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Geometry of canonical bases and mirror symmetry

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Abstract A *decorated surface* S is an oriented surface with boundary and a finite, possibly empty, set of *special* points on the boundary, considered modulo isotopy. Let G be a split reductive group over $\mathbb{Q}$. A pair (G, S) gives rise to a moduli space $\mathcal{A}_{\mathrm{G},S}$, closely related to the moduli space of G-local systems on S. It is equipped with a positive structure (Fock and Goncharov, Publ Math IHES 103:1–212, 2006). So a set $\mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t)$ of its integral tropical points is defined. We introduce a rational positive function $\mathcal{W}$ on the space $\mathcal{A}_{\mathrm{G},S}$, called the *potential*. Its tropicalisation is a function $\mathcal{W}^t : \mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t) \to \mathbb{Z}$. The condition $\mathcal{W}^t \geq 0$ defines a subset of *positive integral tropical points* $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$. For $\mathrm{G} = \mathrm{SL}_2$, we recover the set of positive integral $\mathcal{A}$-laminations on S from Fock and Goncharov (Publ Math IHES 103:1–212, 2006). We prove that when S is a disc with n special points on the boundary, the set $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ parametrises top dimensional components of the fibers of the convolution maps. Therefore, via the geometric Satake correspondence (Lusztig, Astérisque 101–102:208–229, 1983; Ginzburg,1995; Mirkovic and Vilonen, Ann Math (2) 166(1):95–143, 2007; Beilinson and Drinfeld, Chiral algebras. American Mathematical Society Colloquium Publications, vol. 51, 2004) they provide a canonical basis in the tensor product invariants of irreducible modules of the Langlands dual group G^L:

$$(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{\mathrm{G}^L}. \tag{1}$$

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When $\mathrm{G} = \mathrm{GL}_m, n = 3$, there is a special coordinate system on $\mathcal{A}_{\mathrm{G},S}$ (Fock and Goncharov, Publ Math IHES 103:1–212, 2006). We show that it identifies the set $\mathcal{A}^+_{\mathrm{GL_m},S}(\mathbb{Z}^t)$ with Knutson–Tao's hives (Knutson and Tao, The honeycomb model of GL(n) tensor products I: proof of the saturation conjecture, 1998). Our result generalises a theorem of Kamnitzer (Hives and the fibres of the convolution morphism, 2007), who used hives to parametrise top components of convolution varieties for $\mathrm{G} = \mathrm{GL}_m, n = 3$. For $\mathrm{G} = \mathrm{GL}_m, n > 3$, we prove Kamnitzer's conjecture (Kamnitzer, Hives and the fibres of the convolution morphism, 2012). Our parametrisation is naturally cyclic invariant. We show that for any G and $n = 3$ it agrees with Berenstein–Zelevinsky's parametrisation (Berenstein and Zelevinsky, Invent Math 143(1):77–128, 2001), whose cyclic invariance is obscure. We define more general positive spaces with potentials $(\mathcal{A}, \mathcal{W})$, parametrising mixed configurations of flags. Using them, we define a generalization of Mirković–Vilonen cycles (Mirkovic and Vilonen, Ann Math (2) 166(1):95–143, 2007), and a canonical basis in $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$, generalizing the Mirković–Vilonen basis in V_λ. Our construction comes naturally with a parametrisation of the generalised MV cycles. For the classical MV cycles it is equivalent to the one discovered by Kamnitzer (Mirkovich–Vilonen cycles and polytopes, 2005). We prove that the set $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ parametrises top dimensional components of a new moduli space, *surface affine Grasmannian*, generalising the fibers of the convolution maps. These components are usually infinite dimensional. We define their dimension being an element of a $\mathbb{Z}$-torsor, rather then an integer. We define a new moduli space $\mathrm{Loc}_{G^L,S}$, which reduces to the moduli spaces of G^L-local systems on S if S has no special points. The set $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ parametrises a basis in the linear space of regular functions on $\mathrm{Loc}_{G^L,S}$. We suggest that the potential $\mathcal{W}$ itself, not only its tropicalization, is important—it should be viewed as the potential for a Landau–Ginzburg model on $\mathcal{A}_{\mathrm{G},S}$. We conjecture that the pair $(\mathcal{A}_{\mathrm{G},S}, \mathcal{W})$ is the mirror dual to $\mathrm{Loc}_{G^L,S}$. In a special case, we recover Givental's description of the quantum cohomology connection for flag varieties and its generalisation (Gerasimov et al., New integral representations of Whittaker functions for classical Lie groups, 2012; Rietsch, A mirror symmetric solution to the quantum Toda lattice, 2012). We formulate equivariant homological mirror symmetry conjectures parallel to our parametrisations of canonical bases.

Contents

1 Introduction

1.1 Geometry of canonical bases in representation theory

1.1.1 Configurations of flags and parametrization of canonical bases

Let G be a split semisimple simply-connected algebraic group over $\mathbb{Q}$. There are several basic vector spaces studied in representation theory of the Langlands dual group G^L:

1. The weight λ component $U(\mathcal{N}^L)^{(\lambda)}$ in the universal enveloping algebra $U(\mathcal{N}^L)$ of the maximal nilpotent Lie subalgebra in the Lie algebra of G^L.
2. The weight μ subspace $V_\lambda^{(\mu)}$ in the highest weight λ representation V_λ of G^L.
3. The tensor product invariants $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mathrm{G}^L}$.
4. The weight μ subspaces in the tensor products $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$.

Calculation of the dimensions of these spaces, in the cases (1)–(3), is a fascinating classical problem, which led to Weyl's character formula and Kostant's partition function.

The first examples of special bases in finite dimensional representations are Gelfand–Tsetlin's bases [28,29]. Other examples of special bases were given by De Concini–Kazhdan [14].

The *canonical bases* in the spaces above were constructed by Lusztig [59,61]. Independently, canonical bases were defined by Kashiwara [47].

Canonical bases in representations of GL_3, Sp_4 were defined by Gelfand–Zelevinsky–Retakh [27,68].

Closely related, but in general different bases were considered by Nakajima [66,67], Malkin [63], Mirković–Vilonen [65], and extensively studied afterwards. Abusing terminology, we also call them canonical bases.

It was discovered by Lusztig [58] that, in the cases (1)–(2), the sets parametrising canonical bases in representations of the group G are intimately related to the Langlands dual group G^L.

Kashiwara discovered in the cases (1)–(2) an additional *crystal structure* on these sets, and Joseph proved a rigidity theorem [44] asserting that, equipped with the crystal structure, the sets of parameters are uniquely determined.

One of the results of this paper is a uniform geometric construction of the sets parametrizing all of these canonical bases, which leads to a natural uniform construction of canonical bases parametrized by these sets in the cases (2)–(4). In particular, we get a new canonical bases in the case (4), generalizing the Mirković–Vilonen (MV) basis in V_λ. To explain our set-up let us recall some basic notions.

A *positive space* $\mathcal{Y}$ is a space, which could be a stack whose generic part is a variety, equipped with a *positive atlas*. The latter is a collection of rational coordinate systems with subtraction free transition functions between any pair of the coordinate systems. Therefore the set $\mathcal{Y}(\mathbb{Z}^t)$ of the *integral tropical points* of $\mathcal{Y}$ is well defined. We review all this in Sect. 2.1.1.

Let $(\mathcal{Y}, \mathcal{W})$ be a *positive pair* given by a positive space $\mathcal{Y}$ equipped with a positive rational function $\mathcal{W}$. Then one can tropicalize the function $\mathcal{W}$, getting a $\mathbb{Z}$-valued function

$$\mathcal{W}^t : \mathcal{Y}(\mathbb{Z}^t) \longrightarrow \mathbb{Z}.$$

Therefore a positive pair $(\mathcal{Y}, \mathcal{W})$ determines a set of *positive integral tropical points*:

$$\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) := \{l \in \mathcal{Y}(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}. \tag{2}$$

We usually omit $\mathcal{W}$ in the notation and denote the set by $\mathcal{Y}^+(\mathbb{Z}^t)$.

To introduce the positive pairs $(\mathcal{Y}, \mathcal{W})$ which play the basic role in this paper, we need to review some basic facts about flags and decorated flags in G.

Decorated flags and associated characters. Below G is a split reductive group over $\mathbb{Q}$. Recall that the *flag variety* $\mathcal{B}$ parametrizes Borel subgroups in G. Given a Borel subgroup B, one has an isomorphism $\mathcal{B} = G/B$.

Let G' be the adjoint group of G. The group G' acts by conjugation on pairs (U, χ), where $\chi : U \to \mathbb{A}^1$ is an additive character of a maximal unipotent subgroup U in G'. The subgroup U stabilizes each pair (U, χ). A character χ

is *non-degenerate* if U is the stabilizer of (U, χ). The *principal affine space*[1] $\mathcal{A}_{\mathrm{G}'}$ parametrizes pairs (U, χ) where χ is a non-degenerate additive character of a maximal unipotent group U. Therefore there is an isomorphism

$$i_\chi : \mathcal{A}_{\mathrm{G}'} \stackrel{\sim}{\longrightarrow} \mathrm{G}'/\mathrm{U}.$$

This isomorphism is not canonical: the coset $[\mathrm{U}] \in \mathrm{G}'/\mathrm{U}$ does not determine a point of $\mathcal{A}_{\mathrm{G}'}$. To specify a point one needs to choose a non-degenerate character χ. One can determine uniquely the character by using a *pinning*, see Sects. 2.1.2–2.1.3. So writing $\mathcal{A}_{\mathrm{G}'} = \mathrm{G}'/\mathrm{U}$ we abuse notation, keeping in mind a choice of the character χ, or a pinning.

Having said this, one defines the principal affine space $\mathcal{A}_{\mathrm{G}}$ for the group G by $\mathcal{A}_{\mathrm{G}} := \mathrm{G}/\mathrm{U}$. We often write $\mathcal{A}$ instead of $\mathcal{A}_{\mathrm{G}}$. The points of $\mathcal{A}$ are called *decorated flags* in G. The group G acts on $\mathcal{A}$ from the left. For each $\mathrm{A} \in \mathcal{A}$, let U_{A} be its stabilizer. It is a maximal unipotent subgroup of G. There is a canonical projection

$$\pi : \mathcal{A} \longrightarrow \mathcal{B}, \quad \pi(\mathrm{A}) := \text{ the normalizer of } \mathrm{U}_{\mathrm{A}}. \tag{3}$$

The projection $\mathrm{G} \to \mathrm{G}'$ gives rise to a map $p : \mathcal{A}_{\mathrm{G}} \longrightarrow \mathcal{A}_{\mathrm{G}'}$ whose fibers are torsors over the center of G. Let $p(\mathrm{A}) = (\mathrm{U}_{\mathrm{A}}, \chi_{\mathrm{A}})$. Here U_{A} is a maximal unipotent subgroup of G'. It is identified with a similar subgroup of G, also denoted by U_{A}. So a decorated flag A in G provides a non-degenerate character of the maximal unipotent subgroup U_{A} in G:

$$\chi_{\mathrm{A}} : \mathrm{U}_{\mathrm{A}} \longrightarrow \mathbb{A}^1. \tag{4}$$

Clearly, if $u \in \mathrm{U}_{\mathrm{A}}$, then $gug^{-1} \in \mathrm{U}_{g\cdot\mathrm{A}}$, and

$$\chi_{\mathrm{A}}(u) = \chi_{g\cdot\mathrm{A}}(gug^{-1}). \tag{5}$$

Example. A flag for SL_m is a nested collection of subspaces in an m-dimensional vector space V_m equipped with a volume form $\omega \in \det V_m^*$:

$$F_\bullet = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m, \quad \dim F_i = i.$$

A decorated flag for SL_m is a flag $F_\bullet$ with a choice of non-zero vectors $f_i \subset F_i/F_{i-1}$ for each $i = 1, \ldots, m-1$, called *decorations*. For example, $\mathcal{A}_{SL_2}$ parametrises non-zero vectors in a symplectic space (V_2, ω). The subgroup preserving a vector $f \in V_2 - \{0\}$ is given by transformations $u_f(a) : v \longmapsto v + a\omega(f, v)f$. Its character χ_f is given by $\chi_f(u_f(a)) = a$.

[1] Inspite of the name, it is not an affine variety.

Our basic geometric objects are the following three types of configuration spaces:

$$\begin{aligned} \mathrm{Conf}_n(\mathcal{A}) = \mathrm{G}\backslash\mathcal{A}^n, \quad \mathrm{Conf}(\mathcal{A}^n, \mathcal{B}) := \mathrm{G}\backslash(\mathcal{A}^n \times \mathcal{B}), \\ \mathrm{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}) := \mathrm{G}\backslash(\mathcal{B} \times \mathcal{A}^n \times \mathcal{B}). \end{aligned} \tag{6}$$

The potential $\mathcal{W}$. A key observation is that there is a natural rational function

$$\chi^o : \mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathrm{G}\backslash(\mathcal{B} \times \mathcal{A} \times \mathcal{B}) \longrightarrow \mathbb{A}^1.$$

Let us explain its definition. A pair of Borel subgroups $\{\mathrm{B}_1, \mathrm{B}_2\}$ is *generic* if $\mathrm{B}_1 \cap \mathrm{B}_2$ is a Cartan subgroup in G. A pair $\{\mathrm{A}_1, \mathrm{B}_2\} \in \mathcal{A} \times \mathcal{B}$ is generic if the pair $(\pi(\mathrm{A}_1), \mathrm{B}_2)$ is generic. Generic pairs $\{\mathrm{A}_1, \mathrm{B}_2\}$ form a principal homogeneous G-space. Thus, given a triple $\{\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3\} \in \mathcal{B} \times \mathcal{A} \times \mathcal{B}$ such that $\{\mathrm{A}_2, \mathrm{B}_3\}$ and $\{\mathrm{A}_2, \mathrm{B}_1\}$ are generic, there is a unique $u \in \mathrm{U}_{\mathrm{A}_2}$ such that

$$\{\mathrm{A}_2, \mathrm{B}_3\} = u \cdot \{\mathrm{A}_2, \mathrm{B}_1\}. \tag{7}$$

So we define $\chi^o(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) := \chi_{\mathrm{A}_2}(u)$. Using it as a building block, we define a positive rational function $\mathcal{W}$ on each of the spaces (6).

For example, to define the $\mathcal{W}$ on the space $\mathrm{Conf}_n(\mathcal{A})$ we start with a generic collection $\{\mathrm{A}_1, \ldots, \mathrm{A}_n\} \in \mathcal{A}^n$, set $\mathrm{B}_i := \pi(\mathrm{A}_i)$, and define $\mathcal{W}$ as a sum, with the indices modulo n:

$$\mathcal{W} : \mathrm{Conf}_n(\mathcal{A}) \longrightarrow \mathbb{A}^1, \quad \mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n) := \sum_{i=1}^{n} \chi^o(\mathrm{B}_{i-1}, \mathrm{A}_i, \mathrm{B}_{i+1}). \tag{8}$$

Note that the potential $\mathcal{W}$ is well-defined when each adjacent pair $\{\mathrm{A}_i, \mathrm{A}_{i+1}\}$ is generic, meaning that $\{\pi(\mathrm{A}_i), \pi(\mathrm{A}_{i+1})\}$ is generic. Assigning the (decorated) flags to the vertices of a polygon, we picture the potential $\mathcal{W}$ as a sum of the contributions χ_{A} at the A-vertices (shown boldface) of the polygon, see Fig. 1.

By construction, the potential $\mathcal{W}_{\mathrm{G}}$ on the space $\mathrm{Conf}_n(\mathcal{A}_{\mathrm{G}})$ is the pull back of the potential $\mathcal{W}_{\mathrm{G}'}$ for the adjoint group G' via the natural projection $p_{\mathrm{G}\to\mathrm{G}'} : \mathrm{Conf}_n(\mathcal{A}_{\mathrm{G}}) \to \mathrm{Conf}_n(\mathcal{A}_{\mathrm{G}'})$:

$$\mathcal{W}_{\mathrm{G}} = p^*_{\mathrm{G}\to\mathrm{G}'}\mathcal{W}_{\mathrm{G}'}. \tag{9}$$

Potentials for the other two spaces in (6) are defined similarly, as the sums of the characters assigned to the decorated flags of a configuration. A formula similar to (9) evidently holds.

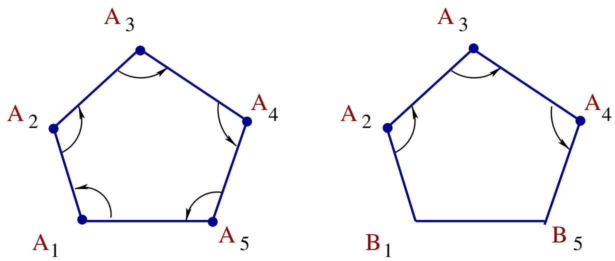


Fig. 1 The potential $\mathcal{W}$ is a sum of the contributions χ_{A} at the A-vertices (*boldface*)

Parametrisations of canonical bases. It was shown in [17] that all of the spaces (6) have natural positive structures. We show that the potential $\mathcal{W}$ is a positive rational function.

We prove that the sets parametrizing canonical bases admit a uniform description as the sets $\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)$ of positive integral tropical points assigned to the following positive pairs $(\mathcal{Y}, \mathcal{W})$. To write the potential $\mathcal{W}$ we use an abbreviation $\chi_{\mathrm{A}_i} := \chi^o(\mathrm{B}_{i-1}, \mathrm{A}_i, \mathrm{B}_{i+1})$, with indices mod n:

1. The canonical basis in $U(\mathcal{N}^L)$:

$$\mathcal{Y} = \mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}), \quad \mathcal{W}(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) := \chi_{\mathrm{A}_2}.$$

2. The canonical basis in V_λ:

$$\mathcal{Y} = \mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B}), \quad \mathcal{W}(\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) := \chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}.$$

3. The canonical basis in invariants of tensor product of n irreducible G^L-modules:

$$\mathcal{Y} = \mathrm{Conf}_n(\mathcal{A}), \quad \mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n) := \sum_{i=1}^{n} \chi_{\mathrm{A}_i}. \tag{10}$$

4. The canonical basis in tensor products of n irreducible G^L-modules:

$$\mathcal{Y} = \mathrm{Conf}(\mathcal{A}^{n+1}, \mathcal{B}), \quad \mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}) := \sum_{i=1}^{n+1} \chi_{\mathrm{A}_i}. \tag{11}$$

Natural decompositions of these sets, like decompositions into weight subspaces in (1) and (2), are easily described in terms of the corresponding configuration space, see Sect. 2.3.2.

Let us emphasize that the canonical bases in tensor products are not the tensor products of canonical bases in irreducible representations. Similarly, in spite of the natural decomposition

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} = \oplus_\lambda V_\lambda \otimes (V_\lambda^* \otimes V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mathrm{G}^L},$$

the canonical basis on the left is not a product of the canonical bases on the right.

Descriptions of the sets parametrizing the canonical bases were known in different but equivalent formulations in the following cases:

In the cases (1)–(2) there is the original parametrization of Lusztig [58].

In the case (3) for $n = 3$, there is Berenstein–Zelevinsky's parametrization [12], referred to as the BZ data. We produce in Appendix of the online version of our paper [39] an isomorphism between our parametrization and the BZ data. The cyclic symmetry, evident in our approach, is obscure for the BZ data.

The description in the $n > 3$ case in (3) seems to be new.

The cases (1), (2) and (4) were investigated by Berenstein and Kazhdan [10,11], who introduced and studied *geometric crystals* as algebraic-geometric avatars of Kashiwara's *crystals*. In particular, they describe the sets parametrizing canonical bases in the form (2), without using, however, configuration spaces. Interpretation of geometric crystals relevant to representation theory as moduli spaces of mixed configurations of flags makes, to our opinion, the story more transparent. See Appendix of the online version of our paper [39]. Kashiwara's crystals were related to affine Grassmannians in [13].

To define canonical bases in representations, one needs to choose a maximal torus in G^L and a positive Weyl chamber. Usual descriptions of the sets parametrizing canonical bases require the same choice. Unlike this, working with configurations we do not require such choices.[2]

Most importantly, our parametrization of the canonical basis in tensor products invariants leads immediately to a similar set which parametrizes a linear basis in the space of functions on the moduli space $\mathrm{Loc}_{\mathrm{G^L},S}$ of G^L-local systems on a decorated surface S. Here the approach via configurations of decorated flags, and in particular its transparent cyclic invariance, are essential. See the example when $G = SL_2$ in Sect. 1.3.1.

Summarizing, we understood the sets parametrizing canonical bases as the sets of positive integral tropical points of various configuration spaces. Let us show now how this, combined with the geometric Satake correspondence [9,34,62,65], leads to a natural uniform construction of canonical bases in the cases (2)–(4).

We explain in Sect. 1.1.2 the construction in the case of tensor products invariants. A canonical basis in this case was defined by Lusztig [61]. However Lusztig's construction does not provide a description of the set parametrizing

[2] We would like to stress that the positive structures and potentials on configuration spaces which we employ for parametrization of canonical bases do not depend on any extra choices, like pinning etc., in the group. See Sect. 6.3.

the basis. Our basis in tensor products is new—it generalizes the MV basis in V_λ. We explain this in Sect. 2.4.

1.1.2 Constructing canonical bases in tensor products invariants

We start with a simple general construction. Let $\mathcal{Y}$ be a positive space, understood just as a collection of split tori glued by positive birational maps [17]. Since it is a birational notion, there is no set of F-points of $\mathcal{Y}$, where F is a field. Let $\mathcal{K} := \mathbb{C}((t))$. In Sect. 2.2.1 we introduce a set $\mathcal{Y}^\circ(\mathcal{K})$. We call it the set of *transcendental $\mathcal{K}$-points of $\mathcal{Y}$*. It is a set making sense of "generic $\mathcal{K}$-points of $\mathcal{Y}$". In particular, if $\mathcal{Y}$ is given by a variety Y with a positive rational atlas, then $\mathcal{Y}^\circ(\mathcal{K}) \subset Y(\mathcal{K})$. The set $\mathcal{Y}^\circ(\mathcal{K})$ comes with a natural *valuation map*:

$$\text{val} : \mathcal{Y}^\circ(\mathcal{K}) \longrightarrow \mathcal{Y}(\mathbb{Z}^t).$$

For any $l \in \mathcal{Y}(\mathbb{Z}^t)$, we define the *transcendental cell* $\mathcal{C}_l^\circ$ assigned to l:

$$\mathcal{C}_l^\circ := \text{val}^{-1}(l) \subset \mathcal{Y}^\circ(\mathcal{K}), \quad \mathcal{Y}^\circ(\mathcal{K}) = \coprod_{l \in \mathcal{Y}(\mathbb{Z}^t)} \mathcal{C}_l^\circ.$$

Let us now go to canonical bases in invariants of tensor products of G^L-modules (1). The relevant configuration space is $\mathrm{Conf}_n(\mathcal{A})$.The tropicalized potential $\mathcal{W}^t : \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) \to \mathbb{Z}$ determines the subset of positive integral tropical points:

$$\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) := \{l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}. \tag{12}$$

We construct a canonical basis in (1) parametrized by the set (12).
Let $\mathcal{O} := \mathbb{C}[[t]]$. In Sect. 2.2.2 we introduce a moduli subspace

$$\mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \subset \mathrm{Conf}_n(\mathcal{A})(\mathcal{K}). \tag{13}$$

We call it the space of *$\mathcal{O}$-integral configurations of decorated flags*. Here are its crucial properties:

1. A transcendental cell $\mathcal{C}_l^\circ$ of $\mathrm{Conf}_n(\mathcal{A})$ is contained in $\mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$ if and only if it corresponds to a positive tropical point. Moreover, given a point $l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$, one has

$$l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) \Longleftrightarrow \mathcal{C}_l^\circ \subset \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \Longleftrightarrow \mathcal{C}_l^\circ \cap \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \neq \emptyset. \tag{14}$$

2. Let $\mathrm{Gr} := \mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O})$ be the affine Grassmannian. It follows immediately from the very definition of the subspace (13) that there is a canonical map

$$\kappa : \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \longrightarrow \mathrm{Conf}_n(\mathrm{Gr}) := \mathrm{G}(\mathcal{K})\backslash(\mathrm{Gr})^n.$$

These two properties of $\mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$ allow us to transport points $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ into the top components of the stack $\mathrm{Conf}_n(\mathrm{Gr})$. Namely, given a point $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$, we define a cycle

$$\mathcal{M}_l := \text{closure of } \mathcal{M}_l^\circ \subset \mathrm{Conf}_n(\mathrm{Gr}), \quad \text{where } \mathcal{M}_l^\circ := \kappa(\mathcal{C}_l^\circ).$$

The cycle $\mathcal{C}_l^\circ$ is defined for any $l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$. However, as it clear from (14), the map κ can be applied to it if and only if l is positive: otherwise $\mathcal{C}_l^\circ$ is not in the domain of the map κ.

We prove that the map $l \longmapsto \mathcal{M}_l$ provides a bijection

$$\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) \stackrel{\sim}{\longrightarrow} \{\text{closures of the top dimensional components of the stack } \mathrm{Conf}_n(\mathrm{Gr})\}. \tag{15}$$

Here the very notion of a "top dimensional" component of a stack requires clarification. For now, we will bypass this question in a moment by passing to more traditional varieties.

We use a very general argument to show the injectivity of the map $l \longmapsto \mathcal{M}_l$. Namely, given a positive rational function F on $\mathrm{Conf}_n(\mathcal{A})$, we define a $\mathbb{Z}$-valued function D_F on $\mathrm{Conf}_n(\mathrm{Gr})$. It generalizes the function on the affine Grassmannian for $\mathrm{G} = \mathrm{GL}_m$ and its products defined by Kamnitzer [45,46]. We prove that the restriction of D_F to $\mathcal{M}_l^\circ$ is equal to the value $F^t(l)$ of the tropicalization F^t of F at the point $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$. Thus the map (15) is injective.

Let us reformulate our result in a more traditional language. The orbits of $\mathrm{G}(\mathcal{K})$ acting on $\mathrm{Gr} \times \mathrm{Gr}$ are labelled by dominant weights of G^L. We write $\mathrm{L}_1 \stackrel{\lambda}{\longrightarrow} \mathrm{L}_2$ if $(\mathrm{L}_1, \mathrm{L}_2)$ is in the orbit labelled by λ. Let $[1]$ be the identity coset in Gr. A set $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of dominant weights of G^L determines a *cyclic convolution variety*, better known as a *fiber of the convolution map*:

$$\begin{aligned}\mathrm{Gr}_{c(\underline{\lambda})} := \{(\mathrm{L}_1, \ldots, \mathrm{L}_n) \mid \mathrm{L}_1 \stackrel{\lambda_1}{\longrightarrow} \mathrm{L}_2 \stackrel{\lambda_2}{\longrightarrow} \cdots \stackrel{\lambda_n}{\longrightarrow} \mathrm{L}_{n+1},\\ \mathrm{L}_1 = \mathrm{L}_{n+1} = [1]\} \subset [1] \times \mathrm{Gr}^{n-1}.\end{aligned} \tag{16}$$

These varieties provide a $\mathrm{G}(\mathcal{O})$-equivariant decomposition

$$[1] \times \mathrm{Gr}^{n-1} = \coprod_{\underline{\lambda}=(\lambda_1,\ldots,\lambda_n)} \mathrm{Gr}_{c(\underline{\lambda})}. \tag{17}$$

Since $\mathrm{G}(\mathcal{O})$ is connected, it preserves each component of $\mathrm{Gr}_{c(\underline{\lambda})}$. Thus the components of $\mathrm{Gr}_{c(\underline{\lambda})}$ live naturally on the stack

$$\mathrm{Conf}_n(\mathrm{Gr}) = \mathrm{G}(\mathcal{O})\backslash([1] \times \mathrm{Gr}^{n-1}).$$

We prove that the cycles $\mathcal{M}_l$ assigned to the points $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ are closures of the top dimensional components of the cyclic convolution varieties. The latter, due to the geometric Satake correspondence, give rise to a canonical basis in (1). We already know that the map (15) is injective. We show that the $\underline{\lambda}$-components of the sets related by the map (15) are finite sets of the same cardinality, respected by the map. Therefore the map (15) is an isomorphism.

Our result generalizes a theorem of Kamnitzer [46], who used hives [54] to parametrize top components of convolution varieties for $\mathrm{G} = \mathrm{GL}_m$, $n = 3$.

Our construction generalizes Kamnitzer's construction of parametrizations of Mirković–Vilonen cycles [45]. At the same time, it gives a coordinate free description of Kamnitzer's construction.

When $\mathrm{G} = \mathrm{GL}_m$, there is a special coordinate system on the space $\mathrm{Conf}_3(\mathcal{A})$, introduced in Section 9 of [17]. We show in Sect. 3 that it provides an isomorphism of sets

$$\mathrm{Conf}_3^+(\mathcal{A})(\mathbb{Z}^t) \stackrel{\sim}{\longrightarrow} \{\text{Knutson–Tao's hives [KT]}\}.$$

Using this, we get a one line proof of Knutson–Tao–Woodward's theorem [55] in Sect. 2.1.6.

For $\mathrm{G} = \mathrm{GL}_m$, $n > 3$, we prove Kamnitzer conjecture [46], describing the top components of convolution varieties via a generalization of hives—we identify the latter with the set $\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ via the special positive coordinate systems on $\mathrm{Conf}_n(\mathcal{A})$ from [17].

1.2 Positive tropical points and top components

1.2.1 Our main example

Denote by $\mathrm{Conf}_n^\times(\mathcal{A})$ the subvariety of $\mathrm{Conf}_n(\mathcal{A})$ parametrizing configurations of decorated flags $(\mathrm{A}_1, \ldots, \mathrm{A}_n)$ such that the flags $(\pi(\mathrm{A}_i), \pi(\mathrm{A}_{i+1}))$ are in generic position for each $i = 1, \ldots, n$ modulo n. The potential $\mathcal{W}$ was defined in (8). It is evidently a regular function on $\mathrm{Conf}_n^\times(\mathcal{A})$.

Let P^+ be the cone of dominant coweights. There are canonical isomorphisms

$$\alpha : \mathrm{Conf}_2^\times(\mathcal{A}) \stackrel{\sim}{\longrightarrow} \mathrm{H}, \quad \mathrm{Conf}_2(\mathrm{Gr}) = \mathrm{P}^+. \tag{18}$$

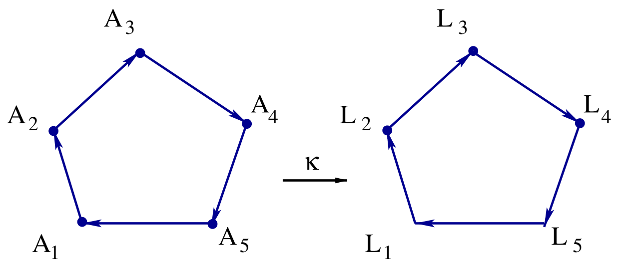


Fig. 2 Going from an $\mathcal{O}$-integral configuration of decorated flags to a configuration of lattices

Configurations $(\mathrm{A}_1, \dots, \mathrm{A}_n)$ sit at the vertices of a polygon, as on Fig. 2. Let $\pi_E : \mathrm{Conf}_n(\mathcal{A}) \to \mathrm{Conf}_2(\mathcal{A})$ be the projection corresponding to a side E of the polygon. Denote by $\pi_E^{\times}$ its restriction to $\mathrm{Conf}_n^{\times}(\mathcal{A})$. The collection of the maps $\{\pi_E^{\times}\}$, followed by the first isomorphism in (18) provides a map

$$\pi : \mathrm{Conf}_n^{\times}(\mathcal{A}) \longrightarrow \mathrm{Conf}_2^{\times}(\mathcal{A})^n \stackrel{\alpha}{=} \mathrm{H}^n.$$

Using similarly the second isomorphism in (18), we get a map

$$\pi_{\mathrm{Gr}} : \mathrm{Conf}_n(\mathrm{Gr}) \longrightarrow \mathrm{Conf}_2(\mathrm{Gr})^n {=} (\mathrm{P}^+)^n.$$

Let $\{\omega_i\}$ be a basis of the cone of positive dominant weights of H. The functions $\pi_E^*\omega_i$ are equations of the irreducible components of the divisor $D := \mathrm{Conf}_n(\mathcal{A}) - \mathrm{Conf}_n^{\times}(\mathcal{A})$:

$$D := \mathrm{Conf}_n(\mathcal{A}) - \mathrm{Conf}_n^{\times}(\mathcal{A}) = \cup_{E,i} D_i^E.$$

Equivalently, the component D_i^E is determined by the condition that the pair of flags at the endpoints of the edge E belongs to the codimension one G-orbit corresponding to the simple reflection $s_i \in W$.[3]

The space $\mathrm{Conf}_n(\mathcal{A})$ has a cluster $\mathcal{A}$-variety structure, described for $\mathrm{G} = SL_m$ in [17, Section 10]. An important fact [21] is that any cluster $\mathcal{A}$-variety $\mathcal{A}$ has a canonical cluster volume form $\Omega_{\mathcal{A}}$, which in any cluster $\mathcal{A}$-coordinate system $(\mathrm{A}_1, \dots, \mathrm{A}_n)$ is given by

$$\Omega_{\mathcal{A}} = \pm d \log \mathrm{A}_1 \wedge \ldots \wedge d \log \mathrm{A}_n.$$

The functions $\pi_E^*\omega_i$ are the *frozen* $\mathcal{A}$*-cluster coordinates* in the sense of Definition 11.5. This is equivalent to the claim that the canonical volume form

[3] Indeed, $\omega_i(\alpha(\mathrm{A}_1, \mathrm{A}_2)) = 0$ if and only if the corresponding pair of flags belongs to the codimension one G-orbit corresponding to a simple reflection s_i.

