

Lifting of elements of Weyl groups

Jeffrey Adams* Xuhua He†
 Department of Mathematics Department of Mathematics
 University of Maryland University of Maryland

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1 Introduction

Let G be a connected reductive group over an algebraically closed field F . Choose a Cartan subgroup $T \subset G$, let $N = \text{Norm}_G(T)$ be its normalizer, and let $W = N/T$ be the Weyl group. We have the exact sequence

$$(1.1) \quad 1 \rightarrow T \rightarrow N \xrightarrow{p} W \rightarrow 1.$$

It is natural to ask what can be said about the orders of lifts of an element $w \in W$ to N . What is the smallest possible order of a lift of w ? In particular, can w be lifted to an element of N of the same order?

Write $o(*)$ for the order of an element of a group, and let $N_w = p^{-1}(w) \subset N$. Define

$$(1.2)(a) \quad \tilde{o}(w, G) = \min_{g \in N_w} o(g).$$

The most important case is for the adjoint group G_{ad} , so define

$$(1.2)(b) \quad \tilde{o}_{\text{ad}}(w) = \tilde{o}(w, G_{\text{ad}}).$$

It is clear that $\tilde{o}(w, G)$ only depends on the conjugacy class \mathcal{C} of w , so write $\tilde{o}(\mathcal{C}, G)$ and $\tilde{o}_{\text{ad}}(\mathcal{C})$ accordingly.

An essential role is played by the Tits group. This is a group which fits in an exact sequence $1 \rightarrow T_2 \rightarrow \mathcal{T} \rightarrow W \rightarrow 1$ where T_2 is a certain subgroup of the elements of T of order (1 or) 2. This implies $\tilde{o}(w, G) = o(w)$ or $2o(w)$, but it can be difficult to determine which case holds.

We also consider the twisted situation. Let δ be an automorphism of G of finite order which preserves a pinning, and set ${}^\delta G = G \rtimes \langle \delta \rangle$. Let ${}^\delta N = \text{Norm}_{{}^\delta G}(T)$ and ${}^\delta W = {}^\delta N/T$. Then conjugacy in $W\delta$ is the same as δ -twisted

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conjugacy in W , and we can ask about the order of lifts of elements of $W\delta$ to ${}^\delta N$. See Section 2 for details.

We say W lifts to G if the exact sequence (1.1) splits, in which case $\tilde{o}(w, G) = \tilde{o}(w)$ for all w . If this is not the case, it may not be practical to give a formula for $\tilde{o}(w, G)$ for all conjugacy classes. Rather, this can be done for several natural families. We say $w \in W\delta$ is elliptic if it has no nontrivial fixed vectors in the reflection representation; in this case all lifts of w are conjugate, so have the same order $\tilde{o}(w, G)$. An element $w \in W\delta$ is said to be regular if it has a regular eigenvector (see Section 7 and [15]).

Let ρ^\vee be one-half the sum of the positive coroots in any positive system. We refer to the element $z_G = (2\rho^\vee)(-1)$ as the *principal involution* in G . It is contained in the center $Z(G)$, is independent of the choice of positive system, and is fixed by every automorphism of G .

Here is a result concerning when W lifts, so $\tilde{o}(w) = o(w)$ for all w .

Theorem A If the characteristic of F is 2, then the Tits group \mathcal{T} is isomorphic to the Weyl group, so the exact sequence (1.1) splits.

Suppose the characteristic of F is not 2, and that G is simple.¹ If G is adjoint of type A_n, B_n, D_n or G_2 then W lifts. The same holds for $\mathrm{SO}(2n)$, and in type A_n if $|Z(G)|$ is odd. For necessary and sufficient conditions for W to lift see Theorem 4.16.

Over \mathbb{C} this is proved in [5], with the exception of some cases in types A_n and D_n .

Theorem B Assume the characteristic of F is not 2.

- (1) Suppose G is simple and $w \in W\delta$ is an elliptic element. Then $\tilde{o}_{\mathrm{ad}}(w) = o(w)$, except in certain cases in type C_n , or G is of type F_4 and w is in the conjugacy class $A_3 + \tilde{A}_1$. See Section 6 for details.
- (2) If $w \in W\delta$ is regular then $\tilde{o}_{\mathrm{ad}}(w) = o(w)$.

The case when w is regular and elliptic is discussed in [13]. The next result gives more detail on $\tilde{o}(w)$ for elliptic conjugacy classes.

Theorem C Assume the characteristic of F is not 2. Suppose G is simple, w is an elliptic element of $W\delta$, and g is a lift of w .

- (1) Suppose G is of type A_n . Then $g^{o(w)} = z_G$.
- (2) Suppose G is of type C_n . If G is simply connected then $g^{o(w)} \neq 1$. The elliptic conjugacy classes are parametrized by partitions of n (cf. Section 5.2). Suppose G is adjoint and w corresponds to a partition (a_1, \dots, a_l) . Then $g^{o(w)} = 1$ if and only if each a_i has the same power of 2 in its prime decomposition.

¹By simple we mean in the sense of algebraic groups: G has no nontrivial, closed, connected, normal subgroups. Some authors use the term *quasi-simple* or *almost simple*.

- (3) Suppose G is of type B_n or D_n . If G is adjoint or $G \simeq \mathrm{SO}(2n)$ then $g^{o(w)} = 1$. Otherwise see Section 9.
- (4) Suppose G is of exceptional type. If G is of type ${}^3D, G_2, E_6, {}^2E_6, E_7^{ad}$ or E_8 then $g^{o(w)} = 1$. The same holds if G is of type F_4 and w is not in the class $A_3 + \tilde{A}_1$.

For a more precise but more technical result see Proposition 8.1.

For brevity we've stated these results over an algebraically closed field. For various weaker conditions see Proposition 2.2.

There are several key tools. The Tits group comes with a canonical set-theoretic splitting $\sigma : W \mapsto \mathcal{T}$, and we make frequent use of an identity in the Tits group: if $o(w) = 2$ then $\sigma(w)^2 = (w\rho^\vee - \rho^\vee)(-1)$ (Lemma 3.1). In particular if w_0 is the longest element of W then $\sigma(w_0)^2 = z_G \in Z(G)$ and $\tilde{o}_{\mathrm{ad}}(w_0) = o(w_0) = 2$. See Section 3. Theorems B and C reduce to this, by an easy calculation in some cases, or using the theory of good elements of conjugacy classes to reduce to principal involutions in Levi factors. See Section 6.

We originally computed $o(\sigma(w))$ for elliptic elements in the exceptional groups *Atlas of Lie Groups and Representations* software [2]. This independently confirms Theorem C (4); the two proofs rely on independent computer calculations.

Sean Rostami has some recent results which overlap these [14].

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2 The Tits group

It is convenient to allow F to be an arbitrary field, and suppose G is a connected, reductive algebraic group defined over F . Furthermore we assume G is split over F , and fix an F -split Cartan subgroup of G . Then $N = \mathrm{Norm}_G(T)$ and $W = N/T$ are defined over F . If F is algebraically closed then all Cartan subgroups of G are conjugate and F -split. We identify G, T, N and W with their F -points $G(F), T(F), N(F)$, and $W(F) = N(F)/T(F)$.

Let $X^*(T)$, or simply X^* , be the character lattice of T , $X_* = X_*(T)$ the co-character lattice, with natural perfect pairing $\langle \cdot, \cdot \rangle : X^* \times X_* \rightarrow \mathbb{Z}$. Write $\Delta \subset X^*$ for the roots of T in G . If B is a Borel subgroup containing T it defines a set of positive roots Δ^+ of T in G , with associated simple roots Π . The Weyl group is generated by $\{s_\alpha \mid \alpha \in \Pi\}$, with the braid relations and $s_\alpha^2 = 1$. If we've numbered the simple roots we write $s_i = s_{\alpha_i}$.

For $\alpha \in \Delta$ let $\alpha^\vee \in X_*$ be the corresponding co-root, and set $m_\alpha = \alpha^\vee(-1)$. The elementary abelian two-group generated by $\{m_\alpha \mid \alpha \in \Pi\}$ is denoted T_2 . If G is simple and simply connected this is the set of elements of order 2 in T . In general it is the image of T_2^{sc} where T^{sc} is a Cartan subgroup of the simply connected cover of the derived group of G .

Now fix a set $\{X_\alpha \mid \alpha \in \Pi\}$ of simple root vectors, so $\mathcal{P} = (T, B, \{X_\alpha\})$ is a pinning. Associated to \mathcal{P} is the Tits group $\mathcal{T} = \mathcal{T}_{\mathcal{P}}$. This is a subgroup of N , generated by elements $\{\sigma_\alpha \mid \alpha \in \Pi\}$, where σ_α is a certain lift of s_α to N . See [17].

Lemma 2.1.

