FROBENIUS SPLITTING AND GEOMETRY OF G-SCHUBERT VARIETIES

XUHUA HE AND JESPER FUNCH THOMSEN

Abstract. Let X be an equivariant embedding of a connected reductive group G over an algebraically closed field k of positive characteristic. Let B denote a Borel subgroup of G. A G-Schubert variety in X is a subvariety of the form $\mathrm{diag}(G) \cdot V$, where V is a $B \times B$-orbit closure in X. In the case where X is the wonderful compactification of a group of adjoint type, the G-Schubert varieties are the closures of Lusztig's G-stable pieces. We prove that X admits a Frobenius splitting which is compatible with all G-Schubert varieties. Moreover, when X is smooth, projective and toroidal, then any G-Schubert variety in X admits a stable Frobenius splitting along an ample divisors. Although this indicates that G-Schubert varieties have nice singularities we present an example of a non-normal G-Schubert variety in the wonderful compactification of a group of type G_2. Finally we also extend the Frobenius splitting results to the more general class of $\mathcal{R}$-Schubert varieties.

1. Introduction

Let G denote a connected and reductive group over an algebraically closed field k, and let B denote a Borel subgroup of G. An equivariant embedding X of G is a $G \times G$-variety which contains $G = (G \times G)/\mathrm{diag}(G)$ as an open $G \times G$-invariant subset, where $\mathrm{diag}(G)$ is the diagonal image of G in $G \times G$. Any equivariant embedding X of G contains finitely many $B \times B$-orbits. In recent years the geometry of closures of $B \times B$-orbits has been studied by several authors. The most general result was obtained in [H-T2] where it was proved that $B \times B$-orbit closures are normal, Cohen-Macaulay and have (F-)rational singularities (actually, even stronger results were obtained). In the present paper we will study (closed) subvarieties in X of the form $\mathrm{diag(G)} \cdot \mathrm{V}$, where V denotes the closure of a $B \times B$-orbit. Subvarieties of equivariant embeddings of G of this form will be called G-Schubert varieties.

When G is a semisimple group of adjoint type there exists a canonical equivariant embedding $\mathbf{X}$ of G which is called the *wonderful compactification*. The wonderful compactifications are of primary interest in this paper. Actually, this work arose from the question of describing the closures of the so-called G-stable pieces of $\mathbf{X}$. The G-stable pieces makes up a decomposition of $\mathbf{X}$ into locally closed subsets. They were introduced by Lusztig in [L] where they were used to construct and study a class of perverse sheaves which generalizes his theory of character sheaves on reductive groups. More precisely, these perverse sheaves are the intermediate extensions of the so-called "character sheaves" on a G-stable piece. This motivates the study of closures of G-stable pieces which turns out to coincide with the set of G-Schubert varieties.

Before discussing the closures of G-stable pieces in details, let us make a short digression and discuss some other motivations for studying G-stable pieces and G-Schubert varieties (in wonderful compactifications):

(1) When G is a simple group, the boundary of the closure of the unipotent subvariety of G in the wonderful compactification $\mathbf{X}$, is a union of certain G-Schubert varieties (see [He] and [H-T]). Thus knowing the geometry of these G-Schubert varieties will help us to understand the geometry of the closure of the unipotent variety within $\mathbf{X}$.

(2) Let $\mathrm{Lie}(G)$ denote the Lie algebra of a simple group G over a field of characteristic zero. Let $\ll, \gg$ denote a fixed symmetric non-degenerate ad-invariant bilinear form. Let $<,>$ be the bilinear form on $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$ defined by

$$<(x,y),(x',y')> = \ll x, x' \gg - \ll y, y' \gg .$$

In [E-L], Evens and Lu showed that each splitting $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) = l \oplus l'$, where l and l' are Lagrangian subalgebras of $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$, gives rise to a Poisson structure $\Pi_{l,l'}$ on $\mathbf{X}$. If moreover, one starts with the Belavin-Drinfeld splitting, then all the G-stable pieces/G-Schubert varieties and $B \times B^-$-orbits of $\mathbf{X}$ are Poisson subvarieties, where B^- is a Borel subgroup opposite to B. Thus to understand the Poisson structure on $\mathbf{X}$ corresponding to the Belavin-Drinfeld splitting, one needs to understand the geometry of the G-stable pieces/G-Schubert varieties. If we start with another splitting, then we obtain a different Poisson structure on $\mathbf{X}$ and in order to understand these Poisson structures, one needs to study the $\mathcal{R}$-stable pieces [L-Y] instead (see Section 12), which generalize both the G-stable pieces and the $B \times B^-$-orbits.

The main technical ingredient in this paper is the positive characteristic notion of Frobenius splitting. Frobenius splitting is a powerful tool which has been proved to be very useful in obtaining strong geometric conclusions for e.g. Schubert varieties and closures of $B \times B$-orbits in equivariant embeddings. In the present paper we obtain two types of results related to G-Schubert varieties over fields of positive characteristic. First of all, if we fix an equivariant embedding X of a reductive group G then we prove that all G-Schubert varieties in X are simultaneously compatibly Frobenius split by a Frobenius splitting of X. Secondly, concentrating on a single G-Schubert variety $\mathcal{X}$, in a smooth projective and toroidal embedding X, we prove that this admits a stable Frobenius splitting along an ample divisor. Statements of this form put strong conditions on the intertwined behavior of cohomology groups of line bundles on X and its G-Schubert varieties. As this is related to geometric properties it therefore seems natural to expect that G-Schubert varieties should have nice singularities. It therefore comes as a complete surprise that G-Schubert varieties, in general, are not even normal. We only provide a single example of this phenomenon (in the wonderful compactification of a group of type G_2), but expect that this absence of normality is the general picture.

In obtaining the Frobenius splitting result mentioned above, we have developed some general theory of how to construct Frobenius splitting of varieties of the form $G \times_P X$ (see Section 4.2 for the definition). This part of the paper is influenced by the theory of B-canonical Frobenius splitting as discussed in [B-K, Chap.4]; in particular the proof of [B-K, Prop.4.1.17]. The presentation we provide is more general and makes it possible to extract even better result from the ideas of B-canonical Frobenius splittings. This theory is presented in Chapter 5 in a generality which is more than necessary for obtaining the described Frobenius splitting results for G-Schubert varieties. However, we hope that this theory could be useful elsewhere and we certainly consider it to be of independent interest.

This paper is organized in the following way. In Section 2 we introduce notation, and in Section 3 we briefly define Frobenius splitting and explain its fundamental ideas. Section 4 is devoted to some results on linearized sheaves which should all be well known. In Section 5 we study the Frobenius splitting of varieties of the form $G \times_P X$ for a variety X with an action by a parabolic subgroup P. The main idea is to decompose the Frobenius morphism on $G \times_P X$ into maps associated to the Frobenius morphism on the base G/P and the fiber X of the natural morphism $G \times_P X \to G/P$. In Section 6 we relate B-canonical Frobenius splittings to the material in Section 5. Section 7 contains applications of Section 5 to general $G \times G$-varieties. In section 8 we define the G-stable pieces and G-Schubert varieties. In Section 9 we apply the material of the previous sections to the class of equivariant embeddings and obtain Frobenius splitting results for G-Schubert varieties. Section 10 contains results related to cohomology of line bundles on G-Schubert varieties. Section 11 contains an example of a non-normal G-Schubert variety. Finally Section 12 contains generalizations and variations of the previous sections.

We would like to thank the referee for a careful reading of this paper and for numerous suggestions concerning the presentation.

2. Notation

We will work over a fixed algebraically closed field k. The characteristic of k will depend on the application. By a variety we mean a reduced and separated scheme of finite type over k. In particular, we allow a variety to have several irreducible components.

2.1. **Group setup.** We let G denote a connected linear algebraic group over k. We fix a Borel subgroup B and a maximal torus $T \subset B$. The notation P is used for a parabolic subgroup of G containing B. The set of T-characters is denoted by $X^*(T)$ and we identify this set with the set $X^*(B)$ of B-characters.

2.2. **Reductive case.** In many cases we will specialize to the case where G is reductive. In this case we will also use the following notation : the set of roots determined by T is denoted by $R \subseteq X^*(T)$ while the set of positive roots determined by (B, T) is denoted by R^+. The simple roots are denoted by $\alpha_1, \dots, \alpha_l$, and we let $\Delta = \{1, \dots, l\}$ denote the associated index set. The simple reflection associated to the simple root α_i is then denoted by s_i. The Weyl group $W = {}^{N_G(T)}/_{T}$ is generated by the simple reflections s_i, for $i \in \Delta$. The length of $w \in W$ will be denoted by $l(w)$. For $J \subset \Delta$, let W_J denote the subgroup of W generated by the simple reflection associated with the elements in J, and let W^J (resp. ${}^J W$) denote the set of minimal length coset representatives for W/W_J (resp. $W_J \backslash W$). The element in W of maximal length will be denoted by w_0, while w_0^J is used for the same kind of element in W_J. For any $w \in W$, we let $\dot{w}$ denote a representative of w in $N_G(T)$. For $J \subset \Delta$, let $P_J \supset B$ denote the corresponding standard parabolic subgroup and $P_J^- \supset B^-$ denote its opposite parabolic. Let $L_J = P_J \cap P_J^-$ be the common Levi subgroup of P_J and P_J^- containing T. Let U_J (resp. U_J^-) denote the unipotent radical of P_J (resp. P_j^-). When $J = \emptyset$ we also use the notation U and U^- for U_J and U_J^- respectively. When G is semisimple and simply connected we may associate a fundamental character ω_i to each simple root α_i. The sum of the fundamental characters is then denoted by ρ. Then ρ also equals half the sum of the positive roots.

3. The relative Frobenius morphism

In this section we collect some results related to the Frobenius morphism and to the concept of Frobenius splitting. Compared to other presentations on the same subject, this presentation differs only in its emphasis on the set $\mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{O}_{X'}\big)$ (to be defined below) and not just the set of Frobenius splittings. Thus, the obtained results are only small variations of already known results as can be found in e.g. [B-K].

3.1. **The Frobenius morphism.** By definition a variety X comes with an associated morphism

$$p_X : X \to \mathrm{Spec}(k),$$

of schemes. Assume that the field k has positive characteristic $p > 0$. Then the Frobenius morphism on $\mathrm{Spec}(k)$ is the morphism of schemes

$$F_k : \mathrm{Spec}(k) \to \mathrm{Spec}(k),$$

which on the level of coordinate rings is defined by $a \mapsto a^p$. As k is assumed to be algebraically closed the morphism F_k is actually an isomorphism and we let F_k^{-1} denote the inverse morphism. Composing p_X with F_k^{-1} we obtain a new variety

$$p'_X : X \to \mathrm{Spec}(k),$$

with underlying scheme X. In the following we suppress the morphism p_X from the notation and simply use X as the notation for the variety defined by p_X. The variety defined by p'_X is then denoted by X'.

The relative Frobenius morphism on X is then the morphism of varieties :

$$F_X : X \to X',$$

which as a morphism of schemes is the identity map on the level of points and where the associated map of sheaves

$$F_X^\sharp : \mathcal{O}_{X'} \to (F_X)_* \mathcal{O}_X,$$

is the p-th power map. A key property of the Frobenius morphism is the relation

$$(F_X)^* \mathcal{L}' \simeq \mathcal{L}^p \tag{1}$$

which is satisfied for every line bundle $\mathcal{L}$ on X (here $\mathcal{L}'$ denotes the corresponding line bundle on X').

3.2. **Frobenius splitting.** A variety X is said to be *Frobenius split* if the $\mathcal{O}_{X'}$-linear map of sheaves :

$$F_X^\sharp : \mathcal{O}_{X'} \to (F_X)_* \mathcal{O}_X,$$

has a section; i.e. if there exists an element

$$s \in \mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{O}_{X'}\big),$$

such that the composition $s \circ F_X^\sharp$ is the identity endomorphism of $\mathcal{O}_{X'}$. The section s will be called *a Frobenius splitting of* X.

3.3. **Compatibility with line bundles and closed subvarieties.** Fix a line bundle $\mathcal{L}$ on X and a closed subvariety Y in X with sheaf of ideals $\mathcal{I}_Y$. Let Y' denote the closed subvariety of X' associated to Y with sheaf of ideals denoted by $\mathcal{I}_{Y'}$. The kernel of the natural morphism

$$\mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big) \to \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*(\mathcal{L}\otimes \mathcal{I}_Y), \mathcal{O}_{Y'}\big),$$

induced by the inclusion $\mathcal{L}\otimes\mathcal{I}_Y \subset \mathcal{L}$ and the projection $\mathcal{O}_{X'} \to \mathcal{O}_{Y'}$, will be denoted by $\mathcal{E}nd_F^{\mathcal{L}}(X,Y)$. The associated space of global sections will be denoted by $\mathrm{End}_F^{\mathcal{L}}(X,Y)$. When $Y = X$ we simply denote $\mathcal{E}nd_F^{\mathcal{L}}(X,Y)$ (resp. $\mathrm{End}_F^{\mathcal{L}}(X,Y)$) by $\mathcal{E}nd_F^{\mathcal{L}}(X)$ (resp. $\mathrm{End}_F^{\mathcal{L}}(X)$). The sheaf $\mathcal{E}nd_F^{\mathcal{L}}(X,Y)$ is a subsheaf of $\mathcal{E}nd_F^{\mathcal{L}}(X)$ consisting of the elements *compatible with* Y. Moreover, there is a natural morphism

$$\mathcal{E}nd_F^{\mathcal{L}}(X,Y)_{|Y} \to \mathcal{E}nd_F^{\mathcal{L}_{|Y}}(Y),$$

where the notation $_{|Y}$ means restriction to Y.

If $Y_1, Y_2, \dots, Y_m$ is a collection of closed subvarieties of X then the notation $\mathcal{E}nd_F^{\mathcal{L}}(X, Y_1, \dots, Y_m)$ (or sometimes $\mathcal{E}nd_F^{\mathcal{L}}(X, \{Y_i\}_{i=1}^m)$) will denote the intersection of the subsheaves $\mathcal{E}nd_F^{\mathcal{L}}(X, Y_i)$ for $i = 1, \dots, m$. The set of global sections of the sheaf $\mathcal{E}nd_F^{\mathcal{L}}(X, Y_1, \dots, Y_m)$ will be denoted by $\mathrm{End}_F^{\mathcal{L}}(X, Y_1, \dots, Y_m)$.

When $\mathcal{L} = \mathcal{O}_X$ we remove $\mathcal{L}$ from all of the above notation. In particular, the vectorspace $\mathrm{End}_F(X)$ denotes the set of morphisms from $(F_X)_*\mathcal{O}_X$ to $\mathcal{O}_{X'}$ and thus contains the set of Frobenius splittings of X. A Frobenius splitting s of X contained in $\mathrm{End}_F(X, \{Y_i\}_i)$ is said to be compatible with the subvarieties $Y_1, \dots, Y_m$. When s is compatible in this sense it induces a Frobenius splitting of each Y_i for $i = 1\dots, m$. In this case we also say that *s compatibly Frobenius splits* $Y_1, \dots, Y_m$. In concrete terms, this is equivalent to

$$s\big((F_X)_*\mathcal{I}_{Y_i}\big) \subset \mathcal{I}_{Y_i'}.$$

for all i.

Lemma 3.1. *Let Y and Z denote closed subvarieties in X and let s denote a global section of $\mathcal{E}nd_F^{\mathcal{L}}(X, Z, Y)$.*

(1) $s \in \mathrm{End}_F^{\mathcal{L}}(X, Y_1)$ for every irreducible component Y_1 of Y.
(2) If the scheme theoretic intersection $Z \cap Y$ is reduced then s is contained in $\mathrm{End}_F^{\mathcal{L}}(X, Y\cap Z)$.

Proof. Let Y_1 denote an irreducible component of Y and let

$$\mathcal{J} = s\big((F_X)_*(\mathcal{I}_{Y_1}\otimes\mathcal{L})\big) \subset \mathcal{O}_{X'}.$$

Let U denote the open complement (in X') of the irreducible components of Y' which are different from Y_1'. Then $\mathcal{I}_{Y_1'}$ coincides with $\mathcal{I}_{Y'}$ on U and consequently $\mathcal{J}_{|U} \subset (\mathcal{I}_{Y'})_{|U}$ as s is compatible with Y. In particular, $\mathcal{J}_{|U} \subset (\mathcal{I}_{Y_1'})_{|U}$. We claim that this implies that $\mathcal{J} \subset \mathcal{I}_{Y_1'}$: let V denote an open subset of X' and let f be a section of $\mathcal{J}$ over V. As $\mathcal{J}$ is a subsheaf of $\mathcal{O}_{X'}$, we may consider f as a function on V, and it suffices to prove that f vanishes on $Y_1' \cap V$. If $Y_1' \cap V$ is empty then this is clear. Otherwise, $U \cap V \cap Y_1'$ is a dense subset of Y_1' and it suffices to prove that f vanishes on this set. But this follows from the inclusion $\mathcal{J}_{|U} \subset (\mathcal{I}_{Y_1'})_{|U}$. As a consequence s is compatible with Y_1. The second claim follows as the sheaf of ideals of the intersection $Z \cap Y$ is $\mathcal{I}_Y + \mathcal{I}_Z$. □

The condition that $Z \cap Y$ is reduced, in Lemma 3.1, only ensures that $Z \cap Y$ is a variety. When $\mathcal{L} = \mathcal{O}_X$ and s is a Frobenius splitting this is always satisfied [B-K, Prop.1.2.1].

3.4. **The evaluation map.** Let $k[X']$ denote the space of global regular functions on X'. Evaluating an element $s : (F_X)_* \mathcal{O}_X \to \mathcal{O}_{X'}$ of $\mathrm{End}_F(X)$ at the constant global function 1 on X defines an element in $k[X']$ which we denote by $\mathrm{ev}_X(s)$. This defines a morphism

$$\mathrm{ev}_X : \mathrm{End}_F(X) \to k[X'],$$

with the property that $\mathrm{ev}_X(s) = 1$ if and only if s is a Frobenius splitting of X.