$\Omega_{\mathcal{A}}$ on $\mathrm{Conf}_n(\mathcal{A})$ has non-zero residues precisely at the irreducible components of the divisor D.[4]

All this data is defined for any split semi-simple group G over $\mathbb{Q}$. Indeed, the form Ω on $\mathrm{Conf}_n(\mathcal{A})$ for the simply-connected group is invariant under the action of the center the group, and thus its integral multiple descends to a form on $\mathrm{Conf}_n(\mathcal{A}_{\mathrm{G}})$. The potential $\mathcal{W}_{\mathrm{G}}$ is defined by pulling back the potential $\mathcal{W}_{\mathrm{G}'}$ for the adjoint group G'. We continue discussion of this example in Sect. 1.4, where it is casted as an example of the mirror symmetry.

The simplest example. Let (V_2, ω) be a two dimensional vector space with a symplectic form. Then $\mathrm{SL}_2 = \mathrm{Aut}(V_2, \omega)$, and $\mathcal{A}_{SL_2} = V_2 - \{0\}$. Next, $\mathrm{Conf}_n(\mathcal{A}_{\mathrm{SL}_2}) = \mathrm{Conf}_n(V_2)$ is the space of configuration $(l_1, \ldots, l_n)$ of n non-zero vectors in V_2. Set $\Delta_{i,j} := \langle \omega, l_i \wedge l_j \rangle$. Then the potential is given by the following formula, where the indices are mod n:

$$\mathcal{W} := \sum_{i=1}^{n} \frac{\Delta_{i,i+2}}{\Delta_{i,i+1}\Delta_{i+1,i+2}}. \tag{19}$$

The boundary divisors are given by equations $\Delta_{i,i+1} = 0$. To write the volume form, pick a triangulation T of the polygon whose vertices a labeled by the vectors. Then, up to a sign,

$$\Omega := \bigwedge_E d \log \Delta_E.$$

where E are the diagonals and sides of the n-gon, and $\Delta_E := \Delta_{i,j}$ if $E = (i, j)$. The function (19) is invariant under $l_i \to -l_i$, and thus descends to $\mathrm{Conf}_n(\mathcal{A}_{\mathrm{PGL}_2}) = \mathrm{Conf}_n(V_2/\pm 1)$.

1.2.2 The general framework

Let us explain main features of the geometric picture underlying our construction in most general terms, which we later on elaborate in details in every particular situation. First, there are three main ingredients:

1. A positive space $\mathcal{Y}$ with a positive rational function $\mathcal{W}$ called the *potential*, and a volume form $\Omega_{\mathcal{Y}}$ with logarithmic singularities. This determines the set $\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)$ of positive integral tropical points—the set parametrizing a canonical basis.[5]

[4] Indeed, it follows from Lemma 11.3 and an explicit description of cluster structure on $\mathrm{Conf}_n(\mathcal{A})$ that the form $\Omega_{\mathcal{A}}$ can not have non-zero residues anywhere else the divisors D_i^E. One can show that the residues at these divisors are non-zero.

[5] The set $\mathcal{Y}(\mathbb{Z}^t)$, the tropicalization $\mathcal{W}^t$, and thus the subset $\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)$ can also be determined by the volume form $\Omega_{\mathcal{Y}}$, without using the positive structure on $\mathcal{Y}$.

2. A subset of $\mathcal{O}$-integral points $\mathcal{Y}^{\mathcal{O}} \subset \mathcal{Y}(\mathcal{K})$. Its key feature is that, given an $l \in \mathcal{Y}(\mathbb{Z}^t)$,

$$l \in \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) \Longleftrightarrow \mathcal{C}^\circ_l \subset \mathcal{Y}^{\mathcal{O}} \Longleftrightarrow \mathcal{C}^\circ_l \cap \mathcal{Y}^{\mathcal{O}} \neq \emptyset. \tag{20}$$

3. A moduli space $\mathrm{Gr}_{\mathcal{Y},\mathcal{W}}$, together with a canonical map

$$\kappa : \mathcal{Y}^{\mathcal{O}} \longrightarrow \mathrm{Gr}_{\mathcal{Y},\mathcal{W}}. \tag{21}$$

These ingredients are related as follows:

- Any positive rational function F on $\mathcal{Y}$ gives rise to a $\mathbb{Z}$-valued function D_F on $\mathrm{Gr}_{\mathcal{Y},\mathcal{W}}$, such that for any $l \in \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)$, the restriction of D_F to $\kappa(\mathcal{C}^\circ_l)$ equals $F^t(l)$.

So we arrive at a collection of irreducible cycles

$$\mathcal{M}^\circ_l := \kappa(\mathcal{C}^\circ_l) \subset \mathrm{Gr}_{\mathcal{Y},\mathcal{W}}, \quad \mathcal{M}_l := \text{closure of } \mathcal{M}^\circ_l, \quad l \in \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t).$$

Thanks to the $\bullet$, the assignment $l \longmapsto \mathcal{M}_l$ is injective.

Consider the set $\{D_c\}$ of all irreducible divisors in $\mathcal{Y}$ such that the residue of the form $\Omega_{\mathcal{Y}}$ at D_c is non-zero. We call them the *boundary divisors* of $\mathcal{Y}$. We define

$$\mathcal{Y}^\times := \mathcal{Y} - \cup D_c. \tag{22}$$

By definition, the form $\Omega_{\mathcal{Y}}$ is regular on $\mathcal{Y}^\times$. In all examples the potential $\mathcal{W}$ is regular on $\mathcal{Y}^\times$.

There is a split torus $\mathbb{H}$, and a positive regular surjective projection

$$\pi : \mathcal{Y}^\times \longrightarrow \mathbb{H}.$$

The map π is determined by the form $\Omega_{\mathcal{Y}}$. For example, assume that each boundary divisor D_c is defined by a global equation $\Delta_c = 0$. Then the regular functions $\{\Delta_c\}$ define the map π, i.e. $\pi(y) = \{\Delta_c(y)\}$.

Next, there is a semigroup $\mathbb{H}^{\mathcal{O}} \subset \mathbb{H}(\mathcal{K})$ containing $\mathbb{H}(\mathcal{O})$, defining a cone

$$\mathbb{P} := \mathbb{H}^{\mathcal{O}}/\mathbb{H}(\mathcal{O}) \subset \mathbb{H}(\mathbb{Z}^t) := \mathbb{H}(\mathcal{K})/\mathbb{H}(\mathcal{O}) = X_*(\mathbb{H}),$$

such that the tropicalization of the map π provides a map $\pi^t : \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) \to \mathbb{P}$, and there is a surjective map $\pi_{\mathrm{Gr}} : \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} \to \mathbb{P}$. Denote by $\pi^{\mathcal{O}}$ restricting of $\pi \otimes \mathcal{K}$ to $\mathcal{Y}^{\mathcal{O}}$. These maps fit into a commutative diagram

$$\begin{array}{ccccc} \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) & \stackrel{\mathrm{val}}{\longleftarrow} & \mathcal{Y}^{\mathcal{O}} & \stackrel{\kappa}{\longrightarrow} & \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} \\ \pi^t \downarrow & & \pi^{\mathcal{O}} \downarrow & & \downarrow \pi_{\mathrm{Gr}} \\ \mathbb{P} & \stackrel{\mathrm{val}}{\longleftarrow} & \mathbb{H}^{\mathcal{O}} & \stackrel{\mathrm{val}}{\longrightarrow} & \mathbb{P} \end{array} \tag{23}$$

We define $\mathrm{Gr}^{(\lambda)}_{\mathcal{Y},\mathcal{W}}$ and $\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)_\lambda$ as the fibers of the maps π_{Gr} and π^t over a $\lambda \in \mathbb{P}$. So we have

$$\mathrm{Gr}_{\mathcal{Y},\mathcal{W}} = \coprod_{\lambda\in\mathbb{P}} \mathrm{Gr}^{(\lambda)}_{\mathcal{Y},\mathcal{W}}, \quad \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) = \coprod_{\lambda\in\mathbb{P}} \mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)_\lambda. \tag{24}$$

The following is a key property of our picture:

- The map $l \longrightarrow \mathcal{M}_l$ provides a bijection

$$\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t)_\lambda \quad \longleftrightarrow \quad \left\{\text{Closures of top dimensional components of } \mathrm{Gr}^{(\lambda)}_{\mathcal{Y},\mathcal{W}}\right\}.$$

Although the space $\mathrm{Gr}_{\mathcal{Y},\mathcal{W}}$ is usually infinite dimensional, it is nice. The map $\pi_{\mathrm{Gr}} : \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} \to \mathbb{P}$ slices it into highly singular and reducible pieces. However the slicing makes the perverse sheaves geometry clean and beautiful. It allows to relate the positive integral tropical points to the top components of the slices.

Example. In our main example, discussed in Sect. 1.1 we have

$$\mathcal{Y} = \mathrm{Conf}_n(\mathcal{A}), \quad \mathcal{Y}^\times = \mathrm{Conf}^\times_n(\mathcal{A}), \quad \mathcal{Y}^{\mathcal{O}} = \mathrm{Conf}^{\mathcal{O}}_n(\mathcal{A}),$$
$$\mathrm{Gr}_{\mathcal{Y},\mathcal{W}} = \mathrm{Conf}_n(\mathrm{Gr}), \quad \mathbb{H} = \mathrm{H}^n, \quad \mathbb{P} = (\mathrm{P}^+)^n.$$

The potential $\mathcal{W}$ is defined in (8), and decomposition (24) is described by cyclic convolution varieties (17).

1.2.3 Mixed configurations and a generalization of Mirković–Vilonen cycles

Let us briefly discuss other examples relevant to representation theory. All of them follow the set-up of Sect. 1.2. The obtained cycles $\mathcal{M}_l$ can be viewed as generalisations of Mirković–Vilonen cycles. Let us list first the spaces $\mathcal{Y}$ and $\mathrm{Gr}_{\mathcal{Y},\mathcal{W}}$. The notation Conf_{w_0} indicates that the pair of the first and the last flags in configuration is in generic position.

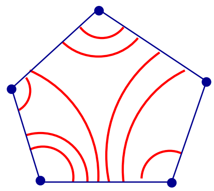


Fig. 3 An integral lamination on a pentagon of type (4, 4, 1, 6, 3)

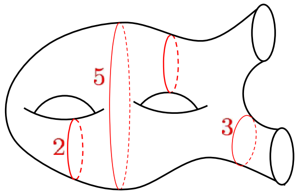


Fig. 4 An integral lamination on a surface with two holes, and no special points

(i) *Generalized Mirković–Vilonen cycles*:

$$\mathcal{Y} := \mathrm{Conf}_{w_0}(\mathcal{A}, \mathcal{A}^n, \mathcal{B}), \quad \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} := \mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B}) = \mathrm{Gr}^n.$$

If $n = 1$, we recover the Mirković–Vilonen cycles in the affine Grassmannian [65].

(ii) *Generalized stable Mirković–Vilonen cycles*:

$$\mathcal{Y} := \mathrm{Conf}_{w_0}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}), \quad \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} := \mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B}) = \mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}^n.$$

If $n = 1$, we recover the stable Mirković–Vilonen cycles in the affine Grassmannian. In our interpretation they are top components of the stack

$$\mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}, \mathcal{B}) = \mathrm{H}\backslash\mathrm{Gr}.$$

(iii) *The cycles providing canonical bases in tensor products*

$$\mathcal{Y} := \mathrm{Conf}(\mathcal{A}^{n+1}, \mathcal{B}), \quad \mathrm{Gr}_{\mathcal{Y},\mathcal{W}} := \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B}) = \mathrm{B}^-(\mathcal{O})\backslash\mathrm{Gr}^n.$$

The spaces $\mathcal{Y}$ in examples (i) and (iii) are essentially the same. However the potentials are different: in the case (iii) it is the sum of contributions of all decorated flags, while in the case (i) we skip the first one. Passing from $\mathcal{Y}$ to $\mathrm{Gr}_{\mathcal{Y},\mathcal{W}}$ we replace those $\mathcal{A}$'s which contribute to the potential by Gr's, but keep the $\mathcal{B}$'s and the $\mathcal{A}$'s which do not contribute to the potential intact.

We picture configurations at the vertices of a convex polygon, as on Fig. 1. Some of the $\mathcal{A}$-vertices are shown boldface. The potential $\mathcal{W}$ is a sum of the characters assigned to the boldface $\mathcal{A}$-vertices, generalizing (8). The decorated polygons in the cases (ii) and (iii) are depicted on the right of Fig. 8 and on Fig. 6. We discuss these examples in detail in Sects. 2.3–2.4.

1.3 Examples related to decorated surfaces

1.3.1 Laminations on decorated surfaces and canonical basis for $\mathrm{G} = SL_2$

1. Canonical basis in the tensor products invariants. This example can be traced back to nineteenth century. We relate it to laminations on a polygon.

Definition 1.1 An integral lamination l on an n-gon P_n is a collection $\{\beta_j\}$ of simple nonselfintersecting intervals ending on the boundary of P_n-vertices, modulo isotopy Fig. 3.

Pick a vertex of P_n, and number the sides clockwise. Given a collection of positive integers $a_1, \ldots, a_n$, consider the set $\mathcal{L}_n(a_1, \ldots, a_n)$ of all integral laminations l on the polygon P_n such that the number of endpoints of l on the k-th side is a_k. Let (V_2, ω) be a two dimensional $\mathbb{Q}$-vector space with a symplectic form. Let us assign to an $l \in \mathcal{L}_n(a_1, \ldots, a_n)$ an SL_2-invariant map

$$\mathbb{I}_l : (\otimes^{a_1} V_2) \otimes \ldots \otimes (\otimes^{a_n} V_2) \longrightarrow \mathbb{Q}.$$

We assign the factors in the tensor product to the endpoints of l, so that the order of the factors match the clockwise order of the endpoints. Then for each interval β in l we evaluate the form ω on the pair of vectors in the two factors of the tensor product labelled by the endpoints of β, and take the product over all intervals β in l. Recall that the SL_2-modules $S^a V_2$, $a > 0$, provide all non-trivial irreducible finite dimensional SL_2-modules up to isomorphism.

Theorem 1.2 *Projections of the maps* $\mathbb{I}_l$, $l \in \mathcal{L}_n(a_1, \ldots, a_n)$, *to* $S^{a_1} V_2 \otimes \ldots \otimes S^{a_n} V_2$ *form a basis in* $\mathrm{Hom}_{SL_2}(S^{a_1} V_2 \otimes \ldots \otimes S^{a_n} V_2, \mathbb{Q})$.

A quantum version was considered by Frenkel-Khovanov [24].

2. Canonical basis in the space of functions on the moduli space of SL_2 -local systems

Definition 1.3 Let S be a surface with boundary. An integral lamination l on S is a collection of simple, mutually non intersecting, non isotopic loops α_i with positive integral multiplicities

$$l = \sum_i n_i[\alpha_i] \quad n_i \in \mathbb{Z}_{>0},$$

considered modulo isotopy. The set of all integral laminations on S is denoted by $\mathcal{L}_{\mathbb{Z}}(S)$ Fig. 4.[6]

[6] Laminations on decorated surfaces were investigated in [17, Section 12], and [19]. However the two types of laminations considered there, the $\mathcal{A}$- and $\mathcal{X}$-laminations, are different then the ones in Definition 1.3. Indeed, they parametrise canonical bases in $\mathcal{O}(\mathcal{X}_{PGL_2,S})$ and, respectively, $\mathcal{O}(\mathcal{A}_{SL_2,S})$, while the latter parametrise a canonical basis in $\mathcal{O}(\mathrm{Loc}_{SL_2,S})$. Notice that a lamination in Definition 1.3 can not end on a boundary circle.

In the case when S is a surface without boundary we get Thurston's integral laminations.

Given an integral lamination l on S, let us define a regular function M_l on the moduli space $\mathrm{Loc}_{SL_2,S}$ of SL_2-local systems on S. Denote by $\mathrm{Mon}_\alpha(\mathcal{L})$ the monodromy of an SL_2-local system $\mathcal{L}$ over a loop α on S. The value of the function M_l on $\mathcal{L}$ is given by

$$M_l(\mathcal{L}) := \prod_i \mathrm{Tr}(\mathrm{Mon}_{\alpha_i}^{n_i}(\mathcal{L})).$$

Theorem 1.4 [17, Proposition 12.2]. *The functions M_l, $l \in \mathcal{L}_{\mathbb{Z}}(S)$, form a linear basis in the space $\mathcal{O}(\mathrm{Loc}_{SL_2,S})$.*

Recall that a *decorated surface* S is an oriented surface with boundary, and a finite, possibly empty, collection $\{s_1, \ldots, s_n\}$ of *special* points on the boundary, considered modulo isotopy.

We define a moduli space $\mathrm{Loc}_{SL_2,S}$ for any decorated surface S, so that laminations on S provide a canonical basis $\mathcal{O}(\mathrm{Loc}_{SL_2,S})$, generalising both Theorem 1.2 (when S is a polygon) and Theorem 10.14, see Sect. 10.3.

Let us discuss now how to generalize constructions of Sect. 1.1.2 to the decorated surfaces.

1.3.2 Positive G*-laminations and top components of surface affine Grassmannians*

A pair (G, S) gives rise to a moduli space $\mathcal{A}_{\mathrm{G},S}$ [17]. Here are two basic examples.

- When S is a disc with n special points on the boundary, we recover the space $\mathrm{Conf}_n(\mathcal{A})$.
- When S is just a surface, without special points, the moduli space $\mathcal{A}_{\mathrm{G},S}$ is a twisted version of the moduli space of G-local systems with unipotent monodromy around boundary components on S equipped with a covariantly constant decorated flag near every boundary component of S.

The space $\mathcal{A}_{\mathrm{G},S}$ has a positive structure [17]. We define in Sect. 10 a *potential* $\mathcal{W}$ on the space $\mathcal{A}_{\mathrm{G},S}$. It is a rational positive function, with the tropicalization $\mathcal{W}^t : \mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t) \longrightarrow \mathbb{Z}$.

The condition $\mathcal{W}^t \geq 0$ determines a subset of *positive integral* G-*laminations on S*:

$$\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t) := \{l \in \mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}. \tag{25}$$

For any decorated surface S, the set $\mathcal{A}^+_{SL_2,S}(\mathbb{Z}^t)$ is canonically isomorphic to the set of integral laminations on S, see Sect. 10.3. An interesting approach to a geometric definition of laminations for $\mathrm{G} = SL_m$, which employs the affine Grassmannian, was suggested by Ian Le [57].

There is a canonical volume form Ω on the space $\mathcal{A}_{\mathrm{G},S}$, which can be defined by using an ideal triangulation of S and the volume forms on $\mathrm{Conf}_n(\mathcal{A})$. When G is simply-connected, it is also the cluster volume form $\Omega_{\mathcal{A}}$.

We also assign to a pair (G, S) a stack $\mathrm{Gr}_{\mathrm{G},S}$, which we call the *surface affine Grassmannian*. When S is a disc with n special points on the boundary, we recover the stack $\mathrm{Conf}_n(\mathrm{Gr})$. In general it is an infinite dimensional stack.

The components of the punctured boundary $\partial S - \{s_1, \ldots, s_n\}$ isomorphic to intervals are called boundary intervals. We define the torus $\mathbb{H}$ and the lattice $\mathbb{P}$ by

$$\mathbb{H} := \mathrm{H}^{\{\text{boundary intervals on} S\}}, \quad \mathbb{P} := (\mathrm{P}^{+})^{\{\text{boundary intervals on } S\}}.$$

The map π is defined by assigning to a boundary interval I the element $i(\mathrm{A}_+, \mathrm{A}_-) \in \mathrm{H}$, see (18), where $(\mathrm{A}_-, \mathrm{A}_+)$ are the decorated flags at the ends of the interval I, ordered by the orientation of S, provided by the very definition of the space $\mathrm{A}_{\mathrm{G},S}$.

Given a point $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$, we define a cycle $\mathcal{M}^o_l \subset \mathrm{Gr}_{G,S}$. Given an element $\lambda \in \mathbb{P}$, we prove that the map $l \longmapsto \mathcal{M}^\circ_l$ gives rise to a bijection of sets

$$\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)_\lambda \stackrel{\sim}{\longrightarrow} \{\text{closures of top dimensional components of } \mathrm{Gr}^{(\lambda)}_{\mathrm{G},S}\}. \quad (26)$$

However in this case we can no longer bypass the question what are the "top components" of an infinite dimensional stack, as we did in Sect. 1.1.2. So we define in Sect. 10.5.1 "dimensions" of certain relevant stacks with values in certain *dimension* $\mathbb{Z}$*-torsors*. As a result, although the "dimension" is no longer an integer, the difference of two "dimensions" from the same dimension $\mathbb{Z}$-torsor is an integer, and so the notion of "top dimensional components" does make sense.

To define the analog of the space of tensor product invariants for a decorated surface S, we introduce in Sect. 10 a moduli space $\mathrm{Loc}_{\mathrm{G}^L,S}$. If S has no special points, it is the moduli space of G^L-local systems on S. If S is a disc with n points on the boundary, it is the space $\mathrm{Conf}_n(\mathcal{A}_{\mathrm{G}^L})$. We prove there that the set $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ parametrizes a linear basis in $\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^L,S})$.

1.4 Canonical pairings and homological mirror symmetry

Below we write $\mathcal{A}$ for $\mathcal{A}_\mathrm{G}$ etc., and use notation $\mathcal{A}_L$ for $\mathcal{A}_{\mathrm{G^L}}$ etc.

For any split reductive group G, the space $\mathcal{O}(\mathcal{A}_L)$ of regular functions on the principal affine space $\mathcal{A}_L$ of G^L is a model of representations of G^L:

every irreducible G^L-module appears there once. This allows us to organize the direct sum of all vector spaces of a given kind where the canonical bases live into a vector space of regular functions on a single space. For example:

$$\bigoplus_{(\lambda_1,\ldots,\lambda_n)\in(\mathrm{P}^+)^n} V_{\lambda_1}\otimes\ldots\otimes V_{\lambda_n} = \mathcal{O}(\mathcal{A}_L^n). \tag{27}$$

$$\bigoplus_{(\lambda_1,\ldots,\lambda_n)\in(\mathrm{P}^+)^n} \left(V_{\lambda_1}\otimes\ldots\otimes V_{\lambda_n}\right)^{\mathrm{G}^L} = \mathcal{O}(\mathcal{A}_L^n)^{\mathrm{G}^L} = \mathcal{O}(\mathrm{Conf}_n(\mathcal{A}_L)). \tag{28}$$

Using this, let us interpret the statement that a canonical basis of a given kind is parametrized by positive integral tropical points of a certain space as existence of a *canonical pairing*.

1.4.1 Tensor product invariants and homological mirror symmetry

For any split reductive group G, the set $\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ parametrizes a canonical basis in the space (28). So there are canonical pairings

$$\mathbf{I}_{\mathrm{G}} : \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)\times\mathrm{Conf}_n(\mathcal{A}_L)\longrightarrow\mathbb{A}^1. \tag{29}$$

$$\mathbf{I}_{\mathrm{G}^\mathrm{L}} : \mathrm{Conf}_n(\mathcal{A})\times\mathrm{Conf}_n^+(\mathcal{A}_L)(\mathbb{Z}^t)\longrightarrow\mathbb{A}^1. \tag{30}$$

So the story becomes completely symmetric. The idea that the set parametrizing canonical bases in tensor product invariants is a subset of $\mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$ goes back to Duality Conjectures from [17]. It is quite surprising that taking into account the potential we get a canonical basis in the space of regular functions on the *same kind of space*, $\mathrm{Conf}_n(\mathcal{A}_L)$, for the Langlands dual group.

To picture this symmetry, consider a convex n-gon P_n on the left of Fig. 5, and assign a configuration $(\mathrm{A}_1,\ldots,\mathrm{A}_n)\in\mathrm{Conf}_n^\times(\mathcal{A})$ to its vertices. The potential $\mathcal{W}$ is a sum of the vertex contributions; so the vertices are shown boldface. The pair of decorated flags at each side is generic; so all sides are dashed. Tropicalizing the data at the vertices, and using the isomorphism $\mathrm{Conf}_2^+(\mathcal{A})(\mathbb{Z}^t)=\mathrm{P}^+$, we assign a dominant weight λ_k of G^L to each side of the left polygon. Consider now the dual n-gon $*P_n$ on the right, and a configuration of decorated flags $(\mathrm{A}'_1,\ldots,\mathrm{A}'_n)$ in G^L at its vertices. The dominant weight λ_k on the left corresponds to the irreducible representation V_{λ_k}, realised in the model $\mathcal{O}(\mathcal{A}_L)$ assigned to the dual vertex of $*P_n$.

Tropical points live naturally at the boundary of a positive space, compactifying the set of its real positive points [20]. An example is given by Thurston's boundary of Teichmüller spaces, realized as the space of projective measured laminations.

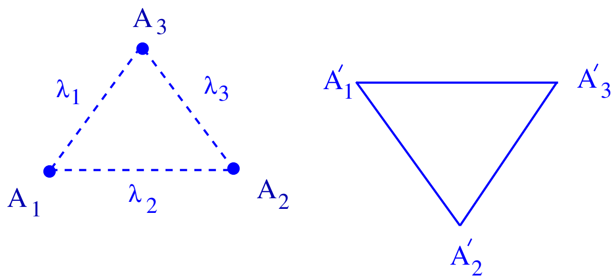


Fig. 5 Duality between configurations spaces of decorated flags for G and G^L. The potential is a sum of contributions at the *boldface* vertices. Pairs of decorated flags at the *dashed sides* are in generic position. No condition on the pairs of decorated flags at the *solid sides*

It is tempting to think that canonical pairings (29) and (30) are manifestations of a symmetry involving both spaces simultaneously, rather then relating the tropical points of one space to the regular functions on the other space. We conjecture that this elusive symmetry is the mirror symmetry, and the function $\mathcal{W}$ is the potential for the Landau–Ginzburg model.

To formulate precise conjectures, let us start with a general set-up.

The A-model. Let $\mathcal{M}$ be a complex affine variety. So it has an affine embedding $i : \mathcal{M} \hookrightarrow \mathbb{C}^N$. The Kahler form $\sum_i dz_i d\bar{z}_i$ on $\mathbb{C}^N$ induces a Kahler form on $\mathcal{M}(\mathbb{C})$ with an exact symplectic form ω. The wrapped Fukaya category $\mathcal{F}_{\rm wr}(\mathcal{M}, \omega)$ [2] does not depend on the embedding i. We denote it by $\mathcal{F}_{\rm wr}(\mathcal{M})$. A potential $\mathcal{W}$ on $(\mathcal{M}, \omega)$ allows to define the wrapped Fukaya–Seidel category $\mathcal{FS}_{\rm wr}(\mathcal{M}) = \mathcal{FS}_{\rm wr}(\mathcal{M}, \omega, \mathcal{W})$. The case of a potential with only Morse singularities is treated in [72]. It also does not depend on the choice of affine embedding. A volume form Ω provides a $\mathbb{Z}$-grading on $\mathcal{FS}_{\rm wr}(\mathcal{M})$ [71].

The positive A-brane. In our examples $\mathcal{M}$ is a positive space over $\mathbb{Q}$. So it has a submanifold $\mathcal{M}(\mathbb{R}_{>0})$ of real positive points. It is a Lagrangian submanifold for the symplectic form ω induced by any affine embedding. The form Ω is defined over $\mathbb{Q}$, and so $\mathcal{M}(\mathbb{R}_{>0})$ is a special Lagrangian submanifold since it restricts to a real volume form on $\mathcal{M}(\mathbb{R}_{>0})$. The potential $\mathcal{W}$ is a positive function on $\mathcal{M}$. So the special Lagrangian submanifold $\mathcal{M}(\mathbb{R}_{>0})$ should give rise to an object of the wrapped Fukaya–Seidel category of $\mathcal{M}$, which we call the *positive A-brane*, denoted by $\mathcal{L}_+$.

The projection/action data. In all our examples we have a mirror dual pair $\mathcal{M} \leftrightarrow \mathcal{M}_L$ equipped with the following data: a projection $\pi : \mathcal{M} \longrightarrow \mathbb{H}$ onto a split torus $\mathbb{H}$, an action of the split torus $\mathbb{T}$ on $\mathcal{M}$ preserving the volume form and the potential, and a similar pair of tori $\mathbb{H}_L, \mathbb{T}_L$ for $\mathcal{M}_L$. These tori are in duality:

$$X_*(\mathbb{T}_L) = X^*(\mathbb{H}), \quad X_*(\mathbb{H}_L) = X^*(\mathbb{T}).$$

This projection/action data gives rise to the following additional structures on the categories.

(i) The group $\mathrm{Hom}(X_*(\mathbb{H}), \mathbb{C}^*) = \widehat{\mathbb{H}}(\mathbb{C})$ of $\mathbb{C}^*$-local systems on the complex torus $\mathbb{H}(\mathbb{C})$ acts on the category $\mathcal{FS}_{\mathrm{wr}}(\mathcal{M})$. Namely, we assume that the objects of the category are given by Lagrangian submanifolds in $\mathcal{M}(\mathbb{C})$ with U(1)-local systems. Then a U(1)-local system $\mathcal{L}$ on $\mathbb{H}(\mathbb{C})$ acts by the tensor product with $\pi^*(\mathcal{L})$, providing an action of the subgroup $\mathrm{Hom}(X_*(\mathbb{H}), \mathrm{U}(1))$ on the category. We assume that the action extends to an algebraic action of the complex torus

$$\mathrm{Hom}(X_*(\mathbb{H}), \mathbb{C}^*) = X^*(\mathbb{H}) \otimes \mathbb{C}^* = X_*(\widehat{\mathbb{H}}) \otimes \mathbb{C}^* = \widehat{\mathbb{H}}(\mathbb{C}).$$

(ii) Let $\mathbb{T}_K$ be the maximal compact subgroup of the torus $\mathbb{T}(\mathbb{C})$. We assume that the action of the group $\mathbb{T}_K$ on the symplectic manifold $(\mathcal{M}, \omega)$ is Hamiltonian.[7] Then any subgroup $S^1 \subset \mathbb{T}_K$ provides a family of symplectic maps r_t, $t \in \mathbb{R}/\mathbb{Z} = S^1$. The map r_1 provides an invertible functorial automorphism of Hom's of the category $\mathcal{FS}_{\mathrm{wr}}(\mathcal{M})$, and thus an invertible element of the center of the category. So the group algebra $\mathbb{Z}[X_*(\mathbb{T})] = \mathcal{O}(\widehat{\mathbb{T}})$ is mapped into the center:

$$\mathcal{O}(\widehat{\mathbb{T}}) \longrightarrow \mathrm{Center}(\mathcal{FS}_{\mathrm{wr}}(\mathcal{M})).$$

(iii) Clearly, there is a map $\mathcal{O}(\mathbb{H}) \longrightarrow \mathrm{Center}(D^b\mathrm{Coh}(\mathcal{M}))$, and the group $\mathbb{T}$ acts on $D^b\mathrm{Coh}(\mathcal{M})$.

The potential/boundary divisors. It was anticipated by Hori–Vafa [43] and Auroux [4] that adding a potential on a space $\mathcal{M}$ amounts to a partial compactification of its mirror $\mathcal{M}_L$ by a divisor. More precisely, denote by $\mathcal{M}^\times$ and $\mathcal{M}_L^\times$ the regular loci of the forms Ω and Ω_L. The potential is a sum $\mathcal{W} = \sum_c \mathcal{W}_c$. Its components $\mathcal{W}_c$ are expected to match the irreducible divisors D_c of $\mathcal{M}_L - \mathcal{M}_L^\times$. The divisors D_c are defined as the divisors on $\mathcal{M}_L$ where $\mathrm{Res}_{D_c}(\Omega_L)$ is non-zero. So we should have

$$\mathcal{W} = \sum_c \mathcal{W}_c, \quad \mathcal{M}_L - \mathcal{M}_L^\times = \cup_c D_c, \quad \mathcal{W}_c \overset{?}{\leftrightarrow} D_c. \tag{31}$$

There are several ways to explain how this correspondence should work.

7 In our main examples the symplectic structure is exact, $\omega = d\alpha$. So averaging the form α by the action of the compact group $\mathbb{T}_K$ we can assume that it is $\mathbb{T}_K$-invariant. Therefore the action is Hamiltonian: the Hamiltonian at x for a one parametric subgroup g^t is given by the formula $\alpha(\frac{d}{dt} g^t(x))$.