(1) *The Tits group \mathcal{T} is given by generators $\{\sigma_\alpha \mid \alpha \in \Pi\}$ and T_2 , and relations*

- (a) $\sigma_\alpha^2 = m_\alpha$,
- (b) *the braid relations,*
- (c) $\sigma_\alpha t \sigma_\alpha^{-1} = s_\alpha(t) \quad (\alpha \in \Pi, h \in T_2)$.

(2) *The map $\sigma_\alpha \rightarrow s_\alpha$ induces an exact sequence*

$$1 \rightarrow T_2 \rightarrow \mathcal{T} \rightarrow W \rightarrow 1.$$

(3) *If $w \in W$, choose a reduced expression $w = s_{\alpha_1} \dots s_{\alpha_n}$ and define $\sigma(w) = \sigma_{\alpha_1} \dots \sigma_{\alpha_n}$. This is independent of the reduced expression, and $w \mapsto \sigma(w)$ is a set-theoretic splitting of the exact sequence in (2).*

We also consider the twisted situation. We say an automorphism of G is distinguished if it fixes a pinning. Suppose δ is a distinguished automorphism of G , of finite order, and it fixes a pinning $\mathcal{P} = (T, B, \{X_\alpha\})$. Define ${}^\delta G = G \rtimes \langle \delta \rangle$ (we identify the automorphism δ of G with the element $(1, \delta)$ of ${}^\delta G$). Then δ induces automorphisms, also denoted δ , of the set of simple roots Π , the Dynkin diagram, N and the Weyl group W . If G is semisimple then δ is determined by an automorphism of the Dynkin diagram. Define ${}^\delta N = \text{Norm}_{{}^\delta G}(T)$ and ${}^\delta W = {}^\delta N/T$. Then ${}^\delta N \simeq N \rtimes \langle \delta \rangle$ and ${}^\delta W \simeq W \rtimes \langle \delta \rangle$.

It is easy to see that δ induces an automorphism of \mathcal{T} , satisfying $\delta(\sigma_\alpha) = \sigma_{\delta(\alpha)}$ ($\alpha \in \Pi$), and $\delta(\sigma(w)) = \sigma(\delta(w))$ ($w \in W$). Define the extended Tits group ${}^\delta \mathcal{T} = \mathcal{T} \rtimes \langle \delta \rangle$, so again there is an exact sequence $1 \rightarrow T_2 \rightarrow {}^\delta \mathcal{T} \rightarrow {}^\delta W \rightarrow 1$. The splitting $\sigma : W \rightarrow \mathcal{T}$ extends to ${}^\delta W$ by setting $\sigma(\delta) = \delta$.

Assume $\delta^2 = 1$. An element $w\delta \in {}^\delta W$ is an involution if and only if $w\delta(w) = 1$, in which case, as in [6], we say w is a δ -twisted involution. More generally if $\delta^r = 1$ then $(w\delta)^r = 1$ if and only if $w\delta(w)\delta^2(w) \dots \delta^{r-1}(w) = 1$.

Some of the main results apply without assuming F is algebraically closed.

Proposition 2.2. *Let F be an arbitrary field.*

(1) *Suppose G is an F -split, connected, reductive algebraic group, and T is an F -split Cartan subgroup. Then Theorems B and C hold.*

(2) *Suppose $F = \mathbb{R}$ and $G(\mathbb{R})$ is compact. Equivalently, suppose G is a compact connected Lie group. Then Theorems A, B and C hold.*

The proofs of Theorems B and C hold only assuming T is F -split. (If $\text{char}(F) = 2$ then Theorem A holds. Otherwise it requires that F contain certain roots of unity.) Statement (2) follows from:

Lemma 2.3. *Suppose G is a connected compact Lie group, and T is a Cartan subgroup. Let $(G(\mathbb{C}), T(\mathbb{C}))$ be the complexification of G and T , and choose a Borel subgroup $B(\mathbb{C})$ containing $T(\mathbb{C})$. Then we can choose a pinning $\mathcal{P} = (T(\mathbb{C}), B(\mathbb{C}), \{X_\alpha\})$ such that the Tits group $\mathcal{T}_{\mathcal{P}}$ is contained in $\text{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))$.*

Proof. This is just a version of the standard result that if G is compact then the $\text{Norm}_G(T)/T \simeq \text{Norm}_{G(\mathbb{C})}(T(\mathbb{C}))/T(\mathbb{C})$. To be precise: choose $\{X_\alpha\}$ so that $[X_\alpha, \sigma(X_\alpha)] = -\alpha^\vee$, where σ is complex conjugation of $\text{Lie}(G(\mathbb{C}))$ with respect to $\text{Lie}(G)$. \square

We dispense here with a case in which it is easy to compute $\tilde{o}(w, G)$.

Lemma 2.4. *Suppose $w \in W\delta$ has odd order. Then $\tilde{o}(w, G) = o(w)$.*

Proof. This is an immediate consequence of the Zassenhaus-Schur Lemma applied to the cyclic group generated by any lift of w . Concretely, let $d = o(w)$, and choose any lift g . If $g^d = 1$ then we are done. Otherwise replace g with g^{d+1} : $(g^{d+1})^d = (g^{2d})^{\frac{d+1}{2}} = 1$. \square

We also mention a basic reduction to simple groups, using the following Lemma, which is proved in the same way as [6, Lemma 2.7].

Lemma 2.5. *Suppose $G = G_1 \times G_1 \times \cdots \times G_1$, with r factors, and δ acts cyclically on the factors, so δ^r is an automorphism of the first factor. Write $W = W_1 \times \cdots \times W_1$ for the Weyl group. Then the twisted Weyl groups ${}^\delta W$ and ${}^{\delta^r} W_1$ are defined, and there is a natural bijection*

$$\{\delta\text{-twisted conjugacy classes in } W\} \longleftrightarrow \{\delta^r\text{-twisted conjugacy classes in } W_1\}.$$

3 Involutions

Suppose G is as in Section 2, δ is a distinguished automorphism of finite order of G , and ${}^\delta W = W \rtimes \langle \delta \rangle$.

Lemma 3.1. *Suppose $w \in W\delta$ is an involution. Then $\sigma(w)^2 = (w\rho^\vee - \rho^\vee)(-1)$. If w_0 is the longest element of W then $(\sigma(w_0\delta))^2 = z_G\delta^2$.*

Proof. For the first assertion, by assumption $\delta(w) = w^{-1}$ and $\delta^2 = 1$, so $(\sigma(w)\delta)^2 = \sigma(w)\delta\sigma(w)\delta = \sigma(w)\sigma(\delta(w))\delta^2 = \sigma(w)\sigma(w^{-1})$. Apply [1, Lemma 5.4]. The second statement follows from this, and the fact that w_0 and $\sigma(w_0)$ are fixed by every distinguished automorphism [1, Lemma 5.3]. \square

Let S be a subset of the simple roots, with corresponding Levi factor $L(S)$ and Weyl group $W(S)$. Then the pinning for G restricts to a pinning for $L(S)$, and the Tits group for L embeds naturally in that for G . If S is δ -stable the same holds for the extended Tits groups. Let $w_0(S)$ be a longest element of the Weyl group $W(S)$. Let $\rho^\vee(S)$ be one-half the sum of the positive coroots of $L(S)$, and let $z_S = z_{L(S)} = (2\rho^\vee(S))(-1)$ be the principal involution in $L(S)$.

The preceding Lemma applied to $L(S)$ gives:

Lemma 3.2. *Suppose $S \subset \Pi$ is a set of simple roots. Then*

$$\sigma(w_0(S))^2 = (2\rho^\vee(S))(-1) = z_S.$$

If δ is a distinguished involution and S is δ -stable then $\delta(\sigma(w_0(S))) = \sigma(w_0(S))$ and $(\sigma(w_0(S))\delta)^2 = z_S\delta^2$.

Lemma 3.3. *Suppose $\delta^2 = 1$ and $w \in W\delta$ acts by inverse on T . Then w is elliptic, and if g is any lift of w then $g^2 = z_G$. Furthermore*

$$\tilde{o}(w, G) = o(\sigma(w)) = \begin{cases} 2 & \rho^\vee \in X_*(T) \\ 4 & \text{otherwise} \end{cases}$$

This is an immediate consequence of Lemma 3.1.

4 Lifting of the Weyl group

In this section we assume F is algebraically closed.

We say W *lifts to* G if the exact sequence (1.1) splits, i.e. there is a group homomorphism $\phi : W \rightarrow N$ satisfying $p(\phi(w)) = w$ for all $w \in W$. If this holds then W is isomorphic to a subgroup of N , and *a fortiori* $\tilde{o}(w, G) = o(w)$ for all $w \in W$, and $o(\sigma(w)) = o(w)$ for all elliptic $w \in W\delta$.

The case of characteristic 2 is easy.