3.5. **Frobenius D-splittings.** Consider an effective Cartier divisor D on X, and let σ_D denote the associated global section of the associated line bundle $\mathcal{O}_X(D)$. A Frobenius splitting s of X is said to be a *Frobenius D-splitting* if s factorizes as

$$s : (F_X)_* \mathcal{O}_X \xrightarrow{(F_X)_* \sigma_D} (F_X)_* \mathcal{O}_X(D) \xrightarrow{s_D} \mathcal{O}_{X'},$$

for some element s_D in $\mathrm{End}_F^{\mathcal{O}_X(D)}\big(X\big)$. We furthermore say that the Frobenius D-splitting s is compatible with a subvariety Y if s_D is compatible with Y. The following result assures that, in this case, the compatibility with closed subvarieties agrees with the usual definition [R, Defn.1.2].

Lemma 3.2. *Assume that s defines a Frobenius D-splitting of X. Then s_D is compatible with Y if and only if (i) s compatibly Frobenius splits Y and (ii) the support of D does not contain any irreducible components of Y.*

Proof. The *if* part of the statement follows from [R, Prop.1.4]. So assume that s_D is compatible with Y. Then s_D induces a morphism

$$\overline{s}_D : (F_Y)_* \mathcal{O}_X(D)_{|Y} \to \mathcal{O}_{Y'},$$

satisfying $\overline{s}_D((\sigma_D)_{|Y})$ is the constant function 1 on Y'. As a consequence $(\sigma_D)_{|Y}$ does not vanish on any of the irreducible components of Y. This proves part (ii) of the statement. Part (i) is clearly satisfied. □

It follows that if s is compatible with Y and, moreover, defines a Frobenius D-splitting of X then $D \cap Y$ makes sense as an effective Cartier divisor on Y and, in this case, s induces a Frobenius $D \cap Y$-splitting of Y.

3.6. **Stable Frobenius splittings along divisors.** Let $X^{(0)} = X$ and define recursively $X^{(n)} = (X^{(n-1)})'$ for $n \geq 1$. Composing the Frobenius morphisms on $X^{(i)}$ for $i = 0, \dots, n$, we obtain a morphism

$$F_X^{(n)} : X \to X^{(n)},$$

with an associated map of sheaves

$$(F_X^{(n)})^\sharp : \mathcal{O}_{X^{(n)}} \to (F_X^{(n)})_* \mathcal{O}_X.$$

Let, as in Section 3.5, D denote an effective Cartier divisor on X with associated canonical section σ_D of $\mathcal{O}_X(D)$. We say that X *admits a stable Frobenius splitting along* D if there exists a positive integer n and an element

$$s \in \mathrm{Hom}_{\mathcal{O}_{X^{(n)}}}\big((F_X^{(n)})_* \mathcal{O}_X(D), \mathcal{O}_{X^{(n)}}\big),$$

such that the composed map

$$\mathcal{O}_{X^{(n)}} \xrightarrow{(F_X^{(n)})^\sharp} (F_X^{(n)})_* \mathcal{O}_X \xrightarrow{(F_X^{(n)})_* \sigma_D} (F_X^{(n)})_* \mathcal{O}_X(D) \xrightarrow{s} \mathcal{O}_{X^{(n)}},$$

is the identity map on $\mathcal{O}_{X^{(n)}}$. The element s is called a *stable Frobenius splitting of* X *along* D. When Y is a closed subvariety of X we say that the stable Frobenius splitting s *is compatible with* Y if

$$s\big((F_X^{(n)})_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D))\big) \subset \mathcal{I}_{Y^{(n)}}.$$

Notice that this condition necessarily implies that the support of D does not contain any of the irreducible components of Y (cf. proof of Lemma 3.2). Notice also that if X admits a Frobenius D-splitting which is compatible with Y then X admits a stable Frobenius splitting along D which is compatible with Y. The following is well known (see e.g. [T, Lem.4.4])

Lemma 3.3. *Let D_1 and D_2 denote effective Cartier divisors on X and let Y denote a closed subvariety of X. Then X admits stable Frobenius splittings along D_1 and D_2 which are compatible with Y if and only if X admits a stable Frobenius splitting along $D_1 + D_2$ which is compatible with Y.*

The following result explains one of the main applications of (stable) Frobenius splitting. Remember that a line bundle $\mathcal{L}$ is nef if $\mathcal{L} \otimes \mathcal{M}$ is ample whenever $\mathcal{M}$ is ample.

Proposition 3.4. *Assume that X admits a stable Frobenius splitting along an effective Cartier divisor D. Then there exists a positive integer*

n such that for each line bundle $\mathcal{L}$ on X we have an inclusion of abelian groups

$$\mathrm{H}^i(X, \mathcal{L}) \subset \mathrm{H}^i(X, \mathcal{L}^{p^n} \otimes \mathcal{O}_X(D)).$$

In particular, if D is ample and $\mathcal{L}$ is nef, then $\mathrm{H}^i(X, \mathcal{L}) = 0$ for $i > 0$. Moreover, if the stable Frobenius splitting of X is compatible with a subvariety Y, D is ample and $\mathcal{L}$ is nef then the restriction morphism

$$\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L}),$$

is surjective.

Proof. Argue as in the proof [R, Prop.1.13(i)]. $\square$

3.7. **Duality for F_X.** By duality (see [Har2, Ex.III.6.10]) for the finite morphism F_X we may to each quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}$ associate an $\mathcal{O}_X$-module denoted by $(F_X)^! \mathcal{F}$ and satisfying

$$(F_X)_*(F_X)^! \mathcal{F} = \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{F}\big).$$

Actually, as F_X is the identity on the level of points we may define $(F_X)^! \mathcal{F}$ as the sheaf of abelian groups

$$\mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{F}\big),$$

with $\mathcal{O}_X$-module structure defined by

$$(g \cdot \phi)(f) = \phi(gf),$$

for $g, f \in \mathcal{O}_X$ and $\phi \in \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{F}\big)$. When $\mathcal{F} = \mathcal{O}_X$ we will also use the notation $\mathcal{E}nd_F^!(X)$ for $(F_X)^! \mathcal{O}_X$. This sheaf is particularly nice when X is smooth as $(F_X)^! \mathcal{O}_X$ then coincides with the line bundle ω_X^{1-p}, where ω_X denotes the dualizing sheaf of X (see e.g. [B-K, Sect.1.3]). If $Y_1, Y_2, \ldots, Y_m$ is a collection of closed subvarieties of X then $\mathcal{E}nd_F^!(X, Y_1, \ldots, Y_m)$ (or $\mathcal{E}nd_F^!(X, \{Y_i\}_{i=1}^m)$) will denote the subsheaf of $\mathcal{E}nd_F^!(X)$ consisting of the elements mapping the sheaf of ideals $\mathcal{I}_{Y_i}$ to $\mathcal{I}_{Y_i'}$ for all $i = 1, \ldots, m$. We say that $\mathcal{E}nd_F^!(X, \{Y_i\}_{i=1}^m)$ is the subsheaf of elements compatible with $Y_1, \ldots, Y_m$.

More generally, duality for F_X implies that we have a natural identification

$$(F_X)_* \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{G}, (F_X)^! \mathcal{F}\big) \simeq \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{G}, \mathcal{F}\big),$$

whenever $\mathcal{G}$ (resp. $\mathcal{F}$) is a quasicoherent sheaf on X (resp. X'). This leads to the identification

$$\mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{G}, (F_X)^! \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{G}, \mathcal{F}\big),$$

where a morphism $\eta : \mathcal{G} \to (F_X)^! \mathcal{F}$ is identified with the composed morphism

$$\eta' : (F_X)_* \mathcal{G} \xrightarrow{(F_X)_* \eta} (F_X)_*(F_X)^! \mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{F}\big) \to \mathcal{F}.$$

Here the latter map is the natural evaluation map at the element 1 in $\mathcal{O}_X$. From now on we will specialize to the case where $\mathcal{F} = \mathcal{O}_{X'}$

and $\mathcal{G}$ equals a line bundle $\mathcal{L}$ on X. In this case, an element in $\mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{L}, \mathcal{E}nd_F^!(X)\big)$ may also be considered as a global section of the sheaf $\mathcal{E}nd_F^!(X) \otimes \mathcal{L}^{-1}$. For later use we emphasize

Lemma 3.5. *Let η be an element in $\mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{L}, \mathcal{E}nd_F^!(X)\big)$ and let η' denote the corresponding element in $\mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big)$ by the above identification. Then η' factors through the morphism*

$$(F_X)_*\mathcal{L} \xrightarrow{(F_X)_*\eta} (F_X)_*\mathcal{E}nd_F^!(X).$$

Moreover, the element η' is compatible with a collection of closed subvarieties $Y_1, \dots, Y_m$ of X if and only if the image of η is contained in $\mathcal{E}nd_F^!(X, Y_1, \dots, Y_m)$.

Proof. The first part of the statement follows directly from the discussion above. To prove the second statement we may assume that $m = 1$. We use the notation $Y = Y_1$. Let σ denote a section of $\mathcal{L}$ over an open subset U of X, and consider $s = \eta(\sigma)$ as a map

$$s : \mathcal{O}_X(U) \to \mathcal{O}_{X'}(U').$$

That s is compatible with Y means that $s(f)$ vanishes on Y' whenever f vanishes on Y for a function f on U. Alternatively, the evaluation of $f \cdot s$ at 1, which coincides with $\eta'(f \cdot \sigma)$, should vanish on Y'. In particular, the image of η is contained in $\mathcal{E}nd_F^!(X, Y)$ if and only if the restriction of η' to $(F_X)_*\big(\mathcal{I}_Y \otimes \mathcal{L}\big)$ maps into $\mathcal{I}_{Y'}$. This ends the proof. □

We will also need the following remark

Lemma 3.6. *Let D denote a reduced effective Cartier divisor on X and $\mathcal{L}$ denote a line bundle on X. Let $\mathcal{M} = \mathcal{O}_X((p-1)D) \otimes \mathcal{L}$ and assume that we have an $\mathcal{O}_X$-linear morphism $\eta : \mathcal{M} \to \mathcal{E}nd_F^!(X)$. Let σ_D denote the canonical section of $\mathcal{O}_X(D)$ and consider the map*

$$\eta_D : \mathcal{L} \to \mathcal{E}nd_F^!(X),$$

induced by σ_D^{p-1}. Then the element

$$\eta'_D \in \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big),$$

induced by η_D, is compatible with the support of D. In particular, the image of η_D is contained in $\mathcal{E}nd_F^!(X, D)$.

Proof. Notice that η'_D is the composition

$$\eta'_D : (F_X)_*\mathcal{L} \xrightarrow{(F_X)_*\sigma_D^{p-1}} (F_X)_*\mathcal{M} \xrightarrow{\eta'} \mathcal{O}_{X'},$$

where η' is the element corresponding to η. Hence, the restriction of η'_D to $\mathcal{L} \otimes \mathcal{O}_X(-D)$ coincides with the map

$$(F_X)_*\big(\mathcal{L} \otimes \mathcal{O}_X(-D)\big) \xrightarrow{(F_X)_*\sigma_D^{p}} (F_X)_*\mathcal{M} \xrightarrow{\eta'} \mathcal{O}_{X'}.$$

But the restriction of η' to (cf. (1))

$$(F_X)_*\big(\mathcal{O}_X(-pD) \otimes \mathcal{M}\big) \simeq \mathcal{O}_{X'}(-D') \otimes (F_X)_*\mathcal{M},$$

maps by linearity into $\mathcal{O}_{X'}(-D')$. The *in particular* part follows by Lemma 3.5. □

3.8. **Push-forward operation.** Assume that $f : X \to Z$ is a morphism of varieties satisfying that the associated map $f^\sharp : \mathcal{O}_Z \to f_*\mathcal{O}_X$ is an isomorphism. Let $f' : X' \to Z'$ denote the associated morphism. Then f'_* induces a morphism

$$f'_*\mathcal{E}nd_F(X) \to \mathcal{E}nd_F(Z).$$

If $Y \subset X$ is a closed subset then the subsheaf $f'_*\mathcal{E}nd_F(X, Y)$ is mapped to $\mathcal{E}nd_F(Z, \overline{f(Y)})$, where $\overline{f(Y)}$ denotes the variety associated to the closure of the image of Y. On the level of global sections this means that every Frobenius splitting s of X induces a Frobenius splitting f'_*s of Z such that when s is compatible with Y then f'_*s is compatible with $\overline{f(Y)}$. Likewise

Lemma 3.7. *With notation as above, let $\mathcal{L}$ denote a line bundle on Z and let s be an element of $\mathrm{End}_F^{f^*(\mathcal{L})}(X)$. Then f'_*s is an element of $\mathrm{End}_F^{\mathcal{L}}(Z)$. Moreover, if s is compatible with a closed subvariety Y of X then f'_*s is compatible with $\overline{f(Y)}$.*

Proof. This follows easily from the fact that the sheaf of ideals of $\overline{f(Y)}$ coincides with $f_*\mathcal{I}_Y$ [B-K, Lem.1.1.8]. □

4. Linearized sheaves

In this section we collect a number of well known facts about linearized sheaves. The chosen presentation follows rather closely the presentation in [Bri, Sect.2].

Let H denote a linear algebraic group over the field k and let X denote a H-variety with H-action defined by $\sigma : H \times X \to X$. We let $p_2 : H \times X \to X$ denote projection on the second coordinate. A *H-linearization* of a quasi-coherent sheaf $\mathcal{F}$ on X is an $\mathcal{O}_{H\times X}$-linear isomorphism

$$\phi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F},$$

satisfying the relation

$$(\mu \times \mathbf{1}_X)^*\phi = p_{23}^*\phi \circ (\mathbf{1}_H \times \sigma)^*\phi, \tag{2}$$

as morphisms of sheaves on $H \times H \times X$. Here $\mu : H \times H \to H$ (resp. $p_{23} : H \times H \times X \to H \times X$) denotes the multiplication on H (resp. the projection on the second and third coordinate). Based on the fact that $\sigma^*\mathcal{O}_X = p_2^*\mathcal{O}_X$ we see that the sheaf $\mathcal{O}_X$ admits a canonical linearization. In the following we will always assume that $\mathcal{O}_X$ is equipped with this canonical linearization.

A morphism $\psi : \mathcal{F} \to \mathcal{F}'$ of H-linearized sheaves is a morphism of $\mathcal{O}_X$-modules commuting with the linearizations ϕ and ϕ' of $\mathcal{F}$ and $\mathcal{F}'$, i.e. $\phi' \circ \sigma^*(\psi) = p_2^*(\psi) \circ \phi$.

Linearized sheaves on X form an abelian category which we denote by $Sh_H(X)$.

4.1. **Quotients and linearizations.** Assume that the quotient $q : X \to {}^X\!/_H$ exists and that q is a locally trivial principal H-bundle. Then for $\mathcal{G} \in Sh({}^X\!/_H)$, $q^*\mathcal{G}$ is naturally a H-linearized sheaf on X. This defines a functor $q^* : Sh({}^X\!/_H) \to Sh_H(X)$. On the other hand, for $\mathcal{F} \in Sh_H(X)$, $q_*\mathcal{F}$ has a natural action of H. Define a functor $q_*^H : Sh_H(X) \to Sh({}^X\!/_H)$ by $q_*^H(\mathcal{F}) = (q_*\mathcal{F})^H$ the subsheaf of H-invariants of $q_*\mathcal{F}$. It is known that the functor $q^* : Sh({}^X\!/_H) \to Sh_H(X)$ is an equivalence of categories with inverse functor q_*^H.

In general, if H is a closed normal subgroup of G and X is a G-variety such that the quotient ${}^X\!/_H$ exists (as above), then ${}^X\!/_H$ is a ${}^G\!/_H$-variety and the functor $q^* : Sh_{G/H}({}^X\!/_H) \to Sh_G(X)$ is an equivalence of categories with inverse functor $q_*^H : Sh_G(X) \to Sh_{G/H}({}^X\!/_H)$.

4.2. **Induction equivalence.** Consider now a connected linear algebraic group G and a parabolic subgroup P in G. Let X denote a P-variety. Then $G \times X$ is a $G \times P$-variety by the action

$$(g,p)(h,x) = (ghp^{-1}, px),$$

for $g, h \in G$, $p \in P$ and $x \in X$. Then the quotient, denoted by $G \times_P X$, of $G \times X$ by P exists and the associated quotient map $q : G \times X \to G \times_P X$ is a locally trivial principal P-bundle. The quotient of $G \times X$ by G also exists and may be identified with the projection $p_2 : G \times X \to X$. In particular, we may apply the above consideration to obtain equivalences of the categories $Sh_P(X)$, $Sh_{G\times P}(G \times X)$ and $Sh_G(G \times_P X)$. Notice that under this equivalence a P-linearized sheaf $\mathcal{F}$ on X corresponds to the G-linearized sheaf $\mathcal{I}nd_P^G(\mathcal{F}) = (q_* p_2^* \mathcal{F})^P$. In particular, the space of global sections of $\mathcal{I}nd_P^G(\mathcal{F})$ equals

$$\begin{aligned} \mathcal{I}nd_P^G(\mathcal{F})(G \times_P X) &= \big(p_2^*\mathcal{F}(G \times X)\big)^P \\ &= \big(k[G] \otimes_k \mathcal{F}(X)\big)^P \\ &= \mathrm{Ind}_P^G(\mathcal{F}(X)), \end{aligned} \tag{3}$$

where the second equality follows by the Künneth formula. This also explains the notation $\mathcal{I}nd_P^G(\mathcal{F})$. Similarly, starting with a G-linearized sheaf $\mathcal{G}$ on $G \times_P X$ then the associated P-linearized line bundle on X equals $\mathcal{G}' = ((p_2)_* q^* \mathcal{G})^G$. However, by [Bri, Lemma 2(1)] the latter also equals the simpler pull back $i^*\mathcal{G}$ by the P-equivariant map

$$i : X \to G \times_P X,$$

sending x to $q(1, x)$. In particular, we conclude that the functor $i^* : Sh_G(G \times_P X) \to Sh_P(X)$ is an equivalence of categories with inverse functor $\mathcal{I}nd_P^G$. Notice also that the space of global sections of $\mathcal{G}$ is G-equivariantly isomorphic to

$$\mathcal{G}(G \times_P X) = \mathrm{Ind}_P^G\big((i^*\mathcal{G})(X)\big),$$

which follows by (3) above.

