A SUBALGEBRA OF 0-HECKE ALGEBRA

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Abstract. Let (W, I) be a finite Coxeter group. In the case where W is a Weyl group, Berenstein and Kazhdan in [BK] constructed a monoid structure on the set of all subsets of I using unipotent χ-linear bicrystals. In this paper, we will generalize this result to all types of finite Coxeter groups (including non-crystallographic types). Our approach is more elementary, based on some combinatorics of Coxeter groups. Moreover, we will calculate this monoid structure explicitly for each type.

1.1. Let W be a Coxeter group generated by the simply reflections s_i (for $i \in I$). Let H be the Iwahori-Hecke algebra associated to W with parameter $q = 0$, i.e., H is a $\mathbb{Q}$-algebra generated by T_{s_i} for $s_i \in I$ with relations $T_{s_i}^2 = -T_{s_i}$ and the braid relations. The algebra H is called 0-Hecke algebra. It was introduced by Norton in [No]. Representations of H were later studied in the work of Carter [Ca], Hivert-Novelli-Thibon [HNT] and etc. More recently, Stembridge [St] used the 0-Hecke algebra to obtain a new proof for the Möbius function of the Bruhat order of W.

1.2. Set $T'_{s_i} = -T_{s_i}$. For $w \in W$, we define $T'_w = T'_{s_{i_1}} \cdots T'_{s_{i_k}}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of w. Tit's theorem implies that T'_w is well defined. Moreover, we have a binary operation $* : W \times W \to W$ such that $T'_x T'_y = T'_{x*y}$ for any $x, y \in W$. It is easy to see that $(W, *)$ is a monoid with unit element 1.

Now we state our main theorem.

Theorem 1. *Let W be a finite Coxeter group. For any subset $J \subset I$, let w_0^J be the maximal element in the subgroup generated by s_j (for $j \in J$). Then $\{w_0^J w_0^I; J \subset I\}$ is a commutative submonoid of $(W, *)$. In other words, there exists a commutative monoid structure $\star_I$ on the set of subsets of I, such that*

$$T'_{w_0^{J_1} w_0^I} T'_{w_0^{J_2} w_0^I} = T'_{w_0^{J_1 \star_I J_2} w_0^I}.$$

Remark. *In the case where W is a Weyl group, this result was discovered by Berenstein and Kazhdan in* [BK, Proposition 2.30]. *Their approach was based on unipotent χ-linear bicrystals. The proof below is more elementary. It is based on some combinatorial properties of*

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Coxeter groups. In the end, we will calculate the operator $\star_I$ explicitly for each type.

Below we introduce a binary operation $\rhd : W \times W \to W$ and study some properties about the operations $*$ and $\rhd$. In the case where W is a finite Weyl group, these properties were proved in [HL] using geometry of flag varieties.

1.3. We denote by l the length function and $\leqslant$ the Bruhat order on W.

By [He, Lemma 1.4(1)], for $x, y \in W$, the subset $\{uy; u \leqslant x\}$ of W contains a unique minimal element which we denote by $x \rhd y$. Moreover, $x \rhd y = u'y$ for some $u' \leqslant x$ and $l(x \rhd y) = l(y) - l(u')$. We also have that if $s_i x > x$, then $(s_i x) \rhd y = \min\{s_i(x \rhd y), x \rhd y\}$.

There is a similar description for $x * y$.

Lemma 1. *Let $x, y \in W$. Then the subset $\{uv; u \leqslant x, v \leqslant y\}$ contains a unique maximal element, which equals $x * y$. Moreover, $x * y = u'y = xv'$ for some $u' \leqslant x$ and $v' \leqslant y$ and $l(x*y) = l(u') + l(y) = l(x) + l(v')$.*

Remark. *A slightly weaker version was proved in* [He, Lemma 1.4 (2)]. *The proof here is similar to loc.cit.*

By definition, $s_i * w = \max\{w, s_i w\}$. Now for $a, b \in W$ and $i \in I$ with $s_i a > a$, we have that

$$T'_{(s_i a)*b} = T'_{s_i a} T'_b = T'_{s_i} T'_a T'_b = T'_{s_i} T'_{a*b}.$$

Hence $(s_i a) * b = \max\{a * b, s_i(a * b)\}$ if $s_i a > a$.

We only prove that $\{uv; u \leqslant x, v \leqslant y\}$ contains a unique maximal element which equals $x * y$ and $x * y = u'y$ for some $u' \leqslant x$ with $l(x * y) = l(u') + l(y)$.