Proposition 4.1. *Suppose F has characteristic 2. Then the Tits group $\mathcal{T} \subset N$ is isomorphic to W .*

Proof. By the exact sequence in Lemma 2.1(2) the kernel of the map from \mathcal{T} to W is T_2 . But T_2 is generated by the elements $\alpha^\vee(-1)$, all of which are trivial in characteristic 2. \square

For the remainder of this section we assume $\text{char}(F) \neq 2$, and determine the simple groups G for which (1.1) splits.

We first address the question of the uniqueness of a splitting. Let $R \subset X^*$ be the root lattice, and $R^\vee \subset X_*$ the coroot lattice. Set $Z = Z(G)$.

Lemma 4.2. *Fix $\mu \in X_*(T)_\mathbb{Q}$. Define*

$$\mu^\perp = \{\gamma \in R \mid \langle \gamma, \mu \rangle = 0\}$$

and

$$S = \bigcap_{\gamma \in \mu^\perp} \ker(\gamma) \subset T.$$

Then S/Z is a (connected) torus. If μ is a coroot then $\dim(S/Z) = 1$.

Proof. It is straightforward to see that $X^*(S) = X^*(T)/\mu^\perp$. If G is adjoint then $X^*(T)$ is the root lattice R . It is obvious that R/μ^\perp is torsion free, which implies S is connected. In general S is the inverse image of a connected torus in G_{ad} , so $S = S_0Z$.

Suppose $\alpha \in \Delta$. After passing to the dual root system if necessary we may assume α is long, and after conjugating by W that it is the highest root. Except in type A_n the highest root is orthogonal to all but 1 simple root. In type A_n ($n \geq 2$) α is orthogonal to $n - 2$ simple roots. If δ, ϵ are the remaining two simple roots then $\langle \delta - \epsilon, \alpha^\vee \rangle = 0$. This proves the final assertion. \square

Lemma 4.3. *Fix $\alpha \in \Delta$. Suppose $t \in T$ satisfies: $\beta(t) = 1$ for all $\beta \in (\alpha^\vee)^\perp$. Then there exists $w \in F^\times$ such that $\alpha^\vee(w)t \in Z$.*

Proof. Let $S = \bigcap_{\gamma \in (\alpha^\vee)^\perp} \ker(\gamma)$. Then $\alpha^\vee(F^\times) \subset S$. By the Lemma above S/Z is a one dimensional torus, so the map $F^\times \xrightarrow{\alpha^\vee} S \rightarrow S/Z$ is surjective. \square

Suppose $\alpha \in \Delta$. It is well known (and easy to check) that, except in type A_n ,

$$(4.4) \quad (\alpha^\vee)^\perp = \mathbb{Z}\{\beta \in \Delta \mid \langle \beta, \alpha^\vee \rangle = 0\}.$$

We need a variant of this. We only need the simply laced case.

Lemma 4.5. *Suppose G is simply laced and no simple factor is of type A_3 or D_4 . Fix $\alpha \in \Delta$. Let $\Delta(\alpha) = \{\beta \in \Delta \mid \langle \beta, \alpha^\vee \rangle = 0\}$. This is a root system. Consider the lattice L spanned by*

$$(4.6)(a) \quad \{\beta \in \Delta(\alpha) \mid \text{the simple factor of } \beta \text{ in } \Delta(\alpha) \text{ is not of type } A_1\}$$

and

$$(4.6)(b) \quad \{2\delta + \alpha \mid \delta \in \Delta, \langle \delta, \alpha^\vee \rangle = -1\}$$

Then $L = (\alpha^\vee)^\perp$.

Proof. The containment $L \subset (\alpha^\vee)^\perp$ is obvious.

Since the statements only involves roots we may assume G is simple. It is easy to check A_1, A_2 directly, so (since A_3 and D_4 are excluded) we may assume G is of type A_4 , or $\text{rank}(G) \geq 5$.

After conjugating we may assume α is the highest root. Assume G is not of type A_n . Then α is orthogonal to all but one simple root, and these are the simple roots of $\Delta(\alpha)$. By (4.4) it is enough to show every simple root of $\Delta(\alpha)$ is in the span of (a) and (b).

In types E_6, E_7 and E_8 , $\Delta(\alpha)$ is connected, the simple factor condition in (a) is trivially satisfied, and the result is immediate. In type D_n , $\Delta(\alpha)$ has type $A_1 \times D_{n-2}$. If $n \geq 5$ there is only one A_1 factor. Taking δ to be the simple root not orthogonal to α , it is easy to see the roots of this factor are in the \mathbb{Z} -span of (a) and (b).

Now suppose G is of type A_n with $n \geq 4$. In this case $\Delta(\alpha)$ is of type A_{n-2} , and there are two simple roots δ, ϵ non-orthogonal to α . Suppose $\gamma \in (\alpha^\vee)^\perp$. Then $\gamma + c(2\delta + \alpha)$ is in the \mathbb{Z} -span of (a) for some choice of integer c . \square

Proposition 4.7. *Suppose either:*

- (1) G is simply laced, and no simple factor is of type A_3 or D_4 , or
- (2) G is simply connected.

Suppose ϕ, ϕ' are two splittings of (1.1). Then there exists $t \in T$ and $\{z_w \in Z \mid w \in W\}$ such that $\phi'(w) = z_w t \phi(w) t^{-1}$ for all $w \in W$.

The elements z_w are determined by $\{z_\alpha = z_{s_\alpha} \mid \alpha \in \Pi\}$, where $z_\alpha \in Z_2$ (the 2-torsion subgroup of Z). If α is conjugate to β then $z_\alpha = z_\beta$. If (1) holds then Z_2 acts simply transitively on the set of splittings.

Proof. Suppose ϕ is a splitting, and set $g_\alpha = \phi(s_\alpha)$ ($\alpha \in \Pi$). Then $\phi'(s_\alpha) = t_\alpha g_\alpha$ for some $t_\alpha \in T$.

First assume (1) holds.

Fix $\alpha \in \Pi$. We claim that $\beta(t_\alpha) = 1$ for all $\beta \in (\alpha^\vee)^\perp$. By Lemma 4.5 it is enough to show $\beta(t_\alpha) = 1$ for all β in 4.6(1) and (2).

First suppose β is in (1). Since β is orthogonal to α , $\{g_\alpha, g_\beta\} = 1$ ($\{, \}$ denotes the commutator). Then $\{t_\alpha g_\alpha, t_\beta g_\beta\} = 1$ if and only if $t_\alpha s_\alpha(t_\beta) = t_\beta s_\beta(t_\alpha)$. Using the fact if $t \in T$ then $s_\alpha(t) = t\alpha^\vee(\alpha(t^{-1}))$, the condition is equivalent to

$$\alpha^\vee(\alpha(t_\beta)) = \beta^\vee(\beta(t_\alpha)).$$

By assumption we can find $\gamma \in \Delta$ such that

$$(4.8) \quad \langle \gamma, \alpha^\vee \rangle = 0 \text{ and } \langle \gamma, \beta^\vee \rangle = -1$$

Apply γ to both sides to conclude $\beta(t_\alpha) = 1$.

Now suppose $\langle \delta, \alpha^\vee \rangle = -1$. Since $g_\alpha^2 = 1$ and $(t_\alpha g_\alpha)^2 = 1$ we conclude $t_\alpha s_\alpha(t_\alpha) = 1$, i.e.

$$t_\alpha^2 \alpha^\vee(\alpha(t_\alpha^{-1})) = 1.$$

Apply δ to both sides to conclude $(2\delta + \alpha)(t_\alpha) = 1$. This proves the claim.

Therefore by Lemma 4.5 we conclude $\mu(t_\alpha) = 1$ for all $\mu \in (\alpha^\vee)^\perp$. By Lemma 4.3 we can find $w_\alpha \in F^\times$ such that $\alpha^\vee(w_\alpha)t_\alpha \in Z$. This holds for all $\alpha \in \Pi$, and we can choose $t \in T$ so that $\alpha(t) = w_\alpha$ for all $\alpha \in \Pi$. Set $z_\alpha = \alpha^\vee(w_\alpha)t_\alpha \in Z$. Then

$$t(t_\alpha g_\alpha)t^{-1} = t s_\alpha(t^{-1}) t_\alpha g_\alpha = \alpha^\vee(\alpha(t)) t_\alpha g_\alpha = \alpha^\vee(w_\alpha) t_\alpha g_\alpha = z_\alpha g_\alpha.$$

Also $(t_\alpha g_\alpha)^2 = g_\alpha^2 = 1$ implies $z_\alpha^2 = 1$.

Now assume (2) holds. Replace (4.6)(1) with the larger set

$$(4.9) \quad \{\beta \in \Delta \mid \langle \beta, \alpha^\vee \rangle = 0\}.$$

The lattice spanned by (4.9) and (4.6)(2) is still equal to $(\alpha^\vee)^\perp$ ((2) is only needed in type A_n). Suppose β is in (4.9). Since G is simply connected, we can

find $\gamma \in X^*(T)$ satisfying (4.8), so as before we conclude $\beta(t_\alpha) = 1$. The rest of the proof is the same.

