BASIC LOCI IN SHIMURA VARIETIES OF COXETER TYPE

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ABSTRACT. This paper is a contribution to the general problem of giving an explicit description of the basic locus in the reduction modulo p of Shimura varieties. Motivated by [29] and [24], we classify the cases where the basic locus is (in a natural way) the union of classical Deligne-Lusztig sets associated to Coxeter elements. We show that if this is satisfied, then the Newton strata and Ekedahl-Oort strata have many nice properties.

INTRODUCTION

Understanding arithmetic properties of Shimura varieties has been a cornerstone in many developments in arithmetic geometry and number theory in the last decades. To a large extent, these arithmetic properties are encoded in the geometric properties of the special fiber of a suitable integral model, and studying these reductions of Shimura varieties has been fruitful in many cases.

A Shimura variety of PEL type can be described as a moduli space of abelian varieties with additional structure (polarization, endomorphisms, level structure). To each such abelian variety we can attach its p-divisible group; it inherits corresponding additional structure. The special fiber at p naturally decomposes into finitely many "Newton strata", which are given by the isogeny class of these p-divisible groups (with additional structure).

There is a unique closed Newton stratum. This is the so-called *basic locus*. For two reasons the basic locus plays a particular role in the study of the geometry of the special fiber. First, it is the only Newton stratum where there is reasonable hope for a complete, explicit description as a variety. Second, it turns out that a good understanding

of basic loci can often be used to prove results about general New-
ton strata, and hence about the whole special fiber, by an induction
process.

Explicit descriptions of the basic locus have been of great importance
in the work of Kudla, Rapoport, Howard, Terstiege and others on the
intersection numbers of special cycles in the special fibers of Shimura
varieties (and their relationship to Fourier coefficients of modular forms
as predicted by the "Kudla program"). Kaiser [14] used the description
of the basic locus in the module space of principally polarized abelian
surfaces in his proof of the twisted fundamental lemma for GSp_4. A
good description of the basic locus is also useful to prove instances
of the "arithmetic fundamental lemma", cf. the paper by Rapoport,
Terstiege and Zhang [25].

While the basic locus is the simplest Newton stratum, it still cannot
be described explicitly in general. For example, the basic locus in the
moduli space of principally polarized abelian varieties of dimension g is
just the supersingular locus, i.e., the closed subvariety of supersingular
abelian varieties.

On the other hand, besides several small rank cases (cf. Section 5.3),
there are a number of families of Shimura varieties of PEL-type where
the basic locus allows for a simple and very explicit description. The
case studied first is probably the "Drinfeld case" where the underly-
ing algebraic group is a division algebra and the basic locus can be
described in terms of Deligne's formal model of Drinfeld's half space.
Other typical cases are those attached to unitary groups $GU(1, n-1)$.
See the papers by Vollaard and Wedhorn [29], and by Rapoport, Ter-
stiege, and Wilson [24]. Roughly speaking, in all those cases the follow-
ing picture emerges: The basic locus is a union of Ekedahl-Oort strata
and admits a stratification by (variants of) classical Deligne-Lusztig
varieties. The index set of the stratification and the closure relations
between strata can be described in terms of the Bruhat-Tits building
of a certain inner form of the underlying group.

The uniformization theorem by Rapoport and Zink [26] allows to de-
scribe the basic locus in terms of a moduli space of p-divisible groups
with additional structure, a so-called Rapoport-Zink space. Roughly
speaking, the set of points of this Rapoport-Zink space can be de-
scribed, using Dieudonné theory, as a space of lattices inside a fixed
L-vector space. Here L is the completion of the maximal unramified
extension of $\mathbb{Q}_p$ (or more generally of a finite extension $F/\mathbb{Q}_p$). The
lattices have to satisfy conditions which ensure that they arise as the
Dieudonné module of a p-divisible group with additional structure as

specified by the moduli problem. In other terminology, these lattices form an *affine Deligne-Lusztig variety* inside the space of all lattices.

In this paper, we are mainly interested in the "Coxeter case", i.e., the cases where the basic locus is a union of Ekedahl-Oort strata and each Ekedahl-Oort stratum is (in a natural way) the union of classical Deligne-Lusztig sets associated to Coxeter elements.

Let F be a finite extension of $\mathbb{Q}_p$ (the mixed characteristic case) or let $F = \mathbb{F}_q((\epsilon))$ (the function field case). Fix a datum (G, μ) of a connected quasi-simple semisimple algebraic group G over F which splits over a tamely ramified extension and a minuscule cocharacter μ (see Section 1.1 and Section 5.1). Let P be a standard rational maximal parahoric subgroup of $G(L)$, where L is the completion of the maximal unramified extension of F. Let σ denote the Frobenius of the extension L/F.

For each $b \in G(L)$ we consider the following union of affine Deligne-Lusztig varieties:

$$X(\mu, b)_P = \{g \in G(L)/P;\ g^{-1}b\sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\mu)} PwP\}$$

(see 7.1, compare also [22] Section 5). Denote by $B(G, \mu)$ the set of σ-conjugacy classes for which this set is non-empty. There is a unique basic σ-conjugacy class in $B(G, \mu)$. If b lies inside this basic σ-conjugacy class, then we call $X(b, \mu)_P$ the *basic locus* attached to the above data. (For a different choice of b we get, up to isomorphism, the same result. In fact, in a large part of the paper we work with all basic b simultaneously, i.e. with "fiber bundles" over the basic σ-conjugacy class, and only later to restrict to single fibers of this bundle.)

In case the data (G, μ, b) corresponds to a Rapoport-Zink space $\mathscr{M}$ (in particular, char $F = 0$), there typically is a bijection $\mathscr{M}(\overline{\mathbb{F}}_p) \cong X(\mu, b)_P$, given by Dieudonné theory; cf. [22].

In Section 5.1 we define a notion of *Ekedahl-Oort elements* which gives rise to a subset $\mathrm{EO}^J(\mu) \subset \tilde{W}$ of the extended affine Weyl group $\tilde{W}$ (here J denotes the type of the parahoric subgroup P). For each b and each $w \in \mathrm{EO}^J(\mu)$, we obtain the EO stratum $X_{J,w}(b)$, cf. (3.4.1), attached to w inside $X(\mu, b)_P$. We say that the basic locus is a union of EO strata, if for all $w \in \mathrm{EO}^J(\mu)$ such that $X_{J,w}(b) \neq \emptyset$ for b basic, we have $X_{J,w}(b') = \emptyset$ for all non-basic b'. For the formal definition of "Coxeter type" see Theorem 5.1.1 and condition (**CC**) in Section 6.

The main results of this paper are summarized below.

Theorem A. *The data (G, μ, P) of Coxeter type are listed in Theorem 5.1.1.*

Theorem B. *If (G, μ, P) is of Coxeter type, then*

(1) *The basic locus*

$$X(\mu, b)_P = \sqcup_\Lambda \mathcal{N}_\Lambda^o,$$

where Λ runs over faces of certain types of the rational Bruhat-Tits building $\mathscr{B}$ of the σ-centralizer $\mathbb{J}_b$ of b, and $\mathcal{N}_\Lambda^o$ is a classical Deligne-Lusztig set associated to a Coxeter element.

(2) *For all non-basic b', the σ-centralizer $\mathbb{J}_{b'}$ acts transitively on $X(\mu, b')_P$. In particular, whenever there is a notion of dimension (in the RZ cases and in the function field case), this implies that $\dim X(\mu, b')_P = 0$.*

In most cases, Λ runs over all vertices in $\mathscr{B}$. In other cases, edges of $\mathscr{B}$ appear. For more details, see Theorem 5.1.1, Theorem 5.2.1 and §7.

Theorem C. *In the function field case, if (G, μ, P) is of Coxeter type, then the closure relation among the strata $\mathcal{N}_\Lambda^o$ can be described explicitly in terms of the simplicial structure of $\mathscr{B}$.*

The key ingredients in this paper are

- The description of affine Deligne-Lusztig varieties in the affine flag variety [10].
- The fine Deligne-Lusztig varieties [20].

In mixed characteristic, in general there is no known scheme structure on affine Deligne-Lusztig varieties. This is the technical difficulty preventing us from extending Theorem C to the mixed characteristic case. However, experience shows that, at least for the basic locus, the descriptions in the mixed characteristic case and in the equal characteristic case are mostly equal (see Sections 5.3, 7.4). Therefore we expect that in those cases which have not been treated in the context of Shimura varieties, our results can serve as guide lines for the precise result to be expected.

We focus on "Coxeter type" in this paper. However, our methods should extend to some other cases where the basic locus is still a union of EO strata but the EO strata there are not of Coxeter type in general. We include one example at the end of this paper.

The paper is organized as follows. In section 1 we fix notation and give a group-theoretic definition of the basic locus. In section 2, we recollect properties of affine Deligne-Lusztig varieties in the affine flag variety. In section 3, we give a stratification, which includes Kottwitz-Rapoport stratification and Ekedahl-Oort stratification as special cases. In section 4, we study the fine affine Deligne-Lusztig varieties in the affine Grassmannian. Theorem A, Theorem B and the first two parts

of Theorem C are stated in section 5 and proofs are given in section 6. Theorem C is proved in section 7. We also study the singularities of $\overline{\mathcal{N}_\lambda^o}$ in section 7.

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After the paper was finished, we received an email from Xinwen Zhu about his joint work in preparation with Liang Xiao [30].They study the basic affine Deligne-Lusztig varieties in the affine Grassmannian for some unramified quasi-split (but not split) groups. This gives a different approach to the basic loci of types $(^2A'_n, \omega_1^\vee, \mathbb{S})$ for n even, $(^2D_n, \omega_1^\vee, \mathbb{S})$ and $(^2A_3, \omega_2^\vee, \mathbb{S})$ in our list in Theorem 5.1.1. They also give a description for some basic loci of non-Coxeter type.

1. PRELIMINARIES

1.1. **Notation.** Let $\mathbb{F}_q$ be the finite field with q elements. Let $\Bbbk$ be an algebraic closure of $\mathbb{F}_q$. Let $F = \mathbb{F}_q((\epsilon))$ or a finite field extension of $\mathbb{Q}_p$ with residue class field $\mathbb{F}_q$ and uniformizer ϵ, and let L be the completion of the maximal unramified extension of F.

Let G be a connected semisimple group over F which splits over a tamely ramified extension of F. Let σ be the Frobenius automorphism of L/F. We also denote the induced automorphism on $G(L)$ by σ.

Let $S \subset G$ be a maximal L-split torus defined over F and let T be its centralizer. Since G is quasi-split over L, T is a maximal torus of G. The *Iwahori-Weyl group* associated to S is

$$\tilde{W} = N_S(L)/T(L)_1.$$

Here N_S denotes the normalizer of S in G, and $T(L)_1$ denotes the unique parahoric subgroup of $T(L)$. For $w \in \tilde{W}$, we choose a representative in $N_S(L)$ and also write it as w.

1.2. **Weyl groups.** We denote by $\mathcal{A}$ the apartment of G_L corresponding to S. We fix a σ-invariant alcove $\mathfrak{a}$ in $\mathcal{A}$, and denote by $I \subseteq G(L)$ the Iwahori subgroup corresponding to $\mathfrak{a}$ over L.

