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# SINGULAR SUPPORTS FOR CHARACTER SHEAVES ON A GROUP COMPACTIFICATION

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*Dedicated to Joseph Bernstein on the occasion of his 60th birthday*

**Abstract.** Let $G$ be a semisimple adjoint group over $\mathbf{C}$ and $\bar{G}$ be the De Concini–Procesi completion of $G$. In this paper, we define a Lagrangian subvariety $\Lambda$ of the cotangent bundle of $\bar{G}$ such that the singular support of any character sheaf on $\bar{G}$ is contained in $\Lambda$.

## Introduction

In the mid-1980s the second author observed that for a connected reductive complex algebraic group $G$ the singular support of any character sheaf on $G$ is contained in a fixed explicit Lagrangian subvariety of the cotangent bundle of $G$. In the present paper this result is extended to character sheaves on the De Concini–Procesi completion of $G$ (assumed to be adjoint). We do not know whether a suitable converse of this property holds (as it does for $G$ itself by results in [MV], [G]).

**1.1** In this paper all algebraic varieties are assumed to be over a fixed algebraically closed field of characteristic 0.

If $X$ is a smooth variety, let $T^*X$ be the cotangent bundle of $X$. For any morphism $\alpha : X \to Y$ of smooth varieties and $x \in X$, we write $\alpha^* : T^*_{\alpha(x)}Y \to T^*_x X$ for the map induced by $\alpha$. If, moreover, $\alpha : X \to Y$ is a locally trivial fibration with smooth connected fibres and $\Lambda$ is a closed Lagrangian subvariety of $T^*Y$, then let $\alpha^{\bigstar}(\Lambda) = \bigcup_{x \in X} \alpha^*(\Lambda \cap T^*_{\alpha(x)}Y) \subset T^*X$. Then

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(a) $\alpha^{\bigstar}(\Lambda)$ *is a closed Lagrangian subvariety of* $T^*X$*. Moreover, the set of irreducible components of* $\Lambda$ *is naturally in bijection with the set of irreducible components of* $\alpha^{\bigstar}(\Lambda)$*.*

Let $X, Y$ be smooth irreducible varieties and let $\alpha : X \to Y$ be a principal $P$-bundle for a free action of a connected linear algebraic group $P$ on $X$.

(b) *If* $\Lambda'$ *is a closed Lagrangian subvariety of* $T^*X$ *stable under the* $P$*-action then* $\Lambda' = \alpha^{\bigstar}(\Lambda)$ *for a unique closed Lagrangian subvariety* $\Lambda$ *of* $T^*Y$*.*

Let $X$ be a smooth irreducible variety and let $i : Y \to X$ be the inclusion of a locally closed smooth irreducible subvariety. Let $\Lambda$ be a closed Lagrangian subvariety of $T^*Y$. Let $i_{\bigstar}(\Lambda)$ be the subset of $T^*X$ consisting of all $\xi \in T_x^*X$ such that $x \in Y$ and the image of $\xi$ under the obvious surjective map $T_x^*X \to T_x^*Y$ belongs to $\Lambda \cap T_x^*Y$. Note that

(c) $i_{\bigstar}(\Lambda)$ *is a locally closed Lagrangian subvariety of* $T^*X$*. Moreover, the set of irreducible components of* $i_{\bigstar}(\Lambda)$ *is naturally in bijection with the set of irreducible components of* $\Lambda$*.*

For an algebraic variety $X$ we write $\mathcal{D}(X)$ for the bounded derived category of constructible $\bar{\mathbf{Q}}_l$-sheaves on $X$ where $l$ is a fixed prime number. For $X$ smooth and $C \in \mathcal{D}(X)$, we denote by $SS(C)$ the singular support of $C$ (a closed Lagrangian subvariety of $T^*X$). Let $A$ be a connected linear algebraic group acting on a smooth variety $X$ and let $B$ be a connected subgroup of $A$. Let $\mu_A : T^*X \to \mathrm{Lie}(A)^*$ be the moment map of the $A$-action on $X$. Consider the diagram $X \xleftarrow{pr_1} A \times X \xrightarrow{pr_2} A \times_B X \xrightarrow{\pi} X$ where $B$ acts on $A \times X$ by $b : (a, x) \mapsto (ab^{-1}, bx)$, $A \times_B X$ is the quotient space and $\pi(a, x) = ax$. Then for any $B$-equivariant perverse sheaf $C$ on $X$ there is a well-defined perverse sheaf $C'$ on $A \times_B X$ such that $pr_2^* C' = pr_1^* C$ up to a shift. We set $\Gamma_B^A(C) = \pi_* C' \in \mathcal{D}(X)$. By [MV, 1.2] we have

(d) $\quad SS(\Gamma_B^A(C)) \subset \overline{A \cdot SS(C)}$.