It is clear that the z_α have order 2 and determine all z_w . For the penultimate assertion, after conjugating by $t \in T$ we may assume $\phi'(w) = z_w \phi(w)$ for some $z_w \in W$. Suppose $\beta = w\alpha$ ($w \in W$). Applying ϕ' to the identity $ws_\alpha w^{-1} = s_\beta$ gives

$$z_w \phi(w) z_\alpha \phi(s_\alpha) \phi(w^{-1}) z_w^{-1} = z_\beta \phi(s_\beta).$$

Then $\phi(ws_\alpha w^{-1}) = s_\beta$ implies $z_\alpha = z_\beta$. The final assertion is now clear. \square

Example 4.10. Let $G = PSL(4)$. Then the conclusion of Proposition 4.7 does not hold. Choose the diagonal Cartan subgroup, the usual simple reflections s_i ($1 \leq i \leq 3$) and choose a fourth root ζ of -1 . Then $\phi(s_i) = g_i$, where

$$g_1 = \begin{pmatrix} 0 & \zeta & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}, \quad g_2 = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}, \quad g_3 = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & \zeta & 0 \end{pmatrix}$$

(the image in $PSL(4)$ of these elements) is a splitting. Also $\phi'(s_i) = g'_i$ where

$$g'_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g'_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g'_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is another splitting, not conjugate to ϕ .

The splitting ϕ is the image of the splitting by permutation matrices into $GL(4)$, composed with the maps $GL(4) \rightarrow GL(4)/Z(GL(4)) \simeq SL(4)/Z(SL(4)) = PSL(4)$. On the other hand ϕ' is the splitting of W into $SO(6)$ discussed below, composed with $SO(6) \rightarrow SO(6)/\pm I \simeq PSL(4)$.

It turns out that this is the *only* (simple) case where the lifting is not unique up to conjugacy and multiplication by Z_2 . See Corollary 4.19.

Before turning to the main result, we dispense with a few cases where it is easy to prove that W does not lift to G .

Lemma 4.11. Suppose $w \in W\delta$ is an elliptic element and $o(\sigma(w)) = 2o(w)$. Then W does not lift to G . If $\rho^\vee \notin X_*(T)$ then W does not lift to G .

This is immediate; the last line is from Lemma 3.3.

Lemma 4.12. The Weyl group does not lift to $SL(2n)$, $Sp(2n)$ or $Spin(n)$.

Proof. In types A_{2n+1} , B_n and C_n ρ^\vee is not in the root lattice, i.e. $X_*(T)$ for the simply connected group.

Suppose $G = Spin(n)$. Associated to the partition $(2, 1, \dots, 1)$ of n there is an elliptic element $w \in W$ (if n is odd) or twisted elliptic element $w \in W\delta$ (if n is even), of order 4 but whose lift has order 8. Therefore W does not lift to $Spin$. \square

Lemma 4.13. *Suppose H is a subgroup of G containing T . If $W(G, T)$ lifts to G then the exact sequence $1 \rightarrow T \rightarrow \text{Norm}_H(T) \rightarrow \text{Norm}_H(T)/T \rightarrow 1$ splits.*

This is also immediate; a splitting of (1.1) restricts to give a splitting. We will use this to eliminate some exceptional cases.

Finally we note a generalization of Lemma 2.4.

Lemma 4.14. *Suppose $A \subset Z$ is a cyclic group of odd order. If W lifts to G/A then W lifts to G .*

Proof. Identifying W with a subgroup of G/Z via a splitting, and taking the inverse image \widetilde{W} in G , we have an exact sequence

$$(4.15) \quad 1 \rightarrow A \rightarrow \widetilde{W} \rightarrow W \rightarrow 1$$

Let $m = |A|$. The exact sequence of trivial W -modules

$$1 \rightarrow A \rightarrow F^\times \xrightarrow{m} F^\times \rightarrow 1$$

gives rise to the exact sequence

$$H^1(W, F^\times) \rightarrow H^2(W, A) \rightarrow H^2(W, F^\times)$$

The middle term is killed by m . On the other hand $H^1(W, F^\times) \simeq \text{Hom}(W, F^\times) \simeq \text{Hom}(W/[W, W], F^\times)$, and this is killed by 2. Also $H^2(W, F^\times)$ is killed by 2 by [11]. Therefore $H^2(W, A) = 1$, so W lifts. \square

Theorem 4.16. *Assume G is simple and $\text{char}(F) \neq 2$. Then (1.1) splits in the following cases, and not otherwise:*

- (1) *Type A_n : $|Z(G)|$ is odd, or $G = SL(4)/\pm I \simeq SO(4)$.*
- (2) *Type B_n : $G = SO(2n + 1)$ (adjoint).*
- (3) *Type C_n : $n \leq 2$ and $G = PSL(2)$ or $PSp(4)$ (adjoint).*
- (4) *Type D_n : $G = SO(2n)$ or $G = PSO(2n)$ (adjoint); also $\text{Semispin}(8) \simeq SO(8)$.*
- (5) *Exceptional groups: G is of type G_2 .*

Implicit in (4) is the assertion that W does not lift to $\text{Semispin}(4m)$, unless $m = 2$.

When $F = \mathbb{C}$ this was proved in [5, Theorem 2], omitting a few cases in types A_n and D_n , using case-by-case calculations in the braid group. Here is a complete proof, including the missing cases, and relying as little as possible on braid group calculations.

Proof of the Proposition. We only consider cases which are not already handled by Lemma 4.11.

$G = PSp(2n)$ (adjoint). If $n = 1$ $G \simeq \text{SO}(3)$, and if $n = 2$ $G \simeq \text{SO}(5)$. In both cases W lifts (see the next case).

Assume $G = PSp(2n)$ (adjoint) and $n \geq 3$. Embed $G_1 = Sp(4) \times \text{SL}(2)^{n-2}$ in $Sp(2n)$ in the usual way. Let w be the Coxeter element of $W(G_1)$. This has order 4 and is elliptic. It is easy to see that if g is a lift of w to $W(G_1)$ then $g^4 \neq -I$, so the image of g in $G_1/\pm I \subset PSp(2n)$ also has order 8. By Lemma 4.11 W does not lift to G . See Section 9; this is the case of the partition $(2, 1, \dots, 1)$ of n .

$G = \text{SO}(n)$ and $PSO(n)$. Let $G = \text{SO}(V)$ where V is a non-degenerate orthogonal space of dimension n . Write $V = X \oplus V_0 \oplus Y$ where X, Y are maximal isotropic subspaces, in duality via the form, and V_0 is isotropic of dimension $r \in \{0, 1\}$. Let $\{e_1, \dots, e_m\}$ be a basis of X , with dual basis $\{f_1, \dots, f_m\}$ of Y . Let $S = \{e_1, \dots, e_m, f_1, \dots, f_m\}$. If $V_0 \neq 0$ choose a nonzero vector $e_0 \in V_0$. Then the subgroup $T \subset G$ stabilizing V_0 and each line $F\langle e_i, f_i \rangle$ is a Cartan subgroup of G . Furthermore the subgroup $\{g \in G \mid g(S) = S, ge_0 = e_0\}$ normalizes T , and is a lifting of W to G .

Therefore *a fortiori* W lifts to the adjoint group.

$G = \text{Semispin}(4n)$

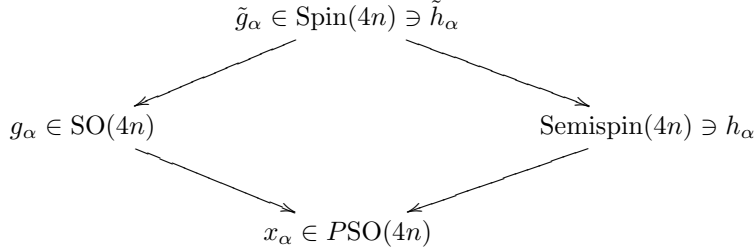
The center of $\text{Spin}(4n)$ is the Klein four-group. Let τ be an automorphism of order 2 of $\text{Spin}(4n)$ coming from an automorphism of the Dynkin diagram (which is unique unless $n = 2$). Write $Z(\text{Spin}(4n)) = \{1, x, y, z\}$ where $\tau(x) = y$ and $\tau(z) = z$. Then $\text{Spin}(4n)/\langle z \rangle \simeq \text{SO}(4n)$. On the other hand $\text{Spin}(4n)/\langle x \rangle \simeq \text{Spin}(4n)/\langle y \rangle$, and this group is denoted *Semispin*(4n).