The Iwahori-Weyl group $\tilde{W}$ is an extension of the *relative Weyl group* $W_0 = N_S(L)/T(L)$ by $X_*(T)_\Gamma$, where $\Gamma = \text{Gal}(\bar{L}/L)$ is the absolute Galois group of L. If we choose a special vertex of $\mathfrak{a}$, we may represent the Iwahori-Weyl group as a semidirect product

$$\tilde{W} = X_*(T)_\Gamma \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(T)_\Gamma, w \in W_0\}.$$

We denote by $\tilde{\mathbb{S}}$ the set of simple reflections of $\tilde{W}$ and denote by $\mathbb{S} \subset \tilde{\mathbb{S}}$ the set of simple reflections of W_0. Both $\tilde{W}$ and $\tilde{\mathbb{S}}$ are equipped with an action of σ.

For any subset J of $\tilde{\mathbb{S}}$, we denote by W_J the subgroup of $\tilde{W}$ generated by the simple reflections in J and by $^J\tilde{W}$ (resp. $\tilde{W}^J$) the set of minimal length elements for the cosets $W_J \backslash \tilde{W}$ (resp. $\tilde{W}/W_J$). We simply write $^J\tilde{W}^K$ for $^J\tilde{W} \cap \tilde{W}^K$.

The subgroup $W_{\tilde{\mathbb{S}}}$ of $\tilde{W}$ is the affine Weyl group and we usually denote it by W_a. Then

$$\tilde{W} = W_a \rtimes \Omega,$$

where Ω is the normalizer of the base alcove $\mathfrak{a}$. We may identify Ω with $\pi_1(G)_\Gamma$.

1.3. σ-**conjugacy classes.** We say that $b, b' \in G(L)$ are σ-conjugate if $b' = g^{-1}b\sigma(g)$ for some $g \in G(L)$. We denote by $B(G)$ the set of σ-conjugacy classes of $G(L)$. The classification of the σ-conjugacy classes is obtained by Kottwitz in [16] and [17]. The description is as follows.

An element $b \in G(L)$ determines a homomorphism $\mathbb{D} \to G_L$, where $\mathbb{D}$ is the pro-algebraic torus whose character group is $\mathbb{Q}$. This homomorphism determines an element $\bar{\nu}_b$ in the closed dominant chamber $X_*(T)_\mathbb{Q}^+$. The element $\bar{\nu}_b$ is called the *Newton point* of b and the map $b \mapsto \bar{\nu}_b$ is called the *Newton map*. Let $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_F}$ be the Kottwitz map [17, §7], where Γ_F is the absolute Galois group of F. By [17, §4.13], the map

$$B(G) \to X_*(T)_\mathbb{Q}^+ \times \pi_1(G)_{\Gamma_F}, \qquad b \mapsto (\bar{\nu}_b, \kappa_G(b))$$

is injective. The set $B(G)$ is equipped with a partial order, see [23] Section 2.3.

1.4. **Straight conjugacy classes.** Following [10], we relate $B(G)$ to the Iwahori-Weyl group $\tilde{W}$.

For any $w \in \tilde{W}$, we consider the element $w\sigma \in \tilde{W} \rtimes \langle \sigma \rangle$. There exists $n \in \mathbb{N}$ such that $(w\sigma)^n = t^\lambda$ for some $\lambda \in X_*(T)_\Gamma$. Let $\bar{\nu}_w$ be the unique dominant element in the W_0-orbit of λ/n. It is known that $\bar{\nu}_w$ is independent of the choice of n and is Γ-invariant. Moreover, $\bar{\nu}_w$ is the Newton point of w when regarding w as an element in $G(L)$.

We say that an element w is σ-*straight* if $\ell((w\sigma)^n) = n\ell(w)$. This is equivalent to saying that $\ell(w) = \langle \bar{\nu}_w, 2\rho \rangle$, where ρ is the half sum of all positive roots in the root system of the affine Weyl group W_a. A σ-conjugacy class of $\tilde{W}$ is called *straight* if it contains a σ-straight element.

The map $N_S(L) \to G(L)$ induces a map $\tilde{W} \to B(G)$. Here $\kappa_G(w)$ is the image of w under the projection $\tilde{W} \to \Omega \cong \pi_1(G)_\Gamma \to \pi_1(G)_{\Gamma_F}$. By [10, §3], the map induces a bijection from the set of straight σ-conjugacy classes of $\tilde{W}$ to $B(G)$.

A σ-conjugacy class $[b]$ of $G(L)$ is called *basic* if $\bar{\nu}_b$ factors through the center of G. Again by [10, §3], a σ-conjugacy class of $G(L)$ is basic if and only if it contains some element of Ω.

1.5. **The variety Z.** Let $\mu \in X_*(T)$ be a minuscule coweight. We denote by λ its image in the coinvariants $X_*(T)_\Gamma$. The *admissible subset* of $\tilde{W}$ associated to μ is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W}; w \leqslant t^{x(\lambda)} \text{ for some } x \in W_0\}.$$

Note that λ is not minuscule in $\tilde{W}$ in general (see also §6.2). We also denote by τ the image of t^λ under the projection map $\tilde{W} = W_a \rtimes \Omega \to \Omega$.

Let $J \subset \tilde{\mathbb{S}}$. Let $P_J \supset I$ be the standard parahoric subgroup corresponding to J. Set $\mathrm{Adm}^J(\mu) = W_J \mathrm{Adm}(\mu) W_{\sigma(J)}$ and

$$Y_J = \cup_{w \in \mathrm{Adm}(\mu)} P_J w P_{\sigma(J)} = \cup_{w \in \mathrm{Adm}^J(\mu)} IwI.$$

Define the action of P_J on $G(L) \times Y_J$ by $p \cdot (g, y) = (gp^{-1}, py\sigma(p)^{-1})$ and denote by Z_J its quotient. Then the map $(g, y) \mapsto (gy\sigma(g)^{-1}, gP_J)$ gives an isomorphism

$$Z_J \cong \{(b, gP_J) \in G(L) \times G(L)/P_J; g^{-1}b\sigma(g) \in Y\}.$$

The image of the projection map $Z_J \to G(L)$ is a union of σ-conjugacy classes of $G(L)$ and we denote it by $B(G, \mu)_J$. In fact, $B(G, \mu)_J$ is independent of the subset J we choose [11]. However, we don't need this fact in this paper.

The basic σ-conjugacy class in $B(G, \mu)_J$ contains the element τ and we denote this σ-conjugacy class by $\mathcal{O}_0$. We have the Newton stratification

$$Z_J = \sqcup_{\mathcal{O} \in B(G,\mu)_J} Z_{J,\mathcal{O}},$$

where $Z_{J,\mathcal{O}} = \{(b, gP_J) \in Z_J; b \in \mathcal{O}\}$. The stratum $Z_{J,\mathcal{O}_0}$ is called the *basic locus in Z_J*.

2. Affine Deligne-Lusztig varieties

2.1. **Affine Deligne-Lusztig varieties.** We first look at the case where $J = \emptyset$. Then $Y_\emptyset = \sqcup_{w \in \mathrm{Adm}(\mu)} IwI$ and we have the Kottwitz-Rapoport stratification

$$Z_\emptyset = \sqcup_{w \in \mathrm{Adm}(\mu)} Z_{\emptyset,w},$$

where $Z_{\emptyset,w} = G(L) \times^I IwI$ for any $w \in \mathrm{Adm}(\mu)$.

Given $w \in \mathrm{Adm}(\mu)$ and $\mathcal{O} \in B(G, \mu)_\emptyset$, the intersection $Z_{\emptyset,w} \cap Z_{\emptyset,\mathcal{O}}$ is a fiber bundle over $\mathcal{O}$ and the fiber over $b \in \mathcal{O}$ is the affine Deligne-Lusztig variety (in the affine flag variety)

$$X_w(b) = \{gI \in G(L)/I; g^{-1}b\sigma(g) \in IwI\} \subset G(L)/I.$$

Moreover, the σ-centralizer $\mathbb{J}_b = \{g \in G(L); g^{-1}b\sigma(g) = b\}$ acts on $X_w(b)$.

In the rest of this section, we recollect some results on affine Deligne-Lusztig varieties. The following result on σ-straight elements is proved in [10, Proposition 4.5 & Theorem 4.8].

Theorem 2.1.1. *If w is σ-straight, then $X_w(b) \neq \emptyset$ if and only if b is σ-conjugate to w. In this case,*

$$X_w(b) \cong X_w(w) \cong \mathbb{J}_w/(\mathbb{J}_w \cap I)$$

(which means that, in the function field case, $X_w(b)$ is 0-dimensional).

2.2. **Support.** For $w \in W_a$, we denote by $\mathrm{supp}(w)$ the support of w, i.e., the set of $i \in \tilde{\mathbb{S}}$ such that s_i appears in some (or equivalently, any) reduced expression of w. We set

$$\mathrm{supp}_\sigma(w\tau) = \bigcup_{n \in \mathbb{Z}} (\tau\sigma)^n(\mathrm{supp}(w)).$$

Then $\mathrm{supp}_\sigma(w\tau)$ is the minimal $\tau\sigma$-stable subset J of $\tilde{\mathbb{S}}$ such that $w\tau\sigma \in W_J \rtimes \langle \tau\sigma \rangle\}$.

If $\ell(w) = \sharp(\mathrm{supp}_\sigma(w\tau)/\langle\tau\sigma\rangle)$, i.e., w is a product of simple reflections in W_a and the simple reflections from each orbit of $\tau\sigma$ appears at most once, then we say that $w\tau$ is a σ-*Coxeter element*.

Proposition 2.2.1. *Let $w \in W_a\tau$ such that $W_{\mathrm{supp}_\sigma(w)}$ is finite. Then*

$$X_w(\tau) = \coprod_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\mathrm{supp}_\sigma(w)})} iY(w),$$

where

$$Y(w) = \{gI \in P_{\mathrm{supp}_\sigma(w)}/I; \ g^{-1}\tau\sigma(g) \in IwI\}$$

is a classical Deligne-Lusztig variety in the finite-dimensional flag variety $P_{\mathrm{supp}_\sigma(w)}/I$.

The proposition follows from the proof of [10, Theorem 4.7]. For the sake of completeness and because the reasoning simplifies in our setting, we reproduce the relevant part of the proof in loc. cit.

Proof. Let $g \in G(L)$ with $g^{-1}\tau\sigma(g) \in IwI$. By [10, Lemma 3.2 & Proposition 4.5], there exists $p \in P_{\mathrm{supp}_\sigma(w)}$, such that $(gp)^{-1}\tau\sigma(gp) = \tau$. Hence $g \in \mathbb{J}_\tau P_{\mathrm{supp}_\sigma(w)}$, and

$$X_w(\tau) = \{gI \in \mathbb{J}_\tau P_{\mathrm{supp}_\sigma(w)}/I; g^{-1}\tau\sigma(g) \in IwI\} \subset \mathbb{J}_\tau P_{\mathrm{supp}_\sigma(w)}/I.$$

Note that $\mathbb{J}_\tau P_{\mathrm{supp}_\sigma(w)}/I = \sqcup_{i\in\mathbb{J}_\tau/(\mathbb{J}_\tau\cap P_{\mathrm{supp}_\sigma(w)})}iP_{\mathrm{supp}_\sigma(w)}/I$ and for any $g \in P_{\mathrm{supp}_\sigma(w)}$ and $i \in \mathbb{J}_\tau$, $(ig)^{-1}\tau\sigma(ig) \in IwI$ if and only if $g^{-1}\tau\sigma(g) \in IwI$. Hence

$$X_w(\tau) = \coprod_{i\in\mathbb{J}_\tau/(\mathbb{J}_\tau\cap P_{\mathrm{supp}_\sigma(w)})} Y(w).$$

$\square$

3. P_J-STABLE PIECES

3.1. **Partial conjugation action.** Let $J \subset \tilde{\mathbb{S}}$. The partial conjugation action of W_J on $\tilde{W}$ defined by $x \cdot_\sigma y = xy\sigma(x)^{-1}$ for $x \in W_J$ and $y \in \tilde{W}$.

