GOOD AND SEMI-STABLE REDUCTIONS OF SHIMURA VARIETIES

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Abstract. We study variants of the local models constructed by the second author and Zhu and consider corresponding integral models of Shimura varieties of abelian type. We determine all cases of good, resp. of semi-stable, reduction under tame ramification hypotheses.

Contents

1. Introduction

The problem of the reduction modulo p of a Shimura variety has a long and complicated history, perhaps beginning with Kronecker. The case of the modular curve (the Shimura variety associated to GL_2) is essentially solved after the work of Igusa, Deligne, Drinfeld and Katz-Mazur. In particular, it is known that the modular curve has good reduction at p if the level structure is prime to p. If the level structure is of $\Gamma_0(p)$-type (in addition to some level structure prime to p), then the modular curve has semi-stable reduction (one even has a global understanding of the reduction modulo p, as the union of two copies of the modular curve with level structure prime to p, crossing transversally at the set of supersingular points). Are there other level structures such that the reduction modulo p is good, resp. is semi-stable?

This is the question addressed in the present paper, in the context of general Shimura varieties. The question can be interpreted in two different ways. One can ask whether there exists *some* model over $\operatorname{Spec}\mathbb{Z}_{(p)}$ which has good, resp. semi-stable reduction. In the case of the modular curve, one can prove that, indeed, the two examples above exhaust

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all possibilities (this statement has to be interpreted correctly, by considering the natural compactification of the modular curve). This comes down to a statement about the spectral decomposition under the action of the Hecke algebra of the ℓ-adic cohomology of modular curves. Unfortunately, the generalization of this statement to other Shimura varieties seems out of reach at the moment.

The other possible interpretation of the question is to ask for good, resp. semi-stable, reduction of a specific class of p-integral models of Shimura varieties. Such a specific class has been established in recent years for Shimura varieties with level structure which is *parahoric* at p, the most general result being due to M. Kisin and the second author [29]. The main point of these models is that their singularities are modeled by their associated *local models*, cf. [39]. These are projective varieties which are defined in a certain sense by linear algebra, cf. [21, 45]. More precisely, for every closed point of the reduction modulo p of the p-integral model of the Shimura variety, there is an isomorphism between the strict henselization of its local ring and the strict henselization of the local ring of a corresponding closed point in the reduction modulo p of the local model. Very often every closed point of the local model is attained in this way. In this case, the model of the Shimura variety has good, resp. semi-stable, reduction if and only if the local model has this property. Even when this attainment statement is not known, we deduce that if the local model has good, resp. semi-stable, reduction, then so does the model of the Shimura variety. Therefore, the emphasis of the present paper is on the structure of the singularities of the local models and our results determine local models which have good, resp. semi-stable reduction.

Let us state now the main results of the paper, as they pertain to local models. See Section 3 for corresponding results for Shimura varieties, and Section 4 for results on Rapoport-Zink spaces. Local models are associated to *local model triples*. Here a LM triple over a finite extension F of $\mathbb{Q}_p$ is a triple $(G, \{\mu\}, K)$ consisting of a reductive group G over F, a conjugacy class of cocharacters $\{\mu\}$ of G over an algebraic closure of F, and a parahoric group K of G. We sometimes write $\mathcal{G}$ for the affine smooth group scheme over O_F corresponding to K. It is assumed that the cocharacter $\{\mu\}$ is minuscule (i.e., any root takes values in $\{0, \pm 1\}$ on $\{\mu\}$). The *reflex field* of the LM triple $(G, \{\mu\}, K)$ is the field of definition of the conjugacy class $\{\mu\}$. One would like to associate to $(G, \{\mu\}, K)$ a *local model* $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$, a flat projective scheme over the ring O_E of integers in the corresponding reflex field E, with action of $\mathcal{G}_{O_E}$. Also, one would like to characterize uniquely this local model.

At this point a restrictive hypothesis enters. Namely, we have to impose throughout most of the paper that the group G splits over a tamely ramified extension. Indeed, only under this hypothesis, X. Zhu and the second author define local models [44] which generalize the local models defined earlier in the concrete situations considered by Arzdorf, de Jong, Görtz, Pappas, Rapoport-Zink, Smithling, comp. [43]. In fact, we slightly modify here the construction in [44] to make sure that $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ always has reduced special fiber, a property that is stable under base change. In [44] this reducedness property was established only when $\pi_1(G_{\mathrm{der}})$ has order prime to p. Our first main result is that the result of the construction in [44] is unique, i.e., is independent of all auxiliary choices. This independence issue was left unexamined in *loc. cit.*. Here, we need uniqueness after base-changing to an unramified extension to even make unambiguous sense of our classification of local models which are smooth or semi-stable. We show:

Theorem 1.1. *Let $(G, \{\mu\}, K)$ be an LM triple such that G splits over a tamely ramified extension of F. The modified local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ of* [44] *is independent of all choices made in its construction. Its generic fiber is G_E-equivariantly isomorphic to the projective homogeneous space $X_{\{\mu\}}$, and its geometric special fiber $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\}) \otimes_{O_E} k$ is reduced and is $\mathcal{G} \otimes_{O_F} k$-equivariantly isomorphic to the* $\{\mu\}$-admissible locus *$\mathfrak{A}_K(G, \{\mu\})$ in an affine partial flag variety over k.*

We refer to the body of the text for undefined items. We conjecture that the properties in Theorem 1.1 uniquely characterize the local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$, cf. Conjecture 2.12.

Local models should exist even without the tameness hypothesis. Levin [33] has achieved some progress on this front by extending the Pappas-Zhu construction to some wild cases. Scholze [49] considers the general case and defines a *diamond local model* over O_E attached to the LM triple $(G, \{\mu\}, K)$. Furthermore, he proves that there is at most one local model whose associated diamond is the diamond local model. Unfortunately, the existence question is still open. Hence in the general situation, Scholze does not have a construction of a local model but has a characterization; under our tameness hypothesis, we have a construction but no characterization. In the case of *classical groups*, the situation is somewhat better: under some additional hypothesis, we then show that the local models of [44] satisfy Scholze's characterizing property, cf. Corollary 2.16.

Our second main result gives a characterization of all cases when Pappas-Zhu local models have good reduction. In its statement, $\breve{F}$ denotes the completion of the maximal unramified extension of F.

Theorem 1.2. *Let $(G, \{\mu\}, K)$ be a LM triple over F such that G splits over a tamely ramified extension of F. Assume that $p \neq 2$. Assume that G_{ad} is F-simple, μ_{ad} is not the trivial cocharacter, and that in the product decomposition over $\breve{F}$, $G_{\mathrm{ad}} \otimes_F \breve{F} = \breve{G}_{\mathrm{ad},1} \times \cdots \times \breve{G}_{\mathrm{ad},m}$, each factor $\breve{G}_{\mathrm{ad},i}$ is* absolutely simple. *Then the local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ is smooth over* $\operatorname{Spec} O_E$ *if and only if K is hyperspecial or (G, μ, K) is an LM triple of exotic good reduction type.*

Here the first alternative, that K be hyperspecial, is the natural generalization of the case of the modular curve with level structure prime to p. There are two cases of the second ("exotic") alternative: The first is a striking discovery of T. Richarz, cf. [1, Prop. 4.16]. He proved that the local model associated to an even, resp. odd, ramified unitary group G, the cocharacter $\{\mu\} = (1, 0, \ldots, 0)$, and the parahoric subgroup which is the stabilizer of a π-modular, resp. almost π-modular, lattice has good reduction (the case of a π-modular lattice is much easier and was known earlier, cf. [42, 5.3]). The second case, which is a new observation of the current paper, is that of the local model associated to an even ramified quasi-split orthogonal group G, the cocharacter $\{\mu\}$ that corresponds to the orthogonal Grassmannian of isotropic subspaces of maximal dimension, and the parahoric K given by the stabilizer of an almost selfdual lattice. We therefore see that in the statement of the theorem both implications are interesting and non-trivial.

Let us comment on the hypotheses in this theorem. The hypothesis that G_{ad} be F-simple is just for convenience. However, the hypothesis that each factor $\breve{G}_{\mathrm{ad},i}$ be absolutely simple is essential to our method. It implies that the translation element associated to $\{\mu\}$ in the extended affine Weyl group for $\breve{G}_{\mathrm{ad},i}$ is not too large and this limits drastically the number of possibilities of LM triples with associated local models of good reduction. Note that the tameness assumption on G is automatically satisfied for $p \geq 5$ under these hypotheses. We

refer to the passage after the statement of Theorem 5.1 for a description of the structure of the proof of Theorem 1.2. Roughly speaking, we eliminate most possibilities by various combinatorial considerations and calculations of Poincare polynomials. Ultimately, we reduce to a few cases that can be examined explicitly, and a single exceptional case (for the quasi-split ramified triality ${}^3D_4$) which is handled by work of Haines-Richarz [22].

Our third main result gives a characterization of all cases when Pappas-Zhu local models have semi-stable reduction.

Theorem 1.3. *Let $(G, \{\mu\}, K)$ be a LM triple over F such that G splits over a tamely ramified extension of F. Assume $p \neq 2$. Assume that G_{ad} is absolutely simple. Then the local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ has semi-stable, but non-smooth, reduction over $\operatorname{Spec} O_E$ if and only if its enhanced Tits datum appears in the table of Theorem 5.6.*

Again, let us comment on the hypothesis in this theorem. We are limited in the hypotheses of this theorem by the same constraints as in the criterion for good reduction–but we have to avoid the product of semi-stable varieties since these are no longer semi-stable: this explains why we make the assumption that G_{ad} be absolutely simple. The enhanced Tits datum of an LM triple is defined in Definition 5.3. In the situation of Theorem 1.3, the enhanced Tits datum determines the LM triple over F up to central isogeny and up to a scalar extension to an unramified extension of F.

Again, as with Theorem 1.2, both implications in Theorem 1.3 are interesting and non-trivial. The semi-stability in the case of the LM triple $(\mathrm{PGL}_n, (1, 0, \dots, 0), K)$, where K is an arbitrary parahoric subgroup has been known for a long time, due to the work of Drinfeld [11]. The case of the LM triple $(\mathrm{PGL}_n, (1^{(r)}, 0^{(n-r)}), K)$, where r is arbitrary and where K is the parahoric subgroup stabilizing two adjacent vertex lattices appears in the work of Görtz [16], although the significance of this case went unnoticed. Related calculations also appear in work of Harris and Taylor [23]. Another interesting case is when G is the adjoint group of a symplectic group with its natural Siegel cocharacter and K is the simultaneous stabilizer of a selfdual vertex lattice and an adjacent almost selfdual vertex lattice. This subgroup K is the so-called "Klingen parahoric" and the semi-stability in this case has been shown by Genestier and Tilouine [14, 6.3]. The case that triggered our interest in the classification of semi-stable local models is the case recently discovered by Faltings [13]. Here G is the adjoint group of the split orthogonal group of even size $2n$, the minuscule coweight is the one which leads to the hermitian-symmetric space given by a quadric, and K is the parahoric subgroup simultaneously stabilizing the selfdual and the selfdual up to a scalar vertex lattices. Faltings' language is different from ours, and it could take the reader some effort to make the connection between our result and his. However, our point of view allows us to view Faltings' result as a corollary of the general results of [29]; see Example 3.7. The list of Theorem 5.6 contains two more cases of LM triples with semi-stable associated local models, both for orthogonal groups, which seem to be new. Using an idea of Milne [36] and results of Vasiu-Zink [52], we show that in most of these cases the corresponding integral models of Shimura varieties are "canonical", see Theorem 3.6.

We refer to the end of Section 5 for a description of the proof of Theorem 1.3. As a consequence of the proof, we obtain the following remarkable fact.

Corollary 1.4. *Let $(G, \{\mu\}, K)$ be a LM triple over F such that G splits over a tamely ramified extension of F. Assume $p \neq 2$. Assume that G is adjoint and absolutely simple.*

Then the local model $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ has semi-stable reduction if and only if the special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ has strictly pseudo semi-stable reduction.

We refer to Definition 6.1 for what it means that the special fiber has strictly pseudo semi-stable reduction. It is a condition that only involves the reduced special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$; the above corollary shows that it implies that the total scheme $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is regular.

Let us now explain the lay-out of the paper. In Section 2, we recall the local models constructed in [44] and show that they are independent (in a sense to be made precise) of the auxiliary data used in their construction; we also introduce the modification of this construction that has reduced special fiber, and compare it with the hypothetical construction of Scholze [49]. In Section 3 we explain the relation between Shimura varieties and local models. Section 4 does the same for Rapoport-Zink spaces. Section 5 contains the statements of the main results on local models. In Section 6 we introduce the concepts of (rationally) strictly pseudo semi-stable reduction and the component count property (CCP condition), and prove that the former condition implies the latter. In Section 7, we give a complete list of all enhanced Coxeter data for which the CCP condition is satisfied. In Section 8, we exclude from this list the cases that do not have rationally strictly pseudo semi-stable reduction. At this point, we have all tools available to prove Theorem 1.2, and this is the content of Section 9. In Section 10, we use Kumar's criterion to eliminate all cases that do not have strictly pseudo semi-stable reduction. At this point, we have all tools available to prove one implication of Theorem 1.3, and this is the content of Section 11, where we also prove Corollary 1.4. In the final long Section 12, we prove the other implication of Theorem 1.3.

Notation: For a local field F, we denote by $\breve{F}$ the completion of its maximal unramified extension (in a fixed algebraic closure). We denote by κ_F the residue field of F and by k the algebraic closure of κ_F which is the residue field of $\breve{F}$. We always denote by p the characteristic of κ_F.

For a reductive group G, we denote by G_{der} its derived group, by G_{sc} the simply-connected covering of G_{der}, and by G_{ad} its adjoint group. If G is defined over the local field F, we denote by $\mathcal{B}(G,F)$ the extended Bruhat-Tits building of $G(F)$; if $S\subset G$ is a maximal F-split torus of G, we denote by $\mathbf{A}(G,S,F)\subset\mathcal{B}(G,F)$ the corresponding apartment. A parahoric subgroup K of $G(F)$ is, by definition, the *connected stabilizer* of a point $x\in\mathcal{B}(G,F)$; by [5], there is a smooth affine group scheme $\mathcal{G}_x$ over O_F with generic fiber G and connected special fiber such that $K=\mathcal{G}_x(O_F)$.

We often write the base change $X\times_{\operatorname{Spec} R}\operatorname{Spec} R'$ as $X\otimes_R R'$, or simply as $X_{R'}$.

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2. Local models

In this section, we discuss the theory of local models, as used in the paper.

2.1. **Local model triples.** Let F be a finite extension of $\mathbb{Q}_p$, with algebraic closure $\overline{F}$. A *local model triple* (LM triple) over F is a triple $(G, \{\mu\}, K)$ consisting of a connected reductive group G over F, a conjugacy class $\{\mu\}$ of cocharacters of $G_{\overline{F}}$, and a parahoric subgroup K of $G(F)$. It is assumed that $\{\mu\}$ is a minuscule cocharacter. We denote by $\mathcal{G} = \mathcal{G}_K$ the extension of G to a smooth group scheme over O_F corresponding to K. Then $\mathcal{G}$ has connected fibers and satisfies $K = \mathcal{G}(O_F)$. We set $\breve{K} = \mathcal{G}(O_{\breve{F}})$. Sometimes we also write $(G, \{\mu\}, \mathcal{G})$ for the LM triple.

Two LM triples $(G, \{\mu\}, K)$ and $(G', \{\mu'\}, K')$ are isomorphic if there exists an isomorphism $G \to G'$ which takes $\{\mu\}$ to $\{\mu'\}$ and $\breve{K}$ to a conjugate of $\breve{K}'$. More generally, a morphism

$$\phi : (G, \{\mu\}, K) \longrightarrow (G', \{\mu'\}, K')$$

of LM triples is a group scheme homomorphism $\phi : G \to G'$ such that $\{\mu'\} = \{\phi \circ \mu\}$ and $\phi(\breve{K}) \subset g'\breve{K}'g'^{-1}$, for some $g' \in G'(\breve{F})$.

Let E be the field of definition of $\{\mu\}$ inside the fixed algebraic closure $\overline{F}$ of F, with its ring of integers O_E. We denote by k the algebraic closure of its residue field κ_E. Denote by $X_{\{\mu\}}$ the partial flag variety over E of G_E associated to $\{\mu\}$.

2.2. **Group schemes.** Let G be a reductive group over F that splits over a tame extension of F. Choose a uniformizer π of F. The theory of [44] starts with the construction of a reductive group scheme $\underline{G}$ over $O_F[u^{\pm}] := O_F[u, u^{-1}]$ which induces by specialization $(O_F[u^{\pm}] \to F, u \mapsto \pi)$ the group G over F. Let G' be the reductive group induced by $\underline{G}$ by specialization along $(O_F[u^{\pm}] \to \kappa_F((u)), \pi \mapsto 0)$.

By [44, Thm 4.1], there exists a smooth affine group scheme $\underline{\mathcal{G}}$ over $O_F[u]$ with connected fibers which restricts to $\underline{G}$ over $O_F[u^{\pm}]$ and which induces the parahoric group scheme $\mathcal{G}$ under the specialization $(O_F[u] \to O_F, u \mapsto \pi)$. It also induces a parahoric group scheme $\mathcal{G}'$ under the specialization $(O_F[u] \to \kappa_F[[u]], \pi \mapsto 0)$, with an identification

$$\mathcal{G} \otimes_{O_F, \pi \longmapsto 0} k = \mathcal{G}' \otimes_{\kappa_F[[u]], u \longmapsto 0} k. \tag{2.1}$$

We denote by $\underline{\breve{G}}$, resp. $\underline{\breve{\mathcal{G}}}$, the group schemes over $O_{\breve{F}}[u^{\pm}]$, resp. $O_{\breve{F}}[u]$, obtained by base change $O_F \to O_{\breve{F}}$.

Let us recall some aspects of the construction of these group schemes. The reader is referred to [44] for more details. For simplicity we abbreviate $O = O_F$, $\breve{O} = O_{\breve{F}}$.

Denote by H (resp. G^*) the corresponding split (resp. quasi-split) form of G over O (resp. F). These forms are each unique up to isomorphism.

Fix, once and for all, a pinning (H, T_H, B_H, e_O) defined over O. As in [44], we denote by Ξ_H the group of automorphisms of the based root datum corresponding to (H, T_H, B_H).

Pick a maximal F-split torus $A \subset G$. By [5, 5.1.12], we can choose an F-rational maximal $\breve{F}$-split torus S in G that contains A and a minimal F-rational parabolic subgroup P which contains $Z_G(A)$. In [44], a triple (A, S, P) as above, is called a *rigidification* of G. Since by Steinberg's theorem, the group $\breve{G} = G \otimes_F \breve{F}$ is quasi-split, $T = Z_G(S)$ is a maximal torus of G which is defined over F.

As in [44, 2.4.2], the indexed root datum of the group G over F gives a Ξ_H-torsor τ over $\mathrm{Spec}(F)$. Then, by [44, Prop. 2.3], we obtain a pinned quasi-split group (G^*, T^*, B^*, e^*) over F and, by the identification of tame finite extensions of F with étale finite covers

of $O[u^{\pm}]$ given by $u \mapsto \pi$, a pinned quasi-split group $(\underline{G}^*, \underline{T}^*, \underline{B}^*, \underline{e}^*)$ over $O[u^{\pm}]$ (see *loc. cit.*, 3.3). As in [44], we denote by $\underline{S}^*$ the maximal split subtorus of $\underline{T}^*$. We have an identification

$$(\underline{G}^*, \underline{T}^*, \underline{B}^*, \underline{e}^*) \otimes_{O[u^{\pm}], u \longmapsto \pi} F = (G^*, T^*, B^*, e^*). \tag{2.2}$$

Remark 2.1. a) The base change $(\underline{G}^*, \underline{T}^*, \underline{B}^*, \underline{e}^*) \otimes_{O[u^{\pm}]} \breve{O}[u^{\pm}]$ is independent of the choice of uniformizer π of F. This follows by the above, since the identification of the tame Galois group of $\breve{F}$ with $\mathbb{Z}'(1) = \prod_{l \neq p} \mathbb{Z}_l(1)$, given by $\gamma \mapsto \gamma(\pi^{1/m})/\pi^{1/m}$, does not depend on the choice of the uniformizer π.

b) It is not hard to see, using [44, 3.3.2], that the Picard group of every finite étale cover of $O[u^{\pm}]$ is trivial. The argument in the proof of [9, Prop. 7.2.12] , then shows that, up to isomorphism, a quasi-split reductive group scheme over $O[u^{\pm}]$ is uniquely determined by a corresponding Ξ_H-torsor over $O[u^{\pm}]$ and therefore obtained by the above construction. In fact, any quasi-split reductive group scheme over $O[u^{\pm}]$ is determined, up to isomorphism, by its base change along $O[u^{\pm}] \to F$, given by $u \mapsto \pi$.

As in [44], we obtain from (2.2) identifications of apartments

$$\mathbf{A}(G^*, S^*, \breve{F}) = \mathbf{A}(\underline{G}^*_{\kappa((u))}, \underline{S}^*_{\kappa((u))}, \kappa((u))), \tag{2.3}$$

for both $\kappa = \breve{F}$, k. Given $x^* \in \mathbf{A}(G^*, S^*, \breve{F}) \subset \mathcal{B}(G^*, \breve{F})$, Theorem 4.1 of [44], produces a smooth connected affine group scheme

$$\underline{\mathcal{G}}^* := \underline{\mathcal{G}}^*_{x^*}$$

over $\breve{O}[u]$ which extends $\underline{G}^* \otimes_{O[u^{\pm}]} \breve{O}[u^{\pm}]$. Using Remark 2.1 we see that $\underline{\mathcal{G}}^*_{x^*}$ does not depend on the choice of the uniformizer. (Notice that $\underline{\mathcal{G}}^*_{x^*}$ might not descend over $O[u]$ since x^* is not necessarily F-rational.)

Now, given $x \in \mathcal{B}(G, F)$ which corresponds to K, choose a rigidification (A, S, P) of G over F, such that $x \in \mathbf{A}(G, S, F)$.

Since $\breve{G} = G \otimes_F \breve{F}$ and $G^* \otimes_F \breve{F}$ are both quasi-split and inner forms of each other, we can choose an inner twist, i.e., a $\mathrm{Gal}(\breve{F}/F)$-stable $G^*_{\mathrm{ad}}(\breve{F})$-conjugacy class of an isomorphism

$$\psi : G \otimes_F \breve{F} \xrightarrow{\sim} G^* \otimes_F \breve{F}.$$

Then the class $[g_\sigma]$ of the 1-cocycle $\sigma \mapsto \mathrm{Int}(g_\sigma) = \psi\sigma\psi^{-1}\sigma^{-1}$ in $\mathrm{H}^1(\hat{\mathbb{Z}}, G^*_{\mathrm{ad}}(\breve{F}))$ maps to the class in $\mathrm{H}^1(\hat{\mathbb{Z}}, \mathrm{Aut}(G^*)(\breve{F}))$ that gives the twist G of G^*. The orbit of $[g_\sigma]$ under the natural action of $\mathrm{Out}(G^*)(F)$ on $\mathrm{H}^1(\hat{\mathbb{Z}}, G^*_{\mathrm{ad}}(\breve{F}))$ only depends on the isomorphism class of G. In [44], it shown that there is a choice of ψ that depends on the rigidification (A, S, P) such that the inclusion

$$\mathcal{B}(G, F) \subset \mathcal{B}(G, \breve{F}) \xrightarrow{\psi_*} \mathcal{B}(G^*, \breve{F})$$

identifies $\mathbf{A}(G, S, \breve{F})$ with $\mathbf{A}(G^*, S^*, \breve{F})$; set $x^* := \psi_*(x)$. In *loc. cit.* the group scheme $\underline{\mathcal{G}}_x$ over $O[u]$ is then constructed such that ψ extends to isomorphisms

$$\underline{\psi} : \underline{\breve{G}} = \underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[u^{\pm}] \xrightarrow{\sim} \underline{\breve{G}}^*,$$

$$\underline{\psi} : \underline{\breve{\mathcal{G}}}_x \xrightarrow{\sim} \underline{\mathcal{G}}^*_{x^*}.$$

A priori, the group scheme $\underline{\mathcal{G}}_x$ depends on several choices, in particular of $\underline{G}$ and of the uniformizer π. However, we now show:

Proposition 2.2. *a) Up to isomorphism, the group scheme* $\breve{\underline{G}} = \underline{G} \otimes_{O[u^{\pm}]} \breve{O}[u^{\pm}]$ *depends only on* $\breve{G} = G \otimes_F \breve{F}$.

b) Up to isomorphism, the group scheme $\breve{\underline{\mathcal{G}}}_x = \underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[u]$ *depends only on* $G \otimes_F \breve{F}$ *and the* $G_{\mathrm{ad}}(\breve{F})$*-orbit of* $x \in \mathcal{B}(G, \breve{F})$.

c) For any $a \in O^\times$, *the group scheme* $\underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[u]$ *supports an isomorphism*

$$R_a : a^*(\underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[u]) \xrightarrow{\sim} \underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[u].$$

that lifts the isomorphism given by $u \mapsto a \cdot u$.

Proof. By the construction, as briefly recalled above, there are isomorphisms

$$\underline{\psi} : \breve{\underline{G}} \xrightarrow{\sim} \breve{\underline{G}}^*, \quad \underline{\psi} : \breve{\underline{\mathcal{G}}}_x \xrightarrow{\sim} \underline{\mathcal{G}}^*_{x^*}.$$

Hence, it is enough to show corresponding independence statements for $\breve{\underline{G}}^*$ and $\underline{\mathcal{G}}^*_{x^*}$. First we notice that by Remark 2.1, $\breve{\underline{G}}^*$ only depends on $G \otimes_F \breve{F}$ and so part (a) follows. Now using the argument in [44, 4.3.1], we see that changing the rigidification (A, S, P) of G, changes the point x^* to another point x'^* of $\mathbf{A}(G^*, S^*, \breve{F})$ in the same $G^*_{\mathrm{ad}}(\breve{F})$-orbit, hence in the same orbit under the adjoint Iwahori-Weyl group $\tilde{W}_{G^*_{\mathrm{ad}}}$. However, each element w of $\tilde{W}_{G^*_{\mathrm{ad}}}$ lifts to an element $\underline{n}$ of $\underline{G}^*_{\mathrm{ad}}(\breve{O}[u^{\pm}])$ that normalizes $\underline{S}^*$. Acting by $\mathrm{Int}(\underline{n})$ gives an isomorphism between the group schemes $\underline{\mathcal{G}}^*_{x^*}$ and $\underline{\mathcal{G}}^*_{x'^*}$. This implies statement (b). To see (c), we first observe that Remark 2.1 implies that there is an isomorphism over $\breve{O}[u^{\pm}]$

$$R_a : a^*(\breve{\underline{G}}^*) \xrightarrow{\sim} \breve{\underline{G}}^*$$

that lifts $u \mapsto a \cdot u$. To check that this extends to an isomorphism over $\breve{O}[u]$ it is enough to check the statement for the corresponding parahoric group scheme over $\breve{F}[[u]]$. This follows by an argument as in the proof of [55, Lemma 5.4]. □

Remark 2.3. Suppose that $G = G^*$ is quasi-split over F. Then, by Remark 2.1 (b), the extension $\underline{G} = \underline{G}^*$ over $O[u^{\pm}]$ is determined by G as the unique, up to isomorphism, quasi-split group scheme that restricts to G after $u \mapsto \pi$. However, the restriction $\underline{G}^* \otimes_{O[u^{\pm}]} F$, by $u \mapsto \pi'$, where $\pi' = a \cdot \pi$ is another choice of uniformizer, is not necessarily isomorphic to G. For example, suppose $G = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{G}_m)$, with $L = \mathbb{Q}_p(p^{1/2})$, p odd. Suppose $\pi = p$. Then,

$$\underline{G} = \mathrm{Res}_{\mathbb{Z}_p[u^{\pm}][X]/(X^2-u)/\mathbb{Z}_p[u^{\pm}]}(\mathbb{G}_m).$$

Specializing this by $u \mapsto \pi' = -p$, gives $\mathrm{Res}_{L'/\mathbb{Q}_p}(\mathbb{G}_m)$, with $L' = \mathbb{Q}_p((-p)^{1/2})$ which is a different torus than G if $p \equiv 1 \bmod 4$.

Therefore, the extension $\underline{G}^*$ depends on both G and π. When we need to be more precise, we will denote it by $\underline{G}^*_\pi$. By the above, we have an isomorphism

$$R^\natural_a : a^*(\underline{G}^*_\pi) \xrightarrow{\sim} \underline{G}^*_{\pi'},$$

where $a : \mathrm{Spec}(O[u^{\pm}]) \to \mathrm{Spec}(O[u^{\pm}])$ is given by $u \mapsto a \cdot u$, which descends R_a above.

2.3. **Weyl groups and the admissible locus.** We continue with the set-up of the last subsection. The group scheme $\breve{\underline{G}}$ admits a chain of tori by closed subgroup schemes $\breve{\underline{S}} \subset \breve{\underline{T}}$ which extend S and T and correspond to $\breve{\underline{S}}^*$, $\breve{\underline{T}}^*$ via $\underline{\psi}$. These define maximal split, resp.

maximal, tori in the fibers $\breve{G} = G \otimes_F \breve{F}$ and $\breve{G}' = G' \otimes_{\kappa_F((u))} k((u))$ of $\underline{\breve{G}}$. By the above constructions, we obtain identifications of *relative Weyl groups*, resp. *Iwahori Weyl groups*,

$$W_0(\breve{G}, \breve{T}) = W_0(\breve{G}', \breve{T}'), \quad \tilde{W}(\breve{G}, \breve{T}) = \tilde{W}(\breve{G}', \breve{T}'), \tag{2.4}$$

cf. [45, §2]. Assume now that we have a conjugacy class $\{\mu\}$ of a minuscule geometric cocharacter of G, so that $(G, \{\mu\}, K)$ is a local model triple over F. Then the above give identifications of $\{\mu\}$-*admissible sets* in the Iwahori Weyl groups

$$\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}'(\{\mu\}), \tag{2.5}$$

cf. [45, §3]. Denoting by $\breve{K}'$ the parahoric subgroup of $G'\big(k((u))\big)$ defined by $\mathcal{G}'$, with corresponding group scheme $\breve{\mathcal{G}}'$, we also obtain an identification of $\{\mu\}$-admissible subsets in the double coset spaces (*cf.* [45, §3]),

$$\mathrm{Adm}_{\breve{K}}(\{\mu\}) = \mathrm{Adm}'_{\breve{K}'}(\{\mu\}) \subset W_{\breve{K}} \backslash \tilde{W} / W_{\breve{K}} = W_{\breve{K}'} \backslash \tilde{W}' / W_{\breve{K}'}. \tag{2.6}$$

We define a closed reduced subset inside the loop group flag variety $\mathcal{F}' = L\breve{G}'/L^+\breve{\mathcal{G}}'$ over k, as the reduced union

$$\mathcal{A}_K(\underline{G}, \{\mu\}) = \bigcup_{w \in \mathrm{Adm}'_{\breve{K}'}(\{\mu\})} S_w. \tag{2.7}$$

Here S_w denotes the $L^+\breve{\mathcal{G}}'$-orbit corresponding to $w \in W_{\breve{K}'} \backslash \tilde{W}' / W_{\breve{K}'}$. We note that, since $\{\mu\}$ is minuscule, the action of $L^+\breve{\mathcal{G}}'$ on $\mathcal{A}_K(\underline{G}, \{\mu\})$ factors through $\mathcal{G}' \otimes_{\kappa_F[[u]]} k$. Via (2.1), we obtain an action of $\mathcal{G} \otimes_{O_F} k$ on $\mathcal{A}_K(\underline{G}, \{\mu\})$.

Corollary 2.4. *Up to isomorphism, the group $\breve{G}'$ over $k((u))$ and its parahoric subgroup $\breve{K}'$ are independent of the choice of the uniformizer π and of $\underline{G}$. The isomorphism can be chosen compatibly with the identification* (2.1)*, and the identifications* (2.5) *of Weyl groups and* (2.6) *of admissible sets. As a consequence, the affine partial flag variety $\mathcal{F}'$ over k and its subscheme $\mathcal{A}_K(\underline{G}, \{\mu\})$ with action of $\mathcal{G} \otimes_{O_F} k$ is independent of the choice of the uniformizer π and of $\underline{G}$.*

Proof. Follows from Proposition 2.2, its proof and the definition of the $\{\mu\}$-admissible set. □

2.4. **Descent.** We continue with the set-up of the previous subsection; we will apply a form of Weil-étale descent from $\breve{O}$ to O. The following result is not needed for the proof of Theorems 1.2 and 1.3 about local models with smooth or semi-stable reduction, see Remark 2.8. However, it is an important part of the argument for the independence result of Theorem 1.1.

Proposition 2.5. *a) The group scheme $\underline{\mathcal{G}}_x \otimes_{O[u]} O[[u]]$ depends, up to isomorphism, only on G, the uniformizer π and the $G_{\mathrm{ad}}(F)$-orbit of $x \in \mathcal{B}(G, F)$. We denote it by $\underline{\mathcal{G}}_{x,\pi} \otimes_{O[u]} O[[u]]$.*

b) If $\pi' = a \cdot \pi$ is another choice of a uniformizer with $a \in O^\times$, then there is an isomorphism of group schemes

$$R^\natural_a : a^*(\underline{\mathcal{G}}_{x,\pi} \otimes_{O[u]} O[[u]]) \xrightarrow{\sim} \underline{\mathcal{G}}^*_{x,\pi'} \otimes_{O[u]} O[[u]]$$

where $a : \operatorname{Spec} O[[u]] \to \operatorname{Spec} O[[u]]$ is given by $u \mapsto a \cdot u$.

Proof. We first show (a). For this we fix the uniformizer π. By Proposition 2.2, the base change $\underline{\mathcal{G}}_x \otimes_{O[u]} \breve{O}[[u]]$ depends only on G and the $G_{\mathrm{ad}}(F)$-orbit of $x \in \mathcal{B}(G,F)$. We will now use descent. By the construction, the group $\underline{\mathcal{G}}_x$ in [44] is given by a (σ-semilinear) Weil descent datum

$$\mathrm{Int}(\underline{g}) \cdot \sigma : \underline{\mathcal{G}}^*_{x^*} \longrightarrow \underline{\mathcal{G}}^*_{x^*}.$$

Here $\underline{g} \in \underline{G}^*_{\mathrm{ad}}(\breve{O}[u^{\pm}])$; this depends on various choices made in [44]. The action of σ is with respect to the rational structure given by the $O[u^{\pm}]$-group $\underline{G}^*_{\pi}$; this depends on our fixed choice of π, see Remark 2.3. We start the proof by giving:

Lemma 2.6. *The automorphism group $\mathscr{A}^* = \mathrm{Aut}(\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]])$ of the group scheme $\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]]$ has the following properties:*

i) It contains the normalizer $\mathscr{N}^*$ of $\underline{\mathcal{G}}^*_{x^*}(\breve{O}[[u]])$ in $\underline{G}^*_{\mathrm{ad}}(\breve{O}((u)))$.

ii) The homomorphism $\mathscr{A}^* \to \mathrm{Aut}(\mathcal{G}^*_{x^*})$ given by $u \mapsto \pi$ is surjective. We have

$$\ker(\mathscr{A}^* \longrightarrow \mathrm{Aut}(\mathcal{G}^*_{x^*})) = \ker(\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \xrightarrow{u \longrightarrow \pi} \mathcal{G}^*_{\mathrm{ad},x^*}(\breve{O}))$$

and this kernel is pro-unipotent.