Example 4.17. Take $n = 1$, so $G = \text{Spin}(4) \simeq \text{SL}(2) \times \text{SL}(2)$, with τ exchanging the factors; set $x = (I, -I), y = (-I, I)$ and $z = (-I, -I)$. Since W does not lift to $\text{SL}(2)$ it obviously does not lift to $\text{Spin}(4)$, or *Semispin*(4) $\simeq \text{PSL}(2) \times \text{SL}(2)$.

If s, t are the simple reflections in the first and second factors, take $g_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\text{diag}(i, -i)$ and set $g_t = \tau(g_s)$. Then $g_s^2 = g_t^2 = (-I, -I) = z$, so W lifts to $\text{Spin}(4)/\langle z \rangle \simeq \text{SO}(4)$.

If $n = 2$, so G is of type D_4 , the three elements x, y, z of $\text{Spin}(8)$ are related by automorphisms of $\text{Spin}(8)$. Since $\text{Spin}(8)/\langle z \rangle \simeq \text{SO}(8)$, we conclude $\text{Semispin}(8) \simeq \text{SO}(8)$, and W lifts by the previous discussion. So assume $n \geq 3$. Fix a simple root α . Let $g_\alpha \in \text{SO}(4n)$ be the image of s_α discussed above. Assume W lifts to $\text{Semispin}(4n)$, and let $h_\alpha \in \text{Semispin}(4n)$ be the image of s_α .

We proceed by contradiction, using Proposition 4.7 and the following diagram, to reduce to the case $n = 1$.



By Proposition 4.7 the images of g_α and h_α in $\text{PSO}(4n)$ are T -conjugate, so after conjugating the splitting into $\text{Semispin}(4n)$ we may assume these are equal.

Now let $\tilde{g}_\alpha, \tilde{h}_\alpha$ be inverse images of g_α, h_α in $\text{Spin}(4n)$. By the preceding discussion these have the same image in $\text{PSO}(4n)$, so they differ by an element of the center. Since the center is a two-group, $\tilde{g}_\alpha^2 = \tilde{h}_\alpha^2$.

Obviously $\tilde{g}_\alpha^2 \in \{1, z\}$ where $z \in Z(\text{Spin}(4n))$ is the nontrivial element of the kernel of the map to $\text{SO}(4n)$. It is enough to show $\tilde{g}_\alpha^2 \neq 1$, for then $\tilde{h}_\alpha^2 = z$, so its image h_α in $\text{Semispin}(4n)$ is nontrivial. This follows by a reduction to $\text{Spin}(4)$.

Take a subgroup $H \simeq \text{SO}(4) \times \text{SO}(4m-4) \subset \text{SO}(4m)$, where the α -root space is contained in the $\text{SO}(4)$ factor. Then, by our choice of splitting of W in $\text{SO}(4m)$ discussed above, $g_\alpha = (u, 1) \in \text{SO}(4) \times \text{SO}(4m-4)$. Let (v, w) be an inverse image of $(u, 1)$ in $\text{Spin}(4m) \times \text{Spin}(4m-4)$. Then $w \in Z(\text{Spin}(4m-4))$, and by Example 4.17 (and Proposition 4.7 again) v^2 is a non-trivial element of the center of $\text{Spin}(4)$. Therefore $(v, w)^2 = (v^2, 1) \neq (1, 1)$. The inverse image of H in $\text{Spin}(4m)$ is isomorphic to $\text{Spin}(4) \times \text{Spin}(4m-4) / \langle z_1, z_2 \rangle$ where z_1, z_2 are non-trivial. It follows that \tilde{g}_α^2 , i.e. the image of $(v^2, 1)$ in $\text{Spin}(4m)$, is non-trivial.

$\text{SL}(n)$:

For $1 \leq i \leq n-1$ let $p_i \in \text{GL}(n)$ be the permutation matrix

$$p_i(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = (x_1, \dots, x_{i+1}, x_i, \dots, x_n).$$

Write s_i for the corresponding simple reflections in W . The map $\phi_{\text{GL}}(s_i) = p_i$, extends to a splitting $W \rightarrow \text{GL}(n)$.

If n is odd then $\phi(s_i) = -p_i$ is a splitting into $\text{SL}(n)$, so assume n is even. We already know W does not lift to $\text{SL}(n)$. If $n=2$ then $G_{\text{ad}} \simeq \text{SO}(3)$, and if $n=4$ $\text{SL}(4)/\pm I \simeq \text{SO}(4)$, so W lifts in these cases, and $\text{PSL}(4)$.

So assume $n \geq 6$, and suppose $A \subset Z(\text{SL}(n))$. We identify A with a subgroup of $\mu_n(F)$. Then ϕ_{GL} factors to a splitting $W \rightarrow \text{GL}(n)/A$. Suppose there exists $z \in F^\times$ such that $\det(zp_i) = 1$ and $(zp_i)^2 \in A$. Then $\phi(s_i) = zp_i A$ is a splitting $W \rightarrow \text{SL}(n)/A$. By Proposition 4.7 this condition is both necessary and sufficient for the existence of a splitting.

The condition holds if and only if there exists $z \in F^\times$ satisfying

$$(4.18) \quad z^2 \in A \text{ and } z^n = -1.$$

Then $(z^2)^n = 1$ so the order of z^2 divides n . Write $n = n_2q$ with $n_2 = 2^k$ (since n is odd $k \geq 1$) and q odd. Thus $(z^2)^{n_2q} = 1$, but $(z^2)^{\frac{n_2}{2}q} = -1$. This implies n_2 divides the order of z^2 , so n_2 divides the order of A . Therefore $|Z/A|$ is odd.

G_2 : Label the simple roots α_1, α_2 . For $i = 1, 2$ the subgroup generated by T and the root groups for $\pm\alpha_i$ is isomorphic to $\text{GL}(2)$, so s_i has a lift to an involution n_i . The long element of the Weyl group is $w_0 = (s_1s_2)^3$. By Lemma 3.1 $(n_1n_2)^6 = (2\rho^\vee)(-1) = 1$. It follows that n_1, n_2 generate a lift of W in G .

For the remaining exceptional groups we choose a subgroup H to be the centralizer of an element of T of order 2, so that $W(H, T)$ does not lift to H . These groups are well understood, for example see [12, Chapter 5, §1]. Then Lemma 4.13 implies $W(G, T)$ does not lift to G .

F_4 : It is well known that F_4 contains a subgroup $H \simeq \text{Spin}(9)$, and we already know W doesn't lift to $\text{Spin}(9)$.

E_6 : The center of the simply connected group is cyclic of order 3, by Lemma 4.14 we may assume G is simply connected. Let H be the subgroup of type $A_1 \times A_5$. Then $H \simeq \text{SL}(2) \times \text{SL}(6)/\langle(-I, -I)\rangle$. Suppose the simple reflection in the first factor lifts to an element of H , with representative $(g, h) \in \text{SL}(2) \times \text{SL}(6)$. Then $g^2 = -I$ so if the image of $(g, h)^2$ is trivial in H then $h^2 = -I$. But clearly $h \in Z(\text{SL}(6))$ and there is no element in $Z(\text{SL}(6))$ with this property.

Since the center of the simply connected group is $\mathbb{Z}/3\mathbb{Z}$, if $W(G_{\text{ad}}, T)$ lifts to G_{ad} then it lifts to the simply connected group by Lemma 4.14.

E_7 : Take H of type A_7 . Then $H \simeq \text{SL}(8)/A$ where A has order 2 or 4, depending on whether E_7 is simply connected or adjoint, so $|Z(H)|$ is 2 or 4, and by (1) of the Proposition $W(H, T)$ does not lift to H .

E_8 : Take H of type D_8 . It is well known that $H \simeq \text{Semispin}(16)$, so $W(H, T)$ does not lift to H .

This concludes the proof of Theorem 4.16. □

Corollary 4.19. *Suppose G is simple, and W lifts to G .*

- (1) *If $G = \text{PSO}(6) \simeq \text{PSL}(4)$ there are two T -conjugacy classes of splittings.*
- (2) *If $G = \text{SO}(2n)$ there are two T -conjugacy classes of splittings, related by multiplication by $-I \in Z$.*
- (3) *In G_2 and all other simply laced cases there is one T -conjugacy class of splittings.*

Proof. Most cases follow from a combination of the Theorem and Proposition 4.7.

Suppose G is of type A_n with $n \neq 3$. By Proposition 4.7 the lift is unique up to conjugacy by T and multiplication by Z_2 . However by Theorem 4.16 the assumption that W lifts to G implies Z_2 is trivial.

If $G = PSL(4)$ then there are two non-conjugate splittings given in Example 4.10. It is straightforward to see these are the only ones up to T -conjugacy, and the lifting to $SO(4)$ is unique up to T -conjugacy and the center.