4.3. **Duality.** Assume that the field k has positive characteristic $p > 0$. Regard X' as a H-variety in the canonical way and let $\mathcal{F}$ denote a H-linearized sheaf on X'. The sheaf $(F_X)^!\mathcal{F}$, defined in Section 3.7, is then naturally a H-linearized sheaf on X. Moreover, the induced H-linearization of $(F_X)_*(F_X)^!\mathcal{F}$ coincides with the natural H-linearization of

$$\mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{O}_X, \mathcal{F}\big).$$

When X is smooth the sheaf $(F_X)^!\mathcal{O}_{X'}$ is canonically isomorphic to ω_X^{1-p} (cf. Section 3.7). We may use this isomorphism to define a H-linearization of ω_X^{1-p}. Alternatively we may consider the natural H-linearization of the dualizing sheaf Ω_X of X and use this to define a H-linearization of ω_X^{1-p}. It may be checked that the two stated ways of defining a H-linearization of ω_X^{1-p} coincide.

5. Frobenius splitting of $G \times_P X$

Let G denote a connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. Let P denote a parabolic subgroup of G and let X denote a P-variety. In this section we want to consider Frobenius splittings of the quotient $Z = G \times_P X$ of $G \times X$ by P. We let $\pi : Z \to {}^G/_P$ denote the morphism induced by the projection of $G \times X$ on the first coordinate. When $g \in G$ and $x \in X$ we use the notation $[g, x]$ to denote the element in Z represented by (g, x).

5.1. **Decomposing the Frobenius morphism.** The Frobenius morphism F_Z admits a decomposition $F_Z = F_b \circ F_f$ where F_b (resp. F_f) is related to the Frobenius morphism on the base (resp. fiber) of π. More precisely, define $\hat{Z}$ and the morphisms $\hat{\pi}$ and F_b as part of the fiber product diagram

(4)
$$\begin{tikzcd} \hat{Z} \arrow[r, "F_b"] \arrow[d, "\hat{\pi}"'] & Z' \arrow[d, "\pi'"] \\ {}^G/_P \arrow[r, "F_{G/P}"] & ({}^G/_P)' \end{tikzcd}$$

A local calculation shows that we may identify $\hat{Z}$ with the quotient $G \times_P X'$, where the P-action on the Frobenius twist X' of X is the natural one. With this identification $\hat{\pi} : G \times_P X' \to {}^G/_P$ is just the map $[g, x'] \mapsto gP$. It also follows that the natural morphism (induced by the Frobenius morphism on X)

$$F_f : G \times_P X \to G \times_P X',$$

makes the following diagram commutative

$$\begin{tikzcd}
Z \arrow[dr, "F_f"] \arrow[drr, bend left, "F_Z"] \arrow[ddr, bend right, "\pi"'] & & \\
& \hat{Z} \arrow[r, "F_b"] \arrow[d, "\hat{\pi}"'] & Z' \arrow[d, "\pi'"] \\
& {}^{G}/_{P} \arrow[r, "F_{G/P}"] & ({}^{G}/_{P})'
\end{tikzcd}$$

5.2. Let $\mathcal{M}$ denote a P-linearized line bundle on X and let $\mathcal{M}_Z = \mathcal{I}nd_P^G(\mathcal{M})$ denote the associated G-linearized line bundle on Z. The main aim of this section is to construct global sections of the sheaf

$$\mathcal{E}nd_F^{\mathcal{M}_Z}(Z) = \mathcal{H}om_{\mathcal{O}_{Z'}}\big((F_Z)_*\mathcal{M}_Z, \mathcal{O}_{Z'}\big).$$

To this end we fix a P-character λ and let $\mathcal{L}$ denote the associated line bundle on ${}^{G}/_{P}$ (cf. Section 4). The pull back $\hat{\pi}^*\mathcal{L}$ of $\mathcal{L}$ to $\hat{Z}$ is then denoted by $\mathcal{L}_{\hat{Z}}$. We then define the following sheaves

$$\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f := \mathcal{H}om_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{M}_Z, \mathcal{O}_{\hat{Z}}\big),$$

$$\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z)_b := \mathcal{H}om_{\mathcal{O}_{Z'}}\big((F_b)_*\mathcal{L}_{\hat{Z}}, \mathcal{O}_{Z'}\big),$$

with spaces of global sections denoted by $\mathrm{End}_F^{\mathcal{M}_Z}(Z)_f$ and $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$. Notice that when $\mathcal{M}$ is substituted with the P-linearized twist $\mathcal{M}(-\lambda) := \mathcal{M} \otimes k_{-\lambda}$ then

$$\mathcal{M}(-\lambda)_Z = \mathcal{M}_Z \otimes \pi^*(\mathcal{L}^{-1}) = \mathcal{M}_Z \otimes (F_f)^*\mathcal{L}_{\hat{Z}}^{-1},$$

and thus by the projection formula

$$\mathcal{E}nd_F^{\mathcal{M}(-\lambda)_Z}(Z)_f = \mathcal{H}om_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{M}_Z, \mathcal{L}_{\hat{Z}}\big). \tag{5}$$

Sections of $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)$ are then constructed as compositions of global sections of the sheaves $\mathcal{E}nd_F^{\mathcal{M}(-\lambda)_Z}(Z)_f$ and $\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$. More precisely, if

$$v \in \mathrm{Hom}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{M}_Z, \mathcal{L}_{\hat{Z}}\big),$$

$$u \in \mathrm{Hom}_{\mathcal{O}_{Z'}}\big((F_b)_*\mathcal{L}_{\hat{Z}}, \mathcal{O}_{Z'}\big),$$

are global sections of the latter sheaves, then the composition $u \circ (F_b)_* v$ defines a global section of $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)$.

5.3. An equivariant setup. We now give equivariant descriptions of the sheaves $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f$ and $\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$.

5.3.1. *A description of* $\mathrm{End}_F^{\mathcal{M}_Z}(Z)_f$. Now $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f$ is a G-linearized sheaf on $\hat{Z} = G \times_P X'$. Let Y denote a P-stable subvariety of X and let $Z_Y = G \times_P Y$ denote the associated subvariety of Z with sheaf of ideals $\mathcal{I}_{Z_Y} \subset \mathcal{O}_Z$. Let $\hat{Z}_Y$ denote the subvariety $G \times_P Y'$ of $\hat{Z}$. Then there is a natural morphism of G-linearized sheaves

$$\mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f \to \mathcal{H}om_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{Z_Y}), \mathcal{O}_{\hat{Z}_Y}\big),$$

induced by the inclusion $\mathcal{I}_{Z_Y} \subset \mathcal{O}_Z$ and the projection $\mathcal{O}_{\hat{Z}} \to \mathcal{O}_{\hat{Z}_Y}$. We let $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f$ denote the kernel of the above map and arrive at a left exact sequence of G-linearized sheaves

(6)

$$0 \to \mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f \to \mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f \to \mathcal{H}om_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{Z_Y}), \mathcal{O}_{\hat{Z}_Y}\big).$$

In particular, the space of global sections of $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f$ is identified with the set of elements in $\mathrm{End}_F^{\mathcal{M}_Z}(Z)_f$ which map $(F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{Z_Y})$ to $\mathcal{I}_{\hat{Z}_Y} \subset \mathcal{O}_{\hat{Z}}$. Using the observations in Section 4.2 we can give another description of the space of global sections of $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f$. Let $i' : X' \to G \times_P X'$ denote the morphism $i'(x') = [1, x']$. Then the functor i' is exact on the category of G-linearized sheaves. We want to apply this fact on the left exact sequence (6) above : notice first that

$$(i')^* \mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f = \mathcal{H}om_{\mathcal{O}_{X'}}\big((i')^*(F_f)_* \mathcal{M}_Z, \mathcal{O}_{X'}\big),$$

where, moreover, $(i')^*(F_f)_* \mathcal{M}_Z = (F_X)_* \mathcal{M}$. Thus $(i')^* \mathcal{E}nd_F^{\mathcal{M}_Z}(Z)_f = \mathcal{E}nd_F^{\mathcal{M}}(X)$. Similarly,

$$(i')^* \mathcal{H}om_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{Z_Y}), \mathcal{O}_{\hat{Z}_Y}\big) = \mathcal{H}om_{\mathcal{O}_{X'}}((F_X)_*(\mathcal{M} \otimes \mathcal{I}_Y), \mathcal{O}_{Y'}).$$

In particular, we see that the P-linearized sheaf on X' corresponding to the G-linearized sheaf $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f$ equals the kernel of the natural map

$$\mathcal{E}nd_F^{\mathcal{M}}(X) \to \mathcal{H}om_{\mathcal{O}_{X'}}((F_X)_*(\mathcal{M} \otimes \mathcal{I}_Y), \mathcal{O}_{Y'}),$$

i.e. it equals $\mathcal{E}nd_F^{\mathcal{M}}(X, Y)$. By Section 4.2 the space of global sections $\mathrm{End}_F^{\mathcal{M}_Z}(Z, Z_Y)_f$ of $\mathcal{E}nd_F^{\mathcal{M}_Z}(Z, Z_Y)_f$ is then G-equivariantly isomorphic to

$$\mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X, Y)\big).$$

Applying the above conclusions to the sheaf $\mathcal{M}(-\lambda)$ we find:

Proposition 5.1. *There exists a G-equivariant isomorphism*

$$\mathrm{End}_F^{\mathcal{M}(-\lambda)_Z}(Z)_f \simeq \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big)$$

such that when Y is a closed P-stable subvariety of X then the subset of elements of $\mathrm{End}_F^{\mathcal{M}(-\lambda)_Z}(Z)_f$ which map $(F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{Z_Y})$ to $(\mathcal{I}_{\hat{Z}_Y} \otimes \mathcal{L}_{\hat{Z}}) \subset \mathcal{L}_{\hat{Z}}$ (cf. equation (5)) is identified with

$$\mathrm{End}_F^{\mathcal{M}(-\lambda)_Z}(Z, Z_Y)_f \simeq \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X, Y) \otimes k_\lambda\big).$$

5.3.2. *A description of* $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$. As π' in the fibre-diagram (4) is flat the natural morphism $(\pi')^*(F_{G/P})_*\mathcal{L} \to (F_b)_*\hat{\pi}^*\mathcal{L}$ is an isomorphism ([Har2, Prop.III.9.3]). Thus there is a natural isomorphism of G-linearized sheaves

$$\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z)_b \simeq (\pi')^*\mathcal{H}om_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\mathcal{L}, \mathcal{O}_{(G/P)'}\big) = (\pi')^*\mathcal{E}nd_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}).$$

Let V denote a closed subvariety of ${}^{G}\!/\!{}_{P}$. Then $\mathcal{E}nd_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}, V)$ is the kernel of the natural map

$$\mathcal{E}nd_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \to \mathcal{H}om_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}), \mathcal{O}_{\mathcal{O}_{V'}}\big).$$

In particular, $(\pi')^*\big(\mathcal{E}nd_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}, V)\big)$ maps into the kernel of the induced morphism

$$\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z)_b \to (\pi')^*\mathcal{H}om_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}), \mathcal{O}_{\mathcal{O}_{V'}}\big). \tag{7}$$

Let $q : G \to {}^{G}\!/\!{}_{P}$ denote the quotient map. Then $\hat{\pi}^{-1}(V)$ identifies with the quotient $q^{-1}(V) \times_P X'$. Moreover, as π' is locally trivial it follows that $\hat{\pi}^*(\mathcal{I}_V) = \mathcal{I}_{q^{-1}(V)\times_P X'}$. In particular, the sheaf

$$(\pi')^*\mathcal{H}om_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}), \mathcal{O}_{\mathcal{O}_{V'}}\big),$$

is isomorphic to

$$\mathcal{H}om_{\mathcal{O}_{Z'}}\big((F_b)_*(\mathcal{I}_{q^{-1}(V)\times_P X'} \otimes \mathcal{L}_{\hat{Z}}), \mathcal{O}_{(q^{-1}(V)\times_P X)'}\big).$$

Thus we see that the kernel of (7) is the subsheaf $\mathcal{E}nd_F^{\mathcal{L}_{\hat{Z}}}(Z, \pi^{-1}(V))_b$ of elements which map $(F_b)_*(\mathcal{I}_{q^{-1}(V)\times_P X'} \otimes \mathcal{L}_{\hat{Z}})$ to $\mathcal{I}_{(q^{-1}(V)\times_P X)'}$. The global sections of this subsheaf is denote by $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z, \pi^{-1}(V))_b$. In conclusion

Proposition 5.2. *The map π' induces a G-equivariant morphism*

$$(\pi')^* : \mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \to \mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z)_b.$$

Moreover, when V is a closed subvariety of ${}^{G}\!/\!{}_{P}$ then $(\pi')^*$ maps the subset $\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}, V)$ into $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z, q^{-1}(V) \times_P X)_b$.

The following is also useful.

Lemma 5.3. *Let Y denote a closed P-stable subvariety of X and fix notation as above. Then each element of $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$ maps $(F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \mathcal{L}_{\hat{Z}})$ to $\mathcal{I}_{(Z_Y)'}$.*

Proof. It suffices to show that the natural morphism

$$\mathcal{H}om_{\mathcal{O}_{Z'}}\big((F_b)_*\mathcal{L}_{\hat{Z}}, \mathcal{O}_{Z'}\big) \to \mathcal{H}om_{\mathcal{O}_{Z'}}\big((F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \mathcal{L}_{\hat{Z}}), \mathcal{O}_{(Z_Y)'}\big)$$

is zero. By linearity, this will follow if the natural morphism

$$\mathcal{I}_{(Z_Y)'} \otimes (F_b)_*\mathcal{L}_{\hat{Z}} \to (F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \mathcal{L}_{\hat{Z}}),$$

is an isomorphism, which can be checked by a local calculation. $\square$

5.4. Conclusions. By Proposition 5.1 an element v in the vectorspace $\mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big)$ defines an element in $\mathrm{End}_F^{\mathcal{M}(-\lambda)_Z}(Z)_f$. Moreover, by Proposition 5.2, an element $u \in \mathrm{End}_F^{\mathcal{L}}({}^G/{}_P)$ defines an element $(\pi')^*(u)$ in $\mathrm{End}_F^{\mathcal{L}_{\hat{Z}}}(Z)_b$. Thus by the discussion in Section 5.2 we obtain a G-equivariant map

$$\Phi^1_{\mathcal{M},\lambda} : \mathrm{End}_F^{\mathcal{L}}({}^G/{}_P) \otimes \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big) \to \mathrm{End}_F^{\mathcal{M}_Z}(G \times_P X),$$

defined by

$$\Phi^1_{\mathcal{M},\lambda}(u \otimes v) = (\pi')^* u \circ (F_b)_* v.$$

We can now prove

Theorem 5.4. *Let X denote a P-variety and $\mathcal{M}$ denote a P-linearized line bundle on X. Let $\mathcal{L}$ denote the equivariant line bundle on ${}^G/{}_P$ associated to the P-character λ. Then the G-equivariant map $\Phi^1_{\mathcal{M},\lambda}$, defined above, satisfies*

(1) When Y is a P-stable closed subvariety of X then the restriction of $\Phi^1_{\mathcal{M},\lambda}$ to the subspace :

$$\mathrm{End}_F^{\mathcal{L}}({}^G/{}_P) \otimes \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X, Y) \otimes k_\lambda\big),$$

maps to $\mathrm{End}_F^{\mathcal{M}_Z}(G \times_P X, G \times_P Y)$.

(2) When V denotes a closed subvariety of ${}^G/{}_P$ then the restriction of $\Phi^1_{\mathcal{M},\lambda}$ to the subspace

$$\mathrm{End}_F^{\mathcal{L}}({}^G/{}_P, V) \otimes \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big),$$

maps to $\mathrm{End}_F^{\mathcal{M}_Z}(G \times_P X, q^{-1}(V) \times_P X)$, where $q : G \to {}^G/{}_P$ denotes the quotient map.

