'(I) = I$.

Let C be the connected component of $V - \cup_{H\in\mathfrak{H}_K} H$ that contains Δ. We claim that $\tilde{w}'(\Delta) \subset C$. Otherwise, there exists $H \in \mathfrak{H}_K$ separating Δ and $\tilde{w}'(\Delta)$. Hence $\ell(s_H\tilde{w}') < \ell(\tilde{w}')$. This contradicts our assumption that $\tilde{w}' \in {}^I\tilde{W}$. Hence $\ell(u) = \sharp\mathfrak{H}_K(\Delta, \tilde{w}\Delta)$ and $\mathfrak{H}_K(\Delta, \tilde{w}'(\Delta)) = \emptyset$.

Since W_I is a finite group and the conjugation by $\tilde{w}$ is a group automorphism on W_K, there exists $n > 0$ such that

$$(\tilde{w}')^n\tilde{w}^{-n} = u^{-1}(\tilde{w}u^{-1}\tilde{w}^{-1})\cdots(\tilde{w}^{n-1}u^{-1}\tilde{w}^{-n+1}) = 1.$$

Hence $(\tilde{w}')^n = \tilde{w}^n$ and there exists $m > 0$ such that $mn\nu_{\tilde{w}} \in Q$ and $(\tilde{w}')^{mn} = \tilde{w}^{mn} = t^{mn\nu_{\tilde{w}}}$.

Note that v and $\tilde{w}(v) = \tilde{w}'(v) = v + \nu_{\tilde{w}}$ are regular points in K. Applying Lemma 2.6, we have $\ell(\tilde{w}') = \langle\bar{\nu}_{\tilde{w}'}, 2\rho\rangle = \langle\bar{\nu}_{\tilde{w}}, 2\rho\rangle$. □

Corollary 2.8. *Let $\tilde{w} \in \tilde{W}$ be of minimal length in its conjugacy class. Then $\tilde{w}$ is of finite order if and only if $\tilde{w} \in W_J \rtimes \langle\delta\rangle$ for some proper subset J of S and $\delta \in \Omega$ with $\delta(J) = J$ such that the corresponding parabolic subgroup W_J is finite.*

Proof. The "if" part is clear.

Now assume that $\tilde{w}$ is of finite order. Let $K = V_{\tilde{w}}$. By Proposition 2.4 and 2.7, there exists an alcove A such that $\tilde{w} \approx \tilde{w}_A$ and $\tilde{w}_A = u\tilde{w}_{K,A}$ for some $u \in W_{I(K,A)}$ and $\tilde{w}_{K,A} \in {}^{I(K,A)}W$ with $\tilde{w}_{K,A}(I(K,A)) = I(K,A)$ and $\ell(\tilde{w}_{K,A}) = \langle\bar{\nu}_{\tilde{w}}, 2\rho\rangle$. Since $\tilde{w}$ is of finite order, $\nu_{\tilde{w}} = 0$ and $\ell(\tilde{w}_{K,A}) = 0$. So $\tilde{w}_{K,A} \in \Omega$.

By definition, $I(K,A)$ is a subset of S such that $W_{I(K,A)}$ is finite. We have that $\tilde{w}_A \in W_{I(K,A)}\tilde{w}_{K,A}$ and $\tilde{w} \approx \tilde{w}_A$. Hence $\tilde{w} \in W_{I(K,A)}\tilde{w}_{K,A}$. □

Now we may prove the main result of this section, generalizing §0.1 (1) and (2) to affine Weyl groups.

Theorem 2.9. *Let $\mathcal{O}$ be a W-conjugacy class in $\tilde{W}$ and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each element $\tilde{w}' \in \mathcal{O}$, there exists $\tilde{w}'' \in \mathcal{O}_{\min}$ such that $\tilde{w}' \to \tilde{w}''$.

(2) Let $\tilde{w}', \tilde{w}'' \in \mathcal{O}_{\min}$, then $\tilde{w}' \sim \tilde{w}''$.

Proof. (1) We fix an element $\tilde{w}$ of $\mathcal{O}$. Set $K = V_{\tilde{w}}$. Then any element in $\mathcal{O}$ is of the form $\tilde{w}_{A'}$ for some alcove A'. By Proposition 2.4, $\tilde{w}_{A'} \to \tilde{w}_A$ for some alcove A such that $\bar{A}$ contains a regular point of K.

If C is a connected component of $V - \cup_{H \in \mathfrak{H}_K} H$, then $\tilde{w}(C)$ is also a connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. We denote by $\ell(C)$ the number of hyperplanes in $\mathfrak{H}_K$ that separate C and $\tilde{w}(C)$. By definition, if C is the connected component that contains A, then $\ell(C) = \sharp \mathfrak{H}_K(A, \tilde{w}A)$. Now by Proposition 2.7, $\ell(\tilde{w}_A) = \ell(C) + \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$.

Let C_0 be a connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ such that $\ell(C_0)$ is minimal among all the connected components of $V - \cup_{H \in \mathfrak{H}_K} H$. Then $\ell(\tilde{w}_{A'}) \geqslant \ell(\tilde{w}_A) \geqslant \ell(C_0) + \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$. In particular, let A_0 be the alcove in C_0 such that $A_0 = uA$ for some $u \in W_K$, then $\bar{A}_0$ contains a regular point of K and $\tilde{w}_{A_0} \in \mathcal{O}_{\min}$.

We have that $\tilde{w}_{A_0} = u' \tilde{w}_A (u')^{-1}$ for some $u' \in W_{K,A}$ and $\tilde{w}_{A_0}$ is a minimal length element in the $W_{K,A}$-conjugacy class $\mathcal{O}' = \{x \tilde{w}_A x^{-1}; x \in W_{K,A}\} \subset \mathcal{O}$. Hence by Theorem 2.5, there exists $\tilde{w}'' \in \mathcal{O}'_{\min}$ such that $\tilde{w}_{A'} \to \tilde{w}_A \to \tilde{w}'' \sim \tilde{w}_{A_0}$. Since $\tilde{w}_{A_0} \in \mathcal{O}_{\min}$, $\tilde{w}'' \in \mathcal{O}_{\min}$. Part (1) is proved.

(2) Let $\tilde{w}' \in \mathcal{O}_{\min}$. We have showed that there exists an alcove $A'_0 \subset C_0$ such that $\bar{A}'_0$ contains a regular point of K and $\tilde{w}' \sim \tilde{w}_{A'_0}$. Now it suffices to prove that $\tilde{w}_{A_0} \sim \tilde{w}_{A'_0}$.

Let $\mathcal{A}_{C_0}$ be the set of all alcoves in C_0 whose closures contain regular points of K. Then $\cup_{A \in \mathcal{A}_{C_0}} \bar{A} \supset K$. Hence there exists a finite sequence of alcoves $A = A_0, \cdots, A_r = A'_0 \in \mathcal{A}_{C_0}$ such that $K_i = \bar{A}_i \cap \bar{A}_{i+1} \cap K \neq \emptyset$ for all $0 \leqslant i < r$. Then there exists $u_i \in W_{K_i}$ such that $A_{i+1} = u_i A_i$. Hence $\tilde{w}_{A_{i+1}} = u'_i \tilde{w}_{A_i} (u'_i)^{-1}$ for some $u'_i \in W_{K_i, A_i}$. Notice that $\tilde{w}_{A_{i+1}}$ and $\tilde{w}_{A_i}$ are minimal length elements in $\{x \tilde{w}_{A_i} x^{-1}; x \in W_{K_i, A_i}\}$. (Actually by the proof of (i), they are of minimal lengths in $\mathcal{O}$.) Thus by Theorem 2.5, $\tilde{w}_{A_{i+1}} \sim \tilde{w}_{A_i}$. Therefore $\tilde{w}_{A_0} \sim \tilde{w}_{A'_0}$. □

As a consequence, we have a similar result for any conjugacy class of $\tilde{W}$, which is a union of W-conjugacy classes.

Corollary 2.10. *Let $\mathcal{O}$ be a conjugacy class of $\tilde{W}$ and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each element $\tilde{w}' \in \mathcal{O}$, there exists $\tilde{w}'' \in \mathcal{O}_{\min}$ such that $\tilde{w}' \to \tilde{w}''$.

(2) Let $\tilde{w}', \tilde{w}'' \in \mathcal{O}_{\min}$, then $\tilde{w}' \tilde{\sim} \tilde{w}''$.

3. Straight conjugacy class

3.1. Following [18], we call an element $\tilde{w} \in \tilde{W}$ a *straight element* if for any $m \in \mathbb{N}$, $\ell(\tilde{w}^m) = m\ell(\tilde{w})$. We call a conjugacy class *straight* if it contains some straight element. It is easy to see that $\tilde{w}$ is straight if and only if $\ell(\tilde{w}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$ (see [11]).

By definition, any basic element of $\tilde{W}$ is straight. Also t^λ is also straight with $\lambda \in Q$. In Proposition 3.1, we'll give some nontrivial examples of straight elements.

3.2. We follow [25, 7.3]. Let $\delta \in \Omega$. For each δ-orbit in S, we pick a simple reflection. Let g be the product of these simple reflections (in any order) and put $c = (g, \delta) \in W \rtimes \langle \delta \rangle$. We call c a *twisted Coxeter element* of $\tilde{W}$. The following result will be used in [15] in the study of basic locus of Shimura varieties.

Proposition 3.1. *Let c be a twisted Coxeter element of $\tilde{W}$. Then c is a straight element.*

Remark. The case where $\delta = 1$ (for any Coxeter group of infinite order) was first obtained by Speyer in [24]. Our method here is different from loc.cit.

Proof. Assume that $c \in W\delta$ for $\delta \in \Omega$. By Proposition 2.4 and 2.7, $c \to ux$ for some $I \subset S$ with W_I finite, $u \in W_I$, and a straight element x with $x(I) = I$. It is easy to see that ux is also a twisted Coxeter element and $c \approx ux$. In particular, $x = w\delta$ for some $w \in W_{S-I}$. For any $s \in I$, $w\delta(s)w^{-1} \in I$. Hence $\delta(s) \in I$ and commutes with w. So $\delta(I) = I$ and $\delta(S - I) = S - I$. Since ux is a twisted Coxeter element of $\tilde{W}$, $x = w\delta$ is a twisted Coxeter element of $W_{S-I} \rtimes \langle \delta \rangle$. On the other hand, w commutes with any element in I. Thus I is a union of connected components of the Dynkin diagram of S. Hence $I = \emptyset$ since W_I is finite. So $u = 1$ and $c \approx x$ is also a straight element. □

3.3. We'll give some algebraic and geometric criteria for straight conjugacy classes. In order to do this, we first make a short digression and discuss another description of $\tilde{W}$.

Let G be a connected complex reductive algebraic group and $T \subset G$ be a maximal torus of G. Let W_0 be the finite Weyl group of G and S_0 the set of simple roots. We denote by Q (resp. P) the coroot lattice (resp. coweight lattice) of T in G. Then $W_G = Q \rtimes W_0$ is an affine Weyl group in 1.3. Set $\tilde{W}_G = P \rtimes W_0$. For the group Ω' of diagram automorphisms of S_0 that induces an action on G, we set $\tilde{W}_{G,\Omega'} = \tilde{W}_G \rtimes \Omega'$. Then $\tilde{W}_{G,\Omega'} = W_G \rtimes \Omega$ for some $\Omega \subset \mathrm{Aut}(W_G, S)$ with $\Omega(S) = S$. It is easy to check that for any affine Weyl group W, $W \rtimes \mathrm{Aut}(W, S) = \tilde{W}_{G,\Omega'}$. Here G is the corresponding semisimple group of adjoint type and Ω' is the group of diagram automorphisms on S_0.

For any $J \subset S_0$, set $\Omega'_J = \{\delta \in \Omega'; \delta(J) = J\}$ and

$$\tilde{W}_J = (P \rtimes W_J) \rtimes \Omega'_J.$$

We call an element in $\tilde{W}_J$ *basic* if it is of length 0 with respect to the length function on $\tilde{W}_J$.

In the rest of this section, we assume that $W = W_G$ and $\tilde{W} = \tilde{W}_{G,\Omega'}$ unless otherwise stated.

