

ON FROBENIUS SPLITTING OF ORBIT CLOSURES OF SPHERICAL SUBGROUPS IN FLAG VARIETIES

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ABSTRACT. Let H be a connected spherical subgroup of a semisimple algebraic group G . In this paper, we give a criterion for H -orbit closures in the flag variety of G to have nice geometric and cohomological properties. Our main tool is the method of Frobenius splitting and of global F-regularity.

1. INTRODUCTION

1.1. Let G be a semisimple algebraic group and let H denote a closed subgroup of G acting with only finitely many orbits on the flag variety G/B associated with G . The group H under this condition is called a spherical subgroup of G . The geometric and cohomological properties of the finitely many H -orbit closures in G/B are of importance in representation theory.

The case where H is a Borel subgroup has been studied in great detail. The H -orbit closures are in this case the set of Schubert varieties which have some remarkable properties: Schubert varieties are normal, Cohen-Macaulay and have rational singularities, all the higher cohomology groups of ample line bundles are zero, etc. They play an important role in representation theory.

Another important case is when H is a symmetric subgroup; i.e. when H is the set of fixed points of an involution of G . The classification and inclusion relation between the orbit closures have in this case been studied in great detail by Richardson and Springer [RS, RS2]. However, the singularities are considerably more complicated than in the case of Schubert varieties and the general picture is far from being fully understood. A non-normal example is constructed by Barbasch and Evens in [BE, 6.9]. A non-normal, non-Cohen-Macaulay example for a spherical H is constructed by Brion in [B1, Example 6].

1.2. In this paper, we give a criterion for H -orbit closures to have nice geometric and cohomological properties. Our main tool is the method of Frobenius splitting and of global F-regularity.

The notion of Frobenius splitting was introduced by Mehta and Ramanathan in [MR]. Any projective Frobenius split variety is weakly normal and the higher cohomology groups of ample line bundle are

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zero. The more restrictive notion of global F-regularity was recently introduced by K. Smith in [S]. Any (projective) globally F-regular variety is normal and Cohen-Macaulay and the higher cohomology groups of nef line bundles are zero.

The main result can be briefly stated as follows

Theorem 1.1. *Let H be a connected reductive subgroup of G and B denote a Borel subgroup of G such that $B_H = B \cap H$ is a Borel subgroup of H . Assume furthermore that (G, H) is a Donkin pair or that the characteristic of the ground field \mathbb{k} is sufficiently large. Let J be a subset of the set I of simple roots of G and let ρ_J (resp. $2\rho_H$) denote the sum of the fundamental weights within J (resp. the sum of the positive roots of H). Then*

(1) *If $2\rho_H - \rho_J$ is dominant for B_H , then ${}^{HP_J/B}$ admits a Frobenius splitting along an ample divisor that is compatible with all subvarieties of the form \overline{HBwB}/B for $w \in W_J$.*

(2) *If moreover $2\rho_H - \rho_J$ is dominant regular for B_H , then \overline{HBwB}/B is globally F-regular for all $w \in W_J$.*

Notice that we do not assume H to be spherical subgroup in the above theorem. However, in many cases the relevant orbit closures \overline{HBwB}/B coincide with closures of orbits under spherical subgroups. For example, if H is the trivial subgroup of G then the theorem applies for $J = I$. In this case the varieties \overline{HBwB}/B , for $w \in W$, are just the set of Schubert varieties. In particular, in this way we obtain the well known results that the flag variety admits a Frobenius splitting along an ample divisor which is compatible with all Schubert varieties and that any Schubert variety is globally F-regular.

Another special case is when (G, H) is in N. Ressayre's list of minimal rank pairs [Re]. In this case one finds that the flag variety admits a Frobenius splitting along an ample divisor that is compatible with all the H -orbit closures.

Notice that one cannot expect the flag variety G/B to be Frobenius split compatible with all the H -orbit closures for a given spherical subgroup H of G . For example, if $(G, H) = (SL_n, SO_n)$, then the scheme theoretic intersection of two H -orbit closures might not be a reduced scheme. Thus the desired Frobenius splitting cannot exist. For more details, see [B1, Introduction].

1.3. Let us make a short digression and discuss another criterion for H -orbit closures to have nice properties.

In [B1], Brion introduced *multiplicity-free* subvarieties of the flag variety. A subvariety is multiplicity-free if it is rationally equivalent to a linear combination of Schubert cycles with coefficients equal to either 0 or 1. In [B2] Brion proved that multiplicity-free subvarieties are normal, Cohen-Macaulay and have nice cohomological properties.

In a recent work [Kn], Knutson proved that given a multiplicity-free divisor X of the flag variety, there exists a Frobenius splitting on the flag variety that is compatible with X . It is still unknown if any multiplicity-free subvariety admits a Frobenius splitting.

The applications of the results in this paper include many multiplicity-free cases, but they also include some non multiplicity-free cases. See the Example in Section 8.2 and 8.3. It is interesting to compare the criterion in this paper with the multiplicity-free criterion.

1.4. The paper is organized as follows. In Section 2 we introduce notation. In Section 3 we give a short introduction to Frobenius splitting and global F-regularity. The main technical result (Theorem 4.1) is presented in Section 4. In Section 5, we discuss the surjectivity condition appearing in Theorem 4.1. In Section 6, we discuss some Frobenius splitting of the flag variety P_J/B , which will be used in Section 7. In Section 7, we prove the main result of this paper and discuss some applications. In Section 8, we discuss some examples and non examples.

2. NOTATION

2.1. Throughout this paper G will denote a connected semisimple and simply connected linear algebraic group over an algebraically closed field \mathbb{k} . Within G we will fix a Borel subgroup B and a maximal torus $T \subset B$.

The set of roots of G determined by T will be denoted by R and the set of positive roots determined by (B, T) will be denoted by R^+ . The simple roots are denoted by α_i , $i \in I$. For $i \in I$, let s_i be the corresponding simple reflection and ω_i be the corresponding fundamental weight. We let ρ denote half the sum of the positive roots or, alternatively defined, the sum of the fundamental weights.

The Weyl group $W = N_G(T)/T$ is generated by the simple reflections s_i , for $i \in I$. The length of $w \in W$ will be denoted by $l(w)$, and the element of maximal length will be denoted by w_0 . By abuse of notation w will sometimes both denote an element $w \in W$ and a corresponding element within the normalizer $N_G(T)$. The set of Schubert varieties in G/B is indexed by the elements in W . We use the notation $X(w)$ for the Schubert variety defined as the closure of BwB/B .

2.2. For $J \subset I$, let $P_J \supset B$ be the corresponding standard parabolic subgroup and $L_J \supset T$ be the corresponding Levi subgroup of P_J . Let U_J be the unipotent radical of P_J . Let W_J denote the parabolic subgroup of W generated by s_j for $j \in J$ and w_0^J denote the element of maximal length in W_J . Let $\rho_J = \sum_{j \in J} \omega_j$. The set of positive roots determined by $B \cap L_J$ in L_J is denoted by R_J^+ .

2.3. By H we will denote a connected reductive subgroup of G . We will assume that B and T are chosen such that $B_H = H \cap B$ is a Borel subgroup of H and $T_H = H \cap T$ is a maximal torus of H .

The roots R_H of H determined by T_H is the set of nonzero restrictions of the roots in R . We consider the character group $X^*(T_H)$ of T_H as embedded inside the tensorproduct $X^*(T_H)_\mathbb{Q} = X^*(T_H) \otimes_{\mathbb{Z}} \mathbb{Q}$. By $\rho_H \in X^*(T_H)_\mathbb{Q}$ we then denote half the sum of the positive roots of H .