Given $w, w' \in \tilde{W}$ and $j \in J$, we write $w \xrightarrow{s_j}_\sigma w'$ if $w' = s_j w s_{\sigma(j)}$ and $l(w') \leqslant l(w)$. If $w = w_0, w_1, \cdots, w_n = w'$ is a sequence of elements in $\tilde{W}$ such that for all k, we have $w_{k-1} \xrightarrow{s_j}_\sigma w_k$ for some $j \in J$, then we write $w \to_{J,\sigma} w'$. We write $w \approx_{J,\sigma} w'$ if $w \to_{J,\sigma} w'$ and $w' \to_{J,\sigma} w$.

The following property is proved in [6, Proposition 3.4].

Proposition 3.1.1. *For any $w \in \tilde{W}$, there exists a minimal length element $w' \in W_J \cdot_\sigma w$ such that $w \to_{J,\sigma} w'$. Moreover, we may take w' to be of the form vw_1 with $w_1 \in {}^J\tilde{W}$ and $v \in W_{I(J,w_1,\sigma)}$. Here $I(J, w_1, \sigma) = \max\{K \subset J; Ad(w_1)\sigma(K) = K\}$.*

3.2. **The subset $P_J \cdot_\sigma IwI$.** By [9, section 2],
   (1) If $w \approx_{J,\sigma} w'$, then $P_J \cdot_\sigma IwI = P_J \cdot_\sigma Iw'I$.
   (2) If $w \xrightarrow{s_i}_\sigma w'$ with $\ell(w') < \ell(w)$, then $P_J \cdot_\sigma IwI = P_J \cdot_\sigma Iw'I \cup P_J \cdot_\sigma Is_iwI$.
   (3) If $w \in {}^J\tilde{W}$ and $x \in W_{I(J,w,\sigma)}$, then $P_J \cdot_\sigma IxwI = P_J \cdot_\sigma IwI$.

A subset of $G(L)$ of the form $P_J \cdot_\sigma IwI$ for some $w \in {}^J\tilde{W}$ is called a P_J-*stable piece*. It is analogous to the G-stable pieces introduced by Lusztig in [19]. It is showed in [21, 1.4] and [9, Proposition 2.5 & 2.6] that

$$G(L) = \sqcup_{w\in{}^J\tilde{W}} P_J \cdot_\sigma IwI.$$

The following result is essentially contained in the proof of [9, Proposition 2.5].

Theorem 3.2.1. *Let $w \in \tilde{W}$. Then*

$$P_J w P_{\sigma(J)} = \sqcup_{x \in W_J w W_{\sigma(J)} \cap {}^J \tilde{W}} P_J \cdot_\sigma IxI.$$

Proof. It is obvious that $\sqcup_{x \in W_J w W_{\sigma(J)} \cap {}^J \tilde{W}} P_J \cdot_\sigma IxI \subset P_J w P_{\sigma(J)}$. Now we prove that $P_J w P_{\sigma(J)} \subset \sqcup_{x \in W_J w W_{\sigma(J)} \cap {}^J \tilde{W}} P_J \cdot_\sigma IxI$.

Notice that $P_J w P_{\sigma(J)} = \sqcup_{w' \in W_J w W_{\sigma(J)}} Iw'I$. We argue by induction that $Iw'I \subset \sqcup_{x \in W_J w W_{\sigma(J)} \cap {}^J \tilde{W}} P_J \cdot_\sigma IxI$ for any $w' \in W_J w W_{\sigma(J)}$.

If w' is a minimal length element in $W_J \cdot_\sigma w'$, then $w' \approx_{J,\sigma} vx$ for some $x \in {}^J \tilde{W}$ and $v \in W_{I(J,x,\sigma)}$. In this case, $x \in W_J w' W_{\sigma(J)} = W_J w W_{\sigma(J)}$. The statement follows from §3.2 (1) & (3).

If w' is not a minimal length element in $W_J \cdot_\sigma w'$, then there exists $w'' \approx_{J,\sigma} w'$ and $i \in J$ such that $\ell(s_i w'' s_{\sigma(i)}) < \ell(w')$. The statement follows from the induction hypothesis and §3.2 (1) & (2). $\qquad\square$

3.3. **A partial order on ${}^J \tilde{W}$.** We introduce $\leqslant_{J,\sigma}$ as follows. For $w \in {}^J \tilde{W}$ and $w' \in \tilde{W}$, we write $w \leqslant_{J,\sigma} w'$ if there exists $x \in W_J$ such that $x w \sigma(x)^{-1} \leqslant w'$. By [6, 4.7], $\leqslant_{J,\sigma}$ gives a partial order on ${}^J \tilde{W}$. For $w, w' \in {}^J \tilde{W}$,

(1) $w \leqslant w'$ implies that $w \leqslant_{J,\sigma} w'$;
(2) $w \leqslant_{J,\sigma} w'$ implies that $\ell(w) \leqslant \ell(w')$.

It is proved in [9, Proposition 2.6] that if $F = \mathbb{F}_q((\epsilon))$, then

$$\overline{P_J \cdot_\sigma IwI} = \sqcup_{x \in {}^J \tilde{W}, x \leqslant_{J,\sigma} w} P_J \cdot_\sigma IxI.$$

3.4. **A stratification of Z_J.** Since $Y_J = \sqcup_{w \in \mathrm{Adm}^J(\mu) \cap {}^J \tilde{W}} P_J \cdot_\sigma IwI$, we have the stratification

$$Z_J = \sqcup_{w \in \mathrm{Adm}^J(\mu) \cap {}^J \tilde{W}} Z_{J,w},$$

where $Z_{J,w} = G(L) \times^{P_J} (P_J \cdot_\sigma IwI)$. This includes as special cases the Kottwitz-Rapoport stratification discussed in §2.1 and the Ekedahl-Oort stratification we will discuss in §5.1. See also [13].

Given $w \in \mathrm{Adm}^J(\mu) \cap {}^J \tilde{W}$ and $\mathcal{O} \in B(G, \mu)_J$, the intersection $Z_{J,w} \cap Z_{J,\mathcal{O}}$ is a fiber bundle over $\mathcal{O}$ and the fiber over $b \in \mathcal{O}$ is

$$(3.4.1) \qquad X_{J,w}(b) := \{gP_J; g^{-1} b\sigma(g) \in P_J \cdot_\sigma IwI\} \subset G(L)/P_J.$$

It is the image of $X_w(b)$ under the projection map $\pi_J : G(L)/I \to G(L)/P_J$. We call $X_{J,w}(b)$ a *fine affine Deligne-Lusztig variety* in $G(L)/P_J$.

4. Fine affine Deligne-Lusztig varieties

4.1. (Coarse) affine Deligne-Lusztig varieties. For any $J \subset \tilde{\mathbb{S}}$, we have another stratification

$$G(L)/P_J = \sqcup_{w \in {}^J \tilde{W}^{\sigma(J)}} \{gP_J; g^{-1}b\sigma(g) \in P_J w P_{\sigma(J)}\}.$$

Each subset $\{gP_J; g^{-1}b\sigma(g) \in P_J w P_{\sigma(J)}\}$ is a union of fine affine Deligne-Lusztig varieties. We call it a *(coarse) affine Deligne-Lusztig variety* in $G(L)/P_J$. Similar to the proof of Proposition 2.2.1, we have

Proposition 4.1.1. *Let $J \subset \tilde{\mathbb{S}}$ and $w \in {}^J\tilde{W}^{\sigma(J)} \cap W_a\tau$ such that $Ad(w)\sigma(J) = J$. If $W_{\mathrm{supp}_\sigma(w) \cup J}$ is finite, then*

$$\{gP_J; g^{-1}\tau\sigma(g) \in P_J w P_{\sigma(J)}\} = \sqcup_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\mathrm{supp}_\sigma(w) \cup J})} iY_J(w),$$

where $Y_J(w) = \{gP_J \in P_{\mathrm{supp}_\sigma(w) \cup J}/P_J; g^{-1}\tau\sigma(g) \in P_J w P_{\sigma(J)}\}$ is a classical Deligne-Lusztig variety in the partial flag variety $P_{\mathrm{supp}_\sigma(w) \cup J}/P_J$.

The main result we prove in this section (see Sections 4.5, 4.6) is the following theorem which relates fine affine Deligne-Lusztig varieties with (coarse) affine Deligne-Lusztig varieties.

Theorem 4.1.2. *For any $J \subset \tilde{\mathbb{S}}$ and $w \in {}^J\tilde{W}$,*
(1) $X_{J,w}(b) \cong \{gP_{I(J,w,\sigma)}; g^{-1}b\sigma(g) \in P_{I(J,w,\sigma)} w P_{\sigma(I(J,w,\sigma))}\}$.
(2) If $F = \mathbb{F}_q((\epsilon))$, then $\dim X_{J,w}(b) = \dim X_w(b)$.

Corollary 4.1.3. *If $J \subset \tilde{\mathbb{S}}$ and $w \in {}^J\tilde{W}$ is a σ-Coxeter element in the finite Weyl group $W_{\mathrm{supp}_\sigma(w)}$, then*

$$X_{J,w}(\tau) \cong \sqcup_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\mathrm{supp}_\sigma(w) \cup I(J,w,\sigma)})} iY_{I(J,w,\sigma)}(w),$$

where $Y_{I(J,w,\sigma)}(w) = \{gP_{I(J,w,\sigma)} \in P_{\mathrm{supp}_\sigma(w) \cup I(J,w,\sigma)}/P_{I(J,w,\sigma)}; g^{-1}\tau\sigma(g) \in P_{I(J,w,\sigma)} w P_{\sigma(I(J,w,\sigma))}\}$.
Furthermore, the projection π_J induces an isomorphism from the classical Deligne-Lusztig variety $\{gI \in P_{\mathrm{supp}_\sigma(w)}/I; g^{-1}\tau\sigma(g) \in IwI\}$ in the flag variety $P_{\mathrm{supp}_\sigma(w)}/I$ to

$$\{gP_{\mathrm{supp}_\sigma(w) \cap J} \in P_{\mathrm{supp}_\sigma(w)}/P_{\mathrm{supp}_\sigma(w) \cap J}; g^{-1}\tau\sigma(g) \in P_{\mathrm{supp}_\sigma(w) \cap J} \cdot_\sigma IwI\}$$
$$\cong Y_{I(J,w,\sigma)}(w),$$

and $Y_{I(J,w,\sigma)}(w)$ has dimension $\ell(w)$.

Proof. The first part of the corollary follows from Theorem 4.1.2 and Proposition 4.1.1.

Furthermore, it is easy to see that $I(J, w, \sigma)$ consists of $j \in J$ such that s_j commutes with s_i for all $i \in \mathrm{supp}_\sigma(w)$. In particular, $I(J, w, \sigma)$ and $\mathrm{supp}_\sigma(w)$ are disconnected in the affine Dynkin diagram. This implies that $\{gI \in P_{\mathrm{supp}_\sigma(w)}/I; g^{-1}\tau\sigma(g) \in IwI\}$ is isomorphic to $Y_{I(J,w,\sigma)}$.