Proposition 3.2. *Let $\mathcal{O}$ be a W_G-conjugacy class of $\tilde{W}$. Then the following conditions are equivalent:*

(1) $\mathcal{O}$ is straight;

(2) For some (or equivalently, any) $\tilde{w} \in \mathcal{O}$, $V_{\tilde{w}} \nsubseteq H$ for any $H \in \mathfrak{H}$;

(3) $\mathcal{O}$ contains a basic element of $\tilde{W}_J$ for some $J \subset S_0$.

In this case, there exist a basic element x in $\tilde{W}_{J_{\mathcal{O}}}$ and $y \in W_0^{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $yxy^{-1} \in \mathcal{O}_{\min}$. Here $\nu_{\mathcal{O}} = \bar{\nu}_{\tilde{w}}$ for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$ and $J_{\mathcal{O}} = \{i \in S_0; \langle \nu_{\mathcal{O}}, \alpha_i \rangle = 0\}$.

Proof. (1) $\Leftrightarrow$ (2). By Proposition 2.4 and Proposition 2.7, there is an alcove A such that $\bar{A}$ contains a regular point of $V_{\tilde{w}}$ and $\tilde{w}_A \in \mathcal{O}_{\min}$. Moreover $\ell(\tilde{w}_A) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle + \sharp \mathfrak{H}_{V_{\tilde{w}}}(A, \tilde{w}A)$.

If $\mathfrak{H}_{V_{\tilde{w}}} = \emptyset$, then $\mathfrak{H}_{V_{\tilde{w}}}(A, \tilde{w}A) = \emptyset$. Hence $\ell(\tilde{w}_A) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle$ and $\tilde{w}$ is a straight element.

If $\mathcal{O}$ is straight, then $\sharp \mathfrak{H}_{V_{\tilde{w}}}(A, \tilde{w}A) = 0$, that is, $\tilde{w}$ fixes the connected component C of $V - \cup_{H \in \mathfrak{H}_{V_{\tilde{w}}}} H$ containing A. Choose $v \in C$ and set $y = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{w}^k(v)$, where $n \in \mathbb{N}$ with $\tilde{w}^n = t^{n\nu_{\tilde{w}}}$. Since C is convex, we have $y \in C \cap V_{\tilde{w}}$, which forces $\mathfrak{H}_{V_{\tilde{w}}}$ to be empty.

(3) $\Rightarrow$ (2). Denote by $\Phi_J \subset \Phi$ the set of roots spanned by α_i for $i \in J$. Assume $\tilde{w} = t^\chi w \delta' \in \mathcal{O}$ is a basic element in $\tilde{W}_J$. Then it is a straight element in $\tilde{W}_J$. By condition (2) for $\tilde{W}_J$, $V_{\tilde{w}} \nsubseteq H_{\alpha,k}$ for any $\alpha \in \Phi_J$ and $k \in \mathbb{Z}$.

Let $\mu \in V$ with $\langle \mu, \alpha_i \rangle = \begin{cases} 0, & \text{if } i \in J \\ 1, & \text{if } i \in S_0 - J \end{cases}$. Since $\delta'(J) = J$, then $\delta'(\mu) = \mu$. Hence $w\delta'(\mu) = \mu$ and $\mathbb{R}\mu + V_{\tilde{w}} = V_{\tilde{w}}$. Therefore $\langle V_{\tilde{w}}, \alpha \rangle = \mathbb{R}$ for any $\alpha \in \Phi - \Phi_J$. Thus $V_{\tilde{w}} \nsubseteq H_{\alpha,k} \in \mathfrak{H}$ with $\alpha \in \Phi - \Phi_J$ and $k \in \mathbb{Z}$.

(1) $\Rightarrow$ (3). By Proposition 2.7 and Condition (2) there exists $\tilde{w} \in \mathcal{O}_{\min}$ such that Δ contains a regular point e of $V_{\tilde{w}}$. Let $y \in W_0^{J_{\mathcal{O}}}$ with $\nu_{\mathcal{O}} = y^{-1}(\nu_{\tilde{w}})$. Set $x = y^{-1}\tilde{w}y$. Then $\nu_x = \nu_{\mathcal{O}}$ is dominant.

Assume that $x = t^\chi w \delta' \in \mathcal{O}$ with $\chi \in P$, $w \in W_0$ and $\delta' \in \Omega'$. Let $n \in \mathbb{N}$ with $x^n = t^{n\nu_{\mathcal{O}}}$. Then

$$t^{n\nu_{\mathcal{O}}+\chi} w \delta' = t^{n\nu_{\mathcal{O}}} x = x t^{n\nu_{\mathcal{O}}} = t^{w\delta'(\nu_{\mathcal{O}})+\chi} w \delta'.$$

Thus $\nu_{\mathcal{O}} = w\delta'(\nu_{\mathcal{O}})$ is the unique dominant element in $W_0 \cdot \delta'(\nu_{\mathcal{O}})$. Hence $\delta'(\nu_{\mathcal{O}}) = \nu_{\mathcal{O}}$ and $w\nu_{\mathcal{O}} = \nu_{\mathcal{O}}$. Therefore $w \in W_{J_{\mathcal{O}}}$ and $\delta'(J_{\mathcal{O}}) = J_{\mathcal{O}}$. Hence $x \in \tilde{W}_{J_{\mathcal{O}}}$.

Let C be the connected component of $V - \cup_k \cup_{\alpha \in \Phi_J} H_{\alpha,k}$ that contains Δ. Since $y \in W_0^{J_{\mathcal{O}}}$, for any $\alpha \in \Phi^+_{J_{\mathcal{O}}}$, $y\alpha \in \Phi^+$ and $0 < (y^{-1}(e), \alpha) = (e, y(\alpha)) < 1$. Hence $y^{-1}(e) \in C$. Moreover,

$$xy^{-1}(e) = y^{-1}\tilde{w}(e) = y^{-1}(e + \nu_{\tilde{w}}) = y^{-1}(e) + \nu_{\mathcal{O}}.$$

Since $\langle \nu_{\mathcal{O}}, \alpha \rangle = 0$ for all $\alpha \in \Phi_J$, we have $y^{-1}(e)$ and $y^{-1}(e) + \nu_{\mathcal{O}}$ are contained in the same connected component of $V - \cup_k \cup_{\alpha \in \Phi_J} H_{\alpha,k}$. Hence

$C \ni y^{-1}(e)$ and $xC \ni xy^{-1}(e)$ are the same connected component of $V - \cup_k \cup_{\alpha \in \Phi_J} H_{\alpha,k}$. Thus there is no hyperplane of the form $H_{\alpha,k}$ with $\alpha \in \Phi_J$ that separates C from xC. So x is a basic element in $\tilde{W}_J$. $\square$

3.4. The next task of this section is to give a parametrization of straight conjugacy classes. Such parametrization coincides with the set of σ-conjugacy classes of p-adic groups [12].

Let P^+ be the set of dominant coweights of G and

$$P^+_{\mathbb{Q}} = \{\lambda \in P \otimes_{\mathbb{Z}} \mathbb{Q}; \langle \lambda, \alpha \rangle \geqslant 0 \text{ for all } \alpha \in \Phi^+\} \subset V.$$

Then we may identify $P^+_{\mathbb{Q}}$ with $(P \otimes_{\mathbb{Z}} \mathbb{Q})/W_0$. For any $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{Q}$, we denote by $\bar{\lambda}$ the unique element in $P^+_{\mathbb{Q}}$ that lies in the W_0-orbit of λ. The group Ω' acts naturally on $P^+_{\mathbb{Q}}$ and on $\tilde{W}_G/W_G \cong P/Q$. Let $\delta' \in \Omega'$.

For $x \in \tilde{W}_G \delta'$, we call $\bar{\nu}_x$ the *dominant Newton point* of x. The map $x \mapsto (x\delta'^{-1} W_G, \bar{\nu}_x)$ induces a natural map

$$f_{\delta'} : \tilde{W}_G \delta' \to (P/Q)_{\delta'} \times P^+_{\mathbb{Q}}.$$

Here $\tilde{W}_G \delta' \in \tilde{W}_G \backslash \tilde{W}$ is a right $\tilde{W}_G$-coset containing δ' and $(P/Q)_{\delta'}$ is the δ'-coinvariants of P/Q. We denote the image by $B(\tilde{W}_G, \delta')$.

Theorem 3.3. *The map $f_{\delta'}$ induces a bijection between the straight $\tilde{W}_G$-conjugacy classes of $\tilde{W}_G \delta'$ and $B(\tilde{W}_G, \delta')$.*

Proof. We first show that

(a) The map $f_{\delta'}$ is constant on each $\tilde{W}_G$-conjugacy class.

Let $\tilde{w} = t^\chi w \delta' \in \tilde{W}$ and $\tilde{u} = t^\lambda u \in \tilde{W}_G$, where $\chi, \lambda \in P$ and $w, u \in W_0$. Then $\tilde{u}\tilde{w}\tilde{u}^{-1} = t^{\lambda + u\chi - (uw\delta' u^{-1}(\delta')^{-1})\delta' \lambda}(uw\delta' u')$. Notice that for any $x \in W_0$ and $\mu \in P$, $x\mu - \mu \in Q$. Hence $t^{\lambda + u\chi - (uw\delta' u^{-1}(\delta')^{-1})\delta' \lambda} \in t^{\lambda + \chi - \delta'(\lambda)} W_G$ and

$$\tilde{u}\tilde{w}\tilde{u}^{-1} \in t^{\lambda + \chi - \delta'(\lambda)} W_G (uw\delta' u'(\delta')^{-1})\delta' = t^{\lambda + u - \delta'(\lambda)} \delta' W_G.$$

Hence the images of $\tilde{u}\tilde{w}\tilde{u}^{-1}$ and $\tilde{w}$ in $(P/Q)_{\delta'}$ are the same. Assume that $n \in \mathbb{N}$ and $\tilde{w}^n = t^{n\nu_{\tilde{w}}}$. Then $(\tilde{u}\tilde{w}\tilde{u}^{-1})^n = \tilde{u} t^{n\nu_{\tilde{w}}} \tilde{u}^{-1} = t^\lambda t^{un\nu_{\tilde{w}}} t^{-\lambda} = t^{un\nu_{\tilde{w}}}$. Therefore $\nu_{\tilde{u}\tilde{w}\tilde{u}^{-1}} = u(\nu_{\tilde{w}})$ and $\bar{\nu}_{\tilde{u}\tilde{w}\tilde{u}^{-1}} = \bar{\nu}_{\tilde{w}}$.

(a) is proved.

Moreover, $t^{n\nu_{\tilde{w}}} = \tilde{w} t^{n\nu_{\tilde{w}}} \tilde{w}^{-1} = t^\chi t^{wn\delta'(\nu_{\tilde{w}})} t^{-\chi} = t^{wn\delta'(\nu_{\tilde{w}})}$. Thus $\nu_{\tilde{w}} = w\delta'(\nu_{\tilde{w}})$. Hence

(b) $\bar{\nu}_{\tilde{w}} = \delta'(\bar{\nu}_{\tilde{w}})$ for all $\tilde{w} \in \tilde{W}_G \delta'$.

By Proposition 2.4 and 2.7, for any $\tilde{w} \in \tilde{W}$, $\tilde{w} \to u\tilde{w}_I$ for some $I \subset S$ with W_I finite, $u \in W_I$, and a straight element $\tilde{w}_I$ with $\tilde{w}_I(I) = I$. By the proof of [11, Proposition 2.2], $f_{\delta'}(\tilde{w}) = f_{\delta'}(u\tilde{w}_I) = f_{\delta'}(\tilde{w}_I)$. So $f_{\delta'}$ is surjective.

Now we prove that $f_{\delta'}$ is injective.

Let $\tilde{w}, \tilde{w}' \in \tilde{W}_G\delta'$ with $f_{\delta'}(\tilde{w}) = f_{\delta'}(\tilde{w}')$. Assume $\tilde{w} = t^\lambda w\delta'$ and $\tilde{w}' = t^{\lambda'} w'\delta'$ for some $\lambda, \lambda' \in P$, $w, w' \in W_G$. Then after conjugating by a suitable element of $\tilde{W}_G$, we can assume further that $\tilde{w}W_G = \tilde{w}'W_G$.