We argue by induction on $l(x)$. For $l(x) = 0$, the statement is clear. Assume that $l(x) > 0$ and that the statement holds for all x' with $l(x') < l(x)$. Then there exists $i \in I$ such that $s_i x < x$. By induction hypothesis, the subset $\{uv; u \leqslant s_i x, v \leqslant y\}$ contains a unique maximal element $(s_i x) * y$ and there exists $u_1 \leqslant s_i x$ such that $(s_i x) * y = u_1 y$ and $l(u_1 y) = l(u_1) + l(y)$.

Set $z = x * y$. Then $z = \max\{u_1 y, s_i u_1 y\}$ and $zy^{-1} = u_1$ or $s_i u_1$. In either case, we have that $zy^{-1} \leqslant x$. If $zy^{-1} = u_1$, then we already know that $l(z) = l(zy^{-1}) + l(y)$. If $zy^{-1} = s_i u_1$, then $s_i u_1 y > u_1 y$ and

$$l(s_i u_1 y) = l(u_1 y) + 1 = l(u_1) + l(y) + 1 \geqslant l(s_i u_1) + l(y) \geqslant l(s_i u_1 y).$$

Thus we must have that $l(s_i u_1) = l(u_1) + 1$ and $l(z) = l(zy^{-1}) + l(y)$.

Now for any $u \leqslant x$ and $v \leqslant y$. By [Lu, Corollary 2.5], $u \leqslant s_i x$ or $s_i u \leqslant s_i x$. By the definition of $(s_i x) * y$, we have that uv or $s_i uv$ is less than or equal to $(s_i x) * y = u_1 y \leqslant z$. By [Lu, Corollary 2.5], $uv \leqslant z$. Therefore z is the unique maximal element in the subset $\{uv; u \leqslant x, v \leqslant y\}$. $\square$

Corollary 1. *Let $x' \leqslant x$ and $y' \leqslant y$. Then $x' * y' \leqslant x * y$.*

Lemma 2. *Let $x' \geqslant x$ and $y' \leqslant y$, then $x' \rhd y' \leqslant x \rhd y$.*

By definition, $x' \rhd y' \leqslant x \rhd y'$. Now we prove that $x \rhd y' \leqslant x \rhd y$ by induction on $l(x)$.

For $l(x) = 0$, $x \rhd y' = y' \leqslant y = x \rhd y$. Now assume that $l(x) > 0$ and that $x_1 \rhd y' \leqslant x_1 \rhd y$ for any x_1 with $l(x_1) < l(x)$. Then there exists $i \in I$ such that $s_i x < x$. By induction hypothesis, $(s_i x) \rhd y' \leqslant (s_i x) \rhd y$.

By [Lu, Corollary 2.5], $x \rhd y' = \min\{(s_i x) \rhd y', s_i((s_i x) \rhd y')\} \leqslant (s_i x) \rhd y, s_i((s_i x) \rhd y)$. Hence $x \rhd y' \leqslant \min\{(s_i x) \rhd y, s_i((s_i x) \rhd y)\} = x \rhd y$. The statement is proved. $\square$

Lemma 3. *The action $(W, *) \times W \to W$, $(x, y) \mapsto x \rhd y$ is a left action of the monoid $(W, *)$.*

By definition, $1 \rhd x = x$ for any $x \in W$.

Let $x, y, z \in W$. Then there exists $u \leqslant x$ and $v \leqslant y$ such that $y \rhd z = vz$ and $x \rhd (vz) = uvz$. By definition $uv \leqslant x * y$. So $(x * y) \rhd z \leqslant x \rhd (y \rhd z)$.

On the other hand, there exists $w \leqslant x$ such that $x * y = wy$ and $l(wy) = l(w) + l(y)$. Then there exists $w' \leqslant wy$ such that $(wy) \rhd z = w'z$. Since $l(wy) = l(w) + l(y)$, we may write w' as $w' = w_1 y_1$ for some $w_1 \leqslant w$ and $y_1 \leqslant y$. Thus $(x * y) \rhd z = w_1(y_1 z)$. Notice that $w_1 \leqslant x$ and $y_1 \leqslant y$ and $y_1 z \geqslant y \rhd z$. By the previous lemma, $x \rhd (y \rhd z) \leqslant w_1(y_1 z) = (x * y) \rhd z$. The lemma is proved. $\square$

Lemma 4. *Assume that W is finite. Then $(x \rhd y) w_0^I = x * (y w_0^I)$.*

By definition, $x \rhd y$ is the unique minimal element in $\{uy; u \leqslant x\}$. Hence $(x \rhd y) w_0^I$ is the unique maximal element in $\{u y w_0; u \leqslant x\}$. By Lemma 1, this unique maximal element is $x * (y w_0^I)$. $\square$

Lemma 5. *Let $J_1, J_2 \subset I$. If $(w_0^{J_1} w_0^I) * (w_0^{J_2} w_0^I) = w_0^{J_3} w_0^I$ for some $J_3 \subset I$, then $(w_0^{J_2} w_0^I) * (w_0^{J_1} w_0^I) = w_0^{J_3} w_0^I$. In other words, if $(w_0^{J_1} w_0^I) \rhd w_0^{J_2} = w_0^{J_3}$ for some $J_3 \subset I$, then $(w_0^{J_2} w_0^I) \rhd w_0^{J_1} = w_0^{J_3}$.*

This was proved in [BK, Proposition 2.30 (c)] by applying the anti-automorphism $w \mapsto w_0^I w^{-1} w_0^I$ on W.