Proof. Let us first study $\mathrm{Aut}(\mathcal{G}^*_{x^*})$: Passing to the generic fiber gives an injection

$$\mathrm{Aut}(\mathcal{G}^*_{x^*}) \subset \mathrm{Aut}(\breve{G}^*).$$

There is also ([9, Prop. 7.2.11]) a (split) exact sequence

$$1 \longrightarrow G^*_{\mathrm{ad}}(\breve{F}) \longrightarrow \mathrm{Aut}(\breve{G}^*) \longrightarrow \mathrm{Out}(\breve{G}^*) \longrightarrow 1.$$

This gives

$$1 \longrightarrow G^*_{\mathrm{ad}}(\breve{F})_{x^*} \longrightarrow \mathrm{Aut}(\breve{G}^*)_{x^*} = \mathrm{Aut}(\mathcal{G}^*_{x^*}) \longrightarrow \mathrm{Out}(\breve{G}^*)_{x^*} \longrightarrow 1$$

where the subscript x^* denotes the subgroup that fixes $x^* \in \mathcal{B}(G^*, \breve{F})$.

Notice here that $G^*_{\mathrm{ad}}(\breve{F})_{x^*}$ is the normalizer in $G^*_{\mathrm{ad}}(\breve{F})$ of the parahoric subgroup $\mathcal{G}^*_{x^*}(\breve{O}) = G^*(\breve{F})^0_{x^*}$. (Indeed, by [5, 5.1.39], the normalizer of the stabilizer of any facet in the Bruhat-Tits building has to also stabilize the facet; this last statement easily follows from that.) We also have

$$1 \longrightarrow G^*_{\mathrm{ad}}(\breve{F})^0_{x^*} \longrightarrow G^*_{\mathrm{ad}}(\breve{F})_{x^*} \longrightarrow \Delta_{x^*} \longrightarrow 1$$

where Δ_{x^*} is the finite abelian group given as the group of connected components of the "stabilizer of x^*" Bruhat-Tits group scheme for G^*_{ad} over $\breve{O}$.

Similarly, we have an injection $\mathscr{A}^* \subset \mathrm{Aut}(\underline{\breve{G}}^*)$. The quasi-split $\underline{G}^*$ carries the pinning $(\underline{T}^*, \underline{B}^*, \underline{e}^*)$ and we can use this to identify $\mathrm{Out}(\underline{\breve{G}}^*) = \mathrm{Out}(\breve{G}^*)$ with a subgroup of the group Ξ_H of "graph" automorphisms. By [9, Prop. 7.2.11], we have

$$\mathrm{Aut}(\underline{\breve{G}}^*) = \underline{G}^*_{\mathrm{ad}}(\breve{O}((u))) \rtimes \mathrm{Out}(\underline{\breve{G}}^*). \tag{2.8}$$

We first show (i), i.e., that every $g \in \mathscr{N}^* \subset \underline{G}^*_{\mathrm{ad}}(\breve{O}((u)))$ naturally induces an automorphism $\mathrm{Int}(g)$ of $\underline{\mathcal{G}}^* \otimes_{\breve{O}[u]} \breve{O}[[u]]$. (For simplicity, we omit the subscript x^* below.) The adjoint action of $g \in \mathscr{N}^*$ gives an ind-group scheme homomorphism $\mathrm{Int}(g) : L\underline{G}^* \to L\underline{G}^*$ which preserves $L^+\underline{\mathcal{G}}^*(\breve{O})$. Using the fact $L^+\underline{\mathcal{G}}^*$ is pro-algebraic and formally smooth over $\breve{O}$, we can easily see that the set of points $L^+\underline{\mathcal{G}}^*(\breve{F})$ with $\breve{F}$ as residue field is dense in

$L^+\underline{\mathcal{G}}^*$. Since $L^+\underline{\mathcal{G}}^*$ is a reduced closed subscheme of the ind-scheme $L\underline{\mathcal{G}}^* = L\underline{G}^*$ over $\breve{O}$, it follows that g induces a group scheme homomorphism

$$\mathrm{Int}(g) : L^+\underline{\mathcal{G}}^* \longrightarrow L^+\underline{\mathcal{G}}^*.$$

In particular, g also normalizes $L^+\underline{\mathcal{G}}^*(\breve{F}) = \mathcal{G}^*(\breve{F}[[u]])$. Since $\mathcal{G}^* \otimes_{\breve{O}[u]} \breve{F}((u))$ is quasi-split and residually split, the $\breve{F}$-valued points are dense in the fiber $\mathcal{G}^* \otimes_{\breve{O}[u]} \breve{F}$ over $u = 0$. Hence, we obtain by [5, 1.7.2] that $\mathrm{Int}(g)$ induces an automorphism of the group scheme $\mathcal{G}^* \otimes_{\breve{O}[u]} \breve{F}[[u]]$. Since $\mathcal{G}^*$ is smooth over $\breve{O}[[u]]$ and $\mathrm{Int}(g)$ gives an automorphism of $\mathcal{G}^* \otimes_{\breve{O}[u]} \breve{O}((u))$, we see that $\mathrm{Int}(g)$ extends to an automorphism of $\mathcal{G}^* \otimes_{\breve{O}[u]} \breve{O}[[u]]$ as desired. This proves (i).

Let us show that $\mathscr{A}^*$ satisfies (ii). Sending $u \mapsto \pi$ gives a homomorphism

$$\mathscr{A}^* \longrightarrow \mathrm{Aut}(\mathcal{G}^*_{x^*}).$$

This restricts to $\mathscr{N}^* \to G^*_{\mathrm{ad}}(\breve{F})_{x^*}$: To see this we use that $L^+\underline{\mathcal{G}}^*(\breve{O}) \to \mathcal{G}^*(\breve{O}) = G^*(\breve{F})^0_{x^*}$ given by $u \mapsto \pi$ is surjective (by smoothness and Hensel's lemma) and that $G^*_{\mathrm{ad}}(\breve{F})_{x^*}$ is the normalizer of $G^*(\breve{F})^0_{x^*}$ in $G^*_{\mathrm{ad}}(\breve{F})$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{N}^* & \longrightarrow & \mathscr{A}^* & \longrightarrow & \mathrm{Out}(\underline{\breve{G}}^*)_{x^*} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G^*_{\mathrm{ad}}(\breve{F})_{x^*} & \longrightarrow & \mathrm{Aut}(\mathcal{G}^*_{x^*}) & \longrightarrow & \mathrm{Out}(\breve{G}^*)_{x^*} & \longrightarrow & 1. \end{array} \tag{2.9}$$

We will show that the left vertical arrow is a surjection with kernel equal to $\mathscr{K}^* := \ker(\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \xrightarrow{u\to\pi} \mathcal{G}^*_{\mathrm{ad},x^*}(\breve{O}))$ and that the right vertical arrow is an isomorphism. This would imply part (ii).

The subgroup $\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \subset \underline{G}^*_{\mathrm{ad}}(\breve{O}((u)))$ is contained in $\mathscr{N}^*$. Mapping $u \mapsto \pi$ followed by taking connected component gives a homomorphism

$$\delta : \mathscr{N}^* \longrightarrow G^*_{\mathrm{ad}}(\breve{F})_{x^*} \to \Delta_{x^*}.$$

We will show that the sequence

$$1 \longrightarrow \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \longrightarrow \mathscr{N}^* \xrightarrow{\delta} \Delta_{x^*} \longrightarrow 1 \tag{2.10}$$

is exact. Since $\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \xrightarrow{u\mapsto\pi} \mathcal{G}^*_{\mathrm{ad},x^*}(\breve{O}) = G^*_{\mathrm{ad}}(\breve{F})^0_{x^*}$ is surjective (by smoothness and Hensel's lemma) this would show that $u \mapsto \pi$ gives a surjective

$$\mathscr{N}^* \longrightarrow G^*_{\mathrm{ad}}(\breve{F})_{x^*} \longrightarrow 1$$

with kernel equal to $\mathscr{K}^* := \ker(\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]) \xrightarrow{u\to\pi} \mathcal{G}^*_{\mathrm{ad},x^*}(\breve{O}))$.

Let us show the exactness of (2.10). The subgroup $\underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]])$ lies in the kernel of δ and we can see that it is actually equal to that kernel: Let $g \in \mathscr{N}^*$ with $\delta(g) = 1$. Since g also normalizes $\mathcal{G}^*(\breve{F}[[u]])$, we see as above, that g lies in $\underline{G}^*_{\mathrm{ad}}(\breve{F}((u)))_{x^*}$. Using the identification of apartments (2.2) we now see that since $\delta(g) = 1$, g is actually in the connected stabilizer $\underline{G}^*_{\mathrm{ad}}(\breve{F}((u)))^0_{x^*} = \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{F}[[u]])$. Since g is also in $\underline{G}^*_{\mathrm{ad}}(\breve{O}((u)))$, we have

$$g \in \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{F}[[u]]) \cap \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}((u))) = \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]]).$$

Therefore, $\ker(\delta) = \underline{\mathcal{G}}^*_{\mathrm{ad},x^*}(\breve{O}[[u]])$. It remains to show that δ is surjective. By [5, Proposition 4.6.28 (ii)], for each $y \in \Delta_{x^*}$, there is an element $n \in N_{\mathrm{ad}}(\breve{F})$ that fixes x^* in the

building so that $\delta(n) = y$. By the identification of the apartments (2.3), we can lift n to $\underline{n} \in \underline{N}_{\mathrm{ad}}(\breve{O}((u)))$ which fixes the point x^* considered in the building over $\breve{F}((u))$. Then $\underline{n}$ normalizes $L^+\underline{\mathcal{G}}^*(\breve{F}) \cap \underline{G}^*(\breve{O}((u))) = L^+\underline{\mathcal{G}}^*(\breve{O})$ so $\underline{n}$ is in $\mathscr{N}^*$.

It remains to show that $\mathrm{Out}(\underline{\breve{G}}^*)_{x^*} \to \mathrm{Out}(\breve{G}^*)_{x^*}$ given by $u \mapsto \pi$ is an isomorphism. The corresponding map $\mathrm{Out}(\underline{\breve{G}}^*) \to \mathrm{Out}(\breve{G}^*)$ is an isomorphism by the construction of $\underline{\breve{G}}^*$ from $\breve{G}^*$. Hence, it is enough to show that $\mathrm{Out}(\underline{\breve{G}}^*)_{x^*} \to \mathrm{Out}(\breve{G}^*)_{x^*}$ is surjective. By definition, $\gamma \in \mathrm{Out}(\breve{G}^*)_{x^*}$ is given by an automorphism of $\breve{G}^*$ preserving the pinning $(\breve{T}^*, \breve{B}^*, \breve{e}^*)$, such that $\gamma(x^*) = \mathrm{Int}(g)(x^*)$, for some $g \in G^*_{\mathrm{ad}}(\breve{F})$. Since $\gamma(x^*)$ and x^* both lie in the apartment for $\breve{S}^* \subset \breve{T}^*$, this implies that $\gamma(x^*) = \mathrm{Int}(n)(x^*)$, for some $N^*_{\mathrm{ad}}(\breve{F})$. As above, we can lift n to $\underline{n} \in N^*_{\mathrm{ad}}(\breve{O}((u)))$. Using the identification of apartments (2.3) we see that γ is in $\mathrm{Out}(\underline{\breve{G}}^*)_{x^*}$. □

We can now resume the proof of Proposition 2.5. We will show that $\underline{\mathcal{G}}_x \otimes_{O[u]} O[[u]]$ is independent, up to isomorphism, of additional choices. Suppose as above that $\underline{g}' \in \underline{G}^*_{\mathrm{ad}}(\breve{O}[u^{\pm}])$ is a second cocycle giving a group scheme $\underline{\mathcal{G}}'_x$; then $\underline{\mathcal{G}}'_x$ is a form of $\underline{\mathcal{G}}_x$. The twisting is obtained by the image of the cocycle given by

$$\underline{c} = \underline{g}' \cdot \underline{g}^{-1} \in \underline{G}^*_{\mathrm{ad}}(\breve{O}[u^{\pm}]).$$

(This is a cocycle for the twisted σ-action on $\underline{G}^*_{\mathrm{ad}}(\breve{O}[u^{\pm}])$ given by $\mathrm{Int}(\underline{g})$.) Notice that the restriction of $\underline{c}$ along $u = \pi$ preserves x^*. Hence, $\underline{c}$ also preserves x^* considered as a point in the building over $\breve{F}((u))$. It follows that $\underline{c}$ lies in the normalizer of the parahoric $\underline{\mathcal{G}}^*_{x^*}(\breve{F}[[u]])$. Using $\breve{O}((u)) \cap \breve{F}[[u]] = \breve{O}[[u]]$, we see that $\underline{c}$ lies in the normalizer $\mathscr{N}^*$ of $\underline{\mathcal{G}}^*_{x^*}(\breve{O}[[u]])$ and it gives a cocycle for the twisted σ-action. The isomorphism class of the form $\underline{\mathcal{G}}'_x \otimes_{O[u]} O[[u]]$ is determined by the class $[\underline{c}]$ in $\mathrm{H}^1(\hat{\mathbb{Z}}, \mathscr{A})$. Here $\mathscr{A} = \mathrm{Aut}(\underline{\mathcal{G}}_x \otimes_{\breve{O}[u]} \breve{O}[[u]])$ which is $\mathscr{A}^*$ but with the twisted σ-action. By Lemma 2.6 (b), $\mathscr{K}^*$ and therefore also the kernel $\mathscr{K} = \ker(\mathscr{A} \to \mathrm{Aut}(\mathcal{G}_x))$ is pro-unipotent. Using this, a standard argument as in the proof of Lemmas 1 and 2, p. 690, of [6], gives that $\mathrm{H}^1(\hat{\mathbb{Z}}, \mathscr{K}) = 0$. Since the specialization of the form $\underline{\mathcal{G}}'_x$ at $u = \pi$ is isomorphic to $\mathcal{G}_x$, the image of the class $\underline{c}$ in $\mathrm{H}^1(\hat{\mathbb{Z}}, \mathrm{Aut}(\mathcal{G}_x))$ is trivial. Hence, by the exact sequence for cohomology, the class $[\underline{c}]$ in $\mathrm{H}^1(\hat{\mathbb{Z}}, \mathscr{A})$ is trivial. Therefore, we obtain $\underline{\mathcal{G}}'_x \otimes_{O[u]} O[[u]] \simeq \underline{\mathcal{G}}_x \otimes_{O[u]} O[[u]]$, where in both, the choice of π remains the same. This proves part (a).

To prove part (b), suppose that $\pi' = a \cdot \pi$, $a \in O^\times$, is another choice of uniformizer. By Proposition 2.2 (c), the group scheme $\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]]$ supports an isomorphism

$$R_a : a^*(\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]]) \xrightarrow{\sim} \underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]].$$

We would like to show that R_a descends to an isomorphism $R_a^\natural : a^*(\underline{\mathcal{G}}_{x,\pi} \otimes_{O[u]} O[[u]]) \xrightarrow{\sim} \underline{\mathcal{G}}_{x,\pi'} \otimes_{O[u]} O[[u]]$. Consider the descent datum $\Phi := \mathrm{Int}(\underline{g}) \cdot \sigma$ for $\underline{\mathcal{G}}_{x,\pi}$ and its "rotation" given as $R_a(\Phi) := R_a(a^*\mathrm{Int}(\underline{g}))\sigma(R_a)^{-1} \cdot \sigma$ for $\underline{\mathcal{G}}_{x,\pi'}$. Consider also a descent datum $\Phi' := \mathrm{Int}(\underline{g}') \cdot \sigma$ for $\underline{\mathcal{G}}_{x,\pi'}$. It is enough to show that Φ' and $R_a(\Phi)$ are cohomologous, i.e., that there is an automorphism h of $\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]]$ such that $h^{-1}R_a(\Phi) = \Phi' \cdot \sigma(h)^{-1}$. Then we can set $R_a^\natural = h^{-1}R_a$ which descends. To show the existence of h, note that R_a is the identity on the maximal reductive quotient of the fiber of $\underline{\mathcal{G}}^*_{x^*} \otimes_{\breve{O}[u]} \breve{O}[[u]]$ over the point (u, π). We have $\underline{\mathcal{G}}_{x,\pi} \simeq \underline{\mathcal{G}}_{x,\pi'}$ modulo (u, π) since they both are isomorphic to $\mathcal{G}_x$ modulo π. Hence, Φ'

and $R_a(\Phi)$ are cohomologous when considered modulo a (connected) pro-unipotent group. An argument similar to the one in the proof of part (a) above then shows the result. □

2.5. **Pappas-Zhu local models.** Let $(G, \{\mu\}, K)$ be a local model triple over F such that G splits over a ramified extension of F. Again we set $O = O_F$.

In [44], there is a construction of a "local model" $M_{\mathcal{G},\mu}$. The local model $M_{\mathcal{G},\mu}$ is a flat projective O_E-scheme equipped with an action of $\mathcal{G}_{O_E}$ such that its generic fiber is G_E-equivariantly isomorphic to $X_{\{\mu\}}$. By definition, $M_{\mathcal{G},\mu}$ is the Zariski closure of $X_{\{\mu\}} \subset \mathrm{Gr}_{\underline{\mathcal{G}}} \otimes_{O[u]} E$ in $\mathrm{Gr}_{\underline{\mathcal{G}},O} \otimes_O O_E$, where $\mathrm{Gr}_{\underline{\mathcal{G}}}$ is the Drinfeld-Beilinson (global) Grassmannian over $O[u]$ for $\underline{\mathcal{G}}$ and $\mathrm{Gr}_{\underline{\mathcal{G}},O} = \mathrm{Gr}_{\underline{\mathcal{G}}} \otimes_{O[u]} O$ is its base change to O by $u \mapsto \pi$. A priori, $M_{\mathcal{G},\mu}$ depends on the group scheme $\underline{\mathcal{G}}$ over $O[u]$ and the choice of the uniformizer π.

Theorem 2.7. *The $\mathcal{G}_{O_E}$-scheme $M_{\mathcal{G},\mu}$ over O_E, depends, up to equivariant isomorphism, only on the local model triple $(G, \{\mu\}, K)$.*

Proof. We first observe that $M_{\mathcal{G},\mu}$ can be constructed starting only with $\{\mu\}$, the base change $\underline{\mathcal{G}} \otimes_{O[u]} O[[u]]$, and the ideal $(u - \pi)$ in $O[[u]]$. Indeed, we first see that $\mathrm{Gr}_{\underline{\mathcal{G}},O}$ only depends on $\underline{\mathcal{G}} \otimes_{O[u]} O[[u]]$, and the ideal $(u - \pi)$ in $O[[u]]$. Set $t = u - \pi$. The base change $\mathrm{Gr}_{\underline{\mathcal{G}},O} = \mathrm{Gr}_{\underline{\mathcal{G}}} \otimes_{O[u]} O$ by $u \mapsto \pi$ has R-valued points for an O-algebra R given by the set of isomorphism classes of $\underline{\mathcal{G}}$-torsors over $R[t]$ with a trivialization over $R[t, 1/t]$. By the Beauville-Laszlo lemma (in the more general form given for example in [44, Lemma 6.1, Prop. 6.2]), this set is in bijection with the set of isomorphism classes of $\underline{\mathcal{G}} \otimes_{O[u]} R[[t]]$-torsors over $R[[t]] = R[[u]]$ together with a trivialization over $R((t)) = R[[u]][(u - \pi)^{-1}]$. To complete the proof we use Proposition 2.5. It gives that $\underline{\mathcal{G}} \otimes_{O[u]} O[[u]]$ only depends on the local model triple and π, hence $\mathrm{Gr}_{\underline{\mathcal{G}},O}$ only depends on the local model triple and π; for clarity, denote it by $\mathrm{Gr}_{\underline{\mathcal{G}},O,\pi}$. Part (b) of Proposition 2.5 with the above then gives that pulling back of torsors along $a : \mathrm{Spec}\, R[[u]] \to \mathrm{Spec}\, R[[u]]$, given by $u \mapsto a \cdot u$, gives an isomorphism

$$\mathrm{Gr}_{\underline{\mathcal{G}},O,\pi} \xrightarrow{\sim} \mathrm{Gr}_{\underline{\mathcal{G}},O,\pi'}.$$

Hence, by the above $\mathrm{Gr}_{\underline{\mathcal{G}},O}$ depends, up to equivariant isomorphism, only on G and K. The result then follows from the definition of $M_{\mathcal{G},\mu}$. □

Remark 2.8. We can obtain directly the independence of the base change $M_{\mathcal{G},\mu} \otimes_{O_E} \breve{O}_E$ via the same argument as above, by using the simpler Proposition 2.2 in place of Proposition 2.5.

2.6. **Pappas-Zhu local models: A variant.** It is not clear that the local models $M_{\mathcal{G},\mu}$ of [44] are well behaved when the characteristic p divides the order of $\pi_1(G_{\mathrm{der}})$. In particular, it is not clear if in this case, their special fiber is reduced and that they are normal. Motivated by an insight of Scholze, we employ z-extensions to slightly modify the definition of *loc. cit.* Suppose that $(G, \{\mu\}, K)$ is an LM triple over F such that G splits over a tame extension of F. Choose a z-extension over F

$$1 \longrightarrow T \longrightarrow \tilde{G} \longrightarrow G_{\mathrm{ad}} \longrightarrow 1. \tag{2.11}$$

In other words, $\tilde{G}$ is a central extension of G_{ad} by an strictly induced torus T and the reductive group $\tilde{G}$ has simply connected derived group, $\tilde{G}_{\mathrm{der}} = G_{\mathrm{sc}}$ (see for example, [37, Prop. 3.1]). (Here, we say that the torus T over F is strictly induced if it splits over a finite Galois extension F'/F and the cocharacter group $X_*(T)$ is a free $\mathbb{Z}[\mathrm{Gal}(F'/F)]$-module.) We can assume that $\tilde{G}$, and then also T, split over a tamely ramified extension of F. By

[37, Applic. 3.4], we can choose a cocharacter $\tilde{\mu}$ of $\tilde{G}$ which lifts μ_{ad} and which is such that the reflex field $\tilde{E}$ of $\{\tilde{\mu}\}$ is equal to the reflex field E_{ad} of $\{\mu_{\mathrm{ad}}\}$. Let $\tilde{K}$ be the unique parahoric subgroup of $\tilde{G}$ which lifts K_{ad}. Then the corresponding group scheme $\tilde{\mathcal{G}}$ fits in a fppf exact sequence of group schemes over O_F,

$$1 \longrightarrow \mathcal{T} \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G}_{\mathrm{ad}} \longrightarrow 1,$$

which extends the z-extension above, comp. [29, Prop. 1.1.4]. We set

$$\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) := M_{\tilde{\mathcal{G}},\tilde{\mu}} \otimes_{O_{E_{\mathrm{ad}}}} O_E$$

which is, again, a flat projective O_E-scheme equipped with an action of $\mathcal{G}_{O_E}$ (factoring through $\mathcal{G}_{\mathrm{ad},O_E}$) with generic fiber G_E-equivariantly isomorphic to $X_{\{\mu\}}$. Indeed, the action of $\tilde{\mathcal{G}}_{O_E}$ on $M_{\tilde{\mathcal{G}},\tilde{\mu}}$ factors through the quotient $\mathcal{G}_{\mathrm{ad},O_E} = \tilde{\mathcal{G}}_{O_E}/\mathcal{T}_{O_E}$ (because it does so on the generic fiber). Since $G \to G_{\mathrm{ad}}$ extends to a group scheme homomorphism $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$, we also obtain an action of $\mathcal{G}_{O_E}$ on $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$.

Remark 2.9. 1) By [44, Thm. 9.1], $M_{\tilde{\mathcal{G}},\tilde{\mu}}$ has reduced special fiber. Therefore, the same is true for the base change $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) = M_{\tilde{\mathcal{G}},\tilde{\mu}} \otimes_{O_{E_{\mathrm{ad}}}} O_E$. By [44, Prop. 9.2], it follows that $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ is a normal scheme.

2) If p does not divide the order of $\pi_1(G_{\mathrm{der}})$ then we have an equivariant isomorphism $M_{\tilde{\mathcal{G}},\tilde{\mu}} \otimes_{O_{E_{\mathrm{ad}}}} O_E \simeq M_{\mathcal{G},\mu}$, cf. [29, Prop. 2.2.7][1]. Therefore, in this case

$$\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) \simeq M_{\mathcal{G},\mu}.$$

We do not know whether this continues to hold when p divides the order of $\pi_1(G_{\mathrm{der}})$.

3) Suppose that $\tilde{G}' \to G_{\mathrm{ad}}$ is another choice of a z-extension as in (2.11) and let $\tilde{\mu}'$ be a cocharacter that also lifts μ_{ad} with reflex field $E = E_{\mathrm{ad}}$. Then the fibered product $H = \tilde{G} \times_{G_{\mathrm{ad}}} \tilde{G}' \to G$ is also a similar z-extension with kernel the direct product $T \times T'$ of the kernels of $\tilde{G} \to G_{\mathrm{ad}}$ and $\tilde{G}' \to G_{\mathrm{ad}}$. We have a cocharacter $\mu_H = (\tilde{\mu}, \tilde{\mu}')$ which also has reflex field E. The parahoric group scheme for H corresponding to $\mathcal{G}$ is $\mathcal{H} = \tilde{\mathcal{G}} \times_{\mathcal{G}_{\mathrm{ad}}} \tilde{\mathcal{G}}'$. We obtain $M_{\mathcal{H},\{\mu_H\}}$ as in [44]. By construction, we obtain

$$M_{\mathcal{H},\{\mu_H\}} \xrightarrow{\sim} M_{\tilde{\mathcal{G}},\{\tilde{\mu}\}}, \quad M_{\mathcal{H},\{\mu_H\}} \xrightarrow{\sim} M_{\tilde{\mathcal{G}}',\{\tilde{\mu}\}},$$

both $\mathcal{H}_{O_E}$-equivariant isomorphisms. Hence, we obtain an isomorphism $M_{\tilde{\mathcal{G}},\{\tilde{\mu}\}} \xrightarrow{\sim} M_{\tilde{\mathcal{G}}',\tilde{\{\mu\}}}$ which is $\mathcal{G}_{\mathrm{ad},O_E}$-equivariant. As a result, $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ is independent of the choice of the z-extension. We can now easily deduce from Theorem 2.5, that $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ also only depends on the local model triple $(G,\{\mu\},K)$.

Theorem 2.10. *The geometric special fiber $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) \otimes_{O_E} k$ is reduced and is $\mathcal{G} \otimes_{O_F} k$-equivariantly isomorphic to $\mathcal{A}_{\tilde{K}}(\tilde{G},\{\tilde{\mu}\})$.*

Proof. This follows from the construction and [44, Thm. 9.1, Thm. 9.3]. □

Note that this implies that the reduced k-scheme $\mathcal{A}_{\tilde{K}}(\tilde{G},\{\tilde{\mu}\})$ is independent of the choice of z-extension and only depends on $(G,\{\mu\},K)$. (This fact can be also seen more directly using Corollary 2.4 and [41, §6].) We call this *the μ-admissible locus* of the local model triple $(G,\{\mu\},K)$ and denote it by $\mathfrak{A}_K(G,\{\mu\})$.

[1] In loc. cit. $F = \mathbb{Q}_p$, but the result holds for general F.

Remark 2.11. It follows from [41, 6.a, 6.b] that $\breve{G}' \to \breve{G}'_{\mathrm{ad}}$ and $\breve{\tilde{G}}' \to \breve{G}'_{\mathrm{ad}}$ induce equivariant morphisms

$$\mathcal{A}_K(G,\{\mu\}) \longrightarrow \mathcal{A}_{K_{\mathrm{ad}}}(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\}), \quad \mathcal{A}_{\tilde{K}}(\tilde{G},\{\tilde{\mu}\}) \longrightarrow \mathcal{A}_{K_{\mathrm{ad}}}(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\})$$

which both induce bijections on k-points. As a result, we have equivariant bijections

$$\mathfrak{A}_K(G,\{\mu\})(k) = \mathcal{A}_K(G,\{\mu\})(k) = \mathcal{A}_{K_{\mathrm{ad}}}(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\})(k).$$

The following conjecture would characterize the local model $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ uniquely.

Conjecture 2.12. *Up to equivariant isomorphism, there exists a unique flat projective O_E-scheme $\mathbb{M}$ equipped with an action of $\mathcal{G}_{O_E}$ and the following properties.*
(a) Its generic fiber is G_E-equivariantly isomorphic to $X_{\{\mu\}}$.
(b) Its special fiber is reduced and there is a $\mathcal{G}\otimes_{O_F} k$-equivariant isomorphism of k-schemes

$$\mathbb{M}\otimes_{O_E} k \simeq \mathfrak{A}_K(G,\{\mu\}).$$

The local models constructed above have the following properties.

Proposition 2.13. *The following hold.*
(i) *If K is hyperspecial, then $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ is smooth over O_E.*
(ii) *If F'/F is a finite unramified extension, then*

$$\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})\otimes_{O_E} O_{E'} \xrightarrow{\sim} \mathbb{M}^{\mathrm{loc}}_{K'}(G\otimes_F F',\{\mu\otimes_F F'\}). \tag{2.12}$$

Note that here the reflex field E' of $(G\otimes_F F',\{\mu\otimes_F F'\})$ is the join of E and F'.
(iii) *If $(G,\{\mu\},K) = (G_1,\{\mu_1\},K_1)\times(G_2,\{\mu_2\},K_2)$, then*

$$\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) = \big(\mathbb{M}^{\mathrm{loc}}_{K_1}(G_1,\{\mu_1\})\otimes_{O_{E_1}} O_E\big)\times\big(\mathbb{M}^{\mathrm{loc}}_{K_2}(G_2,\{\mu_2\})\otimes_{O_{E_2}} O_E\big). \tag{2.13}$$

Note that here the reflex field E of $(G,\{\mu\})$ is the join of the reflex fields E_1 and E_2.
(iv) *If $\phi : (G,\{\mu\},K) \to (G',\{\mu'\},K')$ is a morphism of local model triples such that $\phi: G\to G'$ gives a central extension of G', there is a $\mathcal{G}_{O_E}$-equivariant isomorphism*

$$\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}) \xrightarrow{\sim} \mathbb{M}^{\mathrm{loc}}_{K'}(G',\{\mu'\})\otimes_{O_{E'}} O_E. \tag{2.14}$$

Proof. When K is hyperspecial, we can choose the extension $\underline{\tilde{\mathcal{G}}}$ over $O_F[u]$ to be reductive; then $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ is smooth as required in property (i). By choosing the extension $\underline{\tilde{\mathcal{G}}'} = \underline{\tilde{\mathcal{G}}}\otimes_{O_F[u]} O_{F'}[u]$, we easily obtain (ii). For (iii), we choose the extension $\underline{\tilde{\mathcal{G}}} = \underline{\tilde{\mathcal{G}}}_1\times\underline{\tilde{\mathcal{G}}}_2$ over $O_F[u]$. Finally, (iv) follows by the construction since $G_{\mathrm{ad}} = G'_{\mathrm{ad}}$. □

2.7. **Scholze local models.** Under special circumstances, we can relate the local models above to Scholze local models and give in this way a characterization of them different from Conjecture 2.12. In particular, this gives a different way of proving the independence of all choices in the construction of local models. Recall Scholze's conjecture [49, Conj. 21.4.1] that there exists a flat projective O_E-scheme $\mathbb{M}^{\mathrm{loc,flat}}_{\mathcal{G},\mu}$ with generic fiber $X_{\{\mu\}}$ and reduced special fiber and with an equivariant closed immersion of the associated diamond, $\mathbb{M}^{\mathrm{loc,flat},\diamond}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}O_E}$. Scholze proves that $\mathbb{M}^{\mathrm{loc,flat}}_{\mathcal{G},\mu}$ is unique if it exists, cf. [49, Prop. 18.3.1]. Note that Scholze does not make the hypothesis that G split over a tame extension. We are going to exhibit a class of LM triples $(G,\{\mu\},K)$ (with G split over a tame extension) such that the local models $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ defined above satisfy Scholze's conjecture.

We will say that a pair $(G,\{\mu\})$, consisting of a reductive group over F and a geometric conjugacy class of minuscule coweights is of *abelian type* when there is a similar pair $(G_1,\{\mu_1\})$ with $E_1\subset E\breve{F}$ and with a central isogeny $\phi: G_{1,\mathrm{der}}\to G_{\mathrm{der}}$ which induces an isomorphism $(G_{1,\mathrm{ad}},\{\mu_{1,\mathrm{ad}}\})\simeq(G_{\mathrm{ad}},\{\mu_{\mathrm{ad}}\})$ and is such that there exists a faithful minuscule representation $\rho_1: G_1\hookrightarrow \mathrm{GL}_n$ over F such that $\rho_1\circ\mu_1$ is a minuscule cocharacter μ_d of GL_n. Here by a minuscule representation we mean a direct sum of irreducible minuscules (i.e., with all weights conjugate by the Weyl group.) In this case, we call such a pair $(G_1,\{\mu_1\})$ a *realization* of the pair $(G,\{\mu\})$ of abelian type.

Theorem 2.14. *Let $(G,\{\mu\},K)$ be a LM triple over F such that G splits over a tame extension of F, for which there is an unramified finite extension F'/F such that the base change $(G,\{\mu\})\otimes_F F'$ is of abelian type, with realization $(G_1,\{\mu_1\})$ such that $p\nmid|\pi_1(G_{1,\mathrm{der}})|$. Then the local model $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ defined above satisfies Scholze's conjecture* [49, Conj. 21.4.1].

Proof. We already checked that the flat projective scheme $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has reduced special fiber. To show the conjecture it remains to show that the associated diamond $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})^\diamond$ over $\mathrm{Spd}(O_E)$ embeds via an equivariant closed immersion in $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(O_E)}$ such that its generic fiber identifies with $X^\diamond_{\{\mu\}}$.