Let $J = \{i \in S_0; \langle \bar{\nu}_{\tilde{w}}, \alpha_i \rangle = 0\}$. By (b), $\bar{\nu}_{\tilde{w}} = \delta'(\bar{\nu}_{\tilde{w}})$. Then $\delta'(J) = J$. By Proposition 3.2, after conjugating by some elements in W_G, we may assume that $\tilde{w}, \tilde{w}' \in \tilde{W}_J$ and $\nu = \nu_{\tilde{w}} = \nu_{\tilde{w}'} \in P_{\mathbb{Q}}^+$.

Let $J' = S_0 - J$ and Q_J, $Q_{J'}$ be the sublattices of Q spanned by simple roots of J and J' respectively. Then

$$V = P \otimes_{\mathbb{Z}} \mathbb{R} = Q_J \otimes_{\mathbb{Z}} \mathbb{R} \oplus Q_{J'} \otimes_{\mathbb{Z}} \mathbb{R}.$$

We may write λ and λ' as $\lambda = a_J + a_{J'}$ and $\lambda' = a'_J + a'_{J'}$ with $a_J, a'_J \in Q_J \otimes_{\mathbb{Z}} \mathbb{R}$ and $a_{J'}, a'_{J'} \in Q_{J'} \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\lambda - \lambda' \in Q$, $a_J - a'_J \in Q_J$ and $a_{J'} - a'_{J'} \in Q_{J'}$.

Choose $n \in \mathbb{N}$ such that $(w\delta')^n = (w'\delta')^n = 1$. Then

$$\nu = \frac{1}{n} \sum_{k=0}^{n-1} (w\delta')^k(\lambda) \in \frac{1}{n} \sum_{k=0}^{n-1} (\delta')^k(a_{J'}) + Q_J \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Similarly,

$$\nu \in \frac{1}{n} \sum_{k=0}^{n-1} (\delta')^k(a'_{J'}) + Q_J \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Hence

$$\sum_{k=0}^{n-1} (\delta')^k(a_{J'} - a'_{J'}) = 0.$$

Since $a_{J'} - a'_{J'} \in Q_{J'}$, then $a_{J'} - a'_{J'} = \theta - \delta'(\theta)$ for some $\theta \in Q_{J'}$. Let $\tilde{w}'' = t^\theta \tilde{w}' t^{-\theta}$. By Condition (2) of Theorem 3.2, $\tilde{w}'$ and $\tilde{w}''$ are conjugate to basic elements in $\tilde{W}_J$ by elements in $Q_J \rtimes W_J$. Moreover, $\lambda \in \lambda' + \theta - \delta'(\theta) + Q_J$ and $(Q_J \rtimes W_J)\tilde{w} = (Q_J \rtimes W_J)\tilde{w}'' \in (Q_J \rtimes W_J)\backslash \tilde{W}_J$. Thus $\tilde{w}$ and $\tilde{w}''$ are conjugate to the same basic element of $\tilde{W}_J$ by an element in $Q_J \rtimes W_J$ and $\tilde{w}$ and $\tilde{w}'$ are in the same $\tilde{W}_G$-conjugacy class. $\square$

Combining Proposition 2.4, Proposition 2.7 and the proof of Theorem 2.9, any $\tilde{W}_G$-conjugacy class of $\tilde{W}_G\delta'$ can be "reduced" to the unique straight conjugacy class in the same fiber of $f_{\delta'}$ as follows.

Theorem 3.4. *Let $\mathcal{O}$ be a $\tilde{W}_G$-conjugacy class of $\tilde{W}_G\delta'$ and $\tilde{w} \in \mathcal{O}$. Then there exists $\tilde{w}' \in \mathcal{O}_{\min}$ such that*

(1) $\tilde{w} \to \tilde{w}'$;

(2) There exists a straight element $x \in \tilde{W}_G\delta'$ with $f_{\delta'}(x) = f_{\delta'}(\tilde{w})$, a subset J of S with W_J finite, $x \in {}^J(\tilde{W}_G\delta')$ and $x^{-1}(J) = J$, and an element $u \in W_J$ such that $\tilde{w}' = ux$.

3.5. Let $\tau \in \Omega$. Conjugation by τ gives a permutation on the set of affine simple reflections S of W. We say that τ is *superbasic* if each orbit is a union of connected components of the Dynkin diagram of S.

In this case, any two vertices in the same connected components of S have the same numbers of edges and thus S is a union of affine Dynkin diagrams of type $\tilde{A}$. Hence it is easy to see that $\tau \in \Omega$ is a superbasic element of $\tilde{W}$ if and only if $W = W_1^{m_1} \times \cdots \times W_l^{m_l}$, where W_i is an affine Weyl group of type $\tilde{A}_{n_i-1}$ and τ gives an order $n_i m_i$ permutation on $W_i^{m_i}$.

3.6. We follow the notation in §3.4. Let $\delta' \in \Omega'$. Then any fiber of the map $f_{\delta'} : \tilde{W}_G \delta' \to (P/Q)_{\delta'} \times P_{\mathbb{Q}}^+$ is a union of $\tilde{W}_G$-conjugacy classes. We call a $\tilde{W}_G$-conjugacy class in $\tilde{W}_G \delta'$ *superstraight* if it is a fiber of $f_{\delta'}$. By Theorem 3.3, any fiber contains a straight $\tilde{W}_G$-conjugacy class. Hence a superstraight conjugacy class is in particular straight. Now we give a description of superstraight $\tilde{W}_G$-conjugacy classes which is analogous to Proposition 3.2.

Proposition 3.5. *We keep notations in §3.3. Let $\mathcal{O}$ be a $\tilde{W}_G$-conjugacy class of $\tilde{W}$. Then the following are equivalent:*

(1) $\mathcal{O}$ is a superstraight.

(2) For some (or, equivalently any) $\tilde{w} \in \mathcal{O}$, $H \cap V_{\tilde{w}} = \emptyset$ for any $H \in \mathfrak{H}(\nu_{\tilde{w}})$. Here $H \in \mathfrak{H}(\nu_{\tilde{w}}) = \{H_{\alpha,k} \in \mathfrak{H}; \langle \nu_{\tilde{w}}, \alpha \rangle = 0, k \in \mathbb{Z}\}$.

(3) There exists a superbasic element x in $\tilde{W}_{J_{\mathcal{O}}}$ and $y \in W_G^{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $yxy^{-1} \in \mathcal{O}_{\min}$.

Proof. We assume that $\mathcal{O} \subset \tilde{W}_G \delta'$ for some $\delta' \in \Omega'$.

(1)$\Rightarrow$(3). By Proposition 3.2, there exists a basic element x in $\tilde{W}_{J_{\mathcal{O}}}$ and $y \in W_G^{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $yxy^{-1} \in \mathcal{O}_{\min}$. Assume that x is not superbasic in $\tilde{W}_{J_{\mathcal{O}}}$. Then there exists an x-orbit O such that $C \cap O \subsetneq C$ for each connected component C of the Dynkin diagram $\tilde{W}_{J_{\mathcal{O}}}$.

Note that $C \cap O \subsetneq C$ is the Dynkin diagram of a finite Weyl group. Hence W_O is a finite product of Weyl groups corresponding to $C \cap O$ and hence is finite. By the proof of [11, Proposition 2.2], $f_{\delta'}(wx) = f_{\delta'}(x)$ for all $w \in W_O$. In particular, $s_j x$ and x are in the same fiber of $f_{\delta'}$ for any $j \in O$.

However, $\ell(s_j x) \equiv \ell(x) + 1 \mod 2$. Thus $s_j x$ and x are not in the same conjugacy class. So $\mathcal{O}$ is not superstraight.

(2)$\Rightarrow$(1). Note that $H \in \mathfrak{H}(\nu_{\tilde{w}})$ if $V_{\tilde{w}} \subset H \in \mathfrak{H}$. Hence by Proposition 3.2 (2), $\mathcal{O}$ is straight. Let $\mathcal{O}'$ be another $\tilde{W}_G$-conjugacy class such that $\mathcal{O}'$ and $\mathcal{O}$ are in the same fiber of $f_{\delta'}$. By Proposition 2.4 and Proposition 2.7, $\mathcal{O}'$ contains an element of the form ux, where x is straight, $f_{\delta'}(ux) = f_{\delta'}(x)$ and $u \in W_{V_{ux}}$. By Theorem 3.3, $x \in \mathcal{O}$. By the proof of [11, Proposition 2.2], $\nu_{ux} = \nu_x$.

Let $v \in V_{ux}$. By Lemma 2.2, $ux(v) = v + \nu_{ux} = v + \nu_x \in V_{ux}$. Since $u \in W_{V_{ux}}$, $x(v) = u^{-1}ux(v) = ux(v) = v + \nu_x$ and $v \in V_x$. Thus $V_{ux} \subset V_x$.

Let $H \in \mathfrak{H}_{V_{ux}} \subset \mathfrak{H}(\nu_x)$. Then by our assumption, $H \cap V_{ux} \subset H \cap V_x = \emptyset$, which forces $\mathfrak{H}_{V_{ux}} = \emptyset$. Hence $W_{V_{ux}} = \{1\}$ and $u = 1$. So $\mathcal{O}' = \mathcal{O}$.

(3)⇒(2). Let C be the unique connected component of $V - \cup_{H \in \mathfrak{H}(\nu_{\mathcal{O}})} H$ containing Δ. We call $H \in \mathfrak{H}(\nu_{\mathcal{O}})$ a wall of C if $H \cap \bar{C}$ spans H. Let $\mathfrak{H}(C)$ be the set of walls of C. Note that $xC = C$ for x is basic. Since C is convex, $V_x \cap C \neq \emptyset$.

Suppose that $V_x \cap H' \neq \emptyset$ for some $H' \in \mathfrak{H}(\nu_{\mathcal{O}})$. Let $p \in V_x \cap H'$ and $q \in V_x \cap C$. Then the affine line $L(p, q) \subset V_x$ intersects with the boundary $\partial \bar{C} \subset \cup_{H \in \mathfrak{H}(C)} H$ of $\bar{C}$. Choose $v \in L(p, q) \cap \partial \bar{C}$. Then $v \in H_0$ for some $H_0 \in \mathfrak{H}(C)$. Thus $x^m(v) = v + m\nu_{\mathcal{O}} \in x^m H_0$ for $m \in \mathbb{Z}$. Notice that $\mathbb{R}\nu_{\mathcal{O}} + H = H$ for all $H \in \mathfrak{H}(\nu_{\mathcal{O}})$. Thus $v \in x^m H_0$ for all $m \in \mathbb{Z}$. As x is superbasic, the orbit $O = \{x^i H_0; i \in \mathbb{Z}\}$ is a union of connected components of the Dynkin diagram of $\{s_H; H \in \mathfrak{H}(C)\}$. Hence $v \in \cap_{H \in O} H = \emptyset$. That is a contradiction. □

3.7. In the rest of this section, we'll show that any two straight elements in the same conjugacy class are conjugate by cyclic shift, which is analogous to §0.1 (3) for an elliptic conjugacy class of a finite Coxeter group.

In order to do this, we use the following length formula. The proof is similar to [14, Proposition 2.3] and is omitted here.