Lemma 6. *If $J_2 = K \sqcup K'$ with $s_k s_{k'} = s_{k'} s_k$ for any $k \in K$ and $k' \in K'$. Then for any $J_1 \subset I$, we have that*

$$(w_0^{J_1} w_0^I) \rhd w_0^{J_2} = \big((w_0^{J_1} w_0^I) \rhd w_0^K\big)\big((w_0^{J_1} w_0^I) \rhd w_0^{K'}\big).$$

We fix a reduced expression $w_0^{J_1} w_0^I = s_{i_1} s_{i_2} \cdots s_{i_n}$ for $i_1, i_2, \cdots, i_n \in I$. Assume that $v \leqslant w_0^{J_1} w_0^I$ with $(w_0^{J_1} w_0^J) \rhd w_0^{J_2} = v w_0^{J_2}$. Then $v \leqslant w_0^{J_2}$. Hence $v \in W_{J_2}$. Let $v = s_{i_{l_1}} \cdots s_{i_{l_k}}$ be a reduced subexpression. Then $l_1, \cdots, l_k \in J_2 = K \cup K'$. By assumption on K and K', $v = v_1 v_2$ for $v_1 \in W_K$ and $v_2 \in W_{K'}$. So $v w_0^{J_2} = v_1 v_2 w_0^K w_0^{K'} = (v_1 w_0^K)(v_2 w_0^{K'}) \geqslant \big((w_0^{J_1} w_0^I) \rhd w_0^K\big)\big((w_0^{J_1} w_0^I) \rhd w_0^{K'}\big)$.

On the other hand, assume that $v_1, v_2 \leqslant w_0^{J_1} w_0^I$ with $(w_0^{J_1} w_0^I) \triangleright w_0^K = v_1 w_0^K$ and $(w_0^{J_1} w_0^I) \triangleright w_0^{K'} = v_2 w_0^{K'}$. Then $v_1 \in W_K$ and $v_2 \in W_{K'}$ and there exist $1 \leqslant t_1 < t_2 < \cdots < t_u \leqslant n$ and $1 \leqslant t'_1 < t'_2 < \cdots < t'_{u'} \leqslant n$ such that $v_1 = s_{i_{t_1}} \cdots s_{i_{t_u}}$ is a reduced subexpression of v_1 and $v_2 = s_{i_{t'_1}} \cdots s_{i_{t'_{u'}}}$ is a reduced subexpression of v_2.

Since $K \cap K' = \emptyset$, $\{t_1, \cdots, t_u\}$ and $\{t'_1, \cdots, t'_{u'}\}$ are disjoint subsets of $\{1, \cdots, n\}$. Let v be the element that corresponds to the subexpression $\{t_1, \cdots, t_u\} \sqcup \{t'_1, \cdots, t'_{u'}\}$. Then it is easy to see that $v = v_1 v_2$. Hence $\big((w_0^{J_1} w_0^I) \triangleright w_0^K\big)\big((w_0^{J_1} w_0^I) \triangleright w_0^{K'}\big) = (v_1 w_0^K)(v_2 w_0^{K'}) = v_1 v_2 w_0^K w_0^{K'} = v w_0^{J_1} \leqslant (w_0^{J_1} w_0^I) \triangleright w_0^{J_2}$. $\square$

Lemma 7. *Let $J_1, J_2 \subset I$. Then for any $J_1 \subset J'_1$, we have that*

$$(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = (w_0^{J_1} w_0^{J'_1}) \triangleright \big((w_0^{J'_1} w_0^I) \triangleright w_0^{J_2}\big).$$

Notice that $w_0^{J_1} w_0^I = (w_0^{J_1} w_0^{J'_1})(w_0^{J_2} w_0^I) = (w_0^{J_1} w_0^{J'_1}) * (w_0^{J'_1} w_0^I)$. The lemma follows from Lemma 3. $\square$

Below is the key lemma.

Lemma 8. *Assume that W is a irreducible finite Coxeter group and $i, i' \in I$ are end points of the Coxeter graph of W. Let $J_1 = I - \{i\}$ and $J_2 = I - \{i'\}$. Then $(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_3}$ for some $J_3 \subset J_1 \cap J_2$.*

We will prove this lemma in subsection 1.5. The proof is based on a case-by-case checking. We will also use the result to give an explicit description of the operator $\star_I$ for each type.