Proof. The first statement follows from Proposition 5.1 and Lemma 5.3. The second statement follows from Proposition 5.2 and Lemma 5.5 below. $\square$

Lemma 5.5. *Let V denote a closed subset of ${}^G/{}_P$. Then every element of $\mathcal{E}nd_F^{\mathcal{M}(-\lambda)_Z}(Z)_f$ will map $(F_f)_*(\mathcal{M}_Z \otimes \mathcal{I}_{\pi^{-1}(V)})$ to $\mathcal{I}_{(\hat{\pi})^{-1}(V)} \otimes \mathcal{L}_{\hat{Z}}$.*

Proof. It suffices to prove that the natural morphism

$$\mathcal{I}_{(\hat{\pi})^{-1}(V)} \otimes (F_f)_* \mathcal{M}_Z \to (F_f)_*\big(\mathcal{I}_{\pi^{-1}(V)} \otimes \mathcal{M}_Z\big),$$

is an isomorphism, which can be checked by a local calculation. $\square$

5.5. Identify $\mathrm{Ind}_P^G\big(\mathcal{M}(X)\big)$ with the space of global sections of $\mathcal{M}_Z$ (cf. Equation (3)). Then we can define a G-equivariant morphism

$$\mathrm{End}_F^{\mathcal{M}_Z}(G \times_P X) \otimes \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \to \mathrm{End}_F(G \times_P X), \tag{8}$$

by mapping $s \otimes \sigma$, for σ a global section of $\mathcal{M}_Z$ and $s : (F_Z)_* \mathcal{M}_Z \to \mathcal{O}_{Z'}$, to the element

$$(F_Z)_* \mathcal{O}_Z \xrightarrow{(F_Z)_* \sigma} (F_Z)_* \mathcal{M}_Z \xrightarrow{s} \mathcal{O}_{Z'},$$

in $\mathrm{End}_F(G\times_P X)$. Combining $\Phi^1_{\mathcal{M},\lambda}$ with the morphism in (8) we obtain a G-equivariant map

$$\Phi_{\mathcal{M},\lambda} : \mathrm{End}^{\mathcal{L}}_F({}^G\!/\!_P)\otimes \mathrm{Ind}^G_P\big(\mathrm{End}^{\mathcal{M}}_F(X)\otimes k_\lambda\big)\otimes \mathrm{Ind}^G_P\big(\mathcal{M}(X)\big) \to \mathrm{End}_F(Z),$$

where an element $u\otimes v\otimes \sigma$ in the domain is mapped to the composed map

$$(F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{M}_Z \xrightarrow{(F_b)_*v} (F_b)_*\mathcal{L}_{\hat{Z}} \xrightarrow{(\pi')^*u} \mathcal{O}_{Z'}. \tag{9}$$

Notice that by Lemma 3.5 the map $u\in \mathrm{End}^{\mathcal{L}}_F({}^G\!/\!_P)$ factors as

$$(F_{G/P})_*\mathcal{L} \xrightarrow{(F_{G/P})_*u^!} (F_{G/P})_*\omega^{1-p}_{G/P} \to \mathcal{O}_{(G/P)'}, \tag{10}$$

where $u^!$ is some global section of the line bundle $\check{\mathcal{L}} := \omega^{1-p}_{G/P}\otimes\mathcal{L}^{-1}$ associated to u (cf. Section 3.7), and the rightmost map is the evaluation map with domain $(F_{G/P})_*\omega^{1-p}_{G/P} = \mathrm{End}_F({}^G\!/\!_P)$. It follows that we may extend (9) into a commutative diagram
(11)

$$\begin{tikzcd}
(F_Z)_*\mathcal{O}_Z \arrow[r, "(F_Z)_*\sigma"] \arrow[d, "(F_b)_*\hat{\pi}^*(u^!)"] & (F_Z)_*\mathcal{M}_Z \arrow[r, "(F_b)_*v"] \arrow[d, "(F_b)_*\hat{\pi}^*(u^!)"] & (F_b)_*\mathcal{L}_{\hat{Z}} \arrow[r, "(\pi')^*u"] \arrow[d, "(F_b)_*\hat{\pi}^*(u^!)"'] & \mathcal{O}_{Z'} \\
(F_Z)_*\pi^*\check{\mathcal{L}} \arrow[r] & (F_Z)_*(\mathcal{M}_Z\otimes\pi^*\check{\mathcal{L}}) \arrow[r] & (F_b)_*(\hat{\pi}^*\omega^{1-p}_{G/P}) \arrow[ur] &
\end{tikzcd}$$

where all the vertical maps are induced by multiplication by $\hat{\pi}^*(u^!)$. Likewise the lower horizontal maps are induced from the upper horizontal maps by multiplication with $\hat{\pi}^*(u^!)$. The triangle on the right is induced from (10) by pull-back to Z'.

Theorem 5.6. *Let X denote a P-variety and $\mathcal{M}$ denote a P-linearized line bundle on X. Let $\mathcal{L}$ denote the equivariant line bundle on ${}^G\!/\!_P$ associated to the P-character λ. Then the G-equivariant map $\Phi_{\mathcal{M},\lambda}$, defined above, satisfies*

1. *When Y is a P-stable closed subvariety of X then the restriction of $\Phi_{\mathcal{M},\lambda}$ to the subspace :*
$$\mathrm{End}^{\mathcal{L}}_F({}^G\!/\!_P)\otimes \mathrm{Ind}^G_P\big(\mathrm{End}^{\mathcal{M}}_F(X,Y)\otimes k_\lambda\big)\otimes \mathrm{Ind}^G_P\big(\mathcal{M}(X)\big)$$
maps to $\mathrm{End}_F(G\times_P X, G\times_P Y)$.
2. *When V denotes a closed subvariety of ${}^G\!/\!_P$ then the restriction of $\Phi_{\mathcal{M},\lambda}$ to the subspace :*
$$\mathrm{End}^{\mathcal{L}}_F({}^G\!/\!_P, V)\otimes \mathrm{Ind}^G_P\big(\mathrm{End}^{\mathcal{M}}_F(X)\otimes k_\lambda\big)\otimes \mathrm{Ind}^G_P\big(\mathcal{M}(X)\big)$$
maps to $\mathrm{End}_F(G\times_P X, q^{-1}(V)\times_P X)$, where $q : G\to {}^G\!/\!_P$ denotes the quotient map.

Moreover, let $u \in \mathrm{End}_F^{\mathcal{L}}(G/P)$*,* $v \in \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big)$ *and* $\sigma \in \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big)$*. Then the element* $\Phi_{\mathcal{M},\lambda}(u \otimes v \otimes \sigma)$ *factorizes both as*

$$(F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{M}_Z \xrightarrow{s_1} \mathcal{O}_{Z'},$$

and as

$$(F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*(\sigma\otimes\pi^* u^!)} (F_Z)_*(\mathcal{M}_Z \otimes \pi^*\check{\mathcal{L}}) \xrightarrow{s_2} \mathcal{O}_{Z'},$$

where s_1 *and* s_2 *satisfies*

i) If v *is contained in* $\mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X,Y) \otimes k_\lambda\big)$ *then* s_1 *and* s_2 *are compatible with* $G \times_P Y$*.*

ii) If u *is contained in* $\mathrm{End}_F^{\mathcal{L}}(G/P, V)$ *then* s_1 *is compatible with* $q^{-1}(V) \times_P X$*.*

Proof. Part (1) and (2) follows directly from Theorem 5.4 and the definition of $\Phi_{\mathcal{M},\lambda}$. The existence of s_1 and s_2 follows by the diagram (11). Finally the claims about the compatibility of s_1 and s_2 follows from Theorem 5.4 and Lemma 5.3. $\square$

5.6. We will now describe when an element in the image of $\Phi_{\mathcal{M},\lambda}$ defines a Frobenius splitting of Z. For this we consider the composed map $\mathrm{ev}_Z \circ \Phi_{\mathcal{M},\lambda}$. Recall that an element $s \in \mathrm{End}_F(Z)$ is a Frobenius splitting of Z if and only if $\mathrm{ev}_Z(s)$ is the constant function 1 on Z'.

Let $u \in \mathrm{End}_F^{\mathcal{L}}(G/P)$, $v \in \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big)$ and $\sigma \in \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big)$. By Equation (9) the image of $u \otimes v \otimes \sigma$ under $\mathrm{ev}_Z \circ \Phi_{\mathcal{M},\lambda}$ coincides with the global section of $\mathcal{O}_{Z'}$ determined by the composed map

$$\mathcal{O}_{Z'} \xrightarrow{F_Z^\sharp} (F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{M}_Z \xrightarrow{(F_b)_*v} (F_b)_*\mathcal{L}_{\hat{Z}} \xrightarrow{(\pi')^* u} \mathcal{O}_{Z'}. \tag{12}$$

We may divide this composition into two parts. The first part

$$\mathcal{O}_{Z'} \xrightarrow{F_Z^\sharp} (F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{M}_Z \xrightarrow{(F_b)_*v} (F_b)_*\mathcal{L}_{\hat{Z}}$$

is defined by σ and v and defines a global section of $\mathcal{L}_{\hat{Z}}$. The corresponding map

$$\Phi^2_{\mathcal{M},\lambda} : \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big) \otimes \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \to \mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big),$$

is the map induced by the morphism

$$\mathrm{End}_F^{\mathcal{M}}(X) \otimes \mathcal{M}(X) \to k[X'], \tag{13}$$

mapping $s : (F_X)_*\mathcal{M} \to \mathcal{O}_{X'}$ and τ a global section of $\mathcal{M}$, to $s(\tau)$. Notice that we here identify $\mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big)$ with the space of global sections of $\mathcal{L}_{\hat{Z}}$ (cf. Equation (3)). The second part takes a global section $\tilde{\tau}$ of $\mathcal{L}_{\hat{Z}}$ and an element u in $\mathrm{End}_F^{\mathcal{L}}(G/P)$ to the global section of $\mathcal{O}_{Z'}$ defined by

$$\mathcal{O}_{Z'} \xrightarrow{F_b^\sharp} (F_b)_*\mathcal{O}_{\hat{Z}} \xrightarrow{(F_b)_*\tilde{\tau}} (F_b)_*\mathcal{L}_{\hat{Z}} \xrightarrow{(\pi')^* u} \mathcal{O}_{Z'}.$$

The corresponding map is

$$\Phi_\lambda : \mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big) \to k[Z'],$$

which maps $u \otimes \tilde{\tau}$, to $((\pi')^* u)(\tilde{\tau})$ (cf. Proposition 5.2). The restriction of Φ_λ :

$$\phi_\lambda : \mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(k_\lambda\big) \to k, \tag{14}$$

is the map corresponding to Φ_λ in case X is the one point space $\mathrm{Spec}(k)$ (in which case $k[X']$ is just k). In combination this defines us a commutative diagram

(15)

$$\begin{tikzcd}
\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big) \otimes \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \arrow[r, "\Phi_{\mathcal{M},\lambda}"] \arrow[d, "\mathrm{Id}\otimes\Phi^2_{\mathcal{M},\lambda}"] & \mathrm{End}_F(Z) \arrow[d, "\mathrm{ev}_Z"'] \\
\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big) \arrow[r, "\Phi_\lambda"] & k[Z'] \\
\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(k_\lambda\big) \arrow[u, hook] \arrow[r, two heads, "\phi_\lambda"] \arrow[d, "m_\lambda"'] & k \arrow[u, hook] \\
\mathrm{End}_F\big({}^{G}\!/\!{}_{P}\big) \arrow[ur, "\mathrm{ev}_{G/P}"'] &
\end{tikzcd}$$

where m_λ is the natural map which makes the lower part of the diagram commutative. Notice that when $k[X'] = k$, e.g. if X' is a complete and irreducible variety, then ϕ_λ and Φ_λ coincides. Let χ denote the P-character associated to the canonical G-linearization of $\omega_{G/P}^{-1}$ (cf. Section 4.3). Then as noted earlier (Section 5.5) the G-module $\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P})$ coincides with the space of global sections of $\check{\mathcal{L}} = \omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ and thus coincides with

$$\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) = \mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big), \tag{16}$$

where we abuse notation and write $(p-1)\chi - \lambda$ for the 1-dimensional P-representation associated with the character $(p-1)\chi - \lambda$. It follows that m_λ is the natural multiplication map

$$m_\lambda : \mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big) \otimes \mathrm{Ind}_P^G\big(\lambda\big) \to \mathrm{Ind}_P^G\big((p-1)\chi\big). \tag{17}$$

which is surjective if the domain is nonzero, i.e. if $\mathcal{L}$ and $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ are effective line bundles on ${}^{G}\!/\!{}_{P}$ [R-R, Thm.3].

The commutativity of the diagram (15) then implies:

Proposition 5.7. *Let Ξ denote an element in the domain of $\Phi_{\mathcal{M},\lambda}$, and assume that the image $(\mathrm{Id} \otimes \Phi^2_{\mathcal{M},\lambda})(\Xi)$ is contained in the subspace $\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes \mathrm{Ind}_P^G\big(k_\lambda\big)$ (cf. diagram (15)). Then $\Phi_{\mathcal{M},\lambda}(\Xi)$ is a Frobenius splitting of Z if and only if $\phi_\lambda((\mathrm{Id} \otimes \Phi^2_{\mathcal{M},\lambda})(\Xi))$ equals the constant* 1.

In particular, if $\mathrm{End}_F^{\mathcal{L}}(G/P) \otimes \mathrm{Ind}_P^G\big(k_\lambda\big)$ *is nonzero and* $\mathrm{Ind}_P^G\big(k_\lambda\big)$ *is contained in the image of* $\Phi^2_{\mathcal{M},\lambda}$*, then* Z *admits a Frobenius splitting.*

Proof. The first part of the proof is just a restatement of the fact that the diagram (15) is commutative. The second part follows by the surjectivity of m_λ and the fact that G/P admits a Frobenius splitting.

$\square$

Corollary 5.8. *Assume that* X *is irreducible and complete. If both* $\mathrm{Ind}_P^G\big(\lambda\big)$ *and* $\mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big)$ *are nonzero and* $\Phi^2_{\mathcal{M},\lambda}$ *is surjective, then* Z *admits a Frobenius splitting.*

5.7. In many concrete situation the existence of a P-invariant element in $\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda$ is given. Notice that this is equivalent to a G-invariant element v in $\mathrm{Ind}_P^G\big(\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda\big)$ and thus $\Phi_{\mathcal{M},\lambda}$ defines a G-equivariant map

$$\mathrm{End}_F^{\mathcal{L}}(G/P) \otimes \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \to \mathrm{End}_F(Z), \tag{18}$$
$$u \otimes \sigma \mapsto \Phi_{\mathcal{M},\lambda}(u \otimes v \otimes \sigma).$$

Similarly $\Phi^2_{\mathcal{M},\lambda}$ defines a G-equivariant morphism

$$\mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \to \mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big), \tag{19}$$

which makes the diagram

$$\begin{tikzcd} \mathrm{End}_F^{\mathcal{L}}(G/P) \otimes \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \arrow[r] \arrow[d] & \mathrm{End}_F(Z) \arrow[d, "\mathrm{ev}_Z"'] \\ \mathrm{End}_F^{\mathcal{L}}(G/P) \otimes \mathrm{Ind}_P^G\big(k[X'] \otimes k_\lambda\big) \arrow[r, "\Phi_\lambda"] & k[Z'] \end{tikzcd} \tag{20}$$

commutative. We also note

Corollary 5.9. *Assume that* X *is irreducible and complete and let* v *denote a* P*-invariant element of* $\mathrm{End}_F^{\mathcal{M}}(X) \otimes k_\lambda$*. If the induced map*

$$(\Phi^2_{\mathcal{M},\lambda})_{|v \otimes \mathrm{Ind}_P^G(\mathcal{M}(X))} : \mathrm{Ind}_P^G\big(\mathcal{M}(X)\big) \to \mathrm{Ind}_P^G\big(k_\lambda\big),$$

is surjective then Z *admits a Frobenius splitting. In particular, if* $\mathrm{Ind}_P^G\big(k_\lambda\big)$ *is an irreducible* G*-representation then for* Z *to be Frobenius split it suffices that the latter map is nonzero.*

Proof. Apply Corollary 5.8. $\square$

6. B-Canonical Frobenius splittings

In this section we continue the study of the Frobenius splitting properties of $Z = G \times_P X$. The notation is kept as in Section 5 but we restrict ourselves to the case where G is a connected, semisimple and simply connected linear algebraic group. Moreover, we fix $P = B$, $\mathcal{M} = \mathcal{O}_X$ and $\lambda = -(p-1)\rho$. Recall that, in this setup, the dualizing sheaf $\omega_{G/B}$ is the G-linearized sheaf associated to the B-character 2ρ.

Thus, with the notation in Section 5.6, we have $\chi = -2\rho$. Recall also the G-equivariant identity (see (16))

$$\mathrm{End}_F^{\mathcal{L}}(G/B) \simeq \mathrm{Ind}_B^G((p-1)\chi - \lambda) = \mathrm{Ind}_B^G(\lambda) = \mathrm{Ind}_B^G((1-p)\rho). \tag{21}$$

The latter G-module is called the Steinberg module of G and will be denoted by St. The Steinberg module is a simple and selfdual G-module. A *B-canonical Frobenius splitting* of X is then a B-equivariant map

$$\theta : \mathrm{St} \otimes k_{(p-1)\rho} \to \mathrm{End}_F(X), \tag{22}$$

containing a Frobenius splitting in its image. Notice that a B-canonical Frobenius splitting of X is not a Frobenius splitting as defined in Section 3.2. However, there exists a unique nonzero lowest weight vector v_- of St such that $\theta(v_-)$ is a Frobenius splitting in the sense of Section 3.2. Moreover, as St is a simple G-module the map θ is uniquely determined by $\theta(v_-)$, and we may thus identify θ with $\theta(v_-)$. In this way $\theta(v_-)$ will also be called a B-canonical Frobenius splitting of X.

The importance of B-canonical Frobenius splittings was first observed by O. Mathieu in connection with good filtrations of G-modules. We refer to [B-K, Chapter 4] for a general reference on B-canonical Frobenius splittings.

6.1. Consider a B-canonical Frobenius splitting as in (22). By Frobenius reciprocity this defines a map

$$\mathrm{St} \to \mathrm{Ind}_B^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big),$$

and as $\mathrm{Ind}_B^G\big(k[X]\big)$ contains k we may consider the induced G-equivariant morphism

$$\tilde{\theta} : \mathrm{St} \to \mathrm{Ind}_B^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big) \otimes \mathrm{Ind}_B^G\big(k[X]\big).$$

Composing $\tilde{\theta}$ with the map $\Phi^2_{\mathcal{M},\lambda}$ of Section 5.6 we end up with a map

$$\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta} : \mathrm{St} \to \mathrm{Ind}_B^G\big(k[X'] \otimes k_\lambda\big).$$

We claim

Lemma 6.1. *The composed map $\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta}$ is an isomorphism on its image $\mathrm{Ind}_B^G(\lambda)$.*

Proof. We first prove that the image of $\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta}$ is contained in $\mathrm{Ind}_B^G(\lambda)$. For this let $\mathrm{End}_F(X)_c$ denote the inverse image of $k \subset k[X']$ under the evaluation map ev_X. It suffices to prove that the image of θ is contained in $\mathrm{End}_F(X)_c$. Notice that $\mathrm{End}_F(X)_c$ is a B-submodule of $\mathrm{End}_F(X)$ containing the set of Frobenius splittings of X. In particular, the image of the lowest weight space of St under θ is contained in $\mathrm{End}_F(X)_c$. Moreover, as St is an irreducible G-module it is generated by the lowest weight space as a B-module. Thus, the image of θ will be contained in the B-module $\mathrm{End}_F(X)_c$.