On the other hand, we have

(e) $\quad SS(C) \subset \mu_B^{-1}(0)$.

Indeed, if $p_1 : B \times X \to X$ is the action and $p_2 : B \times X \to X$ is the second projection we have $p_1^*(C) = p_2^*(C)$. Hence $SS(p_1^*(C)) = SS(p_2^*(C))$. Using [KS, 4.1.2], we can rewrite this as $p_1^{\bigstar}(SS(C)) = p_2^{\bigstar}(SS(C))$. Hence if $x \in X$ and $\xi \in T_x^*X \cap SS(C)$ then the image of $\xi$ under the map $T_x^*X \to T_1^*(B)$ induced by $B \to X$, $b \mapsto bx$ is 0. This proves (e).

**1.2** Let $G$ be a connected reductive algebraic group. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}^*$. Let $B$ be a Borel subgroup of $G$. Let $K$ be a closed connected subgroup of $G$ such that $B_K = B \cap K$ is a parabolic subgroup of $K$. Assume that $G$ acts on a smooth variety $X$. Let $C$ be a $B_K$-equivariant perverse sheaf on $X$; assume also that there exists a finite covering $a : \tilde{B} \to B$ such that $C$ is $\tilde{B}$-equivariant for the $\tilde{B}$-action $\tilde{b} : x \mapsto a(\tilde{b})x$ on $X$. By 1.1(e) we have $\mu_{\tilde{B}}(SS(C)) = 0$. Since $\mathrm{Lie}(\tilde{B}) = \mathrm{Lie}(B)$ we then have $\mu_B(SS(C)) = 0$. It follows that $\mu_G(SS(C))$ is contained in the kernel of the obvious map $\mathfrak{g}^* \to \mathrm{Lie}(B)^*$ hence is contained in $\mathcal{N}$. Since $\mathcal{N}$ is stable under the coadjoint action we have $\mu_G(K \cdot SS(C)) = K\mu_G(SS(C)) \subset \mathcal{N}$. Using this together with 1.1(d) and the fact that $\mu_G^{-1}(\mathcal{N})$ is closed in $T^*X$ we see that $SS(\Gamma^K_{B_K}(C)) \subset \mu_G^{-1}(\mathcal{N})$. Applying 1.1(e) to $\Gamma^K_{B_K}(C), K$ instead of $C, B$ we see that $SS(\Gamma^K_{B_K}(C)) \subset \mu_K^{-1}(0) = \mu_G^{-1}(\mathrm{Lie}(K)^\perp)$ where $\mathrm{Lie}(K)^\perp \subset \mathfrak{g}^*$ is the annihilator of $\mathrm{Lie}(K) \subset \mathfrak{g}$. Thus we have

(a) $SS(\Gamma^K_{B_K}(C)) \subset \mu_G^{-1}(\mathrm{Lie}(K)^\perp \cap \mathcal{N})$.

**1.3** We now replace $G, \mathfrak{g}, B, K, B_K, X, C$ by $G\times G, \mathfrak{g}\times\mathfrak{g}, B\times B, G_\Delta, B_\Delta, X', C'$ where $G_\Delta = \{(g,g') \in G\times G;\ g = g'\}$, $B_\Delta = \{(g,g') \in B\times B;\ g = g'\}$, $X'$ is a smooth variety with a given action of $G \times G$ and $C'$ is a $B_\Delta$-equivariant perverse sheaf on $X'$; we assume that there exists a finite covering $a' : \tilde{B}' \to B \times B$ such that $C'$ is $\tilde{B}'$-equivariant for the $\tilde{B}'$-action $\tilde{b}' : x' \mapsto a'(\tilde{b}')x'$ on $X'$. We have the following special case of 1.2(a):

(a) $SS(\Gamma^{G_\Delta}_{B_\Delta}(C')) \subset \mu^{-1}_{G\times G}(\mathcal{N}^-)$.

where

$$\mathcal{N}^- = \{(f,f') \in \mathfrak{g}^* \times \mathfrak{g}^*;\ f + f' = 0\,,\ f, f' \text{ nilpotent}\}\,.$$

**1.4** Let $\mathbf{W}$ be the Weyl group of $G$ and let $\mathbf{I}$ be the set of simple reflections in $\mathbf{W}$. Let $G'$ be a possibly disconnected algebraic group with identity component $G$ and with a given connected component $D$. Now $G \times G$ acts transitively on $D$ by $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$. Hence the moment map $\mu_{G\times G} : T^*D \to \mathfrak{g}^* \times \mathfrak{g}^*$ is well-defined. In [L1, 4.5], a class of perverse sheaves (called character sheaves) on $D$ is introduced. These appear as constituents of some perverse cohomology sheaf of $\Gamma^{G_\Delta}_{B_\Delta}(C')$ for some $C'$ as in 1.3 (with $X' = D$). Hence from 1.3(a) we deduce:

(a) *If $K$ is a parabolic character sheaf on $D$ then $SS(K) \subset \mu^{-1}_{G\times G}(\mathcal{N}^-)$.*

In the case where $G' = G = D$ a statement close to (a) appears in [MV, 2.8] (where it is attributed to the second author) and in [G].