The proof of Theorem 4.1.2 (1) implies that π_J restricts to an isomorphism on $\{gI \in P_{\operatorname{supp}_\sigma(w)}/I; g^{-1}\tau\sigma(g) \in IwI\}$, and that the image of the latter variety can be identified with $Y_{I(J,w,\sigma)}(w)$. $\square$

We follow the approach of [20] and [8] for classical fine Deligne-Lusztig varieties in the partial flag variety.

4.2. **The parahoric subgroup P^Q.** For any $g \in G(L)$ and $H \subset G(L)$, we simply write $^g H$ for gHg^{-1}. Let $G(L)'$ be the subgroup generated by all parahoric subgroups of $G(L)$. For any $J \subset \tilde{\mathbb{S}}$, let $\mathcal{P}_J = \{^g P_J; g \in G(L)'\} \cong G(L)'/P_J$ be the set of parahoric subgroups conjugate to P_J by an element of $G(L)'$. For any $J, K \subset \tilde{\mathbb{S}}$, $P \in \mathcal{P}_J$ and $Q \in \mathcal{P}_K$, we write $\operatorname{pos}(P, Q) = w$ if $w \in {}^J W_a^K$ and there exists $g \in G(L)'$ such that $P = gP_Jg^{-1}$ and $Q = g\dot{w}P_K\dot{w}^{-1}g^{-1}$, where $\dot{w}$ is a representative of w in $G(L)'$.

For any parahoric subgroups P and Q, we set $P^Q = (P \cap Q)U_P$, where U_P is the pro-unipotent radical of P. By [21], §1.1 (see also [8], Lemma 2.3), one shows that P^Q is again a parahoric subgroup. For any $g \in G(L)$,

$$({}^g P)^{({}^g Q)} = {}^g(P^Q).$$

4.3. **Bédard's description of ${}^J W_a$.** For $J \subset \tilde{\mathbb{S}}$, let $\mathcal{T}(J, \tau\sigma)$ be the set of sequences $(J_n, w_n)_{n \geqslant 0}$ such that
 (a) $J_0 = J$,
 (b) $J_n = J_{n-1} \cap w_{n-1}\tau\sigma(J_{n-1})w_{n-1}^{-1}$ for $n \geqslant 1$,
 (c) $w_n \in {}^{J_n} W_a{}^{\tau\sigma(J_n)}$ for $n \geqslant 0$,
 (d) $w_n \in W_{J_n} w_{n-1} W_{\tau\sigma(J_{n-1})}$ for $n \geqslant 1$.
Then for any sequence $(J_n, w_n)_{n \geqslant 0} \in \mathcal{T}(J, \tau\sigma)$, we have that $w_n = w_{n+1} = \cdots$ and $J_n = J_{n+1} = \cdots$ for $n \gg 0$. By [1], the assignment $(J_n, w_n)_{n \geqslant 0} \mapsto w_m$ for $m \gg 0$ defines a bijection $\mathcal{T}(J, \tau\sigma) \to {}^J W_a$.

4.4. **Lusztig's partition of $\mathcal{P}_J$.** Following [20], section 4, we give a partition on $\mathcal{P}_J$.

Let $b \in \tau G(L)'$. To each $P \in \mathcal{P}_J$, we associate a sequence $(P^n, J_n, w_n)_{n \geqslant 0}$ as follows

$$P^0 = P, \quad P^n = (P^{n-1})^{(b\sigma(P^{n-1})b^{-1})} \text{ for } n \geqslant 1,$$

$$J_n \subset I \text{ with } P^n \in \mathcal{P}_{J_n}, \quad w_n = \operatorname{pos}(P^n, b\sigma(P^n)b^{-1}) \qquad \text{for } n \geqslant 0.$$

By [21, §1.4], $(J_n, w_n)_{n \geqslant 0} \in \mathcal{T}(J, \tau\sigma)$. For $w \in {}^J W_a$, let

$$\mathcal{P}_{J,w\tau}(b) = \{P \in \mathcal{P}_J; w_m = w \text{ for } m \gg 0\}.$$

Then $\mathcal{P}_J = \sqcup_{w \in {}^J W_a} \mathcal{P}_{J,w\tau}(b)$.

Proposition 4.4.1. *Let $w \in {}^J W_a$. Then*

$$\mathcal{P}_{J,w\tau}(b) = \{{}^g P_J; g \in G(L)', g^{-1}b\sigma(g) \in P_J \cdot_\sigma Iw\tau I\}.$$

Proof. Notice that $G(L)' \cdot_\sigma b \subset \tau G(L)' = \sqcup_{w \in {}^J W_a} P_J \cdot_\sigma Iw\tau I$. Then any $P \in \mathcal{P}_J$ is of the form $P = {}^g P_J$ for some $g \in G(L)$ with $g^{-1}b\sigma(g) \in Iw\tau I$ for a unique $w \in {}^J W_a$. Let $(P^n, J_n, w_n)_{n \geqslant 0}$ be the sequence associated to ${}^g P_J$. Similar to [8, Lemma 2.4], $w_m = w$ for $m \gg 0$. Hence $P \in \mathcal{P}_{J,w\tau}(b)$. $\square$

4.5. **Part (1) of Theorem 4.1.2.** Let $p_J : \mathcal{P}_\emptyset \to \mathcal{P}_J$ be the projection map. Similarly to [20, §4.2 (c) & (d)], for any $n \geqslant 0$, the map $P \mapsto P^n$ gives an isomorphism $\vartheta_n : \mathcal{P}_{J,w\tau}(b) \to \mathcal{P}_{J_n,w\tau}(b)$ and the inverse map is p_J. In particular, $p_{J_n} = \vartheta_n \circ p_J$.

By [5, Lemma 1.4], $I(J, w\tau, \sigma) = J_m$ for $m \gg 0$. By Lang's theorem, $P_{I(J,w\tau,\sigma)} \cdot_\sigma Iw\tau I = P_{I(J,w\tau,\sigma)}w\tau P_{\sigma(I(J,w\tau,\sigma))}$. Therefore $\mathcal{P}_{J,w\tau}(b)$ is isomorphic to

$$\mathcal{P}_{I(J,w\tau,\sigma),w\tau}(b) = \{{}^g P_{I(J,w\tau,\sigma)} \in \mathcal{P}_{I(J,w\tau,\sigma)}; g^{-1}b\sigma(g) \in P_{I(J,w\tau,\sigma)}w\tau P_{\sigma(I(J,w\tau,\sigma))}\}.$$

We fix $\tau' \in \Omega$. Then

$$X_{J,w\tau}(b) \cap \tau'G(L)'/P_J = \{g\tau'P_J; g \in G(L)', (g\tau')^{-1}b\sigma(g\tau') \in P_J \cdot_\sigma Iw\tau I\}$$
$$= \{gP_{\tau'(J)}\tau'; g \in G(L)', g^{-1}b\sigma(g) \in P_{\tau'(J)} \cdot_\sigma I\tau'w\tau\sigma(\tau')^{-1}I\}$$
$$\cong \{gP_K\tau'; g \in G(L)', g^{-1}b\sigma(g) \in P_K\tau'w\tau\sigma(\tau')^{-1}P_{\sigma(K)}\}$$
$$= \{g\tau'P_{(\tau')^{-1}(K)}; g \in G(L)', g^{-1}b\sigma(g) \in P_{(\tau')^{-1}(K)}w\tau P_{\sigma(\tau')^{-1}(K)}\}$$
$$= X_{I(J,w\tau,\sigma),w\tau}(b) \cap \tau'G(L)'/P_{I(J,w\tau,\sigma)},$$

here $K = I(\tau'(J), \tau'w\tau\sigma(\tau')^{-1}, \sigma) = \tau'(I(J, w\tau, \sigma))$. Part (1) of the Theorem 4.1.2 follows by combining all such τ''s together.

4.6. **Part (2) of Theorem 4.1.2.** In this subsection, we assume that $F = \mathbb{F}_q((\epsilon))$. Suppose that $w_m = w$. Let $\bar{P}_{I(J,w\tau,\sigma)}$ be the reductive quotient of $P_{I(J,w\tau,\sigma)}$ and $\bar{I}$ the image of I in $\bar{P}_{I(J,w\tau,\sigma)}$. The fiber of the map $p_{I(J,w\tau,\sigma)} : \mathcal{P}_{\emptyset,w\tau}(b) \to \mathcal{P}_{I(J,w\tau,\sigma),w\tau}(b)$ is isomorphic to $\{pI \in P_{I(J,w\tau,\sigma)}/I; p^{-1}w\tau\sigma(p) \in Iw\tau I\} = \{pI \in \bar{P}_{I(J,w\tau,\sigma)}/\bar{I}; (\text{Ad}(w\tau) \circ \sigma)(pI) = pI\}$. Here $\text{Ad}(w\tau) \circ \sigma$ is a twisted Frobenius morphism on $P_{I(J,w\tau,\sigma)}/I \cong \bar{P}_{I(J,w\tau,\sigma)}/\bar{I}$. In particular, the fiber of $\pi_{I(J,w\tau,\sigma)}$ is 0-dimensional. Since θ_m is an isomorphism and $p_{I(J,w\tau,\sigma)} = \theta_m \circ p_J$, the fiber of p_J is also 0-dimensional. Hence $\dim X_{J,w\tau}(b) = \dim X_{w\tau}(b)$.

5. Newton strata and Ekedahl-Oort strata

5.1. **Ekedahl-Oort strata.** From now on, we assume that G is absolutely quasi-simple and J is a maximal proper subset of $\tilde{\mathbb{S}}$ and that

$\sigma(J) = J$. Therefore $P_J(F)$ is a rational maximal parahoric subgroup[1] of $G(F)$.

We simply write $\mathrm{EO}^J(\mu)$ for $\mathrm{Adm}^J(\mu) \cap {}^J\tilde{W}$. The elements in $\mathrm{EO}^J(\mu)$ are called *EO elements*. Here EO stands for Ekedahl-Oort. For any $w \in \mathrm{EO}^J(\mu)$, we call $Z_{J,w}$ an *Ekedahl-Oort stratum*.

Let $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$ be the subset of $\mathrm{EO}^J(\mu)$ consists of elements w, where $\mathrm{supp}_\sigma(w)$ is a proper subset of $\tilde{\mathbb{S}}$ and w is a σ-Coxeter element of $W_{\mathrm{supp}_\sigma(w)}$. Since G is absolutely quasi-simple, the affine Dynkin diagram of $\tilde{W}$ is connected. Hence if $w \in \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$, then $W_{\mathrm{supp}_\sigma(w)}$ is a finite Weyl group.

It follows from Viehmann's paper [28], Theorem 1.1, that this notion coincides with the usual notion of Ekedahl-Oort strata, if G is unramified.

We are mainly interested in the case where the basic locus is the union

$$(5.1.1) \qquad Z_{J,\mathcal{O}_0} = \sqcup_{w \in \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)} Z_{J,w}.$$

In other words, the basic locus is a union of Ekedahl-Oort strata and each Ekedahl-Oort stratum is ("in a natural way") the union of classical Deligne-Lusztig varieties attached to a Coxeter element. If (5.1.1) holds, then we say that (G, μ, J) is *of Coxeter type*. The first main result of this paper is the classification theorem.