(i) The potential $\mathcal{W}_c$ determines an element $[\mathcal{W}_c] \in \mathrm{HH}^0(\mathcal{M})$, which defines a deformation of the category $D^b\mathrm{Coh}(\mathcal{M})$ as a $\mathbb{Z}/2\mathbb{Z}$-category. On the dual side it corresponds to a deformation of the Fukaya category obtained by adding to the symplectic form on $\mathcal{M}_L$ a multiple of the 2-form ω_c, whose cohomology class is the cycle class $[D_c] \in H^2(\mathcal{M}_L, \mathbb{Z}(1))$ of the divisor D_c.

(ii) The Landau–Ginzburg potential $\mathcal{W}_c$ should be obtained by counting the holomorphic discs touching the divisor D_c, as was demonstrated by Auroux [4] in examples.

(iii) In the cluster variety set up the correspondence is much more precise, see Sect. 11.

Example. To illustrate the set-up, let us specify the data on the moduli space $\mathrm{Conf}_n(\mathcal{A})$.

- A regular positive function, the potential $\mathcal{W} : \mathrm{Conf}_n^\times(\mathcal{A}) \longrightarrow \mathbb{A}^1$.
- A regular volume form Ω on $\mathrm{Conf}_n^\times(\mathcal{A})$, with logarithmic singularities at infinity.
- A regular projection $\pi : \mathrm{Conf}_n^\times(\mathcal{A}) \longrightarrow \mathbb{H}$ onto a torus $\mathbb{H} := \mathrm{H}^{\{\text{sides of the } n-\text{gon } P_n\}}$.
- An action r of the torus $\mathbb{T} := \mathrm{H}^{\{\text{vertices of } P_n\}}$ on $\mathrm{Conf}_n(\mathcal{A})$ by rescaling decorated flags.

Changing G to G^L we interchanges the action with the projection:

- The torus $\mathbb{T}_L$ is dual to the torus $\mathbb{H}$, i.e. there is a canonical isomorphism $X_*(\mathbb{T}_L) = X^*(\mathbb{H})$.

By construction, the potential is a sum

$$\mathcal{W} = \sum_v \sum_{i \in \mathrm{I}} \mathcal{W}_i^v \tag{32}$$

over the vertices v of the polygon P_n, parametrising configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_n)$, and the set I of simple positive roots for G. Indeed, a non-degenerate character χ of U is naturally a sum $\chi = \sum_i \chi_i$.

On the other hand, the set of irreducible components of the divisor $\mathrm{Conf}_n(\mathcal{A}_L)$–$\mathrm{Conf}_n^\times(\mathcal{A}_L)$ is parametrised by the pairs (E, i) where E are the edges of the dual polygon $*P_n$, see Sect. 1.2.1:

$$\mathrm{Conf}_n(\mathcal{A}_L) - \mathrm{Conf}_n^\times(\mathcal{A}_L) = \cup_E \cup_{i \in \mathrm{I}} D_i^E. \tag{33}$$

Since vertices of the polygon P_n match the sides of the dual polygon $*P_n$, the components of the potential (32) match the irreducible components of the divisor at infinity (33) on the dual space.

We start with the most basic form of our mirror conjectures, which does not involve the potential.

Conjecture 1.5 *For any split semisimple group* G *over* $\mathbb{Q}$*, there is a mirror duality*

$$(\mathrm{Conf}_n^\times(\mathcal{A}), \Omega) \quad \textit{is mirror dual to} \quad (\mathrm{Conf}_n^\times(\mathcal{A}_L), \Omega_L). \tag{34}$$

This means in particular that one has an equivalence of A_∞*-categories*

$$\mathcal{F}_{\mathrm{wr}}(\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{C})) \stackrel{\sim}{\longrightarrow} D^b\mathrm{Coh}(\mathrm{Conf}_n^\times(\mathcal{A}_L)). \tag{35}$$

This equivalence maps the positive A-brane $\mathcal{L}_+$ *to the structure sheaf* $\mathcal{O}$*.*
It identifies the action of the group $\widehat{\mathbb{H}}(\mathbb{C})$ *on the category* $\mathcal{F}_{\mathrm{wr}}(\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{C}))$ *with the action of the group* $\mathbb{T}_L(\mathbb{C})$ *on* $D^b\mathrm{Coh}(\mathrm{Conf}_n^\times(\mathcal{A}_L))$*, and identifies the subalgebras*

$$\mathcal{O}(\widehat{\mathbb{T}}) \subset \mathrm{Center}(\mathcal{F}_{\mathrm{wr}}(\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{C}))) \quad \textit{and}$$
$$\mathcal{O}(\mathbb{H}_L) \subset \mathrm{Center}(D^b\mathrm{Coh}(\mathrm{Conf}_n^\times(\mathcal{A}_L))).$$

The projection/action data for the pair (34) is given by

$$\mathbb{H} = \mathrm{H}^n, \quad \mathbb{H}_L = \mathrm{H}_L^n, \quad \mathbb{T} = \mathrm{H}^n, \quad \mathbb{T}_L = \mathrm{H}_L^n.$$

The pair (34) is symmetric: interchanging the group G with the Langlands dual group G^L amounts to exchanging the A-model with the B-model.

Using the mirror pair (34) as a starting point, we can now turn on the potentials at all vertices of the left polygon P_n. This amounts to a partial compactification of the dual space. Namely, we take the space $\mathrm{Conf}_n(\mathcal{A}_L)$, and consider its affine closure $\mathrm{Conf}_n(\mathcal{A}_L)_{\mathbf{a}} := \mathrm{Spec}(\mathcal{O}(\mathcal{A}_L^n)^{\mathrm{G}^L})$.

Since the action of the group H^n on $\mathrm{Conf}_n^\times(\mathcal{A})$ alters the potential $\mathcal{W}$, and the projection π_L onto H_L^n does not extend to $\mathrm{Conf}_n(\mathcal{A}_L)_{\mathbf{a}}$, the projection/action data for the pair (48) is

$$\mathbb{H} = \mathrm{H}^n, \quad \mathbb{H}_L = \{e\}, \quad \mathbb{T} = \{e\}, \quad \mathbb{T}_L = \mathrm{H}_L^n.$$

Therefore by turning on the potentials we arrive at the following Mirror Conjecture:

Conjecture 1.6 *For any split semisimple group* G *over* $\mathbb{Q}$*, there is a mirror duality*

$$(\mathrm{Conf}_n^\times(\mathcal{A}), \mathcal{W}, \Omega) \quad \textit{is mirror dual to} \quad \mathrm{Conf}_n(\mathcal{A}_L)_{\mathbf{a}}. \tag{36}$$

This means in particular that there is an equivalence of A_∞*-categories*

$$\mathcal{FS}_{\mathrm{wr}}(\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{C}), \mathcal{W}, \Omega) \quad \stackrel{\sim}{\longrightarrow} \quad D^b\mathrm{Coh}(\mathrm{Conf}_n(\mathcal{A}_L)_{\mathbf{a}}). \tag{37}$$

It maps the positive A-brane $\mathcal{L}_+$ *to the structure sheaf* $\mathcal{O}$*, and identifies the action of the group* $\widehat{\mathbb{H}}(\mathbb{C})$ *on the category* $\mathcal{FS}_{\mathrm{wr}}(\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{C}))$ *with the action of* $\mathbb{T}_L(\mathbb{C})$ *on* $D^b\mathrm{Coh}(\mathrm{Conf}_n(\mathcal{A}_L)_{\mathbf{a}})$.

The geometry of mirror dual objects in Conjectures 1.5 and 1.6 is *essentially* dictated by representation theory. Indeed, the tropical points are determined by birational types of the spaces, and canonical bases tell the algebras of functions on the dual affine varieties:[8]

$$\text{The set } \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) \text{ parametrises a canonical basis in } \mathcal{O}(\mathrm{Conf}_n(\mathcal{A}_L)). \tag{38}$$

$$\text{The set } \mathrm{Conf}_n(\mathcal{A}_L)(\mathbb{Z}^t) \text{ should parametrise a canonical basis in } \mathcal{O}(\mathrm{Conf}_n^\times(\mathcal{A})). \tag{39}$$

The potential $\mathcal{W}$ and the projection π define a regular map $(\pi, \mathcal{W})$: $\mathrm{Conf}_n^\times(\mathcal{A}) \longrightarrow \mathbb{H} \times \mathbb{A}^1$. The form Ω on $\mathrm{Conf}_n^\times(\mathcal{A})$ and the canonical volume forms on $\mathbb{H}$ and $\mathbb{A}^1$ provide a volume form $\Omega^{(a,c)}$ at the fiber $F_{a,c}$ of this map over a generic point $(a, c) \in \mathbb{H} \times \mathbb{A}^1$.

More generally, we can turn on only partial potentials at the vertices of the polygon P_n, which amounts on the dual side to taking partial compactifications, and then considering their affine closures. This way we get an array of conjecturally dual pairs, described as follows.

For each vertex v of the polygon P_n parametrising configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_n)$ choose an arbitrary subset $\mathrm{I}_v \subset \mathrm{I}$ of the set parametrising the simple positive roots of G. It determines a partial potential

$$\mathcal{W}_{\{\mathrm{I}_v\}} = \sum_v \mathcal{W}_{\mathrm{I}_v}, \quad \mathcal{W}_{\mathrm{I}_v} := \sum_{i \in \mathrm{I}_v} \mathcal{W}_i^v. \tag{40}$$

On the dual side, subsets $\{\mathrm{I}_v\}$ determine a partial compactification of the space $\mathrm{Conf}_n^\times(\mathcal{A}_L)$, obtained by adding the divisors $D_i^{E_v}$ where $i \in \mathrm{I}_v$. Here E_v is the side of the polygon $*P_n$ dual to the vertex v of P_n:

$$\mathrm{Conf}_n(\mathcal{A}_L)_{\{\mathrm{I}_v\}} := \mathrm{Conf}_n^\times(\mathcal{A}_L) \bigcup \cup_v \cup_{i \in \mathrm{I}_v} D_i^{E_v}. \tag{41}$$

[8] Although the claim (39) is not addressed in the paper, it can be deduced from (38).

For each vertex v of P_n there is a subgroup $\mathrm{H}_{\mathrm{I}_v} \subset \mathrm{H}$ preserving the partial potential $\mathcal{W}_{\mathrm{I}_v}$ at v. On the dual side, let $\mathrm{H}_L^{\mathrm{I}_v}$ be the dual quotient of the Cartan group H_L. So we arrive at the projection/action data

$$\mathbb{H} = \mathrm{H}^n, \quad \mathbb{H}_L = \prod_v \mathrm{H}_L^{\mathrm{I}_v}, \quad \mathbb{T} = \prod_v \mathrm{H}_{\mathrm{I}_v}, \quad \mathbb{T}_L = \mathrm{H}_L^n. \tag{42}$$

So turning on partial potentials we arrive at Conjecture 1.7, interpolating Conjectures 1.5 and 1.6:

Conjecture 1.7 *For any split semisimple group* G *over* $\mathbb{Q}$, *there is a mirror duality*

$$(\mathrm{Conf}_n^\times(\mathcal{A}), \mathcal{W}_{\{\mathrm{I}_v\}}, \Omega) \ \ \textit{is mirror dual to the affine closure of} \ \ \mathrm{Conf}_n(\mathcal{A}_L)_{\{\mathrm{I}_v\}}. \tag{43}$$

Its action/projection data is given by (42).

Needless to say, the positive integral tropical points of the left space parametrise a basis in the space of functions on the right space.

Here is another general principle to generate new mirror dual pairs. We start with a mirror dual pair $(\mathcal{M}, \Omega, \mathcal{W}) \leftrightarrow \mathcal{M}_L$, equipped with the projection/action data which involves a dual pair $(\mathbb{T}, \mathbb{H}_L)$. So $\mathbb{T}$ acts by automorphisms of the triple $(\mathcal{M}, \Omega, \mathcal{W})$, and there is a dual projection $\pi_L : \mathcal{M}_L \to \mathbb{H}_L$.

Choose any subgroup $\mathbb{T}' \subset \mathbb{T}$, and consider the corresponding $\mathbb{T}'$-equivariant category. If the group $\mathbb{T}'$ acts freely, this amounts to taking the quotient of the space with potential $(\mathcal{M}, \mathcal{W})$ by the action of $\mathbb{T}'$. A volume form on $\mathbb{T}'$ gives rise to a volume form on the quotient, obtained by constructing the volume form Ω with the dual polyvector field on $\mathbb{T}'$. The subgroup $\mathbb{T}' \subset \mathbb{T}$ determines by the duality a quotient group $\mathbb{H}_L \longrightarrow \mathbb{H}'_L$, and therefore a projection $\pi'_L : \mathcal{M}_L \to \mathbb{H}'_L$.

- *The quotient stack* $(\mathcal{M}/\mathbb{T}', \mathcal{W})$ *is mirror dual to the family* $\pi'_L : \mathcal{M}_L \to \mathbb{H}'_L$.
  In the examples below $(\mathcal{M}/\mathbb{T}', \mathcal{W})$ is just dual to a fiber ${\pi'_L}^{-1}(a) \subset \mathcal{M}_L$, $a \in \mathbb{H}'_L$.

In particular, starting from a mirror dual pair (43), we can choose any subgroup $\mathbb{T}' \subset \mathbb{T} = \prod_v \mathrm{H}_{\mathrm{I}_v}$ acting on the space with potential on the left. All examples below are obtained this way.

Example. We start with the space $\mathrm{Conf}^\times(\mathcal{A}^{n+1})$ with the potential $\mathcal{W}_{1,\ldots,n}$ given by the sum of the full potentials at all vertices but one, the vertex A_{n+1}.

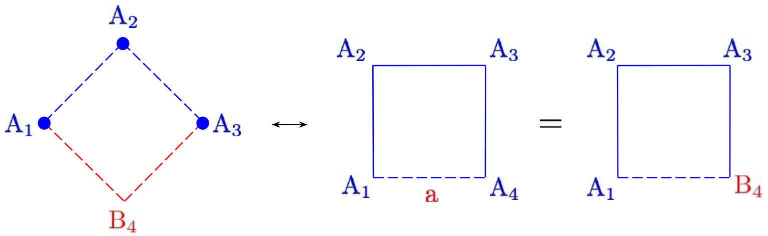


Fig. 6 Dual pairs $(\mathrm{Conf}^{\times}(\mathcal{A}^3, \mathcal{B}), \mathcal{W}_{1,2,3})$ and $\mathrm{Conf}_{w_0}(\mathcal{A}_L^3, \mathcal{B}_L) = \mathcal{A}_L^2$. The H-components of the projection λ sit at the A-decorated *blue dashed edges* on the *left*. The projection μ to H is assigned to the *red* $\mathrm{A}_3\mathrm{B}_4\mathrm{A}_1$ (color figure online)

The action of the group H on the decorated flag A_{n+1} preserves the potential $\mathcal{W}_{1,\ldots,n}$. Applying the above principle, we get a dual pair illustrated on Fig. 6. The fiber over a, illustrated by the middle picture on Fig. 6, is canonically isomorphic to the less symmetrically defined space illustrated on the right.

In the next section we consider this example from a different point of view, starting from representation-theoretic picture, just as we did with our basic example, and arrive to the same dual pairs.

1.4.2 Tensor products of representations and homological mirror symmetry

The set $\mathrm{Conf}^{+}(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t)$ defined using the potential $\mathcal{W}$ from (11) parametrises canonical bases in n-fold tensor products of simple G^L-modules. So using (27) we arrive at a canonical pairing

$$\mathbf{I} : \mathrm{Conf}^{+}(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t) \times \mathcal{A}_L^n \longrightarrow \mathbb{A}^1. \tag{44}$$

Let us present $\mathcal{A}_L^n$ as a configuration space. Recall that $\mathrm{Conf}_{w_0}(\mathcal{A}_L^{n+1}, \mathcal{B}_L)$ parametrises configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ such that the pair $(\mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ is generic. Generic pairs $\{\mathrm{A}, \mathrm{B}\}$ form a G^L-torsor. Let $\{\mathrm{A}^{+}, \mathrm{B}^{-}\}$ be a standard generic pair. Then there is an isomorphism

$$\mathcal{A}_L^n \stackrel{=}{\longrightarrow} \mathrm{Conf}_{w_0}(\mathcal{A}_L^{n+1}, \mathcal{B}_L), \quad \{\mathrm{A}_1, \ldots, \mathrm{A}_n\} \longmapsto (\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{A}^{+}, \mathrm{B}^{-}). \tag{45}$$

The subspace $\mathrm{Conf}^{\times}(\mathcal{A}_L^{n+1}, \mathcal{B}_L)$ parametrises configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ such that the consecutive pairs of flags are generic. It is the quotient of $\mathrm{Conf}_{n+2}^{\times}(\mathcal{A})$ by the action of the group H on the last decorated flag. The projection $\mathrm{Conf}_{n+2}^{\times}(\mathcal{A}) \to \mathrm{H}^{n+2}$ induces a map, see (18),

$$\pi = (\lambda, \mu) : \mathrm{Conf}^{\times}(\mathcal{A}^{n+1}, \mathcal{B}) \longrightarrow \mathrm{H}^n \times \mathrm{H}. \tag{46}$$

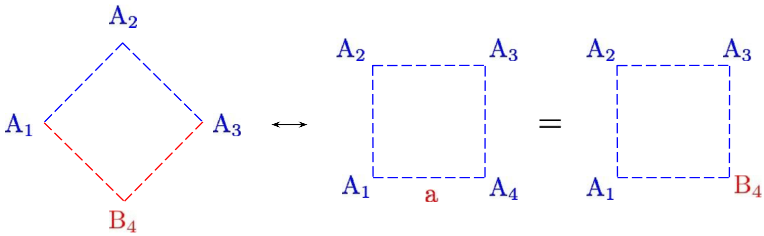


Fig. 7 Dual spaces $\mathrm{Conf}^{\times}(\mathcal{A}^3, \mathcal{B})$ (*left*) and $\mathrm{Conf}^{\times}(\mathcal{A}_L^3, \mathcal{B}_L) =$ (*right*)

$$(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2}) \longmapsto (\alpha(\mathrm{A}_1, \mathrm{A}_2), \ldots, \alpha(\mathrm{A}_n, \mathrm{A}_{n+1})) \\ \times \alpha(\mathrm{A}_{n+1}, \mathrm{B}_{n+2})\alpha(\mathrm{A}_1, \mathrm{B}_{n+2})^{-1}.$$

Then the symmetry is restored, and we can view (44) as a manifestation of a mirror duality:

$$(\mathrm{Conf}^{\times}(\mathcal{A}^{n+1}, \mathcal{B}), \mathcal{W}, \Omega, \pi) \text{ is mirror dual to } (\mathrm{Conf}_{w_0}(\mathcal{A}_L^{n+1}, \mathcal{B}_L), \Omega_L, r_L). \tag{47}$$

Here r_L is the action of H_L^{n+1} by rescaling of the decorated flags. The projection/action data is

$$\mathbb{H} = \mathrm{H}^{n+1}, \quad \mathbb{H}_L = \{e\}, \quad \mathbb{T} = \{e\}, \quad \mathbb{T}_L = \mathrm{H}_L^{n+1},$$

The analog of mirror dual pair (34) and its projection/action data are given by, see Fig. 7,

$$(\mathrm{Conf}^{\times}(\mathcal{A}^{n+1}, \mathcal{B}), \Omega) \quad \text{is mirror dual to} \quad (\mathrm{Conf}^{\times}(\mathcal{A}_L^{n+1}, \mathcal{B}_L), \Omega_L). \tag{48}$$

$$\mathbb{H} = \mathrm{H}^{n+1}, \quad \mathbb{H}_L = \mathrm{H}_L^{n+1}, \quad \mathbb{T} = \mathrm{H}^{n+1}, \quad \mathbb{T}_L = \mathrm{H}_L^{n+1},$$

So we arrived at the two dual pairs and (47) and (48) using canonical pairings as a guideline.

As discussed in the Example in Sect. 1.4, we can get them from the basic dual pairs (36) and (34) using the action/projection duality $\bullet$, which in this case tells that the quotient by the action of the group H on one side is dual to a fiber of the family of spaces over the dual group H_L over a point $a \in \mathrm{H}_L$.

In particular, the dual pair (34) leads to the dual pair illustrated on Fig. 7. Notice that configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_{n+2})$ with $\alpha(\mathrm{A}_{n+1}, \mathrm{A}_{n+2}) = a \in \mathrm{H}$ are in bijection with configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ where the pair $(\mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ is generic. So the two diagrams on the right of Fig. 7 represent isomorphic configuration spaces, and we get the dual pair (48) from (34).

The dual pair (47) is obtained from (48) by adding potentials at the A-vertices, thus allowing arbitrary pairs of flags on the dual sides.

We conjecture that the analogs of Conjectures 1.5 and 1.6 hold for the pairs (48) and (47).

1.4.3 Landau–Ginzburg mirror of a maximal unipotent group U *and it generalisations*

We view Lusztig's dual canonical basis in $\mathcal{O}(\mathrm{U}^L)$ as a canonical pairing, and hence as a mirror duality:

$$\mathbf{I} : \mathrm{U}_\chi^+(\mathbb{Z}^t) \times \mathrm{U}^L \longrightarrow \mathbb{A}^1, \quad (\mathrm{U}^*, \chi) \quad \text{is mirror dual to} \quad \mathrm{U}^L. \tag{49}$$

To define U^*, we realise a maximal unipotent subgroup U as a big Bruhat cell in the flag variety, and intersect it with the opposite big Bruhat cell. The χ is a non-degenerate additive character of U, restricted to U^*. This example is explained and generalised using configurations as follows.

Let $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathcal{A}^n, \mathcal{B})$ be the space parametrising configurations $(\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ such that the pairs $(\mathrm{B}_1, \mathrm{B}_{n+2})$ and $(\mathrm{A}_{n+1}, \mathrm{B}_{n+2})$ are generic, see the right picture on Fig. 8. There is an isomorphism

$$\begin{aligned} \mathrm{U}_L \times \mathcal{A}_L^{n-1} = \mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L^n, \mathcal{B}_L), \quad \{\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n\} \\ \longmapsto (\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n, \mathrm{A}^+, \mathrm{B}^-). \end{aligned} \tag{50}$$

The group H_L^n acts on $\mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L^n, \mathcal{B}_L)$ by rescaling decorated flags.

The subspace $\mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}^n, \mathcal{B})$ parametrises configurations where each consecutive pair of flags is generic. It is depicted on the left of Fig. 8. It is the quotient of $\mathrm{Conf}_{n+2}^\times(\mathcal{A})$ by the action of $\mathrm{H} \times \mathrm{H}$ on the first and last decorated flags. Thus there is a map π, defined similarly to (46):

$$\pi = (\lambda, \mu) : \mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}^n, \mathcal{B}) \to \mathrm{H}^{n-1} \times \mathrm{H}. \tag{51}$$

So the projection/action data in this case is

$$\mathbb{H} = \mathrm{H}^n, \quad \mathbb{H}_L = \{e\}, \quad \mathbb{T} = \{e\}, \quad \mathbb{T}_L = \mathrm{H}_L^n,$$

For example, $\mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathrm{U}^*$, in agreement with U^* in (49).

Conjecture 1.8 *The set* $\mathrm{Conf}^+(\mathcal{B}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t)$ *parametrises a canonical basis in* $\mathcal{O}(\mathrm{U}_L \times \mathcal{A}_L^{n-1})$. *The subset* $(\lambda^t, \mu^t)^{-1}(\lambda_1, \ldots, \lambda_{n-1}; \nu)$ *parametrises a canonical basis in the weight* ν *subspace of*

$$\mathrm{U}(\mathcal{N}^L) \otimes V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n-1}}.$$

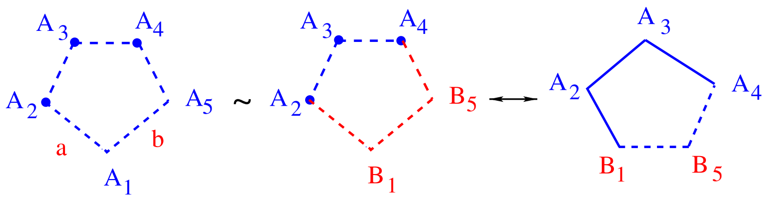


Fig. 8 Duality $\mathrm{Conf}^{\times}(\mathcal{B}^2, \mathcal{A}^3) \leftrightarrow \mathrm{Conf}_{w_0}(\mathcal{B}_L^2, \mathcal{A}_L^3) = \mathrm{U}_L \times \mathcal{A}_L^2$. In the *middle*: the H-components of the map λ sit at the *dashed blue sides*. The map μ is assigned to $\mathrm{A}_2\mathrm{B}_1\mathrm{B}_5\mathrm{A}_4$ (color figure online)

The analogs of Conjectures 1.5 and 1.6 hold for the following mirror dual pairs:

$$(\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}), \Omega) \quad \textit{is mirror dual to} \quad (\mathrm{Conf}^{\times}(\mathcal{B}_L, \mathcal{A}_L^n, \mathcal{B}_L), \Omega_L),$$

$$(\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}), \mathcal{W}, \Omega, \pi) \quad \textit{is mirror dual to} \quad (\mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L^n, \mathcal{B}_L), r_L)$$

These mirror pairs can be obtained from the basic mirror pairs (34) and (36) by trading, using the action / projection principle $\bullet$, the quotient by H_L^2 to the fiber over $(a, b) \in \mathrm{H}^2$ on the dual side, see Fig. 8.

1.4.4 The Landau–Ginzburg mirror of a simple split group G

In this section we interpret a split simple group G as a configuration space, and using this deduce its Landau–Ginzburg mirror from Conjecture 1.6 by using our standard toolbox. The companion conjecture tells that the maximal double Bruhat cells is selfdual, assuming that we change G to G^L.

Denote by $\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A})$ the space parametrising configurations $(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3, \mathrm{A}_4)$ where all four consecutive pairs are generic. There is a potential given by the sum of the potentials at the A-vertices:

$$\mathcal{W}_{2,4}(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3, \mathrm{A}_4) := \chi_{\mathrm{A}_2}(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) + \chi_{\mathrm{A}_4}(\mathrm{B}_3, \mathrm{A}_4, \mathrm{B}_1).$$

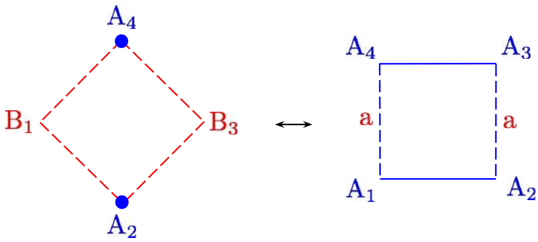
The space with potential is illustrated on the left of Fig. 9. Let us describe its mirror.

Recall the isomorphism $\alpha : \mathrm{Conf}^{\times}(\mathcal{A}, \mathcal{A}) \longrightarrow \mathrm{H}$. Consider the moduli space of configurations

$$\begin{aligned} &(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4) \in \mathrm{Conf}_4(\mathcal{A}_L) \mid (\mathrm{A}_1, \mathrm{A}_4), (\mathrm{A}_2, \mathrm{A}_3) \text{ are generic;} \\ &\quad \alpha(\mathrm{A}_1, \mathrm{A}_4) = \alpha(\mathrm{A}_2, \mathrm{A}_3) = e. \end{aligned} \tag{52}$$

The picture on the right of Fig. 9 illustrates this moduli space.

Fig. 9 The Landau–Ginzburg model (*left*) dual to G^L (*right*)



Lemma 1.9 *The moduli space* (52) *is isomorphic to the group* G^L.

Proof Pick a generic pair $\{A_1, A_2\}$ with $\alpha(A_1, A_4) = e$. Then for each G^L-orbit in (52) there is a unique representative $\{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4\}$ where $\{\mathrm{A}_1, \mathrm{A}_2\}$ is the chosen pair. There is a unique $g \in \mathrm{G}^L$ such that $g\{\mathrm{A}_1, \mathrm{A}_4\} = \{\mathrm{A}_2, \mathrm{A}_3\}$. The map $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4) \to g$ provides the isomorphism. □

Conjecture 1.10 *The mirror to a split semisimple algebraic group* G^L *over* $\mathbb{Q}$ *is the pair*

$$(\mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A}), \mathcal{W}_{2,4}). \tag{53}$$

Example. Let $\mathrm{G}^L = PGL_2$, so $\mathrm{G} = SL_2$. Then $\mathcal{A} = \mathbb{A}^2 - \{0\}$, $\mathcal{B} = \mathbb{P}^1$, and

$$\mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A}) = \{(L_1, v_2, L_3, v_4)\}/SL_2. \tag{54}$$

Here L_1, L_3 are one dimensional subspaces in a two dimensional vector space V_2, and v_2, v_4 are non-zero vectors in V_2. The pairs (L_1, v_2), (v_2, L_3), (L_3, v_4), (v_4, L_1) are generic, i.e. the corresponding pairs of lines are distinct. Pick non-zero vectors $l_1 \in L_1$ and $l_3 \in L_3$. Then

$$\mathcal{W}_{2,4} = \frac{\Delta(l_1, l_3)}{\Delta(l_1, v_2)\Delta(v_2, l_3)} + \frac{\Delta(l_1, l_3)}{\Delta(l_3, v_4)\Delta(l_1, v_4)}.$$

It is a regular function on (54), independent of the choice of vectors l_1, l_3. To calculate it, set

$$l_1 = (1, 0), \quad v_2 = (x, 1/p), \quad l_3 = (1, y/p), \quad v_4 = (0, 1). \tag{55}$$

Then

$$\mathrm{Conf}^\times(\mathcal{B}_L, \mathcal{A}_L, \mathcal{B}_L, \mathcal{A}_L) = \{(x, y, p) \in \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{G}_m - (xy - 1 = 0)\}.$$

$$\mathcal{W}_{2,4} = \frac{y/p}{1/p \cdot (xy/p - 1/p)} + \frac{y/p}{1 \cdot 1} = \frac{yp}{xy - 1} + \frac{y}{p}. \tag{56}$$

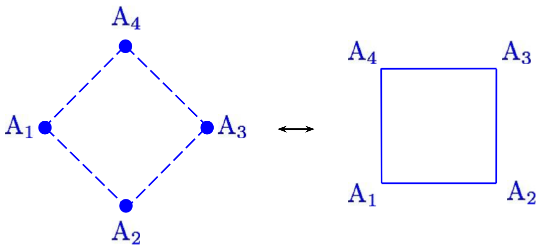


Fig. 10 Duality between configurations of decorated flags for G and G^L

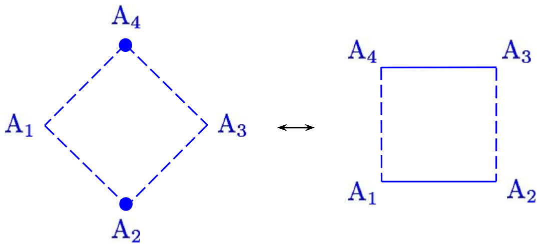


Fig. 11 Dual pair of spaces obtained on Step 2

The case $G = PGL_2$, $G^L = SL_2$ is similar, except that now $\mathcal{A}_{PGL_2} = \mathrm{A}^2 - \{0\}/\pm 1$.

Let us explain how this conjecture can be deduced from our general conjecture.

Step 1. Conjecture 1.6 tells us mirror duality, illustrated on Fig. 10:

$$(\mathrm{Conf}_4^\times(\mathcal{A}), \mathcal{W}_{1,2,3,4}) \leftrightarrow \mathrm{Conf}_4(\mathcal{A}_L).$$

Step 2. We alter the pair $(\mathrm{Conf}_4^\times(\mathcal{A}), \mathcal{W}_{1,2,3,4})$ by removing the potentials at the vertices A_1 and A_3. This reduces the potential $\mathcal{W}_{1,2,3,4}$ to a new potential:

$$\mathcal{W}_{2,4}(A_1, A_2, A_3, A_4) := \chi_{A_2}(B_1, A_2, B_3) + \chi_{A_4}(B_3, A_4, B_1).$$

In the dual picture this amounts to removing two divisors from $\mathrm{Conf}_4(\mathcal{A}_L)$, illustrated by two punctured edges on the right of Fig. 11, dual to the vertices A_1 and A_3 on the left. Precisely, we introduce a subspace $\widetilde{\mathrm{Conf}}_4(\mathcal{A}_L)$ such that the pairs of decorated flags at punctured sides are generic. The obtained dual pair is illustrated on Fig. 11. In particular there is a projection provided by the two punctured sides:

$$\widetilde{\mathrm{Conf}}_4(\mathcal{A}_L) \longrightarrow \mathrm{H}_L^2. \tag{57}$$

Step 3. There is an action of the group $\mathrm{H} \times \mathrm{H}$ on $\mathrm{Conf}_4^\times(\mathcal{A})$ preserves the potential $\mathcal{W}_{2,4}$, given by $(A_1, A_2, A_3, A_4) \longrightarrow (h_1 \cdot A_1, A_2, h_1 \cdot A_3, A_4)$. The quotient is the space (53):

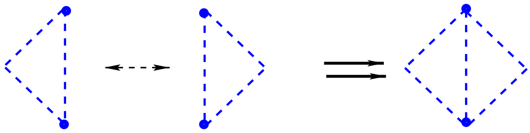


Fig. 12 The (tropicalised) Landau–Ginzburg model dual to G^L is obtained by gluing the two LG models dual to $\mathcal{A}_L$ along their "*vertical sides*", as shown on the *left*

$$(\mathrm{Conf}_4^{\times}(\mathcal{A}), \mathcal{W}_{2,4})/(\mathrm{H} \times \mathrm{H}) = (\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A}), \mathcal{W}_{2,4}).$$

Step 4. The action of the group $\mathrm{H} \times \mathrm{H}$ is dual to the projection (57). The quotient by the $\mathrm{H} \times \mathrm{H}$-action is dual to the fiber over $e \in \mathrm{H}_L \times \mathrm{H}_L$. The fiber is just the space (52). On the level of pictures, this is how we go from Fig. 11 to Fig. 9. This way we arrived at Conjecture 1.10.