Proposition 3.6. *Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}$ be an affine subspace with $\tilde{w}(K) = K$. Let A and A' be two alcoves in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. Assume that $\bar{A} \cap \bar{A}' \cap K$ spans a codimension 1 subspace of K of the form $H_0 \cap K$ for some $H_0 \in \mathfrak{H}$ and $\tilde{w}(H_0 \cap K) \neq H_0 \cap K$. Then*

$$\ell(\tilde{w}_A) = \ell(\tilde{w}_{A'}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho \rangle + \sharp \mathfrak{H}_K(A, \tilde{w}A).$$

Lemma 3.7. *Let $\tilde{w} \in \tilde{W}$. Let $K \subset V_{\tilde{w}}$ be an affine subspace such that $\tilde{w}K = K$. Let A and A' be two alcoves such that $\bar{A} \cap \bar{A}'$ contains a regular point of K and $\tilde{w}_A, \tilde{w}_{A'}$ are straight elements. Then $\tilde{w}_A = \tilde{w}_{A'}$.*

Proof. We may assume that A is the fundamental alcove Δ by replacing $\tilde{w}$ by $\tilde{w}_A$. We simply write I for $I(K, \Delta)$. By Proposition 2.7 and the straightness of $\tilde{w}$, $\tilde{w} \in {}^I\tilde{W}^I$ and $\tilde{w}(I) = I$. Since $\bar{A}' \cap \bar{\Delta}$ contains a regular point of K, $x_{A'} \in W_I$. Thus

$$\tilde{w}_{A'} = x_{A'}^{-1} \tilde{w} x_{A'} = (x_{A'}^{-1} \tilde{w} x_{A'} \tilde{w}^{-1}) \tilde{w}$$

and $x_{A'}^{-1} \tilde{w} x_{A'} \tilde{w}^{-1} \in W_I$. Therefore $\ell(\tilde{w}_{A'}) = \ell(\tilde{w}) + \ell(x_{A'}^{-1} \tilde{w} x_{A'} \tilde{w}^{-1})$. Since $\ell(\tilde{w}_{A'}) = \ell(\tilde{w})$, we have $x_{A'}^{-1} \tilde{w} x_{A'} \tilde{w}^{-1} = 1$ and $\tilde{w}_{A'} = \tilde{w}$. □

Theorem 3.8. *Let $\mathcal{O}$ be a straight W-conjugacy class of $\tilde{W}$. Then for any $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$, $\tilde{w} \approx \tilde{w}'$.*

Proof. Let $\tilde{u} \in \mathcal{O}$ and $K = V_{\tilde{u}}$. Then by Proposition 2.4 and Proposition 2.7, we may assume that $\tilde{w} = \tilde{u}_A$ and $\tilde{w}' = \tilde{u}_{A'}$, where A and A' are two alcoves whose closures contain regular points of K. Let C be the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ that contains A and let A'' be the unique alcove in C such that $\bar{A}' \cap \bar{A}''$ contains a regular point of K. By Proposition 2.7, we have $\tilde{u}_{A''} \in \mathcal{O}_{\min}$. By Lemma 3.7, we have that $\tilde{u}_{A'} = \tilde{u}_{A''}$.

It remains to show that $\tilde{u}_A \approx \tilde{u}_{A''}$. Assume $A \neq A''$. Similar to the proof of [14, Lemma 2.4], there is a sequence of alcoves $A = A_0, A_1, \cdots A_r = A''$ in C such that A_i contains a regular point of K and $\bar{A}_{i-1} \cap \bar{A}_i \cap K$ spans a codimension one affine subspace P_i of K for $i = 1, 2, \cdots, r$. By Proposition 2.7, $\tilde{u}_{A_i} \in \mathcal{O}_{\min}$ for any i.

If $\tilde{u} P_i = P_i$, by Lemma 3.7, we have that $\tilde{u}_{A_{i-1}} = \tilde{u}_{A_i}$. If $\tilde{u}(P_i) \neq P_i$, then there is a sequence of alcoves $A_{i-1} = B_0, B_1, \cdots, B_s = A_i$ in C such that B_{k-1} and B_k share a common face and $\bar{B}_{k-1} \cap \bar{B}_k \cap K$ spans P_i for $k = 1, \cdots, s$. By Proposition 3.6, we have that $\ell(\tilde{u}_{B_{k-1}}) = \ell(\tilde{u}_{B_k})$ and $\tilde{u}_{B_{k-1}} \approx \tilde{u}_{B_k}$ for $k = 1, \cdots, s$. So $\tilde{u}_{A_{i-1}} \approx \tilde{u}_{A_i}$. Hence $\tilde{u}_A \approx \tilde{u}_{A''}$. □

4. Centralizer in W

4.1. Let (W, S) be a Coxeter group and Ω be a group with a group homomorphism to $\mathrm{Aut}(W, S)$. Let $\tilde{W} = W \rtimes \Omega$. Let $\tilde{w} \in \tilde{W}$ be a minimal length element in its conjugacy class. Let $\mathcal{P}_{\tilde{w}}$ be the set of sequences $\mathbf{i} = (s_1, \cdots, s_r)$ of S such that

$$\tilde{w} \xrightarrow{s_1} s_1 \tilde{w} s_1 \xrightarrow{s_2} \cdots \xrightarrow{s_r} s_r \cdots s_1 \tilde{w} s_1 \cdots s_r.$$

We call such sequence a path form $\tilde{w}$ to $s_r \cdots s_1 \tilde{w} s_1 \cdots s_r$. Denote by $\mathcal{P}_{\tilde{w}, \tilde{w}}$ the set of all paths from $\tilde{w}$ to itself. Let $W_{\tilde{w}} = \{x \in W; \ell(x^{-1} \tilde{w} x) = \ell(\tilde{w})\}$ and $Z(\tilde{w}) = \{x \in W; x\tilde{w} = \tilde{w} x\}$.

There is a natural map

$$\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w}} \to W_{\tilde{w}}, \ (s_1, \cdots, s_r) \mapsto s_1 \cdots s_r.$$

which induces a natural map $\tau_{\tilde{w}, \tilde{w}} : \mathcal{P}_{\tilde{w}, \tilde{w}} \to Z(\tilde{w})$.

We call a W-conjugacy class $\mathcal{O}$ of $\tilde{W}$ *nice* if for some (or equivalently, any) $\tilde{w} \in \mathcal{O}_{\min}$, the map $\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w}} \to W_{\tilde{w}}$ is surjective. It is easy to see that $\mathcal{O}$ is nice if and only if properties (1) and (2) below hold for $\mathcal{O}$:

(1) For any $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$, $\tilde{w} \approx \tilde{w}'$;

(2) For any $\tilde{w} \in \mathcal{O}_{\min}$, the map $\tau_{\tilde{w}, \tilde{w}} : \mathcal{P}_{\tilde{w}, \tilde{w}} \to Z(\tilde{w})$ is surjective.

The definition of nice conjugacy classes is inspired by a conjecture of Lusztig [22, 1.2] that property (2) holds for elliptic conjugacy classes of a finite Weyl group.

4.2. For finite Weyl groups, nice conjugacy classes play an important role in the study of Deligne-Lusztig varieties and representations of finite groups of Lie type. Property (1) is a key ingredient to prove that Deligne-Lusztig varieties corresponding to minimal length elements in a nice conjugacy class are universally homeomorphic. Property (2) leads to nontrivial (quasi-)automorphisms on Deligne-Lusztig varieties and their cohomology groups. For more details, see [3] and [22].

Nice conjugacy classes for affine Weyl groups will also play an important role in the study of affine Deligne-Lusztig varieties. See [12].

4.3. The main goal of this section is to classify nice conjugacy classes for both finite Coxeter groups and affine Weyl groups.

We first consider finite Coxeter groups. Let (W_0, S_0) be a finite Coxeter group and $\Omega' \subset \mathrm{Aut}(W_0, S_0)$. Set $\tilde{W}_0 = W_0 \rtimes \Omega'$. Let $z = w\sigma \in \tilde{W}_0$ with $w \in W_0$ and $\sigma \in \Omega'$. We denote by $\mathrm{supp}(w)$ the *support* of w, i.e., the set of simple reflections appearing in a reduced expression of w. Set $\mathrm{supp}(z) = \cup_{n \in \mathbb{Z}} \sigma^n \mathrm{supp}(w)$. We call $\mathrm{supp}(z)$ the support of z. It is a σ-stable subset of S_0.

We call a W_0-conjugacy class $\mathcal{O}$ of $\tilde{W}_0$ *elliptic* if $\mathrm{supp}(\tilde{w}) = S_0$ for any $\tilde{w} \in \mathcal{O}$. An element in an elliptic conjugacy class is called an elliptic element.

We have the following result.

Theorem 4.1. *Any elliptic conjugacy class in a finite Coxeter group is nice.*

4.4. In order to classify nice conjugacy classes for finite Coxeter group, we first recall the geometric interpretation of conjugacy classes and length function in [14].

Let V be a finite dimensional Euclidean vector space over $\mathbb{R}$ and $\mathfrak{H}_0$ be a finite set of hyperspaces of V through the origin such that $s_H(\mathfrak{H}_0) = \mathfrak{H}_0$ for all $H \in \mathfrak{H}_0$. Let $W_0 \subset GL(V)$ be the subgroup generated by s_H for $H \in \mathfrak{H}_0$. Let $\mathfrak{C}(\mathfrak{H}_0)$ be the set of connected components of $V - \cup_{H \in \mathfrak{H}_0} H$. We call an element in $\mathfrak{C}(\mathfrak{H}_0)$ a *chamber*. We fix a fundamental chamber C_0. For any two chambers C, C', we denote by $\mathfrak{H}_0(C, C')$ the set of hyperspaces in $\mathfrak{H}_0$ separating C from C'. Let $S_0 = \{s_H \in W_0; \sharp \mathfrak{H}_0(C_0, s_H C_0) = 1\}$. Then (W_0, S_0) is a finite Coxeter group. Let $\tilde{W}_0 = W_0 \rtimes \Omega'$ where $\Omega' \subset GL(V)$ consists of automorphism preserving S_0. Then $\ell(w) = \sharp \mathfrak{H}_0(C_0, wC_0)$ is the length function on $\tilde{W}_0$.

It is known that W_0 acts simply transitively on the set of chambers. For any chamber C, we denote by x_C the unique element in W_0 with $x_C C_0 = C$. Here C_0 is the fundamental chamber. Then any element in the W_0-conjugacy class of $\tilde{w}$ is of the form $\tilde{w}_C = x_C^{-1} \tilde{w} x_C$ for some chamber C.

For any $\tilde{w} \in \tilde{W}_0$, we denote by $\mathfrak{C}_{\tilde{w}}(\mathfrak{H}_0)$ the set of chambers C such that $\tilde{w}_C$ is of minimal length in its W_0-conjugacy class. We denote by $V_{\mathfrak{H}_0}^{subreg}$ the set of points in V that is contained in at most one hyperplane of $\mathfrak{H}_0$. By [14, Lemma 4.1],

(a) A W_0-conjugacy class $\mathcal{O}$ of $\tilde{W}_0$ is nice if and only if $(\cup_{A \in \mathfrak{C}_{\tilde{w}}(\mathfrak{H}_0)} \bar{A}) \cap V_{\mathfrak{H}_0}^{subreg}$ is connected for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$.

By [10, Lemma 7.2], a W_0-conjugacy class $\mathcal{O}$ of $\tilde{W}_0$ is elliptic if and only if for some (or equivalently, any) element $\tilde{w} \in \mathcal{O}$, the fixed point set $V^{\tilde{w}} \subset V^{W_0}$. Now we introduce the weakly elliptic conjugacy classes.

Proposition 4.2. *Let $\mathcal{O}$ be a W_0-conjugacy class of $\tilde{W}_0$. Then the following conditions are equivalent:*

(1) For some $\tilde{w} \in \mathcal{O}_{\min}$, $\operatorname{supp}(\tilde{w})$ is a union of some connected components of the Dynkin diagram of S_0 and $\tilde{w}$ commutes with any element in $S_0 - \operatorname{supp}(\tilde{w})$.

(2) For any $\tilde{w} \in \mathcal{O}$, $\operatorname{supp}(\tilde{w})$ is a union of some connected components of the Dynkin diagram of S_0 and $\tilde{w}$ commutes with any element in $S_0 - \operatorname{supp}(\tilde{w})$.

(3) For some (or equivalently, any) element $\tilde{w} \in \mathcal{O}$, $s_H V^{\tilde{w}} = V^{\tilde{w}}$ for all $H \in \mathfrak{H}_0$.

In this case, we call $\mathcal{O}$ a weakly elliptic conjugacy class and any element in $\mathcal{O}$ a weakly elliptic element.

Proof. (1)$\Rightarrow$(2). Assume that $\mathcal{O} \subset W_0\sigma$ for $\sigma \in \Omega'$. By our assumption, $\sigma(s) = s$ for all $s \in S_0 - S_0'$ and $\operatorname{supp}(x\tilde{w}x^{-1}) \subset S_0'$ for any $x \in W_0$. Since $\tilde{w}$ is elliptic in $W_{S_0'} \rtimes \langle\sigma\rangle$, then $\operatorname{supp}(x\tilde{w}x^{-1}) = S_0'$ for any $x \in W_0$.