In $SO(2n)$ $|Z_2| = 2$ and in G_2 the center is trivial. The only other exceptional case is D_4 . It follows from a tedious and not very enlightening argument that W lifts to $SO(8)$, uniquely up to T -conjugacy and multiplication by $-I$, and the lifting to $PSO(8)$ is unique up to T -conjugacy. We leave the details to the reader. □

5 Coxeter elements and elliptic conjugacy classes

5.1 Coxeter and twisted Coxeter elements

Choose an ordering $1, \dots, n$ of the simple roots. The corresponding Coxeter element is $\text{Cox} = s_1 s_2 \dots s_n$. All Coxeter elements are conjugate, regular and elliptic.

Now suppose δ is a distinguished automorphism. Write i_1, \dots, i_k for representatives of the δ -orbits on the simple roots. A twisted Coxeter element is defined to be $\text{Cox}' = s_{i_1} \dots s_{i_k} \delta \in W\delta$. These elements are all W -conjugate, elliptic and regular. See [16, Theorem 7.6].

Proposition 5.1.1. *Suppose $g \in G$ is a lift of Cox . Then $g^{o(\text{Cox})} = z_G$. Suppose $g \in G\delta$ is a lift of Cox' . Then $g^{o(\text{Cox}')} = z_G$.*

Since the (twisted) Coxeter elements are regular this follows from Proposition 7.4.

It is convenient to formulate a variant of this in type A , using the fact that we can take -1 for the outer automorphism of the root system.

Set $G = \text{SL}(n, \mathbb{C})$, with the usual diagonal Cartan subgroup and Borel subgroup, and Weyl group W . Set

$$(5.1.2) \quad x = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ (-1)^{n+1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

Then ${}^\delta G = \langle G, \delta \rangle$ where $\delta g \delta^{-1} = x({}^t g^{-1})x^{-1}$ and $\delta^2 = 1$, and similarly ${}^\delta W = \langle W, \delta \rangle$.

Let $\epsilon = x\delta$. Then

$$(5.1.3) \quad {}^\delta G = \langle G, \epsilon \rangle, \quad \epsilon g \epsilon^{-1} = {}^t g^{-1}, \epsilon^2 = z_G.$$

Lemma 5.1.4. *Suppose G is of type A_{n-1} , and let Cox be a Coxeter element of W . If n is odd then $\text{Cox} \cdot \epsilon$ is an elliptic regular element of $W\delta$, $o(\text{Cox} \cdot \epsilon) = 2n$, and if g is any lift of $\text{Cox} \cdot \epsilon$ then $g^{2n} = z_G$.*

5.2 Elliptic conjugacy classes in the classical groups

We use these results to describe the elliptic conjugacy classes in the classical Weyl groups. See [8, Section 3.4] or [7, Section 3] for the untwisted cases, and [6, Section 3 & 4] or [9, Section 7] for the twisted ones.

Type A_{n-1}

The only elliptic conjugacy class of W is that of the Coxeter elements.

For $m \geq 2$ let Cox_m be a Coxeter element of type A_{m-1} , and set $\text{Cox}_1 = 1$. Suppose $\mathcal{P} = (a_1, \dots, a_l)$ is a partition of n with all odd parts. Using (5.1.3) set

$$(5.2.1) \quad \mathcal{E}(\mathcal{P}) = (\text{Cox}_{a_1} \times \cdots \times \text{Cox}_{a_l})\epsilon \in W\delta$$

where $\text{Cox}_{a_1} \times \cdots \times \text{Cox}_{a_l}$ is embedded diagonally as usual. It is immediate that $\mathcal{E}(\mathcal{P})$ is elliptic, and

$$o(\mathcal{E}(\mathcal{P})) = 2 \cdot \text{LCM}(a_1, \dots, a_l).$$

Furthermore the map $\mathcal{P} \rightarrow \mathcal{E}(\mathcal{P})$ gives a bijection between partitions of n with all odd parts and elliptic conjugacy classes of $W\delta$.

Type B_n/C_n Let Cox_n be a Coxeter element of $W(B_n)$. Suppose $\mathcal{P} = (a_1, \dots, a_k)$ is a partition of n , embed $B_{a_1} \times \cdots \times B_{a_k}$ in B_n as usual, and set $\mathcal{E}(\mathcal{P}) = \text{Cox}_{a_1} \times \cdots \times \text{Cox}_{a_k}$ of W . Then $\mathcal{E}(\mathcal{P})$ is elliptic, and the map $\mathcal{P} \rightarrow \mathcal{E}(\mathcal{P})$ defines a bijection between partitions of n and conjugacy classes of elliptic elements of $W(B_n)$.

Exactly the same result holds with type C in place of type B .

Type D_n : Let δ_n be a distinguished automorphism of order 2, and choose the numbering of the simple roots so that root n is not fixed by δ_n . Set

$$\text{Cox}'_n = s_1 s_2 \cdots s_{n-1} \delta_n.$$

This is the twisted Coxeter element of $W(D_n)\delta_n$, and is an elliptic regular element of $W\delta$.

Suppose $\mathcal{P} = (a_1, \dots, a_k)$ is a partition of n , and embed $D_{a_1} \times \cdots \times D_{a_k}$ in D_n as usual. Then $W(D_{a_1})\delta_{a_1} \times \cdots \times W(D_{a_k})\delta_{a_k}$ embeds naturally in $W(D_n)\delta_n$. Set

$$\mathcal{E}(\mathcal{P}) = \text{Cox}'_{a_1} \times \cdots \times \text{Cox}'_{a_k}.$$

Then $\mathcal{E}(\mathcal{P})$ is an elliptic element of $W(D_n)$ if k is even, or $W(D_n)\delta_n$ if n is odd, and $\mathcal{P} \rightarrow \mathcal{E}(\mathcal{P})$ is a bijection between the partitions of n and the union of the elliptic conjugacy classes of $W(D_n)$ and $W(D_n)\delta_n$.

6 Good representatives of conjugacy classes in Weyl groups

Let B^+ be the braid monoid associated to the Coxeter system (W, Π) . Let $j : W \rightarrow B^+$ be the canonical injection identifying the generators of W with the corresponding generators of B^+ and $j(ww') = j(w)j(w')$ for $w, w' \in W$ with $\ell(ww') = \ell(w) + \ell(w')$.

The distinguished automorphism δ of G (and hence of W) induces an automorphism of B^+ , which we still denote by δ . Define the extended Braid monoid ${}^\delta B^+ = B^+ \rtimes \langle \delta \rangle$. The injection j extends in a canonical way to an injection ${}^\delta W \rightarrow {}^\delta B^+$, which we still denote by j .

Following [7], we call $w \in W\delta$ a good element if there exists a strictly decreasing sequence $S_1 \supseteq \cdots \supseteq S_\ell$ of subsets of Π and even positive integers d_1, \dots, d_ℓ such that

$$(6.1) \quad j(w)^{o(w)} = j(w_0(S_1))^{d_1} \cdots j(w_0(S_\ell))^{d_\ell},$$

where $w_0(S_i)$ is the longest element of the parabolic subgroup $W(S_i)$ of W .

Proposition 6.2. *Every conjugacy class of $W\delta$ contains a good element.*

This is proved in [7], [6] and [9] via case-by-case analyses, and a general proof is in [10]. In fact, we may choose a good element having minimal length in the conjugacy class.

If w is written as in (6.1) then the image of $j(w)$ in the Tits group is $\sigma(w)$, so by Lemma 3.2:

$$(6.3) \quad \sigma(w)^{o(w)} = \left(\sum_{i=1}^{\ell} d_i \rho^\vee(S_i) \right) (-1) = \prod_{i=1}^{\ell} z_{L(S_i)}^{d_i/2}.$$

where $z_{L(S_i)}$ is the principal involution in the Levi factor $L(S_i)$.

Assuming we know the d_i and S_i explicitly, this gives a formula for $\sigma(w)^{o(w)}$, and (at least in the elliptic case) $o(\sigma(w))$. Thus we need the explicit formulas of [7], [6] and [9]. See Section 9.

7 Regular Elements

Fix a distinguished automorphism δ of G . Let $\overline{\mathbb{Q}}$ be an algebraic closure of \mathbb{Q} , and set $V = X_* \otimes \overline{\mathbb{Q}}$, and

$$(7.1) \quad V_{\text{reg}} = \{v \in V \mid \langle \alpha, v \rangle \neq 0 \text{ for all } \alpha \in \Delta\}.$$

We say that $w \in W\delta$ is *regular* if it has an eigenvector $v \in V_{\text{reg}}$. In this case if the eigenvalue of v is ζ , we say w is *d-regular* if ζ has order d .

It is obvious that both d and $o(\delta)$ divide $o(w)$. The case of $d = o(w)$ is of particular significance.

Lemma 7.2. *Suppose $w \in W\delta$ is d -regular. Then $o(w) = \text{LCM}(o(\delta), d)$. The following conditions are equivalent:*

- (1) $d = o(w)$,
- (2) $o(\delta)$ divides d ,
- (3) $\langle w \rangle$ acts freely on the roots.