2.4. For any integral weight λ of T , let $\mathbb{k}_{-\lambda}$ be the one-dimensional representation of B with weight $-\lambda$ and $\mathcal{L}(\lambda) = G \times_B \mathbb{k}_{-\lambda}$ be the corresponding G -linearized line bundle on G/B . Let

$$\nabla(\lambda) = \text{Ind}_B^G(\mathbb{k}_{-\lambda}) = \text{H}^0(G/B, \mathcal{L}(\lambda)),$$

denote the dual Weyl G -module with lowest weight $-\lambda$ (if λ is dominant). The restriction of $\mathcal{L}(\lambda)$ to P_J/B will be denoted by $\mathcal{L}_J(\lambda)$ and we define

$$\nabla^J(\lambda) = \text{Ind}_B^{P_J}(\mathbb{k}_{-\lambda}) = \text{H}^0(P_J/B, \mathcal{L}_J(\lambda)).$$

When ν is an integral T_H -weight we similarly write $\mathcal{L}_H(\nu) = H \times_{B_H} \mathbb{k}_{-\nu}$ and

$$\nabla^H(\nu) = \text{Ind}_{B_H}^H(\mathbb{k}_{-\nu}) = \text{H}^0(H/B_H, \mathcal{L}_H(\nu)).$$

When \mathbb{k} is a field of positive characteristic $p > 0$ then the G -module $\text{St} = \nabla((p-1)\rho)$ will play a special role. This module is called the Steinberg module. The Steinberg module is known to be an irreducible and self-dual G -module. When it makes sense we let St_H denote the Steinberg module of H .

2.5. By a variety we mean a reduced and separated scheme of finite type over \mathbb{k} . In particular, we allow a variety to have several irreducible components. When X is a B_H -variety we define an action of B_H on $H \times X$ by $b \cdot (h, x) = (hb^{-1}, b \cdot x)$. The quotient is then denoted by $H \times_{B_H} X$ or sometimes by X_H . This way we obtain an equivalence between the set of quasi-coherent B_H -linearized sheaves on X and quasi-coherent H -linearized sheaves on X_H . The H -linearized sheaf on X_H corresponding to a B_H -linearized sheaf \mathcal{F} on X is denoted by $\text{Jnd}_{B_H}^H(\mathcal{F})$. This notation is explained by the H -equivariant identity

$$\text{H}^0(X_H, \text{Jnd}_{B_H}^H(\mathcal{F})) \simeq \text{Ind}_{B_H}^H(\text{H}^0(X, \mathcal{F})).$$

The sheaf $\text{Jnd}_{B_H}^H(\mathcal{F})$ is characterized as the H -linearized sheaf satisfying that its B_H -linearized restriction to

$$X \simeq \{e\} \times X \subseteq H \times_{B_H} X,$$

is \mathcal{F} .

3. FROBENIUS SPLITTING

3.1. In this section \mathbb{k} denotes an algebraically closed field of positive characteristic p . Let X be a scheme of finite type over \mathbb{k} . The *absolute Frobenius morphism* $F : X \rightarrow X$ on X is the morphism of schemes which on the level of points is the identity map and where the associated map of sheaves

$$F^\sharp : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X,$$

is the p -th power map. Define $\text{End}_F(X)$ to be the \mathbb{k} -vector space which as an abelian group equals $\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, but where the \mathbb{k} -structure is twisted by the map $a \mapsto a^{\frac{1}{p}}$. A *Frobenius splitting* of X is then an element s in $\text{End}_F(X)$ such that the composition $s \circ F^\sharp$ is the identity map.

3.2. Let $\mathbb{k}[X]$ denote the space of global regular functions on X . The evaluating of an element s of $\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ at the constant global function 1 on X defines an element in $\mathbb{k}[X]$. This way we obtain a \mathbb{k} -linear map

$$(1) \quad \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \rightarrow \mathbb{k}[X].$$

Composing (1) with the Frobenius morphism on $\mathbb{k}[X]$ is then a \mathbb{k} -linear map

$$(2) \quad \text{ev}_X : \text{End}_F(X) \rightarrow \mathbb{k}[X],$$

which is called the *evaluation map*.

3.3. Let \mathcal{M} denote a line bundle on X and define $\text{End}_F^{\mathcal{M}}(X)$ to be the \mathbb{k} -vector space which as an abelian group is $\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{M}, \mathcal{O}_X)$ but where the \mathbb{k} -structure is twisted by the map $a \mapsto a^{\frac{1}{p}}$. A *Frobenius \mathcal{M} -splitting* of X is an element $s_{\mathcal{M}}$ of $\text{End}_F^{\mathcal{M}}(X)$ for which there exists a global section \mathfrak{m} of \mathcal{M} such that the composed map

$$(3) \quad F_*\mathcal{O}_X \xrightarrow{F_*\mathfrak{m}} F_*\mathcal{M} \xrightarrow{s_{\mathcal{M}}} \mathcal{O}_X,$$

defines a Frobenius splitting. The construction of (3) from \mathfrak{m} and $s_{\mathcal{M}}$ is a special case of a general \mathbb{k} -linear morphism

$$(4) \quad \text{End}_F^{\mathcal{M}}(X) \otimes H^0(X, \mathcal{M}) \rightarrow \text{End}_F(X).$$

Notice that for (4) to be \mathbb{k} -linear it is necessary that the \mathbb{k} -structure on $\text{End}_F^{\mathcal{M}}(X)$ is chosen in the given way.

In case X is a smooth variety one has the following canonical \mathbb{k} -linear identification

$$(5) \quad \text{End}_F^{\mathcal{M}}(X) \simeq H^0(X, \omega_X^{1-p} \otimes \mathcal{M}^{-1}),$$

where ω_X denotes the dualizing sheaf of X . In this setting the map (4) is just the multiplication map

$$H^0(X, \omega_X^{1-p} \otimes \mathcal{M}^{-1}) \otimes H^0(X, \mathcal{M}) \rightarrow H^0(X, \omega_X^{1-p}).$$

3.4. Let Y denote a closed subscheme of X with sheaf ideals denoted by \mathcal{J}_Y . The subvector space of $\text{End}_F^{\mathcal{M}}(X)$ consisting of the elements $s_{\mathcal{M}}$ satisfying

$$s_{\mathcal{M}}(\mathcal{M} \otimes \mathcal{J}_Y) \subseteq \mathcal{J}_Y,$$

is denoted by $\text{End}_F^{\mathcal{M}}(X, Y)$. If X_i , $i \in \mathfrak{I}$, is a collection of closed subschemes of X then we use the notation $\text{End}_F^{\mathcal{M}}(X, \{X_i\}_{i \in \mathfrak{I}})$ for the intersection of the $\text{End}_F^{\mathcal{M}}(X, X_i)$ for $i \in \mathfrak{I}$. When $\mathcal{M} = \mathcal{O}_X$ we remove \mathcal{M} from all of the above notation.

3.5. Let $s_{\mathcal{M}}$ in $\text{End}_F^{\mathcal{M}}(X, \{X_i\}_{i \in \mathfrak{I}})$ define a Frobenius \mathcal{M} -splitting of X . In this case we say that $s_{\mathcal{M}}$ is compatible with the subschemes X_i , $i \in \mathfrak{I}$. The following result is standard (see e.g. [HT2, Lemma 3.1])

Lemma 3.1. *Let Y and Z be closed subvarieties in X and let s be a global section of $\text{End}_F^{\mathcal{M}}(X, \{Y, Z\})$.*

- (1) $s_{\mathcal{M}} \in \text{End}_F^{\mathcal{M}}(X, Y_1)$ for every irreducible component Y_1 of Y .
- (2) If $Y \cap Z$ denotes the scheme theoretic intersection then $s_{\mathcal{M}}$ is contained in $\text{End}_F^{\mathcal{C}}(X, Y \cap Z)$.