(2)$\Rightarrow$(3). Assume that $\mathcal{O} \subset W_0\sigma$ for some $\sigma \in \Omega'$. By assumption, there exists $S_0' \subset S_0$, which is a union of some connected components of the Dynkin diagram of S_0 such that $\sigma(s) = s$ for all $s \in S_0 - S_0'$ and $\operatorname{supp}(\tilde{w}) = S_0'$ for any $\tilde{w} \in \mathcal{O}$. Then $\mathcal{O}$ is elliptic in $W_{S_0'} \rtimes \langle\sigma\rangle$. Therefore $V^{\tilde{w}} \subset V^{W_{S_0'}}$ for all $\tilde{w} \in \mathcal{O}$. Hence $s_H V^{\tilde{w}} = V^{\tilde{w}}$ for all $H \in \mathfrak{H}_0$.

(3)$\Rightarrow$(1). By [14, Proposition 2.2], there exists $\tilde{w} \in \mathcal{O}_{\min}$ such that $\bar{C}_0$ contains a regular point e of $V^{\tilde{w}}$. Assume that $\tilde{w} = w\sigma$ for $w \in W_0$ and $\sigma \in \Omega'$. Then $w\sigma(e) = e$ and thus $e = \sigma(e) = w(e)$ for $e, \sigma(e) \in \bar{C}_0$ are dominant.

Let $J = \{s \in S_0; s(e) = e\}$. Then $\sigma(J) = J$ and $w \in W_J$. Note that $V^{\tilde{w}} \subset V^{W_J}$ for e is a regular element in $V^{\tilde{w}}$. Hence $\tilde{w}$ is an elliptic element in $W_J \rtimes \langle\sigma\rangle$ and $\operatorname{supp}(\tilde{w}) = J$.

Let $s \in S_0 - J$. Then $s(V^{\tilde{w}}) = V^{\tilde{w}}$. Thus $s(e) = e + \langle e, \alpha\rangle\alpha^\vee \in V^{\tilde{w}}$, where α is the positive root corresponding to s. Since $s \notin J$, we have $s(e) \neq e$ and $\alpha^\vee \in V^{\tilde{w}}$. Hence $\tilde{w}s = s\tilde{w}$. The statement follows from the following Lemma 4.3. $\square$

Lemma 4.3. *Let $\tilde{w} = w\sigma$ with $w \in W_0$ and $\sigma \in \Omega'$. Let $s \in S_0 - \operatorname{supp}(\tilde{w})$. Then $s\tilde{w} = \tilde{w}s$ if and only if $s = \sigma(s)$ and s commutes with each element of $\operatorname{supp}(\tilde{w})$.*

Proof. Let $K = \operatorname{supp}(w)$ and $K' = \{s' \in K; ss' = s's\}$. Write w as $w = abc$ for $a \in W_{K'}$, $c \in W_{\sigma(K')}$ and $b \in {}^{K'}W_0{}^{\sigma(K')}$. Then $sb = b\sigma(s)$. Since $s \notin K$ and each element of $K - K'$ does not commute with s, we have that $b\sigma(s) = sb \in {}^{K-K'}W_0$. Therefore $b \in {}^{K-K'}W_K$ for $\sigma(s) \notin K$. Since $b \in {}^{K'}W_K$, we must have that $b = 1$. Hence $s = \sigma(s)$ and $K = \operatorname{supp}(a) \cup \operatorname{supp}(c) \subset K'$. Hence for any $n \in \mathbb{Z}$ and any $r \in K$,

$s\sigma^n(r) = \sigma^n(sr) = \sigma^n(rs) = \sigma^n(r)s$. Therefore s commutes with every element of $\mathrm{supp}(\tilde{w}) = \cup_{n\in\mathbb{Z}}\sigma^n(K)$. □

Now we classify nice conjugacy classes for $\tilde{W}_0$.

Theorem 4.4. *Let $\mathcal{O}$ be a W_0-conjugacy class of $\tilde{W}_0$. Then $\mathcal{O}$ is nice if and only if it is weakly elliptic.*

Proof. Suppose that $\mathcal{O}$ is nice. Let $\tilde{w} \in \mathcal{O}_{\min}$ and $S_0' = \mathrm{supp}(\tilde{w})$. Then $w_0\tilde{w}w_0 \in \mathcal{O}_{\min}$, where w_0 is the largest element of W_0. Hence there exists a reduced expression $w_0 = s_1s_2\cdots s_r$ such that $\tilde{w}_0 \approx \tilde{w}_1 \approx \cdots \approx \tilde{w}_r$, where $\tilde{w}_i = (s_1\cdots s_i)^{-1}\tilde{w}(s_1\cdots s_i)$. By [10, Lemma 7.4], $\mathrm{supp}(\tilde{w}_i) = \mathrm{supp}(\tilde{w}) = S_0'$ for all i. Therefore $s_{j+1}\tilde{w}_js_{j+1} = \tilde{w}_j$ if $s_{j+1} \notin S_0'$. By Lemma 4.3, $st = ts$ and $\tilde{w}t = t\tilde{w}$ for any $s \in S_0'$ and any $t \in S_0 - S_0'$. Hence $\mathcal{O}$ is weakly elliptic.

Suppose that $\mathcal{O}$ is weakly elliptic. Let $\tilde{w} \in \mathcal{O}_{\min}$ and $J = \mathrm{supp}(\tilde{w})$. Let $x \in W_0$ such that $\ell(x^{-1}\tilde{w}x) = \ell(\tilde{w})$. We may write x as x_2x_1 with some $x_1 \in W_J$ and $x_2 \in W_{S_0-J}$ for J and $S_0 - J$ are unions of connected components of the Dynkin diagram of S_0 and $\tilde{w}$. Then $x^{-1}\tilde{w}x = x_1^{-1}(x_2^{-1}\tilde{w}x_2)x_1 = x_1^{-1}\tilde{w}x_1$. Since $\tilde{w}$ commutes with each simple reflection of $S_0 - J$, x_2 is in the image of $\tau_{\tilde{w}}$. By assumption, $\tilde{w}$ is elliptic in $W_J \rtimes \langle\sigma\rangle$ for some $\sigma \in \Omega'$. Thus by [14, Corollary 4.4], x_1 is in the image of $\tau_{\tilde{w}}$. Hence so is x_2x_1. Thus $\mathcal{O}$ is nice. □

4.5. Now we study affine Weyl groups. We keep the notations in §3.3. Let $\tilde{W}_0 = W_0 \rtimes \Omega'$. Then $\tilde{W} = W_G \rtimes \Omega = P \rtimes \tilde{W}_0$. For any $\tilde{w} \in \tilde{W}$, we denote by $\mathfrak{A}_{\tilde{w}}$ the set of alcoves A such that $\tilde{w}_A$ is of minimal length in its W-conjugacy class. We denote by V^{subreg} the set of points in V that is contained in at most one hyperplane of $\mathfrak{H}$. Similar to the proof of [14, Lemma 4.1],

(a) A W-conjugacy class $\mathcal{O}$ of $\tilde{W}$ is nice if and only if $(\cup_{A\in\mathfrak{A}_{\tilde{w}}}\bar{A}) \cap V^{subreg}$ is connected for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$.

4.6. Let $y \in \tilde{W}$. Let $K \subset V_y$ be an affine subspace such that $yK = K$. Choose $p \in K$. Define $\bar{y} = T_{-\nu_y-p} \circ y \circ T_p \in GL(V)$, where T_v denotes the map of translation by $v \in V$. Then $\bar{y}(0) = 0$ and $\bar{y}$ is the image of y under the map $\tilde{W} = P \rtimes \tilde{W}_0 \to \tilde{W}_0$. In other words, $\bar{y}$ is the finite part of y.

Set $\mathfrak{H}_{K,p} = T_{-p}(\mathfrak{H}_K)$. Then any element in $\mathfrak{H}_{V_y,p}$ is a hyperplane through 0 and contains $V^{\bar{y}} = T_{-p}(V_y)$. Since y preserves $\mathfrak{H}_K$, $\bar{y}$ preserves $\mathfrak{H}_{K,p}$.

The following result relates §4.4 (a) with §4.5 (a).

Lemma 4.5. *Keep notations as above. Assume p is a regular point of K and $A, A' \in \mathfrak{A}_y$ with $p \in \bar{A} \cap \bar{A}'$. Let C (resp. C') be the unique element of $\mathfrak{C}(\mathfrak{H}_{K,p})$ such that $A \subset T_p(C)$ (resp. $A \subset T_p(C')$). Then A and A' are in the same connected component of $(\cup_{A\in\mathfrak{A}_y}\bar{A}) \cap$*

V^{subreg} if and only if C and C' are in the same connected component of $(\cup_{C \in \mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})} \bar C) \cap V^{subreg}_{\mathfrak{H}_{K,p}}$.

Proof. Suppose that C and C' are in the same connected component of $(\cup_{C \in \mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})} \bar C) \cap V^{subreg}_{\mathfrak{H}_{K,p}}$. Then there is a sequence $C = C_1, C_2, \cdots, C_t = C'$ in $\mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})$ such that $\bar C_i \cap \bar C_{i+1}$ spans $H'_i \in \mathfrak{H}_{K,p}$ for $i = 0, \cdots, t-1$. Note that all the numbers $\sharp\mathfrak{H}_{K,p}(C_i, \bar y C_i)$ are the same. Let A_i be the unique alcove in $T_p(C_i)$ whose closure contains p. Then $\bar A_i \cap \bar A_{i+1}$ spans $T_p(H'_i) \in \mathfrak{H}_K$ for each i. By Proposition 2.7,

$$\begin{aligned}\sharp\mathfrak{H}(A_i, yA_i) &= \sharp\mathfrak{H}_K(A_i, yA_i) + \langle \bar\nu_y, 2\rho\rangle = \sharp\mathfrak{H}_{K,p}(C_i, \bar y C_i) + \langle \bar\nu_y, 2\rho\rangle \\ &= \sharp\mathfrak{H}(A, yA).\end{aligned}$$

Hence all A_i's lie in the same connected component of $(\cup_{A \in \mathfrak{A}_y} \bar A) \cap V^{subreg}$.

Suppose that A and A' are in the same connected component of $(\cup_{A \in \mathfrak{A}_y} \bar A) \cap V^{subreg}$. There is a sequence of alcoves $A = A_1, A_2, \cdots, A_r = A'$ in $\mathfrak{A}_y$ such that A_i and A_{i+1} share a common face which spans $H_i \in \mathfrak{H}$ for all i. By Proposition 2.7, $C, C' \in \mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})$. Now we define a sequence of chambers in $\mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})$ as follows. Let $C_1 = C$. Assume that C_i is already defined for $i \geq 1$. Let $j_i = \max\{k; T_{-p}(A_k) \subset C_i\}$. Let C_{i+1} be the unique connected component containing $T_{-p}(A_{j_i+1})$. We obtain a sequence $C = C_1, \cdots, C_s = C'$ in this way.