If $w \in W$ is d -regular then $d = o(w)$.

The elements w satisfying the conditions of the Lemma are called \mathbb{Z} -regular in [13].

Proof. The first assertion is proved in [4] and [13], which gives the equivalence of (1) and (2). The implication (1) implies (3) is proved in [15], following an argument of Kostant for the Coxeter element, and (3) \Rightarrow (2) is proved in [13]. The final assertion is the case $o(\delta) = 1$. \square

The obvious case in which $d < o(w)$ is if $d = 1$, which is easy to handle.

Lemma 7.3. *We have $d = 1$ if and only if w is conjugate to δ .*

Proof. If $d = 1$ then $w\gamma^\vee = \gamma^\vee$ for a regular element γ^\vee . After conjugating by an element of W , we may assume that γ^\vee is in the dominant chamber, which implies $w = \delta$. Conversely, if $w = x\delta x^{-1}$, then w fixes the regular element $x\rho^\vee$, hence w is 1-regular. \square

We have the following result on the d -regular elements.

Proposition 7.4. *Let C be a conjugacy class of d -regular elements in $W\delta$ with $d > 1$. Then C contains an element w so that in the Braid group*

$$j(w)^{o(w)} = j(w_0)^{2o(w)/d}$$

and in the Tits group

$$\sigma(w)^{o(w)} = z_G^{o(w)/d} \in Z(G).$$

Proof. According to [4], Proposition 3.11 and 6.3 (for the untwisted and twisted cases, respectively),

$$j(w)^d = j(w_0)^2 \delta^d.$$

Raise both sides to the power $o(w)/d$, and use the fact that $\delta^{o(w)} = 1$ to conclude the first statement, and the second is an immediate consequence of this. \square

Corollary 7.5. *If w is \mathbb{Z} -regular then have $\sigma(w)^{o(w)} = z_G$, and $o(\sigma(w)) = o(w)$ if and only if $\rho^\vee \in X_*(T)$.*

Remark 7.6. *An example in which $1 < d < o(w)$ is given in [4], Proof of Proposition 6.5. Consider 2A_5 , so δ is the nontrivial diagram automorphism of order 2. Let C be the conjugacy class of $w = (s_1s_3s_5s_2s_4)^2\delta$. It is easy to check that $o(w) = 6$, w is 3-regular, and also that w is good, so by the Proposition $j(w)^6 = j(w_0)^4$ and $\sigma(w)^6 = z_G^2 = 1$.*

Finally, we have

Proposition 7.7. *Let w be a regular element. Then $\tilde{o}_{ad}(w) = o(w)$.*

Proof. Suppose w is a d -regular element. If $d = 1$ then by Proposition 7.3, w is conjugate to δ . By definition of ${}^\delta\mathcal{T}$, the lift of δ to ${}^\delta\mathcal{T}$ has the same order of δ .

Assume $d > 1$. By Proposition 7.4, w is conjugate to an element w' with $\sigma(w')^{o(w)} = 1 \in G_{\text{ad}}$. We take the lifting of w to be a conjugate of $\sigma(w')$. Then the order of that lifting equals $o(w)$. □

8 Theorem 1: Exceptional Cases

We still need to prove Theorem 1(2) for the exceptional groups: if G is simple, adjoint, and exceptional and w is elliptic, then $\tilde{o}(w) = o(w)$, except for the conjugacy class $A_3 + \tilde{A}_1$ in F_4 . We include the case 3D_4 here. We prove a bit more: we calculate $\tilde{o}(w)$ in the non-adjoint simple exceptional groups, i.e. simply connected of type E_6 , E_7 and 3D_4 . We have already treated G_2 (Theorem 4.16).

We use the explicit lists of elliptic conjugacy classes, and formulas for $j(w)^d$, from [7, Section 3] (untwisted) and [6, Section 6] (twisted) and apply (6.1) and (6.3). This is a straightforward case-by-case analysis.

Recall (Lemma 3.2) $j(w_S)^d$ contributes the term

$$(d\rho^\vee(S))(-1) = z_L^{\frac{d}{2}},$$

where $L = L(S)$. This is trivial if and only if $\frac{d}{2}\rho^\vee(S) \in X_*(T)$. In particular if $\rho^\vee \in R^\vee$ we can ignore any term $j(w_I)^d$ (d even). This holds for any adjoint group ($F_4, E_6^{ad}, E_7^{ad}, E_8$) and also in E_6^{sc} and (for any isogeny) 3D_4 .

The same holds for any terms $j(w_S)^d$ provided $4|d$. Here is the example of F_4 . We use notation of [7, 3.5]. The simple roots are $I = \{1, 2, 3, 4\}$ (3, 4 are short). There are 9 elliptic conjugacy classes.

Elliptic class	order	Good representative	$j(w)^{o(w)}$
$4A_1$	2	w_I	$j(w_I)^2$
D_4	8	2323432134	$j(w_I)^2 j(w_{34})^4$
$D_4(a_1)$	4	324321324321	$j(w_I)^2$
$C_3 + A_1$	8	1214321323	$j(w_I)^2 j(w_{12})^4$
$A_2 + A_2$	3	3214321323432132	$j(w_I)^2$
$F_4(a_1)$	6	32432132	$j(w_I)^2$
F_4	12	4321	$j(w_I)^2$
$A_3 + A_1$	4	23234321324321	$j(w_I)^2 j(w_{23})^2$
B_4	8	243213	$j(w_I)^2$

By the preceding discussion all terms are trivial except possibly in the case $A_3 + \tilde{A}_1$, the term $\rho^\vee(\{2, 3\})(-1)$ coming from $j(w_{23})^2$. It is easy to see $\rho^\vee(\{2, 3\}) = \frac{3}{2}\alpha_2^\vee + 2\alpha_3^\vee$, so $o(w) = 4$ and any lift of w has order 8. Alternatively the derived group of $L(\{2, 3\})$ is isomorphic to $Sp(4)$, and $\sigma(w_{23})^2 = z_{Sp(4)}$, which is nontrivial.

The preceding discussion show that in all cases in types E_6 (untwisted) and 3D_4 , $\bar{o}(w, G) = o(w)$. Here is a list of the remaining elliptic conjugacy classes, for which it is not obvious whether $\bar{o}(w) = o(w)$ or $2o(w)$.

G	Elliptic class	order	$j(w)^{o(w)}$
2E_6	4254234565423456	6	$j(w_I)^2 j(w_{2345})^2$
E_7	E_7	18	$j(w_I)^2$
E_7	$E_7(a_1)$	14	$j(w_I)^2$
E_7	$E_7(a_2)$	12	$j(w_I)^6 j(w_{257})^2$
E_7	$E_7(a_3)$	30	$j(w_I)^6 j(w_{24})^4$
E_7	$D_6 + A_1$	10	$j(w_I)^2 j(w_{24})^8$
E_7	A_7	8	$j(w_I)^2 j(w_{257})^2 j(w_2)^4$
E_7	$E_7(a_4)$	6	$j(w_I)^2$
E_7	$D_6(a_2) + A_1$	6	$j(w_I)^2 j(w_{13})^4$
E_7	$A_5 + A_2$	6	$j(w_I)^2 j(w_{2345})^2$
E_7	$D_4 + 3A_1$	6	$j(w_I)^2 j(w_{24567})^4$
E_7	$2A_3 + A_1$	4	$j(w_I)^2 j(w_{257})^2$
E_7	$7A_1$	2	$j(w_I)^2$
E_8	$E_8(a_7)$	12	$j(w_I)^2 j(w_{2345})^2$
E_8	$E_7(a_2) + A_1$	12	$j(w_I)^2 j(w_{2345})^2 j(w_{24})^8$
E_8	$E_6(a_2) + A_2$	12	$j(w_I)^2 j(w_{2345})^6$
E_8	$A_7 + A_1$	8	$j(w_I)^2 j(w_{2345})^2 j(w_{25})^4$
E_8	$E_6(a_2) + A_2$	6	$j(w_I)^2 j(w_{2345})^2$
E_8	$A_5 + A_2 + A_1$	6	$j(w_I)^2 j(w_{234578})^2 j(w_{78})^2$
E_8	$D_5(a_1) + A_3$	12	$j(w_I)^4 j(w_{123456})^2$
E_8	$2A_3 + 2A_1$	4	$j(w_I)^2 j(w_{2345})^2$

Consider types 2E_6 and E_8 . In both cases $\rho^\vee \in R^\vee \subset X_*$, so we can ignore all terms $j(w_I)^d$. The remaining terms are: 2E_6 : $S = \{2, 3, 4, 5\}$, and E_8 : $S = \{2, 3, 4, 5\}, \{2, 3, 4, 5, 7, 8\}, \{7, 8\}$ or $\{1, 2, 3, 4, 5, 6\}$. The Levi factors L are of type $D_4, D_4, D_4 \times A_2, A_2$ or E_6 , respectively. In each case $z_L = 1$, so $\tilde{o}(w, G) = o(w)$ in these cases.