Using étale descent along F'/F and property (ii) of Proposition 2.13, we see that it is enough to show the conjecture for $(G,\{\mu\})\otimes_F F'$; so, we can assume that $(G,\{\mu\})$ is of abelian type to begin with. Let $(G_1,\{\mu_1\})$ be a realization. In fact, we can also assume that $E_1\subset E$. Observe that by using ϕ, we obtain a parahoric subgroup K_1 of G_1 which corresponds to K. By [29, Prop. 1.3.3], $\rho_1: G_1\hookrightarrow\mathrm{GL}_n$ extends to a closed immersion

$$\rho_1:\mathcal{G}'_1\hookrightarrow\mathcal{GL},$$

where $\mathcal{G}'_1$ is the stabilizer (possibly non connected) of a point in the building of $G_1(F)$ that corresponds to K_1 and where $\mathcal{GL}$ is a certain parahoric group scheme for GL_n. In fact, by replacing ρ' by a direct sum $\rho'^{\oplus m}$ we can assume that $\mathcal{GL}$ is GL_n over O_F; we will do this in the rest of the proof. By [49, Prop. 21.4.3], $\mathrm{Gr}_{\mathcal{G}'_1,\mathrm{Spd}(O_E),\mu_1}=\mathrm{Gr}_{\mathcal{G}_1,\mathrm{Spd}(O_E),\mu_1}$, where $\mathcal{G}_1=(\mathcal{G}'_1)^\circ$. This gives a closed immersion $\mathrm{Gr}_{\mathcal{G}_1,\mathrm{Spd}(O_E),\mu_1}\hookrightarrow\mathrm{Gr}_{\mathcal{GL},\mathrm{Spd}(O_E)}$. By [29, Prop. 2.3.7], $\rho_1:\mathcal{G}'_1\hookrightarrow\mathcal{GL}$ induces

$$M_{\mathcal{G}_1,\{\mu_1\}}\hookrightarrow(M_{\mathrm{GL}_n,\{\mu_d\}})_{O_{E_1}}=\mathrm{Gr}(d,n)_{O_{E_1}},$$

which is also an equivariant closed immersion. (Here the local model $M_{\mathrm{GL}_n,\{\mu_d\}}$ is the Grassmannian $\mathrm{Gr}(d,n)$ over O_F.) By the assumption $p\nmid|\pi_1(G_{1,\mathrm{der}})|$, Remark 2.9 above gives that $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})\simeq M_{\mathcal{G}_1,\mu_1}\otimes_{O_{E_1}}O_E$. This allows us to reduce the result to the case of GL_n which is dealt with by [49, Cor. 21.5.10]. □

We view Theorem 2.14 as evidence for the following conjecture.

Conjecture 2.15. *For all local model triples $(G,\{\mu\},K)$ with G split over a tame extension, the local model $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ defined in the last subsection satisfies Scholze's Conjecture* [49, Conj. 21.4.1].

It has in any case the following concrete consequence.[2]

[2]We were recently informed that a similar result, which also covers cases of wildly ramified groups, was obtained by J. Lourenco (forthcoming Bonn thesis).

Corollary 2.16. *Suppose that $(G,\{\mu\},K)$ is an LM triple with G adjoint and classical such that G splits over a tame extension of F. Assume that there exists a product decomposition over $\breve{F}$, $G\otimes_F \breve{F}=\breve{G}_1\times\cdots\times\breve{G}_m$, where each factor $\breve{G}_i$ is* absolutely simple. *If there is a factor for which $(\breve{G}_i,\{\mu_i\})\otimes_{\breve{F}}\bar{F}$ is of type $(D_n,\omega_n^\vee)$ with $n\geq 4$ (i.e., of type $D_n^{\mathbb{H}}$ in Deligne's notation* [10, Tables 1.3.9, 2.3.8]*), also assume that p is odd. Then the local model $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ defined above satisfies Scholze's conjecture* [49, Conj. 21.4.1].

Proof. We will show that such a LM triple $(G,\{\mu\},K)$ is, after an unramified extension, of abelian type. Using our assumption, we can easily reduce to the case that G is absolutely simple, quasi-split and residually split. The possible pairs $(G,\{\mu\})$ with such G and $\{\mu\}$ minuscule, are listed in the first two tables in §4. A case–by–case check gives that, when G is a classical group, we can find a realization $(G_1,\{\mu_1\})$ of $(G,\{\mu\})$ as a pair of abelian type such that $G_{1,\mathrm{der}}$ is simply connected—except when the type of $G_{\bar{F}}$ is D_n. (See [10, Rem. 3.10].) In the latter case we can find a realization with $G_{1,\mathrm{der}}$ simply connected in the case $(D_n,\omega_1^\vee)$ (i.e., of type $D_n^{\mathbb{R}}$ in Deligne's notation), and a realization where $\pi_1(G_{1,\mathrm{der}})$ has order 2 in the case $(D_n,\omega_n^\vee)$ (i.e., of type $D_n^{\mathbb{H}}$ in Deligne's notation). (For types A_n, C_n and $D_n^{\mathbb{H}}$, the minuscule representation ρ_1 is given over $\bar{F}$ by a sum of corresponding standard representations, for types B_n and $D_n^{\mathbb{R}}$, is given by a sum of spin representations.) In all cases, we can pick μ_1 so that $E_1=E$. The result follows from Theorem 2.14. □

3. Shimura varieties

3.1. **Consequences for Shimura varieties.** Let $(\mathbf{G},\mathbf{X})$ be a Shimura datum. We fix a prime $p>2$ such that $G:=\mathbf{G}\otimes_{\mathbb{Q}}\mathbb{Q}_p$ splits over a tamely ramified extension of $\mathbb{Q}_p$. We consider open compact subgroups $\mathbf{K}$ of $\mathbf{G}(\mathbb{A}_f)$ of the form $\mathbf{K}=\mathbf{K}^p\cdot K_p\subset\mathbf{G}(\mathbb{A}_f^p)\times G(\mathbb{Q}_p)$, where $K=K_p$ is a parahoric subgroup of $G(\mathbb{Q}_p)$ and $\mathbf{K}^p$ is sufficiently small. Let $\mathbf{E}$ be the reflex field of $(\mathbf{G},\mathbf{X})$. Fixing an embedding $\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ determines a place $\mathbf{p}$ of $\mathbf{E}$ over p. Let $E=\mathbf{E}_{\mathbf{p}}$. Then E is the reflex field of $(G,\{\mu\})$, where $\{\mu\}$ is the conjugacy class of cocharacters over $\overline{\mathbb{Q}}_p$ associated to $\mathbf{X}$. We denote by the same symbol $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$ the canonical model of the Shimura variety over $\mathbf{E}$ and its base change over E.

Theorem 3.1. *a)* ([29]) *Assume that $(\mathbf{G},\mathbf{X})$ is of abelian type. Then there exists a scheme $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ over O_E with right $\mathbf{G}(\mathbb{A}_f^p)$-action such that:*

1) Any sufficiently small open compact $\mathbf{K}^p\subset\mathbf{G}(\mathbb{A}_f^p)$ acts freely on $\mathcal{S}_K(\mathbf{G},\mathbf{X})$, and the quotient $\mathcal{S}_{\mathbf{K}}(\mathbf{G},\mathbf{X}):=\mathcal{S}_K(\mathbf{G},\mathbf{X})/\mathbf{K}^p$ is a scheme of finite type over O_E which extends $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$. Furthermore

$$\mathcal{S}_K(\mathbf{G},\mathbf{X})=\varprojlim_{\mathbf{K}^p}\mathcal{S}_{\mathbf{K}^pK}(\mathbf{G},\mathbf{X}),$$

where the limit is over all such $\mathbf{K}^p\subset\mathbf{G}(\mathbb{A}_f^p)$.

2) For every closed point x of $\mathcal{S}_K(\mathbf{G},\mathbf{X})$, there is a closed point y of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ such that the strict henselizations of $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ at x and of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ at y are isomorphic.

3) The scheme $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ has the extension property: For every discrete valuation ring $R\supset O_E$ of characteristic $(0,p)$ the map

$$\mathcal{S}_K(\mathbf{G},\mathbf{X})(R)\longrightarrow\mathcal{S}_K(\mathbf{G},\mathbf{X})(R[1/p])$$

is a bijection.

b) ([29]) *Assume that* $(\mathbf{G}, \mathbf{X})$ *is of Hodge type, that* K *is the stabilizer of a point in the Bruhat-Tits building of* G*, and that* p *does not divide* $|\pi_1(G_{\mathrm{der}})|$*. Then the model* $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$ *of (a) above admits a* $\mathbf{G}(\mathbb{A}_f^p)$*-equivariant* local model diagram *over* O_E*,*

$$\begin{array}{ccc} & \tilde{\mathcal{S}}_K(\mathbf{G}, \mathbf{X}) & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \tilde{\varphi}} \\ \mathcal{S}_K(\mathbf{G}, \mathbf{X}) & & \mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\}), \end{array} \tag{3.1}$$

in which π *is a torsor under the group scheme* $\mathcal{G}_{O_E}$*, and* $\tilde{\varphi}$ *is a* $\mathcal{G}_{O_E}$*-equivariant and smooth morphism of relative dimension* $\dim G$*.*

c) ([53, Thm. 8.2], [25, Thm. 4.1]) *Under the assumptions of (b) above, the morphism* $\tilde{\varphi}$ *in the local model diagram (3.1) is surjective.* □

Remark 3.2. Part (a2) appears as [29, Thm. 0.2], but is stated there for the original local models of [44], and under the assumption $p \nmid |\pi_1(G_{\mathrm{der}})|$. The statement above is for the modified local models of this paper and can be deduced by the results in [29]. Part (b) follows from [29, Thm. 4.2.7] and Remark 2.9 (2).

Definition 3.3. Let O be a discrete valuation ring and suppose that X is a locally noetherian scheme over O.

(i) X is said to have *good reduction* over O if X is smooth over O.

(ii) X is said to have *semi-stable reduction* over O if the special fiber is a normal crossings divisor in the sense of [50, Def. 40.21.4].

Both properties are local for the étale topology around each closed point of X and imply that X is a regular scheme with reduced special fiber.

Corollary 3.4. *Assume that* $(\mathbf{G}, \mathbf{X})$ *is a Shimura datum of abelian type. If the local model* $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ *has good, resp. semi-stable, reduction over* O_E*, then so does* $\mathcal{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{X})$*. If* $(\mathbf{G}, \mathbf{X})$ *is of Hodge type and satisfies the assumptions of Theorem 3.1 (b), then the converse also holds.*

Proof. The first assertion follows from Theorem 3.1 (a). The second assertion follows from (b) and (c). □

3.2. **Canonical nature of integral models.** We now explain how an idea of Milne [36] together with results of Vasiu and Zink ([52]) imply that, under some additional assumptions, the integral models $\mathcal{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{X})$ with good or semi-stable reduction over an unramified extension of $\mathbb{Z}_p$, are "canonical". More precisely, we have:

Corollary 3.5. *Assume that* $(\mathbf{G}, \mathbf{X})$ *is a Shimura datum of abelian type. Suppose that* $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ *has good or semi-stable reduction over* O_E*, that* $E/\mathbb{Q}_p$ *is unramified, and that the geometric special fiber* $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\}) \otimes_{O_E} k$ *has no more than* $2p-3$ *irreducible components. Then* $\mathcal{S}_K(\mathbf{G}, \mathbf{X})$ *is, up to isomorphism, the* unique *O_E-faithfully flat* $\mathbf{G}(\mathbb{A}_f^p)$*-equivariant integral model of* $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ *that satisfies (1), (2) and the following stronger version of (3): The bijection*

$$\mathcal{S}_K(\mathbf{G}, \mathbf{X})(R) \xrightarrow{\sim} \mathcal{S}_K(\mathbf{G}, \mathbf{X})(R[1/p])$$

holds for R *any* O_E*-faithfully flat algebra which is either a dvr, or a regular ring which is* healthy *in the sense of* [52].

Proof. Note that under our assumption, by [52, Thm. 3, Cor. 5] (see also loc. cit., p. 594), the scheme $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is *regular healthy*, when the maximum number of transversely intersecting smooth components of its special fiber is $\leq 2p-3$. Then, by Theorem 3.1 (a), the same is true for $\mathcal{S}_K(\mathbf{G},\mathbf{X})$. By the construction of $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ in [29] and [52], it then follows that the limit $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ also satisfies the extension property not just for dvr's but for all regular healthy schemes. The uniqueness part of the statement then also follows (see also [36], [29]). □

Consider the cases of smooth or semi-stable reduction covered by the results in this paper, see Theorems 1.2 and 1.3, for $F=\mathbb{Q}_p$: it turns out that the number r of geometric irreducible components of the special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is ≤ 2 in all cases, except in the first case of Theorem 5.6 (the Drinfeld case). In the latter case, this number r is equal to the number of lattices in the primitive part of the periodic lattice chain. Since we assume that p is odd to begin with, we obtain:

Theorem 3.6. *Assume that $(\mathbf{G},\mathbf{X})$ is a Shimura datum of abelian type such that the corresponding LM triple $(G,\{\mu\},K)$ satisfies the hypothesis of either Theorem 1.2 or Theorem 1.3, with $F=\mathbb{Q}_p$. Then, unless $(G,\{\mu\},K)$ corresponds to the "Drinfeld case" of Theorem 5.6, the model $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ is canonical, i.e., it satisfies the conclusion of Corollary 3.5. If $(G,\{\mu\},K)$ corresponds to the Drinfeld case of Theorem 5.6, then $\mathcal{S}_K(\mathbf{G},\mathbf{X})$ is canonical, provided that K is the connected stabilizer of a facet in the building of* PGL_n *that is of dimension $\leq 2p-4$.* □

Example 3.7. Consider the group $\mathbf{G}=\mathrm{GSpin}(\mathbf{V})$, where $\mathbf{V}$ is a (non-degenerate) orthogonal space of dimension $2n\geq 8$ over $\mathbb{Q}$ of signature $(2n-2,2)$ over $\mathbb{R}$. Take

$$\mathbf{X}=\{v\in\mathbf{V}\otimes_{\mathbb{Q}}\mathbb{C}\mid \langle v,v\rangle=0,\langle v,\bar{v}\rangle<0\}/\mathbb{C}^*.$$

(Here $\langle\ ,\ \rangle$ is the corresponding symmetric bilinear form.) The group $\mathbf{G}(\mathbb{R})$ acts on $\mathbf{X}$ via $\mathbf{G}\to\mathrm{SO}(\mathbf{V})$ and $(\mathbf{G},\mathbf{X})$ is a Shimura datum of Hodge type.

Suppose that there exists a pair (Λ_0,Λ_n) of $\mathbb{Z}_p$-lattices in $\mathbf{V}\otimes_{\mathbb{Q}}\mathbb{Q}_p$, with $\Lambda_0^\vee=\Lambda_0$, $\Lambda_n^\vee=p\Lambda_n$, and $p\Lambda_n\subset\Lambda_0\subset\Lambda_n$. Let $K_p\subset\mathbf{G}(\mathbb{Q}_p)$ be the parahoric subgroup which corresponds to the connected stabilizer of this lattice chain. By combining Theorem 5.6 and the above, we obtain that, for small enough $\mathbf{K}^p$, the Shimura variety $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G},\mathbf{X})$ has a canonical $\mathbb{Z}_p$-integral model with semi-stable reduction. In fact, we can see, using the calculations in Subsection 12.8, that the integral model is locally smoothly equivalent to $\mathbb{Z}_p[x,y]/(xy-p)$. This integral model was found by Faltings [13] as an application of his theory of $\mathcal{MF}$-objects over semi-stable bases.

4. Rapoport-Zink spaces

We consider RZ-spaces of EL-type or PEL-type, cf. [47]. We place ourselves in the situation described in [46, §4].

4.1. **The formal schemes.** In the EL-case, we start with *rational RZ data* of EL-type

$$\mathcal{D}=(F,B,V,G,\{\mu\},[b]).$$

Here F is a finite extension of $\mathbb{Q}_p$, B is a central division algebra over F, V is a finite-dimensional B-module, $G=\mathrm{GL}_B(V)$ as algebraic group over $\mathbb{Q}_p$, $\{\mu\}$ is a conjugacy class of minuscule cocharacters of G, and $[b]\in A(G,\{\mu\})$ is an *acceptable* σ-conjugacy class in $G(\breve{\mathbb{Q}}_p)$. Let $E=E_{\{\mu\}}$ be the corresponding reflex field inside $\overline{\mathbb{Q}}_p$. In addition, we fix

integral RZ data $\mathcal{D}_{\mathbb{Z}_p}$, i.e., a periodic lattice chain of O_B-modules $\Lambda_\bullet$ in V. This lattice chain defines a parahoric group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ with generic fiber G.

In the PEL-case, we start with *rational RZ data* of PEL-type

$$\mathcal{D} = (F, B, V, (\ ,\ ), *, G, \{\mu\}, [b]).$$

Here F, B and V are as in the EL-case, $(\ ,\ )$ is a non-degenerate alternating $\mathbb{Q}_p$-bilinear form on V, $*$ is an involution on B, $G = \mathrm{GSp}_B(V)$ as algebraic group over $\mathbb{Q}_p$, and $\{\mu\}$ and $[b]$ are as before. We refer to [46] for the precise conditions these data have to satisfy. In addition, we fix *integral RZ data* $\mathcal{D}_{\mathbb{Z}_p}$, i.e., a periodic self-dual lattice chain of O_B-modules $\Lambda_\bullet$ in V. In the PEL case we make the following assumptions.

- $p \neq 2$.
- G is connected.
- The stabilizer group scheme $\mathcal{G}$ is a parahoric group scheme over $\mathbb{Z}_p$.

Then in all cases $(G, \{\mu\}, \mathcal{G})$ is a LM triple over $\mathbb{Q}_p$. As in Section 2, we sometimes write the LM triple as $(G, \{\mu\}, K)$ with $K = \mathcal{G}(\mathbb{Z}_p)$.

Let $O_{\breve{E}}$ be the ring of integers in $\breve{E}$ (the completion of the maximal unramified extension of E). In either EL or PEL case, after fixing a *framing object* $\mathbb{X}$ over k (the residue field of $O_{\breve{E}}$), we obtain a formal scheme locally formally of finite type over $\mathrm{Spf}\, O_{\breve{E}}$ which represents a certain moduli problem of p-divisible groups on the category $\mathrm{Nilp}_{O_{\breve{E}}}$. We denote this formal scheme by $\mathcal{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$. The reason for the upper index is that we impose only the *Kottwitz condition* on the p-divisible groups appearing in the formulation of the moduli problem. In particular, $\mathcal{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$ need not be flat over $\mathrm{Spf}\, O_{\breve{E}}$.

Analogously, associated to $\mathcal{D}_{\mathbb{Z}_p}$, there is the local model $\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$, a projective scheme over O_E equipped with an action of $\mathcal{G}_{O_E} = \mathcal{G} \otimes_{\mathbb{Z}_p} O_E$. Furthermore, there is a local model diagram of morphisms of formal schemes over $\mathrm{Spf}\, O_{\breve{E}}$,

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}} & & \\ & \swarrow^{\pi} & & \searrow^{\tilde{\varphi}} & \\ \mathcal{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}} & & & & (\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}})^\wedge, \end{array} \tag{4.1}$$

in which π is a torsor under the group scheme $\mathcal{G}_{O_E}$, and $\tilde{\varphi}$ is a $\mathcal{G}_{O_E}$-equivariant and formally smooth morphism of relative dimension $\dim G$. Here $(\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}})^\wedge$ denotes the completion of $\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}} \otimes_{O_E} O_{\breve{E}}$ along its special fiber.

Lemma 4.1. *Assume that the group G attached to the rational RZ-data $\mathcal{D}$ splits over a tame extension of $\mathbb{Q}_p$. Then the modified PZ-local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ of Subsection 2.6 attached to the LM triple $(G, \{\mu\}, \mathcal{G})$ is a closed subscheme of $\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$, with identical generic fiber.*

Proof. Notice that under our assumptions, since this is always true in the EL case, G is connected. We can see that, under our assumptions, p does not divide $|\pi_1(G_{\mathrm{der}})|$. Indeed, this is clear in the EL case since then $(G_{\mathrm{der}})_{\bar{\mathbb{Q}}_p}$ is a product of special linear groups SL. In the PEL case, $(G_{\mathrm{der}})_{\bar{\mathbb{Q}}_p}$ is the product of groups of types SL, Sp, SO, and our assumptions include that p is odd. It follows from 2.9 (2) that $\mathbb{M}^{\mathrm{loc}}_K(\mathcal{G}, \{\mu\}) \simeq M_{\mathcal{G},\mu}$. By [44, (8.3)],

under the above assumptions again (in particular, the fact that G is connected is used), the local model $M_{\mathcal{G},\mu}$ agrees with the flat closure of the generic fiber of the naive local model $\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$. The result follows. □

We now use the local model diagram (4.1) to define a closed formal subscheme $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ of $\mathcal{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$, defined by an ideal sheaf killed by a power of the uniformizer of O_E. We therefore obtain a local model diagram

$$\begin{array}{ccc} & \widetilde{\mathcal{M}}_{\mathcal{D}_{\mathbb{Z}_p}} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\tilde{\varphi}} \\ \mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}} & & (\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\}))^\wedge, \end{array} \tag{4.2}$$

as in (4.1). Again, the left oblique arrow is a torsor under $\mathcal{G}_{O_E}$, and the right oblique arrow is $\mathcal{G}_{O_E}$-equivariant and formally smooth of relative dimension $\dim G$.

Corollary 4.2. *Assume that the group G attached to the rational RZ-data $\mathcal{D}$ splits over a tame extension of $\mathbb{Q}_p$. If the local model $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has good, resp. semi-stable, reduction over O_E, then so does $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$.*

Proof. This follows by descent from the local model diagram. □

Remark 4.3. In contrast to Corollary 3.4, the converse does not hold in general.

Proposition 4.4. *Assume that the group G attached to the rational RZ-data $\mathcal{D}$ splits over a tame extension of $\mathbb{Q}_p$. Then the formal scheme $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ is flat over $\operatorname{Spf} O_{\breve{E}}$ and normal. Furthermore, it only depends on $\mathcal{D}_{\mathbb{Z}_p}$ through the quadruple $(G,\{\mu\},\mathcal{G},[b])$. Finally,*

$$\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}(k) = \bigcup_{w\in \mathrm{Adm}_{\breve{K}}(\{\mu\})} X_w(b). \tag{4.3}$$

Here $\mathrm{Adm}_{\breve{K}}(\{\mu\}) \subset W_{\breve{K}}\backslash\widetilde{W}/W_{\breve{K}}$ denotes the admissible set. Also $X_w(b)$ denotes for $w \in W_{\breve{K}}\backslash\widetilde{W}/W_{\breve{K}}$ the affine Deligne-Lusztig set

$$X_w(b) = \{g \in G(\breve{\mathbb{Q}}_p)/\breve{K} \mid g^{-1}b\sigma(g) \in \breve{K}w\breve{K}\}$$

where b is a fixed representative of $[b]$.

Proof. Flatness and normality follows via the local model diagram from the corresponding properties of $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$, cf. Remark 2.9 (1). The uniqueness statement follows from [49, Cor. 25.1.3]. The final statement follows from [49, Cor. 25.1.3] and Theorem 2.10 together with (2.6) and the definition [49, Def. 25.1.1] of the v-sheaf $\mathcal{M}^{\mathrm{int}}_{(\mathcal{G},\mu,b)}$ by observing the following: In the definition of $\mathcal{M}^{\mathrm{int}}_{(\mathcal{G},\mu,b)}$ we can take, by Corollary 2.16, the local model $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ to give the "diamond" local model v-sheaf $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ used there. □

4.2. **The RZ tower.** We now pass to the RZ-tower of rigid-analytic spaces $\big(M_K, K \subset G(\mathbb{Q}_p)\big)$, cf. [46, §4.15]. For its formation, we can start with $\mathbb{M}^{\mathrm{naive}}_{\mathcal{D}_{\mathbb{Z}_p}}$ for an arbitrary integral RZ datum $\mathcal{D}_{\mathbb{Z}_p}$ for $\mathcal{D}$; in particular, we need not assume that G is tamely ramified.

Proposition 4.5. *The RZ-tower (M_K) depends only on the rational RZ datum $\mathcal{D}$ through the triple $(G, \{\mu\}, [b])$. Furthermore, if it is non-empty, then $[b] \in B(G, \{\mu\})$. The converse holds if G splits over a tamely ramified extension of $\mathbb{Q}_p$.*

Proof. The first assertion follows from [49, Cor. 24.3.5]. The second assertion is [46, Prop. 4.19]. To prove the converse, using flatness of $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$, it suffices to prove $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}(k) \neq \emptyset$. Via the identification (4.3), this follows from [24]. □

Remark 4.6. The uniqueness statement is conjectured in [46, Conj. 4.16] without the tameness assumption. The converse statement is conjectured in [46, Conj. 4.21], again without the tameness assumption.

5. Statement of the main results

5.1. **Good reduction.** In the following, we call the LM triple $(G, \{\mu\}, K)$ of *exotic good reduction type* if $p \neq 2$ and if the corresponding adjoint LM triple $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\}, K_{\mathrm{ad}})$ is isomorphic to the adjoint LM triple associated to one of the following two LM triples.

1) (*Unitary exotic reduction*)

- $G = \mathrm{Res}_{F'/F} G'$. Here F'/F is an unramified extension, and $G' = U(V)$, with V a $\tilde{F}'/F'$-hermitian vector space of dimension ≥ 3, where $\tilde{F}'/F'$ is a *ramified* quadratic extension.
- $\{\mu\} = \{\mu_\varphi\}_{\varphi\colon F' \to \overline{F}}$, with $\{\mu_\varphi\} = (1, 0, \dots, 0)$ or $\{\mu_\varphi\} = (0, 0, \dots, 0)$, for any φ.
- $K = \mathrm{Res}_{O_{F'}/O_F}(K')$, with $K' = \mathrm{Stab}(\Lambda)$, where Λ is a π-modular or almost π-modular vertex lattice in V, i.e., $\Lambda^\vee = \pi_{\tilde{F}'}^{-1}\Lambda$ if $\dim V$ is even, resp. $\Lambda \subset \Lambda^\vee \subset^1 \pi_{\tilde{F}'}^{-1}\Lambda$ if $\dim V$ is odd.

2) (*Orthogonal exotic reduction*)

- $G = \mathrm{Res}_{F'/F} G'$. Here F'/F is an unramified extension, and $G' = \mathrm{GO}(V)$, with V an orthogonal F'-vector space of even dimension $2n \geq 6$.
- $\{\mu\} = \{\mu_\varphi\}_{\varphi\colon F' \to \overline{F}}$, with $\{\mu_{\mathrm{ad},\varphi}\} = (1^{(n)}, 0^{(n)})_{\mathrm{ad}}$ or $\{\mu_{\mathrm{ad},\varphi}\} = (0, 0, \dots, 0)$, for any φ.
- $K = \mathrm{Res}_{O_{F'}/O_F}(K')$, with $K' = \mathrm{Stab}(\Lambda)$, where Λ is an almost selfdual vertex lattice in V, i.e., $\Lambda \subset^1 \Lambda^\vee \subset \pi_{F'}^{-1}\Lambda$.

Theorem 5.1. *Let $(G, \{\mu\}, K)$ be a triple over F such that G splits over a tame extension of F. Assume $p \neq 2$. Assume that G_{ad} is F-simple, that in the product decomposition over $\breve{F}$,*

$$G_{\mathrm{ad}} \otimes_F \breve{F} = \prod\nolimits_i \breve{G}_{\mathrm{ad},i}$$

each factor is absolutely simple*, and that μ_{ad} is not trivial. Then the local model $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ is smooth over* $\mathrm{Spec}\, O_E$ *if and only if K is hyperspecial or (G, μ, K) is a triple of exotic good reduction type*[3]*.*

We are going to use the following dévissage lemma.

[3]Haines-Richarz [22] gives an alternative explanation for the smoothness of $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ in the case of exotic good reduction type for the *even unitary* case and the *orthogonal* case: in these cases, the special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ can be identified with a Schubert variety attached to a *minuscule* cocharacter in the twisted affine Grassmannian corresponding to the special maximal parahoric K.

Lemma 5.2. *a) Let F'/F be a finite unramified extension contained in $\breve{F}$. Let*

$$(G, \{\mu\}, K) \otimes_F F' = \prod\nolimits_i (G_i, \{\mu_i\}, K_i),$$

where $(G_i, \{\mu_i\}, K_i)$ are LM triples over F'. Then $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ is smooth over $\operatorname{Spec} O_E$ if and only if $\mathbb{M}^{\mathrm{loc}}_{K_i}(G_i, \{\mu_i\})$ is smooth over $\operatorname{Spec} O_{E_i}$ for all i.
b) Let $(G', \{\mu'\}, K') \to (G, \{\mu\}, K)$ be a morphism of triples such that $G' \to G$ gives a central extension. Then $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ is smooth over $\operatorname{Spec} O_E$ if and only $\mathbb{M}^{\mathrm{loc}}_{K'}(G', \{\mu'\})$ is smooth over $\operatorname{Spec} O_{E'}$.

Proof. This follows from properties (ii)–(iv) of Proposition 2.13. □

The lemma implies that, in order to prove Theorem 5.1, we may assume that G_{ad} is absolutely simple and that μ_{ad} is not trivial. That $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ is smooth over $\operatorname{Spec} O_E$ when K is hyperspecial is property (i) of Proposition 2.13. The case of unitary exotic good reduction is treated in [1, Prop. 4.16], comp. [43, Thm. 2.27, (iii)]. The case of orthogonal exotic good reduction is discussed in Subsection 12.11.

The proof of the converse proceeds in three steps. In a first step, we establish a list of all cases in which the special fiber of $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ is irreducible, i.e., $\mathfrak{A}_K(G, \{\mu\})$ is a single Schubert variety in the corresponding affine partial flag variety. This is done in Section 7. In a second step, we go through this list and eliminate the cases when K is not a special maximal parahoric by showing that in those cases the special fiber is not smooth (in fact, not even *rationally smooth*, in the sense explained in Section 6). This is done in Section 8. Finally, we deal with the cases when K is a special maximal parahoric; most of these can be also dealt with by the same methods. In a few cases, we need to refer to certain explicit calculations of the special fibers given in [42], [1], and, in one exceptional type, appeal to the result of Haines-Richarz [22].

5.2. **Weyl group notation.** Recall that simple adjoint groups $\breve{G}$ over $\breve{F}$ are classified up to isomorphism by their associated *local Dynkin diagram*[4], cf. [51, §4]. Recall that to a local Dynkin diagram $\tilde{\Delta}$ there is associated its Coxeter system, cf. [7], which is of affine type. The associated Coxeter group is the affine Weyl group W_a. We denote by $\tilde{W}$ its extended affine Weyl group. Both W_a and $\tilde{W}$ are extensions of the finite Weyl group W_0 by translation subgroups, i.e., finitely generated free $\mathbb{Z}$-modules. We denote by X_* the translation subgroup of $\tilde{W}$.

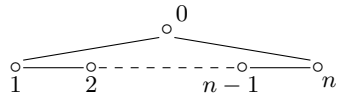
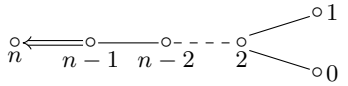
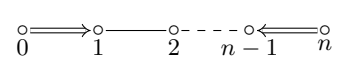
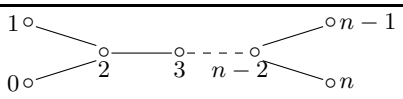
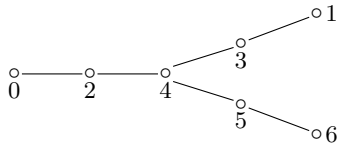
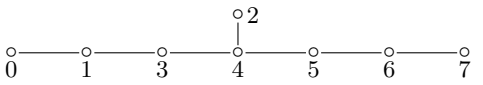
Definition 5.3. (i) An *enhanced Tits datum* is a triple $(\tilde{\Delta}, \{\lambda\}, \tilde{K})$ consisting of a local Dynkin diagram $\tilde{\Delta}$, a W_0-conjugacy class $\{\lambda\}$ of elements in X_*, and a non-empty subset $\tilde{K}$ of the set $\tilde{S}$ of vertices of $\tilde{\Delta}$.
(ii) An *enhanced Coxeter datum* is a triple $\big((W_a, \tilde{S}), \{\lambda\}, \tilde{K}\big)$ consisting of a Coxeter system $(W_a, \tilde{S})$ of affine type, a W_0-conjugacy class $\{\lambda\}$ of elements in X_*, and a non-empty subset $\tilde{K}$ of $\tilde{S}$.

Note that the Coxeter system $(W_a, \tilde{S})$ is given by its associated Coxeter diagram, cf. [7, Chapter VI, §4, Thm. 4]. The Coxeter diagram associated to a local Dynkin diagram is obtained by disregarding the arrows in the local Dynkin diagram. An enhanced Tits datum determines an enhanced Coxeter datum. The natural map from the set of enhanced Tits data to the set of enhanced Coxeter data is not injective.

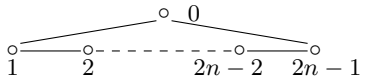
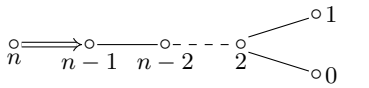
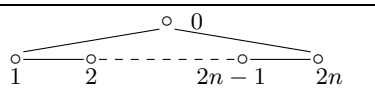
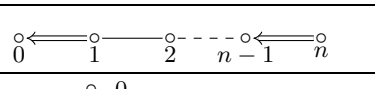
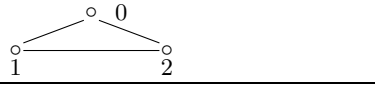
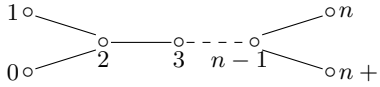
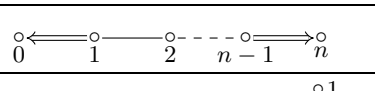
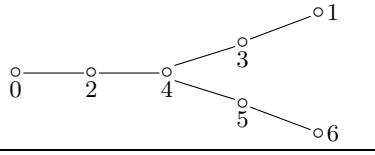
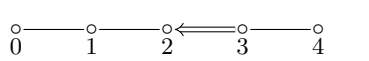
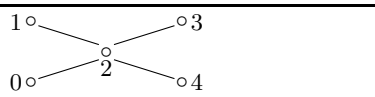
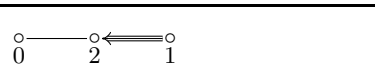
[4]Note that only the first batch of cases on Tits' list is relevant since $\breve{G}$ is automatically residually split.