**1.5** We preserve the setup of 1.4. For any $J \subset \mathbf{I}$ let $\mathcal{P}_J$ be the set of parabolic subgroups of $G$ of type $J$. In particular $\mathcal{P}_\emptyset$ is the set of Borel subgroups of $G$. For $J \subset \mathbf{I}$ let $\mathbf{W}_J$ be the subgroup of $\mathbf{W}$ generated by $J$; let $\mathbf{W}^J$ (resp. ${}^J\mathbf{W}$) be the set of all $w \in \mathbf{W}$ such that $w$ has minimal length among the elements in $\mathbf{W}_J w$ (resp. $w\mathbf{W}_J$). Let $\delta : \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the isomorphism such that $\delta(\mathbf{I}) = \mathbf{I}$ and such that $J \subset \mathbf{I}$, $P \in \mathcal{P}_J$, $g \in D \Rightarrow gPg^{-1} \in \mathcal{P}_{\delta(J)}$. Following [L2, 8.18], for $J, J' \subset \mathbf{I}$ and $y \in {}^{J'}\mathbf{W} \cap \mathbf{W}^J$ such that $\mathrm{Ad}(y)(\delta(J)) = J'$ we set

$$Z_{J,y,\delta} = \big\{(P, P', gU_P)\,;\ P \in \mathcal{P}_J\,,\ P' \in \mathcal{P}_{J'}\,,\ g \in D\,,\ \mathrm{pos}(P', gPg^{-1}) = y\big\}\,.$$

Now $G \times G$ acts (transitively) on $Z_{J,y,\delta}$ by

$$(g_1, g_2) : (P, P', gU_P) \mapsto (g_2Pg_2^{-1}, g_1P'g_1^{-1}, g_1gg_2^{-1})\,.$$

Hence the moment map $\mu_{G\times G} : T^*Z_{J,y,\delta} \to \mathfrak{g}^* \times \mathfrak{g}^*$ is well defined. In [L2, §11], a class of perverse sheaves (called parabolic character sheaves) on $Z_{J,y,\delta}$ is introduced. These appear as constituents of some perverse cohomology sheaf of $\Gamma^{G_\Delta}_{B_\Delta}(C')$ for some $C'$ as in 1.3 (with $X' = Z_{J,y,\delta}$). Hence from 1.3(a) we deduce:

(a) *If $K$ is a parabolic character sheaf on $Z_{J,y,\delta}$ then $SS(K) \subset \mu^{-1}_{G\times G}(\mathcal{N}^-)$.*

When $J = \mathbf{I}$, this reduces to 1.4(a).

**1.6** Assume that $G$ is adjoint. Let $\bar{G}$ be the De Concini–Procesi compactification of $G$. Then $G\times G$ acts naturally on $\bar{G}$ extending continuously the action $(g_1, g_2) : g \mapsto g_1gg_2^{-1}$ of $G \times G$ on $G$. Hence the moment map $\mu_{G\times G} : T^*\bar{G} \to \mathfrak{g}^* \times \mathfrak{g}^*$ is well defined. In [L2] a class of perverse sheaves (called parabolic character sheaves) on $\bar{G}$ is introduced. It has been shown by He [H2] and by Springer (unpublished) that any parabolic character sheaf on $\bar{G}$ appears as a constituent of some perverse cohomology sheaf of $\Gamma^{G_\Delta}_{B_\Delta}(C')$ for some $C'$ as in 1.3 (with $X' = \bar{G}$). Hence from 1.3(a) we deduce:

(a) *If $K$ is a parabolic character sheaf on $\bar{G}$ then $SS(K) \subset \mu^{-1}_{G\times G}(\mathcal{N}^-)$.*

