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Products of Floer Cohomology of Torus Fibers in Toric Fano Manifolds

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Abstract: We compute the ring structure of Floer cohomology groups of Lagrangian torus fibers in some toric Fano manifolds continuing the study of [CO]. Related A_∞-formulas hold for a transversal choice of chains. Two different computations are provided: a direct calculation using the classification of holomorphic discs by Oh and the author in [CO], and another method by using an *analogue of divisor equation* in Gromov-Witten invariants to the case of discs. Floer cohomology rings are shown to be isomorphic to Clifford algebras, whose quadratic forms are given by the Hessians of functions W, which turn out to be the superpotentials of Landau-Ginzburg mirrors. In the case of $\mathbb{C}P^n$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, this proves the prediction made by Hori, Kapustin and Li by B-model calculations via physical arguments. The latter method also provides correspondence between higher derivatives of the superpotential of LG mirror with the higher products of the A_∞(or L_∞)-algebra of the Lagrangian submanifold.

1. Introduction

Floer theory of Lagrangian intersections has been proved to be a powerful technique in symplectic geometry. Also since the "homological mirror symmetry" conjecture by Kontsevich [K], it has become a much more exciting field of mathematics, which yet has a long way to be fully understood. Recently, Fukaya, Oh, Ohta and Ono constructed an A_∞-algebra of a Lagrangian submanifold and Floer homology in a general setting in their beautiful work [FOOO]. But the construction is highly non-trivial to overcome several technical problems. The first problem is the well-definedness of the moduli space of J-holomorphic discs compatible for all homotopy classes. It was observed in [FOOO], that standard Kuranishi perturbation does not produce compatible and transversal moduli space in general. Another problem is that even if moduli spaces of J-holomorphic discs are well-defined, it does not directly produce an A_∞-algebra since one has to work at the chain level.

In [CO], Yong-Geun Oh and the author have explicitly described the moduli space of holomorphic discs in the case of Lagrangian torus fibers in toric Fano manifolds, and used that information to compute Floer cohomology groups. A combinatorial description of a fiber whose Floer cohomology is non-vanishing was found, and for such a fiber, the Floer cohomology was in fact isomorphic to singular cohomology as a module. It was shown that all holomorphic discs in these cases are transversal. To compactify the moduli space, we need an additional assumption regarding the behavior of holomorphic spheres on a toric Fano manifold (see Assumption 3.1). In this paper, we first consider a related A_∞-algebra which is defined transversally. Namely, fiber products with various chains in the Lagrangian submanifold L in the definition of an A_∞-algebra can be made transversal for the generic choice of chains. This gives a partial A_∞-algebra, but products on the cohomology of these A_∞-algebras are shown to be well-defined. How to obtain an actual A_∞-algebra from this partial algebra is an interesting question. With skew-symmetrization in this toric Fano case, these partial A_∞-algebras gives well-defined L_∞-algebras. On the other hand, recently Fukaya has constructed an A_∞-algebra on DeRham complex of Lagrangian submanifolds. A computation in toric Fano case can be carried out in the DeRham setting, which will produce actual A_∞-algebra.

Then we show that Floer cohomology ring $HF^{BM}(L; J_0)$ is isomorphic to a Clifford algebra $Cl(V, Q)$ where Q is a symmetric bilinear form. It is very interesting that the symmetric bilinear form Q we obtained exactly agrees with the Hessian of the superpotential W of the mirror Landau-Ginzburg model studied by Hori and Vafa [HV]. (This is related to homological mirror symmetry conjecture between A-model in Fano manifolds and B-model in Landau-Ginzburg mirror.) In particular, the Floer cohomology of the Clifford torus T^n in $\mathbb{C}P^n$ is isomorphic to the Clifford algebra with n generators as a ring.

Such product structures in the Clifford torus T^n in $\mathbb{C}P^n$ and $T^1 \times T^1$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ have been conjectured by Hori and Kapustin and Li [KL], recently in general by [KL2] from the calculation on B-model side using physical arguments. Mathematical account of the product structure on B-model side looks plausible considering the paper by Orlov [O].

We provide two ways of computing the product structure. First, we provide direct computations exploiting the classification of all holomorphic discs with boundary on L by Oh and the author ([CO]). Another method is by using an analogue of *divisor equation* for discs, which is introduced in section 6. The latter method easily provides the general correspondence between higher derivatives of the superpotential of LG mirror with the higher products of A_∞(or L_∞)-algebra of Lagrangian submanifold. This extends the correspondence proved by Oh and the author in [CO] that obstruction cochain $m_0 = l_0$ agrees with the superpotential itself and non-vanishing of Floer cohomology corresponds to the critical points of the superpotential W. These l_∞-products are invariant under the perturbation of an almost complex structure.

We also provide an explicit filtered chain map between singular cochain complex and Bott-Morse Floer complex in the case of torus fibers L in toric Fano manifolds, which induces an isomorphism in cohomology in case Floer homology is non-vanishing.

2. A_∞-Algebra of Lagrangian Submanifold

In this section we recall the construction of the A_∞-algebra of a Lagrangian submanifold. In fact, we will provide a *transversal* version (partial A_∞-algebra) which is suitable for our purposes. (This version is only suitable for the case when the moduli space is already well-defined.)

The A_∞-algebra in this case naturally arises from the stable map compactification of the moduli spaces of holomorphic discs. The moduli space of a disc with $n+1$ boundary marked points, $\mathcal{M}_{n+1}$, can be seen also as a compactification of a configuration space of $n-2$ points on an interval [0, 1]. (By $Aut(D^2)$, send $n+1, 0, 1^{\text{st}}$ marked points to $1, \infty, 0$ where we identify D^2 with the upper-half plane.) The latter gives the well-known Stasheff Polytope [S1].

We first recall the definition of the (non-unital) A_∞-algebra introduced by Stasheff [S1]. Let $A = \oplus_{i \in \mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded module over R, where R is a commutative ring with unit. As usual, we denote its suspension by $A[1]^i = A^{i+1}$.

Definition 2.1. *A structure of the (non-unital) A_∞-algebra on A is given by a series of R-module homomorphisms $m_k : A^{\otimes n} \to A[2-n]$ for non negative integer k, satisfying quadratic equations*

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{deg\, x_1 + \cdots + deg\, x_{i-1} + i - 1} \tag{2.1}$$

$$m_{k_1}(x_1, \ldots, m_{k_2}(x_i, \ldots, x_{i+k_2-1}), \ldots, x_k) = 0.$$

In the transversal version, the above formula will only hold on a dense transversal sequence of chains for each k.

Now we recall the setting for the objects of the chain complex. We refer readers to [FOOO] Appendix A for a complete explanation about introducing this setup. Let $C^*(L; \Lambda_{nov})$ be the set of currents on L realized by geometric chains as follows: For a given (n-k)-dimensional geometric chain $[P, f]$, we consider the current $T([P, f])$ which is defined as follows: The current $T([P, f])$ is an element in $D'^k(M; \mathbb{R})$, where $D'^k(M; \mathbb{R})$ is the set of distribution valued k-forms on M : For any smooth (n-k)-form ω, we put

$$\int_M T([P, f]) \wedge \omega = \int_P f^* \omega. \tag{2.2}$$

This defines a homomorphism

$$T : S_{n-k}(M; \mathbb{Q}) \to D'^k(M; \mathbb{R}),$$

where $S_{n-k}(M; \mathbb{Q})$ is the set of all (n-k) dimensional geometric chains with $\mathbb{Q}$-coefficient. Let $\overline{S}^k(M, \mathbb{Q})$ be the image of the homomorphism T. We extend the coefficient ring $\mathbb{Q}$ to Λ_{nov}. Then we set

$$C^k(L; \Lambda_{nov}) := \overline{S}^k(M, \Lambda_{nov}). \tag{2.3}$$

Since we consider the elements in the image of T, if the image of the map f of the geometric chain $[P, f]$ is smaller than the expected dimension, then it gives 0 as a current. This fact will be used crucially later on. Also, note that the map T is not injective, hence some elements get identified under the map T. Also note that we take the whole image of T (instead of taking a countable subset of it) as transversality of fiber products in the definition of m_k is achieved by choosing generic chains.

The classical part of the maps $\{m_k\}$ are defined as follows, which is different from that of [FOOO] (In [FOOO], $m_{k,0}$ defines an A_∞-algebra of singular cochains.)

Definition 2.2. *The maps $m_{k,0}$ for $k = 0, 1, \dots$ on $C^*(L; \Lambda_{nov})$ are transversally defined by the following maps. For $[P, f], [Q, g] \in C^*(L; \mathbb{Q})$,*

(1) $m_{0,0} = 0$.
(2) $m_{1,0}([P, f]) = (-1)^n[\partial P, f]$.
(3) $m_{2,0}([P, f], [Q, g]) = (-1)^{degP(degQ+1)}[f(P) \cap g(Q), i] = 0$, *where i is an embedding into L.*
(4) for $k \geq 3$,

$$m_{k,0} \equiv 0. \tag{2.4}$$

We extend the above maps linearly over Λ_{nov}. The notation ∂ here is the usual boundary operator for singular homology.

Now, the quantum contribution part is defined in the same way as in [FOOO].

Definition 2.3 [FOOO].

(1) For a geometric chain $[P, f] \in C^g(L : \mathbb{Q})$ and non-zero β, define

$$m_{0,\beta} = [\mathcal{M}_1(\beta), ev_0], \tag{2.5}$$

$$m_{1,\beta}[P, f] = [\mathcal{M}_2(\beta)\ {}_{ev_1} \times_f P, ev_0]. \tag{2.6}$$

(2) For each $k \geq 2$, non-zero β, for geometric chains

$$[P_1, f_1] \in C^{g_1}(L : \mathbb{Q}), \dots, [P_k, f_k] \in C^{g_k}(L : \mathbb{Q})$$

(i.e. dimension of $[P_i, f_i]$ as a chain is $n - g_i$), define

$$\begin{aligned} m_{k,\beta}([P_1, f_1], \dots, [P_k, f_k]) = (-1)^\epsilon[&\mathcal{M}^{\text{main}}_{k+1}(\beta)\ {}_{(ev_1,\dots,ev_k)} \\ &\times_{(f_1,\dots,f_2)}(P_1 \times \cdots \times P_k), ev_0]. \end{aligned} \tag{2.7}$$

Here ϵ is a sign assigned as follows:

$$\epsilon = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} deg(P_i). \tag{2.8}$$

(3) Then we define the maps m_k $(k \geq 0)$ by

$$\begin{aligned} m_k([P_1, f_1], \dots, [P_k, f_k]) = \sum_{\beta \in \pi_2(M,L)} & m_{k,\beta}([P_1, f_1], \dots, [P_k, f_k]) \\ & \otimes T^{Area(\beta)} q^{\mu(\beta)/2}. \end{aligned}$$

Remark 2.4. Here $\mathcal{M}_k(\beta)$ is a compactified moduli space of J-holomorphic discs with k marked point on ∂D^2. Recall that $\mathcal{M}_k(\beta)$ for $k \geq 3$ has several connected component. By the ordering of the k marked points on ∂D^2 and by $\mathcal{M}^{main}_k(\beta)$ we denote the connected component where marked points $z_1, \dots, z_k$ lie cyclically on ∂D^2 counter-clockwise.