3.6. Let R denote a localizations of a finitely generated \mathbb{k} -algebra and assume, for simplicity, that R is an integral domain. In the following we use the notation $F_* R^e$, $e \in \mathbb{N}$, to denote the R -module which as an abelian group is just R but where the R -structure is twisted by the iterated Frobenius map $r \mapsto r^{p^e}$.

The following notion was introduced by M. Hochster and C. Huneke.

Definition 3.2. The ring R is said to be *strongly F -regular* if for each $r \in R$ there exists an $e \in \mathbb{N}$ and an R -linear map

$$F_*^e R \rightarrow R,$$

which maps r to 1.

Strongly F -regular rings have nice geometric properties; e.g. they are normal and Cohen-Macaulay. It is known that a ring R is strongly F -regular if and only if all its local rings are strongly F -regular. We define an irreducible variety X to be strongly F -regular if all its local rings are strongly F -regular. In that case the coordinate ring of any open affine subvariety of X is also strongly F -regular. The Schubert varieties $X(w)$ are examples of strongly F -regular varieties [LRT].

We now recall the following important notion introduced by Karen Smith [S].

Definition 3.3. Let X denotes an irreducible projective variety and \mathcal{M} denote a ample line bundle on X . If the *section ring*

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{M}^n),$$

is strongly F -regular then X is said to be *globally F -regular*.

It should be noticed that the definition above is independent of the chosen ample line bundle \mathcal{M} . For later use we observe the following result [S, Thm.3.10]

Lemma 3.4. *Let X denote an irreducible projective and strongly F -regular variety. If X admits a Frobenius \mathcal{M} -splitting by an ample line bundle \mathcal{M} , then X is globally F -regular.*

Another useful fact, observed in [LRT, Lemma 1.2], is the following

Lemma 3.5. *Let $f : X \rightarrow Y$ denote a morphism of projective varieties. Assume that the induced map*

$$\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X,$$

is an isomorphism and that X is globally F -regular. Then Y is also globally F -regular.

To apply this result we will later use the following fact

Lemma 3.6. *Let $f : X \rightarrow Y$ denote a surjective morphism of projective varieties. Let X' denote a closed subvariety of X and let $Y' = f(X')$ denote its image. Assume that the map*

$$\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X,$$

induced by f , is an isomorphism and let \mathcal{M} denote an ample line bundle on X . If X admits a Frobenius \mathcal{M} -splitting compatible with X' then the map

$$\mathcal{O}_{Y'} \rightarrow f_*\mathcal{O}_{X'},$$

induced by f , is also an isomorphism.

Proof. Let \mathcal{L} denote an ample line bundle on Y . By [BK, Lemma 3.3.3(b)] it suffices to prove that the induced morphism

$$H^0(Y', \mathcal{L}^n) \rightarrow H^0(X', f^*\mathcal{L}^n),$$

is surjective for sufficiently large n . By the assumption the corresponding statement on X and Y are satisfied. Commutative of the following diagram

$$(6) \quad \begin{array}{ccc} H^0(Y, \mathcal{L}^n) & \xrightarrow{f^*} & H^0(X, f^*\mathcal{L}^n) \\ \text{res}_{Y'}^Y \downarrow & & \downarrow \text{res}_{X'}^X \\ H^0(Y', \mathcal{L}^n) & \xrightarrow{f^*} & H^0(X', f^*\mathcal{L}^n) \end{array}$$

then implies that it suffices to prove that the restriction map

$$H^0(X, f^*\mathcal{L}^n) \rightarrow H^0(X', f^*\mathcal{L}^n),$$

is surjective. As $f^*\mathcal{L}^n$ is globally generated this follows by the ampleness of \mathcal{M} and the assumption that X admits a Frobenius \mathcal{M} -splitting compatible with X' (cf. [BK, Thm.1.4.8(ii)]). \square

4. FROBENIUS SPLITTING OF $H \times_{B_H} X$

In this section we let \mathcal{M} denote a B -linearized line bundle on an irreducible projective B -variety X . The notation \mathcal{M}_H is used to denote the corresponding H -linearized line bundle on $X_H = H \times_{B_H} X$. A linearized line bundle on H/B_H is determined by a B_H -character ν . The pull-back of such a line bundle to $H \times_{B_H} X$ is denoted by $\mathcal{O}(-\nu)$. The tensor product of $\mathcal{O}(\nu)$ and \mathcal{M}_H will be denoted by $\mathcal{M}_H(\nu)$. Recall that we have an H -equivariant identification

$$(7) \quad H^0(X_H, \mathcal{M}_H(\nu)) \simeq \text{Ind}_{B_H}^H (H^0(X, \mathcal{M}) \otimes \mathbb{k}_{-\nu}).$$

We will now consider the following setup : let λ denote a dominant weight of G and $X_i, i \in \mathcal{J}$, denote a collection of closed subvarieties in X . By

$$(8) \quad \theta : \mathbb{k}_\lambda \rightarrow \text{End}_F^{\mathcal{M}}(X, \{X_i\}),$$

and

$$(9) \quad \phi : \nabla(\lambda) \rightarrow H^0(X, \mathcal{M}).$$

we denote B -equivariant maps. By

$$(\theta, \phi) : \nabla(\lambda) \otimes \mathbb{k}_\lambda \rightarrow \text{End}_F(X, \{X_i\}),$$

we denote the map induced by θ, ϕ and the natural B -equivariant morphism

$$\text{End}_F^{\mathcal{M}}(X, \{X_i\}) \otimes H^0(X, \mathcal{M}) \rightarrow \text{End}_F(X, \{X_i\}).$$

Then we can formulate

Theorem 4.1. *Assume that the following conditions are satisfied*

- (1) *(θ, ϕ) contains a Frobenius splitting of X in its image.*
- (2) *The T_H -character $2(p-1)\rho_H - \lambda|_{T_H}$ is dominant.*
- (3) *The H -equivariant restriction morphism*

$$\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H}),$$

is surjective.

Then $H \times_{B_H} X$ admits a Frobenius $\mathcal{M}_H(2(p-1)\rho_H - \lambda|_{T_H})$ -splitting which is compatible with the subvarieties $H \times_{B_H} X_i, i \in \mathcal{J}$.

Proof. Recall that the B_H -character associated to the dualizing sheaf on H/B_H equals $2\rho_H$. In particular, we have an H -equivariant identification (cf. [HT2, Sect. 5])

$$\text{End}_F^{\mathcal{M}_H(\nu)}(X_H, \{(X_i)_H\}) \simeq \text{Ind}_{B_H}^H (\text{End}_F^{\mathcal{M}}(X, \{X_i\}) \otimes \mathbb{k}_{-\lambda}),$$

where ν denotes the B_H -character $2(p-1)\rho_H - \lambda|_{T_H}$, and where we have used the notation $(X_i)_H$ for $H \times_{B_H} X_i$. By application of Frobenius reciprocity this show that θ induces an H -equivariant morphism

$$(10) \quad \mathbb{k} \rightarrow \text{End}_F^{\mathcal{M}_H(\nu)}(X_H, \{(X_i)_H\}).$$

Let g denote a nonzero element inside the image of (10). It then suffices to find a global section s of $\mathcal{M}_H(\nu)$ such that $g(s)$ is nonzero.

To find s we first apply Frobenius reciprocity to (9) to obtain

$$(11) \quad \nabla(\lambda) \rightarrow \text{Ind}_{B_H}^H(\mathbb{H}^0(X, \mathcal{M})) \simeq \mathbb{H}^0(X_H, \mathcal{M}_H),$$

which combined with the natural morphism

$$(12) \quad \nabla^H(\nu) \rightarrow \mathbb{H}^0(X_H, \mathcal{O}(\nu)),$$

defines an H -equivariant morphism

$$(13) \quad \nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow \mathbb{H}^0(X_H, \mathcal{M}_H(\nu)).$$

We claim that we can find the desired s inside the image of (13).