Let $(G, \{\mu\}, K)$ be a LM triple over F such that G is adjoint and absolutely simple. We associate as follows an enhanced Tits datum to $(G, \{\mu\}, K)$. The local Dynkin diagram $\tilde{\Delta}$ is that associated to $\breve{G} = G \otimes_F \breve{F}$. Let $\breve{T}$ be a maximal torus of $\breve{G}$ contained in a Borel subgroup $\breve{B}$ containing $\breve{T}$. We may choose a representative μ of $\{\mu\}$ in $X_*(\breve{T})$ which is dominant for $\breve{B}$. There is a canonical identification of X_* with $X_*(\breve{T})_{\Gamma_0}$ (co-invariants under the inertia group). The second component of the enhanced Tits datum is the image λ of μ in X_*. It is well-defined up to the action of W_0 (this follows, since W_0 is identified with the relative Weyl group of $\breve{G}$ and any two choices of $\breve{B}$ are conjugate under the relative Weyl group). The third component of the enhanced Tits datum is the subset $\tilde{K}$ of vertices of $\tilde{\Delta}$ which describes the conjugacy class under $\breve{G}(\breve{F})$ of the parahoric subgroup $\breve{K}$ of $\breve{G}(\breve{F})$ determined by K.

Given a LM triple, one may compute its associated enhanced Tits datum as follows. First, if G is a split group, with associated Dynkin diagram Δ, then the local Dynkin diagram $\tilde{\Delta}$ is simply the associated affine Dynkin diagram, cf. [3, VI,§2].

Name (Index)	Local Dynkin diagram	Minuscule coweights
A_n (${}^1A^{(1)}_{n,n}$) for $n \geq 2$		$\{\omega_i^\vee\}$, $1 \leq i \leq n$
A_1 (${}^1A^{(1)}_{1,1}$)		$\{\omega_1^\vee\}$
B_n ($B_{n,n}$) for $n \geq 3$		$\{\omega_1^\vee\}$
C_n ($C^{(1)}_{n,n}$) for $n \geq 2$		$\{\omega_n^\vee\}$
D_n (${}^1D^{(1)}_{n,n}$) for $n \geq 4$		$\{\omega_1^\vee, \omega_{n-1}^\vee, \omega_n^\vee\}$
E_6 (${}^1E^0_{6,6}$)		$\{\omega_1^\vee, \omega_6^\vee\}$
E_7 ($E^0_{7,7}$)		$\{\omega_7^\vee\}$

Now let G be quasi-split and residually split. Then the affine root system is calculated following the recipe in [42, §2.3]. This gives the list below. In the column "Local Dynkin diagram", there are two rows associated to each group: the first row gives the local Dynkin diagram of the group G over a (ramified) field extension $\breve{F}'$ of $\breve{F}$ such that G splits over $\breve{F}'$; the second row gives the local Dynkin diagram of the group G over $\breve{F}$. In the column "Coweights", there are two rows: the first row for the minuscule coweight μ; the second row for the corresponding λ realized as a translation element of the associated extended affine Weyl group. Here we put minuscule coweights between braces if they determine the same λ which appears directly below. We follow the notation in [51].

Name (Index)	Local Dynkin diagram	Coweights	
B-C_n $({}^2A^{(1)}_{2n-1,n})$		$\{\omega_i^\vee, \omega_{2n-i}^\vee\}$, $1 \le i \le n$	
for $n \ge 3$		$\omega_i^\vee$	
C-BC_n $({}^2A^{(1)}_{2n,n})$		$\{\omega_i^\vee, \omega_{2n+1-i}^\vee\}$, $1 \le i \le n-1$	$\{\omega_n^\vee, \omega_{n+1}^\vee\}$
for $n \ge 2$		$\omega_i^\vee$	$2\omega_n^\vee$
C-BC_1 $({}^2A^{(1)}_{2,1})$		$\{\omega_1^\vee, \omega_2^\vee\}$	
		$2\omega_1^\vee$	
C-B_n $({}^2D^{(1)}_{n+1,n})$		$\omega_1^\vee$	$\{\omega_n^\vee, \omega_{n+1}^\vee\}$
for $n \ge 2$		$\omega_1^\vee$	$\omega_n^\vee$
F_4^1 $({}^2E^2_{6,4})$		$\{\omega_1^\vee, \omega_6^\vee\}$	
		$\omega_1^\vee$	
G_2^1 $({}^3D_{4,2}$ or ${}^6D_{4,2})$		$\{\omega_1^\vee, \omega_3^\vee, \omega_4^\vee\}$	
		$\omega_2^\vee$	

From this list we deduce the following statement.

Lemma 5.4. *Two LM triples $(G, \{\mu\}, K)$ and $(G', \{\mu'\}, K')$ over F, with G and G' absolutely simple adjoint, define the same enhanced Tits datum if and only if they become isomorphic after scalar extension to an unramified extension of F.* □

Suppose that G and G' are absolutely simple adjoint such that $G \otimes_F \breve{F} \simeq G \otimes_F \breve{F}$. The isomorphism classes of G and G' are distinguished by considering the corresponding action of the automorphism F of the local Dynkin diagram $\tilde{\Delta}$ of $\breve{G} \simeq \breve{G}' \simeq G^* \otimes_F \breve{F}$ given by Frobenius (see [51], [20]). In [20] one can find a very useful list of all possible such actions and of the corresponding forms of the group. The parahoric subgroups K, K' correspond to non-empty F-stable subsets $\tilde{K}$ of the vertices of $\tilde{\Delta}$.

Example 5.5. Consider the enhanced Tits data defined by LM triples of exotic good reduction type, cf. beginning of Subsection 5.1. Assume that $G \otimes_F \breve{F}$ is absolutely simple and adjoint. There are two cases:

1) G is the adjoint group of $U(V)$, where V is the $\tilde{F}/F$-hermitian vector space for a (tamely) ramified quadratic extension $\tilde{F}$ of F. If $\dim V = 2m \geq 4$ is even, then the corresponding enhanced Tits datum is $(B\text{-}C_m, \omega_1^\vee, \{0\})$ for $m \geq 3$ and $(C\text{-}B_2, \omega_2^\vee, \{0\})$ for $m = 2$. If $\dim V = 2m+1 \geq 3$ is odd, then the corresponding enhanced Tits datum is $(C\text{-}BC_m, \omega_1^\vee, \{0\})$ for $m \geq 2$ and $(C\text{-}BC_1, 2\omega_1^\vee, \{0\})$ for $m = 1$.

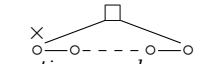
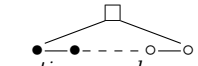
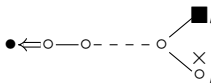
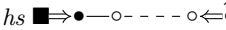
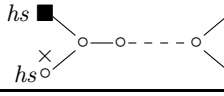
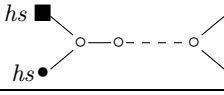
2) G is the adjoint group of $\mathrm{SO}(V)$ where V is an orthogonal F-vector space of dimension $2m+2 \geq 6$. Then V has Witt index m and non-square discriminant. The corresponding enhanced Tits datum is $(C\text{-}B_m, \omega_m^\vee, \{0\})$.

5.3. **Semi-stable reduction.** In the classification problem of all triples $(G, \{\mu\}, K)$ such that $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ has semi-stable reduction, Lemma 5.2 points to two problems. First, the product of semi-stable schemes is semi-stable only when all factors except at most one are smooth. And we can consider the problem of classifying the good reduction cases as solved by Theorem 5.1. Second, the extension of scalars of a semi-stable scheme is again semi-stable only if the base extension is unramified. Therefore, we will consider in the classification problem of semi-stable reduction only triples $(G, \{\mu\}, K)$ such that G is an absolutely simple adjoint group.

Lemma 5.4 justifies classifying local models $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ with semi-stable reduction by the enhanced Tits datum associated to $(G, \{\mu\}, K)$. Indeed, for F'/F unramified, $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\}) \otimes_{O_E} O_{E'} \simeq \mathbb{M}_{K'}^{\mathrm{loc}}(G', \{\mu'\})$, where $G' = G \otimes_F F'$ and $\{\mu'\}$ and K' are induced from $\{\mu\}$ and K, cf. Proposition 2.13, (ii). Furthermore, $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\}) \otimes_{O_E} O_{E'}$ has semi-stable reduction if and only if $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ has semi-stable reduction (this follows because the reflex field E' is an unramified extension of E).

Now we can state the classification of local models with semi-stable reduction.

Theorem 5.6. *Let $(G, \{\mu\}, K)$ be a LM triple over F such that G splits over a tame extension of F. Assume $p \neq 2$. Assume also that the group G is adjoint and absolutely simple. The local model $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ has semi-stable but not smooth reduction over $\mathrm{Spec}(O_E)$ if and only if the enhanced Tits datum corresponding to $(G, \{\mu\}, K)$ appears in the first column of the table below.*

Enhanced Tits datum	*Linear algebra datum*	*Discoverer*
 *All vertices are hyperspecial* $\#\tilde{K} \geq 2$	*Split* SL_n, $r = 1$ *arbitrary chain of lattices of length* ≥ 2	*Drinfeld*
 *All vertices are hyperspecial* μ *is any minuscule coweight*	*Split* SL_n *with* $n \geq 4$ r *arbitrary,* (Λ_0, Λ_1)	*Görtz*
	*Split* SO_{2n+1} *with* $n \geq 3, r = 1, (\Lambda_0, \Lambda_n)$	*new*
	*Split* Sp_{2n} *with* $n \geq 2$, $r = n, (\Lambda_0, \Lambda_1)$	*Genestier-Tilouine*
	*Split* SO_{2n} *with* $n \geq 4$, $r = 1, (\Lambda_0, \Lambda_n)$	*Faltings*
	*Split* SO_{2n} *with* $n \geq 5$, $r = n, \Lambda_1$	*new*

In the second column, we list the linear algebra data that correspond to the LM triple $(G, \{\mu\}, K) \otimes_F \breve{F}$.

In the diagrams above, if not specified, hyperspecial vertices are marked with an hs. In order to also show the coweight $\{\lambda\}$*, a special vertex is specified (marked by a square)*[5] *so that the extended affine Weyl group appears as a semi-direct product of* W_0 *and* X_**. Then* $\{\lambda\}$ *is equal to the fundamental coweight of the vertex marked with* $\times$*. Finally, the subset* $\tilde{K}$ *is the set of vertices filled with black color.*

Note that there are some obvious overlaps between the first two rows.

Remark 5.7. Starting with the table in Theorem 5.6 above, one can also easily list all LM triples $(G, \{\mu\}, K)$ over F, with G adjoint and absolutely simple such that G splits over a tame extension of F and with $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ having semi-stable reduction over O_E (provided $p \neq 2$). These are given by listing the possible conjugacy classes of Frobenius automorphisms in the group $\mathrm{Aut}(\tilde{\Delta}, \tilde{K})$ of automorphisms of the corresponding local Dynkin diagram $\tilde{\Delta}$ that preserve the black subset $\tilde{K}$. B. Gross [20] gives a convenient enumeration of possible Frobenius conjugacy classes in $\mathrm{Aut}(\tilde{\Delta})$.

For example, in the first case of our list, there could be several possible Frobenius actions on the n-gon that stabilize $\tilde{K}$ depending on that set; the corresponding groups are the adjoints of either unitary groups or of $\mathrm{SL}_m(D)$, where D are division algebras and $m|n$ (see [20, p. 15-16]).

In the second case, there is only one possibility of a non-trivial Frobenius action on the n-gon that stabilizes the set of two adjacent vertices: A reflection (F of order 2). Then G is the adjoint group of $U(V)$ where V is a non-degenerate Hermitian space for an unramified quadratic extension of F. Furthermore, when $n = 2m$ is even, F cannot fix a vertex so V *does not* contain an isotropic subspace of dimension m ([20, p. 16]).

In the third and fourth cases, there are no non-trivial automorphisms F that preserve the subset $\tilde{K}$ and so G is split.

In the fifth case, there is also only one possible non-trivial Frobenius action that stabilizes $\tilde{K}$, up to conjugacy in the group $\mathrm{Aut}(\tilde{\Delta}, \tilde{K})$. The corresponding group is the adjoint group of $U(W)$ where W is a non-degenerate anti-Hermitian space over the quaternion division algebra over F; the center of the Clifford algebra is $F \times F$ if n is even and the quadratic unramified extension L/F if n is odd ([20, p. 18-20]).

In the sixth case, there are three possibilities of a non-trivial Frobenius action that stabilizes $\tilde{K}$, up to conjugacy in the group $\mathrm{Aut}(\tilde{\Delta}, \tilde{K})$. In the one case, the group is the adjoint group of $\mathrm{SO}(V)$ where V is a non-degenerate orthogonal space of dimension $2n$, discriminant 1 and Witt index $n-2$. In the other two, the group is the adjoint group of the unramified quasi-split but not split $\mathrm{SO}(V)$ ([20, p. 18-20]).

In all these cases, we can realize K as the parahoric stabilizer of a suitable lattice chain.

Remark 5.8. We note that $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ has semi-stable reduction if and only if the base change $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\}) \otimes_{O_E} \breve{O}_E$ has strictly semi-stable reduction, i.e., the geometric special fiber is a strict normal crossings divisor, in the sense of [50, Def. 40.21.1]: Indeed, both $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\}) \otimes_{O_E} \breve{O}_E$ and all the irreducible components of its special fiber are normal [44], hence unibranch at each closed point x. From this we deduce that each intersection of a subset of irreducible components of the geometric special fiber in the strict henselization

[5]Note that the local Dynkin type *C-BC*$_n$ does not occur here so that all special vertices are conjugate; hence this specification plays no role.

of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ at x (i.e., of "branches"), is isomorphic to the strict henselization of the intersection of a corresponding subset of global irreducible components at x. Therefore, if the geometric special fiber is (étale locally) a normal crossings divisor, it is in fact (globally) a strict normal crossings divisor.

Remarks 5.9. Let us compare this list with the local models investigated in earlier papers. We always assume $p \neq 2$. We use the terminology *rationally smooth, strictly pseudo semi-stable reduction, rationally strictly pseudo semi-stable reduction* introduced in the next section.

(i) Let us consider the LM triples whose first two components are $G = \mathrm{GU}(V)$ where V is a split F'/F-hermitian space of dimension 3 relative to a *ramified* quadratic extension F'/F, and where $\{\mu\} = (1,0,0)$. We identify E with F'. We use the notation for the parahoric subgroups as in [43]. Since G is not unramified, there are no hyperspecial maximal parahoric subgroups. If K is the stabilizer of the self-dual vertex lattice Λ_0, then K is a special maximal parahoric and the special fiber is irreducible, normal with an isolated singularity which is a rational singularity, comp. [43, Thm. 2.24]. The special fiber occurs in the list in [22] of rationally smooth Schubert varieties in twisted affine Grassmannians. The blow-up of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ in the unique singular point of the special fiber has semi-stable reduction, cf. [38, Thm. 4.5], [30]. This is an example of a local model which does not have semi-stable reduction but where the generic fiber has a different model which has semi-stable reduction.

If K is the stabilizer of the non-selfdual vertex lattice Λ_1, then $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is smooth over $\mathrm{Spec}\, O_{F'}$: this case is of exotic good reduction type.

Finally, if K is an Iwahori subgroup, then the local model does not have rationally strictly pseudo semi-stable reduction, comp. [43, Thm. 2.24, (iii)]. And, indeed, this case is eliminated in Subsection 8.13.

(ii) Let us consider $G = \mathrm{GU}(V)$, where V is a split F'/F-hermitian space of arbitrary dimension $n \geq 2$ relative to a ramified quadratic extension F'/F. Let us consider the LM triple $(G,\{\mu\},K)$, where $\{\mu\} = (1,0,\dots,0)$, and where K is the parahoric stabilizer of a self-dual lattice Λ (except when $n = 2$, K is the full stabilizer of Λ, cf. [42, 1.2.3]). If $n = 2$, then $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ has semi-stable reduction, cf. [43, Rm. 2.35]. If $n \geq 3$, the special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is irreducible and has a unique isolated singular point, cf. [38, Thm. 4.5]. Generalizing the previous example, the blow-up of this singular point has semi-stable reduction, cf. [38, 30].

For $n > 3$ with $n = 2m+1$ odd, the associated local Dynkin diagram is of type C-BC_m and the parahoric subgroup K corresponds to the *special* vertex m in the local Dynkin diagram. The special fiber of the local model is a Schubert variety that occurs in the list in [22] of rationally smooth Schubert varieties in twisted affine Grassmannians. Remarkably, Krämer [30, Thm. 5.4] has shown in this case that the *semi-simple Frobenius trace* function is constant equal to 1 on the special fiber, even though the special fiber is singular.

For $n = 2m \geq 4$ even, the associated local Dynkin diagram is of type B-C_m and the parahoric subgroup K corresponds to the *non-special* vertex m in the local Dynkin diagram if $m \geq 3$, or C-B_2 and the *non-special* vertex 1, if $m = 2$. By §8.8.2, resp. Subsection 8.7, the associated Poincaré polynomial is not symmetric and hence the special fiber is not rationally smooth, cf. Lemma 6.2. In this case, Krämer [30, Thm. 5.4] has shown that the semi-simple Frobenius trace function is not constant equal to 1 on the special fiber, but rather has a jump at the singular point.

(iii) Let us consider $G = \mathrm{Res}_{F'/F}(\mathrm{GL}_n)$, where F'/F is a totally ramified (possibly wildly) extension. This is excluded from the above considerations (both for the classification of good reduction and of semi-stable reduction); still, it is interesting to compare this case with the above lists. Let $K = \mathrm{GL}_n(O_{F'})$ and

$$\{\mu\} = \big((1^{(r_\varphi)}, 0^{(n-r_\varphi)})_{\varphi\colon F' \longrightarrow \overline{F}}\big).$$

The singularities of the special fiber are analyzed in [40] by relating the special fiber $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\}) \otimes_{O_E} \kappa_E$ with a Schubert variety in the affine Grassmannian for GL_n. More precisely, the special fiber is irreducible and reduced and there is an isomorphism of closed reduced subschemes

$$\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\}) \otimes_{O_E} \kappa_E \simeq \overline{\mathcal{O}}_{\mathbf{t}}.$$

Here $\overline{\mathcal{O}}_{\mathbf{t}}$ is the Schubert variety associated to the dominant coweight $\mathbf{t} = \mathbf{r}^\vee$ dual to $\mathbf{r} = (r_\varphi)_\varphi$, i.e.,

$$t_1 = \#\{\varphi \mid r_\varphi \geq 1\}, \quad t_2 = \#\{\varphi \mid r_\varphi \geq 2\}, \ \ldots.$$

By [22] (cf. also [34] for the analogue over a ground field of characteristic zero, and [12], [54] for the analogue over $\mathbb{C}$), $\overline{\mathcal{O}}_{\mathbf{t}}$ is smooth if and only if $\mathbf{t}$ is minuscule, i.e., $t_1 - t_n \leq 1$. This holds if and only if there is at most one φ such that $r_\varphi \notin \{0, n\}$. We conclude that $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is smooth only in the trivial case when at most one r_φ is not 0 or n.

(iv) Very similarly to the case above, we can also consider $G = \mathrm{Res}_{F'/F}(H)$, where F' is a totally ramified (possibly wildly) extension, and H is unramified over F' (i.e., quasi-split and split over an unramified extension of F'). Then H extends to a reductive group scheme over $O_{F'}$ which is unique up to isomorphism and which we will also denote by H. Take $K = H(O_{F'})$, let $\{\mu\} = \big((\mu_\varphi)_{\varphi\colon F' \to \overline{F}}\big)$, and consider the LM triple $(G,\{\mu\},K)$.

When F'/F is wildly ramified, the theory of [44] does not apply to $(G,\{\mu\},K)$. However, Levin [33] has extended the construction of [44] to such groups obtained by restriction of scalars and has defined local models $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ for such triples. Assume that p does not divide $|\pi_1(H_{\mathrm{der}})|$. Then, by [33, Thm. 2.3.5], the geometric special fiber $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\}) \otimes_{O_E} k$ is reduced and can be identified with a Schubert variety $\mathrm{Gr}_{H,\lambda}$ of the affine Grassmannian for H over k. Here, λ is given by the sum $\sum_\varphi \mu_\varphi$ of the minuscule coweights μ_φ. By [22], (or [34] for the analogue over a ground field of characteristic zero), $\mathrm{Gr}_{H,\lambda}$ is smooth if and only if λ is minuscule. Therefore, $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is smooth over O_E if and only if at most one of the coweights $\mu_{\varphi,\mathrm{ad}}$ is not trivial.

The proof of Theorem 5.6 proceeds in four steps. In a first step, we establish a list of all cases which satisfy the *component count property* condition (CCP), cf. Section 7. This condition is implied by *strictly pseudo semi-stable reduction*. This last condition, concerns only the special fiber and entails in particular that all irreducible components are smooth, with their intersections smooth of the correct dimension, cf. Section 6. By weakening the condition of smoothness to *rational smoothness*, we arrive at the notion of *rationally strictly pseudo semi-stable reduction*, cf. Section 6. The second step consists in eliminating from the CCP-list all cases which do not have rationally strictly pseudo semi-stable reduction, cf. Section 8. In a third step, we eliminate all cases which have rationally strictly pseudo semi-stable reduction but not strictly pseudo semi-stable reduction, cf. Section 10. In the final step we prove that in all the remaining cases strictly pseudo semi-stable reduction implies semi-stable reduction. This last step is a lengthy case-by-case analysis through linear algebra and occupies Section 12.

6. Strictly pseudo semi-stable reduction and the CCP condition

Definition 6.1. a) A scheme over the spectrum of a discrete valuation ring is said to have *strictly pseudo semi-stable reduction* (abbreviated to *SPSS reduction*) if all irreducible components of the reduced geometric special fiber are smooth and of the same dimension, and the reduced intersection of any i irreducible components is smooth and irreducible and of codimension $i-1$.

b) A scheme over the spectrum of a discrete valuation ring is said to have *rationally strictly pseudo semi-stable reduction* if all irreducible components of the reduced geometric special fiber are rationally smooth and of the same dimension, and the reduced intersection of any i irreducible components is rationally smooth and irreducible and of codimension $i-1$.

Here we recall that an irreducible variety Y of dimension d over an algebraically closed field k is said to be *rationally smooth* if for all closed points y of Y the relative ℓ-adic cohomology (for some $\ell \neq \operatorname{char} k$) satisfies

$$\dim_{\mathbb{Q}_\ell} H^i(Y, Y \setminus \{y\}, \mathbb{Q}_\ell) = \begin{cases} 0 & i \neq 2d \\ 1 & i = 2d. \end{cases}$$

When $k = \mathbb{C}$, this definition (for singular cohomology with coefficients in $\mathbb{Q}$) appears in [28], cf. also [3, 31, 2].

We note that both notions, that of SPSS reduction and that of rationally SPSS reduction, only depend on the geometric special fiber. For instance, they do not imply that the scheme is regular.

Lemma 6.2. *Let Y be a proper irreducible variety of dimension d over an algebraically closed field. If Y is rationally smooth, then the Poincaré polynomial*

$$P(q) = \sum\nolimits_{i=0}^{2d} a_i q^i,$$

of cohomology with $\mathbb{Q}_\ell$-coefficients ($\ell \neq \operatorname{char} k$) is symmetric, i.e., $a_i = a_{2d-i}$, for all i. □

Remark 6.3. It is proved in [8] that for Schubert varieties in the finite and affine flag varieties for split groups, the converse is true. Namely, in this context, a Schubert variety is rationally smooth if and only if its Poincaré polynomial formed with $\mathbb{Q}_\ell$-coefficients is symmetric. Something analogous holds in the Kac-Moody context, cf. [32, 12.2 E(2)].

Notation 6.4. In the rest of this section and also in Sections 7, 8 and 10 we consider the enhanced Tits datum $(\tilde{\Delta}, \{\lambda\}, \tilde{K})$ obtained, as in §5.2, from a local model triple $(G, \{\mu\}, K)$ with G adjoint and absolutely simple.

On the other hand, the enhanced Tits datum $(\tilde{\Delta}, \{\lambda\}, \tilde{K})$ also corresponds to an adjoint, absolutely simple group $G^\flat$ over $k((u))$, a $G^\flat(k((u))^{\mathrm{sep}})$-conjugacy class of a minuscule cocharacter, and a conjugacy class of a parahoric subgroup $K^\flat = \mathcal{G}^\flat(k[[u]])$. In terms of the identifications of §2.3, we have $G^\flat = \breve{G}'$, $K^\flat = \breve{K}'$, $\mathcal{G}^\flat = \underline{\mathcal{G}} \otimes_{O[u]} k[[u]]$, and the class of the cocharacter is the one that corresponds to $\{\mu\}$.

By Theorem 2.10, and the above discussion, the geometric special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ can be identified (up to a radicial morphism) with the union, over the set $\mathrm{Adm}_{\tilde{K}}(\{\lambda\})$, of Schubert varieties in the partial flag variety $LG^\flat / L^+\mathcal{G}^\flat$. In what follows, to ease the notation, we will denote this partial flag variety by $G^\flat/K^\flat$ and its Schubert varieties as $\overline{K^\flat w K^\flat / K^\flat}$.

Below, and also in Sections 7, 8 and 10, we will employ various combinatorial arguments in the extended Weyl group $\tilde{W}$ which only involve $(\tilde{\Delta}, \{\lambda\}, \tilde{K})$; for example, which use cosets for the subgroup $W_{\tilde{K}}$. For these arguments, we will often omit the tilde from the notation. For example, we will simply write W_K instead of $W_{\tilde{K}}$; in any case, this subgroup ultimately only depends on the conjugacy class of the parahoric subgroup $\breve{K} \subset G(\breve{F})$.

Let us first make Lemma 6.2 explicit in the case of interest for us, namely for affine Schubert varieties in the partial flag variety $G^\flat/K^\flat$. Note that for any $w \in \tilde{W}$, we have the projection map

$$\overline{I^\flat v I^\flat/I^\flat} \longrightarrow \overline{K^\flat w K^\flat/K^\flat},$$

where $v = \max(W_K w W_K)$. This map is a fiber bundle with fibers isomorphic to the smooth projective variety $K^\flat/I^\flat$. Hence $\overline{K^\flat w K^\flat/K^\flat}$ is rationally smooth if and only if $\overline{I^\flat v I^\flat/I^\flat}$ is rationally smooth. Thus we may use the Poincaré polynomial of $\overline{I^\flat v I^\flat/I^\flat}$ to determine if $\overline{K^\flat w K^\flat/K^\flat}$ is rationally smooth.

We denote by $\tilde{W}^K$ the set of elements $w \in \tilde{W}$ that are of minimal length in their coset wW_K. For any translation element λ in $\tilde{W}$, we set

$$W_{\leq \lambda, K} = \{v \in \tilde{W}^K \mid v \leq \max\{W_K t^\lambda W_K\}\}. \tag{6.1}$$

The set $W_{\leq\lambda,K}$ contains a unique maximal element, which we denote by $w_{\lambda,K}$. For any $w \in W_{\leq\lambda,K}$, we define the *colength* of w to be $\ell(w_{\lambda,K}) - \ell(w)$, where $\ell(w)$ denotes the length of w.

We have $\overline{K^\flat \lambda K^\flat/K^\flat} = \sqcup_{v \in W_{\leq\lambda,K}} I^\flat v K^\flat/K^\flat$ and the associated Poincaré polynomial is

$$P_{\leq\lambda,K}(q) = \sum\nolimits_{v \in W_{\leq\lambda,K}} q^{\ell(v)}. \tag{6.2}$$

On the other hand, set $v_1 = \max(W_K t^\lambda W_K)$. Then $\overline{I^\flat v_1 I^\flat/I^\flat} = \sqcup_{v \leq W_{\leq\lambda,K}} \sqcup_{x \in W_K} I^\flat v x I^\flat/I^\flat$. The associated Poincaré polynomial is

$$\sum_{v \in W_{\leq\lambda,K}, x \in W_K} q^{\ell(vx)} = P_{\leq\lambda,K}(q) \sum_{x \in W_K} q^{\ell(x)}.$$

As $\sum_{x \in W_K} q^{\ell(x)}$ is symmetric, we deduce that $P_{\leq\lambda,K}(q) \sum_{x \in W_K} q^{\ell(x)}$ is symmetric if and only if $P_{\leq\lambda,K}(q)$ is symmetric. By Lemma 6.2, we have

Proposition 6.5. *If the Schubert variety $\overline{K^\flat \lambda K^\flat/K^\flat}$ is rationally smooth, then $P_{\leq\lambda,K}(q)$ is symmetric.*

Definition 6.6. The LM triple $(G, \{\mu\}, K)$ has the *component count property* (CCP condition) if the following inequality is satisfied,

$$\#\{\text{extreme elements of } \mathrm{Adm}_{\tilde{K}}(\{\lambda\})\} \leq \#\tilde{K}.$$

Proposition 6.7. *If the local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ has rationally SPSS reduction over O_E, then the CCP condition holds for the triple $(G, \{\mu\}, K)$.*

Proof. Let $(\tilde{\Delta}, \{\lambda\}, \tilde{K})$ be the associated enhanced Tits datum. As λ is not central, there exists $\lambda' \in W_0 \cdot \lambda$ such that $\langle \lambda', \alpha \rangle \neq 0$ for some root α of K.

By Theorem 2.10, $\overline{K^\flat \lambda' K^\flat/K^\flat}$ is an irreducible component of the geometric special fiber of $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$. Thus if $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ has rationally SPSS reduction, then $\overline{K^\flat \lambda' K^\flat/K^\flat}$ is

rationally smooth. Therefore by Lemma 6.2, the Poincaré polynomial of $\overline{K^\flat \lambda' K^\flat/K^\flat}$ is symmetric. But this coincides with the Poincaré polynomial of $W_{\leq \lambda',K}$, cf. (6.2), which is therefore symmetric.

Any length one element in $W_{\leq \lambda',K} \subset \tilde{W}^K$ is of the form τs for some $s \in \tilde{K}$, where τ is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda')$. Thus there are at most $\#\tilde{K}$ length one elements in $W_{\leq \lambda',K}$. Hence there are also at most $\#\tilde{K}$ colength one elements of $W_{\leq \lambda',K}$.

Now list the irreducible components of the geometric special fiber as $X_1 = \overline{K^\flat \lambda' K^\flat/K^\flat}$, $X_2, \ldots, X_l$. By the definition of rationally SPSS reduction, for any i with $2 \leq i \leq l$, the intersection $X_1 \cap X_i$ is of the form $\overline{K^\flat w_i K^\flat/K^\flat}$, where $w_i \in W_{\leq \lambda',K}$ with $\dim(K^\flat w_i K^\flat/K^\flat) = \dim(K^\flat \lambda' K^\flat/K^\flat) - 1$. In particular, w_i is a colength one element in $W_{\leq \lambda',K}$. As the intersection of any three irreducible components of the geometric special fiber is of codimension 2, we have $w_i \neq w_j$ for $i \neq j$. In particular, $\{w_2, w_3, \ldots, w_l\} \subset W_{\leq \lambda',K}$ is a subset of colength one elements.

Next we construct another colength one element of $W_{\leq \lambda',K}$. Recall that $w_{\lambda',K}$ is the unique maximal element of $W_{\leq \lambda',K}$. Let $s \notin \tilde{K}$. Then $sw_{\lambda',K} \in W_K t^{\lambda'} W_K$. Therefore, we have either $sw_{\lambda',K} < w_{\lambda',K}$, or $sw_{\lambda',K} > w_{\lambda',K}$ and $sw_{\lambda',K} = w_{\lambda',K} s'$ for some $s' \notin \tilde{K}$.

If $sw_{\lambda',K} > w_{\lambda',K}$ for all $s \notin K$, then $W_K w_{\lambda',K} = w_{\lambda',K} W_K$. Since $w_{\lambda',K} \in W_K t^{\lambda'} W_K$, we get $W_K t^{\lambda'} = t^{\lambda'} W_K$. This contradicts the assumption that $\langle \lambda', \alpha \rangle \neq 0$ for some $\alpha \in \Phi_K$.

Therefore there exists $s \notin K$ such that $sw_{\lambda',K} < w_{\lambda',K}$. Since $w_{\lambda',K} \in \tilde{W}^K$, we have $sw_{\lambda',K} \in \tilde{W}^K$. Hence $sw_{\lambda',K} \in W_{\leq \lambda',K}$ is a colength one element. As $K^\flat(sw_{\lambda',K})K^\flat = K^\flat w_{\lambda',K} K^\flat = K^\flat \lambda' K^\flat$, we have $sw_{\lambda',K} \neq w_i$ for any i.

We now have found at least l distinct colength one elements in $W_{\leq \lambda',K}$, namely $sw_{\lambda',K}$ and $w_2, \ldots, w_l$. Thus we have $l \leq \#\tilde{K}$. The proposition is proved. □

7. Analysis of the CCP condition

7.1. **Statement of the result.** The purpose of this section is to determine for which enhanced Tits data the CCP condition is satisfied. Note that the CCP condition only depends on the associated enhanced Coxeter datum.

Theorem 7.1. *Assume that G is adjoint and absolutely simple. The enhanced Coxeter data satisfying the CCP condition are the following (up to isomorphism):*

(1) Irreducible cases:

- *(a) The parahoric subgroup corresponding to $\tilde{K}$ is maximal special;*
- *(b) The triple $(\tilde{B}_n, \omega_r^\vee, \{n\})$ with $n \geq 3$ and $1 \leq r \leq n-1$;*
- *(c) The triple $(\tilde{C}_n, l\omega_n^\vee, \{i\})$ with $n \geq 2$, $l = 1$ or 2 and $1 \leq i \leq n-1$;*
- *(d) The triple $(\tilde{F}_4, \omega_1^\vee, \{4\})$.*
- *(e) The triple $(\tilde{G}_2, \omega_2^\vee, \{1\})$.*

(2) Reducible cases:

- *(a) The triple $(\tilde{A}_1, 2\omega_1^\vee, \{0,1\})$;*
- *(b) The triple $(\tilde{A}_{n-1}, \omega_1^\vee, \tilde{K})$ with arbitrary $\tilde{K}$ of cardinality ≥ 2;*
- *(c) The triple $(\tilde{A}_{n-1}, \omega_i^\vee, \{0,1\})$ with $n \geq 4$ and $2 \leq i \leq n-2$;*
- *(d) The triple $(\tilde{B}_n, \omega_1^\vee, \{0,n\})$ with $n \geq 3$;*

(e) The triple $(\tilde{B}_n, \omega_n^\vee, \{0,1\})$ *with* $n \geq 3$*;*

(f) The triple $(\tilde{C}_n, \omega_1^\vee, \{0,n\})$ *with* $n \geq 2$*;*

(g) The triple $(\tilde{C}_n, l\omega_n^\vee, \{i,i+1\})$ *with* $n \geq 2$*,* $l = 1$ *or* 2 *and* $0 \leq i \leq \frac{n}{2} - 1$*;*

(h) The triple $(\tilde{D}_n, \omega_1^\vee, \{0,n\})$ *with* $n \geq 4$*;*

(i) The triple $(\tilde{D}_n, \omega_n^\vee, \{0,1\})$ *with* $n \geq 5$*.*

7.2. **Classical types.** We first study the classical types. Let $E = \mathbb{R}^n$ with the canonical basis $(\epsilon_1, \ldots, \epsilon_n)$. We equip E with the scalar product such that this basis is orthonormal and we identify E with E^*.