Also, the fiber products defined above are not always transversal, and we discuss this issue in Sect. 3.

Here we recall the dimension formula of $m_{k,\beta}$ when the involved fiber product is transversal.

Proposition 2.1 ([FOOO] Proposition 13.16). *For non-zero β, when transversal,*

$$m_{k,\beta}((P_1, f_1), \ldots, (P_k, f_k)) \in C_{n-\sum_{i=1}^k g_i+\mu(\beta)-2+k}(L : \mathbb{Q}).$$

Proposition 2.2 (cf. [FOOO]). *These $\{m_k\}$ maps satisfy the A_∞ formulas (2.1) for transversal sequence of chains in $C^*(L; \Lambda_{nov})$.*

Proof. This is essentially the theorem proved in [FOOO]. We recall its proof for the convenience of readers and explain the changes made for $m_{k,0}$.

For simplicity, we recall the proof only for the third A_∞-formula. Consider the moduli space of J-holomorpic discs intersecting chains P and Q (see Fig. 1),

$$m_{2,\beta}(P, Q) = (\mathcal{M}_3^{main}(\beta)_{ev_1,ev_2} \times_{f,g} (P \times Q), ev_0). \tag{2.9}$$

Now, we consider all possible stable map compactification of this moduli space and its image under the evaluation map. The limit configurations of codimension 1 of the image can be written as follows. See Fig. 1, where each figure corresponds to the following terms:

$$m_{2,\beta}(P, Q) \to m_{2,\beta_2}(m_{1,\beta_1}(P), Q), m_{2,\beta_2}(P, (m_{1,\beta_1}(Q),), \tag{2.10}$$

$$m_{3,\beta_2}(P, Q, m_{0,\beta_1}), m_{3,\beta_2}(P, m_{0,\beta_1}, Q), m_{3,\beta_2}(m_{0,\beta_1}, P, Q), m_{1,\beta_1}(m_{2,\beta_2}(P, Q)). \tag{2.11}$$

Degenerations into several (three or more) disc components or sphere bubbles also occur. But if transversalities are satisfied for such singular strata with positivity assumptions on a Lagrangain submanifold, such strata should be of codimension 2 or more, hence they do not contribute to the A_∞ formulas.

Now, these limit configurations can be written into an A_∞-formula up to sign:

$$\begin{aligned}\partial(m_{2,\beta}(P, Q)) &= \pm m_{2,\beta_2}(m_{1,\beta_1}(P), Q) \pm m_{2,\beta_2}(P, (m_{1,\beta_1}(Q),) \\ &\pm m_{3,\beta_2}(P, Q, m_{0,\beta_1}) \pm m_{3,\beta_2}(P, m_{0,\beta_1}, Q) \pm m_{3,\beta_2}(m_{0,\beta_1}, P, Q) \\ &\pm m_{1,\beta_1}(m_{2,\beta_2}(P, Q)).\end{aligned}$$

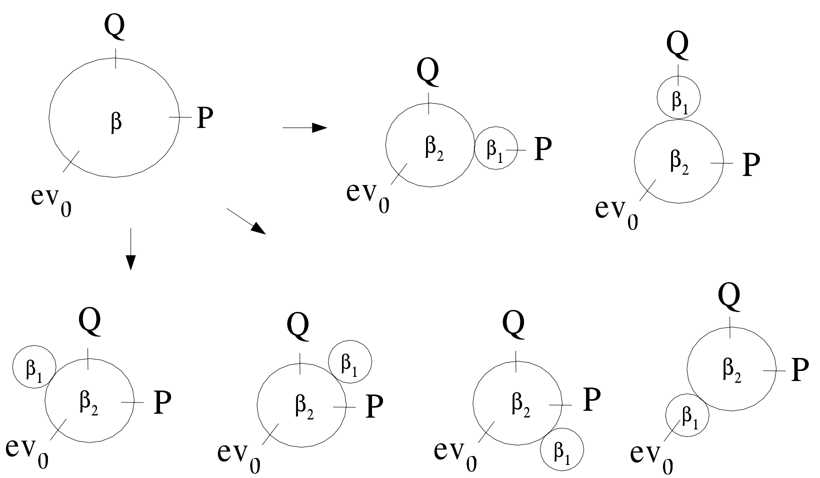

Fig. 1. Limit configurations of (2.9) of codimension 1

This is the third A_∞-formula in (2.1) up to sign, and other formulas can be obtained in a similar fashion by choosing a $m_{k,\beta}(P_1, \ldots, P_k)$ for general k in (2.9).

Now we justify the changes made in the definitions of $m_{k,0} \equiv 0$ for $k \geq 3$. Consider one of the $m_{3,0}$ term appeared in the above configuration when $\beta_2 = 0$, or generally one may consider the geometric chain $m_{3,0}(P, Q, R)$. The dimension of the image under the evaluation map of $m_{3,0}(P, Q, R)$ is always smaller than the virtual dimension of the moduli space: The reason is that the evaluation map of a constant disc forgets the moduli parameter. Namely, before evaluation, there is a parameter describing the position of four marked points on a disc. Recall that the moduli space of 4 marked points on ∂D^2 up to automorphisms of D^2 is diffeomorphic to $\mathbb{R}$ (see [FOh]). But as we evaluate on a constant disc, the image is always a point, while the moduli parameter is lost under the evaluation map. Hence, such a term $m_{3,0}(P, Q, R)$ is of codimension 1 by the virtual dimension, but its actual image is of codimension 2. Hence terms involving $m_{3,0}$ do not appear in the A_∞ formula, which is obtained by considering the codimension 1 boundary of the image of the chain (2.9) under evaluation map.

This phenomenon always happens for $m_{k,0}$ for any $k \geq 3$ because of the same reason. Hence we may set (transversally)

$$m_{k,0} \equiv 0 \quad \text{for } k \geq 3.$$

Note that in [FOOO], the evaluation maps of constant homotopy class are also perturbed by moduli parameters, so that the image has the same dimension as virtual dimension unlike our setting. Also note that $m_{2,0}, m_{1,0}$ does not vanish as there are no moduli parameters in these cases. This proves the proposition. □

Now, because of the presence of m_0 terms, $m_1^2 = 0$ does not always hold. Hence, Floer homology groups are not well-defined in general. Obstructions for the well-definedness of Floer cohomology was studied in [FOOO]. In an unobstructed case, one can deform the chain complex in a suitable way so that $m_1^2 = 0$ holds. For the case of torus fibers in a toric Fano manifold, it is (weakly) obstructed, in which case Floer cohomology itself is well-defined.

A related phenomenon in the language of A_∞-algebra is that m_0 terms disappear from the A_∞-formula.

Proposition 2.3 (compare [FOOO] Proposition 7.1). *Let x_i be an element in $C^*(L; \Lambda_{nov})$ for $i = 0, \ldots, k$, for a Lagrangian torus fiber L in toric Fano manifolds. Then, when transversal, we have*

$$m_{k+1}(x_1, \ldots, [L], \ldots, x_k) = 0, k \geq 2, k = 0,$$

$$m_2([L], x_0) = (-1)^{deg(x_0)} m_2(x_0, [L]) = x_0.$$

Namely, [L] behaves as a strict unit.

Proof. This was proved in [FOOO], except that we do not need to use a homotopy unit argument. Recall that the *forget* maps commute with the evaluation maps obviously in our case, whereas they do not commute in [FOOO] because of the perturbation of evaluation maps at marked points. We give the proof of the proposition here for the convenience of readers. Note that the proposition holds by the definition of $m_{k+1,0}$ for $k \geq 2$. Hence it is enough to show that $m_{k,\beta}(x_1, \ldots, [L], \ldots, x_k) = 0$ for $k \geq 2$ with non-zero $\beta \in \pi_2(M, L)$, and the statement about $m_{2,0}$.

Note that the condition for the image of a marked point to meet the fundamental chain $[L]$ is redundant since it always meets L. Hence,

$$\begin{aligned} & ev_0\big(\mathcal{M}_{k+2}(\beta)_{ev} \times (x_1 \times \cdots \times [L] \times \cdots \times x_k)\big) \\ & \subset ev_0(\mathcal{M}_{k+1}(\beta)_{ev} \times (x_1 \times \cdots \times x_k)). \end{aligned} \tag{2.12}$$

The dimension of RHS is

$$n - \sum deg(x_i) + \mu(\beta) - 2 + k,$$

where as the virtual (expected) dimension of LHS is

$$n - \sum deg(x_i) + \mu(\beta) - 2 + k + 1.$$

Hence, the actual image has smaller dimension than the expected dimension, which becomes zero in the language of currents. The case of $m_{2,0}$ follows from the sign convention of [FOOO]. □

The second formula implies that the Floer cohomology is well-defined in this case as observed in [CO] and [C] Proposition 3.18; in this case it was shown that $m_0(1) = \sum_{i=1}^{N} [L] \otimes T^{e_i} q$ is a multiple of the fundamental chain.

Here, we write the first three A_∞-formulas where m_0 terms are dropped because of the above proposition.

$$0 = m_1 \circ m_1, \tag{2.13}$$

$$0 = m_2(m_1(x), y) + (-1)^{deg(x)+1} m_2(x, m_1(y)) + m_1(m_2(x, y)), \tag{2.14}$$

$$\begin{aligned} 0 = & m_1(m_3(x, y, z)) + m_2(m_2(x, y), z) + (-1)^{deg(x)+1} m_2(x, m_2(y, z)) \\ & + m_3(m_1(x), y, z) + m_3(x, m_1(y), z) + m_3(x, y, m_1(z)). \end{aligned} \tag{2.15}$$

The first equation implies that m_1 defines the cochain complex. The second equation implies that m_2 defines a product of the cohomology up to sign. For $x, y, z \in HF^{BM}(L; J_0)$, we have $m_1(x) = m_1(y) = m_1(z) = 0$. Therefore the third equation implies the associativity of the product up to sign,

$$m_2(m_2(x, y), z) + (-1)^{deg(x)+1} m_2(x, m_2(y, z)) = 0. \tag{2.16}$$

To define an associative product (with correct sign) on cohomology, one should make the following change of signs.