Before proving the lemma, we will show that the key lemma implies the main theorem.

1.4 Proof of Theorem 1. By Lemma 5, if $T'_{w_0^{J_1} w_0^I} T'_{w_0^{J_2} w_0^I} = T'_{w_0^{J_3} w_0^I}$, then $T'_{w_0^{J_2} w_0^I} T'_{w_0^{J_1} w_0^I} = T'_{w_0^{J_3} w_0^I}$. Using Lemma 4, we may reformulate the main theorem as follows:

For any $J_1, J_2 \subset I$, we have that $(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_3}$ for some $J_3 \subset I$.

We argue by induction on the cardinality of I. By Lemma 6 and Lemma 7, it suffices to prove the case where W is irreducible and J_1, J_2 are connected in the Coxeter graph.

It is easy to see that $(w_0^J w_0^I) \triangleright w_0^I = (w_0^I w_0^I) \triangleright w_0^J = w_0^J$ and $(w_0^{\emptyset} w_0^I) \triangleright w_0^J = (w_0^J w_0^I) \triangleright w_0^{\emptyset} = 1 = w_0^{\emptyset}$. Now assume that J_1, J_2 are proper connected subgraph in the Coxeter graph. Then there exists end points $i, i' \in I$ such that $i \notin J_1$ and $i' \notin J_2$. Set $J'_1 = I - \{i\}$ and $J'_2 = I - \{i'\}$. Then $J_1 \subset J'_1$ and $J_2 \subset J'_2$.

By Lemma 7, $(w_0^{J_2} w_0^I) \triangleright w_0^{J'_1} = (w_0^{J_2} w_0^{J'_2}) \triangleright \big((w_0^{J'_2} w_0^I) \triangleright w_0^{J'_1}\big)$. By Lemma 8, $(w_0^{J'_2} w_0^I) \triangleright w_0^{J'_1} = w_0^{J_3}$ for some $J_3 \subset J'_1 \cap J'_2$. By induction hypothesis on $W_{J'_2}$, we have that $(w_0^{J_2} w_0^{J'_2}) \triangleright w_0^{J_3} = w_0^{J_4}$ for some $J_4 \subset J'_1 \cap J_2$. By Lemma 5, $(w_0^{J'_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_4}$.

Again by Lemma 7, $(w_0^{J_1}w_0^I) \triangleright w_0^{J_2} = (w_0^{J_1}w_0^{J_1'}) \triangleright \big((w_0^{J_1'}w_0^I) \triangleright w_0^{J_2}\big) = (w_0^{J_1}w_0^{J_1'}) \triangleright w_0^{J_4}$. By induction hypothesis on $W_{J_1'}$, we have that $(w_0^{J_1}w_0^{J_1'}) \triangleright w_0^{J_4} = w_0^{J_5}$ for some $J_5 \subset J_1 \cap J_2$. $\square$

1.5 Proof of Lemma 8. We use the same labeling of Coxeter graph as in [Bo].

For $1 \leqslant a, b \leqslant n$, set

$$s_{[a,b]} = \begin{cases} s_a s_{a-1} \cdots s_b, & \text{if } a \geqslant b, \\ 1, & \text{otherwise.} \end{cases}$$

Type A_n

We have that $w_0^{I-\{1\}} w_0^I = s_{[n,1]}^{-1}$ and $w_0^{I-\{n\}} w_0^I = s_{[n,1]}$. Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= s_{[n,1]}^{-1} \triangleright w_0^{I-\{1\}} = s_{[n,2]}^{-1} w_0^{I-\{1\}} = w_0^{I-\{1,2\}}, \\
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n\}} &= s_{[n,1]}^{-1} \triangleright w_0^{I-\{n\}} = s_{[n-1,1]}^{-1} w_0^{I-\{n\}} = w_0^{I-\{1,n\}}, \\
(w_0^{I-\{n\}} w_0^I) \triangleright w_0^{I-\{n\}} &= s_{[n,1]} \triangleright w_0^{I-\{n\}} = s_{[n-1,1]} w_0^{I-\{n\}} = w_0^{I-\{n-1,n\}}.
\end{aligned}$$