We regard the Weyl group $W(B_n)$ of type B_n (and also C_n) as the group of permutations σ on $\{\pm 1, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for $1 \leq i \leq n$. The Weyl group $W(A_{n-1})$ of type A_{n-1} is the subgroup of $W(B_n)$ consisting of permutations σ with $\sigma(i) > 0$ for all $1 \leq i \leq n$. We have $W(A_{n-1}) \cong S_n$, the group of permutations on $\{1, 2, \ldots, n\}$. The Weyl group $W(D_n)$ of type D_n is the subgroup of $W(B_n)$ consisting of permutations σ such that $\#\{i; 1 \leq i \leq n, \sigma(i) < 0\}$ is an even number.

7.3. **Type** $\tilde{A}_{n-1}$**.** One may consider the extended affine Weyl group $\mathbb{Z}^n \rtimes S_n$ instead. In this case, one may use the coweight $(1^r, 0^{n-r})$ instead of the fundamental coweight $\omega_r^\vee$.

It is easy to see that the triple $(\tilde{A}_1, 2\omega_1^\vee, \tilde{K})$ with $\tilde{K}$ arbitrary satisfies the CCP condition. Now we assume that $\lambda = \omega_r^\vee$ for some r.

By applying an automorphism, we may assume that $0 \in \tilde{K}$. It is easy to see that the case $\tilde{K} = \{0\}$ satisfies the CCP condition. Now we assume that $\#\tilde{K} \geq 2$. Then $\tilde{K} = \{0, i_1, \ldots, i_{l-1}\}$ with $l \geq 2$ and $i_1 < \cdots < i_l$. Then the action of W_K on $\{1, 2, \ldots, n\}$ stabilizes the subsets $\{1, \ldots, i_1\}$, $\{i_1+1, \ldots, i_2\}$, ..., $\{i_{l-1}+1, \ldots, n\}$. So for $\lambda = (1^r, 0^{n-r})$, the number of extreme elements equals the number of partitions $r = j_1 + \cdots + j_l$, where $0 \leq j_m \leq i_m - i_{m-1}$ for any m. Here by convention, we set $i_0 = 0$ and $i_l = n$. Now the statement of Theorem 7.1 for type $\tilde{A}$ follows from the following result.

Proposition 7.2. *Let* $l \geq 2$ *and* $n, r, n_1, \ldots, n_l$*, be positive integers with* $n = n_1 + \cdots + n_l$ *and* $r < n$*. Set*

$$X = \{(j_1, \ldots, j_l) \mid r = j_1 + \cdots + j_l,\ 0 \leq j_i \leq n_i \text{ for all } i\}.$$

Then $\#X \geq l$ *and equality holds if and only if* $r = 1$*, or* $l = 2$ *and* n_1 *or* n_2 *equals* 1*.*

Proof. Without loss of generality, we may assume that $r \leq n/2$ and $n_1 \geq n_2 \geq \cdots \geq n_l$. Let $t \in \mathbb{N}$ such that $n_1 + \cdots + n_{t-1} < r \leq n_1 + \cdots + n_t$. Note that if $t = l$, then $n - r < n_l \leq n_1 \leq r$, which contradicts our assumption that $r \leq n/2$. Therefore $t < l$.

We have $x_0 = (n_1, \ldots, n_{t-1}, r - n_1 - \cdots - n_{t-1}, 0, \ldots, 0) \in X$. For any $1 \leq i_1 \leq t, t+1 \leq i_2 \leq l$, we obtain a new element in X from the element x_0 by subtracting 1 in the i_1-th factor and adding 1 in the i_2-th factor. In this way, we construct $1 + t(l-t)$ elements in X. Note that $t(l-t) \geq l-1$ and the equality holds if and only if $t = 1$ or $t = l-1$. Therefore, $\#X \geq l$.

Moreover, if $\#X = l$, then $t = 1$ or $t = l-1$, and the elements we constructed above are all the elements in X.

Case (i): $t = 1$. In this case, $x_0 = (r, 0, \ldots, 0)$. By our construction, there is no element of the form $(r-2, j_2, \ldots, j_l)$ in X. This happens only when $r = 1$ or $n_2 + \cdots + n_l = 1$. In the latter case $l = 2$ and $n_2 = 1$.

Case (ii): $t = l-1$. In this case, $x_0 = (n_1, \dots, n_{l-2}, r - n_1 - \cdots - n_{l-1}, 0)$. By our construction,

(1) there is no element in X with 2 in the last factor;

(2) there is no element in X with $r - n_1 - \cdots - n_{l-1} + 1$ in the $l-1$ factor.

Note that (1) happens only when $r = 1$ or $n_l = 1$ and (2) happens only when $l = 2$ or $r = n_1 + \cdots + n_{l-1}$. However, if $r = n_1 + \cdots + n_{l-1}$ and $n_l = 1$, then, since $r \le n/2$, we must have $n = 2$ and $r = 1$. Hence both (1) and (2) happens only when $r = 1$ or $l = 2$ and $n_2 = 1$. □

7.4. **Type $\tilde{B}_n$.** By applying a suitable automorphism, we may assume that if $1 \in \tilde{K}$, then $0 \in \tilde{K}$. Let $\epsilon = \#(\{0, n\} \cap \tilde{K})$.

We have $\{i_1, \dots, i_l\} \subset \tilde{K} \subset \{0, i_1, \dots, i_l, n\}$, where $1 \le i_1 < \cdots < i_l \le n-1$. Then

$$W_K \cong W_1 \times S_{i_2 - i_1} \times \cdots \times S_{i_l - i_{l-1}} \times W_2,$$

where $W_1 = \begin{cases} W(D_{i_1}), & \text{if } 0 \notin \tilde{K} \\ S_{i_1}, & \text{if } 0 \in \tilde{K} \end{cases}$ and $W_2 = \begin{cases} W(B_{n-i_l}), & \text{if } n \notin \tilde{K} \\ S_{n-i_l}, & \text{if } n \in \tilde{K}. \end{cases}$

Case (i): $l = 0$.

In this case $\tilde{K} \subset \{0, n\}$.

If $\tilde{K} = \{0\}$, then K is maximal special and there is a unique extreme element.

If $\tilde{K} = \{n\}$, then $p(W_K)$ is of type D_n and $p(W_K) \backslash W_0$ has cardinality 2. Thus λ is the only extreme element if and only if $p(W_K) W_\lambda = W_0$, i.e., $W_\lambda \nsubseteq p(W_K)$. This happens exactly when $\lambda = \omega_r^\vee$ with $r < n$.

If $\tilde{K} = \{0, n\}$, then $W_K \cong S_n$. In this case, the number of extreme elements equals 2^r, where $\lambda = \omega_r^\vee$. Thus the CCP condition is satisfied exactly when $r = 1$.

Case (ii): $l \ge 1$.

Case (ii)(a): $l \ge 1$ and $r < n$.

By Proposition 7.2, the number of extreme elements with nonnegative entries is at least $l + 1$.

Note that for any m with $2 \le m \le l$, if $\lambda' = (c'_1, \dots, c'_n)$ is an extreme element, then $\lambda'' = (c''_1, \dots, c''_n)$ is another extreme element, where $c''_k = -c'_{i_m + i_{m-1} + 1 - k}$ for any k with $i_{m-1} + 1 \le k \le i_m$, and $c''_k = c'_k$ for all other k's.

In particular, there exists an extreme element with some negative entries among the $\{i_{m-1}+1, \dots, i_m\}$-th entries, and with all the other entries nonnegative. If $0 \in \tilde{K}$ (resp., $n \in \tilde{K}$), then there exists an extreme element with some negative entries among the $\{1, \dots, i_1\}$-th entries (resp., $\{i_l + 1, \dots, n\}$-th entries), and with all the other entries nonnegative. In this way, we construct $l - 1 + \epsilon$ extreme elements.

Therefore the number of extreme elements is at least $l + 1 + l - 1 + \epsilon > l + \epsilon$. The CCP condition does not hold in this case.

Case (ii)(b): $l \ge 1$ and $r = n$.

Note that if $\epsilon = 2$, then $\{0, n\} \subset \tilde{K}$ and there are $2^n \ge n > l$ extreme elements and the CCP condition does not hold in this case.

Now assume that $\epsilon \le 1$. It is easy to see there are at least 2^l extreme elements, whose entries are ± 1, and there are at most one -1 entry in the $\{i_{m-1} + 1, \dots, i_m\}$-th entries for $1 \le m \le l$ (if $0 \in \tilde{K}$), or $2 \le m \le l + 1$ (if $n \in \tilde{K}$). Here we set $i_0 = 0$ and $i_{l+1} = n$. Thus if the CCP condition is satisfied, then $2^l \le l + \epsilon \le l + 1$. Therefore $l = 1$ and $\epsilon = 1$. Hence $\tilde{K} = \{0, i\}$ or $\tilde{K} = \{i, n\}$ for some $1 \le i \le n-1$.

For $\tilde{K} = \{0, i\}$, there are 2^i extreme elements. Thus the CCP condition is satisfied if and only if $i = 1$. For $\tilde{K} = \{i, n\}$, there are $2^{n-i+1} \geq 4$ extreme elements and the CCP condition does not hold in this case.

7.5. **Type $\tilde{C}_n$.** By applying a suitable automorphism, we may assume that if $n \in \tilde{K}$, then $0 \in \tilde{K}$. We have $\{i_1, \ldots, i_l\} \subset \tilde{K} \subset \{0, i_1, \ldots, i_l, n\}$, where $1 \leq i_1 < \cdots < i_l \leq n-1$. Then

$$W_K \cong W_1 \times S_{i_2 - i_1} \times \cdots \times S_{i_l - i_{l-1}} \times W_2,$$

where $W_1 = \begin{cases} W(B_{i_1}), & \text{if } 0 \notin \tilde{K} \\ S_{i_1}, & \text{if } 0 \in \tilde{K} \end{cases}$ and $W_2 = \begin{cases} W(B_{n-i_l}), & \text{if } n \notin \tilde{K} \\ S_{n-i_l}, & \text{if } n \in \tilde{K}. \end{cases}$

Case (i): $l = 0$.

If $\tilde{K} = \{0\}$, then K is maximal special and the number of extreme elements is 1.

If $\tilde{K} = \{0, n\}$, then $W_K \cong S_n$. In this case, the number of extreme elements equals to 2^r, where $\lambda = \omega_r^\vee$. Thus the CCP condition is satisfied exactly when $r = 1$.

Case (ii): $l \geq 1$.

Let $\epsilon = \#(\{0, n\} \cap \tilde{K})$. By the same argument as in Subsection 7.4, if the CCP condition is satisfied, then we must have $\lambda = \omega_n^\vee$ or $2\omega_n^\vee$, and $\epsilon \leq 1$.

Case (ii)(a): $l \geq 1$ and $\epsilon = 0$.

Similarly to the argument in Subsection 7.4, there are at least 2^{l-1} extreme elements. If the CCP condition is satisfied, then $2^{l-1} \leq l$ and hence $l \leq 2$. If $l = 1$, then $\tilde{K} = \{i\}$ for some $1 \leq i \leq n-1$ and there is only one extreme element, i.e., the element λ. If $l = 2$, then $\tilde{K} = \{i_1, i_2\}$ for some $1 \leq i_1 < i_2 \leq n$. The number of extreme elements is $2^{i_2 - i_1}$. In this case, the CCP condition is satisfied if and only if $i_2 = i_1 + 1$.

Case (ii)(b): $l \geq 1$ and $\epsilon = 1$.

By our assumption, $0 \in \tilde{K}$. Similarly to the argument in Subsection 7.4, there are at least 2^l extreme elements. If the CCP condition is satisfied, then $2^l \leq l + 1$ and hence $l = 1$. Thus $\tilde{K} = \{0, i\}$ for some $1 \leq i \leq n-1$. In this case, there are 2^i extreme elements and the CCP condition is satisfied exactly when $i = 1$.

7.6. **Type $\tilde{D}_n$.** By applying a suitable automorphism, we may assume that if $1 \in \tilde{K}$, then $0 \in \tilde{K}$, and if $n-1 \in \tilde{K}$, then $n \in \tilde{K}$, and if $n \in \tilde{K}$, then $0 \in \tilde{K}$.

We have $\{i_1, \ldots, i_l\} \subset \tilde{K} \subset \{0, i_1, \ldots, i_l, n\}$, where $1 \leq i_1 < \cdots < i_l \leq n-1$. Then

$$W_K \cong W_1 \times S_{i_2 - i_1} \times \cdots \times S_{i_l - i_{l-1}} \times W_2,$$

where $W_1 = \begin{cases} W(D_{i_1}), & \text{if } 0 \notin \tilde{K} \\ S_{i_1}, & \text{if } 0 \in \tilde{K} \end{cases}$ and $W_2 = \begin{cases} W(D_{n-i_l}), & \text{if } n \notin \tilde{K} \\ S_{n-i_l}, & \text{if } n \in \tilde{K}. \end{cases}$

Case (i): $l = 0$.

If $\tilde{K} = \{0\}$, then K is maximal special and the number of extreme element is 1.

If $\tilde{K} = \{0, n\}$, then $W_K \cong S_n$. In this case, the number of extreme elements equals to 2^r. Thus the CCP condition is satisfied exactly when $r = 1$.

Case (ii): $l \geq 1$.

Similarly to the argument in Subsection 7.4, if the CCP condition is satisfied, then $\lambda = \omega_n^\vee$ or $\omega_{n-1}^\vee$.

Let $\epsilon = \#(\{0, n\} \cap \tilde{K})$. If $\epsilon = 2$, then there are 2^{n-1} extreme elements. Since $n \geq 4$, we have $2^{n-1} \geq n > l$. Thus the CCP condition does not hold.

If $\epsilon \leq 1$, then similarly to the argument in Subsection 7.4, there are at least 2^l extreme elements. If the CCP condition is satisfied, then $2^l \leq l + \epsilon$. Hence $l = \epsilon = 1$. Thus $\tilde{K} = \{0, i\}$ for some $1 \leq i \leq n-1$. In this case, the number of extreme elements is 2^i. Thus in this case, the CCP condition is satisfied if and only if $\tilde{K} = \{0, 1\}$. Note that $\omega_n^\vee$ and $\omega_{n-1}^\vee$ are permuted by an outer automorphism of the finite Dynkin diagram of D_n, which preserves $\{0, 1\}$.

7.7. **Exceptional types.** For exceptional types, we argue in a different way.

Suppose that the extreme elements are $\lambda_1 = \lambda, \lambda_2, \ldots, \lambda_k$. Then we have $W_0 \cdot \lambda = \sqcup_{i=1}^k W_K \cdot \lambda_i$. We denote by $W_\lambda \subset W_0$ the isotropy group of λ and $W_{K,\lambda_i} \subset W_K$ the isotropy group of λ_i. Then we have

$$\#W_0/W_\lambda = \sum_{i=1}^{k} \#W_K/W_{K,\lambda_i}. \tag{7.1}$$

The trick here is that in most cases, we do not need to compute explicitly the coweights λ_i. Instead, we list the possible cardinalities $\#W_K/W_{K,\lambda_i}$. We then check that, in most cases, $\#W_0/W_\lambda$ does not equal to the sum of at most $\#\tilde{K}$ such numbers. Thus the CCP condition is not satisfied in these cases.

7.8. **Type $\tilde{G}_2$.** Note that $\lambda = \omega_2^\vee$ and $\#W_0/W_\lambda = 6$. Suppose that the CCP condition is satisfied. If $\#\tilde{K} = 1$, then $\#W_K \geq 6$. This implies that $\tilde{K} = \{0\}$ or $\tilde{K} = \{1\}$. One may check directly that these two cases satisfy the CCP condition. If $\#\tilde{K} = 2$, then $\#W_K \geq 3$, which is impossible.

7.9. **Type $\tilde{F}_4$.** Note that $\lambda = \omega_1^\vee$. We refer to [7, Plate VIII] for the explicit description of the root system of type F_4. In particular, we have $\omega_1^\vee = \epsilon_1^\vee + \epsilon_2^\vee$. Below is the list of maximal parahoric subgroups K and $\#W_K/\#W_{K,\lambda}$.

$\tilde{K}$	$\#W_K/W_{K,\lambda}$
$\{0\}$	24
$\{1\}$	2
$\{2\}$	6
$\{3\}$	12
$\{4\}$	24

Thus the CCP condition is satisfied if $\tilde{K} = \{0\}$ or $\{4\}$.

a) Now suppose that $\#\tilde{K} = 2$ and the CCP condition is satisfied. Then the (linear) action W_K on $W_0 \cdot \lambda$ has exactly two orbits: the orbit of λ and the orbit of another element λ' with $\lambda' \in W_0 \cdot \lambda$. Then $\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} = \#W_0/W_\lambda = 24$. In particular, $24 - \#W_K/W_{K,\lambda}$ divides $\#W_K$. This condition fails if $1 \in \tilde{K}$ or $2 \in \tilde{K}$, since in both cases $\#W_K/W_{K,\lambda} \leq 6$. Thus $\tilde{K}$ must be $\{3, 4\}$, or $\{0, 4\}$, or $\{0, 3\}$.

If $\tilde{K} = \{3, 4\}$, then we take $\lambda' = \epsilon_1^\vee - \epsilon_2^\vee$. By direct computation $\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} = 12 + 6 \neq 24$.

If $\tilde{K} = \{0, 4\}$, then we take $\lambda' = -\epsilon_1^\vee + \epsilon_2^\vee$. By direct computation $\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} = 6 + 6 \neq 24$.

If $\tilde{K} = \{0, 3\}$, then $\#W_K/W_{K,\lambda} = 3$ and $24 - 3 = 21$ does not divide $\#W_K$.

b) Now suppose that $\#\tilde{K} = 3$. If the CCP condition is satisfied, then $\#W_K \geq 24/3 = 8$ and thus $\tilde{K} = \{0, 1, 4\}$. However, in this case $W_K = W_{K,\lambda}$ and thus for any λ', λ'', we have $\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} + \#W_K/W_{K,\lambda''} < 24$. That is a contradiction.

c) If $\#\tilde{K} \geq 4$, then $\#\tilde{K} \cdot \#W_K \leq 5 \cdot 2 = 10 < 24$. So the CCP condition is not satisfied in this case.

7.10. **Type $\tilde{E}_6$.** By the definition of a minuscule coweight, λ is conjugate by an element of W_K to the trivial coweight or a minuscule coweight λ' of W_K.

In the table below, we list all the numbers $\#W_K/W_{K,\lambda'}$, where λ' is either trivial or a minuscule coweight of W_K. A direct product of Coxeter groups of type A like $A_{n_1} \times \cdots \times A_{n_k}$ is abbreviated as $A_{n_1,\ldots,n_k}$. We use a double line to separate the subsets $\tilde{K}$ with different cardinalities.

One may check case-by-case that if 27 equals the sum of $\#\tilde{K}$ elements in the list of $\#W_K/W_{K,\lambda'}$ from the table below, then W_K is of type E_6.

Type of W_K	$\#W_K/W_{K,\lambda'}$	Type of W_K	$\#W_K/W_{K,\lambda'}$	Type of W_K	$\#W_K/W_{K,\lambda'}$
E_6	$27, 1$	$A_{5,1}$	$20, 15, 6, 2, 1$	$A_{2,2,2}$	$3, 1$
D_5	$16, 10, 1$	A_5	$20, 15, 6, 1$	$A_{4,1}$	$10, 5, 2, 1$
$A_{3,1,1}$	$6, 4, 2, 1$	$A_{2,2,1}$	$3, 2, 1$		
D_4	$8, 1$	A_4	$10, 5, 1$	$A_{3,1}$	$6, 4, 2, 1$
$A_{2,2}$	$3, 1$	$A_{2,1,1}$	$3, 2, 1$		
A_3	$6, 4, 1$	$A_{2,1}$	$3, 2, 1$	$A_{1,1,1}$	$2, 1$
A_2	$3, 1$	$A_{1,1}$	$2, 1$		
A_1	$2, 1$				

7.11. **Type $\tilde{E}_7$.** In the table below, we list all the numbers $\#W_K/W_{K,\lambda'}$, where λ' is either trivial or a minuscule coweight of W_K.

One may check case-by-case that, if 56 equals the sum of $\#\tilde{K}$ elements in the list of $\#W_K/W_{K,\lambda'}$ from the table below, then W_K is of type E_7, A_7, E_6 or A_6.

If W_K is of type A_7, then $\#W_K/W_{K,\lambda} = 8 \neq 56$.

If W_K is of type E_6, then $\#W_K/W_{K,\lambda} = 1$ and thus there is no λ' with

$$\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} = 56.$$

If W_K is of type A_6, then $\#W_K/W_{K,\lambda} = 7$ or 1. Thus there is no λ' with

$$\#W_K/W_{K,\lambda} + \#W_K/W_{K,\lambda'} = 56.$$

Type of W_K	$\#W_K/W_{K,\lambda'}$	Type of W_K	$\#W_K/W_{K,\lambda'}$	Type of W_K	$\#W_K/W_{K,\lambda'}$
E_7	$56,1$	A_7	$70,56,28,8,1$	$D_6\times A_1$	$32,12,2,1$
$A_{5,2}$	$20,15,6,3,1$	$A_{3,3,1}$	$6,4,2,1$		
E_6	$27,1$	D_6	$32,12,1$	A_6	$35,21,7,1$
$D_5\times A_1$	$16,10,2,1$	$A_{5,1}$	$20,15,6,2,1$	$D_4\times A_1\times A_1$	$8,2,1$
$A_{4,2}$	$10,5,3,1$	$A_{3,3}$	$6,4,1$	$A_{3,2,1}$	$6,4,3,2,1$
$A_{3,1,1,1}$	$6,4,2,1$	$A_{2,2,2}$	$3,1$		
D_5	$16,10,1$	A_5	$20,15,6,1$	$D_4\times A_1$	$8,2,1$
$A_{4,1}$	$10,5,2,1$	$A_{3,2}$	$6,4,3,1$	$A_{3,1,1}$	$6,4,2,1$
$A_{2,2,1}$	$3,2,1$	$A_{2,1,1,1}$	$3,2,1$	$A_{1,1,1,1,1}$	$2,1$
D_4	$8,1$	A_4	$10,5,1$	$A_{3,1}$	$6,4,2,1$
$A_{2,2}$	$3,1$	$A_{2,1,1}$	$3,2,1$	$A_{1,1,1,1}$	$2,1$
A_3	$6,4,1$	$A_{2,1}$	$3,2,1$	$A_{1,1,1}$	$2,1$
A_2	$3,1$	$A_{1,1}$	$2,1$		
A_1	$2,1$				

8. Rationally strictly pseudo semi-stable reduction

In this section, we exclude the cases from the list in Theorem 7.1 that do not have rationally SPSS reduction. By Lemma 6.2, we check if the Poincaré polynomial is symmetric. As we have seen, the Poincaré polynomial depends only on the enhanced Coxeter datum, not the enhanced Tits datum. We start the elimination process with the exceptional types.

8.1. **The case** $(\tilde{G}_2,\omega_2^\vee,\{1\})$**.** Here $t^\lambda=s_0s_2s_1s_2s_1s_2$ and the unique maximal element in $W_{\le\lambda,K}$ is $w_{\lambda,K}=s_2s_0s_2s_1s_2s_1$. The set $W_{\le\lambda,K}$ has a unique element of length 1, which is s_1, but has two elements of length 5, which are $s_2w_{\lambda,K}$ and $s_0w_{\lambda,K}$. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat\lambda K^\flat/K^\flat}$ is not rationally smooth.

8.2. **The case** $(\tilde{F}_4,\omega_1^\vee,\{4\})$**.** Here $t^\lambda=s_0s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1$ and the unique maximal element in $W_{\le\lambda,K}$ is $w_{\lambda,K}=s_1s_2s_3s_2s_1s_0s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4$. The set $W_{\le\lambda,K}$ has a unique element of length 2, which is s_3s_4, but has at least two elements of colength 2, which are $s_2s_1w_{\lambda,K}$ and $s_0s_1w_{\lambda,K}$. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat\lambda K^\flat/K^\flat}$ is not rationally smooth.

8.3. **The case** $(\tilde{F}_4,\omega_1^\vee,\{0\})$ **(special parahoric).** Here the unique maximal element in $W_{\le\lambda,K}$ is $w_{\lambda,K}=t^{-\lambda}=w_0^{\{0,1\}}w_0^{\{n\}}s_0$, where $w_0^{K'}$ is the longest element in $W_{K'}$ for $K'=\{0\}$ or $\{0,1\}$. Thus

$$P_{\le\lambda,K}=1+q\frac{\sum_{x\in W_{\{0\}}}q^{\ell(x)}}{\sum_{x\in W_{\{0,1\}}}q^{\ell(x)}}.$$

Note that $(\sum_{x\in W_{\{0\}}}q^{\ell(x)})(\sum_{x\in W_{\{0,1\}}}q^{\ell(x)})^{-1}$ is a symmetric polynomial and not all the coefficients are equal to 1. Thus $P_{\le\lambda,K}$ is not symmetric and $\overline{K^\flat\lambda K^\flat/K^\flat}$ is not rationally smooth.

8.4. **The classical types.** Now we consider the cases where $\tilde{W}$ is of classical type. Let Φ_{af} be the affine root system of a split group whose associated affine Weyl group is W_a. Let Φ be the set of finite roots. As Φ_{af} comes from a split group, $\Phi_{\mathrm{af}} = \{a + n\delta; a \in \Phi, n \in \mathbb{Z}\}$.

We have the following formula on the length function of an element in $\tilde{W}$ (see [26]).

Lemma 8.1. *For $w \in W_0$ and $\alpha \in \Phi$, set*

$$\delta_w(\alpha) = \begin{cases} 0, & \text{if } w\alpha \in \Phi^+; \\ 1, & \text{if } w\alpha \in \Phi^-. \end{cases}$$

Then for any $x, y \in W_a$ and any translation element $t^{\lambda'}$ in $\tilde{W}$,

$$\ell(xt^{\lambda'}y^{-1}) = \sum_{\alpha \in \Phi^+} | < \lambda', \alpha > +\delta_x(\alpha) - \delta_y(\alpha)|.$$

We also have the following well-known facts on the Bruhat order in $\tilde{W}$.

Lemma 8.2. *Let $w \in \tilde{W}$. If $\alpha \in \Phi_{\mathrm{af}}$ is positive, and $w^{-1}(\alpha)$ is positive (resp., negative), then $s_\alpha w > w$ (resp., $s_\alpha w < w$).*

Lemma 8.3. *Let $w \in \tilde{W}^K$ and $s \in \tilde{S}$. If $sw < w$, then $\ell(sw) = \ell(w) - 1$ and $sw \in \tilde{W}^K$.*

The following result on the maximal element $w_{\lambda,K}$ of $W_{\leq \lambda,K}$ will also be useful in this section.

Lemma 8.4. *The maximal element $w_{\lambda,K}$ in $W_{\leq\lambda,K}$ is $t^{\lambda'}$, where λ' is the unique element in $W_K \cdot \lambda$ such that $\langle \lambda', \alpha \rangle \geq 0$ for any affine simple root α not in $\tilde{K}$.*

Proof. Let w_0^K be the longest element in W_K. We claim that

Claim: *The element $t^{\lambda'} w_0^K$ is the maximal element in $W_K t^{\lambda'} W_K = W_K t^{\lambda} W_K$.*

Let α be a simple affine root that is not in $\tilde{K}$. Then $(t^{\lambda'} w_0^K)^{-1}(\alpha) = (w_0^K)^{-1}(\alpha - \langle \lambda', \alpha \rangle \delta) = (w_0^K)^{-1}(\alpha) - \langle \lambda', \alpha \rangle \delta$ is a negative affine root. Hence by Lemma 8.2, $s_\alpha t^{\lambda'} w_0^K < t^{\lambda'} w_0^K$. Similarly, $(t^{\lambda'} w_0^K)(\alpha) = t^{\lambda'}(w_0^K(\alpha)) = w_0^K(\alpha) + \langle \lambda', w_0^K(\alpha) \rangle \delta$. Note that $w_0^K(\alpha)$ equals to $-\beta$ for some simple affine root β that is not in $\tilde{K}$. Hence $(t^{\lambda'} w_0^K)(\alpha)$ is a negative affine root. By Lemma 8.2, $t^{\lambda'} w_0^K s_\alpha < t^{\lambda'} w_0^K$. The claim is proved.

Note that by the assumption on λ', $t^{\lambda'}(\alpha)$ is a positive affine root for any simple affine root α that is not in $\tilde{K}$. Therefore $t^{\lambda'} \in \tilde{W}^K$.

Since $t^{\lambda'}$ is the unique element contained in both $(t^{\lambda'} w_0^K) W_K$ and $\tilde{W}^K$, and $t^{\lambda'} w_0^K$ is the maximal element in $W_K t^{\lambda} W_K$, $t^{\lambda'}$ is the unique maximal element in $W_{\leq \lambda,K}$. The statement is proved. □

With these facts established, we can now continue our elimination process.

8.5. **The case $(\tilde{B}_n, \omega_i^\vee, \{0\})$ (special parahoric).** Here $n \geq 3$ and $2 \leq i \leq n-1$.

Note that the set $W_{\leq\lambda,K}$ has a unique element of length 2, which is $\tau s_2 s_0$. Here τ, as usual, is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$. By Lemma 8.2, $s_i t^{-\lambda} < t^{-\lambda}$ and $s_{i-1} s_i t^{-\lambda}, s_{i+1} s_i t^{-\lambda} < s_i t^{-\lambda}$. By Lemma 8.3, $s_i t^{-\lambda}$ is a colength-1 element in $W_{\leq\lambda,K}$ and $s_{i-1} s_i t^{-\lambda}, s_{i+1} s_i t^{-\lambda}$ are colength-2 elements in $W_{\leq\lambda,K}$. Hence the set $W_{\leq\lambda,K}$ has at least two elements of colength 2. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.6. **The case** $(\tilde{C}_n, \omega_i^\vee, \{0\})$ **(special parahoric).** Here $n \geq 3$ and $2 \leq i \leq n-1$.

Note that the set $W_{\leq \lambda, K}$ has a unique element of length 2, which is $s_1 s_0$. By Lemma 8.2, $s_i t^{-\lambda} < t^{-\lambda}$ and $s_{i-1} s_i t^{-\lambda}, s_{i+1} s_i t^{-\lambda} < s_i t^{-\lambda}$. By Lemma 8.3, $s_i t^{-\lambda}$ is a colength-1 element in $W_{\leq \lambda, K}$ and $s_{i-1} s_i t^{-\lambda}, s_{i+1} s_i t^{-\lambda}$ are colength-2 elements in $W_{\leq \lambda, K}$. Hence the set $W_{\leq \lambda, K}$ has at least two elements of colength 2. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.7. **The case** $(\tilde{C}_n, l\omega_n^\vee, \{i\})$**.** Here $n \geq 2$, $l = 1$ or 2 and $1 \leq i \leq n-1$.

By Lemma 8.4, $w_{\lambda, K} = t^{\lambda'}$, where the first i entries of λ' are $\frac{l}{2}$ and the last $n-i$ entries of λ' are $-\frac{l}{2}$. Note that the set $W_{\leq \lambda, K}$ has a unique element of length 1, which is τs_i. Here τ, as usual, is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$. By Lemma 8.2, $s_0 t^{\lambda'} < t^{\lambda'}$. By Lemma 8.3, $s_0 t^{\lambda'}$ is a colength-1 element in $W_{\leq \lambda, K}$. Similarly, $s_n t^{\lambda'}$ is a colength-1 element in $W_{\leq \lambda, K}$. Thus the set $W_{\leq \lambda, K}$ has at least two elements of colength 1. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.8. **The case** $(\tilde{B}_n, \omega_r^\vee, \{n\})$**.** Here $n \geq 3$ and $1 \leq r \leq n-1$.

By Lemma 8.4, $w_{\lambda, K} = t^{\lambda'}$, where the first $n-r$ entries of λ' are 0 and the last r entries of λ' are -1.

8.8.1. *The case* $2 \leq r \leq n-2$. Note that the set $W_{\leq \lambda, K}$ has a unique element of length 2, which is $\tau s_{n-1} s_n$, where again τ is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$.

By Lemma 8.2, $s_r t^{\lambda'} < t^{\lambda'}$ and $s_{r-1} s_r t^{\lambda'}, s_{r+1} s_r t^{\lambda'} < s_r t^{\lambda'}$. By Lemma 8.3, $s_r t^{\lambda'}$ is a colength-1 element in $W_{\leq \lambda, K}$ and $s_{r-1} s_r t^{\lambda'}, s_{r+1} s_r t^{\lambda'} \in W_{\leq \lambda, K}$ are colength-2 elements in $W_{\leq \lambda, K}$. Hence the set $W_{\leq \lambda, K}$ has at least two elements of colength 2. Therefore the Poincaré polynomial is not symmetric and $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.8.2. *The case* $r = 1$. By direct computation, $w_{\lambda, K} = \tau(s_{n-1} s_{n-2} \cdots s_2)(s_0 s_1 \cdots s_n)$ and the Poincaré polynomial is $(1 + q + \cdots + q^{2(n-1)})q + q^n + 1$ which is not symmetric. Thus $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.8.3. *The case* $r = n-1$. The set $W_{\leq \lambda, K}$ has a unique element of length 1 which is τs_n, but has at least two elements of colength 1, namely $s_1 w_{\lambda, K}$ and $s_0 w_{\lambda, K}$. The Poincaré polynomial is not symmetric and $\overline{K^\flat \lambda K^\flat / K^\flat}$ is not rationally smooth.

8.9. **The case** $(\tilde{C}_n, 2\omega_n^\vee, \{i, i+1\})$ **with** $0 \leq i \leq \frac{n}{2} - 1$**.** Here $n \geq 2$.

By Lemma 8.4, $w_{\lambda, K} = t^{\lambda'}$, where the first $i+1$ entries of λ' is 1 and the last $n-i-1$ entries of λ' is -1. In this case, $W_{\leq \lambda, K}$ has exactly two elements of length 1 which are s_i and s_{i+1}. Similarly to the argument in §8.7, $s_n t^{\lambda'}$ and $s_0 t^{\lambda'}$ are colength 1 elements in $W_{\leq \lambda, K}$. By Lemma 8.2, $s_{2e_{i+1} + 2\delta} t^{\lambda'} = (i+1, -(i+1)) t^{\lambda''} < t^{\lambda'}$ and $s_{2e_1 + 2\delta} t^{\lambda'} \in \tilde{W}^K$. Here the first i-entries of λ'' is 1 and the last $n-i$-entries of λ'' is -1. By Lemma 8.1, $\ell(s_{2e_{i+1} + 2\delta} t^{\lambda'}) = \ell(t^{\lambda'}) - 1$. Hence $s_{2e_{i+1} + 2\delta} t^{\lambda'}$ is a colength-1 element in $W_{\leq \lambda, K}$. Therefore $W_{\leq \lambda, K}$ contains at least three elements of colength 1 and the Poincaré polynomial of $W_{\leq \lambda, K}$ is not symmetric.