Definition 2.5. *We define*

$$\widetilde{m}_1(P) = (-1)^{deg P} m_1(P), \tag{2.17}$$

$$\widetilde{m}_2(P, Q) = (-1)^{deg P(deg Q+1)} m_2(P, Q). \tag{2.18}$$

Remark 2.6. The first sign appears due to a acohomological sign convention. The second sign appears due to the sign convention of [FOOO].

The resulting A_∞-formulas for the new $\{\widetilde{m}_k\}$ are

$$\widetilde{m}_1(\widetilde{m}_2(x,y)) = \widetilde{m}_2(\widetilde{m}_1(x),y) + (-1)^{deg\, x}\widetilde{m}_2(x,\widetilde{m}_1(y)),$$
$$\widetilde{m}_2(\widetilde{m}_2(x,y),z) = \widetilde{m}_2(x,\widetilde{m}_2(y,z))$$

for $x, y, z \in HF^{BM}(L; J_0)$. Hence $\widetilde{m_2}$ defines an graded associative product on HF^{BM} $(L; J_0)$.

For example, with the new sign, the classical cup product part of $\widetilde{m}_2$ can be written as

$$\widetilde{m}_{2,0}(P_1, P_2) = P_1 \cap P_2.$$

Also associativity in the classical level is just

$$(P_1 \cap P_2) \cap P_3 = P_1 \cap (P_2 \cap P_3).$$

3. Transversality

In this section, we discuss the issues regarding the moduli space of J-holomorphic discs and the transversality of A_∞-algebra.

3.1. Moduli spaces. We first recall the following theorem.

Theorem 3.1 ([CO]). *Holomorphic discs in toric manifolds with boundary on any Lagrangian torus fiber are Fredholm regular, i.e., its linearization map is surjective.*

Hence the moduli space of holomorphic discs (before compactification) is a manifold of the expected dimensions. As we try to compactify the moduli space, we may have strata with sphere bubbles. In general toric Fano manifolds, it is already known that holomorphic spheres are not always Fredholm regular. Hence in the compactification of holomorphic discs, some strata (with sphere bubble) may not have the expected dimension. But since we only evaluate at the boundary of the discs (not on spheres), with the Fano condition, the evaluation image of such strata is always of codimension of two or higher. Hence, it is plausible that these moduli spaces with evaluation maps define currents on L. But to make this precise seems to be a non-trivial problem. A similar problem also has been observed in the case of Gromov-Witten theory if one tries to integrate forms over the pseudo-cycle (see p. 277 of [MS]). The author does not know how to prove it, so we require the following strict assumption on the sympletic manifold so that the moduli chain defines a current.

Assumption 3.1. *The toric Fano manifold M is assumed to be convex. Namely we require that for any genus 0 stable map $f : \Sigma \to M$, f^*T_M is generated by global sections.*

Such an assumption holds in the case of complex projective spaces, and products of complex projective spaces. Except of this rectifiability problem of the compactified moduli chain of holomorphic discs, the results in this paper hold for all toric Fano manifolds. Even when the assumption is not satisfied, the results in Sect. 6 can be understood independently as computations of some invariants (see Proposition 6.5).

We remark about perturbing the standard complex structure to a tame almost complex structure. McDuff and Salamon [MS] showed that for a subset $J_{reg}(M)$ of second category, the moduli spaces of simple J-holomorphic curves become pseudo-cycles. In the case of J-holomorphic discs, it is more complicated since the structure of non-simple J-holomorphic discs can be very complex. But due to the structure theorem proven by Kwon and Oh [KO], a similar proof as in [MS] can be used to show that the moduli space of simple discs are "pseudo-chain" which may be similarly defined as pseudo-cycle. But also in this case, we do not know if these moduli chains would define currents. If these define currents, one can prove the invariance of Floer cohomology ring in a similar way as in [FOOO].

Another approach would be to consider the Kuranishi structure of the moduli space of J-holomorphic discs ([FOOO, FOno]). But as pointed out in [FOOO], it is not (yet) possible to find a Kuranishi perturbation which is compatible for all homotopy classes in $\pi_2(M, L)$. Such compatibility is rather essential since we are interested in the relations between moduli spaces which produce A_∞-formula.

3.2. Transversal A_∞-algebra. Now we explain how to achieve transversality of the fiber product in the definition of A_∞-formulas. First, recall that the ordinary intersection product in the chain level is not well-defined, while the cup product is well-defined on cohomology. Hence, even in the classical level, the A_∞-algebra $(C^*(L; \Lambda_{nov}), m_{k,0})$ is not easy to define, since operations are defined in the chain level. But it is obvious how to define it to work only transversally. A similar problem occurs for $m_{k,\beta}$. For example the fiber product $m_k(P, P, \dots, P)$ is not transversal if $P \neq L$. Hence, the authors of [FOOO] develop a non-trivial technique to overcome such a problem. In this section, we show that if we choose the generic sequence of chains, then the fiber product is transversal, and this transversal A_∞-algebra is enough to determine homology and its ring structure.

Definition 3.2. *A k-tuple $(P_1, \dots, P_k)$ is called a* **transversal sequence** *if the chain $(P_1 \times \cdots \times P_k)$ is transversal to the image of the map ev_β for all $\beta \in \pi_2(M, L)$. For a transversal sequence $(P_1, \dots, P_k)$, the fiber product $m_k(P_1, \dots, P_k)$ is well-defined.*

Recall that a *residual* subset of a space X is one which contains the intersection of countably many dense open subsets.

Lemma 3.2. *For a residual set of $C^*(L; \Lambda_{nov}) \times \cdots \times C^*(L; \Lambda_{nov})$, the k^{th} A_∞-formula (2.1) is well-defined. Namely all the fiber products given in the formula are transversal.*

Proof. It is enough to show that transversality of the chain $(P_1 \times \cdots \times P_k)$ and the image of ev_β from each codimension 1 strata of the moduli space of J-holomorphic discs for all $\beta \in \pi_2(M, L)$, which can be achieved by choosing generic chains P_i's by the standard transversality theorem. □

Corollary 3.3. *$(C^*(L; \Lambda_{nov}), \{m_k\})$ satisfies the A_∞-formula for a dense transversal sequence of chains.*

In fact, in our case it is easy to perturb $(P_1, \dots, P_k)$ to a transversal sequence due to the presence of torus action. Namely, as the torus $(S^1)^n$ acts on L transitively. Hence, for a generic $(t_1, \dots, t_k) \in (S^1)^n \times \cdots \times (S^1)^n$, $(t_1 \cdot P_1) \times \cdots \times (t_k \cdot P_k)$ is a transversal sequence. Also because we have the same torus action on the moduli space of holomorphic discs, we have the following identity:

$$m_k(t \cdot P_1, \dots, t \cdot P_k) = t \cdot m_k(P_1, \dots, P_k). \tag{3.1}$$

Therefore, the transversality of the A_∞-formula also can be achieved by the torus action on each chain: If the m_{k_2} term causes non-transversality to define m_{k_1} in the A_∞-formula, then we can perturb all chains inside m_{k_2} by the same $t \in (S^1)^n$ to make the m_{k_2} term transversal in m_{k_1} by the equality (3.1). Also, it is easy to perturb a Floer-cycle in its cohomology class by the following lemma.

Lemma 3.4. *Let Ψ be the chain map constructed in Definition 4.4. For any cycle P of singular homology, $\Psi(P)$ is a Floer-cycle, i.e. $m_1(\Psi(P)) = 0$. Then for $t \in T^n$, $t \cdot \Psi(P)$ is also a Floer cycle, and we have*

$$\Psi(P) - t \cdot \Psi(P) = (-1)^n m_1 \Psi(H),$$

where the homotopy H is a singular chain with $m_{1,0}(H) = P - t \cdot P$.

Proof. Equation (3.1) for $k = 1$ implies that

$$m_{1,\beta}(t \cdot P) = t \cdot m_{1,\beta} P.$$

Hence the theorem follows. The last statement follows by applying Proposition 4.1 (1) for the chain H with the fact that $t \cdot \Psi(P) = \Psi(t \cdot P)$. □

Proposition 3.5. *$\widetilde{m_2}$ defines a product on the Floer cohomology ring $HF^{BM}(L; J_0)$.*

Proof. To show that the product is well-defined on cohomology, it is enough to show that for $P, Q \in C^*(L; \Lambda_{nov})$ with $m_1(P) = m_1(Q) = 0$, we have

$$m_2(P, t_1 \cdot Q) = m_2(P, t_2 \cdot Q) + m_1(R)$$

for generic $t_1, t_2 \in (S^1)^n$ and for some $R \in C^*(L; \Lambda_{nov})$. First, for any homotopy class $\beta \in \pi_2(M, L)$, note that the the fiber product in $m_{1,\beta}(P)$ of A_∞-algebra is transversal for any chain P since the evaluation map from the moduli space is always submersive due to the torus action. And $m_{2,\beta}(P, Q)$ is transversal if $m_{1,\beta}(P)$ is transversal to Q. Then, for a generic $t \in (S^1)^n$, $m_{1,\beta}(P)$ is transversal to $t \cdot Q$ for any β. Also, for generic $t_1, t_2 \in (S^1)^n$, $m_{1,\beta}(P)$ is transversal to H with $m_1(H) = t_1 \cdot Q - t_2 \cdot Q$ for any β. If not, we can perturb $t_1 \cdot Q, t_2 \cdot Q, H$ by another $t \in (S^1)^n$ to make them transversal. Therefore,

$$m_2(P, t_1 \cdot Q) - m_2(P, t_2 \cdot Q) = m_2(P, m_1(H)) = \pm m_1(m_2(P, H).$$

This finishes the proof. □

4. Bott-Morse Floer Cycles

In [C] and [CO], Oh and the present author have shown that for any such torus fiber $L \subset M$, the Floer homology group $HF(L, L)$ when nonvanishing, is isomorphic to the singular cohomology of the Lagrangian submanifold $H^*(L : \Lambda_{nov})$. Now, we fix a Lagrangian torus fiber L whose Floer cohomology is non-vanishing. The fact that $HF^{BM}(L; J_0)$ and $H^*(L; \Lambda_{nov})$ is isomorphic as a module is a little bit deceiving because a cycle in the singular homology is *not* a cycle in Floer homology. We need to modify a cycle, say P, by adding correction terms, say Q to make it satisfy $m_1(P+Q) = 0$. In the computations of [C] or [CO], it was automatically taken care of by the spectral sequence. We will find exact correction terms for any cycle in Proposition 4.1. Actually we will construct a filtered chain map from the singular chain complex to the Bott-Morse Floer complex.

We start with the following definition and an important example to understand the construction that follows.