**1.7** In the setup of 1.4 let $\Lambda(D) = \mu^{-1}_{G\times G}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(D)$. For $g \in D$ let $I_g$ be the isotropy group at $g$ of the $G \times G$-action on $D$ that is, $I_g = \{(g_1, g_2) \in G \times G; g_2 = g^{-1}g_1g\}$. We have $\mathrm{Lie}(I_g) = \{(y_1, y_2) \in \mathfrak{g} \times \mathfrak{g}; y_2 = \mathrm{Ad}(g)^{-1}(y_1)\}$ and the annihilator of $\mathrm{Lie}(I_g)$ in $\mathfrak{g}^* \times \mathfrak{g}^*$ is $\mathrm{Lie}(I_g)^\perp = \{(z_1, z_2) \in \mathfrak{g}^* \times \mathfrak{g}^*; z_1 + \mathrm{Ad}(g)(z_2) = 0\}$. This may be identified with the fibre of $T^*D$ at $g$. Then

$$\begin{aligned}\Lambda(D) &= \big\{(g, z_1, z_2) \in D \times \mathfrak{g}^* \times \mathfrak{g}^*;\ z_1 + \mathrm{Ad}(g)(z_2) = 0,\ z_1 + z_2 = 0,\ z_2 \in \mathcal{N}\big\}\\ &= \big\{(g, z, -z);\ g \in D,\ z \in \mathcal{N},\ \mathrm{Ad}(g)(z) = z\big\} = \sqcup_{\mathcal{O}} X_{\mathcal{O}}\,,\end{aligned}$$

where $\mathcal{O}$ runs over the (finite) set of $\mathrm{Ad}(G)$-orbits on $\mathcal{N}$ which are normalized by some element of $D$ and $X_{\mathcal{O}} = \{(g, z, -z);\ g \in D,\ z \in \mathcal{O},\ \mathrm{Ad}(g)(z) = z\}$. We pick $\xi \in \mathcal{O}$ and let $\mathcal{Z}' = \{h \in G';\ \mathrm{Ad}(h)\xi = \xi\}$, $\mathcal{Z} = \{h \in G;\ \mathrm{Ad}(h)\xi = \xi\}$. Let $\underline{\mathcal{Z}}'$ (resp. $\underline{\mathcal{Z}}$) be the group of connected components of $\mathcal{Z}'$ (resp. $\mathcal{Z}$). Then $\underline{\mathcal{Z}}'$ is a finite group and $\underline{\mathcal{Z}}$ is a subgroup of $\underline{\mathcal{Z}}'$. Let $\underline{\mathcal{Z}}_1$ be the set of connected components of $\mathcal{Z}'$ that are contained in $D$. Then $\underline{\mathcal{Z}}_1$ is a subset of $\mathcal{Z}'$; also, $\underline{\mathcal{Z}}$ acts on $\underline{\mathcal{Z}}_1$ by conjugation inside $\underline{\mathcal{Z}}'$. Let $F_{\mathcal{O}}^D$ be the set of orbits of this action. Note that $F_{\mathcal{O}}^D$ is independent (up to unique isomorphism) of the choice of $\xi$.

Let $\tilde{X} = \{(g, r) \in D \times G;\ r^{-1}gr \in \mathcal{Z}'\}$. Then $\mathcal{Z}$ acts freely on $\tilde{X}$ by $h : (g, r) \mapsto (g, rh^{-1})$ and we have an isomorphism $\mathcal{Z}\backslash\tilde{X} \xrightarrow{\sim} X_{\mathcal{O}}$, $(g, r) \mapsto (g, \mathrm{Ad}(r)\xi)$. By the change of variable $(g, r) \mapsto (g', r)$, $g' = r^{-1}gr$, $\tilde{X}$ becomes $\{(g', r);\ g' \in \mathcal{Z}' \cap D,\ r \in G\}$. In the new coordinates, the free action of $\mathcal{Z}$ on $\tilde{X}$ is $h : (g', r) \mapsto (hg'h^{-1}, rh^{-1})$. We see that $\tilde{X}$ is smooth of pure dimension $\dim(\mathcal{Z} \times G)$ and its connected components are indexed naturally by $\underline{\mathcal{Z}}_1$ (the connected component containing $(g', r)$ is indexed by the image of $g'$ in $\underline{\mathcal{Z}}_1$). The action of $\mathcal{Z}$ on $\tilde{X}$ permutes the connected components of $\tilde{X}$ according to the action of $\underline{\mathcal{Z}}$ on $\underline{\mathcal{Z}}_1$ considered above. We see that $X_{\mathcal{O}} = \mathcal{Z}\backslash\tilde{X}$ is smooth of pure dimension $\dim G$ and its connected components are indexed naturally by the set $F_{\mathcal{O}}^D$.

We see that $\Lambda(D)$ can be partitioned into finitely many locally closed, irreducible, smooth subvarieties of dimension $\dim G$, indexed by the finite set $F(D) := \sqcup_{\mathcal{O}} F_{\mathcal{O}}^D$. In particular, $\Lambda(D)$ has pure dimension $\dim G$. More precisely, one checks that

(a) $\Lambda(D)$ *is a closed Lagrangian subvariety of* $T^*D$.

**1.8** In the setup of 1.5 we set $\Lambda(Z_{J,y,\delta}) = \mu_{G\times G}^{-1}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(Z_{J,y,\delta})$.