8.10. **The case $(\tilde{C}_n, \omega_n^\vee, \{i, i+1\})$ with $1 \le i \le \frac{n}{2} - 1$.** Here $n \ge 2$.

By Lemma 8.4, $w_{\lambda,K} = t^{\lambda'}$, where the first $i+1$ entries of λ' is $\frac{1}{2}$ and the last $n-i-1$ entries of λ' is $-\frac{1}{2}$. In this case, $W_{\le\lambda,K}$ has exactly two elements of length 1 which are τs_i and τs_{i+1}, where again τ is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$. Similarly to the argument in §8.7, $s_n t^{\lambda'}$ and $s_0 t^{\lambda'}$ are colength 1 elements in $W_{\le\lambda,K}$. By Lemma 8.2, $s_{2e_{i+1}+\delta} t^{\lambda'} = (i+1, -(i+1)) t^{\lambda''} < t^{\lambda'}$ and $s_{2e_1+\delta} t^{\lambda'} \in \tilde{W}^K$. Here the first i-entries of λ'' is $\frac{1}{2}$ and the last $n-i$-entries of λ'' is $-\frac{1}{2}$. By Lemma 8.1, $\ell(s_{2e_{i+1}+\delta} t^{\lambda'}) = \ell(t^{\lambda'}) - 1$. Hence $s_{2e_{i+1}+\delta} t^{\lambda'}$ is a colength-1 element in $W_{\le\lambda,K}$. Therefore $W_{\le\lambda,K}$ contains at least three elements of colength 1 and the Poincaré polynomial of $W_{\le\lambda,K}$ is not symmetric.

8.11. **The case $(\tilde{B}_n, \omega_n^\vee, \{0, 1\})$.** Here $n \ge 3$.

By Lemma 8.4, $w_{\lambda,K} = t^{\lambda'}$, where $\lambda' = (1, -1, -1, \dots, -1)$. In this case, $W_{\le\lambda,K}$ has exactly 3 elements of length 2 which are $\tau s_0 s_1$, $\tau s_2 s_0$ and $\tau s_2 s_1$. Similarly to the argument in Subsection 8.9, $s_n t^{\lambda'}$ and $s_{\epsilon_1+\delta} t^{\lambda'} = (1,-1) t^{-\omega_n^\vee}$ are colength-1 elements in $W_{\le\lambda,K}$.

By Lemma 8.2, $s_{n-1} s_n t^{\lambda'} < s_n t^{\lambda'}$ and $s_n s_{\epsilon_1+\delta} t^{\lambda'} < s_{\epsilon_1+\delta} t^{\lambda'}$. By Lemma 8.3, $s_{n-1} s_n t^{\lambda'}$ and $s_n s_{\epsilon_1+\delta} t^{\lambda'}$ are colength-2 elements in $W_{\le\lambda,K}$. The next lemma produces two more elements of colength 2.

Lemma 8.5. *The elements $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n}$ and $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'}$ are colength 2 elements in $W_{\le\lambda,K}$.*

Proof. We have $s_{\epsilon_1+\delta} t^{\lambda'} \cdot (\epsilon_1 - \epsilon_n) = (1,-1) \cdot (\epsilon_1 - e_n) = -\epsilon_1 - \epsilon_n$. Thus $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} < s_{\epsilon_1+\delta} t^{\lambda'}$. Note that $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} = (1,-1)(1,n) t^{-\omega_n^\vee}$.

For $2 \le i \le n-2$, we have $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} \cdot (\epsilon_i - \epsilon_{i+1}) = (1,-1)(1,n) \cdot (\epsilon_i - \epsilon_{i+1}) = \epsilon_i - \epsilon_{i+1}$. We have $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} \cdot (\epsilon_{n-1} - \epsilon_n) = (1,-1)(1,n) \cdot (\epsilon_{n-1} - \epsilon_n) = \epsilon_1 + \epsilon_{n-1}$. We have $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} \cdot \epsilon_n = (1,-1)(1,n)(\epsilon_n - \delta) = -\epsilon_1 - \delta$. Therefore $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} \in \tilde{W}^K$.

Finally, by Lemma 8.1, we have $\ell(s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n}) = \ell(t^{\lambda'}) - 2$. Therefore $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n}$ is a colength 2 element in $W_{\le\lambda,K}$.

Similarly, we have $(s_{\epsilon_1+\delta} t^{\lambda'})^{-1} \cdot (\epsilon_1 - \epsilon_n + \delta) = -\epsilon_1 - \epsilon_n - \delta$. Thus $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'} < s_{\epsilon_1+\delta} t^{\lambda'}$. Note that $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'} = (1,n)(1,-1) t^{(0,-1,\dots,-1,0)}$.

For $2 \le i \le n-2$, we have $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'} \cdot (\epsilon_i - \epsilon_{i+1}) = \epsilon_i - \epsilon_{i+1}$. We have $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'} \cdot (\epsilon_{n-1} - \epsilon_n) = (1,n)(1,-1) \cdot (\epsilon_{n-1} - \epsilon_n - \delta) = \epsilon_{n-1} - \epsilon_1 - \delta$. We have $s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'} \cdot \epsilon_n = \epsilon_1$. Therefore $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n} \in \tilde{W}^K$.

Finally, by Lemma 8.1, we have $\ell(s_{\epsilon_1-\epsilon_n+\delta} s_{\epsilon_1+\delta} t^{\lambda'}) = \ell(t^{\lambda'}) - 2$. Therefore $s_{\epsilon_1+\delta} t^{\lambda'} s_{\epsilon_1-\epsilon_n}$ is a colength 2 element in $W_{\le\lambda,K}$. □

Therefore $W_{\le\lambda,K}$ contains at least 4 elements of colength 2 and the Poincaré polynomial of $W_{\le\lambda,K}$ is not symmetric.

8.12. **The case $(\tilde{C}_n, \omega_1^\vee, \{0, n\})$.** This case is more complicated than the cases we have discussed earlier. In fact, the geometric special fiber has two irreducible components and the Poincaré polynomials of the irreducible components are both symmetric. However, the Poincaré polynomial of their intersection is not symmetric.

Let $\lambda = (1, 0, \dots, 0)$ and $\lambda' = (0, 0, \dots, 0, -1)$. The irreducible components of the geometric special fibers are $\overline{K^\flat \lambda K^\flat / K^\flat}$ and $\overline{K^\flat \lambda' K^\flat / K^\flat}$.

We set $w_1 = \max(W_K t^\lambda W_K)$ and $w_2 = \max(W_K t^{\lambda'} W_K)$. By direct computation,

$$w_1 = (s_{n-1} s_{n-2} \cdots s_0)(s_1 s_2 \cdots s_n) w_0^K, \qquad w_2 = (s_1 s_2 \cdots s_n)(s_{n-1} s_{n-2} \cdots s_0) w_0^K,$$

where w_0^K is the longest element in W_K. Moreover, the set $\{w' \in \tilde{W}; w' \leq w_1, w' \leq w_2\}$ contains a unique maximal element ww_0^K, where

$$w = (s_1 s_2 \cdots s_{n-1})(s_{n-2} s_{n-3} \cdots s_1) s_0 s_n \in \tilde{W}^K.$$

Set $W_{\leq w,K} = \{v \in \tilde{W}^K; v \leq w\}$. The intersection of $\overline{K^\flat \lambda K^\flat / K^\flat}$ and $\overline{K^\flat \lambda' K^\flat / K^\flat}$ is $\overline{K^\flat w K^\flat / K^\flat}$ and the associated Poincaré polynomial is

$$P_{\leq w,K}(q) = \sum_{v \in W_{\leq w,K}} q^{\ell(v)}.$$

We have

$$W_{\leq w,K} = \{1, v_1 s_0, v_2 s_n, v_3 s_0 s_n\},$$

where

$$v_1 \in W_{\{0,n\}} \cap W^{\{0,1,n\}} = \{1, s_1, s_2 s_1, \ldots, s_{n-1} s_{n-2} \cdots s_1\},$$
$$v_2 \in W_{\{0,n\}} \cap W^{\{0,n-1,n\}} = \{1, s_{n-1}, s_{n-2} s_{n-1}, \ldots, s_1 s_2 \cdots s_{n-1}\},$$
$$v_3 \in W_{\{0,n\}} \cap W^{\{0,1,n-1,n\}}.$$

Note that $W_{\{0,n\}} \cap W^{\{0,1,n-1,n\}} = W(A_{n-1})^{\{1,n-1\}}$, where $W(A_{n-1})$ is the finite Weyl group of type A_{n-1}. Thus

$$\sum_{v_3 \in W_{\{0,n\}} \cap W^{\{0,1,n-1,n\}}} q^{\ell(v_3)} = \frac{\sum_{v \in W(A_{n-1})} q^{\ell(v)}}{\sum_{v \in W(A_{n-3})} q^{\ell(v)}} = (1 + q + \cdots + q^{n-2})(1 + q + \cdots + q^{n-1}).$$

Hence

$$\begin{aligned} P_{\leq w,K} &= 1 + 2(1 + q + \cdots + q^{n-1})q + (1 + q + \cdots + q^{n-2})(1 + q + \cdots + q^{n-1})q^2 \\ &= (1 + 2q + \cdots + nq^{n-1} + (n+1)q^n) + ((n-1)q^{n+1} + (n-2)q^{n+2} + \cdots + q^{2n-1}). \end{aligned}$$

Note that the coefficient of q^{n-1} is n and the coefficient of q^n is $n+1$. Therefore $P_{\leq w,K}$ is not symmetric.

8.13. **The case** $(\tilde{A}_1, 2\omega_1^\vee, \{0,1\})$**.** Finally, we consider the case $(\tilde{A}_1, 2\omega_1^\vee, \{0,1\})$. In this case, $K^\flat = I^\flat$ and $\mathrm{Adm}(\lambda) = \{s_0 s_1, s_1 s_0, s_1, s_0, 1\}$. The irreducible components of the geometric special fiber are $\overline{I^\flat s_0 s_1 I^\flat / I^\flat}$ and $\overline{I^\flat s_1 s_0 I^\flat / I^\flat}$. Their intersection is $\overline{I^\flat s_1 I^\flat / I^\flat} \cup \overline{I^\flat s_0 I^\flat / I^\flat}$ and thus is not irreducible.

9. Proof of Theorem 5.1

In this section we assume $p \neq 2$. As already explained in Subsection 5.1, we may assume that G_{ad} is absolutely simple. By Proposition 2.13 (iv), we may change G arbitrarily, as long as G_{ad} is fixed. Let us check one implication. If K is hyperspecial, then $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ is smooth over O_E, cf. Proposition 2.13, (i). Let $G = \mathrm{GU}(V)$ be the group of unitary similitudes for a hermitian space relative to a ramified quadratic extension $\tilde{F}/F$, and let $\{\mu\} = (1, 0, \ldots, 0)$. If K is the stabilizer of a π-modular lattice Λ if $\dim V$ is even, resp., is the stabilizer of an almost π-modular lattice Λ if $\dim V$ is odd, then $E = \tilde{F}$ and the local model $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ is smooth over O_E, cf. [1, Prop. 4.16], [43, Thm. 2. 27, (iii)]. Now let $G = \mathrm{SO}(V)$ be the orthogonal group associated to an orthogonal F-vector space of even dimension ≥ 6, $\{\mu\}$ the cocharacter corresponding to the orthogonal Grassmannian of isotropic subspaces of maximal dimension, and K the parahoric stabilizer

of an almost selfdual vertex lattice, as in 5.1 (2). After an unramified extension of F, the set-up described in 12.11 applies; by the calculation in 12.11 and Proposition 2.13 (ii) the local model $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is smooth over O_E. The general case of exotic good reduction type (which involves in addition an unramified restriction of scalars) follows.

Conversely, assume that $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is smooth over O_E. Then its geometric special fiber is irreducible, and hence the triple $(G,\{\mu\},K)$ produces enhanced Coxeter data that appear in Theorem 7.1 under the heading (1). By Subsections 8.1 and 8.2, the exceptional types (d) and (e) do not have rationally SPSS reduction. Similarly, Subsection 8.8 eliminates the case (b), and Subsection 8.7 eliminates the case (c). Therefore, the only remaining possibilities are in case (a), i.e., $\breve{K}$ is a special maximal parahoric. Hence the associated enhanced Tits datum $(\tilde{\Delta},\{\lambda\},\tilde{K})$ is such that $\tilde{K}$ consists of a single *special* vertex. From these cases, Subsection 8.3 eliminates $(\tilde{F}_4,\omega_1^\vee,\{0\})$. Subsections 8.5 and 8.6 eliminate $(\tilde{B}_n,\omega_i^\vee,\{0\})$ and $(\tilde{C}_n,\omega_i^\vee,\{0\})$, for $n\geq 3$ and $2\leq i\leq n-1$. (In these cases, the special fiber is irreducible but, again, not rationally smooth). When $\breve{K}$ is hyperspecial and λ is minuscule we have good reduction. When $\breve{K}$ is hyperspecial and λ is not minuscule the reduction is not smooth by [34]. (This reference is for k replaced by $\mathbb{C}$ but the same argument works; see also [22, §6] for an explanation of the passage from $\mathbb{C}$ to k.) It remains to list the remaining cases in which $\breve{K}$ is special but not hyperspecial. Here are these remaining cases:

1) $(\tilde{B}_n,\omega_1^\vee,\{0\})$, $n\geq 3$.

Since we are only considering the cases in which $\{0\}$ is not hyperspecial, the local Dynkin diagram is B-C_n. Since the non-trivial automorphism of B-C_n does not preserve $\{0\}$, the Frobenius has to act trivially (see [20]). The corresponding group is a quasi-split (tamely) ramified unitary group $U(V)$ for V of even dimension $2n$ (e.g. [20, p. 22]). The coweight corresponds to $\{\mu\}=(1,0,\dots,0)$ and K is the parahoric stabilizer of a $\tilde{\pi}$-modular lattice (notations as in 5.1 (1).) This is a case of unitary exotic good reduction.

2) $(\tilde{B}_n,\omega_n^\vee,\{0\})$, $n\geq 3$.

As above, the local Dynkin diagram is B-C_n and the corresponding group a (tamely) ramified unitary group $U(V)$ for V of even dimension $2n$. The subgroup K is the parahoric stabilizer of a $\tilde{\pi}$-modular lattice. In this case, the coweight corresponds to $\{\mu\}=(1^{(n)},0^{(n)})$ so this is the case of signature (n,n). By [42, (5.3)], we see that the geometric special fiber of the local model contains an open affine subscheme which has the following properties: It is the reduced locus C^{red} of an irreducible affine cone C which is defined by homogeneous equations of degree ≥ 2 and which is generically reduced. Then C^{red} is the affine cone over the integral projective variety $(C-\{0\})^{\mathrm{red}}/\sim$, also given by such equations, and is therefore not smooth. We see that, in this case, $\mathbb{M}_K^{\mathrm{loc}}(G,\{\mu\})$ is not smooth.

3) $(\tilde{C}_n,\omega_1^\vee,\{0\})$, $n\geq 2$.

Since we are only considering the case in which $\{0\}$ is not hyperspecial, the local Dynkin diagram is C-BC_n or C-B_n. In both cases, only the trivial automorphism can preserve $\{0\}$ so Frobenius acts trivially. In the case C-BC_n, we have a ramified unitary group $U(V)$ for V of odd dimension $2n+1$; here there are two possibilities for a corresponding enhanced Tits datum. There are three cases overall that also appear as cases (1-a), (1-b), (1-c) in the next section:

a) Ramified quasi-split $U(V)$ for V of odd dimension $2n+1$, $\{\mu\} = (1, 0, \dots, 0)$, and K the parahoric stabilizer of an almost $\tilde{\pi}$-modular lattice. This is a case of unitary exotic good reduction.

b) Ramified quasi-split $U(V)$ for V of odd dimension $2n+1$, $\{\mu\} = (1^{(n-1)}, 0^{(n+2)})$ and K is the parahoric stabilizer of a selfdual lattice. Then the local model has non-smooth special fiber by Subsection 10.3 (or one can employ an argument using [42, (5.2)] as in 2) above).

c) Ramified quasi-split orthogonal group $\mathrm{SO}(V)$ for V of even dimension $2n+2$, μ is the cocharacter that corresponds to the quadric homogeneous space and $\tilde{K}$ is the parahoric stabilizer of an almost selfdual vertex lattice. Then the local model is not smooth; this follows by combining Propositions 12.7 and 12.6 (II).

4) $(\tilde{C}_n, \omega_n^\vee, \{0\})$, $n \geq 2$.

The local Dynkin diagram is C-B_n. As above, we see that we have a ramified quasi-split but not split orthogonal group $SO(V)$ for V of even dimension $2n+2$, $\{\mu\}$ corresponds to the orthogonal Grassmannian of isotropic subspaces of dimension $n+1$ and $\tilde{K}$ is the parahoric stabilizer of an almost selfdual vertex lattice. This is the case of orthogonal exotic good reduction (see also Subsection 12.11).

5) $(\tilde{C}_n, 2\omega_n^\vee, \{0\})$, $n \geq 1$.

The local Dynkin diagram is C-BC_n. We have a ramified unitary group $U(V)$ for V of odd dimension $2n+1$, $\{\mu\} = (1^{(n)}, 0^{(n+1)})$, and K is the parahoric stabilizer of an almost $\tilde{\pi}$-modular lattice. Using [42, (5.2)] and an argument as in 2) above, we see that the special fiber is not smooth when $n > 1$. If $n = 1$, then $\{\mu\} = (1, 0, 0)$ and this is a case of unitary exotic reduction.

6) $(\tilde{G}_2, \omega_2^\vee, \{0\})$.

Again Frobenius is trivial and we have the quasi-split ramified triality group of type ${}^3D_4$. The tameness assumption implies that $p \neq 3$. Therefore, the main result of Haines-Richarz [22] is applicable and implies that the special fiber is not smooth. (In principle, this non-smoothness statement can also be deduced using Kumar's criterion – see Subsection 10.2 below. However, this involves a lengthy calculation that appears to require computer assistance, see the (simpler) case of G_2 in [34, (7.9)-(7.12)].)

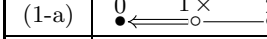
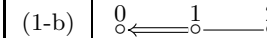
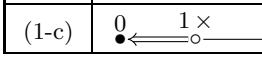
10. Strictly pseudo semi-stable reduction

10.1. **Statement of the result.** Our goal here is to examine smoothness of the affine Schubert varieties contained in the geometric special fiber of $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$. By Theorem 2.10, this fiber can be identified with the admissible locus $\mathfrak{A}_K(G, \{\mu\})$ in the partial affine flag variety for a group $\tilde{G}^\flat$ which is isogenous to $G^\flat$ but which has simply connected derived group. In the rest of this section, we will omit the tilde from the notation; but it is understood that the affine Schubert varieties will be for a group with simply connected derived group. This issue did not appear in our discussion of rational smoothness since this is defined via ℓ-adic cohomology which is insensitive to radicial morphisms. In fact, the rational smoothness of the affine Schubert variety $\overline{I^\flat w I^\flat / I^\flat}$ only depends on the element w in the Iwahori-Weyl group, and does not depend on the reductive group itself. On the other hand, the smoothness of the affine Schubert variety $\overline{I^\flat w I^\flat / I^\flat}$ depends on the reductive group, not only the associated Iwahori-Weyl group. In other words, smoothness

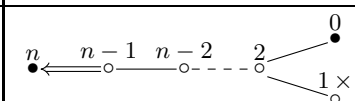
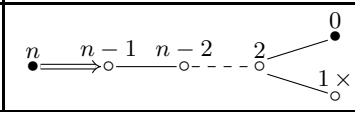
of the affine Schubert varieties in question (assuming simply connected derived group) depends on the enhanced Tits datum, and not only on the enhanced Coxeter datum.

In this section, we consider the enhanced Tits data associated to the enhanced Coxeter data $(\tilde{C}_n, \omega_1^\vee, \{0\})$, $(\tilde{B}_n, \omega_1^\vee, \{0, n\})$ and $(\tilde{C}_n, \omega_n^\vee, \{0, 1\})$. They are as follows:

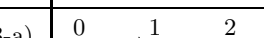
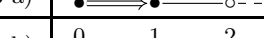
(1) The triple $(\tilde{C}_n, \omega_1^\vee, \{0\})$ with $n \geq 2$:

Label	Enhanced Tits datum	Linear algebra datum
(1-a)		Nonsplit $U_{2n+1}, r = 1, \Lambda_0$
(1-b)		Nonsplit $U_{2n+1}, r = n-1, \Lambda_n$
(1-c)		Nonsplit $SO_{2n+2}, r = 1, \Lambda_0$

(2) The triple $(\tilde{B}_n, \omega_1^\vee, \{0, n\})$ with $n \geq 3$:

Label	Enhanced Tits datum	Linear algebra datum
(2-a)		Split SO_{2n+1}, $r = 1$, (Λ_0, Λ_n)
(2-b)		U_{2n}, $r = 1$, (Λ_0, Λ_n)

(3) The triple $(\tilde{C}_n, \omega_n^\vee, \{0, 1\})$ with $n \geq 2$:

Label	Enhanced Tits datum	Linear algebra datum
(3-a)		Split $Sp_{2n}, r = n, (\Lambda_0, \Lambda_1)$
(3-b)		Nonsplit $SO_{2n+2}, r = n+1, (\Lambda_0, \Lambda_2)$

Here the numbers above the vertices of the Dynkin diagrams are the labelings. The main result of this section is

Proposition 10.1. *The cases (1-b), (1-c), (2-b) and (3-b) are not strictly pseudo semi-stable reduction.*

We prove Proposition 10.1 by showing that at least one of the irreducible components of the geometric fiber is not smooth.

10.2. **Kumar's criterion.** Note that for any $x \in \tilde{W}$ and parahoric subgroup K, the smoothness of $\overline{K^\flat x K^\flat / K^\flat}$ is equivalent to the smoothness of $\overline{I^\flat w I^\flat / I^\flat}$, where we set $w = \max\{W_K x W_K\}$. To study the case (1-b), we use Kumar's criterion [31], which we recall here.

Let Q be the quotient field of the symmetric algebra of the root lattice. Following [3], we fix a reduced expression $\underline{w} = \tau s_{\alpha_1} \cdots s_{\alpha_l}$ of w, where τ is a length-zero element in $\tilde{W}$ and $\alpha_1, \ldots, \alpha_l$ are affine simple roots. For any $x \leq w$, we define

$$e_x X(w) = \sum_{(s_1, \ldots, s_l)} \prod_{j=1}^{l} s_1 \cdots s_j(\alpha_j)^{-1} \in Q, \tag{10.1}$$

where the sum runs over all sequences $(s_1, \dots, s_l)$ such that $s_j = 1$ or s_{α_j} for any j and $s_1 \cdots s_l = x$. We call such sequences the *subexpressions* for x in $\underline{w}$. It is known that $e_x X(w)$ is independent of the choice of the reduced expression $\underline{w}$ of w.

Theorem 10.2. *The Schubert variety $\overline{I^\flat w I^\flat / I^\flat}$ is smooth if and only if*

$$e_1 X(w) = \prod_{\{\alpha \in \Phi_{\mathrm{af}}^+ | s_\alpha \leq w\}} \alpha^{-1}.$$

10.3. **The case (1-b).** Set $\lambda' = (-1, -1, \dots, -1, 0)$. By Lemma 8.4, $t^{\lambda'}$ is the maximal element in $W_{\leq \lambda', K}$. Set $w = \max(W_K t^{\lambda'} W_K)$. By direct computation,

$$w = (s_{n-1} s_{n-2} \cdots s_0)(s_1 s_2 \cdots s_n) w_0^K,$$

where w_0^K is the longest element in W_K.

As s_n is the reflection of a long root, and the other simple reflections are reflections of short roots, in any expression of 1, the simple reflection s_n must appear an even number of times. Note that in a reduced expression of w, the simple reflection s_n appears only once, thus s_n does not appear in the subexpression for 1. Moreover, the reduced expression $\underline{w}$ of w may be chosen to be of the form $\dots s_{n-1} s_n s_{n-1} \dots$. Thus any subexpression $(s_1, \dots, s_j)$ for 1 in $\underline{w}$ is of the form $(\dots, 1, 1, 1, \dots)$, $(\dots, s_{n-1}, 1, s_{n-1}, \dots)$, $(\dots, s_{n-1}, 1, 1, \dots)$, $(\dots, 1, 1, s_{n-1}, \dots)$. Here the three terms in the middle are the subexpressions of $s_{n-1} s_n s_{n-1}$ in which s_n does not appear. A direct computation for the rank-two Weyl groups shows that

$$\begin{aligned} e_1 X(s_{n-1} s_n s_{n-1}) &= -e_{s_{n-1}} X(s_{n-1} s_n s_{n-1}) = \\ &= \frac{-\langle \alpha_n, \alpha_{n-1}^\vee \rangle}{\alpha_n \alpha_{n-1} s_{n-1}(\alpha_n)} = \frac{2}{\alpha_n \alpha_{n-1} s_{n-1}(\alpha_n)}. \end{aligned} \tag{10.2}$$

We rewrite the formula (10.1) as

$$\begin{aligned} e_1 X(w) = &\sum_{(\dots,1,1,1,\dots)} \prod_{j=1}^{l} s_1 \cdots s_j(\alpha_j)^{-1} + \sum_{(\dots,s_{n-1},1,s_{n-1},\dots)} \prod_{j=1}^{l} s_1 \cdots s_j(\alpha_j)^{-1} \\ &+ \sum_{(\dots,s_{n-1},1,1,\dots)} \prod_{j=1}^{l} s_1 \cdots s_j(\alpha_j)^{-1} + \sum_{(\dots,1,1,s_{n-1},\dots)} \prod_{j=1}^{l} s_1 \cdots s_j(\alpha_j)^{-1}. \end{aligned}$$

By (10.2), all coefficients in the first and in the second line are multiples of 2. By Theorem 10.2, $\overline{I^\flat w I^\flat / I^\flat}$ is not smooth.

10.4. **The case (1-c).** The special fiber is irreducible but not smooth. As was also mentioned in Section 9, Case (3), this follows by combining Propositions 12.7 and 12.6 (II).

10.5. **The case (2-b).** Set $\lambda' = (0, 0, \dots, 0, 1)$. By Lemma 8.4, $t^{\lambda'}$ is the maximal element in $W_{\leq \lambda', K}$. Set $w = \max(W_K t^{\lambda'} W_K)$. By direct computation,

$$w = \tau(s_{n-1} s_{n-2} \cdots s_2)(s_0 s_1 \cdots s_n) w_0^K,$$

where τ is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$ and w_0^K is the longest element in W_K.

Note that in a reduced expression of w, the simple reflection s_n appears only once, thus s_n does not appear in the subexpression for 1. Similar to the argument in the subsection 10.3, $\overline{I^\flat w I^\flat/I^\flat}$ is not smooth.

10.6. **The case (3-b).** Set $\lambda' = (-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$. By Lemma 8.4, $t^{\lambda'}$ is the maximal element in $W_{\leq \lambda', K}$. By direct computation,

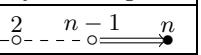
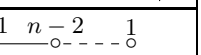
$$\max(W_K t^{\lambda'} W_K) = \tau w_0^{\{n\}} w_0^{\{0,n\}} w_0^K,$$

where τ is the unique length-zero element in $\tilde{W}$ with $\kappa(\tau) = \kappa(\lambda)$ and where $w_0^{K'}$ is the longest element in $W_{K'}$ for $\tilde{K}' = \{0,1\}, \{n\}$ or $\{0,n\}$. Note that

$$\overline{K^\flat t^{\lambda'} K^\flat/K^\flat} = \overline{I^\flat t^{\lambda'} K^\flat/K^\flat} \cong \overline{I^\flat w_0^{\{n\}} w_0^{\{0,n\}} K^\flat/K^\flat} \cong \overline{I^\flat w_0^{\{n\}} w_0^{\{0,n\}} K_1^\flat/K_1^\flat} \subset K_2^\flat/K_1^\flat,$$

where $\tilde{K}_1 = \{0,1,n\}$ and $\tilde{K}_2 = \{n\}$.

Let $U_{K_2^\flat}$ be the pro-unipotent radical of $K_2^\flat$ and $G' = K_2^\flat/U_{K_2^\flat}$ the reductive quotient of $K_2^\flat$. Note that G'_{ad} is the adjoint group of type B_n over k. Let $P = K_1^\flat/U_{K_2^\flat}$. This is a standard parabolic subgroup of G'. We have $K_2^\flat/K_1^\flat \cong G'/P$. This is a partial flag variety of finite type.

Group	Affine/Finite Dynkin diagram
G	
G'	

In the table, the parahoric subgroup $\breve{K}_1$ of G and the parabolic subgroup P of G' correspond to the set of vertices filled with black color in the corresponding diagram.

The finite Dynkin diagram of G' is obtained from the local Dynkin diagram of G by deleting the vertex labeled n. The labeling of the Dynkin diagram is not inherited from the local Dynkin diagram of G, but is the standard labeling of the finite Dynkin diagram in [7]. The reason is that we will apply the smoothness criterion for finite Schubert varieties, and we follow the convention for finite Dynkin diagrams and finite Weyl groups. We identify the finite Weyl group $W_{G'}$ of G' with the group of permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$.

Under the natural isomorphism $K_2^\flat/K_1^\flat \cong G'/P$, the closed subset of the affine partial flag variety $\overline{I^\flat w_0^{\{n\}} w_0^{\{0,n\}} K_1^\flat/K_1^\flat}$ is isomorphic to the closed subvariety $\overline{B' w' P/P}$ of the finite type partial flag variety, where $B' = I^\flat/U_{K_2^\flat}$ is a Borel subgroup of G' and $w' \in W_{G'}$ is the permutation $(1,-n)(2,-(n-1))\cdots$.

Note that the smoothness of $\overline{B' w' P/P}$ is equivalent to the smoothness of $\overline{B' w' w_0^P B'/B'}$, where w_0^P is the longest element in the Weyl group W_P of P. The element $w' w_0^P$ is the permutation of $\{\pm 1, \pm 2, \ldots, \pm n\}$ sending 1 to -2, 2 to $-3, \ldots, n-1$ to $-n$ and n to -1. By the pattern avoidance criterion (see [2, Thm. 8.3.17]), $\overline{B' w' w_0^P B'/B'}$ is not smooth. Hence $\overline{K^\flat \lambda' K^\flat/K^\flat}$ is not smooth.

11. Proof of one implication in Theorem 5.6

Assume that $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ has strictly semi-stable reduction. Inspection of all cases considered in the previous section shows that then $(G, \{\mu\}, K)$ appears in the list of Theorem 5.6. In the remaining section of the paper, we show that indeed for all triples $(G, \{\mu\}, K)$

on this list the corresponding associated local models have semi-stable reduction. As a consequence of this assertion, we obtain the following somewhat surprising result.

Corollary 11.1. *Let $(G,\{\mu\},K)$ be a triple over F such that G splits over a tame extension of F. Assume $p\neq 2$. Assume also that the group G is adjoint and absolutely simple. If $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has strictly pseudo semi-stable reduction, then $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has (strictly) semi-stable reduction, in particular, it is a regular scheme with reduced special fiber.* □

12. Proof of the other implication of Theorem 5.6

In this section, we go through the list of Theorem 5.6, and produce in each case an LM triple $(G,\{\mu\},K)$ in the given central isogeny type which has semi-stable reduction. By Lemma 5.2, we may indeed assume that G is a central extension of the adjoint group appearing in the list of Theorem 5.6. So, for instance, in this section, we work with GL instead of SL, GSp instead of Sp, and, in some instances, with GO instead of SO.

We precede this by the following remarks. The first remark is that the locus where $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has semi-stable reduction is open and $\mathcal{G}$-invariant. Therefore, in order to show that $\mathbb{M}^{\mathrm{loc}}_K(G,\{\mu\})$ has semi-stable reduction, it suffices to check this in a closed point of the unique closed $\mathcal{G}\otimes_{O_F}k$-orbit of the special fiber.

The second remark is that we may always make an unramified field extension F'/F. This implies that, in checking semi-stable reduction, we may assume that in the LM triple $(G,\{\mu\},K)$, the group G is residually split.

The third remark is that in most of the cases which we treat, the LM triples are of "EL or PEL type." Then the corresponding local models of [44] have a more standard/classical description, as closed subschemes of linked classical (i.e., not affine) Grassmannians. This description, which was in fact given in earlier works, is established in [44, 7.2, 8.2]; we use this in our analysis, sometimes without further mention. There are two cases which are different: These are the LM triples for (special) orthogonal groups and the coweight $\omega_1^\vee$ (i.e., $r=1$). The corresponding local models have as generic fiber a quadric hypersurface. These are just of "Hodge type" and, for them, we have to work harder to first establish a standard description. Most of this is done in Subsection 12.7 with the key statement being Proposition 12.7.

12.1. **Preliminaries on GL_n.** In this subsection, we consider the LM triple

$$(G=\mathrm{GL}_n,\{\mu\}=\mu_r:=(1^{(r)},0^{(n-r)}),K_I)$$

for some $r\geq 1$, where K_I is the stabilizer of a lattice chain Λ_I for some non-empty subset $I\subset\{0,1,\dots,n-1\}$. We use the notation (GL_n,μ_r,I). We follow Görtz [16, §4.1] for the description of the local model in this case (the *standard local model*) and of an open subset U around the worst point, cf. [16, Prop.4.5].

The local model $\mathbb{M}^{\mathrm{loc}}_I(\mathrm{GL}_n,\mu_r)$ represents the following functor on O_F-schemes. Write $I=\{i_0<i_1<\cdots<i_{m-1}\}$. Then $\mathbb{M}^{\mathrm{loc}}_I(\mathrm{GL}_n,\mu_r)(S)$ is the set of commutative diagrams

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \cdots & \longrightarrow & \Lambda_{i_{m-1},S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{m-1} & \longrightarrow & \mathcal{F}_0
\end{array} \tag{12.1}$$

where Λ_i is the lattice generated by $e_1^i := \pi^{-1}e_1, \dots, e_i^i := \pi^{-1}e_i, e_{i+1}^i := e_{i+1}, \dots, e_n^i := e_n$, $\Lambda_{i,S}$ is $\Lambda_i \otimes_{O_F} \mathcal{O}_S$, π is a fixed uniformizer of F, and where the $\mathcal{F}_\kappa$ are locally free $\mathcal{O}_S$-submodules of rank r which Zariski-locally on S are direct summands of $\Lambda_{i_\kappa,S}$.