Definition 4.1. *An element* $P = \sum_{i=1}^{k} a_i \; [P_i, f_i] \; T^{e_i} \; q^{\mu_i} \in C^*(L; \Lambda_{nov})$ *is called a Floer-cycle if* $m_1(P) = 0$.

Example 4.2. Consider a Clifford torus T^2 in $\mathbb{C}P^2$. A point $< pt >$ is a cycle in the singular homology of T^2. Let l_0, l_1, l_2 be the cycles in T^2 which are boundaries of holomorphic discs $[z; 1; 1]$, $[1; z; 1]$, and $[1; 1; z]$. These three discs have the same symplectic area which we denote by $\omega(D)$.

Recall from [C] that we have

$$m_1 < pt >= (-1)^n (l_0 + l_1 + l_2) \otimes T^{\omega(D)} q \neq 0.$$

Therefore $< pt >$ is not a Floer-cycle. But, $l_0 + l_1 + l_2$ is homologous to zero. We may choose a 2-chain $Q \subset L$ with $\partial Q = -(l_0 + l_1 + l_2)$. Hence $< pt > +Q \otimes T^{\omega(D)} q$ turns out to be a correct Floer-cycle:

$$\begin{aligned} m_1(< pt > +Q \otimes T^{\omega(D)} q) &= m_{1,2}(< pt >) + m_{1,0}(Q) \otimes T^{\omega(D)} q \\ &= (-1)^n ((l_0 + l_1 + l_2) + \partial Q) \otimes T^{\omega(D)} q = 0. \end{aligned} \tag{4.1}$$

Similarly, we can explicitly construct correction terms as follows for the general toric Fano case. We first recall the usual product structure on the torus $T^n = (S^1)^n$, *i.e.* for $(a_1, \ldots, a_n) \in T^n$, $(b_1, \ldots, b_n) \in T^n$, we have

$$(a_1, \ldots, a_n) \times (b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n).$$

Also for subsets $P \subset T^n$, $Q \subset T^n$, we denote by $P \times Q$

$$P \times Q := \{(p \times q) \in T^n | p \in P, q \in Q\}.$$

We may assign the set $P \times Q$ a product orientation.

Recall from [CO] that we have N holomorphic discs of Maslov index 2 (up to $Aut(D^2)$) with boundary on the Lagrangian torus fiber $L \subset M$, which we denote by $D_1, \ldots, D_N$. We denote the homotopy classes of such discs as $\beta_1, \ldots, \beta_N$. Then we have

$$m_{1,\beta_i}(P) = (-1)^n (\partial D_i) \times P. \tag{4.2}$$

Now, we recall the partition

$$\{1, 2, \ldots, N\} = \coprod_{i=1}^{l} I_i$$

with respect to the symplectic energy of discs, i.e. discs D_j for $j \in I_i$ have the same symplectic area, which we denote as e_i. Nonvanishing of Floer cohomology was shown to be equivalent to the following equality for each $i = 1, \ldots, l$:

$$\left[\sum_{j \in I_i} \partial D_j \right] = 0 \quad \text{in} \quad H^*(T^n).$$

Definition 4.3. *For each i, we denote by Q_i a 2-chain with the following property.*

$$\partial Q_i = -\sum_{j \in I_i} \partial D_j. \tag{4.3}$$

We may choose such a 2-chain since RHS is homologus to zero.

Now, consider the chain complex $C^*(L; \Lambda_{nov})$ defined in (2.3) with two different coboundary operators $m_{1,0}$ and m_1. To distinguish the two chain complex, we label them as $(C_1^*(L, \Lambda_{nov}), m_{1,0})$, whose cohomology is isomorphic to singular cohomology, and $(C_2^*(L, \Lambda_{nov}), m_1)$, whose cohomology is a Bott-Morse Floer cohomology. Now we define a chain map between these two complexes when Floer cohomology is non-vanishing.

Definition 4.4. *Let $P \subset L$ be any singular chain. Define*

$$\begin{aligned}\Psi(P) := P &+ \sum_{i=1}^{l} (Q_i \times P) \otimes T^{e_i} + \sum_{i<j} (Q_i \times Q_j \times P) \\ &\otimes T^{e_i + e_j} q^2 + \cdots + \sum_{i_1 < \cdots < i_k} (Q_{i_1} \times \cdots \times Q_{i_k} \times P) \\ &\otimes T^{\sum_{j=1}^{k} e_{i_j}} q^k + \cdots + (Q_1 \times Q_2 \times \cdots \times Q_l \times P) \otimes T^{\sum_{i=1}^{l} e_i} q^l.\end{aligned}$$

By extending linearly over $C^*(L; \Lambda_{nov})$, we obtain a map

$$\Psi : C_1^*(T^n; \Lambda_{nov}) \to C_2^*(T^n; \Lambda_{nov}).$$

Remark 4.5. For simplicity, we define Ψ for singular chains rather than geometric chains. It can be easily modified to the latter case. We also recall that $C^*(L; \Lambda_{nov})$ has a filtration with respect to energy:

$$\mathcal{F}^{\lambda_0} C^* = \{\sum_i a_i [P_i, f_i] T^{\lambda_i} q^{m_i} | \lambda_i \geq \lambda_0 \text{ for all } i\}.$$

Proposition 4.1. *Let L be a Lagrangian torus fiber in toric Fano manifolds, whose Floer cohomology is non-vanishing. Then, the map Ψ defines a filtered chain map which induces an isomorphism on cohomology.*

$$\Psi : H^*(L; \Lambda_{nov}) \to HF^{BM}(L; J_0).$$

More precisely,

(1) $\Psi(m_{1,0} P) = m_1 \Psi(P)$,
(2) $\Psi(\mathcal{F}^\lambda(C_1)) \subseteq \mathcal{F}^\lambda(C_2)$.

Remark 4.6. Note that Ψ is only defined when Floer homology is non-vanishing since otherwise we can not find chains Q_i in 4.3.

Proof. The second property is clear from the definition, hence we only prove the first statement, which we prove by direct calculation. Recall that $m_{1,k} \equiv 0$ for $k \geq 4$ in toric Fano case (see Proposition 7.2 of [CO]). Hence,

$$m_1(\Psi(P)) = m_{1,0}\Psi(P) + m_{1,2}\Psi(P). \tag{4.4}$$

The first component can be written as

$$\begin{aligned} m_{1,0}\Psi(P) &= (-1)^n \partial\Psi(P) \\ &= (-1)^n\Big(\partial P + \sum_{i=1}^{l} \partial(Q_i \times P) \otimes T^{e_i} + \cdots\Big) \\ &= (-1)^n\Big(\Psi(\partial P) + \sum_{i=1}^{l} \partial(Q_i) \times P \otimes T^{e_i} \\ &\quad + \sum_{i<j} (\partial(Q_i \times Q_j) \times P) \otimes T^{e_i+e_j} q^2 + \cdots\Big). \end{aligned}$$

We used the following formula in the last equality, where there is no sign contribution since Q_i's are 2-chains:

$$\begin{aligned} \partial(Q_{i_1} \times \cdots \times Q_{i_k} \times P) &= (Q_{i_1} \times \cdots \times Q_{i_k}) \times \partial P \\ &\quad + \sum_{j=1}^{k} (Q_{i_1} \times \cdots (\partial Q_{i_j}) \times Q_{i_k}) \times P. \end{aligned}$$

For the second component in (4.4),

$$\begin{aligned} & m_{1,2} \sum_{i_1<\cdots<i_{k-1}} (Q_{i_1} \times \cdots \times Q_{i_{k-1}} \times P) \otimes T^{\sum_{l=1}^{k-1} e_{i_l}} q^{k-1} \\ &= \sum_{j=1}^{N} m_{1,\beta_j} \left(\sum_{i_1<\cdots<i_{k-1}} (Q_{i_1} \times \cdots \times Q_{i_{k-1}} \times P) \right) \otimes T^{e_j} T^{\sum_{l=1}^{k-1} e_{i_l}} q^k \\ &= \sum_{i=1}^{l} \big(-(-1)^n \partial Q_i\big) \times \left(\sum_{i_1<\cdots<i_{k-1}} (Q_{i_1} \times \cdots \times Q_{i_{k-1}} \times P) \right) \otimes T^{e_j} T^{\sum_{l=1}^{k-1} e_{i_l}} q^k \\ &= -(-1)^n \sum_{i_1<\cdots<i_k} \sum_{j=1}^{k} (Q_{i_1} \times \cdots (\partial Q_{i_j}) \times Q_{i_k} \times P) \otimes T^{\sum_{l=1}^{k} e_{i_l}} q^k \\ &= \sum_{i_1<\cdots<i_k} (-(-1)^n) \partial(Q_{i_1} \times \cdots \times Q_{i_k}) \times P \otimes T^{\sum_{l=1}^{k} e_{i_l}} q^k. \end{aligned}$$

In the third equality, we used the identity (4.2), (4.3). Hence, we have

$$m_1(\Psi(P)) = m_{1,0}\Psi(P) + m_{1,2}\Psi(P) = (-1)^n \Psi(\partial P) = \Psi(m_{1,0} P). \quad \square$$

The arguments in this section (hence of the whole paper) can be extended to the case with different spin structures. Extension to the case with flat bundles over a Lagrangian submanifold is possible in the case that non-vanishing Floer cohomology occurs when for each $i = 1, \ldots, l$ the holonomies along discs D_j are equal for all $j \in I_i$ so that we can define Q_j. This includes all the examples we showed in the last section.

5. A Direct Computation of Ring Structure

Now, we provide two different computations of Floer cohomology rings of torus fibers in toric Fano manifolds. In this section, we give a direct computation using the classification of holomorphic discs by Oh and the author in [CO]. For simplicity, we carry out calculations for degree 1 generators, which is enough to see the whole algebraic structure of the ring due to associativity.

First we choose the generators C_i of $H^1(L)$ for $i = 1, \ldots, n$.

Definition 5.1. *Let l_i be a circle $1 \times \cdots S^1 \cdots \times 1$, where S^1 is the i^{th} circle of $(S^1)^n \subset (\mathbb{C}^*)^n$. Then torus action of $(S^1)^n$ on L gives corresponding cycles in L, which we also denote as l_i by abuse of notation. For $i = 1, \ldots, n$, denote by $C_i \in H^1(L)$ the Poincaré dual of the cycle*

$$(-1)^{i-1}(l_1 \times \cdots \times \hat{l_i} \times \cdots l_n).$$

Similarly, we denote by $C_{i,j} \in H^2(L)$ the Poincaré dual of the cycle

$$(l_1 \times \cdots \times \hat{l_i} \times \cdots \times \hat{l_j} \times \cdots l_n)$$

for $i \neq j$, and we also define $C_{i_1,\ldots,i_k} \in H^k(L)$ similarly for the index set $\{i_1, i_2, \ldots, i_k\}$.