Canonical basis motivation. Let us explain how the positive integral tropical points of the space from Conjecture 1.10 parametrise a canonical basis in $\mathcal{O}(\mathrm{G}^L)$. One has $\mathcal{O}(\mathrm{G}^L) = \bigoplus_{\lambda \in \mathrm{P}^+} V_\lambda \otimes V_\lambda^*$.

Recall that $\mathcal{O}(\mathcal{A}_L) = \bigoplus_{\lambda \in \mathrm{P}^+} V_\lambda$. The decomposition of $\mathcal{O}(\mathcal{A}_L)$ into irreducible G^L-modules is provided by the H_L-action on $\mathcal{A}_L$. According to our general picture,

$$\mathcal{A}_L = \mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L, \mathcal{A}_L) \quad \text{is mirror dual to} \quad (\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}, \mathcal{A}), \mathcal{W}_{2,3}).$$

The canonical basis in V_λ is parametrised by the fiber of the projection $\mathrm{Conf}^+(\mathcal{B}, \mathcal{A}, \mathcal{A})(\mathbb{Z}^t) \longrightarrow P^+$ over the $\lambda \in P^+$. This projection is the tropicalisation of the positive rational map $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{A}) \longrightarrow \mathrm{Conf}(\mathcal{A}, \mathcal{A})$. Therefore the tensor product of canonical basis in $V_\lambda \otimes V_\lambda^*$ is parametrised by the fiber over λ of the tropicalisation of the positive rational map $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A}) \longrightarrow \mathrm{Conf}(\mathcal{A}, \mathcal{A})$ (Fig. 12).

Lemma 1.11 *The space* $\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A})$ *is isomorphic to the open double Bruhat cell of* G.

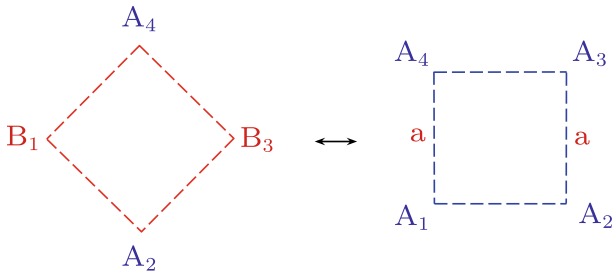
Proof Note that $\mathrm{Conf}^{\times}(\mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{A})$ is isomorphic to the moduli space parametrizing the configurations $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4) \in \mathrm{Conf}_4(\mathcal{A})$ such that $\alpha(\mathrm{A}_1, \mathrm{A}_2) = \alpha(\mathrm{A}_4, \mathrm{A}_3) = e$ and each consecutive pair $(\mathrm{A}_i, \mathrm{A}_{i+1})$ is generic. There is a unique element $g \in \mathrm{G}$ such that $\{g \cdot \mathrm{A}_1, g \cdot \mathrm{A}_2\} = \{\mathrm{A}_4, \mathrm{A}_3\}$. Let $\pi(\mathrm{A}_1) = \mathrm{B}$ and $\pi(\mathrm{A}_2) = \mathrm{B}^-$. Then we have

$$\{\mathrm{A}_1, \mathrm{A}_4\} = \{\mathrm{A}_1, g \cdot \mathrm{A}_1\} \text{ is generic} \quad \Longleftrightarrow \quad g \in \mathrm{B} w_0 \mathrm{B},$$

$$\{\mathrm{A}_2, \mathrm{A}_3\} = \{\mathrm{A}_2, g \cdot \mathrm{A}_2\} \text{ is generic} \quad \Longleftrightarrow \quad g \in \mathrm{B}^- w_0 \mathrm{B}^-.$$

So the space is isomorphic to the open double Bruhat cell $\mathrm{B}w_0\mathrm{B} \bigcap \mathrm{B}^- w_0 \mathrm{B}^-$. □

Conjecture 1.12 *The open double Bruhat cell of* G *is mirror to the open double Bruhat cell of* G^L.



1.4.5 Examples of homological mirror symmetry for stacks

As soon as our space $\mathcal{M}$ is fibered over a split torus $\mathbb{H}$, the mirror dual space $\mathcal{M}_L$ acquires an action of the dual torus $\mathbb{T}_L$. Thus we want to find the mirror of the stack $\mathcal{M}_L/\mathbb{T}_L$. Let us discuss two examples corresponding to the examples in Sects. 1.4.2 and 1.4.3.

Let us look first at the dual pair (47). The subgroup $1 \times \mathrm{H}_L^n$ acts freely on the last n decorated flags in $\mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L^{n+1})$, and the quotient is $\mathcal{B}_L^n$. So one has

$$\mathrm{Conf}_{w_0}(\mathcal{B}_L, \mathcal{A}_L^{n+1})/(\mathrm{H}_L \times \mathrm{H}_L^n) = \mathrm{H}_L \backslash \mathcal{B}_L^n. \tag{58}$$

We start with the problem reflecting the A-model to this stack.

1. Equivariant quantum cohomology of products of flag varieties. There is a way to understand mirror symmetry as an isomorphism of two modules over the algebra of $\hbar$-differential operators $\mathcal{D}_\hbar$: one provided by the quantum cohomology connection, and the other by the integral for the mirror dual Landau–Ginzburg model:

The quantum cohomology $\mathcal{D}_\hbar$-module of a projective (Fano) variety $\mathcal{M} =$

The $\mathcal{D}_\hbar$-module for the Landau–Ginzburg mirror $(\pi : \mathcal{M}^\vee \to \mathbb{H}, \mathcal{W}, \Omega)$, defined by $\int e^{-\mathcal{W}/\hbar}\Omega$.

Here the space $\mathcal{M}^\vee$ is fibered over a torus $\mathbb{H}$, the Ω is a volume form on $\mathcal{M}^\vee$, and $\mathcal{W}$ is a function on $\mathcal{M}^\vee$, called the Landau–Ginzburg potential. The form Ω and the canonical volume form on the torus $\mathbb{H}$ define a volume form $\Omega^{(a)}$ on the fiber of the map π over an $a \in \mathbb{H}$. The integrals $\int e^{-\mathcal{W}/\hbar}\Omega^{(a)}$ over cycles in the fibers are solutions of the $\mathcal{D}_\hbar$-module $\pi_*(e^{-\mathcal{W}/\hbar}\Omega)$ on $\mathbb{H}$.

This approach to mirror symmetry was originated by Givental [35], see also Witten [76] and [16], and developed further in [43] and many other works. See [4–6] for a discussion of examples of mirrors for the complements to anticanonical divisors on Fano varieties.

In our situation $\mathcal{M}$ is a positive space and $\mathcal{W}$ is a positive function, so there is an integral

$$\mathcal{F}_{\mathcal{M}}(a;\hbar) := \int_{\gamma^+(a)} e^{-\mathcal{W}/\hbar}\Omega^{(a)}, \quad \gamma^+(a) := \pi^{-1}(a) \cap \mathcal{M}(\mathbb{R}_{>0}). \tag{59}$$

If it converges, it defines a function on $\mathbb{H}(\mathbb{R}_{>0})$. This function as well as its partial Mellin transforms is a very important object to study. It plays a key role in the story. Below we elaborate some examples related to representation theory.

Let ψ_s be the character of $\mathrm{H}(\mathbb{R}_{>0})$ corresponding to an element $s \in \mathrm{H}_L(\mathbb{R}_{>0})$. Recall the projection $\mu : \mathrm{Conf}^\times(\mathcal{A}^{n+1}, \mathcal{B}) \to \mathrm{H}$ from (46). Consider the integral

$$\mathcal{F}_{\mathrm{Conf}^\times(\mathcal{A}^{n+1},\mathcal{B})}(a, s;\hbar) := \int_{\gamma^+(a)} \mu^*(\psi_s) e^{-\mathcal{W}/\hbar}\Omega^{(a)}, \quad (a, s) \in (\mathrm{H}^n \times \mathrm{H}_L)(\mathbb{R}_{>0}). \tag{60}$$

It is the Mellin transform of the function (59) along the torus $1 \times \mathrm{H} \subset \mathrm{H}^{n+1}$. If $n = 1$, one can identify integral (60) with an integral presentation for the Whittaker–Bessel function of the principal series representation of $\mathrm{G}(\mathbb{R})$ corresponding to the character ψ_s. The latter solves the quantum Toda lattice integrable system [53].

Therefore it provides, generalising Givental's work [37] for $\mathrm{G} = GL_m$ in non-equivariant setting, the integral presentation of the special solution of equivariant quantum cohomology $\mathcal{D}_\hbar$-module for the flag variety $\mathcal{B}_L$ studied in [30–33,36,38,56,58,64,69,70].

Recall the special cluster coordinate system on $\mathrm{Conf}_3(\mathcal{A})$ for $\mathrm{G} = \mathrm{GL}_m$ from [17]. It has a slight modification providing a rational coordinate system on $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathcal{A}, \mathcal{A})$, see Sect. 3.

Theorem 1.13 (i) *Let* $\mathrm{G} = GL_m$. *Then the potential* $\mathcal{W}$, *expressed in the special cluster coordinate system on* $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathcal{A}, \mathcal{A})$, *is precisely Givental's potential from* [37].
The value of the integral $\mathcal{F}_{\mathrm{Conf}^\times(\mathcal{B},\mathcal{A},\mathcal{A})}(a; s, \hbar)$ *at* $s = e$ *coincides with Givental's integral for a solution of the quantum cohomology* $\mathcal{D}_\hbar$*-module* $\mathrm{QH}^*(\mathcal{B}_L)$ [37].

(ii) *For any group* G*, the integral* $\mathcal{F}_{\mathrm{Conf}^{\times}(\mathcal{B},\mathcal{A},\mathcal{A})}(a;s)$ *is a solution of the* $\mathcal{D}_{\hbar}$*-module* $\mathrm{QH}^*_{\mathrm{H_L}}(\mathcal{B}_L)$.

Proof (i) It is proved in Sect. 3.2.

(ii) Since integral (60) provides an integral presentation for the Whittaker function, it is equivalent to the results of [31,69]. Observe that the parameter $a \in \mathrm{H}(\mathbb{C})$ is interpreted as the parameter on $H^2(\mathcal{B}_L, \mathbb{C}^*)$, which is the base of the small quantum cohomology connection, while the parameter $s \in \mathrm{H}_L(\mathbb{R}_{>0})$ is the parameter of the $\mathrm{H_L}$-equivariant cohomology. □

For arbitrary n, integral (60) determines the equivariant quantum cohomology $\mathcal{D}_{\hbar}$-module of $\mathcal{B}^n_L$. The latter lives on $\mathrm{H}^n \times \mathrm{H}_L$, it is a $\mathcal{D}_{\hbar}$-module on H^n, but only $\mathcal{O}$-module along H_L. Integral (60) is a solution of this $\mathcal{D}_{\hbar}$-module.

2. Mirror of equivariant B-model on $\mathcal{B}^n_L$. The integral (60) admits an analytic continuation in s provided by the analytic continuation of the character ψ_s in the integrand. The complex integrand lives on an analytic space defined as follows. Let $\widetilde{\mathrm{H}}(\mathbb{C})$ is the universal cover of $\mathrm{H}(\mathbb{C})$. Denote by $(\mathcal{B} \times \ldots \times \mathcal{B})^{*,a}_n$ the fiber of the map λ in (46) over an $a \in \mathrm{H}^n$. It is a Zariski open subset of $\mathcal{B}^n$. Consider the fibered product

$$
\begin{array}{ccc}
(\mathcal{B} \times \widetilde{\ldots \times} \mathcal{B})^{*,a}_n(\mathbb{C}) & \stackrel{\widetilde{\exp}}{\longrightarrow} & (\mathcal{B} \times \ldots \times \mathcal{B})^{*,a}_n(\mathbb{C}) \\
\widetilde{\mu} \downarrow & & \mu \downarrow \\
\widetilde{\mathrm{H}}(\mathbb{C}) & \stackrel{\exp}{\longrightarrow} & \mathrm{H}(\mathbb{C})
\end{array}
$$

Let $\widetilde{\mathcal{W}}$ and $\widetilde{\Omega}$ be the lifts of $\mathcal{W}$ and Ω by the map $\widetilde{\exp}$. We get a locally constant family of categories $\mathcal{FS}_{\mathrm{wr}}((\mathcal{B} \times \widetilde{\ldots \times} \mathcal{B})^{*,a}_n(\mathbb{C}), \widetilde{\mathcal{W}}, \widetilde{\Omega})$ over $\mathrm{H}^n(\mathbb{C})$. So the fundamental group $\pi_1(\mathrm{H}^n(\mathbb{C}))$ acts on the category for any given a. The group $\pi_1(\mathrm{H}(\mathbb{C}))$ also acts on it by the deck transformations induced from the universal cover $\widetilde{\mathrm{H}}(\mathbb{C}) \longrightarrow \mathrm{H}(\mathbb{C})$.

On the other hand, the Picard group of the stack $\mathrm{H}_L \backslash \mathcal{B}^n_L$,

$$
\mathrm{Pic}(\mathrm{H}_L \backslash \mathcal{B}^n_L) = X^*(\mathrm{H}_L) \times \mathrm{Pic}(\mathcal{B}^n_L) = X^*(\mathrm{H}_L) \times X^*(\mathrm{H}^n_L)
$$

acts by autoequivalences of the category $D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathcal{B}^n_L)$.

Conjecture 1.14 *There is an equivalence of* A_∞*-categories*

$$
\mathcal{FS}_{\mathrm{wr}}((\mathcal{B} \times \widetilde{\ldots \times} \mathcal{B})^{*,a}_n(\mathbb{C}), \widetilde{\mathcal{W}}, \widetilde{\Omega}) \quad \sim \quad D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathcal{B}^n_L). \tag{61}
$$

It intertwines the deck transformation action of $\pi_1(\mathrm{H}(\mathbb{C})) \times$ *the monodromy action of* $\pi_1(\mathrm{H}^n(\mathbb{C}))$ *on the Fukaya–Seidel category with the action of* $X^*(\mathrm{H}_L) \times \mathrm{Pic}(\mathcal{B}^n_L)$ *on the category* $D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathcal{B}^n_L)$.

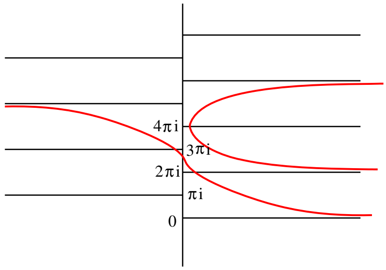


Fig. 13 *Horizontal rays* are the rays of fast decay of the potential. Together with the *vertical line*, they form the Lagrangian skeleton of the Kontsevich model of $\mathcal{FS}_{\mathrm{wr}}(\mathbb{C}; a^{-1}(e^t + e^{-t}), dt)$

The integral (60) over Lagrangian submanifolds supporting objects of the Fukaya–Seidel category is a central charge for a stability condition on the category.

Kontsevich argued [51] that there is a smaller class of stability conditions, which he called "physical stability conditions". Stability conditions above should be from that class.

Examples.

1. Let $n = 1$. Then $\mathcal{B}_1^{*,a}$ is the intersection $\mathcal{B}^*$ of two big Bruhat cells in the flag variety $\mathcal{B}$. It parametrising flags in generic position to two generic flags, say (B^+, B^-).
2. Let $\mathrm{G} = SL_2$, $n = 1$. Then $\mathcal{B}_1^{*,a} = \mathbb{C}^*$ with the coordinate u, $\widetilde{\mathcal{B}}_1^{*,a} = \mathbb{C}$ with the coordinate t, $u = e^t$, and $\widetilde{\mathcal{W}} = a^{-1}(e^t + e^{-t})$ where $a \in \mathbb{C}^*$ is a parameter. Next, $\mathcal{B}_L = \mathbb{CP}^1$, with the natural $\mathbb{C}^*$-action preserving $0, \infty$. Conjecture 1.14 predicts an equivalence

$$\mathcal{FS}_{\mathrm{wr}}(\mathbb{C}; a^{-1}(e^t + e^{-t}), dt) \quad \sim \quad D^b\mathrm{Coh}_{\mathbb{C}^*}(\mathbb{CP}^1), \quad a \in \mathbb{C}^*.$$

The equivalence is a trivial exercise for the experts. It can be checked by using the Kontsevich combinatorial model [1,15,49,73] for the Fukaya–Seidel category as a category of locally constant sheaves on the Lagrangian skeleton for a surface with potential in the case of $(\mathbb{C}, e^t + e^{-t})$, shown on Fig. 13.

Varying the parameter $a \in \mathbb{C}^*$ in the potential we get a locally constant family of the Fukaya–Seidel categories. Its monodromy is an autoequivalence corresponding to the action of a generator of the group $\mathrm{Pic}(\mathbb{P}^1)$. The translation $t \longmapsto t + 2\pi i$ is another autoequivalence corresponding to the action of a generator of the character group $X^*(\mathbb{C}^*) = \mathbb{Z}$ on $D^b\mathrm{Coh}_{\mathbb{C}^*}(\mathbb{CP}^1)$.

Let us consider now the oscillatory integral

$$\int_L \exp\left(\frac{1}{\hbar}(-a^{-1}(e^t + e^{-t}) - st)\right) dt = \int_{\exp(L)} e^{-a^{-1}(u+u^{-1})/\hbar} u^{s/\hbar} \frac{du}{u}.$$

Here L is a path which goes to infinity along the line of fast decay of the integrand. This is an integral for the Bessel function. It defines a family of stability conditions on the Fukaya–Seidel category depending on $s \in \mathbb{C}$—it is the value of the central charge on the K_0-class of the object supported on L. The parameter s reflects the equivariant parameter for the $\mathbb{C}^*$-action.

3. Mirror of equivariant B-model on $\mathcal{B}_L^n \times \mathrm{U}_L$. There is an integral very similar to (60):

$$\mathcal{F}_{\mathrm{Conf}^\times(\mathcal{B},\mathcal{A}^n,\mathcal{B})}(a,s) := \int_{\gamma^+(a)} \mu^*(\psi_s) e^{-\mathcal{W}/\hbar}\Omega^{(a)}, \quad (a,s) \in (\mathrm{H}^{n-1} \times \mathrm{H}_L)(\mathbb{R}_{>0}). \tag{62}$$

Denote by $\lambda_{(51)}$ the map λ onto H_L^{n-1} from (51). The integrand has an analytic continuation in s which lives on the fibered product

$$\begin{array}{ccc} \widetilde{\lambda_{(51)}^{-1}(a)(\mathbb{C})} & \xrightarrow{\widetilde{\exp}} & \lambda_{(51)}^{-1}(a)(\mathbb{C}) \\ \widetilde{\mu} \downarrow & & \mu \downarrow \\ \widetilde{\mathrm{H}}(\mathbb{C}) & \xrightarrow{\exp} & \mathrm{H}(\mathbb{C}) \end{array}$$

There is a conjecture similar to Conjecture 1.14 describing the category $D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathcal{B}_L^{n-1} \times \mathrm{U}_L)$. For example, when $n = 1$ it reads as follows.

Conjecture 1.15 *There is an equivalence of* A_∞*-categories*

$$\mathcal{FS}_{\mathrm{wr}}(\widetilde{\mathrm{U}^*}(\mathbb{C}), \widetilde{\mathcal{W}}, \widetilde{\Omega}) \quad \sim \quad D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathrm{U}_L). \tag{63}$$

It intertwines the deck transformation action of $\pi_1(\mathrm{H}(\mathbb{C}))$ *on the Fukaya–Seidel category with the action of* $X^*(\mathrm{H}_L)$ *on the category* $D^b\mathrm{Coh}_{\mathrm{H}_L}(\mathrm{U}_L)$.

Example. If $\mathrm{G} = SL_2$ and $n = 1$, then $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathbb{C}$ with the $\mathbb{C}^*$-action. On the dual side, $\mathrm{Conf}^\times(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathbb{C}^*$, $\pi = \mu$ is the identity map, $\mathcal{W} = u$, $\Omega = du/u$. The universal cover of $\mathbb{C}^*$ is $\mathbb{C}$ with the coordinate t such that $u = e^t$. The integral is

$$\mathcal{F}(s) = \int_0^\infty e^{-u}u^s du/u = \Gamma(s).$$

The equivalence of categories predicted by Conjecture 1.15 is

$$\mathcal{FS}_{\mathrm{wr}}(\mathbb{C}, e^t, dt) \quad \sim \quad D^b\mathrm{Coh}_{\mathbb{C}^*}(\mathbb{C}). \tag{64}$$

It can be checked by using the Kontsevich combinatorial model [50] for the Fukaya–Seidel category.

1.5 Concluding remarks

1. Mirror dual of the moduli spaces of G^L-local systems on S. The true analog of the moduli space of G^L-local systems for a decorated surface S is the moduli space $\mathrm{Loc}_{\mathrm{G^L},S}$. We view the function $\mathcal{W}$ on the space $\mathcal{A}_{\mathrm{G},S}$ as the Landau–Ginzburg potential on $\mathcal{A}_{\mathrm{G},S}$, and suggest

Conjecture 1.16

$$(\mathcal{A}^{\times}_{\mathrm{G},S}, \mathcal{W}, \Omega, \pi) \textit{ is mirror dual to } (\mathrm{Loc}_{\mathrm{G^L},S}, \Omega_L, r_L). \tag{65}$$

It would be interesting to compare this mirror duality conjecture with the mirror duality conjectures of Kapustin–Witten [48] and Gukov–Witten [42], which do not involve a potential, and refer to families of moduli spaces, which are somewhat different then the moduli spaces we consider. Mirror duality in the case when S is a closed surface without punctures was studied by Hausel and Thaddeus [74]. All these mirror dualities involve Langlands dual groups. However they depend crucially on a choice of a complex structure on S, while in our approach we do not use complex structure on S. See also Kontsevich and Soibelman [52].

Notice also that if each boundary component of S has at least one special point, then $\mathrm{Loc}_{\mathrm{G^L},S} = \mathcal{A}_{\mathrm{G^L},S}$, and so in this case we have a more symmetric picture:

$$(\mathcal{A}^{\times}_{\mathrm{G},S}, \mathcal{W}, \Omega, \pi) \text{ is mirror dual to } (\mathcal{A}_{\mathrm{G^L},S}, \Omega_L, r_L). \tag{66}$$

$$(\mathcal{A}^{\times}_{\mathrm{G},S}, \Omega, \pi) \text{ is mirror dual to } (\mathcal{A}^{\times}_{\mathrm{G^L},S}, \Omega_L, r_L). \tag{67}$$

2. Oscillatory integrals. The analog of integral (59) in the surface case is an integral

$$\mathcal{F}_{\mathrm{G},S}(a) := \int_{\gamma^+(a)/\Gamma_S} e^{-\mathcal{W}/\hbar}\Omega^{(a)}. \tag{68}$$

Since the integrand is Γ_S-invariant, the integration cycles are defined by intersecting the fibers with $\mathcal{A}_{\mathrm{G},S}(\mathbb{R}_{>0})/\Gamma_S$. Notice that $\mathcal{A}_{\mathrm{G},S}(\mathbb{R}_{>0})$ is the decorated Higher Teichmuller space [17]. If $\mathrm{G} = SL_2$, the integral converges. For other groups convergence is a problem.

Notice also that the three convergent oscillatory integrals

$$\mathcal{F}_{\mathrm{Conf}^{\times}_n(\mathcal{A},\mathcal{B},\mathcal{B})}(s), \quad \mathcal{F}_{\mathrm{Conf}^{\times}(\mathcal{A},\mathcal{A},\mathcal{B})}(a;s), \quad \mathcal{F}_{\mathrm{Conf}^{\times}_3(\mathcal{A})}(a_1,a_2,a_3),$$
$$a_i \in \mathrm{H}(\mathbb{R}_{>0}), \ s \in \mathrm{H}_L(\mathbb{R}_{>0})$$

are continuous analogs of the Kostant partition function, weight multiplicities and dimensions of triple tensor product invariants for the Langlands dual group $\mathrm{G}^L(\mathbb{R})$.

3. Relating our dualites to cluster duality conjectures [18]. The latter study dual pairs $(\mathcal{A}, \mathcal{X}^\vee)$, where $\mathcal{A}$ is a cluster $\mathcal{A}$-variety, and $\mathcal{X}^\vee$ is the Langlands dual cluster $\mathcal{X}$-variety:

$$\mathcal{A} \quad \text{is dual to} \quad \mathcal{X}^\vee.$$

There is a discrete group Γ acting by automorphisms of each of the spaces $\mathcal{A}$ and $\mathcal{X}^\vee$, called the *cluster modular group*. So it acts on the sets of tropical points $\mathcal{A}(\mathbb{Z}^t)$ and $\mathcal{X}^\vee(\mathbb{Z}^t)$. Cluster Duality Conjectures predict canonical Γ-equivariant pairings

$$\mathbf{I}_\mathcal{A} : \mathcal{A}(\mathbb{Z}^t) \times \mathcal{X}^\vee \longrightarrow \mathbb{A}^1, \quad \mathbf{I}_\mathcal{X} : \mathcal{A} \times \mathcal{X}^\vee(\mathbb{Z}^t) \longrightarrow \mathbb{A}^1. \tag{69}$$

As the work [41] shows, in general the functions assigned to the tropical points may exist only as formal universally Laurent power series rather then universally Laurent polynomials.

There are cluster volume forms $\Omega_\mathcal{A}$ and $\Omega_\mathcal{X}$ on the $\mathcal{A}$ and $\mathcal{X}$ spaces [21], see Sect. 11.

We suggest that, in a rather general situation, there is a natural Γ-invariant positive potential $\mathcal{W}_\mathcal{A}$ on the space $\mathcal{A}$, a similar potential $\mathcal{W}_\mathcal{X}$ on the space $\mathcal{X}$, and a certain "alterations" $\widehat{\mathcal{X}^\vee}$ and $\widehat{\mathcal{A}^\vee}$ of the spaces $\mathcal{X}^\vee$ and $\mathcal{A}^\vee$ providing mirror dualities underlying canonical pairings (69):

$$(\mathcal{A}, \mathcal{W}_\mathcal{A}, \Omega_\mathcal{A}, \pi_\mathcal{A}) \quad \text{is mirror dual to} \quad (\widehat{\mathcal{X}^\vee}, \Omega_{\mathcal{X}^\vee}, r_{\mathcal{X}^\vee}). \tag{70}$$

$$(\mathcal{X}, \mathcal{W}_\mathcal{X}, \Omega_\mathcal{X}, \pi_\mathcal{X}) \quad \text{is mirror dual to} \quad (\widehat{\mathcal{A}^\vee}, \Omega_{\mathcal{A}^\vee}, r_{\mathcal{A}^\vee}). \tag{71}$$

Canonical pairings (69) should induce canonical pairings related to the potentials and alterations:

$$\mathbf{I}_{(\mathcal{A},\mathcal{W}_\mathcal{A})} : \mathcal{A}^+_{\mathcal{W}_\mathcal{A}}(\mathbb{Z}^t) \times \widehat{\mathcal{X}^\vee} \longrightarrow \mathbb{A}^1, \quad \mathbf{I}_{(\mathcal{X},\mathcal{W}_\mathcal{X})} : \mathcal{X}^+_{\mathcal{W}_\mathcal{X}}(\mathbb{Z}^t) \times \widehat{\mathcal{A}^\vee} \longrightarrow \mathbb{A}^1.$$

This should provide a cluster generalisation of our examples. For instance, there is a split torus $\mathbb{H}_\mathcal{A}$ associated to a cluster variety $\mathcal{A}$, coming with a canonical basis of characters, given by the *frozen $\mathcal{A}$-coordinates*. They describe the projection $\pi_\mathcal{A} : \mathcal{A} \to \mathbb{H}_\mathcal{A}$, see Sect. 11.

An alteration $\widehat{\mathcal{A}}^\vee$, given by a partial compactification of the space $\mathcal{A}^\vee$, and a conjectural definition of the potential $\mathcal{W}_\mathcal{X}$ are given in Sect. 11.2.

4. Conclusion. A parametrisation of a canonical basis, casted as a canonical pairing **I**, should be understood as a manifestation of a mirror duality between a space with a Landau–Ginzburg potential and a similar space for the Langlands dual group.

Our main evidence is that canonical pairing (44) describing a parametrisation of canonical basis in tensor products of n irreducible G^L-modules is related via an integral presentation to the $\mathcal{D}_\hbar$-module describing the equivariant quantum cohomology of $(\mathcal{B}_L)^n$.

There is a remarkable mirror conjecture of Gross-Hacking-Keel [40], who start with a maximally degenerate log Calabi–Yau Y and conjecture that the Gromov–Witten theory of Y gives rise to a commutative ring $R(Y)$, with a basis. Its spectrum is an affine variety which is conjectured to be the mirror of Y.

Notice that in our conjectures we give an *a priori* description of the mirror dual pair of spaces, while in [40] the mirror space is encrypted in the conjecture. For example, mirror conjecture (34) is expected to be an example of the Gross-Hacking-Keel conjecture, but we do not know how to deduce, starting from the pair $(\mathrm{Conf}_n^\times(\mathcal{A}), \Omega)$, the former from the latter, and in particular why the Langlands dual group appears in the description of the mirror.

We want to stress that in our mirror conjectures we usually deal with mirror dual pairs where at least one is a Landau–Ginzburg model, i.e. is represented by a space with a potential. In particular canonical bases in representation theory and their generalisations related to moduli spaces of G-local systems on decorated surfaces S always require the dual space to be a space with a non-trivial potential, unless S is a closed surface without boundary.

Finally, in applications to representation theory we are forced to deal with stacks rather then varieties, as discussed in Sect. 1.4.5. This is a less explored chapter of the homological mirror symmetry. See also a recent paper of Teleman [75] in this direction.

The space $\mathcal{M}(\mathcal{K})$ of $\mathcal{K}$-points of a space $\mathcal{M}$ is a cousin of the loop space $\Omega\mathcal{M}(\mathbb{C})$. Heuristically, the quantum cohomology $\mathcal{D}_\hbar$-module is best seen in the (ill defined) S^1-equivariant Floer cohomology of the loop space $\Omega\mathcal{M}(\mathbb{C})$ [35], which are sort of "semi-infinite cohomology" of the loop space. It would be interesting to relate this to the infinite dimensional cycles $\mathcal{C}_l^\circ \subset \mathcal{M}^\circ(\mathcal{K})$.

It would be very intersecting to relate our approach to the construction of canonical bases via cycles $\mathcal{M}_l^\circ$ to the work in progress of Gross–Hacking–Keel–Kontsevich on construction of canonical bases on cluster varieties via scattering diagrams.

Organization of the paper. In Sect. 2 we present main definitions and results relevant to representation theory. We start from a detailed discussion of the geometry of the tensor product invariants in Sects. 2.1–2.2. We discuss more general examples in Sects. 2.3. In Sect. 2.4 we construct a canonical basis in tensor products of finite dimensional G^L-modules, and its parametrization. In Sects. 2 we give all definitions and complete descriptions of the results, but include a proof only if it is very simple. The only exception is a proof of Theorem 2.38 in Sect. 2.4. The rest of the proofs occupy the next sections. In Sect. 10 we discuss the general case related to a decorated surface. In the Sect. 11 we discuss the volume form and the potential in the cluster set-up.

2 Main definitions and results: the disc case

2.1 Configurations of decorated flags, the potential $\mathcal{W}$, and tensor product invariants

2.1.1 Positive spaces and their tropical points

Below we recall briefly the main definitions, following [18, Section 1].

Positive spaces. A positive rational function on a split algebraic torus T is a nonzero rational function on T which in a coordinate system, given by a set of characters of T, can be presented as a ratio of two polynomials with positive integral coefficients.

A *positive rational morphism* $\varphi : \mathrm{T}_1 \to \mathrm{T}_2$ of two split tori is a morphism such that for each character χ of T_2 the function $\chi \circ \varphi$ is a positive rational function.