Type B_n

We have that $w_0^{I-\{1\}} w_0^I = s_{[n-1,1]}^{-1} s_{[n,1]}$ and $w_0^{I-\{n\}} w_0^I = s_n s_{[n,n-1]}^{-1} \cdots s_{[n,1]}^{-1}$. Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= (s_{[n-1,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{1\}} = s_{[n-1,2]}^{-1} s_{[n,2]} w_0^{I-\{1\}} \\
&= w_0^{I-\{1,2\}}, \\
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_{[n-1,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{n\}} = s_{[n-1,1]}^{-1} \triangleright \big(s_{[n,1]} \triangleright w_0^{I-\{n\}}\big) \\
&= s_{[n-1,1]}^{-1} \triangleright w_0^{I-\{n-1,n\}} = w_0^{I-\{1,n-1,n\}}, \\
(w_0^{I-\{n\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_n s_{[n,n-1]}^{-1} \cdots s_{[n,1]}^{-1}) \triangleright w_0^{I-\{n\}} \\
&= s_{n-1} s_{[n-1,n-2]}^{-1} \cdots s_{[n-1,1]}^{-1} w_0^{I-\{n\}} = 1.
\end{aligned}$$

Type D_n

Set

$$\epsilon = \begin{cases} 1, & \text{if } 2 \nmid n; \\ 0, & \text{if } 2 \mid n. \end{cases}$$

We have that

$$\begin{aligned}
w_0^{I-\{1\}} w_0^I &= s_{[n-2,1]}^{-1} s_{[n,1]}, \\
w_0^{I-\{n-1\}} w_0^I &= s_{n-1}(s_{n-2} s_n) \cdots (s_{[n-2,2]}^{-1} s_{n-\epsilon})(s_{[n-2,1]}^{-1} s_{n-1+\epsilon}), \\
w_0^{I-\{n\}} w_0^I &= s_n(s_{n-2} s_{n-1}) \cdots (s_{[n-2,2]}^{-1} s_{n-1+\epsilon})(s_{[n-2,1]}^{-1} s_{n-\epsilon}).
\end{aligned}$$

Hence

$$\begin{aligned}
(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{1\}} &= (s_{[n-2,1]}^{-1}s_{[n,1]}) \triangleright w_0^{I-\{1\}} = s_{[n-2,2]}^{-1}s_{[n,2]}w_0^{I-\{1\}} \\
&= w_0^{I-\{1,2\}}, \\
(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{n-1\}} &= (s_{[n-2,1]}^{-1}s_{[n,1]}) \triangleright w_0^{I-\{n-1\}} = s_{[n-2,1]}^{-1}s_n s_{[n-2,1]}w_0^{I-\{n-1\}} \\
&= w_0^{I-\{1,n-1,n\}}, \\
(w_0^{I-\{n-1\}}w_0^I) \triangleright w_0^{I-\{n\}} &= \big(s_{n-1}(s_{n-2}s_n)\cdots(s_{[n-2,2]}^{-1}s_{n-\epsilon})(s_{[n-2,1]}^{-1}s_{n-1+\epsilon})\big) \triangleright w_0^{I-\{n\}} \\
&= s_{n-1}s_{n-2}\cdots s_{[n-2+\epsilon,2]}^{-1}s_{[n-1-\epsilon,1]}^{-1}w_0^{I-\{n\}} \\
&= \begin{cases} s_1s_3\cdots s_{n-2}, & \text{if } 2 \nmid n; \\ s_2s_4\cdots s_{n-2}, & \text{if } 2 \mid n. \end{cases} \\
(w_0^{I-\{n\}}w_0^I) \triangleright w_0^{I-\{n\}} &= \big(s_n(s_{n-2}s_{n-1})\cdots(s_{[n-2,2]}^{-1}s_{n-1+\epsilon})(s_{[n-2,1]}^{-1}s_{n-\epsilon})\big) \triangleright w_0^{I-\{n\}} \\
&= s_{[n-1,n-2]}^{-1}\cdots s_{[n-1-\epsilon,2]}^{-1}s_{[n-2+\epsilon,1]}^{-1}w_0^{I-\{n\}} \\
&= \begin{cases} s_2s_4\cdots s_{n-1}, & \text{if } 2 \nmid n; \\ s_1s_3\cdots s_{n-1}, & \text{if } 2 \mid n. \end{cases}
\end{aligned}$$

Applying the automorphism $\sigma : W \to W$ which exchanges s_{n-1} and s_n, we also have that $(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{n\}} = w_0^{I-\{1,n-1,n\}}$ and

$$(w_0^{I-\{n-1\}}w_0^I) \triangleright w_0^{I-\{n-1\}} = \begin{cases} (s_2s_4\cdots s_{n-3})s_n, & \text{if } 2 \nmid n; \\ (s_1s_3\cdots s_{n-3})s_n, & \text{if } 2 \mid n. \end{cases}.$$

For type E, set $x = s_4s_3s_5s_4s_2$.