Now we show that C_i's generate the Floer cohomology ring $HF^{BM}(L; J_0)$.

Proposition 5.1. *Let L be a Lagrangian torus fiber whose Floer cohomology group $HF^{BM}(L; J_0)$ is nonvanishing, thus isomorphic to $H^*(L; \Lambda_{nov})$. Then, for each i, C_i is a Floer-cycle without any correction terms, and Floer cohomology $HF^{BM}(L; J_0)$ is generated by C_i for $i = 1, \ldots, n$ as a ring.*

Proof. From the construction in Definition 4.4, any correction term added to C_i, like $PD(C_i) \times Q_j$, is supposed to have chain dimension $n+1$ or higher. Hence, as a current in L, they are zero. Hence, C_i itself is a Floer-cycle.

To see that $\{C_i\}$ generate the Floer cohomology ring, note that

$$m_2(C_{i_1}, m_2(C_{i_2}, \ldots, m_2(C_{i_{k-1}}, C_{i_k}) \cdots)$$

is a Floer cycle whose index zero part is

$$m_{2,0}(C_{i_1}, m_{2,0}(C_{i_2}, \ldots, m_{2,0}(C_{i_{k-1}}, C_{i_k}) \cdots).$$

Since $m_{2,0}$ is nothing but the cup product, hence the latter equals $C_{i_1,\ldots,i_k} \in H^k(L)$ up to sign. Note that all the other terms (terms containing $m_{2,\beta}$ with non-zero β) are higher order terms with respect to the filtration by T. Hence, these elements generate the ring $HF^{BM}(L; J_0)$. □

Remark 5.2. For the sign convention for the cup product, see [FOOO] Convention 25.14.

Now, we compute the quantum contribution. We first state the following lemma which is a special case of Proposition 2.1.

Lemma 5.2. *Let $\beta \in \pi_2(M, L)$ be a homotopy class. Then the degree (as a cochain) of $m_{2,\beta}(C_i, C_j)$ is given by*

$$deg(C_i) + deg(C_j) - \mu(\beta) = 2 - \mu(\beta).$$

Hence, we have a non-trivial $m_{2,\beta}$ product between the generators C_i for β with $\mu(\beta) = 0$ or 2.

The product when $\mu(\beta) = 0$ is the classical cup product, hence we consider the contributions from homotopy classes with Maslov index two. Let us recall the definition of $m_{2,\beta}$,

$$m_{2,\beta}(C_i, C_j) = (-1)^{n+1}((\mathcal{M}_3^{\text{main}}(\beta_k)\, {}_{ev_1,ev_2} \times (C_i \times C_j), ev_0), \tag{5.1}$$

where i is an embedding of cycles into L. Recall that the *main* component is one of the components of the moduli space of discs with marked points ev_0, ev_1, ev_2 which lie on the disc counter-clockwise direction. The fact that we use only the main component of the moduli space is important, and this makes computation a little cumbersome.

To get an intuitive idea about calculations, we first study the case of $\mathbb{C}P^1$.

5.1. *Example : the equator $L \subset \mathbb{C}P^1$.*

Let L be the equator of $\mathbb{C}P^1$, whose Floer cohomology $HF^(L, L)$ is isomorphic to $H^*(S^1)$. We pick a point p which will be an element of both the singular homology $H_0(L)$ and $HF^1(L, J_0)$. Note that the cup product

$$PD(p) \cup PD(p) = 0,$$

since generically two points do not intersect in S^1. In our case, we choose $t \in S^1$ which is not equal to 1, and consider two points p and $q = t \cdot p$. Then, clearly

$$m_{2,0}(p, q) = 0.$$

Now, we consider products $m_{2,\beta}$ with non-zero β. By Lemma 5.2, we only consider β with $\mu(\beta) = 2$. Recall from [C] that there are only two such holomorphic discs D_u, D_l (up to $Aut(D^2)$) with boundary on L, which are nothing but discs covering the upper(lower)-hemisphere D_u (D_l).

Note that both discs intersect p and q. Then, the product $m_{2,D_u}(p, q)$ is a certain part of the boundary of D_u. More precisely, since we only consider the "main" component of the boundary, where ev_0, p, q is ordered counter-clockwise on the boundary of the disc D_u, we obtain a part of S^1 as in Fig. 2. And similarly, the product $m_{2,D_l}(p, q)$ only takes the "main" component of the boundary, where ev_0, p, q is ordered counter-clockwise on the boundary of the disc D_l. Therefore, after adding these two pieces, we obtain the whole equator:

$$m_2(p, q) = (m_{2,D_u}(p, q) + m_{2,D_l}(p, q))T^{\omega(D)}q = [S^1]T^{\omega(D)}q.$$

Hence, $HF(L, J_0)$ is a Clifford algebra with a generator $[p]$ and a unit $1 = [S^1]$ such that

$$\begin{aligned}\widetilde{m_2}([p], [p]) &= [\widetilde{m_2}(p, q)] = [m_2(p, q)] \\ &= [S^1]T^{\omega(D)}q = 1 \cdot T^{\omega(D)}q \in HF(L, J_0).\end{aligned}$$

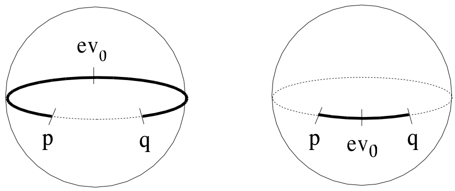

Fig. 2. Main components of evaluation maps for D_u(left) and D_l(right)

5.2. Computation of $m_2(C_i, C_j) + m_2(C_j, C_i)$. The previous example illustrates that the product $m_2(C_i, C_j)$ is a sum of chains in L. Actually this is not a cycle of singular homology in general as seen in Definition 4.4. But we can make the calculation much easier by computing the sum $m_2(C_i, C_j) + m_2(C_j, C_i)$ instead of each individual piece. (The next section generalizes this observation.)

The reason that we compute this sum rather than each part, is that by adding each "main" component we will obtain the whole boundaries of the discs which intersect both C_i and C_j. And computing this sum will be enough to show that the algebra we obtain is a Clifford Algebra. Again the only nontrivial $m_{2,\beta}(C_i, C_j)$ will come from the homotopy classes $\beta_1, \ldots, \beta_N$ of Maslov index 2 by Lemma 5.2.

We recall the relevant fiber product orientation from [C]. This in fact provides the same orientation as in [FOOO], which is described by the orientation of fiber products of Kuranishi structures. (The smooth simplex may be considered as a weakly submersive strongly continuous map from a space with Kuranishi structure with corners where the obstruction bundle is taken to be the normal bundle of the embedding).

Definition 5.3 [C]. *Let X, P, Y be an oriented smooth manifold. Let $f : X \to Y$ and $i : P \to Y$ be a smooth map. We define the orientation of the fibre product $X \times_Y P$ for the case that the map $i : P \to Y$ is an embedding. Let $f : X \to L$ be a submersion and $i : P \to L$ be an embedding. Here we will regard P as a submanifold of L. By x, l, p we denote the dimension of X, L, P. Take a point $q \in f(X) \cap P$. We can choose an oriented basis $< u_1, \ldots, u_l > \in T_qL$ and $< w_1, \ldots, w_p > \in T_qP$ which agrees with the given orientations of L and P. Since f is a submersion, we can choose $< v_1, \ldots, v_l > \in T_pX$ for some $p \in f^{-1}(q)$ such that $(df)_p(v_k) = u_k$ for $k = 1, \ldots, l$. Then, we can choose a basis $< \eta_1, \ldots, \eta_{x-l} > \in Ker(df_p)$ such that $< \eta_1, \ldots, \eta_{x-l}, v_1, \ldots, v_l, >$ is the given orientation of T_pX. Then we define an orientation on the fibre product $X\ {}_f\!\times_i P$ so that $< \eta_1, \ldots, \eta_{x-l}, w_1, \ldots, w_p >$ becomes an oriented basis.*

From now on, [ ] means the oriented frame on its tangent bundle. We remark that we mainly follow the amazing work of the orientation convention in [FOOO]. We may rewrite the following [FOOO].

$$
\begin{aligned}
m_{2,\beta_k}(C_i, C_j) + m_{2,\beta_k}(C_j, C_i) &= (-1)^{n+1}((\mathcal{M}_3(\beta_k)\,{}_{ev_1,ev_2} \times (C_i \times C_j), ev_0) \\
&= (-1)((\mathcal{M}_3(\beta_k)\,{}_{ev_1} \times_i C_i)\,{}_{ev_2} \times_i C_j), ev_0).
\end{aligned}
$$

Recall that

$$
\begin{aligned}
[\mathcal{M}_3(\beta_k)] &= ([\widetilde{\mathcal{M}}(\beta_k)] \times [\partial D_0] \times [\partial D_1] \times [\partial D_2])/PSL(2:\mathbb{R}) \\
&= (-1)([\partial D_0] \times [\partial D_2] \times [\widetilde{\mathcal{M}}(\beta_k)] \times [\partial D_1])/PSL(2:\mathbb{R}) \\
&= (-1)([\partial D_0] \times [\partial D_2] \times [T^n]).
\end{aligned}
$$

Here the last equality follows from [C] Proposition 3.18. By the above definition of fiber product orientation, we have

$$[\mathcal{M}_3(\beta_k)\,{}_{ev_1}\times C_i] = (-1)[\partial D_0]\times[\partial D_2]\times[C_i].$$

Therefore,

$$m_{2,\beta_k}(C_i, C_j) + m_{2,\beta_k}(C_j, C_i) = ([\partial D_0]\times[\partial D_2]\times[C_i])\,{}_{ev_2}\times_i [C_j]).$$

As the marked point travels around the 3rd marked point ∂D_2, its trajectory in L is

$$v_{k1}l_1 + \cdots + v_{kn}l_n.$$

Here, v_k for $k = 1, \ldots, N$ are normal vectors to the codimension 1 faces of the moment polytope for M,

$$\begin{aligned}
[\partial D_2]\times[C_i] &= [v_{k1}l_1 + \cdots + v_{kn}l_n]\times(-1)^{i-1}[l_1\times\cdots\times\hat{l_i}\times\cdots\times l_n]\\
&= (-1)^{i-1}[v_{ki}l_i\times l_1\times\cdots\times\hat{l_i}\times\cdots\times l_n]\\
&= v_{ki}[l_1\times\cdots\times l_n] = v_{ki}[T^n].
\end{aligned}$$

Therefore,

$$\begin{aligned}
m_{2,\beta_k}(C_i, C_j) + m_{2,\beta_k}(C_j, C_i) &= ([\partial D_0]\times[\partial D_2]\times[C_i])\,{}_{ev_2}\times_i [C_j])\\
&= ([\partial D_0]\times[v_{ki}T^n])\,{}_{ev_2}\times_i [C_j]\\
&= v_{ki}([\partial D_0]\times[C_j])\\
&= v_{ki}v_{kj}[T^n]\\
&= v_{ki}v_{kj}\cdot 1.
\end{aligned}$$

Also note that signs of the following cup product works as

$$m_{2,0}(C_i, C_j) = -m_{2,0}(C_j, C_i).$$

Therefore,

Proposition 5.3.

$$\begin{aligned}
m_2(C_i, C_j) + m_2(C_j, C_i) &= \sum_{k=1}^{N}(m_{2,\beta_k}(C_i, C_j) + m_{2,\beta_k}(C_j, C_i))T^{e_k}q\\
&= \sum_{k=1}^{N} v_{ki}v_{kj}T^{e_k}q.
\end{aligned}$$

Now we consider the case when $i = j$. The above formula also works for the case $i = j$ after we perturb C_i by a torus action to $t\cdot C_i$ for some $t \in T^n$. Also we have the following easy lemma.