Theorem 5.1.1. *Let (G, μ, J) be as in §5.1. The following 21 cases is the complete list (up to isomorphisms) of the triples (G, μ, J) of Coxeter type:*[2]

$(A_n, \omega_1^\vee, \mathbb{S})$	$(B_n, \omega_1^\vee, \mathbb{S})$	$(B_n, \omega_1^\vee, \mathbb{S} - \{n\})$
$(B\text{-}C_n, \omega_1^\vee, \mathbb{S})$	$(B\text{-}C_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(C\text{-}B_n, \omega_1^\vee, \mathbb{S})$
$(C\text{-}BC_n, \omega_1^\vee, \mathbb{S})$	$(C\text{-}BC_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$	$(D_n, \omega_1^\vee, \mathbb{S})$
$({}^2A'_n, \omega_1^\vee, \mathbb{S})$	$({}^2B_n, \omega_1^\vee, \mathbb{S} - \{n\})$	$({}^2B\text{-}C_n, \omega_1^\vee, \mathbb{S} - \{n\})$
$({}^2D_n, \omega_1^\vee, \mathbb{S})$	$(A_3, \omega_2^\vee, \mathbb{S})$	$({}^2A'_3, \omega_2^\vee, \mathbb{S})$
$(C_2, \omega_2^\vee, \mathbb{S})$	$(C_2, \omega_2^\vee, \mathbb{S} - \{1\})$	$({}^2C_2, \omega_2^\vee, \tilde{\mathbb{S}} - \{1\})$
$({}^2C\text{-}B_2, \omega_1^\vee, \tilde{\mathbb{S}} - \{1\})$		

The theorem is proved in section 6.

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[1]This assumption excludes some cases to which our method can be applied, for instance the Hilbert-Blumenthal case. In fact, one may consider maximal rational parahoric subgroups instead. However, in some cases, the only maximal rational parahoric subgroups are just rational Iwahori subgroups. Then the resulting stratification is the Kottwitz-Rapoport stratification. It is much harder to achieve a complete classification under this weaker assumption, so we do not consider it here.

[2]Here we use the names in [27, Section 4].

5.2. **Ranked poset.** Recall that a ranked poset is a partially ordered set (poset) equipped with a rank function ρ such that whenever y covers x, $\rho(y) = \rho(x) + 1$. We say that the partial order of a poset is *almost linear* if the poset has a rank function ρ such that for any x, y in the poset, $x < y$ if and only if $\rho(x) < \rho(y)$.

Theorem 5.2.1. *Let (G, μ, J) be as in Theorem 5.1.1. Then*

(1) *Every Newton stratum is a union of Ekedahl-Oort strata. In other words, there is a map $\mathrm{EO}^J(\mu) \to B(G, \mu)_J$, $w \mapsto \mathcal{O}_w$ such that $Z_w \subset Z_{\mathcal{O}_w}$.*

(2) *The partial order of $B(G, \mu)_J$ (inherited from $B(G)$) is almost linear.*

(3) *The partial order $\leqslant_{J,\sigma}$ of $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$ coincides with the usual Bruhat order and is almost linear. Here the rank is the length function.*

(4) *For any $w \in \mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$ and $b' \in \mathcal{O}_w$ the σ-centralizer $\mathbb{J}_{b'}$ acts transitively on $X_{J,w}(b')$. In particular, whenever there is a notion of dimension (in the RZ cases and in the function field case), this implies that $\dim X_{J,w}(b') = 0$.*

(5) *For any $w \in \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$, set $J(w, \sigma) = I(J, w, \sigma) \cup \mathrm{supp}_\sigma(w)$. Then*

$$X_{J,w}(\tau) = \sqcup_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{J(w,\sigma)})} iY(w),$$

where

$$Y(w) = \{gP_{I(J,w,\sigma)} \in P_{J(w,\sigma)}/P_{I(J,w,\sigma)};$$
$$g^{-1}\tau\sigma(g) \in P_{I(J,w,\sigma)}wP_{\sigma(I(J,w,\sigma))}\}.$$

The theorem is proved in section 6.11.

5.3. Many of the above cases have been investigated in the context of Shimura varieties:

The case $(G, \mu, J) = (C_2, \omega_2^\vee, \mathbb{S})$ has been studied by Katsura and Oort [15] and Kaiser [14]; see also the paper [18] by Kudla and Rapoport (where the results are applied to computing intersection numbers of arithmetic cycles).

The case $(G, \mu, J) = (A_n, \omega_1^\vee, \mathbb{S})$ is the $U(1, n)$, p split case in which the basic locus is 0-dimensional. This is the situation considered by Harris and Taylor in [4].

The case $(G, \mu, J) = ({}^2A'_n, \omega_1^\vee, \mathbb{S})$ is the $U(1, n)$, p inert case and is studied by Vollaard and Wedhorn in [29].

The cases $(G, \mu, J) = (B\text{-}C_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\}), ({}^2B\text{-}C_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$ and $(C\text{-}BC_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\})$ are the $U(1, *)$, p ramified cases and are studied

by Rapoport, Terstiege and Wilson in [24]. See Section 7.4 where we discuss these cases in more detail.

The case $(^2A_3', \omega_2^\vee, \mathbb{S})$ is the $U(2,2)$, p inert case and is studied by Howard and Pappas in [12]. They transfer the problem to questions about an orthogonal group. The case $(G, \mu, J) = (A_3, \omega_2^\vee, \mathbb{S})$ is the $U(2,2)$, p split case which has not been written down in detail but which should not be hard to deal with.

The cases $(G, \mu, J) = (B\text{-}C_n, \omega_1^\vee, \mathbb{S})$ and $(C\text{-}BC_n, \omega_1^\vee, \mathbb{S})$ are the "exotic good reduction" cases for ramified $U(1, *)$ and it was conjectured in [24] that the description of basic locus is similar to the cases studied in loc.cit.

The remaining cases (including both PEL and non-PEL types) seem completely new.

6. The study of Ekedahl-Oort elements

6.1. **General strategy.** It is showed in [10, §6] that the nonemptiness pattern of affine Deligne-Lusztig varieties only depends on the data $(\tilde{W}, w, \tau, \sigma)$ and is independent of the choice of G (i.e., independent of the orientation of the local Dynkin diagrams). Hence whether or not (G, μ, J) is of Coxeter type depends only on $(\tilde{W}, \lambda, J, \sigma)$. Here λ is the image of μ in $X_*(T)_\Gamma$.

Let us consider the following two conditions:

Coxeter-Straight Condition (**CSC**): $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu)$ consists of σ-straight elements.

Coxeter Condition (**CC**): If $w \in \mathrm{EO}^J(\mu)$ with $\bar{\nu}_w$ central, then w is a σ-Coxeter element in $W_{\mathrm{supp}_\sigma(w)}$.

By Theorem 2.1.1, (**CSC**) implies that (G, μ, J) is of Coxeter type . If (G, μ, J) is of Coxeter type, then it is obvious that (**CC**) holds. We will show via a case-by-case analysis that (**CC**) implies (**CSC**).

6.2. **The quadruple** $(\tilde{W}, \lambda, J, \sigma)$. We use the same labeling of Coxeter graph as in [2]. If $\omega_i^\vee$ is minuscule, we denote the corresponding element in Ω by τ_i. Let σ_0 be the unique nontrivial diagram automorphism for the finite Dynkin diagram if W_0 is of type A_n, D_n (with $n \geqslant 5$) or E_6. For type D_4, we also denote by σ_0 the diagram automorphism which interchanges α_3 and α_4.

The pairs $(\tilde{W}, \lambda)$ coming from (G, μ) with G absolutely quasi-simple and μ minuscule in G are as follows: $(\tilde{A}_n, \omega_i^\vee)_{1 \leqslant i \leqslant n}$, $(\tilde{B}_n, \omega_i^\vee)_{1 \leqslant i \leqslant n}$, $(\tilde{C}_n, \omega_i^\vee)_{1 \leqslant i \leqslant n}$, $(\tilde{C}_n, 2\omega_n^\vee)$, $(\tilde{D}_n, \omega_i^\vee)_{i=1,n-1,n}$, $(\tilde{E}_6, \omega_i^\vee)_{i=1,6}$, $(\tilde{E}_7, \omega_1^\vee)$, $(\tilde{F}_4, \omega_1^\vee)$, $(\tilde{G}_2, \omega_2^\vee)$.

It was pointed to us by X. Zhu that to compute these tuples one only needs to understand the restriction of μ to a maximal split torus; it is not required to fully "compute" the reduced affine root system the group G gives rise to.

We will show that the basic locus is a union of Ekedahl-Oort strata if and only if the quadruple $(\tilde{W}, \lambda, J, \sigma)$ is one of the following (up to automorphisms of $\tilde{W}$):

$(\tilde{A}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$	$(\tilde{A}_n, \omega_1^\vee, \mathbb{S}, \sigma_0)$	$(\tilde{B}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$
$(\tilde{B}_n, \omega_1^\vee, \mathbb{S} - \{n\}, \mathrm{id})$	$(\tilde{B}_n, \omega_1^\vee, \mathbb{S} - \{n\}, \tau_1)$	$(\tilde{C}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$
$(\tilde{D}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$	$(\tilde{D}_n, \omega_1^\vee, \mathbb{S}, \sigma_0)$	$(\tilde{A}_3, \omega_2^\vee, \mathbb{S}, \mathrm{id})$
$(\tilde{A}_3, \omega_2^\vee, \mathbb{S}, \sigma_0)$	$(\tilde{C}_2, \omega_2^\vee, \mathbb{S}, \mathrm{id})$	$(\tilde{C}_2, \omega_2^\vee, \mathbb{S} - \{1\}, \mathrm{id})$
$(\tilde{C}_2, \omega_2^\vee, \mathbb{S} - \{1\}, \tau_2)$		

Note that this list is shorter than the one in Theorem 5.2.1 because some automorphisms of $\tilde{W}$ will not lift to automorphisms of the affine root system.

If the quadruple $(\tilde{W}, \lambda, J, \sigma)$ is not in the list above, we give an element w such that the Coxeter condition (**CC**) fails for w. If the quadruple $(\tilde{W}, \lambda, J, \sigma)$ is in the list above, we compute $\mathrm{EO}^J(\mu)$, $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$ and the Newton points of the elements in $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$.

We follow [7, 1.5] for the reduced expression of t^μ. For $1 \leqslant a, b \leqslant n$, set

$$s_{[a,b]} = \begin{cases} s_a s_{a-1} \cdots s_b, & \text{if } a \geqslant b, \\ 1, & \text{otherwise.} \end{cases}$$

6.3. **Type** $\tilde{A}_{n-1}$. For simplicity, we consider the extended affine Weyl group of GL_n and $\omega_i^\vee = (1^{(i)}, 0^{(n-i)})$ instead. We may assume that $J = \mathbb{S}$ and $\mu = \omega_i^\vee$ with $1 \leqslant i \leqslant \frac{n}{2}$ (after applying some automorphism of $\tilde{W}$).

Case 1: $\sigma = \mathrm{id}$.

If $2 \leqslant i < \frac{n}{2}$, then (**CC**) fails for $s_0 s_{[n-1, n-\gcd(n,i)]}\tau$.

If $i = \frac{n}{2} \geqslant 3$, then (**CC**) fails for $s_0 s_1 s_{n-1} s_0 \tau$.