Following [L2, 8.18], we consider the partition $Z_{J,y,\delta} = \sqcup_{\mathbf{s}} Z_{J,y,\delta}^{\mathbf{s}}$ where $Z_{J,y,\delta}^{\mathbf{s}}$ are certain locally closed smooth irreducible $G_\Delta$-stable subvarieties of $Z_{J,Y,\delta}$ indexed by the elements $\mathbf{s}$ of a finite set $S(J, \mathrm{Ad}(y)\delta)$ which is in canonical bijection with ${}^{J'}\mathbf{W}$, see [L1, 2.5]. Note that each $\mathbf{s}$ is a sequence $(J_n, J_n', u_n)_{n\ge 0}$ where $J_n, J_n'$ are subsets of $\mathbf{I}$ such that $J_n, J_n'$ are independent of $n$ for large $n$ and $u_n \in \mathbf{W}$ is 1 for large $n$.

We wish to define a Lagrangian subvariety $\Lambda(Z_{J,y,\delta}^{\mathbf{s}})$ of $T^*(Z_{J,y,\delta}^{\mathbf{s}})$.

Assume first that $\mathbf{s}$ is such that $J_n = J$, $J_n' = J'$, $u_n = 1$ for all $n$. In this case we have $J = J'$. Let $P \in \mathcal{P}_J$ and let $L$ be a Levi subgroup of $P$. Then $\mathbf{d_s} = \{g \in D;\ gLg^{-1} = L,\ \mathrm{pos}(P, gPg^{-1}) = y\}$ is a connected component of the algebraic group $N_{G'}(L)$ with identity component $L$. Hence $\Lambda(\mathbf{d_s}) \subset T^*\mathbf{d_s}$

is defined as in 1.7. We have a diagram $Z^{\mathbf{s}}_{J,y,\delta} \xleftarrow{\alpha} G \times \mathbf{d_s} \xrightarrow{pr_2} \mathbf{d_s}$ where $\alpha(h,g) = (hPh^{-1}, hPh^{-1}, U_{hPh^{-1}}hgh^{-1}U_{hPh^{-1}})$. Note that $\alpha$ is a principal $P$-bundle where $P$ acts on $G \times \mathbf{d_s}$ by $p : (h,g) = (hp^{-1}, \bar{p}g\bar{p}^{-1})$ (we denote the canonical homomorphism $P \to L$ by $\bar{p}$). Let $\Lambda' = pr_2^\star \Lambda(\mathbf{d_s}) \subset T^*(G \times \mathbf{d_s})$. By 1.1(a) and 1.7(a), $\Lambda'$ is a closed Lagrangian subvariety of $T^*(G \times \mathbf{d_s})$. It is clearly stable under the natural action of $P$ on $T^*(G \times \mathbf{d_s})$ (since $\Lambda(\mathbf{d_s})$ is $L$-stable). By 1.1(b) there is a unique Lagrangian subvariety $\Lambda''$ of $T^*(Z^{\mathbf{s}}_{J,y,\delta})$ such that $\alpha^\bigstar \Lambda'' = \Lambda'$. We set $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) = \Lambda''$.

We now consider a general $\mathbf{s} = (J_n, J'_n, u_n)_{n\in\mathbf{N}}$. For any $r \in \mathbf{N}$ let $\mathbf{s}_r = (J_n, J'_n, u_n)_{n\geq r}$, $y_r = u_{r-1}^{-1} \dots u_1^{-1}u_0^{-1}y$. Then $Z^{\mathbf{s}_r}_{J_r,y_r,\delta}$ is defined and we have a canonical map $f_r : Z^{\mathbf{s}}_{J,y,\delta} \to Z^{\mathbf{s}_r}_{J_r,y_r,\delta}$ (a composition of affine space bundles, see [L2, 8.20(a)]). Moreover, for sufficiently large $r$, $\mathbf{s}_r, J_r, y_r, f_r$ are independent of $r$; we write $\mathbf{s}_\infty, J_\infty, y_\infty, f_\infty$ instead of $\mathbf{s}_r, J_r, y_r, f_r$. Note also that $\mathbf{s}_\infty, J_\infty, y_\infty$ are of the type considered earlier, so that $\Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$ is defined as above. We set $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) = f_\infty^\bigstar(\Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}))$.

We now define

$$\Lambda'(Z_{J,y,\delta}) = \sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y)\delta)} (i_{\mathbf{s}})_\bigstar/\big(\Lambda(Z^{\mathbf{s}}_{J,y,\delta})\big)$$

where $i_{\mathbf{s}} : Z^{\mathbf{s}}_{J,y,\delta} \to Z_{J,y,\delta}$ is the inclusion. From 1.1(c) we see that $\Lambda'(Z_{J,y,\delta})$ is a finite union of locally closed Lagrangian subvarieties of $T^*(Z_{J,y,\delta})$.