Notice that $H_{j_i} \in \mathfrak{H}_K$ (hence so is yH_{j_i}) and

$$\mathfrak{H}(A_{j_i}, yA_{j_i}) - \{H_{j_i}, yH_{j_i}\} \subset \mathfrak{H}(A_{j_i+1}, yA_{j_i+1}) \subset \mathfrak{H}(A_{j_i}, yA_{j_i}) \cup \{H_{j_i}, yH_{j_i}\}.$$

Since $\sharp\mathfrak{H}(A_{j_i}, yA_{j_i}) = \sharp\mathfrak{H}(A_{j_i+1}, yA_{j_i+1})$, then $\{H_{j_i}, yH_{j_i}\} \cap \mathfrak{H}(A_{j_i+1}, yA_{j_i+1})$ consists of at most one element. Hence

$$\{T_{-p}(H_{j_i}), T_{-p}(yH_{j_i}) = \bar y(T_{-p}(H_{j_i}))\} \cap \mathfrak{H}_{K,p}(C_{i+1}, \bar y C_{i+1})$$

consists of at most one element. Notice that $\sharp\mathfrak{H}_{K,p}(C_1, \bar y C_1)$ is minimal among all the chambers in $\mathfrak{C}(\mathfrak{H}_{K,p})$. Thus

$$\sharp\mathfrak{H}_{K,p}(C_1, \bar y C_1) = \sharp\mathfrak{H}_{K,p}(C_2, \bar y C_2) = \cdots = \sharp\mathfrak{H}_{K,p}(C_s, \bar y C_s)$$

and $C_1, \cdots, C_s \in \mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})$. By our construction $\bar C_i \cap \bar C_{i+1}$ spans $T_{-p}(H_{j_i})$ for $i = 0, \cdots, s-1$. So $C_1, \cdots, C_s$ are in the same connected component of $(\cup_{C \in \mathfrak{C}_{\bar y}(\mathfrak{H}_{K,p})} \bar C) \cap V^{subreg}_{\mathfrak{H}_{K,p}}$. □

Now we classify nice conjugacy classes for affine Weyl groups.

Theorem 4.6. *Let $\mathcal{O}$ be a W-conjugacy class of $\tilde W$ and $\mathcal{O} \subset \tilde W_G \sigma$ for $\sigma \in \Omega'$. Then the following conditions are equivalent:*

(1) $\mathcal{O}$ is nice.

(2) For some (or equivalently, any) $y \in \mathcal{O}$, $s_H(V_y) = V_y$ for any $H \in \mathfrak{H}(\nu_y)$ with $H \cap V_y \neq \emptyset$.

(3) For some (or equivalently, any) $y \in \mathcal{O}$ with $\nu_y = \nu_{\mathcal{O}}$, $\bar y$ is a weakly elliptic element in $W_{J_{\mathcal{O}}} \rtimes \langle \sigma \rangle$.

Proof. (3) $\Rightarrow$ (2). Let $y \in \mathcal{O}$ with $\nu_y = \nu_{\mathcal{O}}$. Let $p \in H \cap V_y$ and $H' = T_{-p}H$. Hence $s_{H'}(V^{\bar{y}}) = V^{\bar{y}}$ by Proposition 4.2 (3) and Theorem 4.4. Thus $s_H(V_y) = V_y$ for $V_y = T_p(V^{\bar{y}})$.

(2) $\Rightarrow$ (3). Let $H = H_{\alpha,0}$ with $\langle \nu_y, \alpha \rangle = 0$. It suffices to show the $s_H(V^{\bar{y}}) = V^{\bar{y}}$. This is trivial if $V^{\bar{y}} \subset H$. Otherwise, $\langle V^{\bar{y}}, \alpha \rangle = \mathbb{R}$. Let $p \in V_y$, then H intersects with $V_y = T_p(V^{\bar{y}})$. By condition (2), $s_H(V_y) = V_y$. Hence $\alpha^\vee \in$ and $s_H(V^{\bar{y}}) = V^{\bar{y}}$.

(1) $\Rightarrow$ (2). Let $H \in \mathfrak{H}(\nu_y)$ such that $K = H \cap V_y \neq \emptyset$. Then K is an affine subspace of V_y of codimension at most 1 and $yK = K$. By Proposition 2.7 and [14, Lemma 2.3], there exists an alcove $A \in \mathfrak{A}_y$ whose closure contains a regular point p of K. Since $\mathcal{O}$ is nice, applying Lemma 4.5 yields, $(\cup_{C \in \mathfrak{C}_{\bar{y}}(\mathfrak{H}_{K,p})} \bar{C}) \cap V^{subreg}_{\mathfrak{H}_{K,p}}$ is connected. By Theorem 4.4, we have $s_{T_{-p}(H)}(V^{\bar{y}}) = V^{\bar{y}}$, that is, $s_H(V_y) = V_y$.

(2) $\Rightarrow$ (1). Let $y \in \mathcal{O}$ with $\nu_y = \nu_{\mathcal{O}}$. By Proposition 2.4, it suffices to prove the following statement:

Let $A, A' \in \mathfrak{A}_y$ such that $\bar{A}$ and $\bar{A}'$ contain regular points of V_y. Then A and A' are in the same connected component of $(\cup_{A \in \mathfrak{A}_y} \bar{A}) \cap V^{subreg}$.

Let C be the connected component of $V - \cup_{H \in \mathfrak{H}_{V_y}} H$ that contains A and A'' be the unique alcove in C such that $\bar{A}' \cap \bar{A}''$ contains a regular point q of V_y. By condition (3) and §4.4 (a), $(\cup_{C \in \mathfrak{C}_{\bar{y}}(\mathfrak{H}_{V_y,q})} \bar{C}) \cap V^{subreg}_{\mathfrak{H}_{V_y,q}}$ is connected. Hence by Lemma 4.5, A' and A'' are in the same connected component of $(\cup_{B \in \mathfrak{A}_y} \bar{B}) \cap V^{subreg}$.

Similar to the proof of [14, Lemma 2.4], there is a sequence of alcoves $A = A_0, A_1, \cdots A_r = A''$ in C such that $\bar{A}_i$ contains a regular point of V_y and $\bar{A}_{i-1} \cap \bar{A}_i \cap V_y$ spans a codimension one affine subspace $P_i = H_i \cap V_y$ of V_y for $1 \leqslant i \leqslant r$.

If $P_i = yP_i$. Then $H_i \in \mathfrak{H}(\nu_y)$. By condition (2), $s_{H_i}(V_y) = V_y$. Since $V_y \nsubseteq H_i$, H_i is the affine hyperplane containing P_i and orthogonal to V_y. Hence H_i is the unique element in $\mathfrak{H}$ whose intersection with V_y is P_i and thus the unique hyperplane separating A_{i-1} from A_i. So A_{i-1} and A_i are in the same connected component of $(\cup_{B \in \mathfrak{A}_y} \bar{B}) \cap V^{subreg}$.

If $yP_i \neq P_i$, then there is a sequence of alcoves $A_{i-1} = B_0, B_1, \cdots, B_s = A_i$ in C such that B_{k-1} and B_k have a common face and $\bar{B}_{k-1} \cap \bar{B}_k \cap V_y$ spans P_i for $k = 1, \cdots, s$. By Proposition 3.6, we see that all $B_k \in \mathfrak{A}_y$. Hence A_{i-1} and A_i are in the same connected component of $(\cup_{B \in \mathfrak{A}_y} \bar{B}) \cap V^{subreg}$.

Hence A and A'' are in the same connected component of $(\cup_{B \in \mathfrak{A}_y} \bar{B}) \cap V^{subreg}$. □

Corollary 4.7. *Let $y \in \tilde{W}$ such that $\bar{y}$ is an elliptic element in W_0. Then the W-conjugacy class of y is nice.*

Now we classify straight nice conjugacy classes.

Proposition 4.8. *Let $\mathcal{O}$ be a straight W-conjugacy class of $\tilde{W}$ and $\mathcal{O} \subset \tilde{W}_G \sigma$ with $\sigma \in \Omega'$. Then $\mathcal{O}$ is nice if and only if there exists $x \in \mathcal{O}$ such that $\nu_x = \nu_{\mathcal{O}}$ and x is superbasic in $\tilde{W}_J$, where $J \subset J_{\mathcal{O}}$ is a union of connected components of Dynkin diagram of $J_{\mathcal{O}}$ and σ fixes each element of $J_{\mathcal{O}} - J$.*

Proof. Assume that $x \in \mathcal{O}$ such that $\nu_x = \nu_{\mathcal{O}}$ and x is superbasic in $\tilde{W}_J$, where $J \subset J_{\mathcal{O}}$ is a union of connected components of Dynkin diagram of $J_{\mathcal{O}}$ and σ fixes each element of $J_{\mathcal{O}} - J$. Let $H = H_{\alpha,k} \in \mathfrak{H}(\nu_x)$. Then $\langle \nu_x, \alpha \rangle = 0$ and α is a linear combination of roots in $J_{\mathcal{O}}$. If α is a linear combination of roots in J, then by Proposition 3.5, $H \cap V_x = \emptyset$. If α is a linear combination of roots in $J_{\mathcal{O}} - J$, then $s_H(V_x) = V_x$ since $\alpha^\vee \in V^{\bar{x}}$. By Theorem 4.6, $\mathcal{O}$ is nice.

Assume that $\mathcal{O}$ is nice. By Proposition 3.2 and Theorem 4.6, there exists a basic element $x \in \tilde{W}_{J_{\mathcal{O}}}$ such that $\nu_x = \nu_{\mathcal{O}}$ and $\bar{x}$ is weakly elliptic in $W_{J_{\mathcal{O}}} \rtimes \langle \sigma \rangle$. Set $J = \operatorname{supp}(\bar{x})$. Then J is a union of connected components of Dynkin diagram of $J_{\mathcal{O}}$ and σ fixes each element of $J_{\mathcal{O}} - J$. Let $H = H_{\alpha,k}$ such that α is a linear combination of roots in J. By assumption, x is elliptic in $W_J \rtimes \langle \sigma \rangle$, i.e., $V^{\bar{x}} \subset V^{W_J}$, we have that $V^{\bar{x}} \subset H_{\alpha,0}$. By Proposition 3.2, $V_x \not\subseteq H$. Notice that $V_x = p + V^{\bar{x}}$ for some $p \in V_x$. Thus $V_x \cap H = \emptyset$. By Proposition 3.5, x is superbasic in $\tilde{W}_J$. □

Corollary 4.9. *Let $\mathcal{O}$ be a superstraight W-conjugacy class of $\tilde{W}$. Then $\mathcal{O}$ is nice.*

5. Class polynomial

5.1. Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The Hecke algebra H associated to W is the associated $\mathcal{A}$-algebra with basis T_w for $w \in W$ and the multiplication law is given by

$$T_x T_y = T_{xy}, \quad \text{if } \ell(x) + \ell(y) = \ell(xy);$$
$$(T_s - v)(T_s + v^{-1}) = 0, \quad \text{for } s \in S.$$

Then $T_s^{-1} = T_s - (v - v^{-1})$ and T_w is invertible in H for all $w \in W$. If δ is an automorphism of W with $\delta(S) = S$, then $T_w \mapsto T_{\delta(w)}$ induces an $\mathcal{A}$-linear automorphism of H which is still denoted by δ. The Hecke algebra associated to $\tilde{W} = W \rtimes \Omega$ is defined to be $\tilde{H} = H \rtimes \Omega$. It is easy to see that $\tilde{H}$ is the associated $\mathcal{A}$-algebra with basis $T_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}$ and multiplication is given by

$$T_{\tilde{x}} T_{\tilde{y}} = T_{\tilde{x}\tilde{y}}, \quad \text{if } \ell(\tilde{x}) + \ell(\tilde{y}) = \ell(\tilde{x}\tilde{y});$$
$$(T_s - v)(T_s + v^{-1}) = 0, \quad \text{for } s \in S.$$

Let $h, h' \in \tilde{H}$, we call $[h, h'] = hh' - h'h$ the *commutator* of h and h'. Let $[\tilde{H}, \tilde{H}]$ be the $\mathcal{A}$-submodule of $\tilde{H}$ generated by all commutators.

For finite Hecke algebras, Geck and Pfeiffer introduced class polynomials in [7]. We'll show in the section that their construction can be generalized to affine Hecke algebra based on Theorem 2.9.

Lemma 5.1. *Let $\tilde{w}, \tilde{w}' \in \tilde{W}$ with $\tilde{w} \tilde{\sim} \tilde{w}'$. Then*

$$T_{\tilde{w}} \equiv T_{\tilde{w}'} \mod [\tilde{H}, \tilde{H}].$$

Proof. By definition of $\tilde{\sim}$, it suffices to prove the case that there exists $x \in \tilde{W}$ such that $\tilde{w}' = x\tilde{w}x^{-1}$ with $\ell(\tilde{w}) = \ell(\tilde{w}')$ and either $\ell(x\tilde{w}) = \ell(x) + \ell(\tilde{w})$ or $\ell(\tilde{w}x^{-1}) = \ell(x) + \ell(\tilde{w})$.