Now $\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta}$ is a map from St to $\mathrm{Ind}_B^G(\lambda) = \mathrm{St}$. Thus, by Frobenius reciprocity, it suffices to prove that $\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta}$ is nonzero which is the case as θ contains a Frobenius splitting in its image. $\square$

Using Lemma 6.1 we can now combine the diagram (15) with the map $\Phi^2_{\mathcal{M},\lambda} \circ \tilde{\theta}$ and obtain a commutative and G-equivariant diagram

(23)
$$
\begin{tikzcd}
\mathrm{St}\otimes\mathrm{St} \arrow[r, "\Theta"] \arrow[d, "\simeq"', "\mathrm{Id}\otimes(\Phi^2_{\mathcal{M},\lambda}\circ\tilde{\theta})"] & \mathrm{End}_F(G\times_B X) \arrow[r, "\mathrm{ev}_Z"] & k[Z'] \\
\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{B})\otimes\mathrm{Ind}_B^G(k_\lambda) \arrow[rr, "\phi_\lambda"] \arrow[dr, "m_\lambda"'] & & k \arrow[u] \\
& \mathrm{End}_F({}^{G}\!/\!{}_{B}) \arrow[ur, "\mathrm{ev}_{G/B}"'] &
\end{tikzcd}
$$

where Θ is the map induced by $\tilde{\theta}$ and $\Phi_{\mathcal{M},\lambda}$. By Proposition 5.7 it follows that $\Theta(\Xi)$, for Ξ in $\mathrm{St}\otimes\mathrm{St}$, is a Frobenius splitting of Z if and only if the image of Ξ under ϕ_λ and $\mathrm{Id}\otimes(\Phi^2_{\mathcal{M},\lambda}\circ\tilde{\theta})$ equals 1. The latter map from $\mathrm{St}\otimes\mathrm{St}$ to k will be denoted by ϕ. By construction ϕ is G-equivariant. Moreover, m_λ is surjective and $\mathrm{ev}_{G/B}$ is nonzero (as ${}^{G}\!/\!{}_{B}$ admits a Frobenius splitting) and thus ϕ is nonzero. As St is a simple G-module it follows that

$$\phi : \mathrm{St}\otimes\mathrm{St} \to k, \tag{24}$$

defines a nondegenerate G-invariant bilinear form on St. By Frobenius reciprocity such a form is uniquely determined up to a nonzero constant. In particular, this provides a very useful way to construct lots of Frobenius splittings of Z.

Corollary 6.2. *Let $\theta : \mathrm{St}\otimes k_{(p-1)\rho} \to \mathrm{End}_F(X)$ denote a B-canonical Frobenius splitting of X. Then the induced morphism (defined above)*

$$\Theta : \mathrm{St}\otimes\mathrm{St} \to \mathrm{End}_F(G\times_B X),$$

satisfies the following

(1) *The image $\Theta(\nu)$ of an element ν in $\mathrm{St}\otimes\mathrm{St}$ defines a Frobenius splitting of $G\times_B X$ up to a nonzero constant if and only if $\phi(\nu)$ is nonzero.*
(2) *If the image of θ is contained in $\mathrm{End}_F(X,Y)$ for a B-stable closed subvariety Y of X, then the image of Θ is contained in $\mathrm{End}_F(G\times_B X, G\times_B Y)$.*
(3) *Let v denote an element of $\mathrm{St} = \mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{B})$ which is compatible with a closed subvariety V of ${}^{G}\!/\!{}_{B}$. For any element $v' \in \mathrm{St}$ we have*

$$\Theta(v\otimes v') \in \mathrm{End}_F(G\times_B X, q^{-1}(V)\times_B X),$$

with $q : G \to {}^{G}\!/\!{}_{B}$ denoting the quotient map.

(4) *Any element of the form $\Theta(v \otimes v')$ factorizes as*

$$(F_Z)_* \mathcal{O}_Z \xrightarrow{(F_Z)_* \pi^* v} (F_Z)_* \pi^* \mathcal{L} \xrightarrow{s} \mathcal{O}_{Z'},$$

where $Z = G \times_B X$ and $\mathcal{L}$ is the line bundle on ${}^G/_B$ associated to the B-character $(1-p)\rho$. Moreover, if the image of θ is contained in $\mathrm{End}_F(X, Y)$ *then s is compatible with $G \times_B Y$.*

Proof. All statements follows directly from Theorem 5.6 and the considerations above. □

The first part (1) and (2) of the above result is well known (see e.g. [B-K, Ex. 4.1.E(4)]). However, the second part (3) and (4) seems to be new.

6.2. B-canonical Frobenius splitting when G is not semisimple. Although Corollary 6.2 is only stated for connected, semisimple and simply connected groups it also applies in other cases : assume that G is a connected linear algebraic group containing a connected semisimple subgroup H such that the induced map ${}^H/_{H \cap B} \to {}^G/_B$ is an isomorphism. E.g. this is satisfied for any parabolic subgroup of a reductive connected linear algebraic group. Let $q_{\mathrm{sc}} : H_{\mathrm{sc}} \to H$ denote a simply connected cover of H. Then X admits an action of the parabolic subgroup $B_{\mathrm{sc}} := q_{\mathrm{sc}}^{-1}(B \cap H)$ of H_{sc}. Furthermore, the natural morphism

$$H_{\mathrm{sc}} \times_{B_{\mathrm{sc}}} X \to G \times_B X,$$

is then an isomorphism. We then say that X admits a B-canonical Frobenius splitting if X, as a B_{sc}-variety, admits a B_{sc}-canonical Frobenius splitting. In this case we may apply Corollary 6.2 to obtain Frobenius splitting properties of $G \times_B X$.

6.3. Restriction to Levi subgroups. Return to the situation where G is connected, semisimple and simply connected. Let J be a subset of the set of simple roots Δ and let G_J denote the commutator subgroup of L_J. Then G_J is a connected, semisimple and simply connected linear algebraic group with Borel subgroup $B_J = G_J \cap B$ and maximal torus $T_J = T \cap G_J$. We let St_J denote the associated Steinberg module. Notice that $\mathrm{St}_J = \mathrm{Ind}_{B_J}^{G_J}((1-p)\rho_J)$ where ρ_J denotes the restriction of ρ to B_J. The following should be well known but we do not know a good reference.

Lemma 6.3. *There exists a G_J-equivariant morphism*

$$\mathrm{St}_J \to \mathrm{St},$$

such that the B_J^--invariant line of St_J *maps surjectively to the B^--invariant line of* St. *In particular, if X is a G-variety admitting a B-canonical Frobenius splitting then X admits a B_J-canonical Frobenius splitting as a G_J-variety.*

Proof. Let M denote the T-stable complement to the B-stable line in St. Then M is B^--invariant and thus also B_J^--invariant. The translate $\dot{w}_0^J M$ is then invariant under B_J and we obtain a B_J-equivariant morphism

$$\mathrm{St} \to \mathrm{St}/(\dot{w}_0^J M) \simeq k_{(1-p)\rho_J}.$$

By Frobenius reciprocity this defines a G_J-equivariant map $\mathrm{St} \to \mathrm{St}_J$ such that the B-stable line of St maps onto the B_J-stable line of St_J. Now apply the selfduality of St_J and St to obtain the desired map. This proves the first part of the statement.

The second part follows easily by composing the obtained morphism $\mathrm{St}_J \to \mathrm{St}$ with the B-canonical Frobenius splitting

$$\mathrm{St} \to \mathrm{End}_F(X) \otimes k_{(1-p)\rho},$$

of X and noticing that the restriction of ρ to B_J is ρ_J.

□

7. Applications to $G \times G$-varieties

In this section we consider a linear algebraic group G satisfying the conditions of Section 6.2, i.e. we assume that G contains a closed connected semisimple subgroup H such that ${}^{H}/_{H\cap B} \to {}^{G}/_{B}$ is an isomorphism. We also let H_{sc} denote the simply connected version of H and let B_{sc} denote the associated Borel subgroup.

7.1. A well known result. Consider for a moment (i.e. in this subsection) the case where G is semisimple and simply connected. Remember that the G-linearized line bundle on ${}^{G}/_{B}$ associated to the B-character 2ρ coincides with the dualizing sheaf $\omega_{G/B}$. Let $\mathcal{L}$ denote the line bundle on ${}^{G}/_{B}$ associated to the B-character $(1-p)\rho$ and recall from Section 6 the notation $\mathrm{St} = \mathrm{Ind}_B^G((1-p)\rho)$ for the Steinberg module. As the Steinberg module is a selfdual G-module we may fix a G-invariant nonzero element v_Δ in the tensorproduct $\mathrm{St} \otimes \mathrm{St}$. We may think of v_Δ as a global section of the line bundle $\mathcal{L} \boxtimes \mathcal{L}$ on $({}^{G}/_{B})^2 = {}^{G}/_{B} \times {}^{G}/_{B}$.

Identify ${}^{G}/_{B} \times {}^{G}/_{B}$ with $G \times_B {}^{G}/_{B}$ by the isomorphism

$$G \times_B {}^{G}/_{B} \to {}^{G}/_{B} \times {}^{G}/_{B},$$

$$[g, hB] \mapsto (gB, ghB),$$

and let D denote the subvariety of ${}^{G}/_{B} \times {}^{G}/_{B}$ corresponding to $G \times_B \partial({}^{G}/_{B})$, where $\partial({}^{G}/_{B})$ denotes the union of the codimension 1 Schubert varieties in ${}^{G}/_{B}$. Then, by [B-K, proof of Thm.2.3.8], the zero scheme of v_Δ equals $(p-1)D$. Consider then the natural morphism :

$$\eta : (\mathcal{L} \boxtimes \mathcal{L}) \otimes (\mathcal{L} \boxtimes \mathcal{L}) \to \omega_{({}^{G}/_{B})^2}^{1-p} = \mathcal{E}nd_F^!(({}^{G}/_{B})^2)$$

and define

$$\eta_D : (\mathcal{L} \boxtimes \mathcal{L}) \to \mathcal{E}nd_F^!(({}^{G}/_{B})^2),$$

as in Lemma 3.6, using the identification $\mathcal{L} \boxtimes \mathcal{L} = \mathcal{O}_{(G/B)^2}\big((p-1)D\big)$. Then by Lemma 3.6 the image of η_D is contained in $\mathcal{E}nd^!_F((G/B)^2, D)$ and thus the associated element

$$\eta'_D \in \mathrm{Hom}_{\mathcal{O}_{((G/B)^2)'}}\big((F_{(G/B)^2})_*(\mathcal{L} \boxtimes \mathcal{L}), \mathcal{O}_{((G/B)^2)'}\big),$$

is compatible with D. It follows

Lemma 7.1. *The element in*

$$\mathrm{End}_F^{\mathcal{L}\boxtimes\mathcal{L}}((G/B)^2) \simeq \mathrm{St} \boxtimes \mathrm{St}$$

defined by v_Δ is compatible with the diagonal $\mathrm{diag}(G/B)$ *in* $G/B \times G/B$.

Proof. We have to prove that η'_D, defined above, is compatible with the diagonal $\mathrm{diag}(G/B)$. As η'_D is compatible with D it suffices to show that $\mathrm{End}_F^{\mathcal{L}\boxtimes\mathcal{L}}((G/B)^2, D)$ is contained in $\mathrm{End}_F^{\mathcal{L}\boxtimes\mathcal{L}}\big((G/B)^2, \mathrm{diag}(G/B)\big)$. This follows by an application of Lemma 3.1 and an argument as at the end of the proof of [B-K, Thm.2.3.1]. $\square$

7.2. We return to the setup as in the beginning of this section. We want to apply the results of the preceding sections to the case when the group equals $G \times G$. So let X denote a $B \times B$-variety and assume that X admits a $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-canonical Frobenius splitting defined by

$$\theta : (\mathrm{St} \boxtimes \mathrm{St}) \otimes (k_{(p-1)\rho} \boxtimes k_{(p-1)\rho}) \to \mathrm{End}_F(X),$$

which is compatible with certain $B \times B$-stable subvarieties $X_1, \ldots, X_m$, i.e. the image of θ is contained in $\mathrm{End}_F(X, X_i)$ for all i. Then

Theorem 7.2. *The variety $(G \times G) \times_{(B\times B)} X$ admits a* $\mathrm{diag}(B_{\mathrm{sc}})$*-canonical Frobenius splitting which compatibly Frobenius splits the subvarieties* $\mathrm{diag}(\mathrm{G}) \times_{\mathrm{diag}(B)} X$ *and $(G \times G) \times_{(B\times B)} X_i$ for all i.*

Proof. It suffices to consider the case where $G = H_{\mathrm{sc}}$ (cf. discussion in Section 6.2). By Corollary 6.2 there exists a $G \times G$-equivariant morphism

$$\Theta : (\mathrm{St} \boxtimes \mathrm{St}) \otimes (\mathrm{St} \boxtimes \mathrm{St}) \to \mathrm{End}_F((G \times G) \times_{(B\times B)} X),$$

satisfying certain compatibility conditions. Let $v_\Delta \in \mathrm{St} \boxtimes \mathrm{St}$ be a nonzero $\mathrm{diag}(G)$-invariant element and let $v \in \mathrm{St} \boxtimes \mathrm{St}$ be arbitrary. Then by Corollary 6.2 and Lemma 7.1 the element $\Theta\big(v_\Delta \otimes v\big)$ is compatible with $\mathrm{diag}(\mathrm{G}) \times_{\mathrm{diag}(B)} X$ and $(G \times G) \times_{(B\times B)} X_i$ for all i. In particular, if we define the $\mathrm{diag}(G)$-equivariant morphism

$$\Theta_\Delta : \mathrm{St} \otimes \mathrm{St} \to \mathrm{End}_F((G \times G) \times_{(B\times B)} X),$$

by $\Theta_\Delta(v) = \Theta\big(v_\Delta \otimes v\big)$, then every element in the image of Θ_Δ is compatible with $\mathrm{diag}(\mathrm{G}) \times_{\mathrm{diag}(B)} X$ and $(G \times G) \times_{(B\times B)} X_i$ for all i. Consider $k_{(p-1)\rho}$ as the highest weight line in St. Then the restriction of Θ_Δ to $\mathrm{St} \otimes k_{(p-1)\rho}$ defines a $\mathrm{diag}(B)$-canonical Frobenius splitting of $(G \times G) \times_{(B\times B)} X$ with the desired properties. $\square$

Notice that by the general machinery of canonical Frobenius splittings (see e.g. [B-K, Prop.4.1.17]) the existence of a Frobenius splitting of $\mathrm{diag(G)} \times_{\mathrm{diag}(B)} X$ follows if X admits a $\mathrm{diag}(B_{\mathrm{sc}})$-canonical Frobenius splitting. In the above setup X only admits a $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-canonical Frobenius splitting which is less restrictive. However, in contrast to the situation when X admits a $\mathrm{diag}(B_{\mathrm{sc}})$-canonical Frobenius splitting, the present Frobenius splitting is not necessarily compatible with subvarieties of the form $\overline{B\dot{w}B} \times_B X$, with w denoting an element of the Weyl group and $\overline{B\dot{w}B}$ denoting the closure of $B\dot{w}B$ in G.

8. G-Schubert varieties in equivariant Embeddings

From now on, unless otherwise stated, we assume that G is a connected reductive group.

8.1. Equivariant embeddings. Consider G as a $G \times G$-variety by left and right translation. An equivariant embedding X of G is then a normal irreducible $G \times G$-variety containing an open dense subset which is $G \times G$-equivariantly isomorphic to G. In particular, we may identify G with an open subset of X, and the complement $X \setminus G$ is then called the boundary. As G is an affine variety the boundary is of pure codimension 1 in X [Har, Prop.3.1]. Any equivariant embedding of G is a spherical variety (with respect to the induced $B \times B$-action) and thus X contains finitely may $B \times B$-orbits.

8.2. Wonderful compactifications. When $G = G_{\mathrm{ad}}$ is of adjoint type there exists a distinguished equivariant embedding $\mathbf{X}$ of G which is called the *wonderful compactification* (see e.g. [B-K, 6.1]).

The boundary $\mathbf{X} \setminus G$ is a union of irreducible divisors $\mathbf{X}_j$, $j \in \Delta$, which intersect transversely. For a subset $J \subset \Delta$, we denote the intersection $\cap_{j \in J} \mathbf{X}_j$ by $\mathbf{X}_J$. As a $(G \times G)$-variety, $\mathbf{X}_J$ is isomorphic to the variety $(G \times G) \times_{P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}} \mathbf{Y}$, where $\mathbf{Y}$ denotes the wonderful compactification of the group of adjoint type associated to $L_{\Delta \setminus J}$. Here the $P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}$-action on $\mathbf{Y}$ is defined by the quotient maps $P_{\Delta \setminus J} \to L_{\Delta \setminus J}$ and $P^-_{\Delta \setminus J} \to L_{\Delta \setminus J}$. In particular, $\mathbf{X}_\Delta$ is $G \times G$-equivariantly isomorphic to the variety ${}^{G}/_{B^-} \times {}^{G}/_{B}$.

8.3. Toroidal embeddings. Let G_{ad} denote the group of adjoint type associated to G, and let $\mathbf{X}$ denote the wonderful compactification of G_{ad}. An embedding X of the reductive group G is then called *toroidal* if the canonical map $G \to G_{\mathrm{ad}}$ admits an extension $X \to \mathbf{X}$.

8.4. G-Schubert varieties. By a *G-Schubert variety* in an equivariant embedding X we will mean a subvariety of the form $\mathrm{diag}(G) \cdot V$, for some $B \times B$-orbit closure V. Notice that $\mathrm{diag}(G) \cdot V$ is the image of $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} V$ under the proper map

$$\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X \to X,$$

$$[g, x] \mapsto g \cdot x,$$

and thus G-Schubert varieties are closed diag(G)-stable subvarieties of X.

If $G = G_{\mathrm{ad}}$ and $X = \mathbf{X}$ is the wonderful compactification then a G-Schubert variety in $\mathbf{X}_\Delta$ is diag(G)-equivariantly isomorphic to a variety of the form $G \times_B X(w)$, where $X(w)$ denotes a Schubert variety in ${}^{G}/_{B}$. In particular, this explains the name *G-Schubert varieties* as this is the name used for varieties of the form $G \times_B X(w)$.

In the rest of this section, we will relate G-Schubert varieties to closures of so-called G-stable pieces. Our primary interest are G-stable pieces in wonderful compactifications but below we will also describe the toroidal case in general.