If $i = \frac{n}{2} = 2$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_0\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_0 s_1 \tau, s_0 s_3 \tau, s_0 s_1 s_3 \tau, s_0 s_1 s_3 s_0 \tau\}$ consists of σ-straight elements. Moreover, $\nu_{s_0 s_1 \tau} = (\frac{2}{3}^{(3)}, 0^{(1)})$, $\nu_{s_0 s_3 \tau} = (1^{(1)}, \frac{1}{3}^{(3)})$, $\nu_{s_0 s_1 s_3 \tau} = (1, \frac{1}{2}^{(2)}, 0)$ and $\nu_{s_0 s_1 s_3 s_0 \tau} = (1^{(2)}, 0^{(2)})$.

If $i = 1$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_0 s_{[n-1,i]}\tau; 2 \leqslant i \leqslant n\}$ consists of σ-straight elements. For $2 \leqslant i \leqslant n$, $\nu_{s_0 s_{[n-1,i]}\tau} = (\frac{1}{i-1}^{(i-1)}, 0^{(n-i+1)})$.

Case 2: $\sigma = \sigma_0$.

If $i \geqslant 3$ or $n > 4$, then (**CC**) fails for $s_0 s_1 s_{n-1} s_0 \tau$.

If $i = 2, n = 4$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_0\tau, s_0 s_1 \tau, s_0 s_3 \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_0 s_1 s_3 \tau, s_0 s_1 s_3 s_0 \tau\}$ consists of σ-straight elements. Moreover, $\nu_{s_0 s_1 s_3 \tau} = (1, \frac{1}{2}^{(2)}, 0)$ and $\nu_{s_0 s_1 s_3 s_0 \tau} = (1^{(2)}, 0^{(2)})$ are all distinct.

If $i = 1$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_0\tau, \cdots, s_0 s_{[n-1, \lceil \frac{n+3}{2} \rceil]} \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_0 s_{[n-1,i]} \tau; 2 \leqslant i \leqslant \lceil \frac{n+1}{2} \rceil\}$ consists of σ-straight elements. For $2 \leqslant i \leqslant \lceil \frac{n+1}{2} \rceil$, $\nu_{s_0 s_{[n-1,i]} \tau} = (\frac{i}{2(i-1)}^{(i-1)}, \frac{1}{2}^{(n-2i+2)}, \frac{i-2}{2(i-1)}^{(i-1)})$.

6.4. Type $\tilde{B}_n$.

Let $J = \tilde{\mathbb{S}} - \{i\}$. If $i \notin \{0, 1, n\}$, then (**CC**) fails for $s^{-1}_{[n,i]} s_{[n-1,i]} \tau$. We may assume that $J = \mathbb{S}$ or $\tilde{\mathbb{S}} - \{n\}$ (after applying some automorphism of $\tilde{W}$).

Case 1: $J = \mathbb{S}$, $\sigma = \mathrm{id}$.

If $\tau = \mathrm{id}$, then $\mu \geqslant \omega^\vee_2$ and (**CC**) fails for $s_0 s^{-1}_{[n,2]} s_{[n-1,2]} s_0$.

If $\tau = \tau_1$ and $\mu > \omega^\vee_1$, then (**CC**) fails for $s_0 s_2 s_1 \tau$.

If $\mu = \omega^\vee_1$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_0\tau, s_0 s_2 \tau, \cdots, s_0 s^{-1}_{[n-1,2]} \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_0 s^{-1}_{[n,2]} \tau s_{[n-1,i]}; 1 \leqslant i \leqslant n\}$ consists of σ-straight elements. For $1 \leqslant i \leqslant n$, $\nu_{s_0 s^{-1}_{[n,2]} \tau s_{[n-1,i]}} = (\frac{1}{i}^{(i)}, 0^{(n-i)})$.

Case 2: $J = \tilde{\mathbb{S}} - \{n\}$, $\sigma = \mathrm{id}$ or τ_1.

If $\mu > \omega^\vee_1$, then (**CC**) fails for $s_n s_{n-1} s_n \tau$.

If $\mu = \omega^\vee_1$ and $\sigma = \mathrm{id}$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_n \tau, s_n s_{n-1} \tau, \cdots, s_{[n,2]} \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_{[n,2]} s_1 \tau, s_{[n,2]} s_0 \tau\} \cup \{s_{[n,0]} s^{-1}_{[i,2]} \tau; 2 \leqslant i \leqslant n-1\}$ consists of σ-straight elements. Moreover, $\nu_{s_{[n,2]} s_1 \tau} = \nu_{s_{[n,2]} s_0 \tau} = (\frac{1}{n}^{(n)})$ and for $2 \leqslant i \leqslant n-1$, $\nu_{s_{[n,0]} s^{-1}_{[i,2]} \tau} = (\frac{1}{n-i}^{(n-i)}, 0^{(i)})$.

If $\mu = \omega^\vee_1$ and $\sigma = \tau_1$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{\tau, s_n \tau, \cdots, s_{[n,2]} \tau\} \cup \{s_{[n,2]} s_1 \tau, s_{[n,2]} s_0 \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s_{[n,0]} s^{-1}_{[i,2]} \tau; 1 \leqslant i \leqslant n-1\}$ consists of σ-straight elements. For $1 \leqslant i \leqslant n-1$, $\nu_{s_{[n,0]} s^{-1}_{[i,2]} \tau} = (\frac{1}{n-i}^{(n-i)}, 0^{(i)})$.

6.5. Type $\tilde{C}_n$.

Let $J = \tilde{\mathbb{S}} - \{i\}$ with $i \leqslant \frac{n}{2}$ (after applying some automorphism of $\tilde{W}$).

Case 1: $\mu \geqslant \omega^\vee_1$.

If $i \neq 0$, then (**CC**) fails for $s_{[i,0]} s_{[1,i]} \tau$. Hence $J = \mathbb{S}$ and $\sigma = \mathrm{id}$.

If $\mu > \omega^\vee_1$, then (**CC**) fails for $s_0 s_1 s_0$.

If $\mu = \omega^\vee_1$, then $\mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{1, s_0, s_0 s_1, \cdots, s^{-1}_{[n-1,0]}\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu) = \{s^{-1}_{[n,0]} s_{[n-1,i]}; 1 \leqslant i \leqslant n\}$ consists of σ-straight elements. For $1 \leqslant i \leqslant n$, $\nu_{s^{-1}_{[n,0]} s_{[n-1,i]}} = (\frac{1}{i}^{(i)}, 0^{(n-i)})$.

Case 2: $\mu = \omega_n^\vee$.

If $0 < i < \frac{n}{2}$, then $\sigma = \mathrm{id}$ and **(CC)** fails for $s_{[n-i,i]}^{-1}\tau$.

If $i = \frac{n}{2} > 1$, then **(CC)** fails for $s_i s_{i+1} s_{i-1} s_i \tau$.

If $i = 0$ and $n > 2$, then **(CC)** fails for $s_0 s_1 s_0 \tau$.

Therefore $n = 2$.

If $J = \mathbb{S}$, then $\sigma = \mathrm{id}$ and $\mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{\tau, s_0\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{s_0 s_1 \tau, s_0 s_1 s_0 \tau\}$ consists of σ-straight elements. Moreover, $\nu_{s_0 s_1 \tau} = (\frac{1}{2}, 0)$ and $\nu_{s_0 s_1 s_0 \tau} = (\frac{1}{2}^{(2)})$.

If $J = \tilde{\mathbb{S}} - \{1\}$ and $\sigma = \mathrm{id}$, then $\mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{\tau, s_1\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{s_1 s_0 \tau, s_1 s_2 s_0 \tau\}$ consists of σ-straight elements. Moreover, $\nu_{s_1 s_2 \tau} = \nu_{s_1 s_0 \tau} = (\frac{1}{2}, 0)$ and $\nu_{s_1 s_2 s_0 \tau} = (\frac{1}{2}^{(2)})$.

If $J = \tilde{\mathbb{S}} - \{1\}$ and $\sigma = \tau_2$, then $\mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{\tau, s_1\tau, s_1 s_2 \tau, s_1 s_0 \tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{s_1 s_2 s_0 \tau\}$ consists of σ-straight elements. Moreover, $\nu_{s_1 s_2 s_0 \tau} = (\frac{1}{2}^{(2)})$.

6.6. Type $\tilde{D}_n$.

Let $J = \tilde{\mathbb{S}} - \{i\}$. After applying some automorphism of $\tilde{W}$, we may assume that $i = 0$ or $2 \leqslant i \leqslant \frac{n}{2}$ and $\mu = \omega_1^\vee$ or $\omega_n^\vee$.

Case 1: $\mu = \omega_1^\vee$.

If $i \neq 0$, then **(CC)** fails for $s_{[n,i]}^{-1} s_{[n-2,i]}\tau$. Hence $J = \mathbb{S}$.

If $\sigma = \mathrm{id}$, then $\mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{\tau, s_0\tau, s_0 s_2\tau, \cdots, s_0 s_{[n-2,2]}^{-1}\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{s_0 s_{[n-2,2]}^{-1} s_{n-1}\tau, s_0 s_{[n-2,2]}^{-1} s_n\tau\} \cup \{s_0 s_{[n,2]}^{-1}\tau s_{[n-2,i]}; 1 \leqslant i \leqslant n-1\}$ consists of σ-straight elements. Moreover, $\nu_{s_0 s_{[n-2,2]}^{-1} s_{n-1}\tau} = \nu_{s_0 s_{[n-2,2]}^{-1} s_n\tau} = (\frac{1}{n}^{(n)})$ and for $1 \leqslant i \leqslant n-1$, $\nu_{s_0 s_{[n,2]}^{-1}\tau s_{[n-2,i]}} = (\frac{1}{i}^{(i)}, 0^{(n-i)})$.

If $\sigma = \sigma_0$, then $\mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{\tau, s_0\tau, \cdots, s_0 s_{[n-1,2]}^{-1}\tau, s_0 s_{[n-2,2]}^{-1} s_n\tau\}$ and $\mathrm{EO}^J(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}^J(\mu) = \{s_0 s_{[n,2]}^{-1}\tau s_{[n-2,i]}; 1 \leqslant i \leqslant n-1\}$ consists of σ-straight elements. Moreover, for $1 \leqslant i \leqslant n-1$, $\nu_{s_0 s_{[n,2]}^{-1}\tau s_{[n-2,i]}} = (\frac{1}{i}^{(i)}, 0^{(n-i)})$.

If $\sigma(1) \neq 1$, then $n = 4$ and **(CC)** fails for $s_0 s_2 s_3$ or $s_0 s_2 s_4$.

Case 2: $\mu = \omega_n^\vee$.

If $2 \leqslant i < \frac{n}{2}$, then **(CC)** fails for $s_{[n-i,i]}^{-1}\tau$.

If $i = \frac{n}{2}$, then **(CC)** fails for $s_i s_{i+1} s_{i-1} s_i \tau$.

Now we consider the case where $J = \mathbb{S}$ and $n > 4$. One may check that **(CC)** fails for $s_0 s_2 s_1 \tau$ or $s_0 s_2 s_1 s_0 \tau$.

6.7. Type $\tilde{E}_6$.

After applying some automorphism of $\tilde{W}$, we may assume that $\mu = \omega_1^\vee$ and $J = \mathbb{S}, \tilde{\mathbb{S}} - \{2\}$ or $\tilde{\mathbb{S}} - \{4\}$.