A *positive atlas* on an irreducible space (i.e. variety/stack) $\mathcal{Y}$ over $\mathbb{Q}$ is given by a non-empty collection $\{\mathbf{c}\}$ of birational isomorphisms over $\mathbb{Q}$

$$\alpha_{\mathbf{c}} : \mathrm{T} \longrightarrow \mathcal{Y},$$

where T is a split algebraic torus, satisfying the following conditions:

- For any pair $\mathbf{c}, \mathbf{c}'$ the map $\varphi_{\mathbf{c},\mathbf{c}'} := \alpha_{\mathbf{c}}^{-1} \circ \alpha_{\mathbf{c}'}$ is a positive birational isomorphism of T.
- Each map $\alpha_{\mathbf{c}}$ is regular on a complement to a divisor given by positive rational function.

A *positive space* is a space with a positive atlas. A split algebraic torus T is the simplest example of a positive space. It has a single positive coordinate system, given by the torus itself.

A *positive rational function* F on $\mathcal{Y}$ is a rational function given by a subtraction free rational function in one, and hence in all coordinate systems of the positive atlas on $\mathcal{Y}$.

A *positive rational map* $\mathcal{Y} \to \mathcal{Z}$ is a rational map given by positive rational functions in one, and hence in all positive coordinate systems.

Tropical points. The tropical semifield $\mathbb{Z}^t$ is the set $\mathbb{Z}$ equipped with tropical addition and multiplication given by

$$a +_t b = \min\{a, b\}, \quad a \cdot_t b = a + b, \quad a, b \in \mathbb{Z}.$$

This definition can be motivated as follows. Consider the semifield $\mathbb{R}_+((t))$ of Laurent series $f(t)$ with *positive* leading coefficients: there is no "−" operation in $\mathbb{R}_+((t))$. Then the valuation map $f(t) \mapsto \mathrm{val}(f)$ is a homomorphism of semifields $\mathrm{val} : \mathbb{R}_+((t)) \to \mathbb{Z}^t$.

Denote by $X_*(\mathrm{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathrm{T})$ and $X^*(\mathrm{T}) = \mathrm{Hom}(\mathrm{T}, \mathbb{G}_m)$ the lattices of cocharacters and characters of a split algebraic torus T. There is a pairing $\langle *, * \rangle : X^*(\mathrm{T}) \times X_*(\mathrm{T}) \to \mathbb{Z}$.

The set of $\mathbb{Z}^t$-points of a split torus T is defined to be its lattice of cocharacters:

$$\mathrm{T}(\mathbb{Z}^t) := X_*(\mathrm{T}).$$

A positive rational function F on T gives rise to its tropicalization F^t, which is a $\mathbb{Z}$-valued function on the set $\mathrm{T}(\mathbb{Z}^t)$. Its definition is clear from the following example:

$$F = \frac{x_1x_2^2 + 3x_2x_3^5}{x_2x_4}, \quad F^t = \min\{x_1 + 2x_2, x_2 + 5x_3\} - \min\{x_2 + x_4\}.$$

Similarly, a positive morphism $\varphi : \mathrm{T} \to \mathrm{S}$ of two split tori gives rise to a piecewise linear morphism $\varphi^t : \mathrm{T}(\mathbb{Z}^t) \to \mathrm{S}(\mathbb{Z}^t)$.

There is a unique way to assign to a positive space $\mathcal{Y}$ a set $\mathcal{Y}(\mathbb{Z}^t)$ of its $\mathbb{Z}^t$-points such that

- Each of the coordinate systems $\mathbf{c}$ provides a canonical isomorphism

$$\alpha_{\mathbf{c}}^t : \mathrm{T}(\mathbb{Z}^t) \stackrel{\sim}{\longrightarrow} \mathcal{Y}(\mathbb{Z}^t).$$

- These isomorphisms are related by piecewise-linear isomorphisms $\varphi_{\mathbf{c},\mathbf{c}'}^t$:

$$\alpha_{\mathbf{c}'}^t(l) = \alpha_{\mathbf{c}}^t \circ \varphi_{\mathbf{c},\mathbf{c}'}^t(l).$$

We raise the above process to the category of positive spaces. It gives us a functor called *tropicalization* from the category of positive spaces to the category of sets of tropical points. For each positive morphism $f : \mathcal{Y} \to \mathcal{Z}$, denote by $f^t : \mathcal{Y}(\mathbb{Z}^t) \to \mathcal{Z}(\mathbb{Z}^t)$ its corresponding tropicalized morphism.

Pick a basis of cocharacters of T. Then, assigning to each positive coordinate system $\mathbf{c}$ a set of integers $(l_1^{\mathbf{c}}, \ldots, l_n^{\mathbf{c}}) \in \mathbb{Z}^n$ related by piecewise-linear isomorphisms $\varphi_{\mathbf{c},\mathbf{c}'}^t$, we get an element

$$l = \alpha_{\mathbf{c}}^t(l_1^{\mathbf{c}}, \ldots, l_n^{\mathbf{c}}) \in \mathcal{Y}(\mathbb{Z}^t).$$

For a variety $\mathcal{Y}$ with a positive atlas, the set $\mathcal{Y}(\mathbb{Z}^t)$ can be interpreted as the set of *transcendental cells* of the infinite dimensional variety $\mathcal{Y}(\mathbb{C}((t)))$, as we will explain in Sect. 2.2.1.

The set of positive tropical points. Let $(\mathcal{Y}, \mathcal{W})$ be a pair given by a positive space $\mathcal{Y}$ equipped with a positive rational function $\mathcal{W}$. Let us tropicalize this function, getting a map

$$\mathcal{W}^t : \mathcal{Y}(\mathbb{Z}^t) \longrightarrow \mathbb{Z}.$$

We define the set of *positive tropical points*:

$$\mathcal{Y}^+_{\mathcal{W}}(\mathbb{Z}^t) := \{l \in \mathcal{Y}(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}.$$

Example. The Cartan group H of G is a split torus and hence has a standard positive structure. The set $\mathrm{H}(\mathbb{Z}^t) = X_*(\mathrm{H})$ is the coweight lattice of G. Let $\{\alpha_i\}$ the set of simple positive roots indexed by I. We define

$$\mathcal{W} : \mathrm{H} \longrightarrow \mathbb{A}^1, \quad h \longmapsto \sum_{i \in I} \alpha_i(h). \tag{72}$$

The set of positive tropical points is the positive Weyl chamber in $X_*(\mathrm{H})$:

$$\mathrm{H}^+(\mathbb{Z}^t) := \mathrm{H}^+_{\mathcal{W}}(\mathbb{Z}^t) = \{\lambda \in X_*(\mathrm{H}) \mid \langle \lambda, \alpha_i \rangle \geq 0, \ \forall i \in I\}.$$

2.1.2 *Basic notations for a split reductive group* G

Denote by H the Cartan group of G, and by H^L the Cartan group of the Langlands dual group G^L. There is a canonical isomorphism $X^*(\mathrm{H}^L) = X_*(\mathrm{H})$. Denote by $\Delta^+ \subset X^*(\mathrm{H})$ the set of positive roots for G, and by $\Pi := \{\alpha_i\} \subset \Delta^+$ the subset of simple positive roots, indexed by a finite set I. We sometimes use P instead of $X_*(\mathrm{H})$. Denote by P^+ the positive Weyl chamber in P. It is also the cone of dominant weights for the dual group G^L. Denote by V_λ the irreducible finite dimensional G^L-modules parametrized by $\lambda \in \mathrm{P}^+$.

Let $\mathrm{U}_i^{\pm}$ $(i \in I)$ be the simple root subgroup of $\mathrm{U}^{\pm}$. Let $\alpha_i^\vee : \mathbb{G}_m \to \mathrm{H}$ be the simple coroot corresponding to the root $\alpha_i : \mathrm{H} \to \mathbb{G}_m$. For all $i \in I$, there are isomorphisms $x_i : \mathbb{G}_a \to \mathrm{U}_i^+$ and $y_i : \mathbb{G}_a \to \mathrm{U}_i^-$ such that the maps

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \longmapsto x_i(a), \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \longmapsto y_i(b), \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto \alpha_i^\vee(t) \tag{73}$$

provide homomorphisms $\phi_i : \mathrm{SL}_2 \to \mathrm{G}$.

Let s_i $(i \in I)$ be the simple reflections generating the Weyl group. Set $\overline{s}_i := y_i(1)x_i(-1)y_i(1)$. The elements $\overline{s}_i$ satisfy the braid relations. So we can associate to each $w \in W$ its representative $\overline{w}$ in such a way that for any reduced decomposition $w = s_{i_1} \dots s_{i_k}$ one has $\overline{w} = \overline{s}_{i_1} \dots \overline{s}_{i_k}$.

Denote by w_0 be the longest element of the Weyl group. Set $s_{\mathrm{G}} := \overline{w}_0^2$. It is an order two central element in G. For $\mathrm{G} = \mathrm{SL}_2$ it is the element $-\mathrm{Id}$. For an arbitrary reductive G the element s_{G} is the image of the element s_{SL_2} under a principal embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{G}$. For example, $s_{\mathrm{SL}_m} = (-1)^{m-1}\mathrm{Id}$. See [17, Section 2.3] for proof.

2.1.3 Lusztig's positive atlas of U *and the character* χ_{A}

Let $w_0 = s_{i_1} \dots s_{i_m}$ be a reduced decomposition. It is encoded by the sequence $\mathbf{i} = (i_1, i_2, \dots, i_m)$. It provides a regular map

$$\phi_{\mathbf{i}} : (\mathbb{G}_m)^m \longrightarrow \mathrm{U}, \quad (a_1, \dots, a_m) \longmapsto x_{i_1}(a_1) \dots x_{i_m}(a_m). \tag{74}$$

The map $\phi_{\mathbf{i}}$ is an open embedding [58], and a birational isomorphism. Thus it provides a rational coordinate system on U. It was shown in *loc.cit.* that the collection of these rational coordinate systems form a positive atlas of U, which we call *Lusztig's positive atlas*. There is a similar positive atlas on U^- provided by the maps y_i.

The choice of the maps x_i, y_i in (73) provides the standard character:

$$\chi : \mathrm{U} \longrightarrow \mathbb{A}^1, \quad x_{i_1}(a_1) \dots x_{i_m}(a_m) \longmapsto \sum_{j=1}^{m} a_j. \tag{75}$$

It is evidently a positive function in Lusztig's positive atlas. Moreover it is independent of the sequence $\mathbf{i}$ chosen. Similarly, there is a character $\chi^- : \mathrm{U}^- \to \mathbb{A}^1$, $y_{i_1}(b_1) \dots y_{i_m}(b_m) \mapsto \sum_{j=1}^m b_j$, which is positive in the positive atlas on U^-.

Let $\mathrm{A} := g \cdot \mathrm{U}$ be a decorated flag. Its stabilizer is $\mathrm{U}_{\mathrm{A}} = g\mathrm{U}g^{-1}$. The associated character is

$$\chi_{\mathrm{A}} : \mathrm{U}_{\mathrm{A}} \longrightarrow \mathbb{A}^1, \quad u \longmapsto \chi(g^{-1}ug).$$

For example, for an $h \in \mathrm{H}$, the character $\chi_{h \cdot \mathrm{U}}$ is given by $x_{i_1}(a_1) \dots x_{i_m}(a_m) \longmapsto \sum_{j=1}^m a_j/\alpha_{i_j}(h)$.

2.1.4 The potential $\mathcal{W}$ on the moduli space $\mathrm{Conf}_n(\mathcal{A})$.

Given a group G and G-sets $X_1, \dots, X_n$, orbits of the diagonal G-action on $X_1 \times \dots \times X_n$ are called *configurations*. Denote by $\{x_1, \dots, x_n\}$ a collection of points, and by $(x_1, \dots, x_n)$ its configuration.

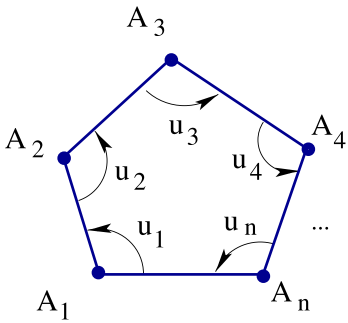


Fig. 14 The potential is a sum of the contribution at the vertices

We usually denote a decorated flag by A_i and the corresponding flag $\pi(\mathrm{A}_i)$ by B_i. Denote the set $\{1, \ldots, n\}$ of consecutive integers by $[1, n]$.

Definition 2.1 A pair $\{\mathrm{B}_1, \mathrm{B}_2\} \in \mathcal{B} \times \mathcal{B}$ of Borel subgroups is *generic* if $\mathrm{B}_1 \cap \mathrm{B}_2$ is a Cartan subgroup in G. A collection $\{\mathrm{A}_1, \ldots, \mathrm{B}_{m+n}\} \in \mathcal{A}^n \times \mathcal{B}^m$ is *generic* if for any distinct $i, j \in [1, m+n]$, the pair $\{\mathrm{B}_i, \mathrm{B}_j\}$ is generic.

Set $\mathrm{Conf}(\mathcal{A}^n, \mathcal{B}^m) := \mathrm{G}\backslash(\mathcal{A}^n \times \mathcal{B}^m)$. Note that if $\{\mathrm{A}_1, \ldots, \mathrm{B}_{m+n}\}$ is generic, then so is $g \cdot \{\mathrm{A}_1, \ldots, \mathrm{B}_{m+n}\}$ for any $g \in \mathrm{G}$. Denote by $\mathrm{Conf}^*(\mathcal{A}^n, \mathcal{B}^m)$ the subset of generic configurations.

Definition 2.2 A frame for a split reductive algebraic group G over $\mathbb{Q}$ is a generic pair $\{\mathrm{A}, \mathrm{B}\} \in \mathcal{A} \times \mathcal{B}$. Denote by $\mathcal{F}_{\mathrm{G}}$ the moduli space of frames.

The space $\mathcal{F}_{\mathrm{G}}$ is a left G-torsor. If $\mathrm{G} = \mathrm{SL}_m$, then a K-point of $\mathcal{F}_{\mathrm{G}}$ is the same thing as a unimodular frame in a vector space over K of dimension m with a volume form. If G is an adjoint group, then a frame is the same thing as a pinning.

Let $\{\mathrm{A}_1, \ldots, \mathrm{A}_n\}$ be a generic collection of decorated flags. For each $j \in [1, n]$, take the triple $\{\mathrm{B}_{j-1}, \mathrm{A}_j, \mathrm{B}_{j+1}\}$. Since $\mathcal{F}_{\mathrm{G}}$ is a G-torsor, there is a unique $u_j \in \mathrm{U}_{\mathrm{A}_j}$ such that

$$\{\mathrm{A}_j, \mathrm{B}_{j+1}\} = u_j \cdot \{\mathrm{A}_j, \mathrm{B}_{j-1}\}. \tag{76}$$

Consider the following rational function on $\mathcal{A}^n$, whose definition is illustrated on Fig. 14:

$$\mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n) := \sum_{j=1}^{n} \chi_{\mathrm{A}_j}(u_j). \tag{77}$$

Lemma 2.3 *For any* $g \in \mathrm{G}$, *we have* $\mathcal{W}(g\mathrm{A}_1, \ldots, g\mathrm{A}_n) = \mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n)$.

Proof Clearly $\{gA_j, gB_{j+1}\} = gu_j g^{-1} \cdot \{gA_j, gB_{j-1}\}$. The Lemma follows from (5). □

Since $\mathcal{W}$ is invariant under the G-diagonal action on $\mathcal{A}^n$, we define

Definition 2.4 The potential $\mathcal{W}$ is a rational function on $\mathrm{Conf}_n(\mathcal{A})$, given by (77).

Theorem 2.5 *The potential $\mathcal{W}$ is a positive rational function on the space $\mathrm{Conf}_n(\mathcal{A})$, $n > 2$.*

Theorem 2.5 is a non-trivial result. It is based on two facts: the character χ is a positive function on U, and the positive structure on $\mathrm{Conf}_n(\mathcal{A})$ is twisted cyclic invariant, see Sect. 2.1.6. We prove Theorem 2.5 in Sect. 6.4.

Therefore we arrive at the set of positive tropical points of $\mathrm{Conf}_n(\mathcal{A})$:

$$\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) := \{l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}, \quad n > 2. \tag{78}$$

Example. Let $\mathrm{G} = \mathrm{SL}_2$. The space $\mathrm{Conf}_3(\mathcal{A})$ parametrizes configurations (v_1, v_2, v_3) of vectors in a two dimensional vector space with a volume form ω. Set $\Delta_{i,j} := \langle v_i \wedge v_j, \omega\rangle$. Then

$$\mathcal{W}(v_1, v_2, v_3) = \frac{\Delta_{1,3}}{\Delta_{1,2}\,\Delta_{2,3}} + \frac{\Delta_{1,2}}{\Delta_{2,3}\,\Delta_{1,3}} + \frac{\Delta_{2,3}}{\Delta_{1,3}\,\Delta_{1,2}}. \tag{79}$$

Therefore tropicalizing the function (79) we get

$$\mathrm{Conf}_3^+(\mathcal{A}_{\mathrm{SL}_2})(\mathbb{Z}^t) = \{a, b, c \in \mathbb{Z} \mid a \geq b + c,\ b \geq a + c,\ c \geq a + b\}.$$

Notice that the inequalities imply $a, b, c \in \mathbb{Z}_{\leq 0}$.

2.1.5 Parametrization of a canonical basis in tensor products invariants

By Bruhat decomposition, for each $(A_1, A_2) \in \mathrm{Conf}_2^*(\mathcal{A})$, there is a unique $h_{A_1,A_2} \in \mathrm{H}$ such that

$$(A_1, A_2) = (\mathrm{U}, h_{A_1,A_2}\overline{w}_0 \cdot \mathrm{U}).$$

It provides an isomorphism, which induces a positive structure on $\mathrm{Conf}_2(\mathcal{A})$:

$$\alpha : \mathrm{Conf}_2^*(\mathcal{A}) \xrightarrow{\sim} \mathrm{H}, \quad (A_1, A_2) \longrightarrow h_{A_1,A_2}. \tag{80}$$

We extend definition (78) to $n = 2$ using the potential (72), so that one has an isomorphism

$$\alpha^t : \mathrm{Conf}_2^+(\mathcal{A})(\mathbb{Z}^t) \xrightarrow{\sim} \mathrm{H}^+(\mathbb{Z}^t) = \mathrm{P}^+.$$

See more details in Sect. 6.3, formula (188), and [17].

The restriction maps π_{ij}. We picture configurations $(\mathrm{A}_1, \ldots, \mathrm{A}_n)$ at the labelled vertices $[1, n]$ of a convex n-gon P_n. Each pair of distinct $i, j \in [1, n]$ gives rise to a map

$$\pi_{ij} : \mathrm{Conf}_n(\mathcal{A}) \longrightarrow \mathrm{Conf}_2(\mathcal{A}), \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longrightarrow \begin{cases} (\mathrm{A}_i, \mathrm{A}_j) & \text{if } i < j, \\ (s_{\mathrm{G}} \cdot \mathrm{A}_i, \mathrm{A}_j) & \text{if } i > j. \end{cases}$$

The maps π_{ij} are positive [17], and therefore can be tropicalized:

$$\begin{array}{ccc} \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) & \xrightarrow{\pi_{ij}^t} & \mathrm{Conf}_2(\mathcal{A})(\mathbb{Z}^t) = \mathrm{P} \\ \cup & & \cup \\ \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) & \xrightarrow{\pi_{ij}^t} & \mathrm{Conf}_2^+(\mathcal{A})(\mathbb{Z}^t) = \mathrm{P}^+ \end{array}$$

The fact that $\pi_{ij}^t(\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)) \subseteq \mathrm{P}^+$ is due to Lemma 6.14.

In particular, the oriented sides of the polygon P_n give rise to a positive map

$$\pi = (\pi_{12}, \pi_{23}, \ldots, \pi_{n,1}) : \mathrm{Conf}_n(\mathcal{A}) \longrightarrow (\mathrm{Conf}_2(\mathcal{A}))^n \simeq \mathrm{H}^n. \quad (81)$$

A decomposition of $\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$. Given $\underline{\lambda} := (\lambda_1, \ldots, \lambda_n) \in (\mathrm{P}^+)^n$, define

$$\mathbf{C}_{\underline{\lambda}} = \{l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) \mid \pi^t(l) = \underline{\lambda}\}. \quad (82)$$

The weights $\underline{\lambda}$ of G^L are assigned to the oriented sides of P_n, as shown on Fig. 15. Such sets provide a canonical decomposition (Fig. 16)

$$\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) = \bigsqcup_{\underline{\lambda} \in (\mathrm{P}^+)^n} \mathbf{C}_{\underline{\lambda}}. \quad (83)$$

Tensor products invariants. Here is one of our main results.

Theorem 2.6 *Let* $\lambda_1, \ldots, \lambda_n \in \mathrm{P}^+$. *The set* $\mathbf{C}_{\lambda_1,\ldots,\lambda_n}$ *parametrizes a canonical basis in the space of invariants* $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{\mathrm{G}^L}$.

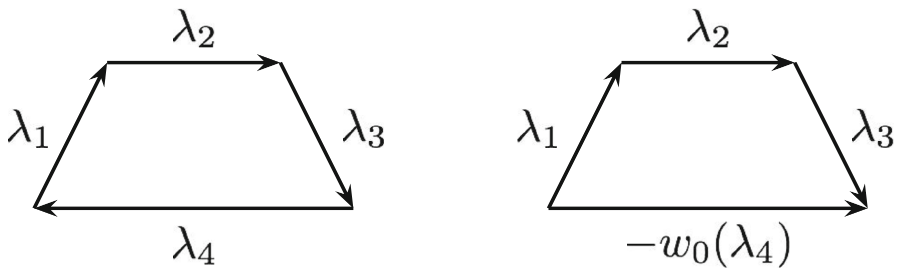


Fig. 15 Dominant weights labels of the polygon sides for the set $\mathbf{C}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$

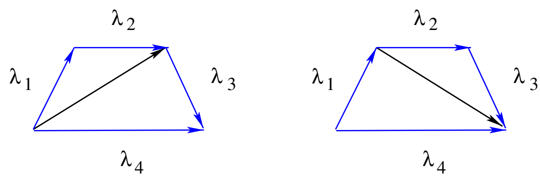


Fig. 16 The associativity

Theorem 2.6 follows from Theorem 2.20 and geometric Satake correspondence, see Sect. 2.2.4.

Alternatively, there is a similar set, defined by reversing the order of the side $(1, n)$:

$$\mathbf{C}^{\lambda_n}_{\lambda_1,\ldots,\lambda_{n-1}} := \{l \in \mathrm{Conf}^+_n(\mathcal{A})(\mathbb{Z}^t) \mid \pi^t_{i,i+1}(l) = \lambda_i,\\ i = 1, \ldots, n-1,\ \pi^t_{1,n}(l) = \lambda_n\}. \tag{84}$$

Then

$$\mathbf{C}_{\lambda_1,\ldots,\lambda_n} = \mathbf{C}^{-w_0(\lambda_n)}_{\lambda_1,\ldots,\lambda_{n-1}}.$$

The set $\mathbf{C}^{\lambda_n}_{\lambda_1,\ldots,\lambda_{n-1}}$ parametrizes a basis in the space of tensor product multiplicities

$$\mathrm{Hom}_{\mathrm{G}^L}(V_{\lambda_n}, V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-1}}). \tag{85}$$

2.1.6 *Some features of the set* $\mathrm{Conf}^+_n(\mathcal{A})(\mathbb{Z}^t)$.

Here are some features of the set $\mathrm{Conf}^+_n(\mathcal{A})(\mathbb{Z}^t)$. All of them follow immediately from the definition of the potential $\mathcal{W}$ and basic facts about the positive structure on $\mathrm{Conf}_n(\mathcal{A})$. One of the most crucial is twisted cyclic invariance, so we start from it.

The twisted cyclic shift. It was proved in [17, Section 8] that the defined there positive atlas on $\mathrm{Conf}_n(\mathcal{A})$ is invariant under the *twisted cyclic shift*

$$t : \mathrm{Conf}_n(\mathcal{A}) \longrightarrow \mathrm{Conf}_n(\mathcal{A}), \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longmapsto (\mathrm{A}_2, \ldots, \mathrm{A}_n, \mathrm{A}_1 \cdot s_{\mathrm{G}}).$$

Its tropicalization is a cyclic shift on the space of the tropical points:

$$t : \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) \longrightarrow \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t). \tag{86}$$

- **Twisted cyclic shift invariance**. *The potential* $\mathcal{W}$ *is evidently invariant under the twisted cyclic shift. Therefore the set* (78) *is invariant under the tropical cyclic shift* (86).

Given a triangle $t = \{i_1 < i_2 < i_3\}$ inscribed into the polygon P_n, there is a positive map

$$\pi_t : \mathrm{Conf}_n(\mathcal{A}) \longrightarrow \mathrm{Conf}_3(\mathcal{A}), \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longmapsto (\mathrm{A}_{i_1}, \mathrm{A}_{i_2}, \mathrm{A}_{i_3}).$$

Each triangulation T of P_n gives rise to a positive injection $\pi_T : \mathrm{Conf}_n(\mathcal{A}) \to \prod_{t \in T} \mathrm{Conf}_3(\mathcal{A})$, where the product is over all triangles t of T. Set its image

$$\mathrm{Conf}_T(\mathcal{A}) := \mathrm{Im}\pi_T \subset \prod_{t \in T} \mathrm{Conf}_3(\mathcal{A}). \tag{87}$$

For each pair (t, d), where $t \in T$ and d is a side of t, there is a map given by obvious projections

$$p(t, d) : \prod_{t \in T} \mathrm{Conf}_3(\mathcal{A}) \xrightarrow{\mathrm{pr}_t} \mathrm{Conf}_3(\mathcal{A}) \xrightarrow{\mathrm{pr}_d} \mathrm{Conf}_2(\mathcal{A}).$$

For each diagonal d of T, there are two triangles, t_1^d and t_2^d, sharing d. A point x of $\mathrm{Conf}_T(\mathcal{A})$ is described by the condition that $p(t_1^d, d)(x) = p(t_2^d, d)(x)$ for all diagonals d of T.

Proposition 2.7 [17] *There is an isomorphism of positive moduli spaces*

$$\pi_T : \mathrm{Conf}_n(\mathcal{A}) \xrightarrow{\sim} \mathrm{Conf}_T(\mathcal{A}).$$

It leads to an isomorphism of sets of their $\mathbb{Z}$-tropical points:

$$\pi_T^t : \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t) \xrightarrow{\sim} \mathrm{Conf}_T(\mathcal{A})(\mathbb{Z}^t). \tag{88}$$

Some important features of the potential $\mathcal{W}$ are the following:

- **Scissor congruence invariance**. *For any triangulation T of the polygon, the potential $\mathcal{W}_n$ on* $\mathrm{Conf}_n(\mathcal{A})$ *is a sum over the triangles t of T*:

$$\mathcal{W}_n = \sum_{t \in T} \mathcal{W}_3 \circ \pi_t. \tag{89}$$

This follows immediately from the fact that $\chi_{\rm A}$ is a character of the subgroup $\rm U_A$. Combining this with the isomorphism (88) we get

- **Decomposition isomorphism**. *Given a triangulation T of P_n, one has an isomorphism*

$$i_T^{t,+} : {\rm Conf}_n^+(\mathcal{A})(\mathbb{Z}^t) \stackrel{\sim}{\longrightarrow} {\rm Conf}_T^+(\mathcal{A})(\mathbb{Z}^t).$$

So one can think of the data describing a point of ${\rm Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ as of a collection of similar data assigned to triangles t of a triangulation T of the polygon, which match at the diagonals. Therefore each triangulation T provides a further decomposition of the set (82). By Lemma 6.14, the weights of $\rm G^L$ assigned to the sides and edges of the polygon are dominant.

Consider an algebra with a linear basis e_λ parametrized by dominant weights λ of $\rm G^L$ with the structure constants given by the cardinality of the set $\mathbf{C}_{\lambda_1,\lambda_2}^{\mu}$:

$$e_{\lambda_1} * e_{\lambda_2} = \sum_{\mu \in {\rm P}^+} |\mathbf{C}_{\lambda_1,\lambda_2}^{\mu}| e_\mu. \tag{90}$$

The following basic property is evident from our definition of the set $\mathbf{C}_{\lambda_1,\lambda_2}^{\mu}$:

- **Associativity**. *The product $*$ is associative*.

The associativity is equivalent to the fact that there are two different decompositions of the set ${\rm Conf}_4^+(\mathcal{A})(\mathbb{Z}^t)$ corresponding to two different triangulations of the 4-gon (Fig. 16).

A simple proof of Knutson–Tao–Woodward's theorem [55]. That theorem asserts the associativity of the similar $*$-product whose structure constants are given by the number of hives. The associativity in our set-up, where the structure constant are given by the number of positive integral tropical points, is obvious for any group G. So to prove the theorem we just need to relate hives to positive integral tropical points for ${\rm G} = {\rm GL}_m$, which is done in Sect. 3.

2.2 Parametrization of top components of fibers of convolution morphisms

2.2.1 Transcendental cells and integral tropical points

For a non-zero $C = \sum_{k \geq p} c_k t^k \in \mathcal{K}$ such that c_p is not zero, we define its *valuation* and *initial term*:

$${\rm val}(C) := p, \quad {\rm in}(C) := c_p.$$

A decomposition of $\mathrm{T}(\mathcal{K})$. For each split torus T, there is a natural projection, which we call the valuation map:

$$\mathrm{val} : \mathrm{T}(\mathcal{K}) \longrightarrow \mathrm{T}(\mathcal{K})/\mathrm{T}(\mathcal{O}) = \mathrm{T}(\mathbb{Z}^t).$$

Given an isomorphism $\mathrm{T} = (\mathbb{G}_m)^k$, the map is expressed as $(C_1, \ldots, C_k) \mapsto (\mathrm{val}(C_1), \ldots, \mathrm{val}(C_k))$.

Each $l \in \mathrm{T}(\mathbb{Z}^t)$ gives rise to a cell

$$\mathrm{T}_l := \{x \in \mathrm{T}(\mathcal{K}) \mid \mathrm{val}(x) = l\}.$$

It is a projective limit of irreducible algebraic varieties: each of them is isomorphic to $(\mathbb{G}_m)^k \times \mathbb{A}^N$. Therefore T_l is an irreducible proalgebraic variety, and $\mathrm{T}(\mathcal{K})$ is a disjoint union of them:

$$\mathrm{T}(\mathcal{K}) = \coprod_{l \in \mathrm{T}(\mathbb{Z}^t)} \mathrm{T}_l.$$

Transcendental $\mathcal{K}$**-points of** T. Let us define an initial term map for $\mathrm{T}(\mathcal{K})$ in coordinates:

$$\mathrm{in} : \mathrm{T}(\mathcal{K}) \longrightarrow \mathrm{T}(\mathbb{C}), \quad (C_1, \ldots, C_k) \longmapsto (\mathrm{in}(C_1), \ldots, \mathrm{in}(C_k)).$$

A subset $\{c_1, \ldots, c_q\} \subset \mathbb{C}$ is *algebraically independent* if $P(c_1, \ldots, c_q) \neq 0$ for any $P \in \mathbb{Q}(X_1, \ldots, X_q)^*$.

Definition 2.8 A point $C \in \mathrm{T}(\mathcal{K})$ is *transcendental* if its initial term $\mathrm{in}(C)$ is algebraically independent as a subset of $\mathbb{C}$. Denote by $\mathrm{T}^\circ(\mathcal{K})$ the set of transcendental points in $\mathrm{T}(\mathcal{K})$. Set

$$\mathrm{T}_l^\circ := \mathrm{T}_l \bigcap \mathrm{T}^\circ(\mathcal{K}).$$

Lemma 2.9 *Let F be a positive rational function on* T. *For any $C \in \mathrm{T}^\circ(\mathcal{K})$, we have*

$$\mathrm{val}\,(F(C)) = F^t\,(\mathrm{val}(C)).$$

Proof It is clear. $\square$

Transcendental $\mathcal{K}$**-cells of a positive space** $\mathcal{Y}$.

Definition 2.10 A birational isomorphism $f : \mathcal{Y} \to \mathcal{Z}$ of positive spaces is a *positive birational isomorphism* if it is a positive morphism, and its inverse is also a positive morphism.

Theorem 2.11 *Let $f : \mathrm{T} \to \mathrm{S}$ be a positive birational isomorphism of split tori. Then*

$$f(\mathrm{T}_l^\circ) = \mathrm{S}^\circ_{f^t(l)}.$$

We prove Theorem 2.11 in Sect. 5. It is crucial that the inverse of f is also a positive morphism. As a counterexample, the map $f : \mathbb{G}_m \to \mathbb{G}_m,\ x \mapsto x+1$ is a positive morphism, but its inverse $x \mapsto x-1$ is not. Let $l \in \mathbb{G}_m(\mathbb{Z}^t) = \mathbb{Z}$. If $l > 0$, then Theorem 2.11 fails: the points in $f(\mathrm{T}_l^\circ)$ are not transcendental since $\mathrm{in}(f(\mathrm{T}_l^\circ)) \equiv 1$.