If $\ell(x\tilde{w}) = \ell(x)+\ell(\tilde{w})$, then $\ell(\tilde{w}'x) = \ell(x\tilde{w}) = \ell(\tilde{w}')+\ell(\delta(x))$. Hence $T_{\tilde{w}'}T_x = T_{\tilde{w}'x} = T_{x\tilde{w}} = T_x T_{\tilde{w}}$ and $T_{\tilde{w}'} = T_x T_{\tilde{w}} T_x^{-1} \equiv T_{\tilde{w}} \mod [\tilde{H}, \tilde{H}]$.

If $\ell(\tilde{w}x^{-1}) = \ell(x) + \ell(\tilde{w})$, then $\ell(x^{-1}\tilde{w}') = \ell(\tilde{w}x^{-1}) = \ell(\tilde{w}') + \ell(x)$. Hence $T_{x^{-1}}T_{\tilde{w}'} = T_{x^{-1}\tilde{w}'} = T_{\tilde{w}x^{-1}} = T_{\tilde{w}}T_{x^{-1}}$ and $T_{\tilde{w}'} = T_{x^{-1}}^{-1} T_{\tilde{w}} T_{x^{-1}} \equiv [\tilde{H}, \tilde{H}]$. □

Now Corollary 2.10 (2) implies that

Corollary 5.2. *Let $\mathcal{O}$ be a conjugacy class of $\tilde{W}$ and $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$. Then*

$$T_{\tilde{w}} \equiv T_{\tilde{w}'} \mod [\tilde{H}, \tilde{H}].$$

Remark. We denote by $T_{\mathcal{O}}$ the image of $T_{\tilde{w}} \in \tilde{H}/[\tilde{H}, \tilde{H}]$ for any $\tilde{w} \in \mathcal{O}_{\min}$.

Theorem 5.3. *Let $\tilde{w} \in \tilde{W}$. Then for any conjugacy class $\mathcal{O}$ of $\tilde{W}$, there exists a polynomial $f_{\tilde{w},\mathcal{O}} \in \mathbb{Z}[v - v^{-1}]$ with nonnegative coefficients such that $f_{\tilde{w},\mathcal{O}}$ is nonzero only for finitely many $\mathcal{O}$ and*

$$\text{(a)} \qquad T_{\tilde{w}} = \sum_{\mathcal{O}} f_{\tilde{w},\mathcal{O}} T_{\mathcal{O}} \in \tilde{H}/[\tilde{H}, \tilde{H}].$$

Proof. We argue by induction on $\ell(\tilde{w})$.

If $\tilde{w}$ is a minimal element in a conjugacy class of $\tilde{W}$, then we set $f_{\tilde{w},\mathcal{O}} = \begin{cases} 1, & \text{if } \tilde{w} \in \mathcal{O} \\ 0, & \text{if } \tilde{w} \notin \mathcal{O} \end{cases}$. In this case, the statement automatically holds.

Now we assume that $\tilde{w}$ is not a minimal element in the conjugacy class of $\tilde{W}$ that contains it and that for any $\tilde{w}' \in \tilde{W}$ with $\ell(\tilde{w}') < \ell(\tilde{w})$, the statement holds for $\tilde{w}'$. By Theorem 2.9, there exist $\tilde{w}_1 \approx \tilde{w}$ and $i \in S$ such that $\ell(s_i \tilde{w}_1 s_i) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. In this case, $\ell(s_i \tilde{w}) < \ell(\tilde{w})$ and we define $f_{\tilde{w},\mathcal{O}}$ as

$$f_{\tilde{w},\mathcal{O}} = (v - v^{-1}) f_{s_i \tilde{w}_1, \mathcal{O}} + f_{s_i \tilde{w}_1 s_i, \mathcal{O}}.$$

By inductive hypothesis, $f_{s_i \tilde{w}_1, \mathcal{O}}, f_{s_i \tilde{w}_1 s_i, \mathcal{O}} \in \mathbb{Z}[v - v^{-1}]$ with nonnegative coefficients. Hence $f_{\tilde{w},\mathcal{O}} \in \mathbb{Z}[v - v^{-1}]$ with nonnegative coefficients. Moreover, there are only finitely many $\mathcal{O}$ such that $f_{s_i \tilde{w}_1, \mathcal{O}} \neq 0$ or $f_{s_i \tilde{w}_1 s_i, \mathcal{O}} \neq 0$. Hence there are only finitely many $\mathcal{O}$ such that $f_{\tilde{w},\mathcal{O}} \neq 0$.

By Lemma 5.1, $T_{\tilde{w}} \equiv T_{\tilde{w}_1} \mod [\tilde{H}, \tilde{H}]$. Now

$$\begin{aligned} T_{\tilde{w}} &\equiv T_{\tilde{w}_1} = T_{s_i} T_{s_i \tilde{w}_1 s_i} T_{s_i} \equiv T_{s_i \tilde{w}_1 s_i} T_{s_i} T_{s_i} \\ &= (v - v^{-1}) T_{s_i \tilde{w}_1 s_i} T_{s_i} + T_{s_i \tilde{w}_1 s_i} = (v - v^{-1}) T_{s_i \tilde{w}_1} + T_{s_i \tilde{w}_1 s_i}. \end{aligned}$$

Hence the image of $T_{\tilde{w}}$ in $\tilde{H}/[\tilde{H}, \tilde{H}]$ is

$$(v - v^{-1}) \sum_{\mathcal{O}} f_{s_i \tilde{w}_1, \mathcal{O}} T_{\mathcal{O}} + \sum_{\mathcal{O}} f_{s_i \tilde{w}_1 s_i, \mathcal{O}} T_{\mathcal{O}} = \sum_{\mathcal{O}} f_{\tilde{w}, \mathcal{O}} T_{\mathcal{O}}.$$

□

5.2. We call $f_{\tilde{w}, \mathcal{O}}$ in the above theorem the *class polynomial* associated to w and $\mathcal{O}$.

Notice that the construction of $f_{\tilde{w}, \mathcal{O}}$ depends on the choice of the sequence of elements in S used to conjugate $\tilde{w}$ to a minimal length element in its conjugacy class. We'll show in the next section that $f_{\tilde{w}, \mathcal{O}}$ is in fact, independent of such choice and is uniquely determined by the identity (a) in Theorem 5.3. For a finite Hecke algebra, similar result was obtained by Geck and Rouquier in [9, Theorem 4.2].

6. Cocenter of $\tilde{H}$

6.1. We call $\tilde{H}/[\tilde{H}, \tilde{H}]$ the *cocenter* of $\tilde{H}$. Now for any conjugacy class $\mathcal{O}$ of $\tilde{W}$, we choose a minimal length representative $\tilde{w}_{\mathcal{O}}$. By Theorem 5.3, $T_{\mathcal{O}}$, where $\mathcal{O}$ ranges over all conjugacy classes of $\tilde{W}$, spans the cocenter of $\tilde{H}$. The main purpose of this section is to show that $T_{\mathcal{O}}$ are linearly independent in the cocenter of $\tilde{H}$ and thus form a basis. We call it the *standard basis* of $\tilde{H}/[\tilde{H}, \tilde{H}]$.

6.2. Following [19], let J be the based ring of W with basis $(t_w)_{w \in W}$. For each $\delta \in \Omega$, the map $t_w \mapsto t_{\delta(w)}$ gives a ring homomorphism of J, which we still denote by δ. Set $\tilde{J} = J \rtimes \Omega$, $J_{\mathcal{A}} = J \otimes_{\mathbb{Z}} \mathcal{A}$ and $\tilde{J}_{\mathcal{A}} = \tilde{J} \otimes_{\mathbb{Z}} \mathcal{A}$.

Let $(c_w)_{w \in W}$ be the Kazhdan-Lusztig basis of H. Then $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$ with $h_{x,y,z} \in \mathcal{A}$. There is a homomorphisms of $\mathcal{A}$-algebras $\varphi : H \to J_{\mathcal{A}}$ defined by $c_w \mapsto \sum_{d \in \mathcal{D}, a(d) = a(x)} h_{w,d,x} t_x$, where $a : \tilde{W} \to \mathbb{N}$ is Lusztig's a-function and $\mathcal{D}$ is the set of distinguished involutions of W. It is easy to see that for each $\delta \in \Omega$

$$\varphi(\delta(c_w)) = \varphi(c_{\delta(w)}) = \sum_{d \in \mathcal{D}, a(d) = a(x)} h_{\delta(w), \delta(d), \delta(x)} t_{\delta(x)} = \delta \varphi(c_w).$$

Hence φ extends in a natural way to a homomorphism $\tilde{\varphi} : \tilde{H} \to \tilde{J}_{\mathcal{A}}$.

6.3. Set $J_{\mathbb{C}} = J \otimes_{\mathbb{Z}} \mathbb{C}$ and $\tilde{J}_{\mathbb{C}} = \tilde{J} \otimes_{\mathbb{Z}} \mathbb{C}$. Set $H_{\mathbb{C}} = H \otimes_{\mathcal{A}} \mathbb{C}[v, v^{-1}]$ and $\tilde{H}_{\mathbb{C}} = \tilde{H} \otimes_{\mathcal{A}} \mathbb{C}[v, v^{-1}]$. For any $q \in \mathbb{C}^{\times}$, let $\mathbb{C}_q$ be the one-dimensional $\mathcal{A}$-module over $\mathbb{C}$ such that v acts as the scalar q. Set $H_q = H \otimes_{\mathcal{A}} \mathbb{C}_q$ and $\tilde{H}_q = \tilde{H} \otimes_{\mathcal{A}} \mathbb{C}_q$.

Let E be a $\tilde{J}_{\mathbb{C}}$-module. Through the homomorphism

$$\tilde{\varphi}_q = \tilde{\varphi} \mid_{v=q} : \tilde{H}_q \to \tilde{J}_{\mathbb{C}},$$

it is endowed with an $\tilde{H}_q$-module structure. We denote this $\tilde{H}_q$-module by E_q.

For an associative $\mathbb{C}$-algebra R, we denote by $\mathrm{K}(R)$ the Grothendieck group of all finite dimensional representations of R over $\mathbb{C}$.

Thus the map $E \mapsto E_q$ induces a homomorphism $(\tilde{\varphi}_q)_* : \mathrm{K}(\tilde{J}_{\mathbb{C}}) \to \mathrm{K}(\tilde{H}_q)$. We have that

Lemma 6.1. *The map $(\tilde{\varphi}_q)_* : K(\tilde{J}_{\mathbb{C}}) \to K(\tilde{H}_q)$ is surjective.*

Proof. The proof is similar to [20, Lemma 1.9]. Since Ω preserves a-function, we associate to each $\tilde{H}_q$-module M an integer a_M such that $c_w M = 0$ whenever $a(w) > a_M$ and $c_{w'} M \neq 0$ for some $w' \in W$ with $a(w') = a_M$. For each simple $\tilde{H}_q$-module M, we construct as in the proof of [20, Lemma 1.9] a finite dimensional $\tilde{J}_{\mathbb{C}}$-module $\tilde{M}$ and a nonzero $\tilde{H}_q$-morphism $p : \tilde{M}_q \to M$ with $M' = \ker p$ such that $a_{M'} < a_M$. Since the a-function is bounded, it follows easily that $(\tilde{\varphi}_q)_*$ is surjective. □

6.4. Let $\mathrm{K}^*(\tilde{H}_q) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\tilde{H}_q), \mathbb{C})$ be the space of linear functions on $\mathrm{K}(\tilde{H}_q)$. The map $\tilde{H}_q \to \mathrm{K}^*(\tilde{H}_q)$ sending $h \in \tilde{H}_q$ to the function $M \mapsto Tr_M(h)$ on $\mathrm{K}(\tilde{H}_q)$ induces a map

$$\Psi_q : \tilde{H}_q / [\tilde{H}_q, \tilde{H}_q] \to \mathrm{K}^*(\tilde{H}_q).$$

It is proved in [17, Theorem B] and [4, Main Theorem] that Ψ_q is injective if q is a power of prime. We'll show below that Ψ_1 is also injective. Here $\tilde{H}_1 = \mathbb{C}[\tilde{W}]$ is the group algebra.