We state the following result:

PROPOSITION 1.9. *We have $\Lambda(Z_{J,y,\delta}) = \Lambda'(Z_{J,y,\delta})$. In particular, $\Lambda'(Z_{J,y,\delta})$ is closed in $T^*(Z_{J,y,\delta})$ and $\Lambda(Z_{J,y,\delta})$ is a Lagrangian subvariety of $T^*(Z_{J,y,\delta})$.*

Let $x \in Z^{\mathbf{s}}_{J,y,\delta}$. We will show that

$$\Lambda(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta}) = \Lambda'(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta})\,.$$

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via a $G$-invariant symmetric bilinear form. Choose an element $g_0$ in $D$ that normalizes $B$ and a maximal torus $T$ of $B$. Let $P_J$ be the unique element in $\mathcal{P}_J$ that contains $B$. For $w \in W$, we choose a representative $\dot{w}$ of $w$ in $N(T)$. Set $h_{J,y,\delta} = (P_J, {}^{\dot{y}^{-1}}P_{J'}, U_{{}^{\dot{y}^{-1}}P_{J'}} g_0 U_{P_J}) \in Z_{J,y,\delta}$. The element $\mathbf{s} \in S(J, \mathrm{Ad}(y)\delta)$ corresponds to an element $w' \in {}^{J'}W$ under the bijection in [L1, 2.5]. Set $w = (w')^{-1}y \in W^{\delta(J)}$. By [H1, 1.10],

$$Z^{\mathbf{s}}_{J,y,\delta} = G_\Delta(P_{J_\infty}\dot{w}, 1) \cdot h_{J,y,\delta} = G_\Delta(L_{J_\infty}\dot{w}, 1) \cdot h_{J,y,\delta}\,.$$

Note that $\Lambda(Z_{J,y,\delta})$ and $\Lambda'(Z_{J,y,\delta})$ are stable under the action of $G_\Delta$. Then we may assume that $x = (l_1\dot{w}, 1) \cdot h_{J,y,\delta}$ for some $l_1 \in L_{J_\infty}$. By [H1, 1.10(2)], $(P_{J_\infty})_\Delta \cdot (L_{J_\infty}, 1)x = (P_{J_\infty}, 1) \cdot x$, where $(P_{J_\infty})_\Delta = \{(g,g') \in P_{J_\infty} \times P_{J_\infty};\, g = g'\}$. For $u \in U_{P_{J_\infty}}$, $(u,1) \cdot x = (u'l, u') \cdot x$ for some $u' \in U_{P_{J_\infty}}$ and $l \in L_{J_\infty}$. Note that $(U_{P_{J_\infty}}, U_{P_{J_\infty}}) \cdot (l,1) \cdot x = (U_{P_{J_\infty}}l, 1) \cdot x$ and $(P_{J_\infty}, 1) \cdot x \cong U_{P_{J_\infty}} \times (L_{J_\infty}, 1) \cdot x$. Thus $(l,1) \cdot x = x$ and

$(U_{P_{J_\infty}},1)\cdot x\subset (P_{J_\infty})_\Delta\cdot x$. Therefore, for $(f,-f)\in\mu_{G\times G}(T_x^*(Z_{J,y,\delta}))$, we have that $(f,-f)(\mathrm{Lie}(U_{P_{J_\infty}}),0)=(f,-f)(\mathrm{Lie}(P_{J_\infty})_\Delta)=0$, i.e. $f\in\mathrm{Lie}(P_{J_\infty})$. In particular, $f$ is nilpotent if and only if the image of $f$ under $\mathrm{Lie}(P_{J_\infty})\to\mathrm{Lie}(P_{J_\infty})/\mathrm{Lie}(U_{P_{J_\infty}})\cong\mathrm{Lie}(L_{J_\infty})$ is nilpotent.

Hence $\mu_{G\times G}(\Lambda(Z_{J,y,\delta})\cap T_x^*(Z_{J,y,\delta}))$ consists of elements of the form $(u+l,-u-l)$ with $u\in\mathrm{Lie}(U_{P_{J_\infty}})$, $l$ nilpotent in $\mathrm{Lie}(L_{J_\infty})$ and $(u,-u)I_x=(l,-l)I_x=0$, where $I_x$ is the Lie subalgebra of the isotropic subgroup of $G\times G$ at point $x$.