If $J = \mathbb{S}$, then **(CC)** fails for $s_0 s_2 s_4 s_3 s_1 \tau$.

If $J = \tilde{\mathbb{S}} - \{2\}$, then **(CC)** fails for $s_2 s_4 s_5 \tau$ or $s_2 s_4 s_3 \tau$.

If $J = \tilde{\mathbb{S}} - \{4\}$, then (**CC**) fails for $s_4 s_3 s_5 s_4 \tau$.

6.8. **Type** $\tilde{E}_7$. Here $\mu = \omega_1^\vee$. After applying some automorphism of $\tilde{W}$, $J = \tilde{\mathbb{S}} - \{i\}$ for $0 \leqslant i \leqslant 4$.

If $J = \mathbb{S}$, then (**CC**) fails for $s_0 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_0 \tau$.

If $J = \tilde{\mathbb{S}} - \{1\}$, then (**CC**) fails for $s_1 s_3 s_4 s_5 s_6 \tau$.

If $J = \tilde{\mathbb{S}} - \{2\}$, then (**CC**) fails for $s_2 s_4 s_3 s_5 s_4 s_2 \tau$.

If $J = \tilde{\mathbb{S}} - \{3\}$, then (**CC**) fails for $s_3 s_4 s_5 \tau$.

If $J = \tilde{\mathbb{S}} - \{4\}$, then (**CC**) fails for $s_4 s_2 s_5 s_4 \tau$.

6.9. **Type** $\tilde{F}_4$. Here $\mu = \omega_1^\vee$. If $J = \mathbb{S}$, then (**CC**) fails for $s_0 s_1 s_2 s_3 s_2 s_1$. If $J = \tilde{\mathbb{S}} - \{1\}$, then (**CC**) fails for $s_1 s_2 s_3 s_2$. If $J = \tilde{\mathbb{S}} - \{2\}$, then (**CC**) fails for $s_2 s_3 s_2$. If $J = \tilde{\mathbb{S}} - \{3\}$, then (**CC**) fails for $s_3 s_2 s_3$. If $J = \tilde{\mathbb{S}} - \{4\}$, then (**CC**) fails for $s_4 s_3 s_2 s_3$.

6.10. **Type** $\tilde{G}_2$. Here $\mu = \omega_2^\vee$. If $J = \mathbb{S}$, then (**CC**) fails for $s_0 s_2 s_1 s_2$. If $J = \tilde{\mathbb{S}} - \{1\}$, then (**CC**) fails for $s_1 s_2 s_1 s_0$. If $J = \tilde{\mathbb{S}} - \{2\}$, then (**CC**) fails for $s_2 s_1 s_2 s_0$.

6.11. **Proof of Theorem 5.2.1.** Part (1) and (2) follow from the Coxeter-Straight condition. Part (3) follows from the explicit computation in each case. Part (4) follows from Theorem 2.1.1. Part (5) follows from Theorem 4.1.2 and Proposition 4.1.1. Part (6) follows from the explicit description of $\text{EO}_{\sigma,\text{cox}}^J(\mu)$. Part (7) follows from §3.3 (1), (2) and the fact that for all the cases we are considering, $\ell(x) < \ell(y)$ implies that $x < y$.

7. Closure relations

7.1. **The $\tau\sigma$-orbits on $\tilde{\mathbb{S}}$.** In this section we assume that $F = \mathbb{F}_q((\epsilon))$ and that (G, μ, J) is as in Theorem 5.1.1. Let $\mathcal{N}$ be the fiber over τ for the map $Z_{\mathcal{O}_0} \to \mathcal{O}_0$; with the notation of [22], this means $\mathcal{N} = X(\mu, \tau)_{P_J}$. We usually refer to $\mathcal{N}$ as the *basic locus*. In this section, we study the stratification of $\mathcal{N}$ by classical Deligne-Lusztig varieties in more detail.

This description proceeds as follows: first we describe the set $\mathcal{I}$ of EO strata which occur in the basic locus $\mathcal{N}$ (in terms of the Dynkin diagram). Second, we will describe the set of strata within each EO stratum (in terms of the Bruhat-Tits building of $\mathbb{J}_\tau$). Finally we will discuss the closure relations between strata.

Identify $\tilde{\mathbb{S}}$ with the set of vertices of the affine Dynkin diagram, and for any vertex v, denote by $d(v)$ the distance between v and the unique vertex not contained in J.

Let

$$\mathfrak{I} = \{\Sigma \subset \tilde{\mathbb{S}};\ \emptyset \neq \Sigma \text{ is } \tau\sigma\text{-stable and } \forall v, v' \in \Sigma \colon d(v) = d(v')\}$$

Clearly every $\tau\sigma$-orbit is an element of $\mathfrak{I}$. In some cases, there is also one further element which is the union of two orbits, see Section 7.4. We denote by $d(\Sigma)$ the value of $d(v)$ for any $v \in \Sigma$.

Given $\Sigma \in \mathfrak{I}$, we denote by $\Sigma^\flat$ the union of all the $\tau\sigma$-orbits Σ' with $d(\Sigma') \leqslant d(\Sigma)$ and $\Sigma' \nsubseteq \Sigma$ and denote by $\Sigma^\sharp$ the union of all the $\tau\sigma$-orbits Σ' with $d(\Sigma') > d(\Sigma)$.

From the explicit computation in §6, we get the following properties:

Proposition 7.1.1. *Let $\Sigma \in \mathfrak{I}$.*

(1) *We have $\tilde{\mathbb{S}} = \Sigma \sqcup \Sigma^\flat \sqcup \Sigma^\sharp$.*

(2) *The subsets $\Sigma^\flat$ and $\Sigma^\sharp$ are disconnected in the affine Dynkin diagram.*

(3) *For any $\Sigma \in \mathfrak{I}$, there is exactly one element $w \in \mathrm{EO}^J_{\sigma,\mathrm{cox}}(\mu)$ such that $\mathrm{supp}_\sigma(w) = \Sigma^\flat$. We denote this element by w_Σ. We have $\ell(w_\Sigma) = d(\Sigma)$.*

(4) *We have $I(J, w_\Sigma, \sigma) = \Sigma^\sharp$ and $J(w_\Sigma, \sigma) = \tilde{\mathbb{S}} - \Sigma$ (cf. Prop. 3.1.1, Theorem 5.2.1).*

Example 7.1.2. As an example, consider the case $(\tilde{B}_n, \omega_1^\vee, \tilde{\mathbb{S}} - \{n\}, \tau_1)$. In this case, the possible triples $(\Sigma, \Sigma^\flat, \Sigma^\sharp)$ are $(\{i\}, \{m, m-1, \ldots, i+1\}, \{i-1, \ldots, 1, 0\})$ for $i = m, \ldots, 2$, and $(\{0\}, \{m, m-1, \ldots, 1\}, \emptyset)$, $(\{1\}, \{m, m-1, \ldots, 2, 0\}, \emptyset)$ and $(\{0, 1\}, \{m, m-1, \ldots, 2\}, \emptyset)$. In particular, in the last case, Σ has more than one element.

7.2. **A stratification of $\mathcal{N}$.** Set $\mathcal{N}_\Sigma = \mathcal{N} \cap Z_{J,w_\Sigma} = X_{J,w_\Sigma}(\tau)$, the Ekedahl-Oort stratum attached to Σ. Then $\mathcal{N} = \sqcup_{\Sigma \in \mathfrak{I}} \mathcal{N}_\Sigma$. By Theorem 5.2.1 (5),

$$(7.2.1) \qquad \mathcal{N}_\Sigma = \sqcup_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})} iY(w_\Sigma),$$

where by Corollary 4.1.3

$$Y(w_\Sigma) = \{gP_{\Sigma^\sharp} \in P_{\tilde{\mathbb{S}}-\Sigma}/P_{\Sigma^\sharp}; g^{-1}\tau\sigma(g) \in P_{\Sigma^\sharp}wP_{\sigma(\Sigma^\sharp)}\}$$

$$\cong \{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw_\Sigma I\}.$$

Since $\mathrm{supp}_\sigma(w_\Sigma) = \Sigma^\flat$, $Y(w_\Sigma)$ is connected and $\mathcal{N}_\Sigma = \sqcup_{i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})} iY(w_\Sigma)$ is the decomposition of $\mathcal{N}_\Sigma$ into connected components. Each of them has dimension $\ell(w_\Sigma)$.

Now we describe the closure of each stratum in $\mathcal{N}$.

Theorem 7.2.1. *Let $\Sigma \in \mathfrak{I}$. Then for any $i \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})$,*

$$\overline{iY(w_\Sigma)} = \sqcup_{(\Sigma')^\flat \subset \Sigma^\flat} \sqcup_{j \in \mathbb{J}_\tau/(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'}); i \cap j \neq \emptyset} jY(w_{\Sigma'}).$$

The intersection $i \cap j$ is understood as the intersection inside $\mathbb{J}_\tau$ of the cosets given by i and j.

Proof. It suffices to consider the case where $i = 1$. Since π_J is proper, $\overline{Y(w_\Sigma)} = \pi_J(\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in \overline{Iw_\Sigma I}\}) = \cup_{w' \leqslant w_\Sigma}\pi_J(\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw'I\})$. Any element $w' \leqslant w_\Sigma$ is of the form $w_{\Sigma'}w''$ for some Σ' with $(\Sigma')^\flat \subset \Sigma^\flat$ and $w'' \in W_{(\Sigma')^\sharp}$. Since $(\Sigma')^\sharp \subset J$, $P_J \cdot_\sigma Iw'I = P_J \cdot_\sigma Iw_{\Sigma'}I$ and $\pi_J(\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw'I\}) = \pi_J(\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw_{\Sigma'}I\})$.

Similarly to the proof of Proposition 2.2.1, $\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw_{\Sigma'}I\} = \bigsqcup_{j \in (\mathbb{J}_\tau \cap P_{\Sigma^\flat})/(\mathbb{J}_\tau \cap P_{(\Sigma')^\flat})}\{jgI \in P_{(\Sigma')^\flat}/I; g^{-1}\tau\sigma(g) \in Iw_{\Sigma'}I\}$ and $\pi_J(\{gI \in P_{\Sigma^\flat}/I; g^{-1}\tau\sigma(g) \in Iw_{\Sigma'}I\}) = \bigsqcup_{j \in (\mathbb{J}_\tau \cap P_{\Sigma^\flat})/(\mathbb{J}_\tau \cap P_{(\Sigma)^\flat} \cap P_{\tilde{\mathbb{S}}-\Sigma'})}jY(w_{\Sigma'})$. Note that $\tilde{\mathbb{S}} - \Sigma = \Sigma^\flat \sqcup \Sigma^\sharp$ and $\Sigma^\sharp \subset \tilde{\mathbb{S}} - \Sigma'$. Hence $(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})/(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma-\Sigma'}) \cong (\mathbb{J}_\tau \cap P_{\Sigma^\flat})/(\mathbb{J}_\tau \cap P_{(\Sigma)^\flat} \cap P_{\tilde{\mathbb{S}}-\Sigma'})$. The theorem is proved. $\square$

Another way to describe the closure of strata is via the Bruhat-Tits building of the group $\mathbb{J}_\tau$ over F. This reproduces precisely the descriptions in [29] and [24].