Type E_6

We have that

$$\begin{aligned}
w_0^{I-\{1\}}w_0^I &= s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}, \\
w_0^{I-\{2\}}w_0^I &= x^{-1}s_{[6,1]}^{-1}s_{[5,1]}x, \\
w_0^{I-\{6\}}w_0^I &= s_{[6,1]}s_{[4,6]}^{-1}s_{[3,5]}^{-1}s_2s_4s_3s_1.
\end{aligned}$$

Hence

$$\begin{aligned}
(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{1\}} &= (s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}) \triangleright w_0^{I-\{1\}} \\
&= (s_{[6,3]}^{-1}x^{-1}s_{[6,2]}^{-1})w_0^{I-\{1\}} = w_0^{\{2,4,5\}},
\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{2\}} &= \big(s_1 s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}\big)\triangleright w_0^{I-\{2\}}\\ &= \big(s_1 s_{[6,3]}^{-1}s_4s_3s_5s_4\big)\triangleright\Big(\big(s_1s_{[6,3]}^{-1}\big)\triangleright w_0^{I-\{2\}}\Big)\\ &= \big(s_1 s_{[6,3]}^{-1}s_4s_3s_5s_4\big)\triangleright w_0^{I-\{1,2\}}\\ &= s_{[6,3]}^{-1}s_4s_3s_5s_4w_0^{I-\{1,2\}} = s_4s_6,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{6\}} &= \big(s_1 s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}\big)\triangleright w_0^{I-\{6\}}\\ &= s_1 s_{[5,3]}^{-1}x^{-1}s_{[5,1]}^{-1}w_0^{I-\{6\}} = w_0^{\{3,4,5\}},\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{2\}}w_0^I)\triangleright w_0^{I-\{2\}} &= \big(x^{-1}s_{[6,1]}^{-1}s_{[5,1]}x\big)\triangleright w_0^{I-\{2\}}\\ &= (x^{-1})\triangleright\Big(\big(s_1s_{[6,3]}^{-1}s_{[5,3]}s_1s_4s_3s_5s_4\big)\triangleright w_0^{I-\{2\}}\Big)\\ &= x^{-1}\triangleright(s_3s_5) = 1.\end{aligned}$$

Applying the nontrivial diagram automorphism, we also have that $(w_0^{I-\{6\}}w_0^I)\triangleright w_0^{I-\{6\}} = w_0^{\{2,3,4\}}$ and $(w_0^{I-\{6\}}w_0^I)\triangleright w_0^{I-\{2\}} = s_1s_4$.

Type E_7
We have that

$$\begin{aligned}w_0^{I-\{1\}}w_0^I &= s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1,\\ w_0^{I-\{2\}}w_0^I &= s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2,\\ w_0^{I-\{7\}}w_0^I &= s_{[7,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{1\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\triangleright w_0^{I-\{1\}}\\ &= \big(s_3s_4s_2s_{[5,3]}\big)\triangleright\Big(\big(s_{[6,2]}s_{[6,4]}^{-1}s_{[7,2]}xs_{[6,3]}\big)\triangleright w_0^{I-\{1\}}\Big)\\ &= \big(s_3s_4s_2s_{[5,3]}\big)\triangleright w_0^{\{2,4,5,7\}} = s_2s_5s_7,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{2\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\triangleright w_0^{I-\{2\}}\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\triangleright\Big(\big(s_{[7,1]}xs_{[6,3]}s_1\big)\triangleright w_0^{I-\{2\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\triangleright\big(s_1s_4s_{[6,3]}s_1\big) = 1,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{7\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\triangleright w_0^{I-\{7\}}\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\triangleright\Big(\big(s_{[6,1]}xs_{[6,3]}s_1\big)\triangleright w_0^{I-\{7\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\triangleright w_0^{I-\{6,7\}}\\ &= s_1s_3s_4s_2s_{[5,3]}s_1s_{[5,2]}s_{[5,4]}^{-1}w_0^{I-\{6,7\}} = w_0^{\{3,4,5\}},\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{2\}}w_0^I)\rhd w_0^{I-\{2\}} &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2\big)\rhd w_0^{I-\{2\}}\\ &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}x\big)\rhd\Big(\big(s_{[7,1]}xs_{[6,3]}s_{[7,4]}\big)\rhd w_0^{I-\{2\}}\Big)\\ &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}x\big)\rhd\big(s_1s_4s_6\big)=1,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{2\}}w_0^I)\rhd w_0^{I-\{7\}} &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2\big)\rhd w_0^{I-\{7\}}\\ &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}\big)\rhd\Big(\big(s_{[5,1]}xs_{[6,1]}xs_{[6,3]}s_{[6,4]}s_2\big)\rhd w_0^{I-\{7\}}\Big)\\ &= \big(s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}\big)\rhd w_0^{\{1,3,4,6\}}=1,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{7\}}w_0^I)\rhd w_0^{I-\{7\}} &= \big(s_{[7,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}\big)\rhd w_0^{I-\{7\}}\\ &= \big(s_{[6,1]}xs_{[6,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)w_0^{I-\{7\}}=w_0^{\{2,3,4,5\}}.\end{aligned}$$