Applying the natural morphism

$$\text{End}_F^{\mathcal{M}_H(\nu)}(X_H, \{(X_i)_H\}) \otimes \mathbb{H}^0(X_H, \mathcal{M}_H) \rightarrow \text{End}_F^{\mathcal{O}(\nu)}(X_H, \{(X_i)_H\}),$$

we see that g and (11) defines an H -equivariant map

$$(14) \quad \Phi : \nabla(\lambda) \rightarrow \text{End}_F^{\mathcal{O}(\nu)}(X_H, \{(X_i)_H\}).$$

By construction (14) is the map induced by (θ, ϕ) and the identification

$$\text{End}_F^{\mathcal{O}(\nu)}(X_H, \{(X_i)_H\}) \simeq \text{Ind}_{B_H}^H(\text{End}_F(X, \{X_i\}) \otimes \mathbb{k}_{-\lambda}),$$

in particular, the following diagram is commutative

$$(15) \quad \begin{array}{ccc} \nabla(\lambda) & \xrightarrow{\Phi} & \text{Ind}_{B_H}^H(\text{End}_F(X, \{X_i\}) \otimes \mathbb{k}_{-\lambda}) & \xrightarrow{\text{Ind}_{B_H}^H(\text{ev}_X)} & \nabla^H(\lambda|_{T_H}) \\ & \searrow^{(\theta, \phi) \otimes \mathbb{k}_{-\lambda}} & \downarrow & & \downarrow \\ & & \text{End}_F(X, \{X_i\}) \otimes \mathbb{k}_{-\lambda} & \xrightarrow{\text{ev}_X} & \mathbb{k}_{-\lambda} \end{array}$$

Notice that the lower horizontal part of the diagram (15) is a B -equivariant morphism and thus it must, up to a constant, coincide with the projection map onto the lowest weight space of $\nabla(\lambda)$. By assumption (1) this map is nonzero. Thus the composed upper horizontal morphism $\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H})$ must, up to a non-zero constant, be the natural restriction map. In particular, the composed upper horizontal map $\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H})$ is surjective by assumption (3).

Consider next the natural morphism

$$(16) \quad \text{End}_F^{\mathcal{O}(\nu)}(X_H, \{(X_i)_H\}) \otimes \mathbb{H}^0(X_H, \mathcal{O}(\nu)) \rightarrow \text{End}_F(X_H, \{(X_i)_H\}).$$

Composing (14) with (12) and Φ defines the following commutative diagram (17)

$$\begin{array}{ccc}
\nabla(\lambda) \otimes \nabla^H(\nu) & & \\
\Phi \otimes \mathbf{1} \downarrow & \searrow \psi & \\
\text{End}_F^{\mathcal{O}(\nu)}(X_H, \{(X_i)_H\}) \otimes \nabla^H(\nu) & \longrightarrow & \text{End}_F(X_H, \{(X_i)_H\}) \\
\text{Ind}_{B_H}^H(\text{ev}_X) \otimes \mathbf{1} \downarrow & & \text{Ind}_{B_H}^H(\text{ev}_X) \downarrow \quad \searrow \text{ev}_{X_H} \\
\nabla^H(\lambda|_{T_H}) \otimes \nabla^H(\nu) & \xrightarrow{m} & \nabla^H(2(p-1)\rho_H) \xrightarrow{\text{ev}_{H/B_H}} \mathbb{k}
\end{array}$$

where m is the multiplication map and ψ is the map induced by g . In this notation we have to show that the image of ψ contains a Frobenius splitting or, equivalently, that the map

$$\nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow \mathbb{k},$$

from the upper left corner in (17) to the lower right corner is surjective. First of all the composed vertical map in (17) is surjective by the above observations. Moreover, m is surjective by assumption (2) and [RR, Thm. 3] while ev_{H/B_H} is surjective because H/B_H admits a Frobenius splitting. This ends the proof. \square

Remark. The statement of Theorem 4.1 provides us with a morphism

$$(18) \quad F_*\mathcal{M}_H(2(p-1)\rho_H - \lambda|_{T_H}) \xrightarrow{g} \mathcal{O}_{X_H},$$

and a global section s of the line bundle $\mathcal{M}_H(2(p-1)\rho_H - \lambda|_{T_H})$ such that the composition of (18) with

$$(19) \quad F_*\mathcal{O}_{X_H} \xrightarrow{F_*s} F_*\mathcal{M}_H(2(p-1)\rho_H - \lambda|_{T_H}),$$

defines a Frobenius splitting of X_H . Actually the proof of Theorem 4.1 provides us with more precise information. It defines three H -equivariant maps ($\nu = 2(p-1)\rho_H - \lambda$):

$$\begin{aligned}
f_1 &: \nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow \text{End}_F(X_H, \{(X_i)_H\}), \\
f_2 &: \nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow H^0(X_H, \mathcal{M}_H(\nu)), \\
f_3 &: \nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow k.
\end{aligned}$$

related in the following way: let $s = f_2(x)$, for some $x \in \nabla(\lambda) \otimes \nabla^H(\nu)$. Then $g \circ F_*s = f_1(x)$ defines a Frobenius splitting of X_H up to a nonzero constant if and only if $f_3(x)$ is nonzero. Moreover, the map f_3 may be explicitly described as the composed morphism

$$\nabla(\lambda) \otimes \nabla^H(\nu) \rightarrow \nabla^H(\lambda|_{T_H}) \otimes \nabla^H(\nu) \xrightarrow{m} \nabla(2\rho_H(p-1)) \xrightarrow{\text{ev}_{H/B_H}} \mathbb{k},$$

induced by the restriction map $\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H})$, while f_2 is the tensorproduct of (11) and (12).

It follows that we may choose s to be a product of the form $s_1 s_2$, where s_1 and s_2 denote global sections of the line bundles \mathcal{M}_H and $\mathcal{O}(2(p-1)\rho_H - \lambda|_{T_H})$ respectively. Moreover, s_1 can be chosen inside the image of (11) or, even more specific, if $\nabla(\lambda)$ is generated by an element v as an H -module then s_1 can be chosen to be the image of v under (11). Part of the outcome of this is that (19) factors through the morphism

$$(20) \quad F_* \mathcal{O}_{X_H} \xrightarrow{F_* s_1} F_* \mathcal{M}_H$$

and consequently X_H also admits a Frobenius \mathcal{M}_H -splitting compatible with all the subvarieties $(X_i)_H$. Even though this conclusion seems weaker than Theorem 4.1 it will be useful for us later.

Corollary 4.2. *Assume that H is semisimple and simply connected and that $(p-1)\rho_H - \lambda|_{T_H}$ is dominant as a T_H -weight. Then there exists a B_H -equivariant map*

$$\mathrm{St}_H \otimes (p-1)\rho_H \rightarrow \mathrm{End}_F(X_H, \{(X_i)_H\}),$$

containing a Frobenius splitting in its image; i.e. X_H admits a B_H -canonical Frobenius splitting compatible splitting each $(X_i)_H$ (see [M]).