Lemma 6.2. *Let $\tilde{w}, \tilde{w}' \in \tilde{W}$. If $\tilde{w}$ and $\tilde{w}'$ are not in the same conjugacy class of $\tilde{W}$, then there exists $n > 0$ such that the images of $\tilde{w}$ and $\tilde{w}'$ in $\tilde{W}/nP$ are not in the same conjugacy class.*

Proof. It suffices to consider the case where $\tilde{W} \subset W \rtimes \mathrm{Aut}(W, S)$. Then we have that $\tilde{W}/W$ is finite. Notice that Q is a normal subgroup of $\tilde{W}$ and $\tilde{W}/mQ$ is a quotient group of $\tilde{W}$. Since $\tilde{W}/W$ and W/Q are both finite groups, $\tilde{W}/Q$ is also a finite group. We choose a representative x_i for each coset of Q. Set $\tilde{w}_i = x_i \tilde{w} x_i^{-1}$. Then any element conjugate to $\tilde{w}$ is of the form $t^{\lambda} \tilde{w}_i t^{-\lambda}$ for some i and $\lambda \in Q$.

Now we show that

(a) for any i, there exists $n_i > 0$ such that the images of $\tilde{w}_i$ and $\tilde{w}'$ are not conjugate by an element in $n_i Q$.

Otherwise, there exists i such that $\tilde{w}' = t^\chi \tilde{w}_i$ for some $\chi \in Q$ such that $\chi \in (1-\tilde{w}_i)Q + nQ$ for all $n > 0$. Therefore the image of χ in $Q/(1-\tilde{w}_i)Q$ is divisible by all the positive integer n. So the image of χ in $Q/(1-\tilde{w}_i)Q$ is 0 and $\chi = \lambda - \tilde{w}_i \lambda$ for some $\lambda \in Q$. Hence $\tilde{w}' = t^\lambda \tilde{w}_i t^{-\lambda}$ is conjugate to $\tilde{w}$ in $\tilde{W}$. That is a contradiction.

(a) is proved.

Now set $n = \Pi_i n_i$. If the images of $\tilde{w}$ and $\tilde{w}'$ in $\tilde{W}/nQ$ are in the same conjugacy class, then there exists i such that the images of $\tilde{w}_i$ and $\tilde{w}'$ in $\tilde{W}/nQ$ are conjugate by an element in Q. Hence their images in $\tilde{W}/nQ \to \tilde{W}/n_i Q$ are conjugate by an element in Q. That contradicts (a). □

Proposition 6.3. *Let $\tilde{w}_1, \cdots, \tilde{w}_r \in \tilde{W}$ be elements in distinct conjugacy classes of $\tilde{W}$. The the elements $\Psi_1(\tilde{w}_1), \cdots, \Psi_1(\tilde{w}_r)$ are linearly independent functions on $K(\mathbb{C}[\tilde{W}])$.*

Proof. Again, it suffices to consider the case where $\tilde{W}/Q$ is finite.

For any $1 \leqslant i < j \leqslant r$, there exists $n_{ij} > 0$ such that the image of $\tilde{w}_i$ and $\tilde{w}_j$ in $\tilde{W}/n_{ij}Q$ are not in the same conjugacy class.

Set $n = \Pi_{1 \leqslant i < j \leqslant r} n_{ij}$ and $F = \tilde{W}/nP$. Then F is a finite group. Let $\underline{\tilde{w}}_i$ be the image of $\tilde{w}_i$ in F. If $\underline{\tilde{w}}_i$ and $\underline{\tilde{w}}_j$ are in the same conjugacy class of F, then their images in $\tilde{W}/n_{ij}P$ under the map $F \to \tilde{W}/n_{ij}P$ are still in the same conjugacy class. That is a contradiction. Hence $\underline{\tilde{w}}_1, \cdots, \underline{\tilde{w}}_r$ are in distinct conjugacy classes of F.

The surjection $\tilde{W} \to F$ induces an injection $\mathrm{K}(\mathbb{C}[F]) \to \mathrm{K}(\mathbb{C}[\tilde{W}])$ and a surjection $\mathrm{K}^*(\mathbb{C}[\tilde{W}]) \to \mathrm{K}^*(\mathbb{C}[F])$. We have the following commutative diagram

$$\begin{tikzcd} \tilde{W} \arrow[r, "\Psi_1"] \arrow[d] & \mathrm{K}^*(\mathbb{C}[\tilde{W}]) \arrow[d] \\ F \arrow[r, "\Psi"] & \mathrm{K}^*(\mathbb{C}[F]) \end{tikzcd}$$

Here $\Psi : F \to \mathrm{K}^*(\mathbb{C}[F])$ is defined in the same way as Ψ_q in §6.4.

Since F is a finite group and $\underline{\tilde{w}}_1, \cdots, \underline{\tilde{w}}_r$ are in distinct conjugacy classes of F, $\Psi(\underline{\tilde{w}}_1), \cdots, \Psi(\underline{\tilde{w}}_r)$ are linearly independent functions on $\mathrm{K}(\mathbb{C}[F])$. Hence $\Psi_1(\tilde{w}_1) \cdots, \Psi_1(\tilde{w}_r)$ are linearly independent functions on $\mathrm{K}(\mathbb{C}[\tilde{W}])$. □

Corollary 6.4. *The map $\Psi_1 : \mathbb{C}[\tilde{W}]/[\mathbb{C}[\tilde{W}], \mathbb{C}[\tilde{W}]] \to K^*(\mathbb{C}[\tilde{W}])$ is injective.*

Now we prove the main result of this section.

Theorem 6.5. *Let $\tilde{w}_1, \cdots, \tilde{w}_r \in \tilde{W}$ be elements in distinct conjugacy classes of $\tilde{W}$. Then the image of $T_{\tilde{w}_1}, \cdots, T_{\tilde{w}_r}$ in $\tilde{H}_{\mathbb{C}}/[\tilde{H}_{\mathbb{C}}, \tilde{H}_{\mathbb{C}}]$ are linearly independent.*

Proof. Set $\mathcal{A}_\mathbb{C} = \mathbb{C}[v, v^{-1}]$. Assume that $\sum_{i=1}^r c_i T_{\tilde{w}_i} \in [\tilde{H}_\mathbb{C}, \tilde{H}_\mathbb{C}]$ with all $c_i \in \mathcal{A}_\mathbb{C}$. Suppose that not all c_i's are 0. Then there exist $c \in \mathcal{A}_\mathbb{C}$ and $c'_i \in \mathcal{A}_\mathbb{C}$ for $1 \leqslant i \leqslant r$ such that $c_i = cc'_i$ and there exists $1 \leqslant s \leqslant r$ with $c'_s \mid_{v=1} \neq 0$.

Let E be a $\tilde{J}$-module over $\mathbb{C}$. Set $E_\mathbb{C} = E \otimes_\mathbb{C} \mathcal{A}_\mathbb{C}$. Then $E_\mathbb{C}$ is a $\tilde{J}_{\mathcal{A}_\mathbb{C}}$-module over $\mathcal{A}_\mathbb{C}$. Via the homomorphism $\tilde{\varphi} : \tilde{H} \to \tilde{J}_\mathcal{A}$, $E_\mathbb{C}$ admits an $\tilde{H}_\mathbb{C}$-module structure over $\mathcal{A}_\mathbb{C}$. We denote it by E_φ. For $h \in \tilde{H}$, let $\mathrm{Tr}_E(h) \in \mathcal{A}_\mathbb{C}$ be the trace of the endomorphism h on E_φ over $\mathcal{A}_\mathbb{C}$. Then $\mathrm{Tr}_E(\sum_{i=1}^r c_i T_{\tilde{w}_i}) = c(\sum_{i=1}^r c'_i \mathrm{Tr}_E(T_{\tilde{w}_i})) = 0 \in \mathcal{A}_\mathbb{C}$. Since $\mathcal{A}_\mathbb{C}$ is an integral domain and $c \neq 0$, we have that $\sum_{i=1}^r c'_i \mathrm{Tr}_E(T_{\tilde{w}_i}) = 0$ for all $\tilde{J}$-modules E. Set $v = 1$, then

$$\text{(a)} \qquad \sum_{i=1}^r c'_i \mid_{v=1} \mathrm{Tr}_E(T_{\tilde{w}_i}) = 0$$

for all $\tilde{J}$-module E. Hence by Lemma 6.1, (a) holds for all $\tilde{W}$-modules. Now by Proposition 6.3,, $c'_i \mid_{v=1} = 0$ for all i. That is a contradiction. □

Corollary 6.6. *The class polynomial $f_{\tilde{w},\mathcal{O}}$ is uniquely determined by the identity (a) in Theorem 5.3.*

Now combining Theorem 5.3 and Theorem 6.5, we have that

Theorem 6.7. *The set $\{T_\mathcal{O}\}$, where $\mathcal{O}$ ranges over the conjugacy classes of W, is a $\mathcal{A}$-basis of $\tilde{H}/[\tilde{H}, \tilde{H}]$.*

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References

[1] N. Bourbaki, *Lie groups and Lie algebras, Chapters 4–6*, Elements of Mathematics, Springer- Verlag, Berlin, 2002.

[2] C. Bonnafé and R. Rouquier, *Affineness of Deligne-Lusztig varieties for minimal length elements*, J. Algebra **320** (2008), no. 3, 1200–1206.

[3] F. Digne and J. Michel, *Endomorphisms of Deligne-Lusztig varieties*, Nagoya Math. J. **183** (2006), 35–103.

[4] Y.Z. Flicker, *Bernstein's isomorphism and good forms*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Amer. Math. Soc. 58 (1995), 171–196.

[5] U. Görtz, X. He and S. Nie, *P-alcoves and nonemptiness of affine Deligne-Lusztig varieties*, arXiv:1211.3784.

[6] M. Geck, S. Kim, and G. Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Coxeter groups*, J. Algebra **229** (2000), no. 2, 570–600.

[7] M. Geck and G. Pfeiffer, *On the irreducible characters of Hecke algebras*, Adv. Math. **102** (1993), no. 1, 79–94.

[8] ———, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press Oxford University Press, New York, 2000.

[9] M. Geck and R. Rouquier, *Centers and simple modules for Iwahori–Hecke algebras*, Finite reductive groups (Luminy, 1994), Progr. Math., 141, pp. 251–272, Birkhäuser, Boston, 1997.

[10] X. He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. **215** (2007), no. 2, 469–503.

[11] ———, *Minimal length elements in conjugacy classes of extended affine Weyl group*, arXiv:1004.4040. A shorter version for $\tilde{A}$ is available at `http://www.math.ust.hk/~maxhhe/affA.pdf`.

[12] ———, *Geometric and cohomological properties of affine Deligne-Lusztig varieties*, Ann. Math. 179 (2014), 367–404.

[13] X. He and G. Lusztig, *A generalization of Steinberg's cross-section*, J. Amer. Math. Soc. **25** (2012), 739–757.

[14] X. He and S. Nie, *Minimal length elements of finite Coxeter groups*, Duke Math. J., **161** (2012), 2945–2967.

[15] X. He and T. Wedhorn, *On parahoric reductions of Shimura varieties of PEL type*, in preparation.

[16] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 5–48.

[17] D. Kazhdan, *Representations of groups over close local fields*, J. Analyse Math. 47 (1986), 175–179.

[18] D. Krammer, *The conjugacy problem for Coxeter groups*, Groups Geom. Dyn. 3 (2009), no. 1, 71–171.

[19] G. Lusztig, *Cells in affine Weyl groups, II*, J. Alg. 109 (1987), 536-548.

[20] ———, *Cells in affine Weyl groups, III*, J. Fac. Sci. Univ. Tokyo. **34** (1987), 223–243.

[21] ———, *From conjugacy classes in the Weyl group to unipotent classes*, Represent. Theory **15** (2011),494–530.

[22] ———, *On certain varieties attached to a Weyl group element*, Bull.Inst.Math.Acad. Sinica (N.S.) **6** (2011), 377–414.

[23] S. Orlik and M. Rapoport, *Deligne-Lusztig varieties and period domains over finite fields*, J. Algebra **320** (2008), no. 3, 1220–1234.

[24] D. E. Speyer, *Powers of Coxeter elements in infinite groups are reduced*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1295–1302.

[25] T.A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–198.

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