12.2. **The case** $(\mathrm{GL}_n, r = 1, I)$**,** I **arbitrary.** That in this case we have semi-stable reduction is well-known and follows from[6] [16, Prop. 4.13].

12.3. **Preliminaries on** $(\mathrm{GL}_n, r, \{0, \kappa\})$**,** r **arbitrary.** Note that to verify the remaining case for GL_n, we only need the case $\kappa = 1$, cf. Subsection 12.4. However, as we will see later, in order to verify the cases for other classical groups, we need to describe the incidence relation between 0 and κ for some other κ. So we discuss arbitrary κ here.

In terms of the bases $\{e_1^i, \dots, e_n^i\}$ of $\Lambda_{i,S}$, the transition maps $\Lambda_{0,S} \to \Lambda_{\kappa,S}$, resp., $\pi\colon \Lambda_{\kappa,S} \to \Lambda_{0,S}$ are given by the diagonal matrices

$$\phi_{0,\kappa} = \mathrm{diag}(\pi^{(\kappa)}, 1^{(n-\kappa)}), \text{ resp., } \phi_{\kappa,0} = \mathrm{diag}(1^{(\kappa)}, \kappa^{(n-\kappa)}). \tag{12.2}$$

For the open subset U around the worst point we take the pair of subspaces $\mathcal{F}_0$ of $\Lambda_{0,S}$, resp., $\mathcal{F}_\kappa$ of $\Lambda_{\kappa,S}$, given by the $n \times r$-matrices

$$\mathcal{F}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{11}^0 & a_{12}^0 & \cdots & a_{1r}^0 \\ \vdots & \vdots & & \vdots \\ a_{n-r,1}^0 & a_{n-r,2}^0 & \cdots & a_{n-r,r}^0 \end{pmatrix},$$

$$\mathcal{F}_\kappa = \begin{pmatrix} a_{n-r-\kappa+1,1}^\kappa & a_{n-r-\kappa+1,2}^\kappa & \cdots & a_{n-r-\kappa+1,r}^\kappa \\ \vdots & \vdots & & \vdots \\ a_{n-r,1}^\kappa & a_{n-r,2}^\kappa & \cdots & a_{n-r,r}^\kappa \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{11}^\kappa & a_{12}^\kappa & \cdots & a_{1r}^\kappa \\ \vdots & \vdots & & \vdots \\ a_{n-r-\kappa,1}^\kappa & a_{n-r-\kappa,2}^\kappa & \cdots & a_{n-r-\kappa,r}^\kappa \end{pmatrix}.$$

Then the incidence relation from 0 to κ is given by

$$\phi_{0,\kappa} \cdot \mathcal{F}_0 = \mathcal{F}_\kappa \cdot N_0,$$

and the incidence relation from κ to 0 is given by

$$\phi_{\kappa,0} \cdot \mathcal{F}_\kappa = \mathcal{F}_0 \cdot N_\kappa,$$

where $N_0, N_\kappa \in \mathrm{GL}_r(\mathcal{O}_S)$ are uniquely defined matrices. These equations can now be evaluated and lead to the following description of U:

[6] More precisely, in loc. cit., the case $I = \{0, \dots, n-1\}$ is considered, but the case of an arbitrary subset I is the same.

Proposition 12.1 ([16, §4.4.5]). *Let $\kappa \leq r$. Let*

$$A = (a^0_{i,j})_{i,j=1,\dots,\kappa}, B = (a^\kappa_{i,j})_{i=1,\dots,\kappa, j=r-\kappa+1,\dots,r}$$

be $\kappa \times \kappa$-matrices of indeterminates. Then

$$U \cong \operatorname{Spec} O_F[A,B]/(BA-\pi, AB-\pi) \times V,$$

where

$$V = \operatorname{Spec} O_F[a^0_{i,j}]_{i=1,\dots,r, j=\kappa+1,\dots,n-r} \times \operatorname{Spec} O_F[a^\kappa_{i,j}]_{i=1,\dots,r-\kappa, j=1,\dots,\kappa}$$

is an affine space $\mathbb{A}^{(n-r)r-\kappa^2}$ over O_F. □

Something analogous holds in the case when $\kappa > r$, cf. loc. cit..

12.4. **The case** $(\mathrm{GL}_n, r = 1, I = \{i, i+1\})$**,** r **arbitrary.** After changing the basis, we may assume that $i = 0$. Then the above proposition implies that U is a product of $\operatorname{Spec} O_F[X,Y]/(XY-\pi)$ and an affine space $\mathbb{A}^{(n-r)r-1}$. Hence U is regular and the special fiber is the union of two smooth divisors crossing normally along a smooth subscheme of codimension 2. Hence U has semi-stable reduction.

Remark 12.2. In contrast to the case of a general subset I, in this case the incidence condition from $\mathcal{F}_0$ to $\mathcal{F}_1$ automatically implies the incidence relation from $\mathcal{F}_1$ to $\mathcal{F}_0$.

12.5. **The case** $(\mathrm{GSp}_{2n}, r = n, \{0,1\})$**.** In the case of GSp_{2n} there is only one non-trivial minuscule coweight $\{\mu\} = \mu_n$. Let $e_1, \dots, e_{2n}$ be a symplectic basis of F^{2n}, i.e., $\langle e_i, e_{2n-j+1}\rangle = \pm\delta_{ij}$ for $i,j \leq 2n$ (with sign $+$ if $i = j \leq n$ and sign $-$ if $n+1 \leq i = j$). Then the standard lattice chain is self-dual, i.e., Λ_i and Λ_{2n-1} are paired by a perfect pairing. In this case, a parahoric subgroup K is the stabilizer of a *selfdual* periodic lattice chain Λ_I , i.e., I satisfies $i \in I \iff 2n-i \in I$. In this case, the local model is contained in the closed subscheme $\mathbb{M}^{\mathrm{naive}}_I(\mathrm{GSp}_{2n}, \mu_n)$ of the local model $\mathbb{M}^{\mathrm{loc}}_I(\mathrm{GL}_{2n}, \mu_n)$ given by the condition that

$$\mathcal{F}_i = \mathcal{F}^\perp_{2n-i}, \quad i \in I. \tag{12.3}$$

In fact, by [17], it is equal to this closed subscheme but we will not use this fact.

Now let $I_0 = \{0, 1, 2n-1\}$. Then, since $\mathcal{F}_{2n-1}$ is determined by $\mathcal{F}_1$ via the identity (12.3), we obtain a closed embedding $\mathbb{M}^{\mathrm{naive}}_{I_0}(\mathrm{GSp}_{2n}, \mu_n) \subset \mathbb{M}^{\mathrm{loc}}_{\{0,1\}}(\mathrm{GL}_{2n}, \mu_n)$.

As open subset U of the worst point we take the scheme of $(\mathcal{F}_0, \mathcal{F}_1)$, where

$$\mathcal{F}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

$$\mathcal{F}_1 = \begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix}.$$

The condition that $\mathcal{F}_0$ be a totally isotropic subspace of $\Lambda_{0,S}$ is expressed by

$$c_{\mu\nu} = c_{n-\nu+1,n-\mu+1}. \tag{12.4}$$

The incidence relation from $\mathcal{F}_0$ to $\mathcal{F}_1$ is given by the following system of equations,

$$\begin{pmatrix} \pi & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \\ c_{11} & c_{12} & \dots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{n1} & a_{n2} & \dots & a_{nn} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ c_{11} & c_{12} & c_{13} & \dots & c_{1n} \end{pmatrix}$$

The first row of this matrix identity gives

$$a_{nn} c_{11} = \pi, \tag{12.5}$$

and allows one to eliminate $a_{n1}, \dots, a_{n,n-1}$. The last $n-1$ entries of the $n+2$-th row allow one to eliminate $a_{11}, \dots, a_{1,n-1}$, the last $n-1$ entries of the $n+3$-th row allow one to eliminate $a_{21}, \dots, a_{2,n-1}$, etc., until the last $n-1$ entries of the $2n$-th row eliminate $a_{n-1,1}, \dots, a_{n-1,n-1}$. Finally, the first column of these rows allows one to eliminate $c_{21}, , \dots, c_{n1}$. All in all, we keep the entries $a_{1n}, \dots, a_{nn}$, c_{11}, and $c_{\mu\nu}$ for $\mu \geq \nu > 1$, which are subject to equation (12.5).

Let $\mathrm{Grass}^{\mathrm{lagr}}(\Lambda_0) \times \mathrm{Grass}(\Lambda_1)$ be the product of the Grassmannian variety of Lagrangian subspaces of Λ_0 and of the Grassmannian variety of subspaces of dimension n of Λ_1. Let $\mathbb{M}$ denote its closed subscheme of elements $(\mathcal{F}_0, \mathcal{F}_1)$ such that $\mathcal{F}_0$ is incident to $\mathcal{F}_1$. Note that $\mathbb{M}$ and $\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GSp}_{2n}, \mu_n)$ have identical generic fibers. We have a chain of closed embeddings

$$\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GSp}_{2n}, \mu_n) \subset \mathbb{M}^{\mathrm{naive}}_{I_0}(\mathrm{GSp}_{2n}, \mu_n) \subset \mathbb{M}. \tag{12.6}$$

But we just proved that $\mathbb{M}$ has semi-stable reduction, and is therefore flat over O_F. Hence all inclusions are equalities, since we can identify all three schemes with the flat closure of the generic fiber of $\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GSp}_{2n},\mu_n)$ in $\mathbb{M}^{\mathrm{loc}}_{\{0,1\}}(\mathrm{GL}_{2n},\mu_n)$. In particular, $\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GSp}_{2n},\mu_n)$ has semi-stable reduction (in fact, with special fiber the union of two smooth divisors meeting transversally in a smooth subscheme of codimension 2).

Remark 12.3. Again, as in the case of GL_n, in this case the incidences from $\mathcal{F}_1$ to $\mathcal{F}_{2n-1}$ and from $\mathcal{F}_{2n-1}$ to $\mathcal{F}_0$ are automatic.

12.6. **The case** $(\mathbf{split}\,\mathrm{GO}_{2n}, r = n, \{1\})$**.** In this subsection, we assume $p \neq 2$. Let $e_1,\dots,e_{2n}$ be a Witt basis of F^{2n}, i.e., $\langle e_i, e_{2n-j+1}\rangle = \delta_{ij}$ for $i,j \leq 2n$. Then the standard lattice chain is self-dual, i.e., Λ_i and Λ_{2n-i} are paired by a perfect pairing. In this case K is the parahoric stabilizer of a *selfdual* periodic lattice chain Λ_I , i.e., I has the property $i \in I \iff 2n-i \in I$. In this case, by [44, 8.2.3], the local model is contained in the closed subscheme $\mathbb{M}^{\mathrm{naive}}_I(\mathrm{GO}_{2n},\mu_n)$ of the local model $\mathbb{M}^{\mathrm{loc}}_I(\mathrm{GL}_{2n},\mu_n)$ given by the condition that

$$\mathcal{F}_i = \mathcal{F}_{2n-i}^{\perp}, \quad i \in I. \tag{12.7}$$

Now let $I_0 = \{1, 2n-1\}$. Then, since $\mathcal{F}_{2n-1}$ is determined by $\mathcal{F}_1$ via the identity (12.24), we obtain a closed embedding $\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GO}_{2n},\mu_n) \subset \mathbb{M}^{\mathrm{loc}}_{\{1\}}(\mathrm{GL}_{2n},\mu_n)$.

As open subset U of the worst point we take the scheme of $(\mathcal{F}_1, \mathcal{F}_{2n-1})$, where

$$\mathcal{F}_1 = \begin{pmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,n} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix}.$$

$$\mathcal{F}_{2n-1} = \begin{pmatrix} 0 & 1 & 0\dots & 0 \\ 0 & 0 & 1\dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0\dots & 1 \\ b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

The condition that $\mathcal{F}_1$ and $\mathcal{F}_{2n-1}$ be orthogonal is expressed by

$$b_{\mu\nu} = -a_{n-\nu+1,n-\mu+1}. \tag{12.8}$$

Recall the spin condition on $\mathcal{F}_1$, cf. [42, 7.1, 8.3]. This is a set of conditions stipulating the vanishing of certain linear forms on $\wedge^n\Lambda_1$ on the line $\wedge^n\mathcal{F}_1$ in $\wedge^n\Lambda_{1,S}$. These linear forms are enumerated by certain subsets $E \subset \{1,\dots,2n\}$ of order n. For a subset $E \subset \{1,\dots,2n\}$ of order n, set $E^{\perp} = (2n+1-E)^c$. Also, to such a subset E is associated a permutation σ_E of S_{2n}, cf. [42, 7.1.3]. We call the *weak spin condition* the vanishing of the linear forms

corresponding to subsets E with the property

$$E = E^{\perp}, \quad |E \cap \{2, 3, \dots, n+1\}| = n-1, \quad \operatorname{sgn} \sigma_E = 1. \tag{12.9}$$

It is easy to see that there are precisely the following subsets satisfying this condition: $\{1, \dots, n\}$ and $\{2, \dots, n-1, n+1, 2n\}$.

Lemma 12.4. *The weak spin condition on $\mathcal{F}_1$ implies $a_{n-1,n-1} = a_{nn} = 0$.*

Proof. Indeed, the linear forms for $E = \{1, \dots, n\}$, resp., $E = \{2, \dots, n-1, n+1, 2n\}$, are the minors of size n consisting of the rows $\{1, \dots, n\}$, resp., $\{2, \dots, n-1, n+1, 2n\}$. □

The incidence relation between $\mathcal{F}_{2n-1}$ and $\mathcal{F}_1$ is given by the following system of equations,

$$\begin{pmatrix} 0 & \pi & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ b_{11} & b_{12} & \dots & & b_{1n} \\ \vdots & & & & \vdots \\ b_{n1} & b_{n2} & \dots & & b_{nn} \\ \pi & 0 & \dots & & 0 \end{pmatrix} = \begin{pmatrix} a_{n1} & a_{n2} & \dots & a_{nn} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \end{pmatrix}$$

Using (12.8) and Lemma 12.4, we may also write the equations for the closed sublocus U^{wspin} where, in addition to the incidence relation from $\mathcal{F}_0$ to $\mathcal{F}_1$ and the isotropy condition on $\mathcal{F}_0$, the weak spin condition is satisfied, as an identity of $n \times n$-matrices,

$$\begin{pmatrix} 0 & \pi & \dots & 0 \\ -a_{n,n-2} & -a_{n-1,n-2} & \dots & -a_{1,n-2} \\ -a_{n,n-3} & -a_{n-1,n-3} & \dots & -a_{1,n-3} \\ \vdots & & & \vdots \\ -a_{n,1} & -a_{n-1,1} & \dots & -a_{11} \\ \pi & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} a_{n1} & \dots & a_{n,n-1} & 0 \\ a_{11} & \dots & a_{1,n-1} & a_{1,n} \\ a_{21} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & & & \vdots \\ a_{n-2,1} & \dots & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,1} & \dots & 0 & a_{n-1,n} \end{pmatrix}$$
$$\cdot \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & -a_{n-1,n} & -a_{n-2,n} & \dots & -a_{1,n} \\ -a_{n,n-1} & 0 & -a_{n-2,n-1} & \dots & -a_{1,n-1} \end{pmatrix}$$

It implies (look at the $(1,2)$ entry, which also equals the $(n,1)$ entry)

$$a_{n-1,n} \cdot a_{n,n-1} = -\pi. \tag{12.10}$$

Let us call E_{ij} the polynomial identity among the $a_{\mu\nu}$ that is given by the entry i, j of the above matrix identity. Then $E_{1,j}$ for $3 \le j \le n$ is of the form

$$a_{n,j-2} = P_{1,j}(a_{\bullet,n-1}, a_{\bullet,n}).$$

The identities $E_{n,j}$ for $3 \le j \le n$ are of the form

$$a_{n-1,j} = P_{n,j}(a_{\bullet,n-1}, a_{\bullet,n}).$$

The identities $E_{i,1}$ for $2 \leq i \leq n-1$ are of the form

$$a_{n,n-i} = P_{i,1}(a_{\bullet,n-1}, a_{\bullet,n}).$$

The identities $E_{i,2}$ for $2 \leq i \leq n-1$ are of the form

$$a_{n-1,n-i} = P_{i,2}(a_{\bullet,n-1}, a_{\bullet,n}).$$

The identities $E_{i,j}$ for $2 \leq i \leq n-1$ and $3 \leq j \leq n$ are of the form

$$a_{i-1,j-2} + a_{n+1-j,n-i} = P_{i,j}(a_{\bullet,n-1}, a_{\bullet,n}).$$

We also note the following identities

$$\begin{aligned} E_{i,j} =& E_{n+2-j,n+2-i}, \quad \text{for } i \in [2, n-1], j \in [3, n]; \\ E_{1,j} =& E_{n+2-j,1}, \qquad\qquad\quad \text{for } j \in [3, n-2]; \\ E_{n,j} =& E_{n-j,2}, \qquad\qquad\quad \text{for } j \in [3, n-2]. \end{aligned}$$

We keep the $2(n-2)$ variables $a_{\bullet,n-1}$ and $a_{\bullet,n}$, but eliminate $a_{n,j}$ and $a_{n-1,j}$, for $j \in [1, n-2]$. Then the remaining variables $a_{i,j}$ with $i, j \in [1, n-2]$ satisfy the identities

$$a_{i,j} + a_{n-1-j,n-1-i} = Q_{i,j}(a_{\bullet,n-1}, a_{\bullet,n}).$$

It follows that

$$U^{\mathrm{wspin}} \simeq \operatorname{Spec} O_F[X,Y]/(XY - \pi) \times \mathbb{A}^{\frac{n(n-1)}{2}-1}. \tag{12.11}$$

We obtain the semi-stability of $\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GO}_{2n}, \mu_n)$ as in the case of the symplectic group via the chain of closed embeddings

$$\mathbb{M}^{\mathrm{loc}}_{I_0}(\mathrm{GO}_{2n}, \mu_n) \subset \mathbb{M}^{\mathrm{wspin}}_{I_0}(\mathrm{GO}_{2n}, \mu_n) \subset \mathbb{M}^{\mathrm{loc}}_{\{1\}}(\mathrm{GL}_{2n}, \mu_n).$$

Remark 12.5. Again, as in the case of GL_n, in this case we can see, using flatness, that the further incidence relation from $\mathcal{F}_1$ to $\mathcal{F}_{2n-1}$ and from $\mathcal{F}_{2n-1}$ to $\mathcal{F}_1$, as well as the full spin condition are automatically satisfied.

12.7. **Quadric local models.** Let V be an F-vector space of dimension $d = 2n$ or $2n+1$, with a non-degenerate symmetric F-bilinear form $\langle\ ,\ \rangle$. Assume that $d \geq 5$ and that $p \neq 2$. We will consider a minuscule coweight μ of $\mathrm{SO}(V)(F)$ (i.e., defined over F) that corresponds to cases with $r = 1$.

Recall the notion of a *vertex lattice* in V: this is an O_F-lattice Λ in V such that $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$. We will say that an orthogonal vertex lattice Λ in V is *self-dual* if $\Lambda = \Lambda^\vee$, and *almost self-dual* if the length $\lg(\Lambda^\vee/\Lambda) = 1$. We list this as two cases:

I) Λ is self-dual, i.e., $\Lambda^\vee = \Lambda$.

II) We have $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$, and $\lg(\Lambda^\vee/\Lambda) = 1$.

Now let us consider the following cases:

a) $d = 2n+1$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, and $\Lambda = \oplus_{i=1}^d O_F \cdot e_i$, so that $\Lambda^\vee = \Lambda$.

b) $d = 2n$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, and $\Lambda = \oplus_{i=1}^d O_F \cdot e_i$, so that we have $\Lambda^\vee = \Lambda$.

c) $d = 2n+1$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \pi\delta_{ij}$, and $\Lambda = (\oplus_{i=1}^n O_F \cdot \pi^{-1} e_i) \oplus (\oplus_{i=n+1}^d O_F \cdot e_i)$, so that $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$.

d) $d = 2n$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, if $i, j \neq n, n+1$, and $\langle e_n, e_n\rangle = \pi$, $\langle e_{n+1}, e_{n+1}\rangle = 1$, $\langle e_n, e_{n+1}\rangle = 0$, and $\Lambda = \oplus_{i=1}^d O_F \cdot e_i$, so that $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$.

In all these cases, we take $\mu : \mathbb{G}_m \to \mathrm{SO}(V)$ to be given by $\mu(t) = \mathrm{diag}(t^{-1}, 1, \dots, 1, t)$ (under the embedding into the group of matrices by giving the action on the basis).

It follows from the classification of quadratic forms over local fields [15] that for each $(V, \langle\ ,\ \rangle, \Lambda)$ with $\lg(\Lambda^\vee/\pi\Lambda) \le 1$, and μ as in the beginning of this subsection, there is an unramified finite field extension F'/F such that the base change of $(V, \langle\ ,\ \rangle, \Lambda)$ to F' affords a basis as in one of the cases (a)-(d), and μ is given as above. In fact, we can also consider similarly cases of $(V, \langle\ ,\ \rangle, \Lambda)$ with $\pi\Lambda \subset \Lambda^\vee \subset \Lambda$, with $\lg(\Lambda^\vee/\pi\Lambda) \le 1$, by changing the form $\langle\ ,\ \rangle$ to $\pi\langle\ ,\ \rangle$; these two forms have the same orthogonal group.

In what follows, for simplicity we set $O_F = O$.

12.7.1. *Quadrics.* We will now consider the quadric $Q(\Lambda)$ over $\operatorname{Spec} O$ which, by definition, is the projective hypersurface in $\mathbb{P}^{d-1}_O$ whose R-valued points parametrize isotropic lines, i.e., R-locally free rank 1 direct summands

$$\mathcal{F} \subset \Lambda_R := \Lambda \otimes_O R,$$

with $\langle \mathcal{F}, \mathcal{F} \rangle_R = 0$. Here $\langle\ ,\ \rangle_R$ is the symmetric R-bilinear form $\Lambda_R \times \Lambda_R \to R$ obtained from $\langle\ ,\ \rangle$ restricted to $\Lambda \times \Lambda$ by base change.

Proposition 12.6. *Set* $\mathbb{P}^{d-1}_{\breve{O}} = \mathrm{Proj}(\breve{O}[x_1, \dots, x_d])$.

In case (I), $Q(\Lambda) \otimes_O \breve{O}$ *is isomorphic to the closed subscheme of* $\mathbb{P}^{d-1}_{\breve{O}}$ *given by*

$$\sum\nolimits_{i=1}^{d} x_i x_{d+1-i} = 0$$

and the scheme $Q(\Lambda)$ *is smooth over* O.

In case (II), $Q(\Lambda) \otimes_O \breve{O}$ *is isomorphic to the closed subscheme of* $\mathbb{P}^{d-1}_{\breve{O}}$ *given by*

$$\sum\nolimits_{i=1}^{d-1} x_i x_{d-i} + \pi x_d^2 = 0.$$

Then $Q(\Lambda)$ *is regular with normal special fiber which is singular only at the point* $(0; \dots; 1)$*; this point corresponds to* $\mathcal{F}_0 = \pi\Lambda^\vee/\pi\Lambda \subset \Lambda/\pi\Lambda = \Lambda \otimes_O k$.

Proof. Follows from the classification of quadratic forms over $\breve{F}$, by expressing $Q(\Lambda)$ in cases (a)-(d). (To get the equations in the statement we have to rearrange the basis vectors.) □

12.7.2. *Quadrics and PZ local models.* We can now extend our data to $O[u, u^{-1}]$. We set $\mathbb{V} = \oplus_{i=1}^d O[u, u^{-1}] \cdot \underline{e}_i$ with $\langle\ ,\ \rangle : \mathbb{V} \times \mathbb{V} \to O[u, u^{-1}]$ a symmetric $O[u, u^{-1}]$-bilinear form for which $\langle \underline{e}_j, \underline{e}_j \rangle$ is given as $\langle e_i, e_j \rangle$ above, but with π replaced by u. We define $\underline{\mu} : \mathbb{G}_m \to \mathrm{SO}(\mathbb{V})$ by $\underline{\mu}(t) = \mathrm{diag}(t^{-1}, 1, \dots, 1, t)$ by using the basis $\underline{e}_i$.

Similarly, we define $\mathbb{L}$ to be the free $O[u]$-submodule of $\mathbb{V}$ spanned by $\underline{e}_i$ (or $u^{-1}\underline{e}_i$ and $\underline{e}_j$) as above, following the pattern of the definition of Λ in each case. Then, the base change of $(\mathbb{V}, \langle\ ,\ \rangle, \mathbb{L})$ from $O[u, u^{-1}]$ to F given by $u \mapsto \pi$ are $(V, \langle\ ,\ \rangle, \Lambda)$.

We can now define the local model $\mathbb{M}^{\mathrm{loc}} = \mathbb{M}^{\mathrm{loc}}(\Lambda) = \mathbb{M}^{\mathrm{loc}}_K(\mathrm{SO}(V), \{\mu\})$ for the LM triple $(\mathrm{SO}(V), \{\mu\}, K)$ where K is the parahoric stabilizer of Λ, as in [44]. Consider the smooth affine group scheme $\underline{\mathcal{G}}$ over $O[u]$ given by $g \in \mathrm{SO}(\mathbb{V})$ that also preserve $\mathbb{L}$ and $\mathbb{L}^\vee$. Base changing by $u \mapsto \pi$ gives the Bruhat-Tits group scheme $\mathcal{G}$ of $\mathrm{SO}(V)$ which is the stabilizer of the lattice chain $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$. This is a hyperspecial subgroup when $\Lambda^\vee = \Lambda$. If $\lg(\Lambda^\vee/\Lambda) = 1$, we can see that $\mathcal{G}$ has special fiber with $\mathbb{Z}/2\mathbb{Z}$ as its group of connected components. The corresponding parahoric group scheme is its connected component $\mathcal{G}^0$.

The construction of [44] produces the group scheme $\underline{\mathcal{G}}^0$ that extends $\mathcal{G}^0$. By construction, there is a group scheme immersion $\underline{\mathcal{G}}^0 \hookrightarrow \underline{\mathcal{G}}$.

As in [44], one can see that the Beilinson-Drinfeld style ("global") affine Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}},O[u]}$ over $O[u]$ represents the functor that sends the $O[u]$-algebra R given by $u \mapsto r$ to the set of projective finitely generated $R[u]$-submodules $\mathcal{L}$ of $\mathbb{V} \otimes_O R$ which are R-locally free such that $(u-r)^N \mathbb{L} \subset \mathcal{L} \subset (u-r)^{-N}\mathbb{L}$ for some $N >> 0$ and satisfy $\mathcal{L}^\vee = \mathcal{L}$ in case (I), resp.,

$$u\mathcal{L}^\vee \overset{d-1}{\subset} \mathcal{L} \overset{1}{\subset} \mathcal{L}^\vee \overset{d-1}{\subset} u^{-1}\mathcal{L}$$

in case (II), with all graded quotients R-locally free and of the indicated rank.

By definition, the PZ local model $\mathbb{M}^{\mathrm{loc}}$ is a closed subscheme of the base change $\mathrm{Gr}_{\mathcal{G}^0,O} = \mathrm{Gr}_{\underline{\mathcal{G}}^0,O[u]} \otimes_{O[u]} O$ by $O[u] \to O$ given by $u \mapsto \pi$. Consider the O-valued point $[\mathcal{L}(0)]$ given by

$$\mathcal{L}(0) = \underline{\mu}(u-\pi)\mathbb{L}.$$

By definition, the PZ local model $\mathbb{M}^{\mathrm{loc}}$ is the reduced Zariski closure of the orbit of $[\mathcal{L}(0)]$ in $\mathrm{Gr}_{\mathcal{G}^0,O}$; it inherits an action of the group scheme $\mathcal{G}^0 = \underline{\mathcal{G}}^0 \otimes_{O[u]} O$. As in [44, 8.2.3], we can see that the natural morphism $\mathrm{Gr}_{\underline{\mathcal{G}}^0,O[u]} \to \mathrm{Gr}_{\underline{\mathcal{G}},O[u]}$ induced by $\underline{\mathcal{G}}^0 \hookrightarrow \underline{\mathcal{G}}$ identifies $\mathbb{M}^{\mathrm{loc}}$ with a closed subscheme of $\mathrm{Gr}_{\underline{\mathcal{G}},O} := \mathrm{Gr}_{\underline{\mathcal{G}},O[u]} \otimes_{O[u]} O$.

Proposition 12.7. *In each of the above cases (a)-(d) with* $\lg(\Lambda^\vee/\Lambda) \leq 1$*, there is a* $\mathcal{G}$*-equivariant isomorphism*

$$\mathbb{M}^{\mathrm{loc}}(\Lambda) \xrightarrow{\sim} Q(\Lambda)$$

between the PZ local model as defined above and the quadric.

Proof. Note that since, by definition, $\mathcal{G}$ maps to $\mathrm{GL}(\Lambda)$ and preserves the form $\langle\ ,\ \rangle$, it acts on the quadric $Q(\Lambda)$. By the definition of $\mathcal{L}(0)$, we have

$$(u-\pi)\mathbb{L} \subset \mathcal{L}(0) \cap \mathbb{L} \begin{array}{ccc} & \mathbb{L} & \\ \curvearrowright & & \curvearrowleft \\ \curvearrowleft & & \curvearrowright \\ & \mathcal{L}(0) & \end{array} \mathbb{L} + \mathcal{L}(0) \subset (u-\pi)^{-1}\mathbb{L},$$

where the quotients along all slanted inclusions are O-free of rank 1. Consider the subfunctor M of $\mathrm{Gr}_{\mathcal{G},O_F}$ parametrizing $\mathcal{L}$ such that

$$(u-\pi)\mathbb{L} \subset \mathcal{L} \subset (u-\pi)^{-1}\mathbb{L}.$$

Then M is given by a closed subscheme of $\mathrm{Gr}_{\mathcal{G},O}$ which contains the orbit of $\mathcal{L}(0)$; therefore the local model $\mathbb{M}^{\mathrm{loc}}$ is a closed subscheme of M and $\mathbb{M}^{\mathrm{loc}}$ is the reduced Zariski closure of its generic fiber in M.

We now consider another projective scheme $P(\Lambda)$ which parametrizes pairs $(\mathcal{F}, \mathcal{F}')$ where $\mathcal{F} \subset \Lambda_R$, $\mathcal{F}' \subset \Lambda_R^\vee$ are both R-lines, such that $\mathcal{F}$ is isotropic for $\langle\ ,\ \rangle_R$ and $\mathcal{F}'$ is isotropic for $\pi\langle\ ,\ \rangle_R$, and such that $\mathcal{F}$, $\mathcal{F}'$ are linked by both natural R-maps $\Lambda_R \to \Lambda_R^\vee$ and $\pi : \Lambda_R^\vee \to \Lambda_R$.

Lemma 12.8. *In case (I), the forgetful morphism* $f : P(\Lambda) \xrightarrow{\sim} Q(\Lambda)$ *is an isomorphism. In case (II), denote by* $P(\Lambda)^{\mathrm{fl}}$ *the flat closure of* $P(\Lambda)$*. Then the forgetful morphism* $f : P(\Lambda)^{\mathrm{fl}} \to Q(\Lambda)$*, given by* $(\mathcal{F}, \mathcal{F}') \mapsto \mathcal{F}$*, can be identified with the blow-up of* $Q(\Lambda)$ *at the unique singular closed point of its special fiber. In particular, it is an isomorphism away from the closed point given by* $\mathcal{F} =$ *the radical of the form* $\langle\ ,\ \rangle$ *on* Λ_{κ_F}*. If* $\mathcal{F}$ *is the radical,*

then $\mathcal{F}'$ lies in the radical of the form $\pi\langle\ ,\ \rangle$ on $\Lambda^\vee \otimes_O \kappa_F$. Since this radical has dimension $d-1$, the exceptional locus is isomorphic to $\mathbb{P}^{d-2}_{\kappa_F}$.