Proposition 7.2.2. *Let $\Sigma, \Sigma' \in \mathcal{I}$ and $j, j' \in \mathbb{J}_\tau$. The following are equivalent:*

(1) $j(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma}) \cap j'(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'}) \neq \emptyset$,
(2) $j(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})j^{-1} \cap j'(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'})(j')^{-1}$ *contains a (rational) Iwahori subgroup of $\mathbb{J}_\tau$,*
(3) *The faces in the rational building of $\mathbb{J}_\tau$ corresponding to $j(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})j^{-1}$ and $j'(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'})(j')^{-1}$ are neighbors (i.e., there exists an alcove which contains both of them).*

Proof. We may and will assume throughout that $j' = 1$.

The Iwahori subgroups of $j(\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma})j^{-1}$ are of the forms $jg(\mathbb{J}_\tau \cap I)g^{-1}j^{-1}$ for some $g \in \mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma}$ and the Iwahori subgroups of $\mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'}$ are of the forms $g'(\mathbb{J}_\tau \cap I)(g')^{-1}$ for some $g' \in \mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'}$. Hence (2) is equivalent to saying that for some $g \in \mathbb{J}_\tau \cap P_{\Sigma^\flat}$ and $g' \in \mathbb{J}_\tau \cap P_{\tilde{\mathbb{S}}-\Sigma'}$, $jg(\mathbb{J}_\tau \cap I)g^{-1}j^{-1} = g'(\mathbb{J}_\tau \cap I)(g')^{-1}$, i.e., $(g')^{-1}jg \in I$. The latter condition is equivalent to condition (1).

By definition of the simplicial structure of the Bruhat-Tits building, (2) and (3) are equivalent . $\square$

7.3. Singularities of the closures of strata.
Consider (G, μ, J) as in Theorem 5.1.1.

Proposition 7.3.1. *Each fine Deligne-Lusztig variety $Y(w_\Sigma)$ occurring in the stratification (7.2.1) is smooth. Its closure $\check{Y}$ inside the flag variety $P_{\Sigma^\flat}/I$ is smooth. In particular, the projection $\mathcal{P}_\emptyset \to \mathcal{P}_J$ restricts to a resolution of singularities $\tilde{Y} \to \overline{Y(w_\Sigma)}$ of the closure of $Y(w_\Sigma)$ in $\mathcal{P}_J$ (or equivalently in $\mathcal{N}$).*

Proof. The smoothness of $\tilde{Y}$ is equivalent to the smoothness of the Schubert variety inside $P_{\Sigma^\flat}/I$ attached to w_Σ. Since the latter is a Coxeter element, this Schubert variety is isomorphic to its Bott-Samelson resolution, and hence is smooth. $\qquad\square$

Proposition 7.3.2. *If the triple $(\tilde{W}, \lambda, J, \sigma)$ belongs to the following list*

$(\tilde{A}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$ (*)	$(\tilde{A}_n, \omega_1^\vee, \mathbb{S}, \sigma_0)$ (*)	$(\tilde{B}_n, \omega_1^\vee, \mathbb{S} - \{n\}, \mathrm{id})$
$(\tilde{B}_n, \omega_1^\vee, \mathbb{S} - \{n\}, \tau_1)$	$(\tilde{C}_n, \omega_1^\vee, \mathbb{S}, \mathrm{id})$	$(\tilde{A}_3, \omega_2^\vee, \mathbb{S}, \mathrm{id})$ (*)
$(\tilde{A}_3, \omega_2^\vee, \mathbb{S}, \sigma_0)$ (*)	$(\tilde{C}_2, \omega_2^\vee, \mathbb{S}, \mathrm{id})$ (*)	$(\tilde{C}_2, \omega_2^\vee, \mathbb{S} - \{1\}, \mathrm{id})$ (*)
$(\tilde{C}_2, \omega_2^\vee, \mathbb{S} - \{1\}, \tau_2)$		

then

1. *for all $\Sigma \in \mathcal{I}$, the closure $\overline{Y(w_\Sigma)}$ of $Y(w_\Sigma)$ inside $\mathcal{N}$ has at most isolated singularities, and*
2. *the closure $\overline{Y(w_\Sigma)}$ is smooth if and only if $\tau(J) \neq J$ or $\ell(w_\Sigma) \leqslant 1$.*

The cases where all closures $\overline{Y(w_\Sigma)}$ are smooth are marked (*) in the table.

Proof. We write $w := w_\Sigma$ and assume that $\ell(w) > 0$. Denote by F the twisted Frobenius $g \mapsto \tau\sigma(g)\tau^{-1}$; it acts on $\mathrm{supp}_\sigma(w)$ and hence on the flag variety $P_{\mathrm{supp}_\sigma(w)}/I$. Let $Q := P_{\mathrm{supp}_\sigma(w) \cap J}$ and $w' := w\tau^{-1}$, and denote by

$$X_Q(w') := \{gQ \in P_{\mathrm{supp}_\sigma(w)}/Q; \ g^{-1}\mathrm{F}(g) \in Qw\mathrm{F}(Q)\}$$

the Deligne-Lusztig variety attached to w' inside $P_{\mathrm{supp}_\sigma(w)}/Q$ with respect to F. Recall that w is σ-Coxeter which by definition means that w' is a twisted Coxeter element for F.

Using Cor. 4.1.3, we identify $Y(w)$ with

$$\{gQ \in P_{\mathrm{supp}_\sigma(w)}/Q; \ g^{-1}\mathrm{F}(g) \in Q \cdot_\sigma Iw'I\}.$$

Let $\overline{Y(w)}$, $\overline{X_Q(w')}$ denote the closures inside $P_{\mathrm{supp}_\sigma(w)}/Q$. Note that $\overline{Y(w)}$ is isomorphic to the closure of $Y(w)$ inside $\mathcal{N}$. Clearly we have

(7.3.1) $$\overline{Y(w)} \subseteq \overline{X_Q(w')}.$$

Furthermore $\overline{X_Q(w')}$ is irreducible by the well-known criterion for irreducibility of Deligne-Lusztig varieties and $\dim \overline{X_Q(w')} = \dim Qw'\mathrm{F}(Q)/\mathrm{F}(Q)$.

By Corollary 4.1.3, we have $\dim Y(w) = \ell(w)$. Thus we see that the inclusion (7.3.1) is an equality if and only if $\ell(w) = \dim Qw'\mathrm{F}(Q)/\mathrm{F}(Q)$.

Let w_0 denote the longest element in $W_{\mathrm{supp}_\sigma(w)\cap J}$. Since $w \in {}^J\tilde{W}$, we have $\ell(w_0 w') = \ell(w_0) + \ell(w') = \ell(w') + \dim Q/I$, and we conclude that the inclusion (7.3.1) is an equality if and only if $w_0 w'$ is the longest element inside $W_{\mathrm{supp}_\sigma(w)\cap J} w' W_{\mathrm{F}(\mathrm{supp}_\sigma(w)\cap J)}$. In the cases listed in the statement of the proposition, the Dynkin type of $\mathrm{supp}_\sigma(w) \cap J$ is type A, and it is easily checked that in those cases, $w_0 w'$ indeed is the longest element inside this double coset.

Now suppose that $\tau(J) \neq J$, or equivalently that $\mathrm{F}(Q) \neq Q$. Since $\tilde{\mathbb{S}} - J \subseteq \mathrm{supp}(w)$, we have $w' \in W_{\mathrm{F}(\mathrm{supp}_\sigma(w)\cap J)}$, so $X_Q(w') = X_Q(\mathrm{id})$ is smooth and closed in $P_{\mathrm{supp}_\sigma(w)}/Q$.

On the other hand, if $\mathrm{F}(Q) = Q$, then the above implies that $\overline{Y(w)} = \overline{X_Q(w')} = X_Q(w') \sqcup X_Q(\mathrm{id})$. Since $\dim X_Q(\mathrm{id}) = 0$, part (1) follows. Finally, since closures of Deligne-Lusztig varieties are always normal, $\overline{Y(w)}$ is smooth if $\ell(w) \leqslant 1$. The remaining assertion in part (2) follows from Proposition 3.3 and Proposition 4.4 in [3] by Brion and Polo (note that $\ell(w) < \dim P_{\mathrm{supp}_\sigma(w)}/Q$ whenever $\ell(w) > 1$, hence the boundary in the sense of loc. cit. is $X_Q(\mathrm{id})$). $\square$

7.4. **Examples.** As examples, we discuss the three cases treated by Rapoport, Terstiege and Wilson in [24] and an example of non-Coxeter type.

7.4.1. $(C\text{-}BC_m, \omega_1^\vee, \tilde{\mathbb{S}} - \{m\})$. This case arises from $GU(1, 2m)$, p ramified. Note here that the vertices 0 and m are conjugate under the extended affine Weyl group, but not conjugate under the diagram automorphism for the relative local Dynkin diagram. In this case there exist $m + 1$ different EO strata in the basic locus, of dimensions 0, ..., m. In the terminology of [24], the EO stratum of dimension i is the union of all strata attached to vertex lattices of type $2i$. The stratification is indexed by the vertices of the Bruhat-Tits building of $\mathbb{J}_\tau$.

7.4.2. $({}^2B\text{-}C_m, \omega_1^\vee, \tilde{\mathbb{S}} - \{m\})$. This case arises from $GU(1, 2m - 1)$, p ramified, the hermitian form C is split. In this case there exist $m + 2$ EO strata in the basic locus; one each of dimension 0, ..., $m - 1$, and two of dimension m. For $i = 0, \ldots, m-1$, the EO stratum of dimension i is the union of all strata attached to vertex lattices of type $2i$. On the other hand, the union of all strata attached to vertex lattices of

type $2m$ is equal to the union of the two m-dimensional EO strata (the corresponding EO strata are related by the conjugation action of $GU(1, 2m - 1)$). In this case, the action of $\sigma\tau$ on the affine Dynkin diagram is trivial. There are two orbits with the same distance to the vertex m (namely the vertices 0 and 1). See Example 7.1.2. The union of these two orbits occurs as an element $\Sigma \in \mathfrak{I}$; it corresponds to the $m - 1$-dimensional EO stratum. The index set for the Deligne-Lusztig varieties inside this EO stratum is the set of edges of type $\{0, 1\}$ inside the Bruhat-Tits building of $\mathbb{J}_\tau$. In the description in [24] this is reflected by Prop. 3.4.

7.4.3. $(B\text{-}C_m, \omega_1^\vee, \tilde{\mathbb{S}} - \{m\})$. This case arises from $GU(1, 2m - 1)$, p ramified, the hermitian form C being nonsplit. In this case basic EO strata correspond bijectively to types of vertex lattices, and the index set of the stratification is the set of vertices of the Bruhat-Tits building of $\mathbb{J}_\tau$.

7.4.4. $(C\text{-}BC_2, \omega_1^\vee, \tilde{\mathbb{S}} - \{1\})$. This case arises from $GU(1, 3)$, p ramified. However, the level structure is different from the one considered in [24]. One can show that the basis locus is still the union of EO strata. There are 6 EO strata in the basic locus, which correspond to the elements 1, s_1, $s_1 s_0$, $s_1 s_2$, $s_1 s_0 s_1$ and $s_1 s_2 s_1$ respectively. One is of dimension 0, one is of dimension 1, two of dimension 2 and two of dimension 3. The elements $s_1 s_0 s_1$ and $s_1 s_2 s_1$ are not Coxeter elements.

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