Denote by $N_x$ the stalk at point $x$ of the conormal bundle $N^*_{Z^{\mathbf{s}}_{J,y,\delta}}(Z_{J,y,\delta})$. Since $Z^{\mathbf{s}}_{J,y,\delta}=G_\Delta(P_{J_\infty},1)h_{J,y,\delta}$, we have

$$\mu_{G\times G}(N_x)=\left\{(u,-u);\, u\in\mathrm{Lie}(U_{P_{J_\infty}}),\,(u,-u)I_x=0\right\}.$$

Let $p_x:T_x^*(Z_{J,y,\delta})\to T_x^*(Z^{\mathbf{s}}_{J,y,\delta})\cong T_x^*(Z_{J,y,\delta})/N_x$ be the obvious surjective map. Then $\Lambda(Z_{J,y,\delta})\cap T_x^*(Z_{J,y,\delta})=p_x^{-1}(p_x(\Lambda(Z_{J,y,\delta})\cap T_x^*(Z_{J,y,\delta})))$. Note that

$$I_x=\big\{(u_1+\mathrm{Ad}(l_1\dot w g_0)l,u_2+l)\,;\quad u_1\in\mathrm{Ad}(l_1\dot w\dot y^{-1})\,\mathrm{Lie}(U_{P_{J'}}),u_2\in\mathrm{Lie}(U_{P_J}),l\in\mathrm{Lie}(L_J)\big\}.$$

Thus, for $l\in\mathrm{Lie}(L_{J_\infty})$, $(l,-l)I_x=0$ if and only if $\mathrm{Ad}(l_1\dot w g_0)l=l$.

We identify $T_x^*(Z_{J,y,\delta})$ with $\mu_{G\times G}(T_x^*(Z_{J,y,\delta}))\subset\mathfrak{g}\times\mathfrak{g}$ and regard $T_x^*(Z^{\mathbf{s}}_{J,y,\delta})$ as a subspace of $(\mathfrak{g}\times\mathfrak{g})/N_x$. Set $M=\{(l,-l);l\in\mathrm{Lie}(L_{J_\infty}),\mathrm{Ad}(l_1\dot w g_0)l=l\}\subset\mathfrak{g}\times\mathfrak{g}$. Then

$$p_x(\Lambda(Z_{J,y,\delta})\cap T_x^*(Z_{J,y,\delta}))=(M+N_x)/N_x\,. \tag{1}$$

Now consider the commuting diagram

$$\begin{array}{ccc}(P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J,y,\delta} & \xrightarrow{i_1} & Z^{\mathbf{s}}_{J,y,\delta}\\ {\scriptstyle f'_\infty}\big\downarrow & & {\scriptstyle f_\infty}\big\downarrow\\ (P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J_\infty,y_\infty,\delta} & \xrightarrow{i_2} & Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}\,,\end{array}$$

where $i_1,i_2$ are inclusions and $f'_\infty$ is the restriction of $f_\infty$. Let $x'=f_\infty(x)\in Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}$. Since $f'_\infty$ is $P_{J_\infty}\times P_{J_\infty}$-invariant, we have the following commuting diagram

$$\begin{array}{ccc}T^*_{x'}\big((P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J_\infty,y_\infty,\delta}\big) & \xrightarrow{{}^1\mu_{P_{J_\infty}\times P_{J_\infty}}} & \mathrm{Lie}(P_{J_\infty})^*\times\mathrm{Lie}(P_{J_\infty})^*\\ {\scriptstyle (f'_\infty)^*}\big\downarrow & & {\scriptstyle id}\big\downarrow\\ T^*_{x}\big((P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J,y,\delta}\big) & \xrightarrow{{}^2\mu_{P_{J_\infty}\times P_{J_\infty}}} & \mathrm{Lie}(P_{J_\infty})^*\times\mathrm{Lie}(P_{J_\infty})^*,\end{array}$$

where ${}^1\mu_{P_{J_\infty}\times P_{J_\infty}}$ and ${}^2\mu_{P_{J_\infty}\times P_{J_\infty}}$ are the moment maps. Since the actions of $P_{J_\infty}\times P_{J_\infty}$ on $(P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J,y,\delta}$ and $(P_{J_\infty})_\Delta(L_{J_\infty}\dot w,1)\cdot h_{J_\infty,y_\infty,\delta}$ are transitive, ${}^1\mu_{P_{J_\infty}\times P_{J_\infty}}$ and ${}^2\mu_{P_{J_\infty}\times P_{J_\infty}}$ are injective.

Set $\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}) = \Lambda(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta}) \cap T^*_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$. Then

$$^2\mu_{P_{J_\infty}\times P_{J_\infty}}\big(i_2^*(\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})\big) = {}^1\mu_{P_{J_\infty}\times P_{J_\infty}}\big((f'_\infty)^* i_2^*(\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})\big) = M\,.$$

Here we identify $\mathrm{Lie}(P_{J_\infty})^*$ with $\mathrm{Lie}(P^-_{J_\infty})$ via the symmetric bilinear form. Moreover, $i_1^* f_\infty^* = (f'_\infty)^* i_2^*$ maps $\Lambda_{x'}(Z^{\mathbf{s}_\infty}_{J_\infty,y_\infty,\delta})$ bijectively onto its image. Therefore, ${}^1\mu_{P_{J_\infty}\times P_{J_\infty}}(i_1^*(\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta}))) = M$ and $i_1^*$ maps $\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta})$ bijectively onto its image. In other words,

$$\Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta}) = (M + N_x)/N_x\,. \tag{2}$$

Combining (1) and (2), $p_x(\Lambda(Z_{J,y,\delta}) \cap T^*_x(Z_{J,y,\delta})) = \Lambda(Z^{\mathbf{s}}_{J,y,\delta}) \cap T^*_x(Z^{\mathbf{s}}_{J,y,\delta})$. The proposition is proved.