Type E_8

We have that

$$\begin{aligned}w_0^{I-\{1\}}w_0^I &= s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[6,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_1,\\ w_0^{I-\{2\}}w_0^I &= s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_{[7,4]}s_2,\\ w_0^{I-\{8\}}w_0^I &= s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[8,4]}^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\rhd w_0^{I-\{1\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\rhd\\ &\quad\Big(\big(s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[6,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_1\big)\rhd w_0^{I-\{1\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\rhd\big(s_3s_5s_7\big)\\ &=1,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\rhd w_0^{I-\{2\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1s_{[8,1]}xs_{[6,3]}s_1\big)\rhd\\ &\quad\Big(\big(s_{[7,2]}s_{[6,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_1\big)\rhd w_0^{I-\{2\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1s_{[8,1]}xs_{[6,3]}s_1\big)\rhd\big(s_1s_3s_{[6,3]}s_1\big)\\ &=1,\end{aligned}$$

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I)\rhd w_0^{I-\{8\}} &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\rhd\\ &\quad\Big(\big(s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[6,4]}^{-1}s_{[8,1]}xs_{[6,3]}s_1\big)\rhd w_0^{I-\{8\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1\big)\rhd w_0^{\{2,3,4,5,6\}}\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\rhd\Big(\big(s_{[7,1]}xs_{[6,3]}s_1\big)\rhd w_0^{\{2,3,4,5,6\}}\Big)\\ &= \big(s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}\big)\rhd w_0^{\{3,4,5\}}=1,\end{aligned}$$

$$
\begin{aligned}
(w_0^{I-\{2\}}w_0^I)\triangleright w_0^{I-\{2\}} &= (s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1})\triangleright \\
&\quad \big((s_{[8,1]}xs_{[6,3]}s_{[7,4]}s_2)\triangleright w_0^{I-\{2\}}\big) \\
&= (s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2s_{[8,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1})\triangleright \\
&\quad (s_1s_4s_6s_{[7,3]}s_1) \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
(w_0^{I-\{2\}}w_0^I)\triangleright w_0^{I-\{8\}} &= (s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]})\triangleright \\
&\quad (xs_{[6,3]}s_{[6,4]}s_2s_{[7,1]}xs_{[6,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2w_0^{I-\{8\}}) \\
&= (s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]})\triangleright w_0^{\{1,5,6\}} = 1,
\end{aligned}
$$

$$
\begin{aligned}
(w_0^{I-\{8\}}w_0^I)\triangleright w_0^{I-\{8\}} &= s_{[6,1]}xs_{[6,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}w_0^{I-\{8\}} \\
&= w_0^{\{2,3,4,5\}}.
\end{aligned}
$$

Type F_4

We have that $w_0^{I-\{1\}}w_0^I = s_{[4,1]}^{-1}s_2s_3s_2s_1s_2s_3s_2s_{[4,1]}$ and $w_0^{I-\{4\}}w_0^I = s_{[4,1]}s_3s_2s_3s_4s_3s_2s_3s_{[4,1]}^{-1}$. Hence

$$
\begin{aligned}
(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{1\}} &= (s_{[4,1]}^{-1}s_2s_3s_2s_1s_2s_3s_2s_{[4,1]})\triangleright w_0^{I-\{1\}} = 1, \\
(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{4\}} &= (s_{[4,1]}^{-1}s_2s_3s_2s_1s_2s_3s_2s_{[4,1]})\triangleright w_0^{I-\{4\}} = 1.
\end{aligned}
$$

Applying the nontrivial diagram automorphism, we also have that $(w_0^{I-\{4\}}w_0^I)\triangleright w_0^{I-\{4\}} = 1$.

Type H_3

We have that $w_0^{I-\{1\}}w_0^I = s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2s_1$ and $w_0^{I-\{3\}}w_0^I = s_3s_2s_1s_2s_1s_3s_2s_1s_2s_3$. Hence it is easy to see that $(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{3\}} = (w_0^{I-\{3\}}w_0^I)\triangleright w_0^{I-\{3\}} = 1$.

Type H_4

We have that $w_0^{I-\{1\}}w_0^I = s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2s_1(s_{[4,1]}s_2s_1s_{[3,1]}s_2s_3)^4s_3s_2$ and $w_0^{I-\{4\}}w_0^I = (s_{[4,1]}s_2s_1s_{[3,1]}s_2s_3)^4s_4$. Hence it is easy to see that $(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{4\}} = (w_0^{I-\{4\}}w_0^I)\triangleright w_0^{I-\{4\}} = 1$.

Type I_m

It is easy to see that $(w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}}w_0^I)\triangleright w_0^{I-\{2\}} = (w_0^{I-\{2\}}w_0^I)\triangleright w_0^{I-\{2\}} = 1$.