Consider type E_7 . After reducing each d modulo 4 we have to determine if $\rho^\vee \in X_*$ or $\rho^\vee + \rho^\vee(S) \in X_*$ where $S = \{2, 5, 7\}$ or $\{2, 3, 4, 5\}$. Using notation of [3] we have:

$$\begin{aligned}\rho^\vee &= 17\alpha_1^\vee + \frac{49}{2}\alpha_2^\vee + 33\alpha_3^\vee + 48\alpha_4^\vee + \frac{75}{2}\alpha_5^\vee + 26\alpha_6^\vee + \frac{27}{2}\alpha_7^\vee \\ \rho^\vee(\{2, 5, 7\}) &= \frac{1}{2}\alpha_2^\vee + \frac{1}{2}\alpha_5^\vee + \frac{1}{2}\alpha_7^\vee \\ \rho^\vee(\{2, 3, 4, 5\}) &= 3\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_5^\vee\end{aligned}$$

Since $\rho^\vee(\{2, 3, 4, 5\}) \in R^\vee$ we can ignore these terms. Also ρ^\vee is in the coweight lattice, and $\rho^\vee \equiv \rho^\vee(\{2, 5, 7\}) \pmod{R^\vee}$. Therefore in type E_7^{ad} , ρ^\vee and $\rho^\vee + \rho^\vee(\{2, 5, 7\})$ are contained in X_* (the coweight lattice), so $\tilde{o}_{ad}(w) = \tilde{o}(w, G) = o(w)$ in all cases.

Finally in type E_7^{sc} we have $X_* = R^\vee$, and we see $\rho^\vee \notin R^\vee$, $\rho^\vee + \rho^\vee(\{2, 5, 7\}) \in R^\vee$. We conclude that $\tilde{o}(w, G) = 2o(w)$ *except* for the classes $E_7(a_2)$, A_7 and $2A_3 + A_1$. Here is the conclusion.

Proposition 8.1. *Suppose G is simple, exceptional, and $w \in W\delta$ is an elliptic element.*

- If G is type G_2 then W lifts, so $\tilde{o}(w) = o(w)$.
- In types ${}^3D_4, E_6, {}^2E_6$ and E_8 every term z_L occurring is trivial, so $\tilde{o}(w) = o(w)$.
- In type F_4 every term z_L occurring is trivial except for the conjugacy class $A_3 + \tilde{A}_1$, and

$$\tilde{o}(w) = \begin{cases} 2o(w) & \text{conjugacy class } A_3 + \tilde{A}_1 \\ o(w) & \text{otherwise} \end{cases}$$

- In type E_7

$$\sigma(w)^{o(w)} = \begin{cases} 1 & \text{conjugacy classes } E_7(a_2), A_7 \text{ and } 2A_3 + A_1 \\ z_G & \text{otherwise} \end{cases}$$

In particular if G is adjoint then $\tilde{o}(w) = o(w)$. If G is simply connected then (since $z_G \neq 1$ in G), $\tilde{o}(w) = o(w)$ only in the three conjugacy classes $E_7(a_2), A_7$ and $2A_3 + A_1$, and $\tilde{o}(w) = 2o(w)$ otherwise.

This completes the proof of Theorem 1(2).

9 Proof of Theorem 1

For the classical groups we use the description of the (twisted) elliptic conjugacy classes (Section 5.2).

Type A_{n-1} .

The only elliptic conjugacy class is that of the Coxeter elements, and by Proposition 5.1.1, $\sigma(\text{Cox})^{o(\text{Cox})} = z_G$.

Now consider the twisted case, so δ is the non-trivial distinguished involution. Suppose (a_1, \dots, a_l) is a partition of n with all odd parts. Let $w = (\text{Cox}_{a_1} \times \dots \times \text{Cox}_{a_l})\epsilon \in W\delta$ (see (5.2.1)). Set $d = \text{LCM}(a_1, \dots, a_l)$. Since d is odd it is easy to see that $w^d = \epsilon$, and $o(w) = 2d$. Choose a representative $g \in G$ of $\text{Cox}_{a_1} \times \dots \times \text{Cox}_{a_l}$. Without loss of generality we may assume ${}^t g^{-1} = g$. By Proposition 5.1.1 (applied to each factor) $g^{2d} = I$. Then

$$\begin{aligned} (g\epsilon)^{2d} &= (g\epsilon g\epsilon^{-1}\epsilon^2)^d \\ &= (g({}^t g^{-1})(-I)^{n+1})^d \\ &= g^{2d}(-I)^{d(n+1)} \\ &= (-I)^{n+1} \\ &= z_G. \end{aligned}$$

Note that this is independent of w , and $\sigma(w)^{o(w)} = 1$ if and only if $\rho^\vee \in X_*$.

Type C_n . Suppose $\mathcal{P} = (a_1, \dots, a_l)$ is a partition of n and w is in the corresponding elliptic conjugacy class $\mathcal{E}(\mathcal{P})$ (cf. Section 5.2). Set $e = \text{LCM}(a_1, \dots, a_l)$. Since $o(\text{Cox}_n) = 2n$, and $Sp(2a_1) \times \dots \times Sp(2a_l)$ embeds in $Sp(2n)$, it is easy to see that

$$o(w) = 2e.$$

Recall (Proposition 5.1.1) $\sigma(\text{Cox}_n)^{2n} = (2\rho^\vee)(-1)$. It follows easily that if we set

$$\tau^\vee = \left(\frac{e}{a_1}\rho^\vee(C_{a_1}) \times \dots \times \frac{e}{a_l}\rho^\vee(C_{a_l})\right)$$

then it follows that

$$\sigma(w)^{o(w)} = (2\tau^\vee)(-1).$$

and this is trivial if and only if $\tau^\vee \in X_*(T)$. At least one term e/a_i is odd. It follows that if G is simply connected then $\tau^\vee \notin X_*(T)$, and if G is adjoint this holds if and only if all e/a_i are odd, or equivalently if and only if each a_i has the same power of 2 in its prime decomposition.

Type B_n/D_n . Suppose $\mathcal{P} = (a_1, \dots, a_l)$ is a partition of n with $a_1 \geq a_2 \geq \dots \geq a_l \geq 1$, and w is an element of the corresponding elliptic conjugacy class

$\mathcal{E}(\mathcal{P})$ (cf. Section 5.2). Then $o(w) = 2\text{LCM}(a_1, \dots, a_l)$. For $1 \leq i \leq l$ set

$$\begin{aligned}\Sigma(\mathcal{P}, i) &= \sum_{k=0}^{i-1} a_k \\ [a, b] &= \{a, a+1, \dots, b\} \\ S_i &= \begin{cases} [\Sigma(\mathcal{P}, i) + 1, n] & \text{type } B_n \\ [\Sigma(\mathcal{P}, i) + 1, n] & \text{type } D_n, \Sigma(\mathcal{P}, i) \leq n-2 \\ \emptyset & \text{type } D_n, \Sigma(\mathcal{P}, i) > n-2 \end{cases} \\ e_i &= 2o(w)/a_i \in 2\mathbb{Z}.\end{aligned}$$

There exists an element w in the corresponding elliptic conjugacy class with

$$j(w)^{o(w)} = j(w_0)^{e_1} j(w_0(S_2))^{e_2 - e_1} \dots j(w_0(S_l))^{e_l - e_{l-1}}.$$

Set $e_0 = 0$ and

$$\tau^\vee = \sum_{i=1}^{\ell} \frac{e_i - e_{i-1}}{2} \rho^\vee(S_i).$$

Then $\sigma(w)^{o(w)} = 1$ if and only if $\tau^\vee \in X_*(T)$. This is automatic if G is adjoint or $\text{SO}(2n)$.

Example 9.1. Consider the partition $(2, 1, \dots, 1)$ of $n \geq 3$. Then $o(w) = 4$, $e_1 = 2$, $e_2 = e_3 = \dots = e_{n-1} = 4$, $S_1 = \Pi$, $S_2 = \{3, 4, \dots, n\}$. Since $e_i - e_{i-1} = 0$ for $i \geq 2$ we get $\tau^\vee = \rho^\vee + \rho^\vee(\{3, \dots, n\})$, i.e. in the standard coordinates

$$\tau^\vee = (n-1, n-2, 2(n-3), 2(n-4), \dots, 2, 0).$$

The sum of the coordinates of τ^\vee is odd. Therefore τ^\vee is in $X_*(T)$ if G is adjoint or $G = \text{SO}(2n)$, but not if $G = \text{Spin}(2n)$. By Lemma 4.11 this implies W does not lift to $\text{Spin}(2n)$.

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