Proof. With notation as in the remark above we have, by assumption, a surjective multiplication map

$$\nabla^H((p-1)\rho_H - \lambda|_{T_H}) \otimes \nabla^H((p-1)\rho_H) \rightarrow \nabla^H(\nu).$$

Composing this map with f_1 defines a morphism

$$\nabla(\lambda) \otimes \nabla^H((p-1)\rho_H - \lambda|_{T_H}) \otimes \mathrm{St}_H \rightarrow \mathrm{End}_F(X_H, \{(X_i)_H\}),$$

and by the description of f_3 it suffices to find a B_H semi-invariant element v in

$$\nabla(\lambda) \otimes \nabla^H((p-1)\rho_H - \lambda|_{T_H}),$$

mapping to a (nonzero) highest weight vector under the natural map

$$\nabla(\lambda) \otimes \nabla^H((p-1)\rho_H - \lambda|_{T_H}) \rightarrow \nabla^H((p-1)\rho_H).$$

Let v_- denote a lowest weight vector in $\nabla(\lambda)$ and w_0^H denote the longest element in the Weyl group of H . Then $w_0^H v_-$ will be B_H semi-invariant. Let now v^+ denote a highest weight vector in $\nabla^H((p-1)\rho_H - \lambda|_{T_H})$. Then the product $v = v^+(w_0^H v_-)$ has the desired property. \square

Lemma 4.3. *Assume that X is contained in a G -variety Z and that the line bundle \mathcal{M} is the restriction of an ample G -linearized line bundle \mathcal{M}_Z on Z . Then $\mathcal{M}_H(\nu)$ is an ample line bundle for every regular dominant weight ν of H .*

Proof. Consider the H -equivariant morphism

$$\psi : X_H = H \times_{B_H} X \rightarrow Z,$$

defined by

$$\psi(h, x) = h \cdot x.$$

The pull-back $\psi^*(\mathcal{M}_Z)$ is then an H -linearized line bundle on X_H . Consider X as a B_H -stable subvariety of X_H in the natural way. Then, by assumption, the restriction of $\psi^*(\mathcal{M}_Z)$ to X coincides with the B_H -linearized line bundle \mathcal{M} on X . In particular, \mathcal{M}_H must coincide with $\psi^*(\mathcal{M}_Z)$ as H -linearized line bundles.

As Z is an H -variety we have an identification

$$(21) \quad \begin{aligned} H \times_{B_H} Z &\simeq {}^H/B_H \times Z, \\ (h, z) &\mapsto (hB_H, h \cdot z). \end{aligned}$$

Moreover, by the assumptions on ν and \mathcal{M}_Z , the external tensor product $\mathcal{L}_H(\nu) \boxtimes \mathcal{M}_Z$ is an ample line bundle on ${}^H/B_H \times Z$. The conclusion now follows as $\mathcal{M}_H(\nu)$ is the pull-back of $\mathcal{L}_H(\nu) \boxtimes \mathcal{M}_Z$ under the closed inclusion

$$H \times_{B_H} X \subseteq H \times_{B_H} Z,$$

composed with the identification (21) above \square

Corollary 4.4. *Consider a setup as in Theorem 4.1. Assume, moreover, that $2(p-1)\rho_H - \lambda|_{T_H}$ is a regular weight of H and that \mathcal{M} is the restriction of a G -linearized ample line bundle on a G -variety which contains X as a closed B -stable subvariety. If X is a strongly F -regular projective variety then $H \times_{B_H} X$ is globally F -regular.*

Proof. By Lemma 4.3 and Theorem 4.1 we know that X_H admits a Frobenius splitting along an ample divisor. Moreover, as X is strongly F -regular the same is the case for X_H . Now apply Lemma 3.4. \square

5. SURJECTIVITY CONDITION

In this section, we discuss one of the conditions in Theorem 4.1 about the surjectivity of the restriction map

$$(22) \quad \nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H}).$$

The first observation is the following

Lemma 5.1. *Assume that the restriction map (22) is surjective for all the fundamental weights ω_i . Then the restriction map (22) is surjective for any dominant weight λ .*

Proof. We may assume that $\lambda = \sum_{i \in I} m_i \omega_i$, where $m_i \geq 0$. Consider the following commutative diagram

$$\begin{array}{ccc} \nabla(\omega_1)^{\otimes m_1} \otimes \cdots \otimes \nabla(\omega_n)^{\otimes m_n} & \xrightarrow{m} & \nabla(\lambda) \\ f_1^{\otimes m_1} \otimes \cdots \otimes f_n^{\otimes m_n} \downarrow & & \downarrow f \\ \nabla^H(\omega_1|_{T_H})^{\otimes m_1} \otimes \cdots \otimes \nabla^H(\omega_n|_{T_H})^{\otimes m_n} & \xrightarrow{m'} & \nabla^H(\lambda|_{T_H}), \end{array}$$

where f, f_1, \dots, f_n are restriction maps and m, m' are multiplication maps. By [RR, Theorem 3], m and m' are surjective and by assumption the left vertical map is also surjective. This proves the claim. \square

The above results applies in case $\nabla^H(\omega_1|_{T_H}), \dots, \nabla^H(\omega_n|_{T_H})$ are irreducible H -modules. In particular, it applies in case the characteristic p of \mathbb{k} is sufficiently large; e.g.

Lemma 5.2. *Assume that $\langle \rho_H + \omega_i |_{T_H}, \beta^\vee \rangle \leq p$ for all fundamental weights ω_i of G and any positive root β of H . Then the restriction map (22) is surjective for all dominant weights λ of T .*

5.1. Below we give a criterion on the surjectivity of (22) valid for all characteristics. We first recall the definition and some known results on Donkin pairs.

An ascending chain

$$0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = V$$

of submodules of a G -module V is called a *good filtration* if for any i , $V_i/V_{i-1} \cong \bigoplus_\lambda \nabla(\lambda) \otimes_{\mathbb{k}} A(\lambda, j)$ for some trivial G -modules $A(\lambda, j)$, where λ runs over the set of dominant integral weights of T .

We say that (G, H) is a *Donkin pair* if for any G -module M with a good filtration, the H -module $\text{res}_H^G(M)$ also has a good filtration. The following are some examples of Donkin pairs that will be used in this paper :

(1) If H is a Levi subgroup of G , then (G, H) is a Donkin pair. This is proved by Donkin in [Do] for almost all cases and later by Mathieu in [M] in full generality.

(2) If H is the centralizer of a graph automorphism of G or the centralizer of an involution of G and the characteristic of \mathbb{k} is at least 3, then (G, H) is a Donkin pair. This is conjectured by Brundan in [Bru] and proved by Van der Kallen in [V].

Lemma 5.3. *Let (G, H) be a Donkin pair. Let $\lambda \in X^*(T)$ be a dominant weight. Then the restriction map $\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H})$ is surjective for all dominant weights λ of T .*

Proof. By definition, $\nabla(\lambda)$ has a good filtration. Hence $\text{res}_H^G \nabla(\lambda)$ also has a good filtration

$$0 = M_0 \subset \cdots \subset M_{n-1} \subset M_n = \text{res}_H^G \nabla(\lambda).$$

As $\text{res}_H^G \nabla(\lambda)$ is finite dimensional we may furthermore assume that the quotients M_i/M_{i-1} are isomorphic to $\nabla^H(\nu_i)$ for certain dominant T_H -weights ν_i . Moreover, as $-\lambda$ is the (unique) lowest weight vector of $\nabla(\lambda)$ we must have $\nu_i \leq \lambda|_{T_H}$.

Now use that $\nabla(\lambda) \rightarrow \nabla^H(\lambda|_{T_H})$ is nonzero to find a minimal j such that the induced map $M_j \rightarrow \nabla^H(\lambda|_{T_H})$ is nonzero. In particular, we obtain a nonzero map

$$(23) \quad \nabla^H(\nu_j) \rightarrow \nabla^H(\lambda|_{T_H}).$$

By Frobenius reciprocity this implies that $\nu_j \geq \lambda|_{T_H}$ and thus that $\nu_j = \lambda|_{T_H}$. Another use of Frobenius reciprocity now implies that (23) is the identity map which suffices to end the proof. \square

6. FROBENIUS SPLITTING OF P_J/B

We eventually want to apply the result in Section 4 to the case where X is B -variety of the form P_J/B . In particular, we need a good description of the Frobenius splitting properties of P_J/B . As a variety P_J/B is just the flag variety associated to the Levi subgroup L_J and as such we already have a detailed knowledge about its Frobenius splitting properties. The aim of this section is to formulate this knowledge in a B -equivariant way.