Definition 2.12 Let $\alpha_{\mathbf{c}} : \mathrm{T} \to \mathcal{Y}$ be a coordinate system from a positive atlas on $\mathcal{Y}$. The set of transcendental $\mathcal{K}$-points of $\mathcal{Y}$ is

$$\mathcal{Y}^\circ(\mathcal{K}) := \alpha_{\mathbf{c}}(\mathrm{T}^\circ(\mathcal{K})).$$

For each $l \in \mathcal{Y}(\mathbb{Z}^t)$, the transcendental l-cell[9] of $\mathcal{Y}$ is

$$\mathcal{C}_l^\circ := \alpha_{\mathbf{c}}(\mathrm{T}^\circ_{\beta^t(l)}), \quad \text{where } \beta = \alpha_{\mathbf{c}}^{-1}.$$

By Theorem 2.11, this definition is independent of the coordinate system $\alpha_{\mathbf{c}}$ chosen. Similarly one can upgrade the valuation map to positive spaces: given a positive space $\mathcal{Y}$, there is a unique map

$$\mathrm{val} : \mathcal{Y}^\circ(\mathcal{K}) \longrightarrow \mathcal{Y}(\mathbb{Z}^t) \tag{91}$$

such that

$$\mathcal{C}_l^\circ = \{y \in \mathcal{Y}^\circ(\mathcal{K}) \mid \mathrm{val}(y) = l\}.$$

The valuation map (91) is functorial under positive birational isomorphisms of positive spaces. Therefore the transcendental cells are also functorial under positive birational isomorphisms.

Thus there is a canonical decomposition parametrized by the set $\mathcal{Y}(\mathbb{Z}^t)$:

$$\mathcal{Y}^\circ(\mathcal{K}) = \bigsqcup_{l \in \mathcal{Y}(\mathbb{Z}^t)} \mathcal{C}_l^\circ.$$

Thanks to the following Lemma, one can identify each tropical point l with $\mathcal{C}_l^\circ$.

[9] By abuse of notation, such a cell will always be denoted by $\mathcal{C}_l^\circ$. The tropical point l tells which space it lives.

Lemma 2.13 *Let F be a positive rational function on $\mathcal{Y}$. For any $C \in \mathcal{Y}^{\circ}(\mathcal{K})$, we have*

$$\mathrm{val}\,(F(C)) = F^t\,(\mathrm{val}(C))\,.$$

Proof It follows immediately from Lemma 2.9 and Theorem 2.11. □

2.2.2 $\mathcal{O}$-integral configurations of decorated flags and the affine Grassmannian

Recall the affine Grassmannian Gr. Recall the moduli space $\mathcal{F}_{\mathrm{G}}$ of frames from Definition 2.2.

Lemma-Construction 2.14 *There is a canonical onto map*

$$\mathrm{L}: \mathcal{F}_{\mathrm{G}}(\mathcal{K}) \longrightarrow \mathrm{Gr}, \quad \{\mathrm{A}_1, \mathrm{B}_2\} \longmapsto \mathrm{L}(\mathrm{A}_1, \mathrm{B}_2) \tag{92}$$

Proof Let $\{\mathrm{U}, \mathrm{B}^-\} \in \mathcal{F}_{\mathrm{G}}(\mathbb{Q})$ be a standard frame. There is a unique $g_{\{\mathrm{A}_1,\mathrm{B}_2\}} \in \mathrm{G}(\mathcal{K})$ such that

$$\{\mathrm{A}_1, \mathrm{B}_2\} = g_{\{\mathrm{A}_1,\mathrm{B}_2\}} \cdot \{\mathrm{U}, \mathrm{B}^-\}.$$

It provides an isomorphism $\mathcal{F}_{\mathrm{G}}(\mathcal{K}) \stackrel{\sim}{\rightarrow} \mathrm{G}(\mathcal{K})$. Composing it with the projection $[\cdot]: \mathrm{G}(\mathcal{K}) \rightarrow \mathrm{Gr}$,

$$\mathrm{L}(\mathrm{A}_1, \mathrm{B}_2) := [g_{\{\mathrm{A}_1,\mathrm{B}_2\}}] \in \mathrm{Gr}. \tag{93}$$

Note that $\mathcal{F}_{\mathrm{G}}(\mathbb{Q})$ is a $\mathrm{G}(\mathbb{Q})$-torsor. So choosing a different frame in $\mathcal{F}_{\mathrm{G}}(\mathbb{Q})$ we get another representative of the coset $g_{\{\mathrm{A}_1,\mathrm{B}_2\}} \cdot \mathrm{G}(\mathbb{Q})$. Since $\mathrm{G}(\mathbb{Q}) \subset \mathrm{G}(\mathcal{O})$, the resulting lattice (93) will be the same. Therefore the map L is canonical. □

Symmetric space and affine Grassmannian. The affine Grassmannian is the non-archimedean version of the symmetric space $\mathrm{G}(\mathbb{R})/\mathrm{K}$, where K is a maximal compact subgroup in $\mathrm{G}(\mathbb{R})$. A generic pair of flags $\{\mathrm{B}_1, \mathrm{B}_2\}$ over $\mathbb{R}$ gives rise to an $\mathrm{H}(\mathbb{R}_{>0})$-torsor in the symmetric space—the projection of $\mathrm{B}_1 \cap \mathrm{B}_2$.[10] Notice that $\mathrm{H}(\mathbb{R}_{>0}) = \mathrm{H}(\mathbb{R})/(\mathrm{H}(\mathbb{R}) \cap \mathrm{K})$. A generic pair $\{\mathrm{A}_1, \mathrm{B}_2\}$ determines a point[11] $Q(\mathrm{A}_1, \mathrm{B}_2) \in \mathrm{G}(\mathbb{R})/\mathrm{K}$. So we get the archimedean analog of the map (92):

[10] Here is a non-archimedean analog: A generic pair of flags $\{\mathrm{B}_1, \mathrm{B}_2\}$ over $\mathcal{K}$ gives rise to an $H(\mathcal{K})/H(\mathcal{O})$-torsor in the affine Grassmannian—the projection of $\mathrm{B}_1(\mathcal{K}) \cap \mathrm{B}_2(\mathcal{K})$ to $\mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O})$.

[11] In the archimedean case, a maximal compact subgroup K is defined by using the Cartan involution. A generic pair $\{\mathrm{A}, \mathrm{B}\}$ determines a pinning, and hence a Cartan involution.

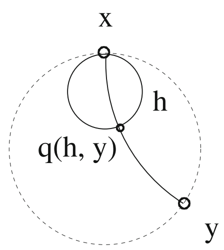


Fig. 17 The metric $q(h, y)$ determined by a horocycle h and a boundary point y

$$Q : \mathcal{F}_{\mathrm{G}}(\mathbb{R}) \longrightarrow \mathrm{G}(\mathbb{R})/\mathrm{K}, \quad \{\mathrm{A}_1, \mathrm{B}_2\} \longmapsto Q(\mathrm{A}_1, \mathrm{B}_2). \tag{94}$$

Decorated flags and horospheres. For the adjoint group G', the principal affine space $\mathcal{A}$ can be interpreted as the moduli space of horospheres in the symmetric space $\mathrm{G}'(\mathbb{R})/K$ in the archimedean case, or in the affine Grassmannian Gr. The horosphere $\mathcal{H}_{\mathrm{A}}$ assigned to a decorated flag A is an orbit of the maximal unipotent subgroup U_{A}. Let $\mathcal{B}^*_{\mathrm{A}}$ be the open Schubert cell of flags in generic position to a given decorated flag A. Then there is an isomorphism

$$\mathcal{B}^*_{\mathrm{A}} \longrightarrow \mathcal{H}_{\mathrm{A}}, \quad \mathrm{B} \longmapsto \mathrm{L}(\mathrm{A}, \mathrm{B}) \quad \text{or} \quad \mathrm{B} \longmapsto Q(\mathrm{A}, \mathrm{B}).$$

Examples. (1) Let $\mathrm{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. Its maximal compact subgroup $\mathrm{K} = \mathrm{SO}_2(\mathbb{R})$. The symmetric space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is the hyperbolic plane $\mathcal{H}^2$. A decorated flag $\mathrm{A}_1 \in \mathcal{A}_{\mathrm{PGL}_2}(\mathbb{R})$ corresponds to a horocycle h based as a point x at the boundary. A flag B_2 corresponds to another point y at the boundary. Let $g(x, y)$ be the geodesic connecting x and y. The point $Q(\mathrm{A}_1, \mathrm{B}_2)$ is the intersection of h and $g(x, y)$, see Fig. 17:

$$q(h, y) := h \cap g(x, y) \in \mathcal{H}^2.$$

(2) Let $\mathrm{G} = \mathrm{GL}_n$. Recall that a flag $F_\bullet$ in an n-dimensional vector space V_n over a field is a data $F_1 \subset \ldots \subset F_n$, $\dim F_i = i$. A generic pair of flags $(F_\bullet, G_\bullet)$ in V_n is the same thing as a decomposition of V_n into a direct sum of one dimensional subspaces

$$V_n = L_1 \oplus \ldots \oplus L_n, \tag{95}$$

where $L_i = F_i \cap G_{n+1-i}$. Conversely, $F_a = L_1 \oplus \ldots \oplus L_a$ and $G_b = L_{n-b+1} \oplus \ldots \oplus L_n$.

Over the field $\mathbb{R}$, this decomposition gives rise to a $(\mathbb{R}^*_{>0})^n$-torsor in the symmetric space, given by a family of positive definite metrics on V_n with the principal axes $(L_1, \ldots, L_n)$:

$$a_1x_1^2 + \ldots + a_nx_n^2, \quad a_i > 0. \tag{96}$$

Here $(x_1, \dots, x_n)$ is a coordinate system for which the lines L_i are the coordinate lines.

A decorated flag A in V_n is a flag $F_\bullet$ plus a collection of non-zero vectors $l_i \in F_i/F_{i-1}$. A frame in $\mathcal{F}_{\mathrm{GL}_n}$ is equivalent to a generic pair of flags $(F_\bullet, G_\bullet)$ and a decorated flag A over the flag $F_\bullet$. It determines a basis $(e_1, \dots, e_n)$ in V_n and vice verse. Here $e_i \in L_i$ and $e_i = l_i$ under the projection $L_i \longrightarrow F_i/F_{i-1}$. This basis determines a metric—the positive definite metric with the principal axes L_i such that the vectors e_i are unit vectors.

(3) Over the field $\mathcal{K}$, decomposition (95) gives rise to an $\mathrm{H}(\mathcal{K})/\mathrm{H}(\mathcal{O}) = \mathbb{Z}^n$-torsor in Gr, given by the following collection of lattices in V_n.

$$\mathcal{O}t^{k_1}e_1 \oplus \dots \oplus \mathcal{O}t^{k_n}e_n, \quad k_i \in \mathbb{Z}.$$

These lattices are the non-archimedean version of the unit balls of the metrics (96).

$\mathcal{O}$-integral configurations of decorated flags.

Definition 2.15 A collection of decorated flags $\{\mathrm{A}_1, \dots, \mathrm{A}_n\}$ over $\mathcal{K}$ is $\mathcal{O}$-integral if it is generic and for any $i \in [1, n]$ the lattice $\mathrm{L}(\mathrm{A}_i, \mathrm{B}_j)$ does not depend on the choice of j different than i.

Let $g \in \mathrm{G}(\mathcal{K})$. Note that $\mathrm{L}(g\mathrm{A}_i, g\mathrm{B}_j) = g \cdot \mathrm{L}(\mathrm{A}_i, \mathrm{B}_j)$. Therefore if $\{\mathrm{A}_1, \dots, \mathrm{A}_n\}$ is $\mathcal{O}$-integral, so is $g \cdot \{\mathrm{A}_1, \dots, \mathrm{A}_n\}$. Thus we define

Definition 2.16 A configuration in $\mathrm{Conf}_n(\mathcal{A})(\mathcal{K})$ is $\mathcal{O}$-integral if it is a $\mathrm{G}(\mathcal{K})$-orbit of an $\mathcal{O}$-integral collection of decorated flags. Denote by $\mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$ the space of such configurations.

The archimedean version of Definition 2.16 is trivial. For example, let $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. Then there are no horocycles (h_1, h_2, h_3) such that their boundary points (x_1, x_2, x_3) are distinct, and the intersection of the horocycle h_i with the geodesic $g(x_i, x_j)$ do not depend on $j \neq i$.

In contrast with this, we demonstrate below that the non-archimedean version is very rich. The difference stems from the fact that in the archimedean case the intersection $K \cap \mathrm{U} = e$ is trivial, while in the non-archimedean $\mathrm{G}(\mathcal{O}) \cap \mathrm{U}(\mathcal{K}) = \mathrm{U}(\mathcal{O})$.

Transcendental cells and $\mathcal{O}$-integral configurations. The following fact is crucial.

Theorem 2.17 *If $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$, then there is an inclusion $\mathcal{C}_l^\circ \subset \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$. Otherwise $\mathcal{C}_l^\circ \cap \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$ is an empty set.*

Theorem 2.17 gives an alternative conceptual definition of the set of positive integral tropical points of the space $\mathrm{Conf}_n(\mathcal{A})$, which refers neither to the potential $\mathcal{W}$, nor to a specific positive coordinate system. However to show that the set $\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ is "big", or even non-empty, we use the potential $\mathcal{W}$ and its properties, which imply, for example, that the set $\mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ is obtained by amalgamation of similar sets assigned to triangles of a triangulation of the polygon. We prove Theorem 2.17 in Sect. 6.4.

2.2.3 *The canonical map κ and cycles on* $\mathrm{Conf}_n(\mathrm{Gr})$

The canonical map κ. Recall the configuration space

$$\mathrm{Conf}_n(\mathrm{Gr}) := \mathrm{G}(\mathcal{K})\backslash(\mathrm{Gr} \times \ldots \times \mathrm{Gr}).$$

Given an $\mathcal{O}$-integral collection $\{\mathrm{A}_1, \ldots, \mathrm{A}_n\}$ of decorated flags, we get a collection of lattices $\{\mathrm{L}_1, \ldots, \mathrm{L}_n\}$ by setting $\mathrm{L}_i := \mathrm{L}(\mathrm{A}_i, \mathrm{B}_j)$ for some $j \neq i$. By definition, the lattice L_i is independent of j chosen. This construction descends to configurations, providing a canonical map

$$\kappa : \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \longrightarrow \mathrm{Conf}_n(\mathrm{Gr}), \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longmapsto (\mathrm{L}_1, \ldots, \mathrm{L}_n). \tag{97}$$

The map is evidently cyclic invariant, and commutes with the restriction to subconfigurations:

$$\kappa(\mathrm{A}_{i_1}, \ldots, \mathrm{A}_{i_k}) = (\mathrm{L}_{i_1}, \ldots, \mathrm{L}_{i_k}) \quad \text{for any } 1 \leq i_1 < \cdots < i_k \leq n.$$

The cycles $\mathcal{M}_l$ in $\mathrm{Conf}_n(\mathrm{Gr})$. Let $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$. Thanks to Theorem 2.17, we can combine the inclusion there with the canonical map (97):

$$\mathcal{C}_l^\circ \subset \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \xrightarrow{\kappa} \mathrm{Conf}_n(\mathrm{Gr}). \tag{98}$$

Definition 2.18 The cycle $\mathcal{M}_l \subset \mathrm{Conf}_n(\mathrm{Gr})$ is a substack given by the closure of $\kappa(\mathcal{C}_l^\circ)$:

$$\mathcal{M}_l := \overline{\mathcal{M}_l^\circ}, \quad \mathcal{M}_l^\circ := \kappa(\mathcal{C}_l^\circ) \subset \mathrm{Conf}_n(\mathrm{Gr}), \qquad l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t). \tag{99}$$

Lemma 2.19 *The cycle $\mathcal{M}_l$ is irreducible.*

Proof For a split torus T, the cycle T_l is irreducible. So the cycles $\mathcal{C}_l^\circ$ and $\mathcal{M}_l$ are irreducible. □

In other words, $\mathcal{M}_l$ is a $\mathrm{G}(\mathcal{K})$-invariant closed subspace in Gr^n. There is a bijection

$$\{\mathrm{G}(\mathcal{K})\text{-orbits in} \mathrm{Gr}^n\} \stackrel{1:1}{\longleftrightarrow} \{\mathrm{G}(\mathcal{O})\text{-orbits in} [1] \times \mathrm{Gr}^{n-1}\}. \tag{100}$$

Therefore one can also view the cycles $\mathcal{M}_l$ as $\mathrm{G}(\mathcal{O})$-invariant closed subspaces in $[1] \times \mathrm{Gr}^{n-1}$. Let us describe them using this point of view.

2.2.4 *Top components of the fibers of the convolution morphism*

Given $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathrm{P}^+)^n$, recall the cyclic convolution variety

$$\mathrm{Gr}_{c(\underline{\lambda})} := \{(\mathrm{L}_1, \dots, \mathrm{L}_n) \in \mathrm{Gr}^n \mid \mathrm{L}_1 \xrightarrow{\lambda_1} \mathrm{L}_2 \xrightarrow{\lambda_2} \dots \xrightarrow{\lambda_n} \mathrm{L}_{n+1},\\ \mathrm{L}_1 = \mathrm{L}_{n+1} = [1]\}.$$

It is a finite dimensional reducible variety of top dimension

$$\mathrm{ht}(\underline{\lambda}) := \langle \rho, \lambda_1 + \dots + \lambda_n \rangle.$$

It is the fiber of the convolution morphism, and therefore, thanks to the geometric Satake correspondence [34,62,65], there is a canonical isomorphism

$$\mathrm{IH}^{\mathrm{ht}(\underline{\lambda})}(\mathrm{Gr}_{c(\underline{\lambda})}) = \left(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}\right)^{\mathrm{G}^L}. \tag{101}$$

Each top dimensional component of $\mathrm{Gr}_{c(\underline{\lambda})}$ provides an element in the space (101). These elements form a canonical basis in (101). Let $\mathbf{T}_{\underline{\lambda}}$ be the set of top dimensional components of $\mathrm{Gr}_{c(\underline{\lambda})}$. Recall the set $\mathbf{C}_{\underline{\lambda}}$ of positive tropical points (82), and the cycle $\mathcal{M}_l$ from Definition 2.18.

Theorem 2.20 *Let $l \in \mathbf{C}_{\underline{\lambda}}$. Then the cycle $\mathcal{M}_l$ is the closure of a top dimensional component of* $\mathrm{Gr}_{c(\underline{\lambda})}$. *The map $l \longmapsto \mathcal{M}_l$ provides a canonical bijection from $\mathbf{C}_{\underline{\lambda}}$ to $\mathbf{T}_{\underline{\lambda}}$.*

Theorem 2.20 is proved in Sect. 9.4. It implies Theorem 2.6.

2.2.5 *Constructible equations for the top dimensional components*

We have defined the cycles $\mathcal{M}_l$ as the closures of the images of the cells $\mathcal{C}_l^\circ$. Now let us define the cycles $\mathcal{M}_l$ by equations, given by certain constructible functions on the space $\mathrm{Conf}_n(\mathrm{Gr})$. These functions generalize Kamnitzer's functions $H_{i_1,\dots,i_n}$ for $\mathrm{G} = \mathrm{GL}_m$ [46].

Constructible function D_F. Let R be a reductive algebraic group over $\mathbb{C}$. We assume that there is a rational left algebraic action of R on $\mathbb{C}^n$. Let $\mathbb{C}(x_1, \dots, x_n)$ be the field of rational functions on $\mathbb{C}^n$. We get a right algebraic action of R on $\mathbb{C}(x_1, \dots, x_n)$ denoted by $\circ$.

Let $\mathcal{K}(x_1, \dots, x_n)$ be the field of rational functions with $\mathcal{K}$-coefficients. The valuation of $\mathcal{K}^\times$ induces a natural valuation map

$$\mathrm{val}: \quad \mathcal{K}(x_1, \ldots, x_n)^\times \longrightarrow \mathbb{Z}.$$

Let $F, G \in \mathcal{K}(x_1, \ldots, x_n)^\times$. The valuation map has two basic properties

$$\mathrm{val}(FG) = \mathrm{val}(F) + \mathrm{val}(G), \tag{102}$$

$$\mathrm{val}(F + G) = \mathrm{val}(F), \quad \text{if } \mathrm{val}(F) < \mathrm{val}(G). \tag{103}$$

The group $R(\mathcal{K})$ acts on $\mathcal{K}(x_1, \ldots, x_n)$ on the right. We have the following

Lemma 2.21 *Let* $F \in \mathcal{K}(x_1, \ldots, x_n)^\times$. *If* $h \in R(\mathcal{O})$, *then* $\mathrm{val}(F \circ h) = \mathrm{val}(F)$.

Proof For any $k \in \mathcal{K}^\times$, we have $(kF) \circ h = k(F \circ h)$. Therefore it suffices to prove the case when $\mathrm{val}(F) = 0$.

Note that the group R is reductive. It is generated by

$$x_i(a) \in \mathrm{U}, \quad y_i(b) \in \mathrm{U}^-, \quad \alpha(c) \in \mathrm{H}, \quad \text{where } i \in I \text{ and } \alpha \in \mathrm{Hom}(\mathbb{G}_m, \mathrm{H}).$$

Since the action of R is algebraic, for any $f \in \mathbb{C}(x_1, \ldots, x_n)^\times$, we have $f \circ x_i(a) \in \mathbb{C}(x_1, \ldots, x_n, a)^\times$. Note that $f \circ x_i(0) = f$. Therefore we get

$$f \circ x_i(a) = \frac{f + af_1 + \ldots + a^l f_l}{1 + ag_1 + \ldots + a^m g_m}, \quad \text{where } f_j, g_j \in \mathbb{C}(x_1, \ldots, x_n). \tag{104}$$

If $a \in \mathbb{C}$, then $f \circ x_i(a) \in \mathbb{C}(x_1, \ldots, x_n)$. Moreover $f \circ x_i(a)$ is non zero. Otherwise, $f = (f \circ x_i(a)) \circ x_i(-a) = 0$. If $a \in t\mathcal{O}$, then by the basic property (103), we get $\mathrm{val}(f \circ x_i(a)) = \mathrm{val}(f) = 0$.

Let $a = a_0 + b = a_0 + a_1 t + a_2 t^2 + \ldots \in \mathcal{O}$. Then $f \circ x_i(a) = (f \circ x_i(a_0)) \circ x_i(b)$.

Note that $f \circ x_i(a_0) \in \mathbb{C}(x_1, \ldots, x_n)^\times$ and $b \in t\mathcal{O}$. Combining the above arguments we get

$$\mathrm{val}(f \circ x_i(a)) = \mathrm{val}(f \circ x_i(a_0)) = 0 = \mathrm{val}(f), \quad \forall a \in \mathcal{O}. \tag{105}$$

Now let $F \in \mathcal{K}(x_1, \ldots, x_n)^\times$ such that $\mathrm{val}(F) = 0$. Then F can be expressed as

$$F = \frac{f_0 + b_l f_1 + \ldots + b_l f_l}{1 + c_1 g_1 + \ldots + c_m g_m}.$$

Here $f_0, f_p, g_q \in \mathbb{C}(x_1, \ldots, x_n)^\times$, $b_p, c_q \in \mathcal{K}^\times$, $\mathrm{val}(b_p) > 0$, $\mathrm{val}(c_q) > 0$,. By definition, we have

$$F \circ x_i(a) = \frac{f_0 \circ x_i(a) + b_1 f_1 \circ x_i(a) + \ldots + b_l f_l \circ x_i(a)}{1 + c_1 g_1 \circ x_i(a) + \ldots + c_m g_m \circ x_i(a)}.$$

Let $a \in \mathcal{O}$. By (105), we get

$$\begin{aligned} \mathrm{val}(f_0 \circ x_i(a)) &= 0, \\ \mathrm{val}(b_p f_p \circ x_i(a)) &= \mathrm{val}(b_p) + \mathrm{val}(f_p \circ x_i(a)) = \mathrm{val}(b_p) > 0, \\ \mathrm{val}(c_q g_q \circ x_i(a)) &= \mathrm{val}(c_q) + \mathrm{val}(g_q \circ x_i(a)) = \mathrm{val}(c_q) > 0. \end{aligned}$$

By the basic property (103), we get $\mathrm{val}(F \circ x_i(a)) = \mathrm{val}(f_0 \circ x_i(a)) = 0$. Hence we prove that

$$\mathrm{val}(F \circ x_i(a)) = \mathrm{val}(F), \quad \forall a \in \mathcal{O}.$$

By the same argument, we show that

$$\mathrm{val}(F \circ y_i(b)) = \mathrm{val}(F), \quad \forall b \in \mathcal{O}, \quad \mathrm{val}(F \circ \alpha(c)) = \mathrm{val}(F), \quad \forall c \in \mathcal{O}^\times.$$

Note that $R(\mathcal{O})$ is generated by the elements $x_i(a), y_i(b), \alpha(c), \quad a, b \in \mathcal{O}, c \in \mathcal{O}^\times$. The Lemma is proved. □

Let $\mathfrak{X}$ be rational space over $\mathbb{C}$, i.e., $\mathbb{C}(\mathfrak{X}) \stackrel{\sim}{=} \mathbb{C}(x_1, \ldots, x_n)$. Similarly, there is a valuation map $\mathrm{val} : \mathcal{K}(\mathfrak{X})^\times \to \mathbb{Z}$. We assume that there is left algebraic action of R on $\mathfrak{X}$. Lemma 2.21 implies

Lemma 2.22 *Let $F \in \mathcal{K}(\mathfrak{X})^\times$. If $h \in R(\mathcal{O})$, then* $\mathrm{val}(F \circ h) = \mathrm{val}(F)$.

Constructible equations for top components. Let $\mathfrak{X} := \mathcal{A}^n$ and let $R := \mathrm{G}^n$. Let $F \in \mathbb{C}(\mathcal{A}^n)$ and let $(g_1, \ldots, g_n) \in \mathrm{G}^n$. Then G^n acts on $\mathbb{C}(\mathcal{A}^n)$ on the right:

$$\begin{aligned} &(F \circ (g_1, \ldots, g_n))(\mathrm{A}_1, \ldots, \mathrm{A}_n) := F(g_1 \cdot \mathrm{A}_1, \ldots, g_n \cdot \mathrm{A}_n), \\ &\quad \forall (\mathrm{A}_1, \ldots, \mathrm{A}_n) \in \mathcal{A}^n. \end{aligned} \tag{106}$$

By definition, a nonzero rational function $F \in \mathbb{C}(\mathrm{Conf}_n(\mathcal{A}))$ is also a G-diagonal invariant function on $\mathcal{A}^n$

$$F(g\mathrm{A}_1, \ldots, g\mathrm{A}_n) = F(\mathrm{A}_1, \ldots, \mathrm{A}_n).$$

There is a $\mathbb{Z}$-valued function

$$D_F : \mathrm{G}(\mathcal{K})^n \longrightarrow \mathbb{Z}, \quad D_F(g_1(t), \dots, g_n(t)) := \mathrm{val}\,(F \circ (g_1(t), \dots, g_n(t)))\,. \tag{107}$$

Lemma-Construction 2.23 *The function D_F is invariant under the left diagonal action of the group* $\mathrm{G}(\mathcal{K})$ *on* $\mathrm{G}(\mathcal{K})^n$*, and the right action of the subgroup* $\mathrm{G}(\mathcal{O})^n \subset \mathrm{G}(\mathcal{K})^n$*. Therefore D_F descends to a function* $\mathrm{Conf}_n(\mathrm{Gr}) \rightarrow \mathbb{Z}$ *which we also denote by D_F.*

Proof The first property is clear since $F \in \mathbb{C}(\mathcal{A}^n)^{\mathrm{G}}$. The second property is by Lemma 2.22. □

Let $\mathbb{Q}_+(\mathrm{Conf}_n(\mathcal{A}))$ be the semifield of positive rational functions on $\mathrm{Conf}_n(\mathcal{A})$. Take a non-zero function $F \in \mathbb{Q}_+(\mathrm{Conf}_n(\mathcal{A}))$. Therefore it gives rise to a function D_F on $\mathrm{Conf}_n(\mathrm{Gr})$. Meanwhile, its tropicalization F^t is a function on $\mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$.

Theorem 2.24 *Let $l \in \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)$ and $F \in \mathbb{Q}_+(\mathrm{Conf}_n(\mathcal{A}))$. Then $D_F(\kappa(\mathcal{C}_l^\circ)) \equiv F^t(l)$.*

Theorem 2.24 is proved in Sect. 8. It implies that the map in Theorem 2.20 is injective. It can be reformulated as follows:

$$\text{For any } l \text{ and } F \text{ as above, the generic value of } D_F \text{ on the cycle } \mathcal{M}_l \text{ is } F^t(l). \tag{108}$$

When $\mathrm{G} = \mathrm{GL}_m$, one can describe the set $\mathbf{C}_{\underline{\lambda}}$ by using the special collection of functions on the space $\mathrm{Conf}_n(\mathcal{A})$ defined in Section 9 of [17]. The obtained description coincides with Kamnitzer's generalization of hives [46]. He conjectured in [46] that the latter set parametrizes the components of the convolution variety for GL_m. Therefore Theorems 2.20 and 2.24 imply Conjecture 4.3 in [46].

2.3 Mixed configurations and a generalization of Mirković–Vilonen cycles

In this section we discuss several other examples. Each of them fits in the general scheme of Sect. 1.2. We show how to encode all the data in a polygon.

2.3.1 Mixed configurations and the map κ

Definition 2.25 (i) Given a subset $\mathrm{I} \subset [1, n]$, the moduli space $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ parametrizes configurations $(x_1, \dots, x_n)$, where $x_i \in \mathcal{A}$ if $i \in \mathrm{I}$, otherwise $x_i \in \mathcal{B}$.

(ii) Given subsets $\mathrm{J} \subset \mathrm{I} \subset [1, n]$, the moduli space $\mathrm{Conf}_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$ parametrizes configurations $(x_1, \ldots, x_n)$ where

$$x_i \in \mathrm{Gr} \quad \text{if} \quad i \in \mathrm{J}, \quad x_i \in \mathcal{A}(\mathcal{K}) \quad \text{if} \quad i \in \mathrm{I} - \mathrm{J}, \quad x_i \in \mathcal{B}(\mathcal{K}) \quad \text{otherwise}.$$

We set $\mathrm{Conf}_{\mathrm{I}}(\mathrm{Gr}; \mathcal{B}) := \mathrm{Conf}_{\mathrm{I}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{B})$.

A positive structure on the space $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ is defined in Sect. 6.3. This positive structure is invariant under a cyclic twisted shift. See Lemma 6.8 for the precise statement.

Definition 2.26 Let $\mathrm{J} \subset \mathrm{I} \subset [1, n]$. A configuration in $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathcal{K})$ is called $\mathcal{O}$-integral relative to J if

1. For all $j \in \mathrm{J}$ and $k \neq j$, the pairs $(\mathrm{A}_j, \mathrm{B}_k)$ are generic. Here $\mathrm{B}_k = \pi(\mathrm{A}_k)$ if $k \in \mathrm{I}$.
2. The lattices $\mathrm{L}_j := \mathrm{L}(\mathrm{A}_j, \mathrm{B}_k)$ given by the above pairs only depend on j.

Denote by $\mathrm{Conf}^{\mathcal{O}}_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})$ the moduli space of such configurations.

By the very definition, there is a canonical map

$$\kappa : \mathrm{Conf}^{\mathcal{O}}_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B}) \longrightarrow \mathrm{Conf}_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B}). \tag{109}$$

It assigns to A_j the lattice L_j when $j \in \mathrm{J}$ and keeps the rest intact.
Recall $u_j \in \mathrm{U}_{\mathrm{A}_j}$ in (76). The potential $\mathcal{W}_{\mathrm{J}}$ on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ is a function

$$\mathcal{W}_{\mathrm{J}} := \sum_{j\in\mathrm{J}} \chi_{\mathrm{A}_j}(u_j). \tag{110}$$

Positivity of $\mathcal{W}_{\mathrm{J}}$ is proved in Sect. 6.4.
Next Theorem generalizes Theorem 2.17. Its proof is the same. See Sect. 6.4.

Theorem 2.27 *Let $l \in \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$. A configuration in $\mathcal{C}^\circ_l$ is $\mathcal{O}$-integral relative to* J *if and only if $\mathcal{W}^t_{\mathrm{J}}(l) \geq 0$.*

Denote by $\mathrm{Conf}^{+}_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$ the set of points $l \in \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$ such that $\mathcal{W}^t_{\mathrm{J}}(l) \geq 0$. Set

$$\mathcal{M}^\circ_l := \kappa(\mathcal{C}^\circ_l) \subset \mathrm{Conf}_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B}), \quad l \in \mathrm{Conf}^{+}_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t). \tag{111}$$

These cycles generalize the Mirković–Vilonen cycles, as we will see in Sect. 2.3.3.