8.5. G-stable pieces in the wonderful compactification. Let $G = G_{\mathrm{ad}}$ denote a group of adjoint type and let $\mathbf{X}$ denote its wonderful compactification. Let $J \subset \Delta$ and identify $\mathbf{X}_J$ with $(G \times G) \times_{P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}} \mathbf{Y}$ as in Section 8.2. Using this identification it easily follows that there exists a unique element in $\mathbf{X}_J$ which is invariant under $U_J^- \times U_J$ and diag(L_J). We denote this element by $\mathbf{h}_J$ and note that as an element of $(G \times G) \times_{P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}} \mathbf{Y}$ it equals $[(e, e), e_J]$, where e (resp. e_J) denotes the identity element of G (resp. the adjoint group associated to $L_{\Delta \setminus J}$). For $w \in W^{\Delta \setminus J}$, we then let

$$\mathbf{X}_{J,w} = \mathrm{diag}(G)(Bw, 1) \cdot \mathbf{h}_J,$$

and call $\mathbf{X}_{J,w}$ a G-stable piece of $\mathbf{X}$. A G-stable piece is a locally closed subset of $\mathbf{X}$ and by [L, section 12] and [He, section 2], we can use them to decompose $\mathbf{X}$ as follows

$$\mathbf{X} = \bigsqcup_{\substack{J \subset \Delta \\ w \in W^{\Delta \setminus J}}} \mathbf{X}_{J,w}.$$

Moreover, by the proof of [He2, Theorem 4.5], any G-Schubert variety is a finite union of G-stable pieces. In particular, we may think of G-Schubert varieties as closures of G-stable pieces.

8.6. G-stable pieces in arbitrary toroidal embeddings. We fix a toroidal embedding X of G. The irreducible components of the boundary $X \setminus G$ will be denoted by $X_1, \dots, X_n$. For each $G \times G$-orbit closure Y in X we then associate the set

$$K_Y = \{i \in \{1, \dots, n\} \mid Y \subset X_i\},$$

where by definition $K_Y = \varnothing$ when $Y = X$. Then by [B-K, Prop.6.2.3], $Y = \cap_{i \in K_Y} X_i$. Moreover, we define

$$\mathcal{I} = \{K_Y \subset \{1, \dots, n\} \mid Y \text{ a } G \times G\text{-orbit closure in } X\ \},$$

and write $X_K := \cap_{i \in K} X_i$ for $K \in \mathcal{I}$. Then $(X_K)_{K \in \mathcal{I}}$ are the set of closures of $G \times G$-orbits in X. Let now $\pi_X : X \to \mathbf{X}$ denote the given extension of $G \to G_{\mathrm{ad}}$. Then the closure of $\pi_X(X_K)$ equals $\mathbf{X}_{P(K)}$ for some unique subset $P(K)$ of Δ. This defines a map $P : \mathcal{I} \to \mathcal{P}(\Delta)$, where $\mathcal{P}(\Delta)$ denotes the set of subsets of Δ.

As in [H-T2, 5.4], for $K \in \mathcal{I}$ we may choose a base point h_K in the open $G \times G$-orbit of X_K which maps to $\mathbf{h}_{P(K)}$. By [H-T2, Proposition 5.3], X_K is then naturally isomorphic to $(G \times G) \times_{P^-_{\Delta\setminus J} \times P_{\Delta\setminus J}} \overline{L_{\Delta\setminus J} \cdot h_K}$, where $J = P(K)$ and $\overline{L_{\Delta\setminus J} \cdot h_K}$ is a toroidal embedding of a quotient ${}^{(L_{\Delta\setminus J})}/_{H}$ by some subgroup H of the center of $L_{\Delta\setminus J}$.

For $K \in \mathcal{I}$ and $w \in W^{\Delta\setminus p(K)}$, we then define

$$X_{K,w} = \mathrm{diag}(G)(Bw, 1) \cdot h_K,$$

and call $X_{K,w}$ a G-stable piece of X. One can then show, in the same way as in [He2, 4.3], that

$$X = \bigsqcup_{\substack{K \in \mathcal{I} \\ w \in W^{\Delta\setminus P(K)}}} X_{K,w}.$$

Also similar to the proof of [He2, Theorem 4.5], for any $B \times B$-orbit closure V in X, the G-Schubert variety $\mathrm{diag}(G) \cdot V$ is a finite union of G-stable pieces. In particular, G-Schubert varieties are closures of G-stable pieces.

9. Frobenius splitting of G-Schubert varieties

In this section, we assume that X is an equivariant embedding of G. Let G_{sc} denote a simply connected cover of the semisimple commutator subgroup (G, G) of G. We fix a Borel subgroup B_{sc} of G_{sc} which is compatible with the Borel subgroup B in G. Similarly we fix a maximal torus $T_{\mathrm{sc}} \subset B_{\mathrm{sc}}$.

Let $X_1, \dots, X_n$ denote the boundary divisors of X. The closure within X of the $B \times B$-orbit $Bs_jw_0B \subset G$ will be denoted by D_j. Then D_j is of codimension 1 in X. The translate $(w_0, w_0)D_j$ of D_j will be denoted by $\tilde{D}_j$.

By earlier work we know

Theorem 9.1. [H-T2, Prop.7.1] *The equivariant embedding X admits a $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-canonical Frobenius splitting which compatibly Frobenius splits the closure of every $B \times B$-orbit.*

As a direct consequence of Theorem 7.2 we then obtain

Corollary 9.2. *The variety $(G \times G) \times_{(B\times B)} X$ admits a* $\mathrm{diag}(B_{\mathrm{sc}})$*-canonical Frobenius splitting which is compatible with all subvarieties of the form $(G \times G) \times_{(B\times B)} Y$ and $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} Y$, for a $B \times B$-orbit closure Y in X.*

Proposition 9.3. *The equivariant embedding X admits a* $\mathrm{diag}(B_{\mathrm{sc}})$*-canonical Frobenius splitting which compatibly splits all G-Schubert varieties in X.*

Proof. By Corollary 9.2 the variety $Z = \mathrm{diag}(G) \times_{\mathrm{diag}(B)} X$ admits a $\mathrm{diag}(B_{\mathrm{sc}})$-canonical Frobenius splitting which is compatible with all subvarieties of the form $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} Y$, with Y denoting a $B \times B$-orbit closure in X. As X is a $\mathrm{diag}(G)$-stable we may identify Z with ${}^{G}/_{B} \times X$ using the isomorphism

$$G \times_B X \to {}^{G}/_{B} \times X,$$

$$[g, x] \mapsto (gB, gx).$$

In particular, we see that the morphism

$$\pi : Z = \mathrm{diag}(G) \times_{\mathrm{diag}(B)} X \to X,$$

$$[g, x] \mapsto g \cdot x,$$

is projective and that $\pi_*(\mathcal{O}_Z) = \mathcal{O}_X$. As a consequence (see Section 3.8) the $\mathrm{diag}(B_{\mathrm{sc}})$-canonical Frobenius splitting of Z induces a $\mathrm{diag}(B_{\mathrm{sc}})$-canonical Frobenius splitting of X which is compatible with all subvarieties of the form

$$\pi(\mathrm{diag}(G) \times_{\mathrm{diag}(B)} Y) = \mathrm{diag}(G) \cdot Y,$$

i.e. with all the G-Schubert varieties in X. This ends the proof. $\square$

As a direct consequence of Proposition 9.3, we conclude the following vanishing result (see [B-K, Theorem 1.2.8]).

Corollary 9.4. *Let X denote a projective equivariant embedding of G. Let $\mathcal{X}$ denote a G-Schubert variety in X and let $\mathcal{L}$ denote an ample line bundle on $\mathcal{X}$. Then*

$$\mathrm{H}^i(\mathcal{X}, \mathcal{L}) = 0, i > 0.$$

Moreover, if $\tilde{\mathcal{X}} \subset \mathcal{X}$ is another G-Schubert variety, then the restriction map

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}) \to \mathrm{H}^0(\tilde{\mathcal{X}}, \mathcal{L}),$$

is surjective.

Later (i.e. Cor. 10.5) we will generalize the vanishing part of this result to nef line bundle.

9.1. F-splittings along ample divisors. In this subsection we assume that X is toroidal. The following structural properties of toroidal embeddings can all be found in [B-K, Sect.6.2]. Let X_0 denote the complement in X of the union of the subsets $\overline{Bs_iB^-}$ for $i \in \Delta$. If we let $\bar{T}$ denote the closure of T in X, then X_0 admits a decomposition defined by the following isomorphism

$$U \times U^- \times (\bar{T} \cap X_0) \to X_0, \ (x, y, z) \mapsto (x, y) \cdot z. \tag{25}$$

Moreover, every $G \times G$-orbit in X intersects $(\bar{T} \cap X_0)$ in a unique orbit under the left action of T. Notice here that as T is commutative the $T \times T$-orbits and the (left) T-orbit in $\overline{T}$ will coincide.

Lemma 9.5. *Let X denote a projective toroidal equivariant embedding of G and let Y denote a $G \times G$-orbit closure in X. Let K denote the subset of $\{1, \dots, n\}$ consisting of those j such that Y is contained in the boundary component X_j. Then*

$$Y \cap (\bigcup_{j \notin K} X_j \cup \bigcup_{i \in \Delta} (1, w_0) D_i),$$

has pure codimension 1 in Y and contains the support of an ample effective Cartier divisor on Y.

Proof. Let $X^K = \cup_{j \notin K} X_j$. We claim that $Y \setminus X^K$ coincides with the open $G \times G$-orbit Y_0 of Y. Clearly Y_0 is contained in $Y \setminus X^K$. On the other hand, let U be a $G \times G$-orbit in $Y \setminus X^K$. Then X_j contains U if and only if $j \notin K$. But every $G \times G$-orbit closure in X is the intersection of those X_j which contain it [B-K, Prop.6.2.3]. It follows that the closure of Y_0 and U coincide and thus $U = Y_0$.

As X is normal we may choose a $G \times G$-linearized very ample line bundle $\mathcal{L}$ on X. Then $\mathrm{H}^0(Y, \mathcal{L})$ is a finite dimensional (nonzero) representation of $G \times G$, and it thus contains a nonzero element v which is $B \times B^-$-invariant up to constants. The support of v is then the union of $B \times B^-$-invariant divisors on Y. As $Y_0 \cap (\bar{T} \cap X_0)$ is a single $T \times T$-orbit it follows that

$$Y_0 \cap X_0 \simeq U \times U^- \times (Y_0 \cap (\bar{T} \cap X_0)),$$

is an affine variety and a single $B \times B^-$-orbit. In particular, the support of v is contained in

$$Y \setminus (Y_0 \cap X_0) = Y \cap (X^K \cup \bigcup_{i \in \Delta} (1, w_0) D_i).$$

This shows the second part of the statement. The first part follows as $Y_0 \cap X_0$ is affine [Har, Prop.3.1]. $\square$

Let now X denote a smooth projective toroidal embedding of G. As the line bundles $\mathcal{O}_X(D_i)$ and $\mathcal{O}_X(\tilde{D}_i)$ are isomorphic it follows by [B-K, Prop.6.2.6] that

$$\omega_X^{-1} \simeq \mathcal{O}_X\big(\sum_{i \in \Delta} (D_i + \tilde{D}_i) + \sum_{j=1}^{n} X_j\big). \tag{26}$$

Recall that a X is normal and G is semisimple and simply connected, any line bundle on X will admit a unique $G_{\mathrm{sc}}^2 = G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-linearization. In particular, if we let τ_i denote the canonical section of the line bundle $\mathcal{O}_X(D_i)$, then we may consider τ_i as a $B_{\mathrm{sc}}^2 = B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-eigenvector of the space of global sections of $\mathcal{O}_X(D_i)$. As in the proof of [B-K, Prop.6.1.11] we find that the associated weight of τ_i equals $\omega_i \boxtimes -w_0 \omega_i$, where ω_i denotes the i-th fundamental weight. Similarly, we may consider the canonical section σ_j of $\mathcal{O}_X(X_j)$ as a G_{sc}^2-invariant element.

Let V denote a $B \times B$-orbit closure in X. As V is $B \times B$-stable the subset $Y = (G \times G) \cdot V$ is closed in X. Thus we may consider Y as the smallest $G \times G$-invariant subvariety of X containing V. Now define K as in Lemma 9.5 and let $\mathcal{M}$ denote the line bundle

$$\mathcal{M} = \mathcal{O}_X\big((p-1)(\sum_{i \in \Delta} \tilde{D}_i + \sum_{j \notin K} X_j)\big).$$

By Equation (26) and Lemma 3.6 it then follows that multiplication with τ_i^{p-1}, for $i \in \Delta$, and σ_j^{p-1}, for $j \in K$, defines a morphism of B_{sc}^2-linearized line bundles

$$\mathcal{M} \to \mathcal{E}nd_F^!\big(X, \{D_i, X_j\}_{i \in \Delta, j \in K}\big) \otimes k_{\lambda \boxtimes \lambda},$$

where $\lambda = (1-p)\rho$. By [H-T2, Prop.6.5] and Lemma 3.1 any element in $\mathcal{E}nd_F^!(X)$ which is compatible with the closed subvarieties D_i, $i \in \Delta$, and X_j, $j \in K$, is also compatible with V and Y. In particular, we have defined a B_{sc}^2-equivariant map

$$\eta : \mathcal{M} \to \mathcal{E}nd_F^!(X, Y, V\big) \otimes k_{\lambda \boxtimes \lambda}, \tag{27}$$

which, by Lemma 3.5, is the same as a B_{sc}^2-invariant element η' in $\mathrm{End}_F^{\mathcal{M}}\big(X, Y, V\big) \otimes k_{\lambda \boxtimes \lambda}$. In particular, this defines us an element

$$v \in \mathrm{Ind}_{B_{\mathrm{sc}}^2}^{G_{\mathrm{sc}}^2}\big(\mathrm{End}_F^{\mathcal{M}}\big(X, Y, V\big) \otimes k_{\lambda \boxtimes \lambda}\big), \tag{28}$$

which is G_{sc}^2-invariant. We are then ready to use the ideas explained in Section 5.7. First we use (18) to construct a morphism

$$\mathrm{End}_F^{\mathcal{L} \boxtimes \mathcal{L}}\big(({}^{G_{\mathrm{sc}}}/_{B_{\mathrm{sc}}})^2\big) \otimes \mathcal{M}(X) \to \mathrm{End}_F\big(G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} X\big), \tag{29}$$

$$(u, \sigma) \mapsto \Phi_{\mathcal{M}, \lambda \boxtimes \lambda}(u \otimes v \otimes \sigma),$$

where $\mathcal{L}$ is the G_{sc}-linearized line bundle on ${}^{G_{\mathrm{sc}}}/_{B_{\mathrm{sc}}}$ associated to the character $\lambda = (1-p)\rho$. Notice that we here have used that $\mathcal{M}(X)$ is a G_{sc}^2-module.

Lemma 9.6. *There exists a G_{sc}^2-equivariant map*

$$\mathrm{St} \boxtimes \mathrm{St} \to \mathcal{M}(X), \tag{30}$$

which maps the $B_{\mathrm{sc}}^- \times B_{\mathrm{sc}}^-$-invariant line in $\mathrm{St} \boxtimes \mathrm{St}$ *to a nonzero multiple of the global section*

$$\tilde{\sigma} = \prod_{i \in \Delta} \tilde{\tau}_i^{p-1} \prod_{j \notin K} \sigma_j^{p-1} \in \mathcal{M}(X),$$

where $\tilde{\tau}_i$ denotes the canonical section of $\mathcal{O}_X(\tilde{D}_i)$.

Proof. As $\mathcal{O}_X(\tilde{D}_i)$ and $\mathcal{O}_X(D_i)$ are isomorphic as line bundles we may consider the element

$$\sigma = \prod_{i \in \Delta} \tau_i^{p-1} \prod_{j \notin K} \sigma_j^{p-1}$$

as a global section of $\mathcal{M}$. Then σ is a $B_{\rm sc}^2$-eigenvector in $\mathcal{M}(X)$ of weight $(p-1)\rho \boxtimes (p-1)\rho$. In particular, σ induces a $B_{\rm sc} \times B_{\rm sc}$-equivariant map

$$k_{(p-1)\rho} \boxtimes k_{(p-1)\rho} \to \mathcal{M}(X).$$

Applying Frobenius reciprocity and the selfduality of the Steinberg module St, this defines the desired map

$$\mathrm{St} \boxtimes \mathrm{St} \to \mathcal{M}(X),$$

with the stated properties. □

Combining the map (29) with the map (30) in Lemma 9.6 we obtain a $G_{\rm sc}^2$-equivariant map

$$\Theta : \mathrm{End}_F^{\mathcal{L}\boxtimes\mathcal{L}}\big(({}^{G_{\rm sc}}/_{B_{\rm sc}})^2\big) \otimes \big(\mathrm{St} \boxtimes \mathrm{St}\big) \to \mathrm{End}_F\big(G_{\rm sc}^2 \times_{B_{\rm sc}^2} X\big), \tag{31}$$

We will now study when the map (31) describes a Frobenius splitting of $G_{\rm sc}^2 \times_{B_{\rm sc}^2} X$. Consider the $G_{\rm sc}^2$-equivariant map

$$\mathcal{M}(X) \to \mathrm{St} \boxtimes \mathrm{St}, \tag{32}$$

$$\sigma \mapsto \Phi^2_{\mathcal{M},\lambda\boxtimes\lambda}(v \otimes \sigma),$$

defined as the map (19) in Section 5.7. We claim

Lemma 9.7. *The composition of the map (30) in Lemma 9.6 and the map in (32) is an isomorphism on* $\mathrm{St} \boxtimes \mathrm{St}$.