Lemma 5.4.

$$[m_2(C_i, t\cdot C_i)] = [m_2(t\cdot C_i, C_i)] \;\; \textit{in } HF^*(L, J_0).$$

Corollary 5.5.

$$m_2(C_i, t \cdot C_i) = \sum_{k=1}^{N} \frac{1}{2} v_{ki}^2 \otimes T^{e_k} q.$$

Now, we recall the definition of the Clifford algebra.

Definition 5.4. *Let V be a $\mathbb{Q}$-vector space with a non-degenerate symmetric bilinear form Q on V. The Clifford Algebra $Cl(V, Q)$ is defined as*

$$Cl(V, Q) = T(V)/I(V, Q),$$

where $T(V)$ is the tensor algebra

$$T(V) = \bigoplus_{k=0} V^k,$$

and $I(V, Q)$ is the ideal in $T(V)$ generated by elements

$$v \otimes v - \frac{1}{2} Q(v, v)1 \ \ \textit{for} \ \ v \in V.$$

Alternatively, one may define Cl(V,Q) with the relation

$$v \cdot w + w \cdot v = Q(v, w).$$

In our case, we consider a universal Novikov ring Λ_{nov} instead of $\mathbb{Q}$ as a coefficient. Now Proposition 5.3 and Corollary 5.5 imply our main theorem.

Theorem 5.6. *Let $L \subset M$ be a Lagrangian torus fiber in the Fano toric manifold whose Floer cohomology is non-vanishing. Then, the Floer cohomology ring $(HF^{BM}(L; J_0), \widetilde{m_2})$ has a Clifford Algebra structure with generators given by C_i for $i = 1, \ldots, n$, and its relations as*

$$\widetilde{m_2}(C_i, C_j) + \widetilde{m_2}(C_j, C_i) = Q(C_i, C_j),$$

where the symmetric bilinear form Q is given by

$$Q(C_i, C_j) = \sum_{k=1}^{N} v_{ki} v_{kj} T^{e_k} q.$$

Furthermore, this Q agrees with the Hessian of the superpotential $W(\Theta)$ of the mirror Landau-Ginzburg model of toric Fano manifold (upon the substitution "$T^{2\pi} = e^{-1}$").

Proof. We only need to check the last statement. Recall that the superpotential is given as (see for example [HV, CO])

$$W(\Theta) = \sum_{k=1}^{N} e^{-y_k - <\Theta, v_k>}.$$

Hence, it is easy to see that

$$\frac{\partial W(\Theta)}{\partial \Theta_i} = -\sum_{k=1}^{N} v_{ki} e^{-y_k - <\Theta, v_k>},$$

and

$$\frac{\partial^2 W(\Theta)}{\partial \Theta_i \partial \Theta_j} = \sum_{k=1}^{N} v_{ki} v_{kj} e^{-y_k - <\Theta, v_k>}.$$

Here Θ is a the coordinate on the mirror Landau-Ginzburg model, and it is related to the toric manifold M as follows. The real part of the variable Θ is given by $(a_1, \ldots, a_n) \in P$, which is the image point of the Lagrangian torus fiber L in the moment polytope P, whereas the imaginary part is given by the holonomy of the flat line bundle along L. When the Floer cohomology of L is non-vanishing, the corresponding Θ becomes the critical point of W as shown in [CO], and 2π times its exponent $(y_k + < \Theta, v_k >)$ becomes the area of holomorphic discs which we denoted as e_k in this paper. If we ignore harmless grading q, and with the equivalence "$T^{2\pi} = e^{-1}$",

$$e^{-y_k - <\Theta, v_k>} = T^{e_k}.$$

Hence, this proves the claim. □

More correspondences will be given in the next section.

6. Analogue of Divisor Equation for Discs

In this section, we introduce an analogue of the divisor equation and this will explain how Clifford algebra structure naturally arises for Floer cohomology rings of Lagrangian submanifolds, as this section provides the alternative proof of results in the previous section. To state the result, it is better to write down the formula in terms of the L_∞-algebra (strong homotopy Lie algebra) maps. Recall that every A_∞-algebra has an underlying L_∞-algebra structure by the following relation (this is similar to the fact that the commutator of an associative algebra A defines a Lie algebra on A).

Theorem 6.1 ([LM, LS](or see [Fu2])). *An A_∞-structure $\{m_k : \otimes^k V \to V\}$ on the graded vector space V induces an L_∞-structure $\{l_k : \otimes^k V \to V\}$, where for all non-negative integer k, $\beta \in \pi_2(M, L)$,*

$$l_{k,\beta}(v_1 \otimes \cdots \otimes v_k) = \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} m_{k,\beta}(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}), \tag{6.1}$$

$$\textit{with } \epsilon(\sigma) = \sum_{i,j \text{ with } i<j, \sigma(i)>\sigma(j)} (deg(v_i) + 1)(deg(v_j) + 1).$$

Namely, the l_k map is a skew-symmetrization of the m_k map. For example,

$$l_{2,\beta}(x, y) = m_{2,\beta}(x, y) + (-1)^{(x+1)(y+1)} m_{2,\beta}(y, x).$$

The following is the Divisor equation of Gromov-Witten invariants. For a general equation involving gravitational descendents, see [H1].

Proposition 6.2 ([KM]). *Let M be a convex algebraic manifold, For $\alpha \in H_2(M)$, let $I^M_{g,m,\alpha} : H^*(V)^{\otimes n} \to H^*(\overline{M}_{g,n})$ be the Gromov-Witten invariants. Then, for $\gamma_1 \in H^2(M)$, and $\pi_n : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$, we have*

$$\pi_{n*}(I^M_{g,n,\alpha}(\gamma_1 \otimes \cdots \otimes \gamma_n) = (\alpha \cdot \gamma_1) I^M_{g,n-1,\alpha}(\gamma_2 \otimes \cdots \otimes \gamma_n).$$

Now, we state an **analogue of the divisor equation for discs**.

Proposition 6.3. *If P_i is a cycle of cohomology degree 1 in L, then for $k \geq 1$,*

$$l_{k,\beta}(P_1, \ldots, P_k) = (P_i \cdot \partial\beta)\, l_{k-1,\beta}(P_1, \ldots, \widehat{P_i}, \ldots, P_k), \tag{6.2}$$

where $\partial : \pi_2(M, L) \to \pi_1(L)$, and $\widehat{P_i}$ means that P_i term is omitted.

Remark 6.1. Here is the sign convention for the intersection of two chains P, Q of complementary degree in L. At each transversal intersection $p \in P \cap Q$, for a basis $[T_p P]$ of tangent space $T_p P$, and similarly for $[T_p Q]$ and $[T_p L]$, if $[T_p P][T_p Q]$ has the same orientation as $[T_p L]$ then it is counted as $(+1)$, otherwise it is counted as (-1).

Before we prove the proposition, we show how to prove the results in the previous section using the analogue of the divisor equation for discs. Recall that by $\beta_k \in \pi_2(M, L)$ for $k = 1, \ldots, N$, we denote the homotopy class of a holomorphic disc of Maslov index two corresponding to N codimension one facets of the moment polytope([CO]).

By definition, we have

$$\begin{aligned} l_{0,\beta_k} &= m_{0,\beta_k} = T^{e_k} q, \\ l_{1,\beta_k}(P) &= m_{1,\beta_k}(P). \end{aligned}$$

For degree 1 generators C_i, C_j of $H^*(L)$ which are defined in Definition 5.1, we apply the divisor equation for discs repeatedly

$$\begin{aligned} l_{2,\beta_k}(C_i, C_j) &= (C_i \cdot \partial\beta_k) l_{1,\beta_k}(C_j) \\ &= (C_i \cdot \partial\beta_k)(C_j \cdot \partial\beta_k) l_{0,\beta_k} \\ &= (v_{ki})(v_{kj}) \otimes T^{e_k} q. \end{aligned}$$

The last equality follows from the definitions that

$$C_i = (-1)^{i-1}(l_1 \times \cdots \times \hat{l_i} \times \cdots l_n),$$

$$\partial\beta_k = v_{k1} l_1 + \cdots + v_{kn} l_n.$$

Hence, it is easy to see that

$$C_i \cdot \partial\beta_k = (-1)^n v_{ki}.$$

Hence, we obtain Lemma 5.3, as we have $l_2(C_i, C_j) = m_2(C_i, C_j) + m_2(C_j, C_i)$.

In general, we have

Corollary 6.4. *For any Lagrangian torus fiber L in the toric Fano manifold M(whose Floer cohomology may be vanishing), we have*

$$\begin{aligned} l_m(C_{i_1}, \dots, C_{i_m}) &= \sum_{k=1}^{N} l_{m,\beta_k}(C_{i_1}, \dots, C_{i_m}) \\ &= (-1)^{nm} \sum_{k=1}^{N} v_{ki_1} \cdots v_{ki_m} \otimes T^{e_k} q \\ &= (-1)^{(n-1)m} \frac{\partial^m W(\Theta)}{\partial \Theta_{i_1} \cdots \partial \Theta_{i_m}}, \end{aligned}$$

where $W(\Theta)$ is the superpotential of the Landau-Ginzburg mirror model of M.

This corollary extends the correspondence observed in [CO], $m_0 = l_0 = W(\Theta)$. Note that such correspondence, considered at every Lagrangian torus fiber with flat line bundles, may be used to recover the superpotential $W(\Theta)$ of the Landau-Ginzburg mirror. But the above corollary indicates that in fact one Lagrangian torus fiber with a fixed flat line bundle (whose Floer cohomology may be vanishing) in M is enough to recover the superpotential in this case: It is because the superpotential is a holomorphic function on $(\mathbb{C}^*)^n$ and all its partial derivatives at the corresponding point on the mirror is given from the products of the L_∞-algebra by the above correspondence.