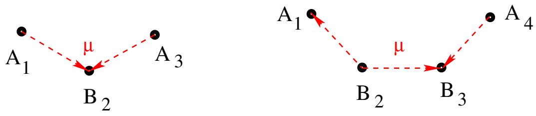


Fig. 18 The invariants $\mu(\mathrm{A}_1, \mathrm{B}_2, \mathrm{A}_3) \in \mathrm{H}$ and $\mu(\mathrm{A}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{A}_4) \in \mathrm{H}$

2.3.2 *Basic invariants*

Recall the isomorphism (80):

$$\alpha : \mathrm{Conf}^*(\mathcal{A}, \mathcal{A}) \xrightarrow{\sim} \mathrm{H}, \quad \alpha(\mathrm{A}_1 \cdot h_1, \mathrm{A}_2 \cdot h_2) = h_1^{-1} w_0(h_2)\alpha(\mathrm{A}_1, \mathrm{A}_2). \tag{112}$$

Given a generic triple $\{\mathrm{A}_1, \mathrm{B}_2, \mathrm{A}_3\}$, we choose a decorated flag A_2 over the flag B_2, and set

$$\mu(\mathrm{A}_1, \mathrm{B}_2, \mathrm{A}_3) := \alpha(\mathrm{A}_1, \mathrm{A}_2)\alpha(\mathrm{A}_3, \mathrm{A}_2)^{-1} \in \mathrm{H}.$$

Due to (112), it does not depend on the choice of A_2. We illustrate the invariant μ by a pair of red dashed arrows on the left in Fig. 18.

Given a generic configuration $(\mathrm{A}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{A}_4)$, see the right of Fig. 18, choose decorated flags $\mathrm{A}_2, \mathrm{A}_3$ over the flags $\mathrm{B}_2, \mathrm{B}_3$, and set

$$\mu(\mathrm{A}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{A}_4) := \alpha(\mathrm{A}_2, \mathrm{A}_1)\alpha_2(\mathrm{A}_2, \mathrm{A}_3)^{-1}\alpha(\mathrm{A}_4, \mathrm{A}_3) \in \mathrm{H}.$$

These invariants coincide with a similar H-valued μ-invariants from Sect. 1.4.

There are canonical isomorphisms:

$$\begin{aligned} \pi_{\mathrm{Gr}} &: \mathrm{Conf}(\mathrm{Gr}, \mathrm{Gr}) \xrightarrow{=} \mathrm{P}^+, \\ \alpha_{\mathrm{Gr}} &: \mathrm{Conf}(\mathcal{A}, \mathrm{Gr}) \xrightarrow{=} \mathrm{P}, \\ \alpha'_{\mathrm{Gr}} &: \mathrm{Conf}(\mathrm{Gr}, \mathcal{A}) \xrightarrow{=} \mathrm{P}. \end{aligned} \tag{113}$$

The first map uses the decomposition $\mathrm{G}(\mathcal{K}) = \mathrm{G}(\mathcal{O}) \cdot \mathrm{H}(\mathcal{K}) \cdot \mathrm{G}(\mathcal{O})$:

$$\mathrm{Conf}(\mathrm{Gr}, \mathrm{Gr}) = \mathrm{G}(\mathcal{O})\backslash\mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O}) = W\backslash\mathrm{H}(\mathcal{K})/\mathrm{H}(\mathcal{O}) = \mathrm{P}^+.$$

The second map uses the Iwasawa decomposition $\mathrm{G}(\mathcal{K}) = \mathrm{U}(\mathcal{K}) \cdot \mathrm{H}(\mathcal{K}) \cdot \mathrm{G}(\mathcal{O})$:

$$\begin{aligned} \mathrm{Conf}(\mathcal{A}, \mathrm{Gr}) &= \mathrm{G}(\mathcal{K})\backslash\, (\mathrm{G}(\mathcal{K})/\mathrm{U}(\mathcal{K}) \times \mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O})) = \mathrm{U}(\mathcal{K})\backslash\mathrm{G}(\mathcal{K})/\mathrm{G}(\mathcal{O}) \\ &= \mathrm{H}(\mathcal{K})/\mathrm{H}(\mathcal{O}) = \mathrm{P}. \end{aligned}$$

The third map is a cousin of the second one:

$$\alpha'_{\mathrm{Gr}}(\mathrm{L}, \mathrm{A}) := -w_0\left(\alpha_{\mathrm{Gr}}(\mathrm{A}, \mathrm{L})\right).$$

Remark. These isomorphisms parametrize $\mathrm{G}(\mathcal{O})$, $\mathrm{U}(\mathcal{K})$ and $\mathrm{U}^-(\mathcal{K})$-orbits of Gr. Each coweight $\lambda \in \mathrm{P} = \mathrm{H}(\mathbb{Z}^t) = \mathrm{H}(\mathcal{K})/\mathrm{H}(\mathcal{O})$ corresponds to an element t^λ of Gr. Then

$$\begin{aligned}
&\pi_{\mathrm{Gr}}([1], g \cdot t^\lambda) = \lambda, \quad \forall g \in \mathrm{G}(\mathcal{O});\\
&\alpha_{\mathrm{Gr}}(\mathrm{U}, u \cdot t^\lambda) = \lambda, \quad \forall u \in \mathrm{U}(\mathcal{K});\\
&\alpha'_{\mathrm{Gr}}(v \cdot t^{-\lambda}, \overline{w}_0 \cdot \mathrm{U}) = \lambda, \quad \forall v \in \mathrm{U}^-(\mathcal{K}).
\end{aligned} \tag{114}$$

We define Grassmannian versions of μ-invariants:

$$\mu_{\mathrm{Gr}} : \mathrm{Conf}(\mathrm{Gr}, \mathcal{B}, \mathrm{Gr}) \longrightarrow \mathrm{P}, \quad \mu_{\mathrm{Gr}} : \mathrm{Conf}(\mathrm{Gr}, \mathcal{B}, \mathcal{B}, \mathrm{Gr}) \longrightarrow \mathrm{P}$$

$$\mu_{\mathrm{Gr}}(\mathrm{L}_1, \mathrm{B}_2, \mathrm{L}_3) := \alpha'_{\mathrm{Gr}}(\mathrm{L}_1, \mathrm{A}_2) - \alpha'_{\mathrm{Gr}}(\mathrm{L}_3, \mathrm{A}_2) \in \mathrm{P}.$$

$$\mu_{\mathrm{Gr}}(\mathrm{L}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{L}_4) := \alpha_{\mathrm{Gr}}(\mathrm{A}_2, \mathrm{L}_1) - \mathrm{val} \circ \alpha(\mathrm{A}_2, \mathrm{A}_3) + \alpha'_{\mathrm{Gr}}(\mathrm{L}_4, \mathrm{A}_3) \in \mathrm{P}.$$

Let $\mathrm{pr} : \mathrm{B}^-(\mathcal{K}) \to \mathrm{H}(\mathcal{K}) \to \mathrm{P}$ be the composite of standard projections. The first map has an equivalent description:

$$\mu_{\mathrm{Gr}}([b_1], \mathrm{B}^-, [b_2]) = \mathrm{pr}(b_1^{-1} b_2), \quad b_1, b_2 \in \mathrm{B}^-(\mathcal{K}).$$

More generally, take a chain of flags starting and ending by a decorated flag, pick an alternating sequence of arrows, and write an alternating product of the α-invariants. We get regular maps

$$\mu : \mathrm{Conf}^*(\mathcal{A}, \mathcal{B}^{2n+1}, \mathcal{A}) \longrightarrow \mathrm{H}, \tag{115}$$

$$(\mathrm{A}_1, \mathrm{B}_2, \ldots, \mathrm{B}_{2n+2}, \mathrm{A}_{2n+3}) \longmapsto \frac{\alpha(\mathrm{A}_1, \mathrm{A}_2)}{\alpha(\mathrm{A}_3, \mathrm{A}_2)} \frac{\alpha(\mathrm{A}_3, \mathrm{A}_4)}{\alpha(\mathrm{A}_5, \mathrm{A}_4)} \cdots \frac{\alpha(\mathrm{A}_{2n+1}, \mathrm{A}_{2n+2})}{\alpha(\mathrm{A}_{2n+3}, \mathrm{A}_{2n+2})}.$$

$$\mu : \mathrm{Conf}^*(\mathcal{A}, \mathcal{B}^{2n}, \mathcal{A}) \longrightarrow \mathrm{H}, \tag{116}$$

$$(\mathrm{A}_1, \mathrm{B}_2, \ldots, \mathrm{B}_{2n+1}, \mathrm{A}_{2n+2}) \longmapsto \frac{\alpha(\mathrm{A}_2, \mathrm{A}_1)}{\alpha(\mathrm{A}_2, \mathrm{A}_3)} \frac{\alpha(\mathrm{A}_4, \mathrm{A}_3)}{\alpha(\mathrm{A}_4, \mathrm{A}_5)} \cdots \alpha(\mathrm{A}_{2n+2}, \mathrm{A}_{2n+1}).$$

Given a cyclic collection of an even number of flags, there is an invariant which for $n = 2$ and $\mathrm{G} = \mathrm{SL}_2$ recovers the cross-ratio:

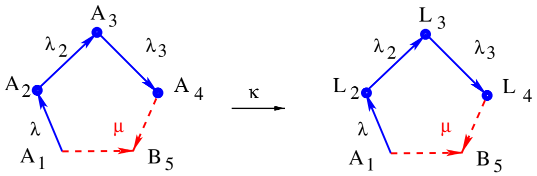


Fig. 19 Generalized MV cycles $\mathcal{M}_l \subset \mathrm{Gr}^3 = \mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^3, \mathcal{B})$

Fig. 20 One has $\lambda \in \mathrm{P}^+$ and $\mu \in \mathrm{R}^+$

$$\mathrm{Conf}^*_{2n}(\mathcal{B}) \longrightarrow \mathrm{H}, \quad (\mathrm{B}_1, \ldots, \mathrm{B}_{2n}) \longmapsto \frac{\alpha(\mathrm{A}_1, \mathrm{A}_2)}{\alpha(\mathrm{A}_3, \mathrm{A}_2)} \frac{\alpha(\mathrm{A}_3, \mathrm{A}_4)}{\alpha(\mathrm{A}_5, \mathrm{A}_4)} \ldots \frac{\alpha(\mathrm{A}_{2n-1}, \mathrm{A}_{2n})}{\alpha(\mathrm{A}_1, \mathrm{A}_{2n})}.$$

One gets Grassmannian versions by replacing $\mathcal{A}$ by Gr, and α by one of the maps (113).

These invariants provide decompositions for both spaces in (111).

Let us encode all the data in a polygon, as illustrated on Fig. 19. Let $l \in \mathrm{Conf}^+_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$. We show on the left an element of $\mathcal{C}^\circ_l$. Flags or decorated flags are assigned to the vertices of a convex polygon. The vertices labeled by J are boldface. Note that although we order the vertices by choosing a reference vertex, due to the twisted cyclic invariance the story does not depend on its choice.

The solid blue sides are labeled by a pair of decorated flags. There is an invariant $\lambda_E \in \mathrm{P}$ assigned to such a side E. It is provided by the tropicalization of the isomorphism (112) evaluated on l. The collection of dashed edges determines an invariant $\mu \in \mathrm{P}$.

Recall the cone $\mathrm{R}^+ \subset \mathrm{P}$ generated by positive coroots. The $\mathcal{O}$-integrality imposes restrictions on basic invariants, summarized in Lemma 2.28, and illustrated on Fig. 20.

Lemma 2.28 (i) *Let* $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) \in \mathcal{C}^\circ_l \subset \mathrm{Conf}^{\mathcal{O}}(\mathcal{A}, \mathcal{A}, \mathcal{B})$. *Then* $\mathrm{val} \circ \alpha(\mathrm{A}_1, \mathrm{A}_2) \in \mathrm{P}^+$.

(ii) *Let* $(\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) \in \mathcal{C}^\circ_l \subset \mathrm{Conf}^{\mathcal{O}}(\mathcal{B}, \mathcal{A}, \mathcal{B})$. *Then* $\mathrm{val} \circ \mu(\mathrm{A}_2, \mathrm{B}_1, \mathrm{B}_3, \mathrm{A}_2) \in \mathrm{R}^+$.

Proof Here (i) follows from Lemma 6.14, and (ii) follows from Lemmas 5.3 & 6.4(4). □

Applying the map κ, we replace the decorated flag at each boldface vertex by the corresponding lattice. Others remain intact. We use the notation $\underline{\mathcal{A}}$ for the decorated flags which do not contribute the character χ_{A} to the potential—they are assigned to the unmarked vertices. For example, we associate to the polygons on Fig. 19 the following maps

$$\begin{aligned}
&\kappa : \mathcal{C}_l^\circ \longrightarrow \mathrm{Conf}(\mathcal{A}, \mathrm{Gr}^3, \mathcal{B}), \quad l \in \mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}^3, \mathcal{B})(\mathbb{Z}^t).\\
&\pi : \mathrm{Conf}^*(\underline{\mathcal{A}}, \mathcal{A}^3, \mathcal{B}) \longrightarrow \mathrm{H}^3, \quad \mu : \mathrm{Conf}^*(\underline{\mathcal{A}}, \mathcal{A}^3, \mathcal{B}) \longrightarrow \mathrm{H},\\
&(\pi^t, \mu^t) : \mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}^3, \mathcal{B})(\mathbb{Z}^t) \longrightarrow \mathrm{P} \times (\mathrm{P}^+)^2 \times \mathrm{P},\\
&(\pi_{\mathrm{Gr}}, \mu_{\mathrm{Gr}}) : \mathrm{Conf}(\underline{\mathcal{A}}, \mathrm{Gr}^3, \mathcal{B}) \longrightarrow \mathrm{P} \times (\mathrm{P}^+)^2 \times \mathrm{P}.
\end{aligned} \tag{117}$$

It is easy to check that the targets of the invariants assigned to configurations of flags are the same as the targets of their Grassmannian counterparts.

2.3.3 Generalized Mirković–Vilonen cycles

Let us recall the standard definition of *Mirković–Vilonen cycles* following [3,45,65].

For $w \in W$, let $\mathrm{U}_w = w\mathrm{U}w^{-1}$. For $w \in W$ and $\mu \in \mathrm{P}$ define the *semi-infinite cells*

$$\mathrm{S}_w^\mu := \mathrm{U}_w(\mathcal{K})t^\mu. \tag{118}$$

Let $\lambda, \mu \in \mathrm{P}$. The closure $\overline{\mathrm{S}_e^\lambda \cap \mathrm{S}_{w_0}^\mu}$ is non-empty if and only if $\lambda - \mu \in \mathrm{R}^+$. In that case, it is also well known that $\overline{\mathrm{S}_e^\lambda \cap \mathrm{S}_{w_0}^\mu}$ has pure dimension $\mathrm{ht}(\lambda - \mu) := \langle \rho, \lambda - \mu \rangle$.

Definition 2.29 A component of $\overline{\mathrm{S}_e^\lambda \cap \mathrm{S}_{w_0}^\mu} \subset \mathrm{Gr}$ is called an *MV cycle* of coweight (λ, μ).

Since H normalizes U_w, for each $h \in \mathrm{H}(\mathcal{K})$ such that $[h] = t^\nu$, we have $h \cdot \mathrm{S}_w^\mu = \mathrm{S}_w^{\mu+\nu}$. Therefore if V is an MV cycle of coweight (λ, μ), then $h \cdot V$ is an MV cycle of coweight $(\lambda + \nu, \mu + \nu)$. The $\mathrm{H}(\mathcal{K})$-orbit of an MV cycle of coweight (λ, μ) is called a *stable MV cycle* of coweight $\lambda - \mu$.

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathrm{P}^+)^n$. Consider the convolution variety

$$\mathrm{Gr}_{\underline{\lambda}} = \{(\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n) \mid [1] \xrightarrow{\lambda_1} \mathrm{L}_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_n} \mathrm{L}_n\} \subset \mathrm{Gr}^n. \tag{119}$$

Let $\mathrm{pr}_n : \mathrm{Gr}^n \to \mathrm{Gr}$ be the projection onto the last factor. Set

$$\mathrm{Gr}_{\underline{\lambda}}^\mu := \mathrm{Gr}_{\underline{\lambda}} \cap \mathrm{pr}_n^{-1}\left(\mathrm{S}_{w_0}^\mu\right). \tag{120}$$

When n = 1, under the geometric Satake correspondence, the components of Gr_λ^μ give a basis (the MV basis) for the weight space $V_\lambda^{(\mu)}$, see [65, Corollary

7.4]. It is easy to see that they are precisely MV cycles of coweight (λ, μ) contained in $\overline{\mathrm{Gr}_{\lambda}}$, see [3, Proposition 3].

Now we restrict constructions in preceding subsections to four main examples associated to an (n + 2)-gon. The $n = 1$ case recovers the above three versions of MV cycles. In this sense, the following can be viewed as a generalization of MV cycles.

Example 1. $\mathrm{J} = [2, n+1] \subset \mathrm{I} = [1, n+1]$. Let $\mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B}) \subset \mathrm{Conf}_{\mathrm{J} \subset \mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$ be the substack parametrizing configurations $(\mathrm{A}_1, \mathrm{L}_2, \ldots, \mathrm{L}_{n+1}, \mathrm{B}_{n+2})$ where $(\mathrm{A}_1, \mathrm{B}_{n+2})$ is generic.

Recall $\mathcal{F}_{\mathrm{G}}$ in Definition 2.2. Then

$$\mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B}) = \mathrm{G}(\mathcal{K}) \backslash \left(\mathcal{F}_{\mathrm{G}}(\mathcal{K}) \times \mathrm{Gr}^n\right).$$

Since $\mathcal{F}_{\mathrm{G}}$ is a G-torsor, we get an isomorphism

$$i : \mathrm{Gr}^n \xrightarrow{=} \mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B}), \quad (\mathrm{L}_1, \ldots, \mathrm{L}_n) \longmapsto (\mathrm{U}, \mathrm{L}_1, \ldots, \mathrm{L}_n, \mathrm{B}^-). \tag{121}$$

From now on we identify Gr^n with $\mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B})$.

There is a map, whose construction is illustrated on the right of Fig. 19:

$$\pi_{\mathrm{Gr}} : \mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}^n, \mathcal{B}) \longrightarrow \mathbb{P} := \mathrm{P} \times (\mathrm{P}^+)^{n-1} \times \mathrm{P}.$$

Its fibers are finite dimensional subvarieties $\mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}}$:

$$\mathrm{Gr}^n = \coprod \mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}}, \quad \text{where} \quad (\lambda, \underline{\lambda}, \mu) \in \mathrm{P} \times (\mathrm{P}^+)^{n-1} \times \mathrm{P}. \tag{122}$$

By (114) we see that

$$\mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}} = \left\{ (\mathrm{L}_1, \ldots, \mathrm{L}_n) \in \mathrm{Gr}^n \mid \mathrm{L}_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_n} \mathrm{L}_n, \ \mathrm{L}_1 \in \mathrm{S}^{\lambda}_{e}, \ \mathrm{L}_n \in \mathrm{S}^{\mu}_{w_0} \right\},$$
$$\underline{\lambda} := (\lambda_2, \ldots, \lambda_n).$$

When $n = 1$, it is the intersection $\mathrm{S}^{\lambda}_{e} \cap \mathrm{S}^{\mu}_{w_0}$. Note that the very notion of MV cycles depends on the choice of the pair $\mathrm{H} \subset \mathrm{B}$. We transport the MV cycles to $\mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}, \mathcal{B})$ by the isomorphism (121). It is then independent of the pair chosen. In general we define

Definition 2.30 The irreducible components of $\overline{\mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}}}$ are called the *generalized Mirković–Vilonen cycles* of coweight $(\lambda, \underline{\lambda}, \mu)$.

Similarly the left of Fig. 19 provides a map

$$\pi^t : \mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t) \longrightarrow \mathrm{P} \times (\mathrm{P}^+)^{n-1} \times \mathrm{P}. \tag{123}$$

Let $\mathbf{P}^{\mu}_{\lambda;\underline{\lambda}} := \mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t)^{\mu}_{\lambda;\underline{\lambda}}$ be the fiber of map (123) over $(\lambda, \underline{\lambda}, \mu)$. Then

$$\mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t) = \coprod \mathbf{P}^{\mu}_{\lambda;\underline{\lambda}} \quad \text{where} \quad (\lambda, \underline{\lambda}, \mu) \in \mathrm{P} \times (\mathrm{P}^+)^{n-1} \times \mathrm{P}. \tag{124}$$

By definition $\pi^t \circ \mathrm{val}$ and $\pi_{\mathrm{Gr}} \circ \kappa$ deliver the same map from $\mathcal{C}_l^{\circ}$ to $\mathbb{P}$. Thus we arrive at

$$\mathcal{M}_l := \overline{\mathcal{M}_l^{\circ}} \subset \overline{\mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}}}, \quad l \in \mathbf{P}^{\mu}_{\lambda;\underline{\lambda}} := \mathrm{Conf}^+_{w_0}(\mathcal{A}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t)^{\mu}_{\lambda;\underline{\lambda}}. \tag{125}$$

Theorem 2.31 *The cycles* (125) *are precisely the generalized MV cycles of coweight* $(\lambda, \underline{\lambda}, \mu)$.

Example 2. $\mathrm{J} = \mathrm{I} = [2, n+1]$. Let $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B}) \subset \mathrm{Conf}_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$ be the substack parametrizing configurations $(\mathrm{B}_1, \mathrm{L}_2, \ldots, \mathrm{L}_{n+1}, \mathrm{B}_{n+2})$ where $(\mathrm{B}_1, \mathrm{B}_{n+2})$ is generic.

Similarly, we get an isomorphism of stacks

$$\begin{aligned} i_s : \mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}^n &\stackrel{=}{\longrightarrow} \mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B}), \\ (\mathrm{L}_1, \ldots, \mathrm{L}_n) &\longmapsto (\mathrm{B}, \mathrm{L}_1, \ldots, \mathrm{L}_n, \mathrm{B}^-). \end{aligned} \tag{126}$$

Here the group $\mathrm{H}(\mathcal{K})$ acts diagonally on Gr^n. Let $h \in \mathrm{H}(\mathcal{K})$. If $[h] = t^{\mu}$, then $h \cdot \overline{\mathrm{Gr}^{\nu}_{\lambda;\underline{\lambda}}} = \overline{\mathrm{Gr}^{\nu+\mu}_{\lambda+\mu;\underline{\lambda}}}$. It provides an isomorphism between the sets of components of both varieties.

Definition 2.32 The $\mathrm{H}(\mathcal{K})$-orbit of a generalized MV cycle of coweight $(\lambda, \underline{\lambda}, \nu)$ is called a *generalized stable MV cycle* of coweight $(\underline{\lambda}, \lambda - \nu)$.

When $n = 1$, it recovers the usual stable MV cycles. The generalized stable MV cycles live naturally on the stack $\mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}^n$. The isomorphism (126) transports them to $\mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B})$.

The solid blue arrows and the triple of dashed reds on Fig. 21 provide a canonical projection

$$(\pi^t, \mu^t) : \mathrm{Conf}^+(\mathcal{B}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t) \longrightarrow (\mathrm{P}^+)^{n-1} \times \mathrm{P}.$$

Let $\mathbf{A}^{\mu}_{\underline{\lambda}} := \mathrm{Conf}^+(\mathcal{B}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t)^{\mu}_{\underline{\lambda}}$ be its fiber over $(\underline{\lambda}, \mu)$. Then

$$\mathrm{Conf}^+(\underline{\mathcal{B}}, \mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t) = \coprod \mathbf{A}^{\mu}_{\underline{\lambda}} \quad \text{where} \quad \underline{\lambda} \in (\mathrm{P}^+)^{n-1}, \quad \mu \in \mathrm{P}. \tag{127}$$

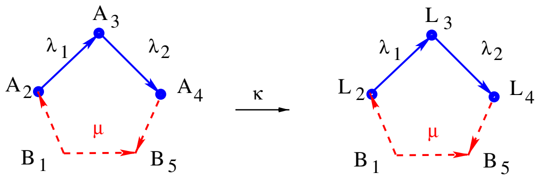


Fig. 21 Generalized stable MV cycles $\mathcal{M}_l \subset \mathrm{Conf}(\mathcal{B}, \mathrm{Gr}^3, \mathcal{B}) = \mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}^3$

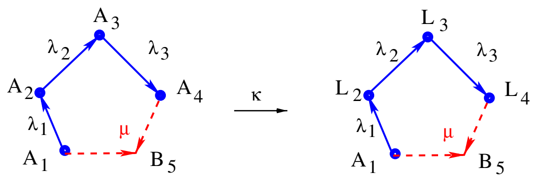


Fig. 22 Generalized MV cycles $\mathcal{M}_l \subset \mathrm{Conf}(\mathrm{Gr}^4, \mathcal{B}) = \mathrm{B}^-(\mathcal{O})\backslash\mathrm{Gr}^3$

On the other hand, our general construction provides us with the irreducible cycles

$$\mathcal{M}_l := \overline{\mathcal{M}_l^\circ} \subset \mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}^n = \mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B}), \quad l \in \mathbf{A}^{\mu}_{\underline{\lambda}}. \tag{128}$$

Theorem 2.33 *The cycles* (128) *are precisely the generalized stable MV cycles of coweight* $(\underline{\lambda}, \mu)$.

Example 3. $\mathrm{J} = \mathrm{I} = [1, n+1]$. By Iwasawa decomposition we get an isomorphism

$$\begin{aligned} i_b : \mathrm{B}^-(\mathcal{O})\backslash\mathrm{Gr}^n &\stackrel{=}{\longrightarrow} \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B}), \\ (\mathrm{L}_1, \ldots, \mathrm{L}_n) &\longmapsto ([1], \mathrm{L}_1, \ldots, \mathrm{L}_n, \mathrm{B}^-). \end{aligned} \tag{129}$$

There are two projections, illustrated on Fig. 22:

$$(\pi_{\mathrm{Gr}}, \mu_{\mathrm{Gr}}) : \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B}) \longrightarrow (\mathrm{P}^+)^n \times \mathrm{P}, \tag{130}$$

$$(\pi^t, \mu^t) : \mathrm{Conf}^+(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t) \longrightarrow (\mathrm{P}^+)^n \times \mathrm{P}. \tag{131}$$

Their fibers over $(\underline{\lambda}, \mu) \in (\mathrm{P}^+)^n \times \mathrm{P}$ provide decompositions

$$\mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B}) = \coprod_{\underline{\lambda}, \mu} \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B})^{\mu}_{\underline{\lambda}}. \tag{132}$$

$$\mathrm{Conf}^+(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t) = \coprod_{\underline{\lambda}, \mu} \mathrm{Conf}^+(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t)^{\mu}_{\underline{\lambda}}. \tag{133}$$

By definition, these decompositions are compatible under the map κ.

We get irreducible cycles

$$\mathcal{M}_l := \overline{\mathcal{M}_l^\circ} \subset \mathrm{B}^-(\mathcal{O})\backslash \mathrm{Gr}^n = \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B}),$$
$$l \in \mathbf{B}_{\underline{\lambda}}^{\mu} := \mathrm{Conf}^+(\mathcal{A}^{n+1}, \mathcal{B})(\mathbb{Z}^t)_{\underline{\lambda}}^{\mu}. \qquad (134)$$

The connected group $\mathrm{B}^-(\mathcal{O})$ acts diagonally Gr^n. It preserves components of subvarieties $\overline{\mathrm{Gr}_{\underline{\lambda}}^{\mu}}$ in (120). Hence these components live naturally on the stack $\mathrm{B}^-(\mathcal{O})\backslash \mathrm{Gr}^n$. We transport them to $\mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B})$ by (129).

Theorem 2.34 *The cycles* (134) *are precisely the components of* $\mathrm{B}^-(\mathcal{O})\backslash\overline{\mathrm{Gr}_{\underline{\lambda}}^{\mu}}$.

Example 4. $\mathrm{J} = \mathrm{I} = [1, n+2]$. There is an isomorphism

$$i_g : \mathrm{G}(\mathcal{O})\backslash \mathrm{Gr}^{n+1} \stackrel{=}{\longrightarrow} \mathrm{Conf}_{n+2}(\mathrm{Gr}),$$
$$(\mathrm{L}_1, \ldots, \mathrm{L}_{n+1}) \longrightarrow ([1], \mathrm{L}_1, \ldots, \mathrm{L}_{n+1}). \qquad (135)$$

We arrive at irreducible cycles defined in Definition 2.18:

$$\mathcal{M}_l := \overline{\mathcal{M}_l^\circ} \subset \mathrm{G}(\mathcal{O})\backslash \mathrm{Gr}^{n+1} = \mathrm{Conf}_{n+2}(\mathrm{Gr}) \quad l \in \mathbf{C}_{\underline{\lambda}} := \mathrm{Conf}_n^+(\mathcal{A})(\mathbb{Z}^t)_{\underline{\lambda}}.$$

This example recovers Theorem 2.20.

Specializing Theorems 2.31–2.34 to $n = 1$, we get

Theorem 2.35 *(1) Mirković–Vilonen cycles of coweight* (λ, μ) *are precisely the cycles*

$$\mathcal{M}_l \subset \mathrm{Gr}, \quad l \in \mathbf{P}_\lambda^\mu := \mathrm{Conf}^+(\underline{\mathcal{A}}, \mathcal{A}, \mathcal{B})(\mathbb{Z}^t)_\lambda^\mu \quad \textit{for} \quad \mathcal{W} = \chi_{\mathrm{A}_2}.$$

(2) Stable Mirković–Vilonen cycles of coweight μ *are precisely the cycles*

$$\mathcal{M}_l \subset \mathrm{H}(\mathcal{K})\backslash \mathrm{Gr}, \quad l \in \mathbf{A}_\mu := \mathrm{Conf}^+(\mathcal{B}, \mathcal{A}, \mathcal{B})(\mathbb{Z}^t)^\mu \quad \textit{for} \quad \mathcal{W} = \chi_{\mathrm{A}_2}.$$

(3) Mirković–Vilonen cycles of coweight (λ, μ) *which lie in* $\overline{\mathrm{Gr}}_\lambda \subset \mathrm{Gr}$ *are precisely the cycles*

$$\mathcal{M}_l \subset \mathrm{B}^-(\mathcal{O})\backslash \mathrm{Gr}, \quad l \in \mathbf{B}_\lambda^\mu := \mathrm{Conf}^+(\mathcal{A}, \mathcal{A}, \mathcal{B})(\mathbb{Z}^t)_\lambda^\mu$$
$$\textit{for} \quad \mathcal{W} = \chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}$$

Theorem 2.35 is proved in Sect. 9.1.

Note that there is a positive birational isomorphism $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) \stackrel{\sim}{=} \mathrm{U}$. Thus we identify $\mathrm{Conf}^+(\mathcal{B}, \mathcal{A}, \mathcal{B})(\mathbb{Z}^t)$ with the subset of $\mathrm{U}(\mathbb{Z}^t)$ used by Lusztig [58,59] to parametrize the canonical basis in Lemma 5.1. Then Theorem 2.35 is equivalent to the main results of Kamnitzer's paper [45]. Our

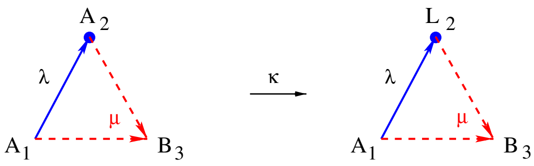


Fig. 23 Mirković–Vilonen cycles $\mathcal{M}_l \subset \mathrm{Conf}_{w_0}(\mathcal{A}, \mathrm{Gr}, \mathcal{B}) = \mathrm{Gr}$

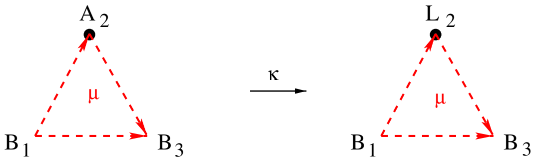


Fig. 24 Stable Mirković–Vilonen cycles $\mathcal{M}_l \subset \mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}, \mathcal{B}) = \mathrm{H}(\mathcal{K})\backslash\mathrm{Gr}$

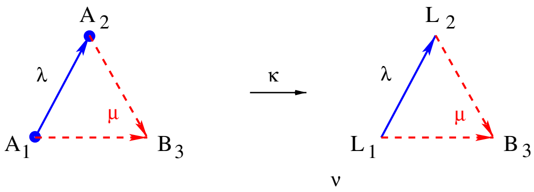


Fig. 25 MV cycles which lie in Gr_λ are the cycles $\mathcal{M}_l^\circ \subset \mathrm{Conf}(\mathrm{Gr}, \mathrm{Gr}, \mathcal{B})_\lambda = \mathrm{B}^-(\mathcal{O})\backslash\mathrm{Gr}_\lambda$

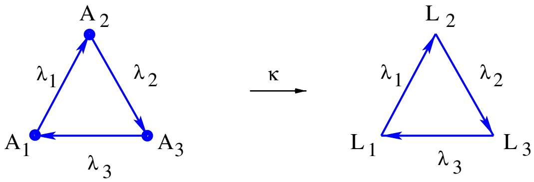


Fig. 26 Generalized MV cycles $\mathcal{M}_l \subset \mathrm{Conf}(\mathrm{Gr}, \mathrm{Gr}, \mathrm{Gr})$

approach, using the moduli space $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B})$ rather than U, makes parametrization of the MV cycles more natural and transparent, and puts it into the general framework of this paper.