Lemma 6.1.

$$\sum_{\alpha \in R_J^+} \alpha = \rho_J - w_0^J \rho_J.$$

Proof. Define

$$\lambda^J = \sum_{\beta \in R^+ \setminus R_J^+} \beta, \quad \lambda_J = \sum_{\alpha \in R_J^+} \alpha.$$

Then $2\rho = \lambda^J + \lambda_J$. Now recall the following identities

$$w_0^J(R_J^+) = -R_J^+,$$

$$w_0^J(R^+ \setminus R_J^+) = R^+ \setminus R_J^+.$$

It follows that

$$2w_0^J \rho = w_0^J(\lambda^J + \lambda_J) = \lambda^J - \lambda_J.$$

and thus

$$\rho - w_0^J \rho = \lambda_J.$$

On the other hand

$$w_0^J \rho = w_0^J(\rho_J + \rho_{I \setminus J}) = w_0^J \rho_J + \rho_{I \setminus J},$$

and thus

$$\lambda_J = \rho - w_0^J \rho = \rho_J - w_0^J \rho_J.$$

□

The trivial P_J -linearization on $\mathcal{O}_{P_J/B}$ induces a P_J -linearization of the line bundle $\omega_{P_J/B}$ which is associated to B -character

$$\sum_{\alpha \in R_J^+} \alpha.$$

In particular, we have a P_J -equivariant identification

$$(24) \quad \nabla^J((p-1) \sum_{\alpha \in R_J^+} \alpha) = H^0(P_J/B, \omega_{P_J/B}^{1-p}) \simeq \text{End}_F(P_J/B).$$

By Lemma 6.1 this leads to the following central morphism

Proposition 6.2. *There exists a surjective P_J -equivariant morphism*

$$(25) \quad \nabla^J((p-1)\rho_J) \otimes \nabla^J((1-p)w_0^J\rho_J) \rightarrow \text{End}_F(P_J/B).$$

Composing (25) with the evaluation map defines a P_J -equivariant map

$$\nabla^J((p-1)\rho_J) \otimes \nabla^J((1-p)w_0^J\rho_J) \rightarrow k,$$

which defines an P_J -equivariant isomorphism

$$(26) \quad \nabla^J((p-1)\rho_J) \rightarrow \nabla^J((1-p)w_0^J\rho_J)^*,$$

between irreducible P_J -representations.

Proof. The map (25) is just the surjective multiplication map

$$\nabla^J((p-1)\rho_J) \otimes \nabla^J((1-p)w_0^J\rho_J) \rightarrow \nabla^J((p-1)(\rho_J - w_0^J\rho_J)),$$

composed with the identification (24) using Lemma 6.1.

For the second part, it suffices to prove the irreducibility claim and that (26) is non-zero. That (26) is non-zero follows as (25) is surjective and as P_J/B admits a Frobenius splitting. Consider next the simply connected commutator group $G_J = (L_J, L_J)$. As G_J -modules both tensor-factors on the left hand side of (25) are equal to the associated Steinberg module St_J of G_J . In particular, they are irreducible as G_J -modules and thus also as P_J -modules. \square

Let v_J^\dagger denote a highest weight vector of the P_J -module $\nabla^J((1-p)w_0^J\rho_J)$. The weight of v_J^\dagger is then $(p-1)\rho_J$. The element v_J^\dagger vanishes with multiplicity $(p-1)$ along the union of the codimension 1 Schubert varieties in P_J/B . In particular, we obtain (cf. [BK, Prop.1.3.11])

Corollary 6.3. *The restriction of (25) to the highest weight space defines a B -equivariant morphism*

$$\nabla^J((p-1)\rho_J) \otimes \mathbb{k}_{(p-1)\rho_J} \rightarrow \text{End}_F(P_J/B, \{X(w)\}_{w \in W_J}),$$

where the image is compatible with all Schubert varieties contained in P_J/B and contains a Frobenius splitting of P_J/B .

6.1. Frobenius \mathcal{M} -splitting. We now want to formulate a slightly more precise statement based on the observations above. Define $\mathcal{M} = \mathcal{L}_J((p-1)\rho_J)$ and start by observing the P_J -equivariant identification

$$(27) \quad \text{End}_F^{\mathcal{M}}(P_J/B) \simeq H^0(P_J/B, \omega_{P_J/B}^{(1-p)} \otimes \mathcal{M}^{-1}) \simeq \nabla^J((1-p)w_0^J\rho_J).$$

As

$$(28) \quad H^0(P_J/B, \mathcal{M}) \simeq \nabla^J((p-1)\rho_J),$$

we may relate (25) with (27) by the natural map

$$H^0(P_J/B, \mathcal{M}) \otimes \text{End}_F^{\mathcal{M}}(P_J/B) \rightarrow \text{End}_F(P_J/B).$$

The statement in Corollary 6.3 then means that the highest weight line in $\text{End}_F^{\mathcal{M}}(P_J/B)$ is compatible with any Schubert variety $X(w)$, $w \in W_J$. More precisely we find the following result related to the setup in Section 4 :

Proposition 6.4. *There exists a B -equivariant map*

$$(29) \quad \theta_J : \mathbb{k}_{(p-1)\rho_J} \rightarrow \text{End}_F^{\mathcal{M}}(P_J/B, \{X(w)\}_{w \in W_J}),$$

such that when

$$\phi_J : \nabla((p-1)\rho_J) \rightarrow \nabla^J((p-1)\rho_J) \simeq H^0(P_J/B, \mathcal{M}),$$

denotes the restriction map, then the induced B -equivariant map

$$(\theta_J, \phi_J) : \nabla((p-1)\rho_J) \otimes \mathbb{k}_{(p-1)\rho_J} \rightarrow \text{End}_F(P_J/B, \{X(w)\}_{w \in W_J}),$$

contains a Frobenius splitting of P_J/B in its image.

7. FROBENIUS SPLITTING $H \times_{B_H} X(w)$.

We are now ready to apply the results in Section 4 and Section 6.

Theorem 7.1. *Let \mathcal{M} denote the line bundle $\mathcal{L}_J((p-1)\rho_J)$ on P_J/B . If the following conditions are satisfied*

- (1) *The T_H -weight $2\rho_H - \rho_J|_{T_H}$ is dominant,*
- (2) *The H -equivariant restriction morphism*

$$\nabla((p-1)\rho_J) \rightarrow \nabla^H((p-1)\rho_J|_{T_H}),$$

is surjective,

then the variety $H \times_{B_H} P_J/B$ admits a Frobenius $\mathcal{M}_H((p-1)(2\rho_H - \rho_J|_{T_H}))$ -splitting which is compatible with the subvarieties $H \times_{B_H} X(w)$, $w \in W_J$. If, moreover, $2\rho_H - \rho_J$ is regular then the line bundle $\mathcal{M}_H((p-1)(2\rho_H - \rho_J|_{T_H}))$ is ample and as a consequence the varieties $H \times_{B_H} X(w)$, $w \in W_J$, are globally F -regular.