Also note that the above product does not depend on the choice of cycles C_* since it is determined by the intersection numbers which only depend on the homology class of C_*. These are also invariants with respect to the change of an almost complex structure. By J_0 we denote the standard complex structure of the toric Fano manifold M, and denote the corresponding l_m products by $l_m^{J_0}$.

Proposition 6.5. *Let L be any Lagrangian torus fiber of the toric Fano manifold M. Let $J_1 \in J_{reg}(M)$ be a tame almost complex structure such that all simple J-holomorphic discs are Fredholm regular. Then, for $k = 1, \dots, N$, we have*

$$l^{J_0}_{m,\beta_k}(C_{i_1}, \dots, C_{i_m}) = l^{J_1}_{m,\beta_k}(C_{i_1}, \dots, C_{i_m})$$

in $H^*(L; \Lambda_{nov})$.

Proof. As in [MS], one can prove that the subset $J_{reg}(M)$ is of the second category and path connected. Since any J-holomorphic disc with Maslov index two is simple and its homotopy class β_k is minimal, the moduli space $\mathcal{M}(\beta_k; J_t)$ of J_t holomorphic discs is in fact a manifold without boundary. Then, by choosing a path $J_t \in J_{reg}(M)$, we set

$$\mathcal{M}_{m+1}(\beta_k; \mathcal{J}) = \cup_{t\in[0,1]}\big(\{t\} \times \mathcal{M}_{m+1}(\beta_k; J_t)\big).$$

Then we have

$$\begin{aligned} \partial\big(\mathcal{M}_{m+1}(\beta_k; \mathcal{J})_{ev} \times (\prod_{j=1}^{m} C_{i_m})\big) &= \mathcal{M}(\beta_k; J_1)_{ev} \\ &\quad \times (\prod_{j=1}^{m} C_{i_m}) - \mathcal{M}(\beta_k; J_0)_{ev} \times (\prod_{j=1}^{m} C_{i_m}) \end{aligned}$$

which proves the proposition. □

Now we begin the proof of Proposition 6.3

Proof. A rough idea is that if P_i is a cycle of codimension 1, then it always intersects with the boundary of a J-holomorphic disc of homotopy class β with $(P_i \cdot \partial\beta)$ number of times (counted with sign). Hence, if P_i is dropped from the argument of m_k, the resulting image should be the same up to a multiple of the intersection number. While this is the same idea as the "divisor equation" in Gromov-Witten theory, there are a few differences. First, the m_k map records only part of the boundaries of J-holomorphic discs as it is defined by using only the main component $\mathcal{M}_k^{main}$. But note that intersection of P_i and the disc may occur at an arbitrary point of the domain ∂D^2. Hence we consider the L_∞-algebra map, l_k, which will be shown to record the whole boundaries of discs. Then, the next step involves delicate sign analysis in the case that the parameter P_i is dropped from the $m_k(P_1, \ldots, P_k)$ to obtain $m_{k-1}(P_1, \ldots, \widehat{P_i}, \ldots, P_k)$ for a codimension 1 cycle P_i.

Suppose there exists an element $((D^2, \vec{z}), h) \in \mathcal{M}_{k+1}^{main}(\beta)$, where $h : D^2 \to M$ is a J-holomorphic map. We also assume that for fixed chains $P_1, \ldots, P_k$ in L, we have $h(z_i) \in P_i$ for each $i = 1, \ldots, k$. Boundary marked points $z_1, \ldots, z_k$ (z_0 is omitted here) separate ∂D^2 into k connected pieces. And only the component between the k^{th} and 1^{th} marked point contributes to the chain $m_k(P_1, \ldots, P_k)$, as it is obtained as an evaluation of 0^{th} marked point which lies between those two marked point in $\mathcal{M}_{k+1}^{main}$. Now, it is easy to see that up to sign, other connected components will contribute to the chains

$$m_k(P_2, \ldots, P_k, P_1), m_k(P_3, \ldots, P_1, P_2), \cdots, m_k(P_k, P_1, \cdots, P_{k-1}),$$

and $((D^2, \vec{z}), h)$ will not contribute to other terms of $l_k(P_1, \ldots, P_k)$ generically due to the ordering of marked points. Now, we show that signs in (6.1) are needed to have a coherent sign in the images of the above chains.

We recall the following lemma from [FOOO].

Lemma 6.6 (FOOO, Lemma 25.3). *Let σ be the transposition element $(i, i+1)$ in the k^{th} symmetric group S_k. Then the action of σ on $\mathcal{M}_1(\beta, P_1, \ldots, P_i, P_{i+1}, \ldots, P_k)$ by changing the order of marked points is described by the following:*

$$\begin{aligned}&\sigma(\mathcal{M}_1(\beta, P_1, \ldots, P_i, P_{i+1}, \ldots, P_k))\\ &\quad = (-1)^{(deg\ P_i+1)(deg\ P_{i+1}+1)}\mathcal{M}_1^\sigma(\beta, P_1, \ldots, P_{i+1}, P_i, \ldots, P_k).\end{aligned}$$

Remark 6.2. In the first term, $\mathcal{M}_1(\beta, P_1, \ldots, P_i, P_{i+1}, \ldots, P_k))$ is defined by using the moduli space with boundary marked points lying cyclically, whereas in the second term, $\mathcal{M}_1^\sigma(\beta, P_1, \ldots, P_{i+1}, P_i, \ldots, P_k)$ is defined by using the moduli space $\mathcal{M}_k^\sigma$ with boundary marked points lying in the order $z_0, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_k$. Namely, in the latter case, only the labeling of two marked point is changed from the first case.

Let $\sigma \in S_n$ be a permutation denoted by $(1, 2, \ldots, k)$ (i.e. $1 \to 2, 2 \to 3, \cdots, k \to 1$). Then, by applying the above lemma repeatedly, we have

$$\sigma(\mathcal{M}_1(\beta, P_1, \ldots, P_k)) = (-1)^{\epsilon(\sigma)}\mathcal{M}_1^\sigma(\beta, P_2, \ldots, P_k, P_1),$$

where $\epsilon(\sigma)$ is the same sign as appeared in (6.1). Now, it is not hard to check that the latter has the same sign as $(-1)^{\epsilon(\sigma)}\mathcal{M}(\beta; P_2, \ldots, P_k, P_1)$. Here, $\mathcal{M}_1^\sigma(\beta, P_2, \ldots, P_k, P_1)$ and $\mathcal{M}(\beta; P_2, \ldots, P_k, P_1)$ have different images coming from the same set of J-holomorphic discs. This is because marked points in the former case lie on the circle in the order

$0, k, 1, 2, \ldots, k-1$ and in the latter case marked points lie on the circle in the order $k, 0, 1, 2, \ldots, k-1$. Hence, as we evaluate at 0^{th} marked point, their images come from the neighboring connected components of ∂D^2 separated by marked points. Hence, this proves that with the sign given as in (6.1), the image of the boundary of discs can be glued in the l_k map.

Now, we explain the second step which computes the change of sign as the argument P_i is dropped from the $m_k(P_1, \ldots, P_k)$ to obtain $m_{k-1}(P_1, \ldots, \widehat{P_i}, \ldots, P_k)$ for a codimension 1 cycle P_i. In the computation, we will calculate the i^{th} fiber product with P_i to remove the term from the fiber product.

Let $((D^2, \vec{z}), h) \in \mathcal{M}_k^{main}(\beta)$, where $h : D^2 \to M$ is a J-holomorphic map. We also assume that for fixed chains $P_1, \ldots, \widehat{P_i}, \ldots, P_k$ in L, we have $h(z_i) \in P_i$ for each $i = 1, \ldots, \widehat{i}, \ldots, k$. And let P_i be a cycle of codimension one in L. If $[P_i] \cdot \partial\beta$ is not zero, then, a generic cycle P_i should intersect with $h(\partial D^2)$ transversally. Hence, we obtain a corresponding element $((D^2, \vec{z'}), h) \in \mathcal{M}_{k+1}^{main}(\beta)$ with $h(z'_i) \in P_i$ for each $i = 1, \ldots, k$.

We recall that the moduli space $\mathcal{M}_{k+1}^{main}(\beta)$ is oriented as

$$\big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C}),$$

where $[\partial D_i^2]$ denotes the tangent vector corresponding to the counterclockwise rotation of the i^{th} marked point. If we take $[\partial D_i^2]$ to the last, we have

$$= (-1)^{s_1}\Big(\big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big) \times [\partial D_i^2],$$

where $s_1 = k - i + 1$. Now we write

$$\begin{aligned}\mathcal{M}(\beta; P_1, \ldots, P_k) &= (-1)^{s_2}\mathcal{M}_{k+1}^{\text{main}}(\beta)_{ev_1,\ldots,ev_k} \times (P_1 \times \cdots \times P_k)\\ &= (-1)^{s_3}\Big(\cdots\big(\mathcal{M}_{k+1}^{\text{main}}(\beta)_{ev_1} \times P_1\big)\cdots_{ev_k} \times P_k\Big),\end{aligned}$$

where $s_2 = (n+1)\sum_{l=1}^{k-1}\sum_{j=1}^{l} deg(P_j)$, $s_3 = \sum_{l=1}^{k-1}\sum_{j=1}^{l} deg(P_j)$.