COROLLARY 1.10. *The set of irreducible components of $\Lambda(Z_{J,y,\delta})$ is in natural bijection with $\sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y)\delta)} F(\mathbf{d}_{\mathbf{s}_\infty})$ (notation of 1.7, 1.8).*

**1.11** In the setup of 1.6 let $\Lambda(\bar G) = \mu^{-1}_{G\times G}(\mathcal{N}^-)$. We want to describe the variety $\Lambda(\bar G)$. As in [L2, 12.3], we have $\bar G = \sqcup_{J\subset \mathbf{I}} G_J$ where $G_J$ are the various $G\times G$-orbits in $\bar G$; moreover we may identify $G_J = T_J \backslash Z_{J,y_J,1}$ where $y_J$ is the longest element in $\mathbf{W}^J$ and $T_J$ is a torus acting freely on $Z_{J,y_J,1}$. Let $a_J : Z_{J,y_J,1} \to G_J$ be the canonical map.

For each $J$ let $\mu_{G\times G;J} : T^*G_J \to \mathfrak{g}^* \times \mathfrak{g}^*$ be the moment map of the restriction of the $G\times G$-action on $\bar G$ to $G_J$. Let $i_J : G_J \to \bar G$ be the inclusion. From the definitions, we have

(a) $\Lambda(\bar G) = \sqcup_{J\subset\mathbf{I}} (i_J)_\bigstar \mu^{-1}_{G\times G;J}(\mathcal{N}^-)$

and $\Lambda(Z_{J,y_J,1}) = a_J^\bigstar(\mu^{-1}_{G\times G;J}(\mathcal{N}^-))$.

Since $\Lambda(Z_{J,y_J,1})$ is a $T_J$-stable Lagrangian subvariety of $T^*(Z_{J,y_J,1})$ (see 1.9), it follows that $\mu^{-1}_{G\times G;J}(\mathcal{N}^-)$ is a Lagrangian subvariety of $T^*(G_J)$. Hence, using 1.1(c), we see that $(i_J)_\bigstar \mu^{-1}_{G\times G;J}(\mathcal{N}^-)$ is a Lagrangian subvariety of $T^*\bar G$. Using this and (a) we see that

(b) *$\Lambda(\bar G)$ is a Lagrangian subvariety of $T^*\bar G$.*

From the previous proof we see that

(c) *The set of irreducible components of $\Lambda(\bar G)$ is in natural bijection with $\sqcup_{J\subset\mathbf{I}} \sqcup_{\mathbf{s}\in S(J,\mathrm{Ad}(y_J))} F(\mathbf{d}_{\mathbf{s}_\infty})$ (notation of 1.7, 1.8).*

**1.12** Let $X = Z_{J,y,\delta}$ or $\bar G$. There is a well-defined map from the irreducible components of $\Lambda(X)$ to the nilpotent conjugacy classes of $\mathfrak{g}^*$ which sends the irreducible component $C$ of $\Lambda(X)$ to the nilpotent conjugacy class $\mathcal{O}$, where $\mu_{G\times G}(C) \cap \{(f,-f) \in \mathcal{O}\times\mathcal{O}\}$ is dense in $\mu_{G\times G}(C)$.

## References

[G] V. Ginzburg, Admissible modules on a symmetric space, Astérisque 173-174 (1989), 199–256.

[H1] X. He, The $G$-stable pieces of the wonderful compactification, math.RT/0412302.

[H2] X. He, The character sheaves on the group compactification, math.RT/0508068.

[KS] M. Kashiwara, P. Schapira, Sheaves on Manifolds, Springer-Verlag, Berlin, 1994.

[L1] G. Lusztig, Parabolic character sheaves I, Moscow Math. J. 4 (2004), 153–179.

[L2] G. Lusztig, Parabolic character sheaves II, Moscow Math. J. 4 (2004), 869–896.

[MV] I. Mirković, K. Vilonen, Characteristic varieties of character sheaves, Invent. Math. 93 (1988), 405–418.

[S] T.A. Springer, Intersection cohomology of $B \times B$-orbit closures in group compactifications, J. Alg. 258 (2002), 71–111.

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