Proof. The first part of the statement follows by an application of Theorem 4.1 and Proposition 6.4. The second part follows from Corollary 4.4 by using that Schubert varieties are strongly F -regular. \square

By applying the natural morphism

$$(30) \quad \pi_J : H \times_{B_H} P_J/B \rightarrow {}^H P_J/B \subseteq G/B,$$

we may sometimes transfer the statements in Theorem 7.1 into statements about H -orbit closures in G/B . For this to work we however need (30) to be separable, which is easily seen to be equivalent to the following condition on the level of the Lie algebras

$$(31) \quad \text{Lie}(H) \cap \text{Lie}(P_J) = \text{Lie}(H \cap P_J).$$

Corollary 7.2. *Assume that the relation (31) as well as condition (1) and (2) of Theorem 7.1 are satisfied. Then ${}^H P_J/B$ admits a Frobenius $\mathcal{L}((p-1)\rho_J)|_{{}^H P_J/B}$ -splitting which is compatible with all subvarieties of the form $H \cdot X(w)$, $w \in W_J$. If, moreover, $2\rho_H - \rho_J|_{T_H}$ is regular then each $H \cdot X(w)$, $w \in W_J$, is globally F -regular.*

Proof. Use the notation \mathcal{M} to denote the line bundle $\mathcal{L}_J((p-1)\rho_J)$. Applying the remark in Section 4 we find, as in Theorem 7.1, that the variety $H \times_{B_H} P_J/B$ admits a Frobenius \mathcal{M}_H -splitting which is compatible with the subvarieties $H \times_{B_H} X(w)$, $w \in W_J$; i.e. there exists a map

$$(32) \quad F_*\mathcal{M}_H \rightarrow \mathcal{O}_{H \times_{B_H} P_J/B},$$

compatible with all subvarieties $H \times_{B_H} X(w)$, $w \in W_J$, and a global section s of \mathcal{M}_H such the composition of (32) with

$$(33) \quad F_*\mathcal{O}_{H \times_{B_H} P_J/B} \xrightarrow{F_*s} F_*\mathcal{M}_H,$$

defines a Frobenius splitting of $H \times_{B_H} P_J/B$. As observed in the remark in Section 4 we may even assume that s is contained in the image of the morphism

$$(34) \quad \nabla((p-1)\rho_J) \rightarrow \text{Ind}_{B_H}^H(H^0(P_J/B, \mathcal{M})) = \text{Ind}_{B_H}^H(\nabla^J((p-1)\rho_J)),$$

i.e. we may assume that s is the pull-back $\pi_J^*(s')$ of a global section s' of the line bundle $\mathcal{L}((p-1)\rho_J)|_{HP_J/B}$. In this connection we notice the identity $\mathcal{M}_H \simeq \pi_J^*(\mathcal{L}((p-1)\rho_J))$ which follows by arguing as in the proof of Lemma 4.3.

We now claim that the morphism

$$(35) \quad \mathcal{O}_{HP_J/B} \rightarrow (\pi_J)_*\mathcal{O}_{H \times_{B_H} P_J/B},$$

induced by π_J , is an isomorphism. If so, we may apply $(\pi_J)_*$ to the composition of (32) and (33) to obtain a map

$$F_*\mathcal{O}_{HP_J/B} \xrightarrow{F_*s'} F_*\mathcal{L}((p-1)\rho_J)|_{HP_J/B} \rightarrow \mathcal{O}_{HP_J/B},$$

defining a Frobenius $\mathcal{L}((p-1)\rho_J)|_{HP_J/B}$ -splitting of HP_J/B compatible with all the subvarieties (cf. [BK, Lemma 1.1.8])

$$HBwB/B = \pi_J(H \times_{B_H} X(w)), w \in W_J.$$

This is exactly the first part of the statement of this corollary.

To prove the claim consider the natural morphism

$$(36) \quad H/P_J \cap H \rightarrow G/P_J.$$

As relation (31) is assumed to be satisfied the morphism (36) is a closed embedding. In particular, the pull back HP_J/B of the closed subvariety $H/P_J \cap H$ by the morphism

$$G/B = G \times_{P_J} P_J/B \rightarrow G/P_J,$$

is isomorphic to $H \times_{P_J \cap H} P_J/B$. The claim now follows as the natural morphism

$$H \times_{B_H} P_J/B \rightarrow H \times_{P_J \cap H} P_J/B,$$

is a locally trivial $P_J \cap H/B_H$ -bundle. This ends the proof of the first part of the statement.

As to the second statement the global F -regularity of ${}^{HP_J/B}$ follows from Theorem 7.1 using the claim above and the fact that global F -regularity is preserved by push forward (Lemma 3.5). The global F -regularity of $H \cdot X(w)$, $w \in W_J$, now follows in the same way by applying Lemma 3.6. \square

Remark. In case G/B contains a dense H -orbit HgB and $\nabla((p-1)\rho_J)$ is irreducible as a G -module we may, in the proof of the above corollary, choose the section s to be the image under (34) of the element gv_+ . Here v_+ denotes a the highest weight vector in $\nabla((p-1)\rho_J)$. This follows from the remark in Section 4 as gv_+ generates $\nabla((p-1)\rho_J)$ as an H -module. In this case the zero divisor associated to s' is the sum

$$D = (p-1) \sum_{i \in J} (g \cdot X(w_0 s_i) \cap {}^{HP_J/B}).$$

The Frobenius splitting in Corollary 7.2 may therefore also be considered as a Frobenius D -splitting (cf. [BK, § 1.4]).

Remark. The assumption in Corollary 7.2 that the relation (31) is satisfied is necessary to make the proof work. This assumption does not follow from the rest of the assumptions as can be seen by the following example : Consider the case $G = SL_2 \times SL_2$ and $H = \{(g, F(g)); g \in SL_2\} \subset G$. If P_J is chosen to be the set of pair $(g_1, g_2) \in G$ with the condition that g_2 upper triangular, then the natural morphism (36) is not a closed embedding. However, the rest of the assumption in Corollary 7.2 are satisfied.

As relation (31) is always satisfied in case $J = I$ we find

Corollary 7.3. *If $2\rho_H - \rho|_{T_H}$ is a dominant T_H -weight and if the restriction map*

$$\nabla((p-1)\rho) \rightarrow \nabla^H((p-1)\rho|_{T_H}),$$

is surjective, then there exists a Frobenius $\mathcal{L}((p-1)\rho)$ -splitting of G/B that is compatibly with all subvarieties over the form $H \cdot X(w)$, $w \in W$. In particular, for any dominant weight λ of T and any $w \in W$ we have

- (1) $H^i(H \cdot X(w), \mathcal{L}(\lambda)) = 0$ for $i \geq 1$.
- (2) *The restriction map*

$$H^0(G/B, \mathcal{L}(\lambda)) \rightarrow H^0(H \cdot X(w), \mathcal{L}(\lambda)),$$

is surjective.

Corollary 7.4. *We keep the assumption in Corollary 7.3. Assume furthermore that H is a spherical subgroup of G . Let $w \in W$ and $g \in G$ such that HgB/B is open dense in $H \cdot X(w)$. Let λ be a dominant weight of T and $V(\lambda)$ be the Weyl module $\nabla(\lambda)^*$. Let v_λ be a nonzero vector of weight λ in $V(\lambda)$. Then $H^0(H \cdot X(w), \mathcal{L}(\lambda))^*$ is isomorphic to the H -submodule of $V(\lambda)$ generated by gv_λ .*

Proof. We follow the idea of the proof of [BK, Cor.3.3.11]. Let M be the H -module generated by gv_λ . By Corollary 7.3, the restriction map

$$\gamma : \nabla(\lambda) \rightarrow H^0(H \cdot X(w), \mathcal{L}(\lambda)),$$

is surjective. As HgB/B is dense in $H \cdot X(w)$ we have

$$(37) \quad \begin{aligned} \ker(\gamma) &= \{f \in \nabla(\lambda) : f|_{HgB/B} = 0\} \\ &= \{f \in V(\lambda)^* : ((g^{-1}h)f)(eB) = 0, \text{ for all } h \in H\}. \end{aligned}$$

The central point is now that the B -equivariant map $\nabla(\lambda) \rightarrow \mathbb{k}_{-\lambda}$, coincides with the map $f \mapsto f(eB)$. In particular, f is zero at eB if and only if $v_\lambda(f) = 0$. Thus, by (37), f is contained in $\ker(\gamma)$ if and only if $((hg)v_\lambda)(f)$ is zero for all $h \in H$. The kernel of γ is therefore $(V(\lambda)/M)^*$. This ends the proof. \square