Proof. By Frobenius reciprocity it suffices to show that the described composed map is nonzero. In particular, it suffices to show that

$$\Phi^2_{\mathcal{M},\lambda\boxtimes\lambda}(v \otimes \tilde{\sigma}) \neq 0,$$

where $\tilde{\sigma}$ denotes the global section of $\mathcal{M}$ defined in Lemma 9.6. For this we use the fact that the global section

$$\Big(\prod_{i \in \Delta} (\tau_i \tilde{\tau}_i) \prod_{j=1}^{n} \sigma_j\Big)^{p-1},$$

of ω_X^{1-p} defines a Frobenius splitting of X (see e.g. [B-K, proof of Thm.6.2.7]). As a consequence $\eta(\tilde{\sigma})$ is a Frobenius splitting of X, where η is the map defined in (27). Equivalently , the natural $G_{\rm sc}^2$-equivariant morphism

$$\mathrm{End}_F^{\mathcal{M}}(X) \otimes \mathcal{M}(X) \to k[X'] = k,$$

defined in (13), will map $\eta' \otimes \tilde{\sigma}$ to 1. This induces a commutative diagram

$$\begin{tikzcd} \mathrm{Ind}_{B_{\rm sc}^2}^{G_{\rm sc}^2}\big(\mathrm{End}_F^{\mathcal{M}}\big(X\big) \otimes k_{\lambda\boxtimes\lambda}\big) \otimes \mathcal{M}(X) \arrow[r, "\Phi^2_{\mathcal{M},\lambda\boxtimes\lambda}"] \arrow[d] \arrow[dr] & \mathrm{St} \otimes \mathrm{St} \arrow[d] \\ \mathrm{End}_F^{\mathcal{M}}(X) \otimes k_{\lambda\boxtimes\lambda} \otimes \mathcal{M}(X) \arrow[r] & k_{\lambda\boxtimes\lambda} \end{tikzcd} \tag{33}$$

where the image of $v \otimes \tilde{\sigma}$ under the diagonal map is nonzero. This ends the proof. $\square$

Proposition 9.8. *Let Θ denote the map defined in (31). The image $\Theta(\nu)$ of an element ν defines, up to a nonzero constant, a Frobenius splitting of $G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} X$ if and only if the image of ν under the map*

$$\phi_{\lambda \boxtimes \lambda} : \mathrm{End}_F^{\mathcal{L} \boxtimes \mathcal{L}}\big((G_{\mathrm{sc}}/B_{\mathrm{sc}})^2\big) \otimes \big(\mathrm{St} \boxtimes \mathrm{St}\big) \to k, \tag{34}$$

defined in Section 5.6, is nonzero.

Proof. Apply Proposition 5.7 and Lemma 9.7. $\square$

With the identification $\mathrm{End}_F^{\mathcal{L} \boxtimes \mathcal{L}}\big((G_{\mathrm{sc}}/B_{\mathrm{sc}})^2\big) \simeq \mathrm{St} \boxtimes \mathrm{St}$ the map $\phi_{\lambda \boxtimes \lambda}$, defined in (34), must necessarily (up to a nonzero constant) be the G_{sc}^2-invariant form on $\mathrm{St} \boxtimes \mathrm{St}$ mentioned in Section 6.1. Let v_Δ denote the $\mathrm{diag}(G)$-invariant element in $\mathrm{End}_F^{\mathcal{L} \boxtimes \mathcal{L}}\big((G_{\mathrm{sc}}/B_{\mathrm{sc}})^2\big)$ defined in Section 7.1. Then the $\mathrm{diag}(G)$-equivariant map

$$\mathrm{St} \otimes \mathrm{St} \to k,$$
$$\nu \mapsto \phi_{\lambda \boxtimes \lambda}(v_\Delta \otimes \nu),$$

is nonzero and thus it must coincide (up to a nonzero constant) with the G_{sc}-invariant form ϕ on St defined in (24).

Proposition 9.9. *Fix notation as above and let D denote the effective Cartier divisor*

$$(p-1)\Big(\sum_{i \in \Delta}(1, w_0) D_i + \sum_{j \notin K} X_j\Big),$$

on X. Then X admits a Frobenius D-splitting which is compatible with the subvariety Y and the G-Schubert variety $\mathrm{diag}(G) \cdot V$.

Proof. Consider the $\mathrm{diag}(G)$-equivariant morphism

$$\Theta_\Delta : \mathrm{St} \boxtimes \mathrm{St} \to \mathrm{End}_F\big(G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} X\big),$$
$$\nu \mapsto \Theta(v_\Delta \otimes \nu),$$

where Θ is the map in (31). By Lemma 9.8 the image $\Theta_\Delta(\nu)$ of an element $\nu \in \mathrm{St} \otimes \mathrm{St}$ is a Frobenius splitting, up to a nonzero constant, if and only if $\phi(\nu)$ is nonzero. Here ϕ is the the map defined in (24).

Let v_+ (resp. v_-) denote a nonzero B (resp. B^-)-eigenvector of St and let $\nu = v_+ \otimes v_-$. After possibly multiplying v_+ with a constant we may assume that $s = \Theta_\Delta(\nu)$ defines a Frobenius splitting of $Z = G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} X$. As v is compatible with Y and V (cf. (28)) it follows by Theorem 5.6 and Lemma 7.1 that s factorizes as

$$s : (F_Z)_* \mathcal{O}_Z \xrightarrow{(F_Z)_* \sigma} (F_Z)_* \mathcal{M}_Z \xrightarrow{s_1} \mathcal{O}_{Z'}, \tag{35}$$

where s_1 is compatible with the subvarieties $G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} V$, $G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} Y$ and $\mathrm{diag}(G_{\mathrm{sc}}) \times_{\mathrm{diag}(B_{\mathrm{sc}})} X$. Here $\mathcal{M}_Z$ is the G_{sc}^2-linearized line bundle on Z

associated with the B_{sc}^2-linearized line bundle $\mathcal{M}$ on X as explained in Section 5.2, and σ is the global section of $\mathcal{M}_Z$ defined as the image of ν under the map (30) in Lemma 9.6. Notice that as $\mathcal{M}$ is a G_{sc}^2-linearized line bundle on the G_{sc}^2-variety X we may identify the global sections of $\mathcal{M}$ and $\mathcal{M}_Z$. Actually , as X is a G_{sc}^2-variety the morphism

$$G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} X \to {}^{G_{\mathrm{sc}}}\!/\!_{B_{\mathrm{sc}}} \times {}^{G_{\mathrm{sc}}}\!/\!_{B_{\mathrm{sc}}} \times X,$$
$$[(g_1, g_2), x] \mapsto (g_1 B, g_2 B, (g_1, g_2) \cdot x),$$

is an isomorphism. Moreover, under this isomorphism, the line bundle $\mathcal{M}_Z$ is just the pull back of $\mathcal{M}$ under projection p_X on the third coordinate. Thus, by Lemma 9.6 it follows that σ is the pull back from X of the effective Cartier divisor

$$D = (p-1)\big(\sum_{i\in\Delta}(1, w_0)D_i + \sum_{j\notin K} X_j\big).$$

Applying the functor $(p_X)_*$ to (35) we obtain the Frobenius D-splitting

$$(p_X)_* s : (F_X)_* \mathcal{O}_X \xrightarrow{(F_X)_* \sigma_D} (F_X)_* \mathcal{O}(D) \xrightarrow{(p_X)_* s_1} \mathcal{O}_{X'}$$

of X where $(p_X)_* s_1$ is compatible with the subvarieties $p_X(G_{\mathrm{sc}}^2 \times_{B_{\mathrm{sc}}^2} Y) = Y$ and $p_X(\mathrm{diag}(G_{\mathrm{sc}}) \times_{\mathrm{diag}(B_{\mathrm{sc}})} V) = \mathrm{diag}(G) \cdot V$ (by Lemma 3.7). This ends the proof. □

Corollary 9.10. *Let $\mathcal{X}$ denote a G-Schubert variety in a smooth projective toroidal embedding of a reductive group G. Then $\mathcal{X}$ admits a stable Frobenius splitting along an ample divisor.*

Proof. Apply Proposition 9.9, Lemma 9.5 and Lemma 3.3. □

10. Cohomology of line bundles

The main aim of this section is to obtain a generalizing the vanishing part of Corollary 9.4 to nef line bundles. The concept of a rational morphism is here central and for this we use [B-K, Sect.3.3] as a general reference. First we recall :

Definition 10.1. A morphism $f : Y \to Z$ of varieties is a called a *rational morphism* if the induced map $f^\sharp : \mathcal{O}_Z \to f_* \mathcal{O}_Y$ is an isomorphism and $\mathrm{R}^i f_* \mathcal{O}_Y = 0$, $i > 0$.

The following criterion for a morphism to be rational will be very useful ([R, Lem.2.11]).

Lemma 10.2. *Let $f : Y \to Z$ denote a projective morphism of irreducible varieties and let $\hat{Y}$ denote a closed irreducible subvariety of Y. Consider the image $\hat{Z} = f(\hat{Y})$ as a closed subvariety of Z. Let $\mathcal{L}$ denote an ample line bundle on Z and assume*

(1) $f^\sharp : \mathcal{O}_Z \to f_* \mathcal{O}_Y$ is an isomorphism.
(2) $\mathrm{H}^i(Y, f^* \mathcal{L}^n) = \mathrm{H}^i(\hat{Y}, f^* \mathcal{L}^n) = 0$, for $i > 0$ and $n \gg 0$.

(3) The restriction map $\mathrm{H}^0(Y, f^*\mathcal{L}^n) \to \mathrm{H}^0(\hat{Y}, f^*\mathcal{L}^n)$ is surjective for $n \gg 0$.

Then the induced map $\hat{f} : \hat{Y} \to \hat{Z}$ is a rational morphism.

10.1. Toric variety. An equivariant embedding Z of the (reductive) group T is called a toric variety (wrt. T). Notice that, as T is commutative, we may consider the $T \times T$-action on Z as just a T-action. The following result should be well known but, as we do not know a good reference, we include a proof.

Lemma 10.3. *Let $f : Y \to Z$ denote a projective surjective morphism of equivariant embeddings of T. Let $T \cdot z$ denote a T-orbit in Z and let $T \cdot y$ denote a T-orbit in $f^{-1}(T \cdot z)$ of minimal dimension. Then the map $T \cdot y \to T \cdot z$, induced by f, is an isomorphism.*

Proof. Let $\overline{T \cdot z}$ and $\overline{T \cdot y}$ denote the closures of $T \cdot z$ and $T \cdot y$ in Z and Y respectively. Then the induced map

$$\hat{f} : \overline{T \cdot y} \to \overline{T \cdot z},$$

is a projective morphism. Moreover, by the minimality assumption on $T \cdot y$, the inverse image $\hat{f}^{-1}(T \cdot z)$ equals $T \cdot y$. In particular, the induced morphism : $T \cdot y \to T \cdot z$ is projective. But any T-orbit in a toric variety (wrt. to T) is isomorphic to a torus T_1 satisfying that the cokernel of the induced map of character groups $X^*(T_1) \to X^*(T)$ is a free abelian group ([Ful, Sect.3.1]). In particular, the varieties $T \cdot y$ and $T \cdot z$ are tori and the cokernel of the induced map of character groups $X^*(T \cdot z) \to X^*(T \cdot y)$ is a free abelian group. But $T \cdot y \to T \cdot z$ is an affine projective morphism and thus it must be a finite morphism. Thus the cokernel of $X^*(T \cdot z) \to X^*(T \cdot y)$ is a finite group and, as it is already a free group, it must be trivial. This ends the proof as tori are determined by their character groups. □

Lemma 10.4. *Let X denote a projective embedding of a reductive group G and let Y denote a $G \times G$-orbit closure of X. Then there exists a smooth toroidal embedding $\hat{X}$ of G, a projective G-equivariant morphism $f : \hat{X} \to X$ and a $G \times G$-orbit closure $\hat{Y}$ in $\hat{X}$ such that the induced morphism $f : \hat{Y} \to Y$ is a rational morphism.*

Proof. Assume first that X is toroidal. By [B-K, Prop.6.2.5] there exists a smooth toroidal embedding $\hat{X}$ of G with a projective morphism $f : \hat{X} \to X$. Let X_0 denote the open subset of X introduced in the beginning of Section 9.1, and let $\hat{X}_0$ denote the corresponding subset of $\hat{X}$. Then the inverse image $f^{-1}(X_0)$ coincides with $\hat{X}_0$ [B-K, Prop.6.2.3(i)]. Let $\overline{T}$ (resp. $\hat{T}$) denote the closure of T in X (resp. $\hat{X}$). Then $\overline{T}$ and $\hat{T}$ are toric varieties [B-K, Prop.6.2.3], and the induced map $f : \hat{T} \to \overline{T}$ is a projective morphism of toric varieties. Thus also the induced map

$$\hat{X}_0 \cap \hat{T} \to X_0 \cap \overline{T},$$

is a projective morphism of toric varieties. As mentioned in Section 9.1 every $G \times G$-orbit in X will intersect $X_0 \cap \overline{T}$ in a unique T-orbit. We let $T \cdot x$ denote the open T-orbit in the intersection of Y with $X_0 \cap \overline{T}$. By Lemma 10.3 we may find a T-orbit $T \cdot \hat{x}$ in $\hat{X}_0 \cap \hat{T}$ which by f is isomorphic to $T \cdot x$, and we then define $\hat{Y}$ to be the closure of the $G \times G$-orbit through $\hat{x}$. By the isomorphism (25) we then conclude that f induces a projective birational morphism $\hat{Y} \to Y$. By [H-T2, Cor.8.4] the orbit closure Y is normal and thus, by Zariski's main theorem, we conclude $f_* \mathcal{O}_{\hat{Y}} = \mathcal{O}_Y$. By Lemma 10.2 (used on the morphism $\hat{Y} \to Y$ and the closed non-proper subvariety $\hat{Y}$ of $\hat{Y}$) it now suffices to prove that

$$\mathrm{H}^i(\hat{Y}, f^* \mathcal{L}) = 0, \ i > 0,$$

for a very ample line bundle $\mathcal{L}$ on Y. This follows from [H-T2, Prop.7.2] and ends the proof in the case when X is toroidal.

Consider now an arbitrary projective equivariant embedding X of G. Let $\hat{X}$ denote the normalization of the closure of the image of the natural $G \times G$-equivariant embedding

$$G \to X \times \mathbf{X},$$

where $\mathbf{X}$ denotes the wonderful compactification of G_{ad}. Then $\hat{X}$ is a toroidal embedding of G with an induced projective equivariant morphism $f : \hat{X} \to X$. Let $\hat{Y}$ denote any $G \times G$-orbit closure in $\hat{X}$. Then $f : \hat{Y} \to f(\hat{Y})$ is a rational morphism [H-T2, Lem.8.3]. In particular, we may find a $G \times G$-orbit closure $\hat{Y}$ of $\hat{X}$ with an induced rational morphism $f : \hat{Y} \to Y$. Finally we may apply the first part of the proof to $\hat{Y}$ and $\hat{X}$ and use that a composition of rational morphisms is again a rational morphism. □

Corollary 10.5. *Let X denote a projective embedding of a reductive group G and let $\mathcal{X}$ denote a G-Schubert variety in X. Let $Y = (G \times G) \cdot \mathcal{X}$ denote the minimal $G \times G$-orbit closure of X containing $\mathcal{X}$. When $\mathcal{L}$ is a nef line bundle on $\mathcal{X}$ then*

$$\mathrm{H}^i(\mathcal{X}, \mathcal{L}) = 0, \ i > 0.$$

Moreover, when $\mathcal{L}$ is a nef line bundle on Y then the restriction morphism

$$\mathrm{H}^0(Y, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L}),$$

is surjective.

Proof. Assume first that X is smooth and toroidal. Then by Proposition 9.9, Lemma 9.5 and Lemma 3.3 the variety Y admits a stable Frobenius splitting along an ample divisor which is compatibly with $\mathcal{X}$. Thus the statement follows in this case by Proposition 3.4.

Let now X denote an arbitrary projective equivariant embedding of G. Choose, using Lemma 10.4, a smooth projective toroidal embedding $\hat{X}$ with a projective equivariant morphism $f : \hat{X} \to X$ onto X, and a $G \times G$-orbit closure $\hat{Y}$ in $\hat{X}$ with an induced rational morphism onto Y. Let V denote a $B \times B$-orbit closure in Y such that $\mathcal{X} = \mathrm{diag}(G) \cdot V$. As Y is the minimal $G \times G$-orbit closure containing $\mathcal{X}$ it follows that V will intersect the open $G \times G$-orbit of Y. In particular, there exists a $B \times B$-orbit closure $\hat{V}$ in $\hat{X}$ which intersects the open $G \times G$-orbit of $\hat{Y}$ and which maps onto V. In particular,

$$\hat{\mathcal{X}} := \mathrm{diag}(G) \cdot \hat{V},$$

is a G-Schubert variety in $\hat{X}$ which by f maps onto $\mathcal{X}$. Moreover, $\hat{Y}$ is the minimal $G \times G$-orbit closure containing $\hat{\mathcal{X}}$.

We claim that the induced morphism $\hat{\mathcal{X}} \to \mathcal{X}$ is a rational morphism. To prove this we apply Lemma 10.2 to the rational morphism $f : \hat{Y} \to Y$. Choose an ample line bundle $\mathcal{M}$ on Y. Then it suffices to prove that

$$\mathrm{H}^i(\hat{Y}, f^*\mathcal{M}^n) = \mathrm{H}^i(\hat{\mathcal{X}}, f^*\mathcal{M}^n) = 0,\ i > 0,\ n > 0, \tag{36}$$

and that the restriction map

$$\mathrm{H}^0(\hat{Y}, f^*\mathcal{M}^n) \to \mathrm{H}^0(\hat{X}, f^*\mathcal{M}^n), \tag{37}$$

is surjective for $n > 0$. But $\mathcal{M}^n$ is an ample, and thus nef, line bundle on Y and therefore the pull back $f^*\mathcal{M}^n$ is a nef line bundle on $\hat{Y}$ ([Laz, Ex. 1.4.4]). As $\hat{X}$ is smooth and toroidal, the conclusion of the first part of this proof then shows that conditions (36) and (37) are satisfied.