Proof. In case (I), we have $\mathcal{F}' = \mathcal{F}$, so $P(\Lambda) \simeq Q(\Lambda)$. Assume we are in case (II). Using the universal property of the blow-up, we see it is enough to show the statement after base changing to $\breve{O}$. For convenience we rearrange the basis of V such that $\Lambda^\vee/\Lambda$ is generated by $\pi^{-1}e_d$. Set

$$\mathcal{F} = (\sum_{i=1}^{d} x_i e_i), \quad \mathcal{F}' = (\sum_{i=1}^{d-1} y_i e_i + y_d \pi^{-1} e_d).$$

Since $\mathcal{F}$ maps to $\mathcal{F}'$ by $\Lambda_R \to \Lambda_R^\vee$ and $\mathcal{F}'$ maps to $\mathcal{F}$ by $\pi : \Lambda_R^\vee \to \Lambda_R$, there are λ, $\mu \in R$ such that $\lambda\mu = \pi$ and

$$x_1 = \lambda y_1, \ldots, x_{d-1} = \lambda y_{d-1}, \quad \pi x_d = \lambda y_d$$

$$\pi y_1 = \mu x_1, \ldots, \pi y_{d-1} = \mu x_{d-1}, \quad y_d = \mu x_d.$$

The isotropy conditions are

$$\sum_{i=1}^{d-1} x_i x_{d-i} + \pi x_d^2 = 0, \quad \sum_{i=1}^{d-1} \pi y_i y_{d-i} + y_d^2 = 0.$$

Let us consider the inverse image of the affine chart with $x_d = 1$ under the forgetful morphism $f : P(\Lambda) \to Q(\Lambda)$. We obtain $y_d = \mu$ and the equations become

$$\lambda(\sum_{i=1}^{d-1} x_i y_{d-i} + \mu) = 0, \quad \mu(\sum_{i=1}^{d-1} x_i y_{d-i} + \mu) = 0.$$

The flat closure $P(\Lambda)^{\mathrm{fl}}$ is given by $\lambda\mu = \pi$ and $\sum_{i=1}^{d-1} x_i y_{d-i} + \mu = 0$. Since $x_i = \lambda y_i$, we get $\lambda \sum_{i=1}^{d-1} y_i y_{d-i} + \mu = 0$. Eliminating μ gives

$$-\lambda^2(\sum_{i=1}^{d-1} y_i y_{d-i}) = \pi.$$

An explicit calculation shows that this coincides with the blow-up of the affine chart $x_d = 1$ in the quadric $Q(\Lambda)$ at the point $(0 : \ldots : 1)$ of its special fiber. In fact, we see that $P(\Lambda)^{\mathrm{fl}}$ is regular and that the special fiber $P(\Lambda)^{\mathrm{fl}}_k$ has two irreducible components: The smooth blow up of the special fiber $Q(\Lambda) \otimes_O \kappa_F$ at the singular point and the exceptional locus $\mathbb{P}^{d-2}_{\kappa_F}$ for $\lambda = 0$; they intersect along a smooth quadric of dimension $d-3$ over κ_F. The exceptional locus has multiplicity 2 in the special fiber of $P(\Lambda)^{\mathrm{fl}}$. The rest of the statements follow easily. □

We now continue with the proof of Proposition 12.7. Assume that $(\mathcal{F}, \mathcal{F}')$ gives an R-valued point of $P(\Lambda)$. The pair $(\mathcal{F}, \mathcal{F}')$ uniquely determines lattices $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}'(\mathcal{F}')$ with

$$(u-\pi)\mathbb{L} \subset \mathcal{L}'(\mathcal{F}') \overset{1}{\subset} \mathbb{L} \overset{1}{\subset} \mathcal{L}(\mathcal{F}) \subset (u-\pi)^{-1}\mathbb{L},$$

by

$$\mathcal{L}(\mathcal{F}) = \text{ the inverse image of } \mathcal{F} \text{ under } u-\pi : (u-\pi)^{-1}\mathbb{L} \longrightarrow \mathbb{L}/(u-\pi)\mathbb{L} = \Lambda_R,$$

$$\mathcal{L}'(\mathcal{F}') = \text{ the inverse image of } \mathcal{F}'^{\perp} \subset \Lambda_R \text{ under } \mathbb{L} \longrightarrow \mathbb{L}/(u-\pi)\mathbb{L} = \Lambda_R.$$

The other conditions translate to $(u-\pi)\mathcal{L}(\mathcal{F}) \subset \mathcal{L}'(\mathcal{F}') \subset \mathcal{L}(\mathcal{F})$ and

$$\mathcal{L}'(\mathcal{F}') \subset \mathcal{L}(\mathcal{F})^\vee \subset u^{-1}\mathcal{L}'(\mathcal{F}'),$$

$$\mathcal{L}(\mathcal{F}) \subset \mathcal{L}'(\mathcal{F}')^\vee \subset u^{-1}\mathcal{L}(\mathcal{F}).$$

Note that we obtain a symmetric R-bilinear form by interpreting the value $\langle\ ,\ \rangle$ in $(u-\pi)^{-1}R[u]/R[u] \simeq R$,

$$h : \mathcal{L}(\mathcal{F})/\mathcal{L}'(\mathcal{F}') \times \mathcal{L}(\mathcal{F})/\mathcal{L}'(\mathcal{F}') \longrightarrow R$$

Consider the scheme $Z \to P(\Lambda)^{\mathrm{fl}} \subset P(\Lambda)$ classifying isotropic lines in the rank 2 symmetric space $\mathcal{L}(\mathcal{F}^{\mathrm{univ}})/\mathcal{L}'(\mathcal{F}'^{\mathrm{univ}})$ over $P(\Lambda)$. One of these isotropic lines is always $\mathbb{L}/\mathcal{L}'(\mathcal{F}')$. Suppose $R = k$. When $\mathcal{F}$ is the radical in Λ_k, then $\mathcal{L}(\mathcal{F}) = \mathbb{L}^\vee$. Then $\mathbb{L}/\mathcal{L}'(\mathcal{F}')$ is the radical of h and gives the unique isotropic line. If $\mathcal{F}$ is not the radical in Λ_k, then h is non-degenerate and there are two distinct isotropic lines, one of which is $\mathbb{L}/\mathcal{L}'(\mathcal{F}')$.

We first consider case (I), i.e., $\Lambda = \Lambda^\vee$. Then $\mathcal{F} = \mathcal{F}'$, and $P(\Lambda) = Q(\Lambda)$. The form h is perfect (everywhere non-degenerate) and Z is the disjoint union $Z = Z^0 \sqcup Z^1$, where Z^0 is the component where the isotropic line is $\mathbb{L}/\mathcal{L}'(\mathcal{F}')$. Each component Z^i projects isomorphically to $Q(\Lambda)$. We can give a morphism $g : Z^1 \simeq P(\Lambda) = Q(\Lambda) \to \mathbb{M}^{\mathrm{loc}}$ by sending $\mathcal{F}$ to $\mathcal{L}$ characterized by the condition that $\mathcal{L}/\mathcal{L}'(\mathcal{F}') \subset \mathcal{L}(\mathcal{F})/\mathcal{L}'(\mathcal{F}')$ is the tautological isotropic line over Z^1. The morphism g is the desired equivariant isomorphism $Q(\Lambda) \simeq \mathbb{M}^{\mathrm{loc}}(\Lambda)$.

We now consider case (II), i.e., $\lg(\Lambda^\vee/\Lambda) = 1$. Then the scheme $Z \to P(\Lambda)^{\mathrm{fl}}$ has two irreducible components Z^0, Z^1, where Z^0 is the irreducible component over which the isotropic line is $\mathbb{L}/\mathcal{L}(\mathcal{F}')$ and where Z^1 is the irreducible component over which the isotropic line generically is not $\mathbb{L}/\mathcal{L}(\mathcal{F}')$. By the above, the two components intersect over the exceptional locus of the blow-up $P(\Lambda)^{\mathrm{fl}} \to Q(\Lambda)$. Each irreducible component maps isomorphically to $P(\Lambda)^{\mathrm{fl}}$. (Note that $P(\Lambda)^{\mathrm{fl}}$ is normal and each morphism $Z^i \to P(\Lambda)^{\mathrm{fl}}$ is clearly birational and finite.) We can now produce a morphism $g : Z^1 \simeq P(\Lambda)^{\mathrm{fl}} \to \mathbb{M}^{\mathrm{loc}}$ by sending $(\mathcal{F}, \mathcal{F}')$ to $\mathcal{L}$ characterized by the condition that $\mathcal{L}/\mathcal{L}(\mathcal{F}') \subset \mathcal{L}(\mathcal{F})/\mathcal{L}(\mathcal{F}')$ is the tautological isotropic line over Z^1,

$$(u-\pi)\mathbb{L} \subset \mathcal{L}'(\mathcal{F}') \begin{array}{ccc} & \mathbb{L} & \\ \subset & & \supset \\ \supset & & \subset \\ & \mathcal{L} & \end{array} \mathcal{L}(\mathcal{F}) \subset (u-\pi)^{-1}\mathbb{L}.$$

When $\mathcal{L} \neq \mathbb{L}$, then $\mathcal{L}(\mathcal{F}) = \mathbb{L} + \mathcal{L}$ and $\mathcal{L}'(\mathcal{F}') = \mathbb{L} \cap \mathcal{L}$ and so $(\mathcal{F}, \mathcal{F}')$ is uniquely determined by its image $\mathcal{L}$ in $\mathbb{M}^{\mathrm{loc}}$. Hence, g is an isomorphism over the open subscheme of $\mathbb{M}^{\mathrm{loc}}$ where $\mathcal{L} \neq \mathbb{L}$. When $\mathcal{L} = \mathbb{L}$, $\mathbb{L}/\mathcal{L}(\mathcal{F}')$ is isotropic in $\mathcal{L}(\mathcal{F})/\mathcal{L}(\mathcal{F}')$ and, as above, $\mathcal{F}$ = the radical of the form on Λ_k. This shows that the inverse image of $g : P(\Lambda) \to \mathbb{M}^{\mathrm{loc}}$ over $[\mathbb{L}]$ agrees with exceptional locus of the blow-up $f : P(\Lambda)^{\mathrm{fl}} \to Q(\Lambda)$ over the point $\mathcal{F}$ given by the radical. Since $Q(\Lambda)$, $\mathbb{M}^{\mathrm{loc}}$ are both normal, we can conclude that the birational map $f \circ g^{-1} : \mathbb{M}^{\mathrm{loc}} \dashrightarrow Q(\Lambda)$ is an isomorphism; it is $\mathcal{G}$-equivariant since this is true on the generic fibers. This completes the proof of Proposition 12.7. □

12.7.3. *More orthogonal local models.* We will now consider orthogonal local models associated to the self-dual chains generated by two lattices Λ_0, Λ_n and their duals, where Λ_0, Λ_n are both self-dual or almost self-dual vertex lattices, Λ_0 for the form $\langle\ ,\ \rangle$ and Λ_n for

its multiple $\pi\langle\ ,\ \rangle$. In all cases, the self-dual lattice chain has the form

$$\cdots \subset \Lambda_0 \overset{r}{\subset} \Lambda_0^\vee \subset \Lambda_n \overset{s}{\subset} \pi^{-1}\Lambda_n^\vee \subset \pi^{-1}\Lambda_0 \subset \cdots$$

with each r, s either 0 or 1. Again, after an unramified extension of F, we can reduce to the following cases:

1) $d = 2n+1$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, $\Lambda_0 = \oplus_{i=1}^d O\cdot e_i$ so that $\Lambda_0^\vee = \Lambda_0$ and $\Lambda_n = (\oplus_{i=1}^n O\cdot \pi^{-1}e_i) \oplus (\oplus_{i=n+1}^d O\cdot e_i)$ so that $\Lambda_n \subsetneq \pi^{-1}\Lambda_n^\vee$.

2) $d = 2n$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, $\Lambda_0 = \oplus_{i=1}^d O\cdot e_i$ so that $\Lambda_0^\vee = \Lambda_0$, and $\Lambda_n = (\oplus_{i=1}^n O\cdot \pi^{-1}e_i) \oplus (\oplus_{i=n+1}^d O\cdot e_i)$ so that $\Lambda_n = \pi^{-1}\Lambda_n^\vee$.

3) $d = 2n$, there is a basis e_i with $\langle e_i, e_{d+1-j}\rangle = \delta_{ij}$, if $i,j \neq n, n+1$, and $\langle e_n, e_n\rangle = \pi$, $\langle e_{n+1}, e_{n+1}\rangle = 1$, $\langle e_n, e_{n+1}\rangle = 0$, and $\Lambda_0 = \oplus_{i=1}^d O\cdot e_i$ so that $\Lambda_0 \subsetneq \Lambda_0^\vee \subset \pi^{-1}\Lambda_0$ and $\Lambda_n = (\oplus_{i=1}^n O\cdot \pi^{-1}e_i) \oplus (\oplus_{i=n+1}^d O\cdot e_i)$ so that $\Lambda_n^\vee \subset \Lambda_n \subsetneq \pi^{-1}\Lambda_n^\vee$.

We extend $(V, \langle\ ,\ \rangle)$ and Λ_j to $(\mathbb{V}, \langle\ ,\ \rangle)$ and $\mathbb{L}_j$ over $O[u]$ as in 12.7.2. We consider the (smooth, affine) group scheme $\underline{\mathcal{G}} = \underline{\mathcal{G}}_{\mathbb{L}_\bullet}$ over $O[u]$ given by $g \in \mathrm{SO}(\mathbb{V})$ that also preserve the chain

$$\mathbb{L}_\bullet : \cdots \subset \mathbb{L}_0 \subset \mathbb{L}_0^\vee \subset \mathbb{L}_n \subset u^{-1}\mathbb{L}_n^\vee \subset u^{-1}\mathbb{L}_0 \subset \cdots$$

The base change of $\underline{\mathcal{G}}$ by $u \mapsto \pi$ is the the Bruhat-Tits group scheme $\mathcal{G}$ for $\mathrm{SO}(V)$ that preserves the chain

$$\Lambda_\bullet : \cdots \subset \Lambda_0 \subset \Lambda_0^\vee \subset \Lambda_n \subset \pi^{-1}\Lambda_n^\vee \subset \pi^{-1}\Lambda_0 \subset \cdots .$$

This is connected and hence parahoric in cases 1) and 2), since it is contained in the hyperspecial stabilizer of Λ_0. In case 3), the group of connected components of its special fiber is $\mathbb{Z}/2\mathbb{Z}$. The corresponding parahoric is the connected component of $\mathcal{G}$. We can now see that the diagonal embedding gives a closed immersion

$$\underline{\mathcal{G}}_{\mathbb{L}_\bullet} \hookrightarrow \underline{\mathcal{G}}_{\mathbb{L}_0} \times \underline{\mathcal{G}}_{\mathbb{L}_n}$$

of group schemes over $O[u]$. Similarly, we have a compatible closed immersion of the global affine Grassmannian for $\underline{\mathcal{G}}_{\mathbb{L}_\bullet}$ into the product of the ones for $\underline{\mathcal{G}}_{\mathbb{L}_0}$ and $\underline{\mathcal{G}}_{\mathbb{L}_n}$.

The global affine Grassmannian for $\underline{\mathcal{G}}_{\mathbb{L}_\bullet}$ represents the functor which sends the $O[u]$-algebra R given by $u \mapsto r$ to the set of pairs of projective finitely generated $R[u]$-submodules $(\mathcal{L}_0, \mathcal{L}_n)$ of $\mathbb{V}\otimes_O R$ which are R-locally free, such that $(u-r)^N\mathbb{L} \subset \mathcal{L}_i \subset (u-r)^{-N}\mathbb{L}$ for some $N >> 0$, and

$$\mathcal{L}_0 \overset{r}{\subset} \mathcal{L}_0^\vee \overset{n-r}{\subset} \mathcal{L}_n \overset{s}{\subset} u^{-1}\mathcal{L}_n^\vee \overset{n-s}{\subset} u^{-1}\mathcal{L}_0$$

with all graded quotients R-locally free and of the indicated rank. From this and the discussion before Proposition 12.7 it easily follows that there is an equivariant closed embedding of local models

$$\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet) \hookrightarrow \mathbb{M}^{\mathrm{loc}}(\Lambda_0) \times \mathbb{M}^{\mathrm{loc}}(\Lambda_n)$$

which restricts to the diagonal morphism on the generic fibers. Proposition 12.7 now gives equivariant isomorphisms $\mathbb{M}^{\mathrm{loc}}(\Lambda_0) \simeq Q(\Lambda_0, \langle\ ,\ \rangle)$ and $\mathbb{M}^{\mathrm{loc}}(\Lambda_n) \simeq Q(\Lambda_n, \pi\langle\ ,\ \rangle)$. In fact, we can now see that the construction of these isomorphisms in the proof of this proposition implies that the image of the resulting closed embedding

$$\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet) \hookrightarrow Q(\Lambda_0, \langle\ ,\ \rangle) \times Q(\Lambda_n, \pi\langle\ ,\ \rangle)$$

lies in the closed subscheme of the product of the two quadrics where the two lines $(\mathcal{F}_0, \mathcal{F}_n)$ are linked in the same manner as for the corresponding local model for GL_d, i.e., $\mathcal{F}_0 \subset \Lambda_{0,R}$

maps to $\mathcal{F}_n \subset \Lambda_{n,R}$ via $\Lambda_{0,R} \to \Lambda_{n,R}$ induced by $\Lambda_0 \subset \Lambda_n$ and $\mathcal{F}_n$ maps to $\mathcal{F}_0$ under $\Lambda_{n,R} \to \Lambda_{0,R}$ induced by $\pi : \Lambda_n \to \Lambda_0$. Indeed, the reason is that $\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet)$, as a closed subscheme of the Beilinson-Drinfeld Grassmannian, classifies $(\mathcal{L}_0, \mathcal{L}_n)$ which in particular satisfy $\mathcal{L}_0 \subset \mathcal{L}_n \subset u^{-1}\mathcal{L}_0$. Therefore, we have that $\mathbb{L}_0 + \mathcal{L}_0 \subset \mathbb{L}_n + \mathcal{L}_n \subset u^{-1}(\mathbb{L}_0 + \mathcal{L}_0)$. But, as the proof shows, on the open dense non-singular part of the quadrics, the sums $\mathbb{L}_0 + \mathcal{L}_0$ and $\mathbb{L}_n + \mathcal{L}_n$ determine the lines $\mathcal{F}_0$ and $\mathcal{F}_n$ and we easily see that the linkage inclusions as above are satisfied.

12.8. **The case** (**split** $\mathrm{SO}_{2n}, r = 1, \{0, n\}$)**.** This corresponds to case 2) in 12.7.3. We continue to assume $p \neq 2$. We have $V = \oplus_{i=1}^{2n} F \cdot e_i$ with symmetric F-bilinear form determined by $\langle e_i, e_{2n+1-j} \rangle = \delta_{ij}$. For $1 \leq j \leq n$, set

$$\Lambda_j = (\pi^{-1}e_1, \ldots, \pi^{-1}e_j, e_{j+1}, \ldots, e_{2n}) \simeq O^{2n} \subset V;$$

Then $\Lambda_0 = \Lambda_0^\vee$, $\pi\Lambda_n = \Lambda_n^\vee$. In this case, the local model is contained in the closed subscheme $\mathbb{M}^{\mathrm{naive}}_{\{0,n\}}(\mathrm{SO}_{2n}, \mu_1)$ of the local model $\mathbb{M}^{\mathrm{loc}}_{\{0,n\}}(\mathrm{GL}_{2n}, \mu_1)$ given by the condition

$$\mathcal{F}_0 \subset \mathcal{F}_0^\perp, \quad \mathcal{F}_n \subset \mathcal{F}_n^\perp. \tag{12.12}$$

As open subset U of the worst point we take the scheme of $(\mathcal{F}_0, \mathcal{F}_n)$, where

$$\mathcal{F}_0 = (e_1 + a_1 e_2 + \cdots + a_{2n-1}e_{2n}),$$

$$\mathcal{F}_n = (b_n\pi^{-1}e_1 + \cdots + b_{2n-1}\pi^{-1}e_n + e_{n+1} + b_1 e_{n+1} + \cdots + b_{n-1}e_{2n}).$$

The incidences from $\mathcal{F}_0$ to $\mathcal{F}_n$, resp., from $\mathcal{F}_n$ to $\mathcal{F}_0$ are given by the following matrix relations

$$\begin{pmatrix} \pi \\ \pi a_1 \\ \vdots \\ \pi a_{n-1} \\ a_n \\ \vdots \\ a_{2n-1} \end{pmatrix} = \begin{pmatrix} b_n \\ b_{n+1} \\ \vdots \\ b_{2n-1} \\ 1 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \cdot a_n, \quad \begin{pmatrix} b_n \\ \vdots \\ b_{2n-1} \\ \pi \\ \pi b_1 \\ \vdots \\ \pi b_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_n \\ \vdots \\ a_{2n-1} \end{pmatrix} \cdot b_n.$$

We deduce that

$$a_n b_n = \pi, \tag{12.13}$$

and the following identities for $i = 1, \ldots, n-1$,

$$\begin{aligned} a_n b_i &= a_{n+i}, & b_n a_i &= b_{n+i}, \\ a_n b_{n+i} &= \pi a_i, & b_n a_{n+i} &= \pi b_i. \end{aligned} \tag{12.14}$$

The isotropy conditions on $\mathcal{F}_0$, resp., $\mathcal{F}_n$, are given by the following equations,

$$\begin{aligned} a_{2n-1} + a_1 a_{2n-2} + \cdots + a_{n-1}a_n &= 0, \\ b_{2n-1} + b_1 b_{2n-2} + \cdots + b_{n-1}b_n &= 0. \end{aligned} \tag{12.15}$$

We use the first lines of (12.14) to eliminate $a_{n+1}, \ldots, a_{2n-1}$ and $b_{n+1}, \ldots, b_{2n-1}$. Then the second lines in (12.14) are automatically satisfied (use (12.13)),

$$a_n b_{n+i} - \pi a_i = a_n b_n a_i - \pi a_i = a_i(a_n b_n - \pi) = 0;$$

$$b_n a_{n+i} - \pi b_i = b_n a_n b_i - \pi b_i = b_i(a_n b_n - \pi) = 0.$$

Expressing $a_{n+1}, \dots, a_{2n-1}$ in terms of $b_1, \dots, b_{n-1}$ in the first equation of (12.15), we obtain the equation

$$a_n(b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-2}b_{n-1} + a_{n-1}) = 0.$$

Similarly, the second equation of (12.15) gives

$$b_n(b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-2}b_{n-1} + a_{n-1}) = 0.$$

These equations also hold in the generic fiber of U; but by (12.13), both a_n and b_n are units in the generic fiber, and hence we obtain the following equation, first in the generic fiber but then by flatness on all of U,

$$b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-2}b_{n-1} + a_{n-1} = 0. \tag{12.16}$$

We can now eliminate b_{n-1} and remain only with equation (12.13) among the indeterminates $a_1, \dots, a_n, b_1, \dots, b_{n-2}, b_n$. Hence

$$U \simeq \operatorname{Spec} O_F[X,Y]/(XY - \pi) \times \mathbb{A}^{2n-3}$$

has semi-stable reduction.

12.9. **The case** (**split** $\mathrm{SO}_{2n+1}, r = 1, \{0, n\}$)**.** This corresponds to case 1) in 12.7.3. We continue to assume $p \neq 2$. Again we denote by $e_1, \dots, e_{2n+1}$ a Witt basis, i.e., $\langle e_i, e_{2n+2-j}\rangle = \delta_{ij}$ for $i, j \leq 2n+1$.

The local model is contained in the closed subscheme $\mathbb{M}^{\mathrm{naive}}_{\{0,n\}}(\mathrm{SO}_{2n+1}, \mu_1)$ of the local model $\mathbb{M}^{\mathrm{loc}}_{\{0,n\}}(\mathrm{GL}_{2n+1}, \mu_1)$ given by the condition

$$\mathcal{F}_0 \subset \mathcal{F}_0^{\perp}, \quad \mathcal{F}_n \subset \mathcal{F}_n^{\perp} \tag{12.17}$$

where the second $\perp$ is for the form $\pi\langle\ ,\ \rangle$ on $\Lambda_{n,R}$.

As open subset U of the worst point we take the scheme of $(\mathcal{F}_0, \mathcal{F}_n)$, where

$$\mathcal{F}_0 = (e_1 + a_1 e_2 + \cdots + a_{2n} e_{2n+1}),$$

$$\mathcal{F}_n = (b_1\pi^{-1}e_1 + \cdots + b_n\pi^{-1}e_n + e_{n+1} + b_{n+1}e_{n+1} + \cdots + b_{2n}e_{2n}).$$

The incidences from $\mathcal{F}_0$ to $\mathcal{F}_n$, resp., from $\mathcal{F}_n$ to $\mathcal{F}_0$ are given by the following matrix relations

$$\begin{pmatrix} \pi \\ \pi a_1 \\ \vdots \\ \pi a_{n-1} \\ a_n \\ \vdots \\ a_{2n} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ 1 \\ b_{n+1} \\ \vdots \\ b_{2n} \end{pmatrix} \cdot a_n, \qquad \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ \pi \\ \pi b_{n+1} \\ \vdots \\ \pi b_{2n} \end{pmatrix} = \begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_n \\ \vdots \\ a_{2n} \end{pmatrix} \cdot b_1.$$

We deduce that

$$a_n b_1 = \pi, \tag{12.18}$$

and the following identities ,

$$\begin{aligned} a_n b_{n+i} &= a_{n+i}, \text{ for } i = 1, \dots, n & b_1 a_i &= b_{i+1}, \text{ for } i = 1, \dots, n-1 \\ a_n b_{i+1} &= \pi a_i, \text{ for } i = 1, \dots, n-1 & b_1 a_{n+i} &= \pi b_{n+i} \text{ for } i = 1, \dots, n\ . \end{aligned} \tag{12.19}$$

The isotropy conditions on $\mathcal{F}_0$, resp., $\mathcal{F}_n$, are given by the following equations,

$$\begin{aligned} 2a_{2n} + 2a_1a_{2n-1} + \cdots + 2a_na_{n+1} + a_n^2 = 0, \\ \pi + 2b_1b_{2n} + 2b_2b_{2n-1} + \cdots + 2b_nb_{n+1} = 0. \end{aligned} \tag{12.20}$$

We use the first lines of (12.19) to eliminate $a_{n+1}, \ldots, a_{2n}$ and $b_2, \ldots, b_n$. Then the second lines in (12.19) are automatically satisfied (use (12.13)),

$$a_nb_{i+1} - \pi a_i = a_nb_1a_i - \pi a_i = a_i(a_nb_1 - \pi) = 0;$$

$$b_1a_{n+i} - \pi b_{n+i} = b_1a_nb_{n+i} - \pi b_{n+i} = b_{n+i}(a_nb_1 - \pi) = 0.$$

Expressing $a_{n+1}, \ldots, a_{2n-1}$ in terms of $b_1, \ldots, b_{n-1}$ in the first equation of (12.20), we obtain the equation

$$a_n(2b_n + 2a_1b_{2n-1} + \cdots + 2a_{n-1}b_{n+1} + a_n) = 0.$$

Similarly, the second equation of (12.14) gives

$$b_1(2b_{2n} + a_1b_{2n-1} + \cdots + 2a_{n-1}b_{n+1} + a_n) = 0.$$

These equations also hold in the generic fiber of U; but by (12.13), both a_n and b_1 are units in the generic fiber, and hence we obtain the following equation, first in the generic fiber but then by flatness on all of U,

$$2b_{2n} + a_1b_{2n-1} + \cdots + 2a_{n-1}b_{n+1} + a_n = 0. \tag{12.21}$$

We can now eliminate b_{2n} and remain only with equation (12.13) among the indeterminates $a_1, \ldots, a_n, b_1, b_{n+1}, \ldots, b_{2n-1}$. Hence

$$U \simeq \operatorname{Spec} O_F[X,Y]/(XY - \pi) \times \mathbb{A}^{2n-2}$$

has semi-stable reduction.

12.10. **The case** (**nonsplit** $\mathrm{SO}_{2n}, r = 1, \{0, n\}$)**.** This corresponds to case 3) in 12.7.3. We continue to assume $p \neq 2$. Considering this case is not essential for the proof of the main result, since it has already been excluded in Subsection 10.6. However, we include it here since it fits the pattern of the previous cases.

We have $V = \oplus_{i=1}^{2n} F \cdot e_i$ with symmetric F-bilinear form determined by

$$(e_i, e_{2n+1-j}) = \delta_{ij}, \text{ for } i, j \neq n, n+1, \ (e_n, e_n) = \pi, \ (e_{n+1}, e_{n+1}) = 1, \ (e_n, e_{n+1}) = 0.$$

Here, $\Lambda_0 \subset \Lambda_0^\vee \subset \pi^{-1}\Lambda_0$, $\pi\Lambda_n \subset \Lambda_n^\vee \subset \Lambda_n$ with the quotients $\Lambda_0^\vee/\Lambda_0$, $\Lambda_n^\vee/\pi\Lambda_n$ both of length one.

We consider the functor which to an O-algebra R, associates the set of $\mathcal{F}_0 \subset \Lambda_0 \otimes_O R$, $\mathcal{F}_n \subset \Lambda_n \otimes_O R$, both R-locally direct summands of rank 1 that are isotropic for the symmetric forms induced by $(\ ,\ )$ on $\Lambda_0 \otimes_O R$, resp., by $\pi(\ ,\ )$ on $\Lambda_n \otimes_O R$, and which are linked, i.e., $\Lambda_0 \otimes_O R \to \Lambda_n \otimes_O R$ maps $\mathcal{F}_0$ to $\mathcal{F}_n$ and $\pi : \Lambda_n \otimes_O R \to \Lambda_0 \otimes_O R$ maps $\mathcal{F}_n$ to $\mathcal{F}_0$. This functor is represented by a closed subscheme

$$\mathbb{M}^{\mathrm{naive}}_{\Lambda_\bullet}(\mathrm{SO}(V), \mu_1) \subset Q(\Lambda_0, (\ ,\ )) \times Q(\Lambda_n, \pi(\ ,\ ))$$

of the product of the two quadrics. The local model $\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet)$ is the flat closure of the generic fiber of this subscheme. Set

$$\mathcal{F}_0 = (\sum_{i=1}^{2n} x_ie_i), \qquad \mathcal{F}_n = (\sum_{i=1}^{n} y_i\pi^{-1}e_i + \sum_{i=n+1}^{2n} y_ie_i).$$

The isotropy conditions translate to:

$$x_1x_{2n} + \cdots + x_{n-1}x_{n+2} + \pi x_n^2 + x_{n+1}^2 = 0, \tag{12.22}$$

$$y_1y_{2n} + \cdots + y_{n-1}y_{n+2} + y_n^2 + \pi y_{n+1}^2 = 0. \tag{12.23}$$

(Here $(x_1; \ldots; x_{2n})$, $(y_1; \ldots; y_{2n})$ are homogeneous coordinates.) Linkage translates to the existence of λ, $\mu \in R$ with

$$\sum_{i=1}^{n} x_ie_i + \sum_{i=n+1}^{2n} x_ie_i = \lambda \cdot (\sum_{i=1}^{n} y_i\pi^{-1}e_i + \sum_{i=n+1}^{2n} y_ie_i),$$

$$\sum_{i=1}^{n} y_ie_i + \sum_{i=n+1}^{2n} \pi y_ie_i = \mu \cdot (\sum_{i=1}^{n} x_ie_i + \sum_{i=n+1}^{2n} x_ie_i).$$

This gives

$$\pi x_1 = \lambda y_1, \ldots, \pi x_n = \lambda y_n,\ x_{n+1} = \lambda y_{n+1}, \ldots, x_{2n} = \lambda y_{2n},$$

$$y_1 = \mu x_1, \ldots, y_n = \mu x_n,\ \pi y_{n+1} = \mu x_{n+1}, \ldots, \pi y_{2n} = \mu x_{2n}.$$

We obtain $\lambda\mu = \pi$. Now the two isotropy conditions become:

$$x_1\lambda y_{2n} + \cdots + x_n\lambda y_n + \lambda^2 y_{n+1}^2 = 0,$$

$$x_1\mu y_{2n} + \cdots + \mu^2 x_n^2 + x_{n+1}\mu y_{n+1} = 0.$$

These give:

$$\lambda \cdot (x_1y_{2n} + \cdots + x_ny_n + \lambda y_{n+1}^2) = \mu \cdot (x_1y_{2n} + \cdots + \mu x_n^2 + x_{n+1}y_{n+1}) = 0.$$

Since $x_{n+1} = \lambda y_{n+1}$, $y_n = \mu x_n$, the expressions in both parentheses are the same, and are equal to

$$x_1y_{2n} + \cdots + x_ny_n + x_{n+1}y_{n+1}.$$

By flatness, $x_1y_{2n} + \cdots + x_ny_n + x_{n+1}y_{n+1} = 0$ holds on $\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet)$. In fact, the worst point lies in the affine chart U with $x_n = 1$ and $y_{n+1} = 1$. Then $\mu = y_n$ and $\lambda = x_{n+1}$ and we can see

$$U \simeq \operatorname{Spec} O_F[X, x_1, \ldots, x_{n-1}, y_{n+2}, \ldots, y_{2n}]/(X(X + x_1y_{2n} + \cdots + x_{n-1}y_{n+2}) + \pi)$$

with $X = x_{n+1}$. The special fiber U_{κ_F} has two irreducible components that are both isomorphic to $\mathbb{A}^{2n-2}_{\kappa_F}$. Their intersection is isomorphic to

$$\operatorname{Spec} \kappa_F[x_1, \ldots, x_{n-1}, y_{n+2}, \ldots, y_{2n}]/(x_1y_{2n} + \cdots + x_{n-1}y_{n+2}),$$

which is singular. Therefore, in this case, the local model $\mathbb{M}^{\mathrm{loc}}(\Lambda_\bullet)$ indeed does not have pseudo semi-stable reduction.

Note that $\mathbb{M}^{\mathrm{naive}}_{\Lambda_\bullet}(\mathrm{SO}(V), \mu_1)$ is not flat; the special fiber contains $\lambda = \mu = 0$ and $x_{n+1} = \cdots = x_{2n} = 0$, $y_1 = \cdots = y_n = 0$. This shows that there is an extra irreducible component isomorphic to $\mathbb{P}^{n-1}_{\kappa_F} \times \mathbb{P}^{n-1}_{\kappa_F}$ given by $(x_1; \ldots; x_n, 0; \ldots; 0) \times (0; \ldots; 0, y_{n+1}; \ldots; y_{2n})$. On this component, the equation $x_1y_{2n} + \cdots + x_ny_n + x_{n+1}y_{n+1} = 0$ becomes $x_1y_1 + \cdots + x_{n-1}y_{n+2} = 0$ and it is not satisfied.

12.11. **The case (nonsplit $\mathrm{SO}_{2n}, r = n, \{0\}$).** Here $n \geq 2$. We have $V = \oplus_{i=1}^{2n} F \cdot e_i$ with symmetric F-bilinear form determined by

$$(e_i, e_{2n+1-j}) = \delta_{ij}, \text{ for } i,\, j \neq n, n+1,\ (e_n, e_n) = \pi,\ (e_{n+1}, e_{n+1}) = 1,\ (e_n, e_{n+1}) = 0.$$

Then $\Lambda_0 \subset^1 \Lambda_0^\vee \subset \pi^{-1}\Lambda_0$, the quotient $\Lambda_0^\vee/\Lambda_0$ is of length one. In this case, K is the parahoric stabilizer of the selfdual periodic lattice chain

$$\cdots \subset \pi\Lambda_0^\vee \subset \Lambda_0 \subset \Lambda_0^\vee \subset \pi^{-1}\Lambda_0 \subset \cdots$$

The reflex field E is the ramified quadratic extension of F obtained by adjoining the square root of π.

Set $I = \{0, 1\}$. In this case, by [44, 8.2.3], the local model is contained in the closed subscheme $\mathbb{M}_I^{\mathrm{naive}}(\mathrm{GO}_{2n}, \mu_n)$ of the local model $\mathbb{M}_I^{\mathrm{loc}}(\mathrm{GL}_{2n}, \mu_n) \otimes_O O_E$ described by

$$\mathcal{F}_1 = \mathcal{F}_0^\perp. \tag{12.24}$$

(Note that the group GO_{2n} is not connected and so the discussion in [44, p. 215] applies.) As open subset U of the worst point we take the scheme of $(\mathcal{F}_0, \mathcal{F}_1) = (\mathcal{F}_0, \mathcal{F}_0^\perp)$, where

$$\mathcal{F}_0 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}.$$

We can now see that U is a subscheme of the closed subscheme of $\mathrm{Spec}(O_E[a_{i,j}]_{1 \leq i,j \leq n})$ defined by the equations

$$a_{1n}^2 = \pi, \quad \text{and,}$$

$$a_{n+1-i,j} + a_{n+1-j,i} + a_{1i}a_{1j} = 0,$$

(if at least one of i or j is not equal to n). This has two irreducible components defined by setting $a_{1n} = \sqrt{\pi}$, or $a_{1n} = -\sqrt{\pi}$ respectively. As we can see from the equations, each component is isomorphic to affine space over O_E in the coordinates $a_{i,j}$ with $i + j \leq n$, and is therefore smooth over O_E. The generic fiber $U \otimes_{O_E} E$ has two isomorphic connected components, given by the generic fibers of these two irreducible components and the two irreducible components above are the Zariski closures of these two connected components. Our discussion implies that the corresponding local model, which has an open affine given by the Zariski closure of one of these connected components, is smooth.

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