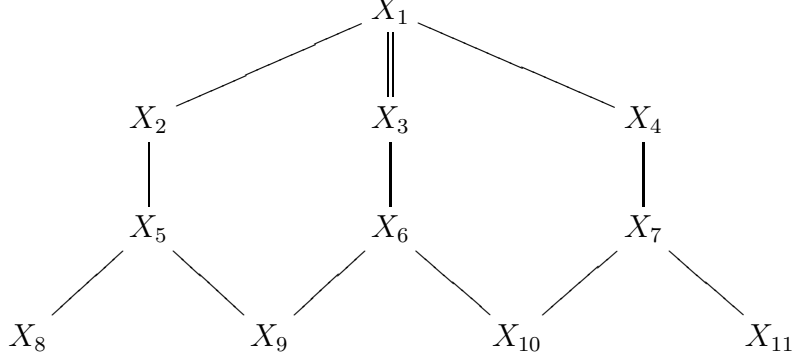
8. EXAMPLES

8.1. In this subsection, we discuss some cases where there is a splitting on G/B that is compatible with all the H -orbit closures. Let (G, H) be one of the following: (H, H) , $(H \times H, H_{\text{diag}})$, (A_{2n+1}, C_n) , (D_n, B_{n-1}) , (E_6, F_4) , (B_3, G_2) . Let B be a Borel subgroup of G such that $B_H = H \cap B$ is a Borel subgroup of H . It is known that all such B are conjugated by H (see [Re, Prop 2.2]) and that all the H -orbit closure in G/B are of the form \overline{HBwB}/B for some $w \in W$. By §5.1(2), (G, H) is a Donkin pair. It is also easy to check that $2\rho_H - \rho|_{T_H}$ is dominant for B_H . Hence (G, H, B) is admissible. By Corollary 7.3, there exists a Frobenius $\mathcal{L}((p-1)\rho)$ -splitting of G/B that is compatible with all H -orbit closures.

It is proved by N. Ressayre in [Re, Theorem A] that if G is a complex reductive group and H a closed quasi-simple subgroup of G , then (G, H) is of minimal rank if and only if (G, H) is (up to isomorphism) one of the pairs above.

However, in positive characteristic, the pair (G, H) is also of minimal rank, where $G = SL_2 \times SL_2$ and $H = \{(g, F(g)); g \in SL_2\} \subset G$. The closed H -orbit in $G/B \cong \mathbb{P}^1 \times \mathbb{P}^1$ is defined by the equation $X^pW - Y^pV = 0$, where X, Y are the coordinates of the first \mathbb{P}^1 and V, W are the coordinates of the second \mathbb{P}^1 . There is no Frobenius splitting on G/B that compatibly splits the closed H -orbit.

8.2. Let $G = Sp_4$ and B a Borel subgroup of G . Let α be the short simple root and β be the long simple root. We denote by s_1 and s_2 the simple reflections corresponding to α and β respectively. Let H be the standard Levi subgroup corresponding to α . The H -orbit closures on the flag variety of G are described in the the following graph



Here X_8 is the H -orbit of $B_1 = B$, X_9 is the H -orbit of $B_2 = s_2Bs_2$, X_{10} is the H -orbit of $B_3 = s_2s_1Bs_1s_2$ and X_{11} is the H -orbit of $B_4 = s_2s_1s_2Bs_2s_1s_2$. It is easy to see that $2\rho_H - \rho|_{T_i \cap H}$ is dominant for $B_1 \cap H$ and $B_4 \cap H$ but is not dominant for $B_2 \cap H$ and $B_3 \cap H$. By Corollary 7.2,

(1) There exists a Frobenius $\mathcal{L}((p-1)\rho)$ -splitting of the flag variety G/B that compatibly splits X_2 , X_5 and X_8 . We may even apply the second part of Corollary 7.2 to obtain global F -regularity of X_2 , X_5 and X_8 . In fact, X_2 , X_5 and X_8 is just a subset of the set of Schubert varieties so this is a well known result.

(2) Similarly, there exists a Frobenius splitting of the flag variety G/B_4 that compatible splits certain H -orbit closures X'_4 , X'_7 and X'_{11} (which are also Schubert varieties in G/B_4). By the natural identification of G/B_4 with G/B this leads to a Frobenius $\mathcal{L}((p-1)\rho)$ -splitting of the flag variety G/B that compatibly splits X_4 , X_7 and X_{11} . The varieties X_4 , X_7 and X_{11} are not Schubert varieties in G/B .

(3) Let P_1 and P_2 denote the minimal parabolic subgroups containing B_2 . Fix notation such that P_1 corresponds to the short simple root. Then, by Corollary 7.2, the variety ${}^{HP_2}/B_2$ in G/B_2 admits a Frobenius \mathcal{M}_2 -splitting compatible with the orbit closure ${}^{HB_2}/B_2$. Here \mathcal{M}_2 is some ample line bundle on ${}^{HP_2}/B_2$ which can be explicitly determined. In this case we cannot apply the strong part of Corollary 7.2. Focusing on P_1 instead we may conclude that ${}^{HP_1}/B_2$ admits a Frobenius \mathcal{M}_1 -splitting compatible with ${}^{HB_2}/B_2$. Again \mathcal{M}_1 is some ample line bundle. In this case we may apply the strong part of Corollary 7.2 to obtain global F -regularity of both ${}^{HP_1}/B_2$ and ${}^{HB_2}/B_2$. Transferring this information into G/B it means that X_6 as well as X_5 admits a Frobenius splitting, along an ample line bundle, which is compatible with X_9 . Moreover, as X_6 corresponds to ${}^{HP_1}/B_2$ we may conclude global F -regularity of X_6 . Notice that X_6 and X_9 are not multiplicity-free in the sense of [B2].

(4) Similar statement as for the pairs (X_5, X_9) and (X_6, X_9) are also satisfied for the pairs (X_7, X_{10}) and (X_6, X_{10}) .

We do not know if X_3 admits a Frobenius splitting.

8.3. Let $(G, H) = (SL_n, SO_n)$ and $p \geq 3$. Then (G, H) is a Donkin pair. Let $I = \{1, 2, \dots, n-1\}$ be the set of simple roots of G and

$$J = \begin{cases} I - \{\frac{n}{2}\}, & \text{if } 2 \mid n; \\ I - \{\frac{n-1}{2}\} \text{ or } I - \{\frac{n+1}{2}\}, & \text{if } 2 \nmid n. \end{cases}$$

Then $2\rho_H - \rho_J|_{T_H}$ is dominant. By Corollary 7.2, ${}^H P_J/B$ admits a Frobenius splitting along an ample divisor which is compatible with all subvarieties $\overline{{}^H B w B}/B$ for $w \in W_J$.

Notice that none of the codimension one H -orbit closures in G/B are multiplicity-free.

8.4. Let $(G, H) = (H \times H \times H, H_{\text{diag}})$. Then $\rho|_{T_H} = 3\rho_H$ and $2\rho_H - \rho|_{T_H}$ is not dominant. Moreover, (G, H) satisfies the pairing criterion (see [V, Example 8]). It is easy to see that the subvarieties $\overline{{}^H B(w_0^H, 1, 1)B}/B$, $\overline{{}^H B(1, w_0^H, 1)B}/B$, $\overline{{}^H B(1, 1, w_0^H)B}/B$ are the partial diagonals of $G/B = H/B_H \times H/B_H \times H/B_H$. By [BK, Exercise 3.5.3], the flag variety of G does not admit a Frobenius splitting which is compatible with all the partial diagonals. This proves the necessity in Corollary 7.3 of something like the condition that $2\rho_H - \rho|_{T_H}$ is dominant.

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