Then, if we look at the term $\big(\mathcal{M}_{k+1}^{main}(\beta)_{ev_1} \times P_1\big)$, it can be oriented as

$$\begin{aligned}&(-1)^{s_1}\Big(\Big(\big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big) \times [\partial D_i^2]\Big)_{ev_1}\\ &\quad \times [P_1]\\ &= (-1)^{s_4}\Big(\Big(\big([\widetilde{\mathcal{M}}^o(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big) \times [\partial D_i^2]\\ &\quad \times [L]\Big)_{ev_1} \times [P_1]\\ &= (-1)^{s_4}\Big(\Big(\big([\widetilde{\mathcal{M}}^o(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big)\\ &\quad \times [\partial D_i^2] \times [P_1]\\ &= (-1)^{s_5}\Big(\Big(\big([\widetilde{\mathcal{M}}^o(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big) \times [P_1]\\ &\quad \times [\partial D_i^2]\\ &= (-1)^{s_6}\Big(\Big(\big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\big)/PSL(2;\mathbb{C})\Big)_{ev_1} \times [P_1]\Big)\\ &\quad \times [\partial D_i^2],\end{aligned}$$

where $[\widetilde{\mathcal{M}}^o(\beta)][L] = [\widetilde{\mathcal{M}}(\beta)]$ and $s_4 = n(k+2) + s_1$, $s_5 = s_4 + p_1 = s_4 + dim(P_1)$ and $s_6 = s_5 + n(k+1)$. Hence $s_6 = n + p_1 + k - i + 1 = deg(p_1) + k - i + 1$. Now we repeat this process up to P_{i-1} and

$$\left(\cdots\left(\mathcal{M}_{k+1}^{main}(\beta)_{ev_1} \times P_1\right)\cdots \times P_{i-1}\right)$$

is oriented as

$$(-1)^{s_7}\left(\cdots\left(\left([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\right)/[PSL(2;\mathbb{C})]\right)_{ev_1} \right.$$
$$\left. \times [P_1]\right) \times \cdots \times [P_{i-1}]\Big) \times [\partial D_i^2]$$

with $s_7 = deg(P_2) + \cdots + deg(P_{i-1}) + k - i + 1$. Then,

$$\left(\cdots\left(\mathcal{M}_{k+1}^{\text{main}}(\beta)_{ev_1} \times P_1\right)\cdots \times P_{i-1}\right)_{ev_i} \times P_i$$

is oriented as

$$\begin{aligned}
&(-1)^{s_8}\Big(\cdots\Big(\Big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\Big)/[PSL(2;\mathbb{C})]\Big)_{ev_1} \\
&\quad \times [P_1]\Big) \times \cdots \times [P_{i-1}]\Big)^o \times [\partial D_i^2][P_i] \\
&= (-1)^{s_9}\Big(\cdots\Big(\Big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\Big)/[PSL(2;\mathbb{C})]\Big)_{ev_1} \\
&\quad \times [P_1]\Big) \times \cdots \times [P_{i-1}]\Big)^o \times [P_i][\partial D_i^2] \\
&= (-1)^{s_{10}}\Big(\cdots\Big(\Big([\widetilde{\mathcal{M}}(\beta)] \times [\partial D_0^2] \times \cdots \widehat{[\partial D_i^2]} \times \cdots \times [\partial D_k^2]\Big)/[PSL(2;\mathbb{C})]\Big)_{ev_1} \\
&\quad \times [P_1]\Big) \times \cdots \times [P_{i-1}]\Big),
\end{aligned}$$

where $s_8 = \sum_{j=1}^{i-1} deg(P_j) + n$, $s_9 = s_8 + (n-1)\cdot 1$, $s_{10} = s_9 + \epsilon$. The last equality follows from the sign of the intersection $[P_i][\partial\beta] = (-1)^\epsilon[L]$. Hence $s_{10} = \epsilon + (n-1) + n + \sum_{j=1}^{i-1} deg(P_j) + k - i + 1$.

Now, the last expression can be considered as an orientation of

$$\left(\cdots\left(\mathcal{M}_k^{\text{main}}(\beta)_{ev_1} \times P_1\right)\cdots_{ev_{i-1}} \times P_{i-1}\right).$$

Hence, orientation of

$$\mathcal{M}(\beta; P_1, \ldots, P_k)$$

corresponds to

$$\begin{aligned}
&(-1)^{s_3}\left(\cdots\left(\mathcal{M}_{k+1}^{main}(\beta)_{ev_1} \times P_1\right)\cdots_{ev_k} \times P_k\right), \\
&\subset (-1)^{s_{11}}\left(\cdots\left(\mathcal{M}_k^{main}(\beta)_{ev_1} \times P_1\right)\cdots \times \widehat{P_i}\right) \times \cdots_{ev_k} \times P_k\Big), \\
&= (-1)^{s_{12}}\mathcal{M}(\beta; P_1, \ldots, \widehat{P}_i, \ldots, P_k),
\end{aligned}$$

where $s_{11} = s_3 + s_{10}$, s_{12} is obtained in a similar way as s_3 and we have $s_{12} = \epsilon + k - i + (k-i)deg(P_i) = \epsilon$, since $deg(P_i) = 1$.

This proves that if P_i intersect with the J-holomorphic disc at several boundary points, then at each intersection, the sign change between $m_k(P_1, \ldots, P_k)$ and $m_{k-1}(P_1, \ldots, \widehat{P}_i, \ldots, P_k)$ of the contribution from this J-holomorphic disc, is given by $(-1)^\epsilon$, where

ϵ is the sign of the intersection between $[P_i]$ and the J-holomorphic disc at each intersection point.

Now, we prove the proposition. Note that in the expression $l_k(P_1, \ldots, P_k)$, a term $m_k(P_1, \ldots, P_k)$ carries the same sign as the term $m_k(P_1, \ldots, \widehat{P_i}, \ldots, P_k, P_i)$ or any other term which is obtained by moving P_i around if $deg(P_i) = 1$. Consider the J-holomorphic disc contributing non-trivially to the expression $m_{k-1,\beta}(P_1, \ldots, \widehat{P_i}, \ldots, P_k)$. Then the whole boundary of this disc contributes to $l_{k-1,\beta}(P_1, \ldots, \widehat{P_i}, \ldots, P_k)$. Generically, P_i may intersect with $\partial\beta$ at arbitrary points of the domain ∂D^2. Suppose such disc intersect P_i between j and $j+1^{\text{th}}$ marked point with intersection sign $(-1)^\epsilon$ for $j > i+1$ without loss of generality. Then, the whole boundary of this disc would contribute to the terms (while divided into several pieces)

$$\begin{aligned}
&m_{k,\beta}(P_1, \ldots, \widehat{P_i}, \ldots, P_{j-1}, P_i, P_j, \ldots, P_k) \\
&\quad + m_{k,\beta}(P_2, \ldots, \widehat{P_i}, \ldots, P_{j-1}, P_i, P_j, \ldots, P_k, P_1) \\
&\quad + \cdots + m_{k,\beta}(P_k, P_1, \cdots, \widehat{P_i}, \ldots, P_{j-1}, P_i, P_j, \ldots, P_{k-1}).
\end{aligned}$$

By applying the sign analysis, the above terms correspond to terms in $l_k(P_1, \ldots, \widehat{P_i}, \ldots, P_k)$ with multiplicity $(-1)^\epsilon$. By adding up all the possibilities of intersections between P_i and $\partial\beta$, we obtain the proposition. $\square$

7. Examples

In what follows we omit area terms $T^{e_i} q$ for simplicity.

7.1. The Clifford torus $T^2 \subset \mathbb{C}P^2$. In [CO], it is shown that the Clifford torus is the only Lagrangian torus fiber whose Floer cohomology is non-vanishing, which is isomorphic to $H^*(T^2; \Lambda_{nov})$. Hence, by Theorem 5.6, $HF^*(T^2, T^2)$ as a ring is a Clifford algebra with two generators C_1 and C_2. Using its moment polytope data, one can immediately compute the matrix of the symmetric bilinear form

$$Q = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}.$$

(See [KL] for computations of the B-model by physical arguments and the predictions made for the Clifford torus case.)

But it is also instructive to compute $m_2(C_1, C_1)$ and $m_2(C_1, C_2)$ directly. Consider T^2 as a rectangle whose edges are glued accordingly. Let us assume that its edges are cycles l_1, l_2 as given in Definition 5.1. Then by definition, we have

$$C_1 = l_2, C_2 = -l_1.$$

First we consider $m_2(C_1, C_1) = m_2(l_2, l_2)$. As before, we pick $t \in T^2$ so that l_2 and tl_2 do not intersect. Then,

$$m_{2,0}(l_2, tl_2) = 0.$$

Recall that there exist 3 holomorphic discs (up to $Aut(D^2)$) with boundary trajectory as

$$\partial D_0 = -l_1 - l_2, \ \partial D_1 = l_1, \ \partial D_2 = l_2.$$

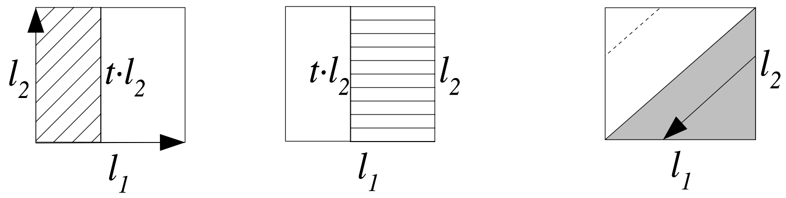

Fig. 3. $m_{2,\beta_0}(l_2, tl_2) + m_{2,\beta_1}(l_2, tl_2) = L, \quad m_{2,\beta_0}(l_1, l_2)$

For $m_{2,\beta}(l_2, tl_2)$, holomorphic discs D_0, D_1 contribute nontrivially. Since we only consider the *main* components as in Fig. 3, we have

$$m_{2,\beta_0}(l_2, tl_2) + m_{2,\beta_1}(l_2, tl_2) = [L].$$

Therefore, we have

$$\widetilde{m_2}([C_1], [C_1]) = [m_2(C_1, tC_1)] = [L]T^{\omega(D)}q.$$

This agrees with Corollary 5.5. From Fig. 3, it is easy to see that the product $m_2(l_2, tl_2)$ is independent of $t \in T^2$.

Now, we consider the product $m_2(C_1, C_2)$,

$$m_2(C_1, C_2) = m_{2,0}(C_1, C_2) + \sum_{\beta} m_{2,\beta}(C_1, C_2)T^{Area(\beta)}q.$$

Here $m_{2,0}(C_1, C_2)$ is a cup product which is nothing but the Poincaré dual of the intersection $C_1 \cap C_2 = point$. But as we discussed in Example 4.2, the point itself is not a Floer-cycle. The needed correction term Q is obtained in this case from the quantum contribution $m_{2,\beta}$.

It is easy to see that only the β_0 disc contributes to the product $m_{2,\beta}$, since other discs generically do not intersect both C_1, C_2. Now, $m_{2,\beta_0}(C_1, C_2)$ is not a cycle but a chain as drawn in Fig. 3, since we only evaluate on the main component. This is the chain Q that we added to make $< pt >$ a Floer cycle in Definition 4.4,

$$\begin{aligned} m_2(C_1, C_2) &= m_{2,0}(C_1, C_2) + m_{2,\beta_0}(C_1, C_2)T^e q \\ &= < pt > + QT^e q. \end{aligned}$$

7.2. $\mathbb{C}P^1 \times \mathbb{C}P^1$. Consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ whose moment map image is a rectangle. For the equator $S^1 \in \mathbb{C}P^1$, $S^1 \times S^1 \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ has nontrivial Floer cohomology and its product structure is given by

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

7.3. $\mathbb{C}P^n$. The example $\mathbb{C}P^2$ easily generalizes to $\mathbb{C}P^n$. The Floer cohomology of the Clifford torus $T^n \subset \mathbb{C}P^n$ becomes the Clifford Algebra with n generators with symmetric bilinear form as

$$Q = \begin{pmatrix} 2 & 1/2 & \cdots & 1/2 \\ 1/2 & 2 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 2 \end{pmatrix}.$$

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