1.6. Now we will calculate the operator $\star_I$ explicitly for each type. This is based on the previous subsection, the equalities $J_1\star_I J_2 = J_2\star_I J_1$ (see Lemma 5), $J_1'\star_I J_2 = J_1'\star_{J_1}(J_1\star_I J_2)$ for any $J_1'\subset J_1$ (see Lemma

7) and the inequality $J_1 \star_I J_2 \subset J_1' \star_I J_2'$ for $J_1 \subset J_1'$ and $J_2 \subset J_2'$ (see Lemma 2). We just list below the cases where J_1 and J_2 are proper connected subgraph of the Coxeter graph of W.

For the case where J_1 and J_2 are not necessarily connected and $J_1', \cdots, J_l'$ (resp. $J_1'', \cdots, J_{l'}''$) are the connected components of J_1 (resp. J_2), we have that $J_1 \star_I J_2 = \sqcup_{1 \leqslant i \leqslant l, 1 \leqslant i' \leqslant l'} (J_i' \star_I J_i'')$ (see Lemma 6).

Type A_n: Let $J_1 = \{a, a+1, \cdots, n-b\}$, $J_2 = \{a', a'+1, \cdots, n-b'\}$. Then

$$J_1 \star_I J_2 = \{a + a' - 1, a + a', \cdots, n - b - b'\}.$$

Type B_n: Let $J_1 = \{a, a+1, \cdots, n-b\}$, $J_2 = \{a', a'+1, \cdots, n-b'\}$. Then

$$J_1 \star_I J_2 = \begin{cases} \{a + a' - 1, a + a', \cdots, n\}, & \text{if } b = b' = 0; \\ \{a + a' - 1, a + a', \cdots, n - b' - 1\}, & \text{if } b = 0, b' \geqslant 1; \\ \emptyset, & \text{if } b, b' \geqslant 1. \end{cases}$$

Type D_n: If $I - J_1 = n - 1$ or n and $I - J_2 = n - 1$ or n, then $J_1 \star_I J_2$ was already calculated in the previous subsection.

If $\{n-1, n\} \not\subseteq J_1$ and $\{n-1, n\} \not\subseteq J_2$ and J_1 or J_2 contains at most $n-2$ elements, then $J_1 \star_I J_2 = \emptyset$.

Otherwise, we may assume that $\{n-1, n\} \subset J_1$, i.e. $J_1 = \{a, a + 1, \cdots, n\}$ for some a. If $\{n-1, n\} \subset J_2$, i.e. $J_2 = \{a', a'+1, \cdots, n\}$ for some a', then $J_1 \star_I J_2 = \{a + a' - 1, a + a', \cdots n\}$.

If $\{n-1, n\} \not\subseteq J_2$, we may assume without loss of generality that $n \notin J_2$, i.e. $J_2 = \{a', a'+1, \cdots, n - b'\}$ for some a', b' with $b' \geqslant 1$. Then $J_1 \star_I J_2 = \{a + a' - 1, a + a', \cdots, n - b' - 1\}$.

Type E_6: If $2 \notin J_1$ and $2 \notin J_2$, then $J_1 \star_I J_2 = \emptyset$.

If $2 \in J_1$ and $2 \notin J_2$, then

$$J_1 \star_I J_2 = \begin{cases} \{4, 6\}, & \text{if } J_1 = I - \{1\}, J_2 = I - \{2\}; \\ \{1, 4\}, & \text{if } J_1 = I - \{6\}, J_2 = I - \{2\}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $2 \in J_1 \cap J_2$, then

$$J_1 \star_I J_2 = \begin{cases} \{2, 4, 5\}, & \text{if } J_1 = J_2 = I - \{1\}, \\ \{2, 3, 4\}, & \text{if } J_1 = J_2 = I - \{6\}, \\ \{3, 4, 5\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, J_2 = I - \{6\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, I - \{1, 6\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{6\}, I - \{1, 6\}\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type E_7: For proper subsets J_1 and J_2, we have that

$$J_1 \star_I J_2 = \begin{cases} \{2,5,7\}, & \text{if } J_1 = J_2 = I - \{1\}, \\ \{2,3,4,5\}, & \text{if } J_1 = J_2 = I - \{7\}, \\ \{3,4,5\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, I - \{7\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{7\}, I - \{1,7\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{7\}, I - \{6,7\}\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type E_8: For proper subsets J_1 and J_2, we have that

$$J_1 \star_I J_2 = \begin{cases} \{2,3,4,5\}, & \text{if } J_1 = J_2 = I - \{8\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type F, H and I: For proper subsets J_1, J_2 of I, we always have that $J_1 \star_I J_2 = \emptyset$.

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