Now both $\hat{\mathcal{X}} \to \mathcal{X}$ and $\hat{Y} \to Y$ are rational morphisms. In particular, we have identifications

$$\mathrm{H}^i(\hat{Y}, f^*\mathcal{L}) \simeq \mathrm{H}^i(Y, \mathcal{L}),\ i \geq 0,$$

$$\mathrm{H}^i(\hat{\mathcal{X}}, f^*\mathcal{L}) \simeq \mathrm{H}^i(\mathcal{X}, \mathcal{L}),\ i \geq 0,$$

for any line bundle $\mathcal{L}$ on Y or, in the second equation, on X. When $\mathcal{L}$ is a nef line bundle the pull back $f^*\mathcal{L}$ is also nef ([Laz, Ex. 1.4.4]). Thus as we have already completed the proof of the statement for smooth toroidal embeddings, in particular for $\hat{X}$, this now ends the proof. □

By the proof of the above result we also find that any G-Schubert variety $\mathcal{X}$ in a projective equivariant embedding of G, will admit a G-equivariant rational morphism $f : \hat{\mathcal{X}} \to \mathcal{X}$ by a G-Schubert variety $\hat{\mathcal{X}}$ of some smooth projective toroidal embedding of G.

Remark 10.6. *When $X = \mathbf{X}$ is the wonderful compactification of a group G of adjoint type and $\mathcal{L}$ is a nef line bundle on $\mathbf{X}$, then the restriction morphism*

$$\mathrm{H}^0(\mathbf{X}, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L}),$$

to any closed $G \times G$-stable irreducible subvariety Y of $\mathbf{X}$ is surjective. In particular, also the restriction morphism

$$\mathrm{H}^0(\mathbf{X}, \mathcal{L}) \to \mathrm{H}^0(\mathfrak{X}, \mathcal{L}),$$

to any G-Schubert variety $\mathfrak{X}$ is surjective by the above result. We do not know if the latter is true for arbitrary equivariant embeddings.

11. Normality questions

The obtained Frobenius splitting properties of G-Schubert varieties in Section 9 and the cohomology vanishing results in Corollary 10.5 should be expected to have strong implications on the geometry of these varieties. However, in this section we provide an example of a G-Schubert variety in the wonderful compactification of a group of type G_2 which is not even normal. In fact, it seems that there are plenty of such examples.

11.1. Some general theory. We keep the notations as in Section 8.5. For $J \subset \Delta$ and $w \in W^{\Delta \setminus J}$, we let $\overline{\mathbf{X}_{J,w}}$ denote the closure of $\mathbf{X}_{J,w}$ in $\mathbf{X}$. Let

$$K = \max\{K' \subset \Delta \setminus J; wK' \subset K'\}.$$

By [He2, Prop. 1.12], we have a diag(G)-equivariant isomorphism

$$\mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} (P_K \dot{w}, P_K)\mathbf{h}_J \simeq \mathbf{X}_{J,w}$$

induced by the inclusion of $(P_K \dot{w}, P_K)\mathbf{h}_J$ in $\mathbf{X}$. Let V denote the closure of $(P_K \dot{w}, P_K)\mathbf{h}_J$ within $\mathbf{X}$. Then V is the closure of a $B \times B$-orbit and we find that the induced map

$$f : \mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} V \to \overline{\mathbf{X}_{J,w}}, \tag{38}$$

is a birational and projective morphism. Thus, by Zariski's Main Theorem, a necessary condition for $\overline{\mathbf{X}_{J,w}}$ to be normal is that the fibers of f are connected. Actually, in positive characteristic, connectedness of the fibers is also sufficient for $\overline{\mathbf{X}_{J,w}}$ to be normal. This follows as $\overline{\mathbf{X}_{J,w}}$ is Frobenius split (Prop. 9.3) and thus weakly normal [B-K, Prop.1.2.5].

11.2. An example of a non-normal closure. Let now, furthermore, G be a group of type G_2. Let α_1 denote the short simple root and α_2 denote the long simple root. The associated simple reflections are denoted by s_1 and s_2. Let $J = \{\alpha_2\}$ and $w = s_1 s_2 \in W^{\Delta \setminus J}$. In this case $K = \emptyset$ and we obtain a birational map

$$f : \mathrm{diag}(G) \times_{\mathrm{diag}(B)} V \simeq \overline{\mathbf{X}_{J,w}}$$

where V is the closure of $(B\dot{w}, B)\mathbf{h}_J$. By [Sp, Prop. 2.4], the part of V which intersect the open $G \times G$-orbit of $\mathbf{X}_J$ equals

$$\bigcup_{w \leq w'} (B\dot{w}', B)\mathbf{h}_J \cup \bigcup_{ws_1 \leq w'} (B\dot{w}', B\dot{s}_1)\mathbf{h}_J. \tag{39}$$

In particular, $x := (\dot{v}, 1)\mathbf{h}_J$ is an element of V, where $v = s_2 s_1 s_2$. We claim that the fiber of f over x is not connected. To see this let y denote a point in the fiber over x. Then we may find $g \in G$ and $\tilde{x} \in V$ such that

$$y = [g, \tilde{x}].$$

By (39), $\tilde{x} = (b\dot{w}', b')\mathbf{h}_J$ for some $b \in B$, $b' \in P_{\Delta \setminus J}$ and $w' \geq w$. Then

$$(gb\dot{w}', gb')\mathbf{h}_J = (\dot{v}, 1)\mathbf{h}_J.$$

It follows that $(\dot{v}^{-1} gb\dot{w}', gb')$ lies in the stabilizer of $\mathbf{h}_J$. In particular, $gb' \in P_{\Delta \setminus J}$ and thus also $g \in P_{\Delta \setminus J}$. If $g \in B$ then $y = [1, x]$. So assume that $g = u_1(t)\dot{s}_1$ where u_1 is the root homomorphism associated to α_1. Assume that $t \neq 0$. Then we may find $b_1 \in B$ and $s \in k$ such that $g = u_{-1}(s)b_1$ where u_{-1} is the root homomorphism associated to $-\alpha_1$. Thus

$$\begin{aligned} \tilde{x} &= (g^{-1}, g^{-1})(\dot{v}, 1)\mathbf{h}_J \\ &= (b_1^{-1} u_{-1}(-s)\dot{v}, g^{-1})\mathbf{h}_J \\ &= (b_1^{-1}\dot{v}, g^{-1})\mathbf{h}_J \\ &\in (B\dot{v}, B\dot{s}_1)\mathbf{h}_J \end{aligned}$$

where the third equality follows as $\dot{v}^{-1} u_{-1}(-s)\dot{v}$ is contained in the unipotent radical of $P^-_{\Delta \setminus J}$. But $(B\dot{v}, B\dot{s}_1)\mathbf{h}_J$ has empty intersection with V (by (39)) which contradicts the assumption that $t \neq 0$. It follows that the only possibilities for y are $[1, x]$ and $[\dot{s}_1, (\dot{s}_1^{-1}\dot{v}, \dot{s}_1^{-1})\mathbf{h}_J]$. As $(\dot{s}_1^{-1}\dot{v}, \dot{s}_1^{-1})$ is contained in V (by (39)) we conclude that the fiber of f over x consists of 2 points; in particular the fiber is not connected and thus $\overline{\mathbf{X}_{J,w}}$ is not normal.

Remark 11.1. *It seems likely that normalizations of G-Schubert varieties should have nice singularities : If we let $\mathcal{Z}_{J,w}$ denote the normalization of the closure of $\mathbf{X}_{J,w}$, then the map (38) induces a birational and projective morphism*

$$\tilde{f} : \mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} V \to \mathcal{Z}_{J,w}.$$

We expect that $\tilde{f}$ can be used to obtain global F-regularity of $\mathcal{Z}_{J,w}$ (see [S] *for an introduction to global F-regularity). In fact, by the results in* [H-T2] *the $B \times B$-orbit closure V is globally F-regular. Thus* $\mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} V$ *is locally strongly F-regular, and as*

$$\tilde{f}_* \mathcal{O}_{\mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} V} = \mathcal{O}_{\mathcal{Z}_{J,w}},$$

it seems likely that $\mathcal{Z}_{J,w}$ is also locally strongly F-regular. Moreover, similarly to Corollary 9.10 one may conclude that $\mathcal{Z}_{J,w}$ admits a stable Frobenius splitting along an ample divisor. Thus $\mathcal{Z}_{J,w}$ is globally F-regular if it is locally strongly F-regular. At the moment we do not know if $\mathcal{Z}_{J,w}$ is locally strongly F-regular.

12. Generalizations

Fix notation as in Section 2. An admissible triple of $G \times G$ is by definition a triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$ consisting of $J_1, J_2 \subset \Delta$, a bijection $\delta : J_1 \to J_2$ and an isomorphism $\theta_\delta : L_{J_1} \to L_{J_2}$ that maps T to T and the root subgroup U_{α_i} to the root subgroup $U_{\alpha_{\delta(i)}}$ for $i \in J_1$. To each admissible triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$, we associate the subgroup $\mathcal{R}_\mathcal{C}$ of $G \times G$ defined by

$$\mathcal{R}_\mathcal{C} = \{(p, q) : p \in P_{J_1}, q \in P_{J_2}, \theta_\delta(\pi_{J_1}(p)) = \pi_{J_2}(q)\},$$

where $\pi_J : P_J \to L_J$, for a subset $J \subset \Delta$, denotes the natural quotient map.

Let X denote an equivariant embedding of the reductive group G. A $\mathcal{R}_\mathcal{C}$-Schubert variety of X is then a subset of the form $\mathcal{R}_\mathcal{C} \cdot V$ for some $B \times B$-orbit closure V in X. When $G = G_{\mathrm{ad}}$ is a group of adjoint type and $X = \mathbf{X}$ is the associated wonderful compactification the set of $\mathcal{R}_\mathcal{C}$-Schubert varieties coincides with closures of the set of $\mathcal{R}_\mathcal{C}$-stable pieces. By definition [L-Y, section 7], a $\mathcal{R}_\mathcal{C}$-stable piece in the wonderful compactification $\mathbf{X}$ of G_{ad} is a subvariety of the form $\mathcal{R}_\mathcal{C} \cdot Y$, where $Y = (Bv_1, Bv_2) \cdot \mathbf{h}_J$ for some $J \subset \Delta$, $v_1 \in W^J$ and $v_2 \in {}^{J_2}W$ (notation as in Section 8.5). Notice that when $J_1 = J_2 = \Delta$ and θ_δ is the identity map then a $\mathcal{R}_\mathcal{C}$-stable piece is the same as a G-stable piece. On the other hand, when $J_1 = J_2 = \emptyset$, then a $\mathcal{R}_\mathcal{C}$-stable piece is the same as a $B \times B$-orbit. Moreover, any $\mathcal{R}_\mathcal{C}$-Schubert variety is a finite union of $\mathcal{R}_\mathcal{C}$-stable pieces [L-Y, Section 7].

The following is a generalization of Proposition 9.3 and Proposition 9.9.

Proposition 12.1. *Let $\mathcal{C} = (J_1, J_2, \theta_\delta)$ denote an admissible triple of $G \times G$ and let X denote an equivariant embedding of G. Then X admits a Frobenius splitting which compatible splits all $\mathcal{R}_\mathcal{C}$-Schubert varieties in X. If, moreover, X is a smooth, projective and toroidal embedding and $Y = X_K = (G \times G) \cdot V$, for some $B \times B$-orbit closure V in X, then X admits a Frobenius splitting along the Cartier divisor*

$$D = (p-1)\Big(\sum_{i \in \Delta} (w_0^{J_1}, 1)\tilde{D}_i + \sum_{j \notin K} X_j\Big),$$

which is compatibly with Y and $\mathcal{R}_\mathcal{C} \cdot V$.

Proof. As the proof is similar to the proof of Proposition 9.3 and Proposition 9.9 we only sketch the proof. In the following G_J, for a subset $J \subset \Delta$, denotes the commutator of the Levi subgroup in G_{sc} associated to J. The Borel subgroup $G_J \cap B_{\mathrm{sc}}$ of G_J is denoted by B_J. Define $X_\mathcal{C}$ to be the $G_{J_1}^2$-variety which as a variety is X but where the action is twisted by the morphism

$$G_{J_1} \times G_{J_1} \xrightarrow{\mathbf{1} \times \theta_\delta} G_{J_1} \times G_{J_2}.$$

Then the $B_{J_1} \times B_{J_2}$-canonical Frobenius splitting of X defined by Theorem 9.1 and Lemma 6.3 induces a $B_{J_1}^2$-canonical Frobenius splitting of $X_{\mathcal{C}}$. In particular, all subvarieties of $X_{\mathcal{C}}$ which corresponds to $B \times B$-orbit closures in X will be compatibly Frobenius split by this canonical Frobenius splitting. Now apply an argument as in the proof of Proposition 9.3 and use the identification of $\mathcal{R}_{\mathcal{C}} \cdot V \subset X$ with $\mathrm{diag}(G_{J_1}) \cdot V \subset X_{\mathcal{C}}$. This ends the proof of the first statement.

Assume now that X is a smooth, projective and toroidal embedding and consider the B_{sc}^2-equivariant morphism

$$\eta : \mathcal{M} \to \mathcal{E}nd_F^!(X, Y, V) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho},$$

defined in (27). Let $Y_{\mathcal{C}}$ and $V_{\mathcal{C}}$ be defined similar to $X_{\mathcal{C}}$. Then η induces a $B_{J_1}^2$-equivariant morphism

$$\eta_{\mathcal{C}} : \mathcal{M} \to \mathcal{E}nd_F^!(X_{\mathcal{C}}, Y_{\mathcal{C}}, V_{\mathcal{C}}) \otimes k_{(1-p)\rho_{J_1} \boxtimes (1-p)\rho_{J_1}}.$$

Similar to the definition of v in (28) we obtain from $\eta_{\mathcal{C}}$ an element

$$v_{\mathcal{C}} \in \mathrm{Ind}_{B_{J_1}^2}^{G_{J_1}^2} \big(\mathrm{End}_F(X_{\mathcal{C}}, Y_{\mathcal{C}}, V_{\mathcal{C}}) \otimes k_{(1-p)\rho_{J_1} \boxtimes (1-p)\rho_{J_1}}\big),$$

and from this a $G_{J_1}^2$-equivariant morphism

$$\mathrm{End}_F^{\mathcal{L}_{J_1} \boxtimes \mathcal{L}_{J_1}}\big(({}^{G_{J_1}}/_{B_{J_1}})^2\big) \otimes \mathcal{M}(X_{\mathcal{C}}) \to \mathrm{End}_F\big(G_{J_1}^2 \times_{B_{J_1}^2} X_{\mathcal{C}}\big), \tag{40}$$

similar to (29). Here $\mathcal{L}_{J_1}$ is the line bundle on ${}^{G_{J_1}}/_{B_{J_1}}$ associated to the character $(1-p)\rho_{J_1}$. Combining Lemma 6.3 and Lemma 9.6 we also obtain a map

$$\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1} \to \mathcal{M}(X_{\mathcal{C}}), \tag{41}$$

with properties similar to the ones described in Lemma 9.6. As in (32) we may also use $v_{\mathcal{C}}$ to construct a morphism

$$\mathcal{M}(X_{\mathcal{C}}) \to \mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1},$$

such that the composition with (41) is an isomorphism on $\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1}$. Finally we may construct

$$\Theta_{\mathcal{C}} : \mathrm{End}_F^{\mathcal{L}_{J_1} \boxtimes \mathcal{L}_{J_1}}\big(({}^{G_{J_1}}/_{B_{J_1}})^2\big) \otimes (\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1}) \to \mathrm{End}_F\big(G_{J_1}^2 \times_{B_{J_1}^2} X_{\mathcal{C}}\big),$$

similar to (31). In particular, a statement equivalent to Proposition 9.8 is satisfied for $\Theta_{\mathcal{C}}$. Let $v_+^{J_1}$ (resp. $v_-^{J_1}$) denote a highest (resp. lowest) weight vector in St_{J_1} and let $v_\Delta^{J_1}$ denote the $\mathrm{diag}(G_{J_1})$-invariant element of $\mathrm{End}_F^{\mathcal{L}_{J_1} \boxtimes \mathcal{L}_{J_1}}\big(({}^{G_{J_1}}/_{B_{J_1}})^2\big)$. Imitating the proof of Proposition 9.9 we then find that $\Theta_{\mathcal{C}}(v_\Delta^{J_1} \otimes (v_+^{J_1} \otimes v_-^{J_1}))$ is a Frobenius splitting of $G_{J_1}^2 \times_{B_{J_1}^2} X_{\mathcal{C}}$ (up to a nonzero constant). Moreover, the push forward of this Frobenius splitting to X has the desired properties. We only have to note that the effective Cartier associated to the image of $v_+^{J_1} \otimes v_-^{J_1}$ under the map (41) equals

$$D = (p-1)\big(\sum_{i \in \Delta} (w_0^{J_1}, 1)\tilde{D}_i + \sum_{j \notin K} X_j\big).$$

This ends the proof. $\square$

We may also argue as in Corollary 10.5 to obtain

Corollary 12.2. *Let X denote a projective embedding of a reductive group G and let V denote the closure of a $B \times B$-orbit in X. Let $Y = (G \times G) \cdot V$ and $\mathcal{X}_{\mathcal{C}} = \mathcal{R}_{\mathcal{C}} \cdot V$. When $\mathcal{L}$ is a nef line bundle on $\mathcal{X}_{\mathcal{C}}$ then*

$$\mathrm{H}^i(\mathcal{X}_{\mathcal{C}}, \mathcal{L}) = 0, \ i > 0.$$

Moreover, when $\mathcal{L}$ is a nef line bundle on Y then the restriction morphism

$$\mathrm{H}^0(Y, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{C}}, \mathcal{L}),$$

is surjective.

Remark 12.3. *In the case where $k = \mathbb{C}$ and X is the wonderful compactification, the subvarieties $(w_0^{J_1}, 1)\tilde{D}_i$, X_j and all the $\mathcal{R}_{\mathcal{C}}$-Schubert varieties are Poisson subvarieties with respect to the Poisson structure on X corresponding to the splitting*

$$\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) = l_1 \oplus l_2,$$

where $l_1 = \mathrm{Lie}(\mathcal{R}_{\mathcal{C}})$ and l_2 is a certain subalgebra of $\mathrm{Ad}(w_0^{J_1})\mathrm{Lie}(B^-) \oplus \mathrm{Lie}(B^-)$. *See* [L-Y2, 4.5].

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA

E-mail address: hugo@math.sunysb.edu

Institut for matematiske fag, Aarhus Universitet, 8000 Århus C, Denmark

E-mail address: funch@imf.au.dk