To summarize, there are four different versions of the cycles relevant to representation theory related to mixed configurations of triples, as illustrate on Figs. 23, 24, 25, 26.

2.3.4 *Constructible equations for the cycles* $\mathcal{M}_l^\circ$

Let F be a rational function on the stack $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$. We generalize the construction of D_F from Sect. 2.2.5. As an application, it implies that the cycles $\mathcal{M}_l^\circ$ in (111) are disjoint.

Given $\mathrm{J} \subset \mathrm{I} \subset [1, n]$, let m be the cardinality of J. We assume $\mathrm{J} = \{j_1, \ldots, j_m\}$.

Consider the space

$$\mathfrak{X} := X_1 \times \ldots \times X_n, \quad \text{where } X_i = \begin{cases} \mathrm{G} & \text{if } i \in \mathrm{J}, \\ \mathcal{A} & \text{if } i \in \mathrm{I} - \mathrm{J}, \\ \mathcal{B} & \text{otherwise.} \end{cases}$$

Let $\mathfrak{X}_*$ be its subset consisting of collections $\{x_1, \ldots, x_n\}$ whose subcollections $\{x_{i_1}, \ldots, x_{i_{n-m}}\}$, $i_s \notin \mathrm{J}$, are generic.

Given a rational function F on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$, each $x = \{x_1, \ldots, x_n\} \in \mathfrak{X}_*(\mathcal{K})$ provides a function F_x on $\mathcal{A}^m$, whose value on $\{\mathrm{A}_{j_1}, \ldots, \mathrm{A}_{j_m}\} \in \mathcal{A}^m$ is

$$F_x(\mathrm{A}_{j_1}, \ldots, \mathrm{A}_{j_m}) := F(x'_1, \ldots, x'_n) \in \mathcal{K}, \quad x'_i = \begin{cases} x_j \cdot \mathrm{A}_j & \text{if } j \in \mathrm{J}, \\ x_i & \text{otherwise.} \end{cases} \tag{136}$$

Then $F_x \in \mathcal{K}(\mathcal{A}^m)$

Recall the map $\mathrm{val} : \mathcal{K}(\mathcal{A}^m)^\times \to \mathbb{Z}$. We get a $\mathbb{Z}$-valued function

$$D_F : \mathfrak{X}_*(\mathcal{K}) \longrightarrow \mathbb{Z}, \quad D_F(x) := \mathrm{val}(F_x). \tag{137}$$

Recall the right action of G^m on $\mathbb{C}(\mathcal{A}^m)$. Thanks to Lemma 2.22 and the fact that $F \in \mathbb{Q}(\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}))$, we have

$$\forall g \in \mathrm{G}(\mathcal{K}),\ \forall h \in \mathrm{G}(\mathcal{O})^m, \quad \mathrm{val}(F_{g \cdot x} \circ h) = \mathrm{val}(F_x). \tag{138}$$

Thus D_F descends to

$$D_F : \mathrm{Conf}^*_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B}) \longrightarrow \mathbb{Z}. \tag{139}$$

Here $\mathrm{Conf}^*_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$ is a subspace of $\mathrm{Conf}_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$ consisting of the configurations whose subconfigurations of flags and decorated flags are generic.

By definition, $\mathcal{M}_l^\circ$ in (111) are contained in $\mathrm{Conf}^*_{\mathrm{J}\subset\mathrm{I}}(\mathrm{Gr}; \mathcal{A}, \mathcal{B})$. The following Theorem is a generalization of Theorem 2.24. See Sect. 8 for its proof.

Theorem 2.36 *Let* $l \in \mathrm{Conf}^+_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$. *Let* $F \in \mathbb{Q}_+(\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}))$. *Then* $D_F(\mathcal{M}_l^\circ) \equiv F^t(l)$.

2.4 Canonical bases in tensor products and $\mathrm{Conf}(\mathcal{A}^n, \mathcal{B})$

Recall that a collection of dominant coweights $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ gives rise to a convolution variety $\mathrm{Gr}_{\underline{\lambda}} \subset \mathrm{Gr}^n$. It is open and smooth. Its dimension is calculated inductively:

$$\dim \mathrm{Gr}_{\underline{\lambda}} = 2\mathrm{ht}(\underline{\lambda}) := 2\langle \rho, \lambda_1 + \cdots + \lambda_n \rangle. \tag{140}$$

The subvarieties $\mathrm{Gr}_{\underline{\lambda}}$ form a stratification $\mathcal{S}$ of Gr^n. Let $\mathrm{IC}_{\underline{\lambda}}$ be the IC-sheaf of $\overline{\mathrm{Gr}_{\underline{\lambda}}}$. By the geometric Satake correspondence,

$$\mathrm{H}^*(\mathrm{IC}_{\underline{\lambda}}) = V_{\underline{\lambda}} := V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}. \tag{141}$$

Let $\mathrm{pr}_n : \mathrm{Gr}^n \to \mathrm{Gr}$ be the projection onto the last factor. Recall the point $t^\mu \in \mathrm{Gr}$. Set

$$\mathrm{S}_\mu := \mathrm{pr}_n^{-1}(\mathrm{U}(\mathcal{K})t^\mu) \subset \mathrm{Gr}^n, \quad \mathrm{T}_\mu := \mathrm{pr}_n^{-1}(\mathrm{U}^-(\mathcal{K})t^\mu) \subset \mathrm{Gr}^n.$$

The sum of positive coroots is a cocharacter $2\rho^\vee : \mathbb{G}_m \to \mathrm{H}$. It provides an action of the group $\mathbb{G}_m$ on Gr^n given by the action on the last factor. The subvarieties S_μ and T_μ are attracting and repulsing subvarieties for this action. Set

$$\mathrm{Gr}_{\underline{\lambda}}^\mu := \mathrm{Gr}_{\underline{\lambda}} \cap \mathrm{S}_\mu.$$

Lemma 2.37 *If* $\mathrm{Gr}_{\underline{\lambda}}^\mu$ *is non-empty, then it is a subvariety of pure dimension*

$$\dim \mathrm{Gr}_{\underline{\lambda}}^\mu = \mathrm{ht}(\underline{\lambda}; \mu) := \langle \rho, \lambda_1 + \ldots + \lambda_n + \mu \rangle. \tag{142}$$

Denote by $\mathrm{Irr}(X)$ the set of top dimensional components of a variety X, and by $\mathbb{Q}[\mathrm{Irr}(X)]$ the vector space with the bases parametrised by the set $\mathrm{Irr}(X)$.

Theorem 2.38 *There are canonical isomorphisms*

$$\mathrm{H}^*(\mathrm{Gr}^n, \mathrm{IC}_{\underline{\lambda}}) = \oplus_\mu \mathrm{H}_c^{2\mathrm{ht}(\mu)}(\mathrm{S}_\mu, \mathrm{IC}_{\underline{\lambda}}) = \oplus_\mu \mathbb{Q}[\mathrm{Irr}(\overline{\mathrm{Gr}_{\underline{\lambda}}^\mu})].$$

Proof Theorem 2.38 for $n = 1$ is proved in [65, Section 3]. The proof for arbitrary n follows the same line. For convenience of the reader we provide a complete proof.

Let $m : \mathbb{C}^* \times X \to X$ be a map defining an action of the group $\mathbb{C}^*$ on X. Let $\mathcal{D}(X)$ be the bounded derived category of constructible sheaves on X. An object $\mathcal{F} \in \mathcal{D}(X)$ is *weakly* $\mathbb{C}^*$*-equivariant*, if $m^*\mathcal{F} = L \boxtimes \mathcal{F}$ for some locally constant sheaf L on $\mathbb{C}^*$.

Recall the action of G_m on Gr^n defined above. Denote by $\mathrm{P}_\mathcal{S}(\mathrm{Gr}^n)$ the category of weakly $\mathbb{C}^*$-equivariant perverse sheaves on Gr^n which are constructible with respect to the stratification $\mathcal{S}$. □

Lemma 2.39 *The sheaf* $\mathrm{IC}_{\underline{\lambda}}$ *is locally constant along the stratification* $\mathcal{S}$*. It belongs to the category* $\mathrm{P}_\mathcal{S}(\mathrm{Gr}^n)$.

Proof Given a subgroup $\mathrm{G}' \subset \mathrm{G}$, denote by $\mathrm{G}'_{[k,n]} \subset \mathrm{G}^n$ the subgroup of elements $(e, \ldots, e, g, \ldots, g)$, with $(n-k+1)$ of $g \in \mathrm{G}'$. Denote by $\mathrm{G}(L)$ the subgroup stabilising a point $L \in \mathrm{Gr}$. The group $\mathrm{G}(L)_{[k,n]}$ preserves the category $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$. Take two collections $(L_1, \ldots, L_n), (M_1, \ldots, M_n) \in \mathrm{Gr}^n$, with $L_1 = M_1 = [1]$ and in the same stratum. We can move $(L_1, \ldots, L_n)$ by an element of $\mathrm{G}(L_1)_{[1,n]}$, getting $(M_1, M_2, L'_3, \ldots, L'_n)$. Then we move it by an element of $\mathrm{G}(M_2)_{[2,n]}$, getting $(M_1, M_2, M_3, \ldots, L''_n)$, and so on, using subgroups $\mathrm{G}(L_n)_{[k,n]}$ for $k = 3, 4, \ldots n-1$. In the last step we get $(M_1, \ldots, M_n)$. The $\mathbb{C}^*$-equivariance is evident. □

Proposition 2.40 *For all $\mathcal{P} \in \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$ we have a canonical isomorphism*

$$\mathrm{H}^k_c(\mathrm{S}_\mu, \mathcal{P}) \stackrel{\sim}{\longrightarrow} \mathrm{H}^k_{\mathrm{T}_\mu}(\mathrm{Gr}^n, \mathcal{P}). \tag{143}$$

Both sides vanish if $k \neq 2\mathrm{ht}(\mu)$. The functors $F_\mu := \mathrm{H}_c^{2\mathrm{ht}(\mu)}(S_\mu, -) : \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n) \longrightarrow \mathrm{Vect}$ are exact.

Proof Isomorphism (143) follows from the hyperbolic localisation theorem of Braden [7]. Let us briefly recall how it works.

Let X be a normal complex variety on which the group $\mathbb{C}^*$ acts. Let F be the stable points variety. It is a union of components $F_1, \ldots, F_k$. Consider the attracting and repulsing subvarieties

$$X^+_k = \{x \in X \mid \lim_{t\to 0} t \cdot x \in F_k\}, \quad X^-_k = \{x \in X \mid \lim_{t\to\infty} t \cdot x \in F_k\},$$

Let X^+ (resp. X^-) be the disjoint union of all the X^+_k (resp. X^-_k). There are projections

$$\pi^\pm : X^\pm \to F, \quad \pi^+(x) = \lim_{t\to 0} t \cdot x, \quad \pi^-(x) = \lim_{t\to\infty} t \cdot x.$$

Let $g^\pm : X^\pm \hookrightarrow X$ be the natural inclusions. Given an object $\mathcal{F} \in \mathcal{D}(X)$, define hyperbolic localisation functors

$$\mathcal{F}^{!*} := (\pi^+)_!(g^+)^*\mathcal{F}, \quad \mathcal{F}^{*!} := (\pi^-)_*(g^-)^!\mathcal{F}.$$

Combining Theorem 1 and Section 3 of [7], we have the following result, which implies (143). □

Proposition 2.41 *If $\mathcal{F}$ is weakly $\mathbb{C}^*$-equivariant, the natural map $\mathcal{F}^{!*} \to \mathcal{F}^{*!}$ is an isomorphism.*

Let us prove the vanishing. One has $\mathrm{H}^k_c(\mathrm{Gr}^\mu_{\underline{\lambda}}, \mathbb{Q}) = 0$ for $k > 2\mathrm{dimGr}^\mu_{\underline{\lambda}} = 2\mathrm{ht}(\underline{\lambda}; \mu)$. Due to perversity, the restriction of any $\mathcal{P} \in \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$ to $\mathrm{Gr}_{\underline{\lambda}}$ lies in degrees $\leq -\mathrm{dimGr}_{\underline{\lambda}} = -2\mathrm{ht}(\underline{\lambda})$. So

$$\mathrm{H}_c^k(\mathrm{Gr}_{\underline{\lambda}}^{\mu}, \mathcal{P}) = 0 \quad \text{if } k > 2\mathrm{ht}(\mu). \tag{144}$$

Although S_μ is infinite dimensional, we can slice it by its intersections with the strata $\mathrm{Gr}_{\underline{\lambda}}$. Since the estimate (144) on each strata does not depend on $\underline{\lambda}$, a devissage using exact triangles $j_! j^* \mathcal{A} \to \mathcal{A} \to i_! i^* \mathcal{A}$ tells that

$$\mathrm{H}_c^k(\mathrm{S}_\mu, \mathcal{P}) = 0 \quad \text{if } k > 2\mathrm{ht}(\mu).$$

Applying the duality, and using the fact that $*\mathcal{P} = \mathcal{P}$, we get the dual estimate

$$\mathrm{H}_{\mathrm{T}_\mu}^k(\mathrm{Gr}^n, \mathcal{P}) = 0 \quad \text{if } k < 2\mathrm{ht}(\mu).$$

Combining with the isomorphism (143), we get the proof. The last claim is then obvious.

Proposition 2.42 *We have natural equivalence of functors*

$$\mathrm{H}^* \stackrel{\sim}{=} \oplus_{\mu \in \mathrm{P}} \mathrm{H}_c^{2\mathrm{ht}(\mu)}(S_\mu, -) : \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n) \longrightarrow \mathrm{Vect}.$$

Proof The proof of Theorem 3.6 in [65] works in our case. Namely, the two filtrations of Gr^n by the closures of S_μ and T_μ give rise to two filtrations of H^*, given by the kernels of $\mathrm{H}^* \to \mathrm{H}_c^*(\overline{\mathrm{S}}_\mu, -)$ and the images of $\mathrm{H}^*_{\overline{\mathrm{T}}_\mu}(\mathrm{Gr}^n, -) \to \mathrm{H}^*$. The vanishing implies $\mathrm{H}_c^{2\mathrm{ht}(\mu)}(\overline{\mathrm{S}}_\mu, -) = \mathrm{H}_c^{2\mathrm{ht}(\mu)}(\mathrm{S}_\mu, -)$ and $\mathrm{H}_{\overline{\mathrm{T}}_\mu}^{2\mathrm{ht}(\mu)}(\mathrm{Gr}^n, -) = \mathrm{H}_{\mathrm{T}_\mu}^{2\mathrm{ht}(\mu)}(\mathrm{Gr}^n, -)$, and the composition $\mathrm{H}_{\mathrm{T}_\mu}^{2\mathrm{ht}(\mu)}(\mathrm{Gr}^n, -) \to \mathrm{H}^{2\mathrm{ht}(\mu)} \to \mathrm{H}_c^{2\mathrm{ht}(\mu)}(\mathrm{S}_\mu, -)$ is an isomorphism. So the two filtrations split each other. □

Corollary 2.43 *The global cohomology functor* $\mathrm{H}^* : \mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n) \longrightarrow \mathrm{Vect}$ *is faithful and exact.*

Denote by $\mathrm{H}_{\mathrm{per}}^p \mathcal{F}$ the cohomology of an $\mathcal{F} \in D_{\mathcal{S}}^b(\mathrm{Gr}^n)$ for the perverse t-structure. Let $j : \mathrm{Gr}_{\underline{\lambda}} \hookrightarrow \overline{\mathrm{Gr}_{\underline{\lambda}}}$ be the natural embedding, $\mathcal{J}_!(\underline{\lambda}, \mathbb{Q}) := \mathrm{H}_{\mathrm{per}}^0(j_! \mathbb{Q}[\dim \mathrm{Gr}_{\underline{\lambda}}])$, and $\mathcal{J}_*(\underline{\lambda}, \mathbb{Q}) := \mathrm{H}_{\mathrm{per}}^0(j_* \mathbb{Q}[\dim \mathrm{Gr}_{\underline{\lambda}}])$. The following Lemma is a generalisation of Lemma 7.1 of [65].

Lemma 2.44 *The category* $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$ *is semi-simple. The sheaves* $\mathcal{J}_!(\underline{\lambda}, \mathbb{Q})$, $\mathcal{J}_*(\underline{\lambda}, \mathbb{Q})$, *and* $\mathcal{J}_{!*}(\underline{\lambda}, \mathbb{Q})$ *are isomorphic.*

Proof Let us prove first the parity vanishing for the stalks of the sheaf $\mathcal{J}_{!*}(\underline{\lambda}, \mathbb{Q})$: the stalks could have non-zero cohomology only at even degrees. For $n = 1$ it is proved in [62]. It can also be proved by using the Bott–Samelson resolution of the Schubert cells in the affine (i.e. Kac-Moody) case, as was explained to us by A. Braverman. Let $\mathcal{F}$ be a Kac-Moody flag variety. Take an element $w = w_1 \dots w_n$ of the affine Weyl group such that $l(w) = l(w_1) + \cdots + l(w_n)$. Denote by $\mathcal{F}_{w_1,\dots,w_n}$ the variety parametrising flags $(F_1 = [1], F_2, \dots, F_n)$ such that the pair (F_i, F_{i+1}) is in the incidence relation w_i. Choose reduced decompositions $[w_1], \dots, [w_n]$ of the elements $w_1, \dots, w_n$. Their product is a reduced decomposition $[w]$ of w. It gives rise to the Bott-Samelson variety $X_{[w]}$. By its very definition, it is a tower of fibrations

$$X([w_1], \dots, [w_n]) \longrightarrow X([w_1], \dots, [w_{n-1}]) \longrightarrow \dots \longrightarrow X([w_1]).$$

The Bott-Samelson resolution of the affine Schubert cell Gr_λ is a smooth projective variety X_λ with a map $\beta_\lambda : X_\lambda \to \mathrm{Gr}_\lambda$ which is $1:1$ at the open stratum, and which, according to [25,26], has the following property. For each of the strata $\mathrm{Gr}_\mu \subset \mathrm{Gr}_\lambda$, there exists a point $p_\mu \in \mathrm{Gr}_\mu$ such that the fiber $\beta_\lambda^{-1}(p_\mu)$ of the Bott-Samelson resolution has a cellular decomposition with the cells being complex vector spaces. Therefore the stalk of the push forward $\beta_{\lambda *}\mathbb{Q}_{X_\lambda}$ of the constant sheaf on X_λ at the point p_μ satisfies the parity vanishing. By the decomposition theorem [8], the sheaf IC_λ is a direct summand of the push forward $\beta_{\lambda *}\mathbb{Q}_{X_\lambda}$ of the constant sheaf on X_λ. Indeed, the latter is a direct sum of shifts of perverse sheaves, and it is the constant sheaf over the open stratum. Therefore the stalk of the sheaf IC_λ at the point p_μ satisfies the parity vanishing. Since the cohomology of IC_λ is locally constant over each of the stratum Gr_μ, we get the parity vanishing. The general case of $\mathrm{Gr}_{\underline{\lambda}}$ is treated very similarly to the case of Gr_λ.

The rest is pretty standard, and goes as follows. The strata $\mathrm{Gr}_{\underline{\lambda}}$ are simply connected: this is well known for $n = 1$, and the strata $\mathrm{Gr}_{\underline{\lambda}}$ is fibered over $\mathrm{Gr}_{\underline{\lambda'}}$ with the fiber Gr_{λ_n}, where $\underline{\lambda} = (\underline{\lambda'}, \lambda_n)$. Since the strata are even dimensional over $\mathbb{R}$, this plus the parity vanishing implies that there are no extensions between the simple objects in $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$. Indeed, by devissage this claim reduces to calculation of extensions between constant sheaves concentrated on two open strata. Thus there are no extensions in the category $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$, i.e. it is semi-simple.

Let us show now that $\mathcal{J}_!(\underline{\lambda}, \mathbb{Q}) = \mathcal{J}_{!*}(\underline{\lambda}, \mathbb{Q})$. Since $\mathrm{H}^p_{\mathrm{per}}(j_!\mathbb{Q}_{\mathrm{Gr}_{\underline{\lambda}}}) = 0$ for $p > 0$, there is a map $j_!\mathbb{Q}_{\mathrm{Gr}_{\underline{\lambda}}} \to \mathrm{H}^0_{\mathrm{per}}(j_!\mathbb{Q}_{\mathrm{Gr}_{\underline{\lambda}}}) = \mathcal{J}_!(\underline{\lambda}, \mathbb{Q})$. If $\mathcal{J}_!(\underline{\lambda}, \mathbb{Q}) \neq \mathcal{J}_{!*}(\underline{\lambda}, \mathbb{Q})$, since the category $\mathrm{P}_{\mathcal{S}}(\mathrm{Gr}^n)$ is semisimple, there is a non-zero direct summand $\mathcal{B}$ of $\mathcal{J}_!(\underline{\lambda}, \mathbb{Q})$ supported at a lower stratum. Composing these two maps, we get a non-zero map $j_!\mathbb{Q}_{\mathrm{Gr}_{\underline{\lambda}}} \to \mathcal{B}$. On the other hand, given a space X and complexes of sheaves $\mathcal{A}$ and $\mathcal{B}$ supported at disjoint subsets A and B respectively, one has $\mathrm{Hom}(j_!\mathcal{A}, \mathcal{B}) = 0$, where $j : A \hookrightarrow X$. Contradiction. The statement about $\mathcal{J}_*$ follows by the duality. $\square$

Lemma 2.45 *There are canonical isomorphisms*

$$F_\mu[\mathcal{J}_!(\underline{\lambda}, \mathbb{Q})] = \mathbb{Q}[\mathrm{Irr}(\overline{\mathrm{Gr}^\mu_{\underline{\lambda}}})] = F_\mu[\mathcal{J}_*(\underline{\lambda}, \mathbb{Q})].$$

Proof We prove the first claim. The second is similar. We follow closely the proof of Proposition 3.10 in [65]. Set $\mathcal{F} := \mathcal{J}_!(\underline{\lambda}, \mathbb{Q})$. Let $\mathrm{Gr}_{\underline{\eta}}$ be a stratum in the closure of $\mathrm{Gr}_{\underline{\lambda}}$. Let $i_{\underline{\eta}} : \mathrm{Gr}_{\underline{\eta}} \hookrightarrow \overline{\mathrm{Gr}_{\underline{\lambda}}}$ be the natural embedding. Then $i^*_{\underline{\eta}}\mathcal{F} \in D^{\leq -\mathrm{dim}\mathrm{Gr}_{\underline{\eta}}-2}(\mathrm{Gr}_{\underline{\eta}})$. Indeed, we use $i^*_\eta j_!\mathbb{Q} = 0$, and $\mathrm{H}^p_{\mathrm{per}} j_!\mathbb{Q}[\mathrm{dim}\mathrm{Gr}_{\underline{\lambda}}] = 0$ for $p > 0$ and apply $i^*_{\underline{\eta}}$ to the exact triangle

$$\longrightarrow \tau^{\leq -1}_{\mathrm{per}}(j_!\mathbb{Q}[\mathrm{dim}\mathrm{Gr}_{\underline{\lambda}}]) \longrightarrow j_!\mathbb{Q}[\mathrm{dim}\mathrm{Gr}_{\underline{\lambda}}] \longrightarrow \mathrm{H}^0_{\mathrm{per}}(j_!\mathbb{Q}[\mathrm{dim}\mathrm{Gr}_{\underline{\lambda}}]) \longrightarrow \ldots.$$

Due to dimension counts (140) and (142), we have $\mathrm{H}^k_c(\mathrm{Gr}_{\underline{\eta}} \cap \mathrm{S}_\mu, \mathcal{F}) = 0$ if $k > 2\mathrm{ht}(\mu) - 2$. Thus the devissage associated to the filtration of Gr^n by $\mathrm{Gr}_{\underline{\eta}}$ tells that there is no contribution from the lower strata $\mathrm{Gr}_{\underline{\eta}}$ to $\mathrm{H}^{2\mathrm{ht}(\mu)}_c$, i.e. $\mathrm{H}^{2\mathrm{ht}(\mu)}_c(\mathrm{S}_\mu, \mathcal{F}) = \mathrm{H}^{2\mathrm{ht}(\mu)}_c(\mathrm{Gr}_{\underline{\lambda}} \cap \mathrm{S}_\mu, \mathcal{F})$. Now we can conclude:

$$\mathrm{H}^{2\mathrm{ht}(\mu)}_c(\mathrm{Gr}^\mu_{\underline{\lambda}}, \mathcal{F}) = \mathrm{H}^{2\mathrm{ht}(\mu)+2\mathrm{ht}(\underline{\lambda})}_c(\mathrm{Gr}^\mu_{\underline{\lambda}}, \mathbb{Q}) = \mathrm{H}^{2\mathrm{dim}(\mathrm{Gr}^\mu_{\underline{\lambda}})}_c(\mathrm{Gr}^\mu_{\underline{\lambda}}, \mathbb{Q}).$$

The last cohomology group has a basis given by the top dimensional components of $\mathrm{Gr}^\mu_{\underline{\lambda}}$.

Lemma 2.45 implies that there is a canonical isomorphism $\mathrm{H}^{2\mathrm{ht}(\mu)}_c(\mathrm{S}_\mu, \mathrm{IC}_{\underline{\lambda}}) = \mathbb{Q}[\mathrm{Irr}(\overline{\mathrm{Gr}^\mu_{\underline{\lambda}}})]$. Combined with Proposition 2.42 we arrive at Theorem 2.38. □

Parametrisation of a canonical basis. Since the group $\mathrm{B}(\mathcal{O})$ is connected, the projection

$$p : \mathrm{Gr}^\mu_{\underline{\lambda}} \longrightarrow \mathrm{B}(\mathcal{O})\backslash\mathrm{Gr}^\mu_{\underline{\lambda}} = \mathrm{Conf}(\mathrm{Gr}^{n+1}, \mathcal{B})^\mu_{\underline{\lambda}}$$

identifies the top components. So Theorem 2.34 tells that the cycles $p^{-1}(\mathcal{M}^\circ_l)$, $l \in \mathbf{B}^\mu_{\underline{\lambda}}$, see (134), are the top components of $\mathrm{Gr}^\mu_{\underline{\lambda}}$. Theorem 2.38 plus (141) implies that they give rise to classes $[p^{-1}(\mathcal{M}^\circ_l)] \in V_{\underline{\lambda}}$. Moreover, the μ is the weight of the class in $V_{\underline{\lambda}}$. So we get the following result.

Theorem 2.46 *The set $\mathbf{B}^\mu_{\underline{\lambda}}$ parametrises a canonical basis in the weight μ part $V^{(\mu)}_{\underline{\lambda}}$ of the representation $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ of* G^L. *This basis is given by the classes $[p^{-1}(\mathcal{M}_l)]$, $l \in \mathbf{B}^\mu_{\underline{\lambda}}$.*

3 The potential $\mathcal{W}$ in special coordinates for GL_m

3.1 Potential for $\mathrm{Conf}_3(\mathcal{A})$ and Knutson–Tao's rhombus inequalities

Recall that a flag $F_\bullet$ for GL_m is a collection of subspaces in an m-dimensional vector space V_m:

$$F_\bullet = F_0 \subset F_1 \subset \ldots \subset F_{m-1} \subset F_m, \quad \dim F_i = i.$$

A decorated flag for GL_m is a flag $F_\bullet$ with a choice of non-zero vectors $f_i \in F_i/F_{i-1}$ for each $i = 1, \ldots, m$, called *decorations*. It determines a collection of decomposable k-vectors

$$f_{(1)} := f_1, \quad f_{(2)} := f_1 \wedge f_2, \quad \ldots, \quad f_{(m)} := f_1 \wedge \cdots \wedge f_m.$$

A decorated flag is determined by a collection of decomposable k-vectors such that each divides the next one. A linear basis $(f_1, \ldots, f_m)$ in the space V_m determines a decorated flag by setting $F_k := \langle f_1, \ldots, f_k \rangle$, and taking the projections of f_k to F_k/F_{k-1} to be the decorations.

Recall the notion of an m-triangulation of a triangle [17, Section 9]. It is a graph whose vertices are parametrized by a set

$$\Gamma_m := \{(a, b, c) \mid a + b + c = m, \quad a, b, c \in \mathbb{Z}_{\geq 0}\}. \tag{145}$$

Let $(\mathrm{F}, \mathrm{G}, \mathrm{H}) \in \mathrm{Conf}_3(\mathcal{A})$ be a generic configuration of three decorated flags, described by a triple of linear bases in the space V_m:

$$\mathrm{F} = (f_1, \ldots, f_m), \ \mathrm{G} = (g_1, \ldots, g_m), \ \mathrm{H} = (h_1, \ldots, h_m).$$

Let $\omega \in \det V_m^*$ be a volume form. Then each vertex $(a, b, c) \in$ (145) gives rise to a function

$$\Delta_{a,b,c}(\mathrm{F}, \mathrm{G}, \mathrm{H}) = \langle f_{(a)} \wedge g_{(b)} \wedge h_{(c)}, \omega \rangle.$$

There is a one dimensional space $\mathrm{L}_a^{b,c} := \mathrm{F}_{a+1} \cap (\mathrm{G}_b \oplus \mathrm{H}_c)$.

Let $e_a^{b,c} \in \mathrm{L}_a^{b,c}$ such that $e_a^{b,c} - f_{a+1} \in \mathrm{F}_a$. It is easy to see that $e_a^{b+1,c-1} - e_a^{b,c} \in \mathrm{L}_{a-1}^{b+1,c}$. Therefore there exists a unique scalar $\alpha_a^{b,c}$ such that $e_a^{b+1,c-1} - e_a^{b,c} = \alpha_a^{b,c} e_{a-1}^{b+1,c}$.

Lemma 3.1 *One has*

$$\alpha_a^{b,c} = \frac{\Delta_{a-1,b+1,c}\Delta_{a+1,b,c-1}}{\Delta_{a,b,c}\Delta_{a,b+1,c-1}}. \tag{146}$$

Proof Set

$$\alpha := \alpha_a^{b,c}, \quad \beta := \frac{\Delta_{a,b,c}}{\Delta_{a+1,b,c-1}}, \quad \gamma := \frac{\Delta_{a,b+1,c-1}}{\Delta_{a+1,b,c-1}}. \tag{147}$$

By definition,

$$\begin{aligned} f_{(a)} &= f_{(a-1)} \wedge e_{a-1}^{b,c+1}, \\ f_{(a+1)} &= f_{(a)} \wedge e_a^{b,c} = f_{(a)} \wedge e_a^{b-1,c+1}, \\ g_{(b)} \wedge h_{(c)} &= \beta e_a^{b,c} \wedge g_{(b)} \wedge h_{(c-1)}, \\ g_{(b+1)} \wedge h_{(c-1)} &= \gamma e_a^{b+1,c-1} \wedge g_{(b)} \wedge h_{(c-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_{(b+1)} \wedge h_{(c)} &= \gamma e_a^{b+1,c-1} \wedge g_{(b)} \wedge h_{(c)} \\ &= \beta\gamma e_a^{b+1,c-1} \wedge e_a^{b,c} \wedge g_{(b)} \wedge h_{(c-1)} \\ &= \beta\gamma (e_a^{b+1,c-1} - e_a^{b,c}) \wedge e_a^{b,c} \wedge g_{(b)} \wedge h_{(c-1)} \\ &= \beta\gamma\alpha e_{a-1}^{b+1,c} \wedge e_a^{b,c} \wedge g_{(b)} \wedge h_{(c-1)}. \end{aligned}$$

So

$$f_{(a-1)} \wedge g_{(b+1)} \wedge h_{(c)} = \alpha\beta\gamma f_{(a+1)} \wedge g_{(b)} \wedge h_{(c-1)}.$$

Therefore,

$$\alpha\beta\gamma = \frac{\Delta_{a-1,b+1,c}}{\Delta_{a+1,b,c-1}}.$$

Go back to (147), the Lemma is proved. □

As shown on Fig. 27, each zig-zag path p provides a basis E_p for F. For example,

$$\mathrm{E}_l := \left\{ e_0^{0,n}, e_1^{0,n-1}, \ldots, e_{n-1}^{0,1} \right\}, \quad \mathrm{E}_r := \left\{ e_0^{n,0}, e_1^{n-1,1}, \ldots, e_{n-1}^{1,0} \right\}$$

are the bases provided by the very left and very right paths.

Given two zig-zag paths, say p and q, there is a unique unipotent element u_{pq} stabilizing F, transforming E_p to E_q. Recall the character χ_{F} in section 1. For each triple (p, q, r) of zig-zag paths, we have

$$\begin{aligned} \chi_{\mathrm{F}}(u_{pq}) &= -\chi_{\mathrm{F}}(u_{qp}), \\ \chi_{\mathrm{F}}(u_{pr}) &= \chi_{\mathrm{F}}(u_{pq}) + \chi_{\mathrm{F}}(u_{qr}). \end{aligned}$$

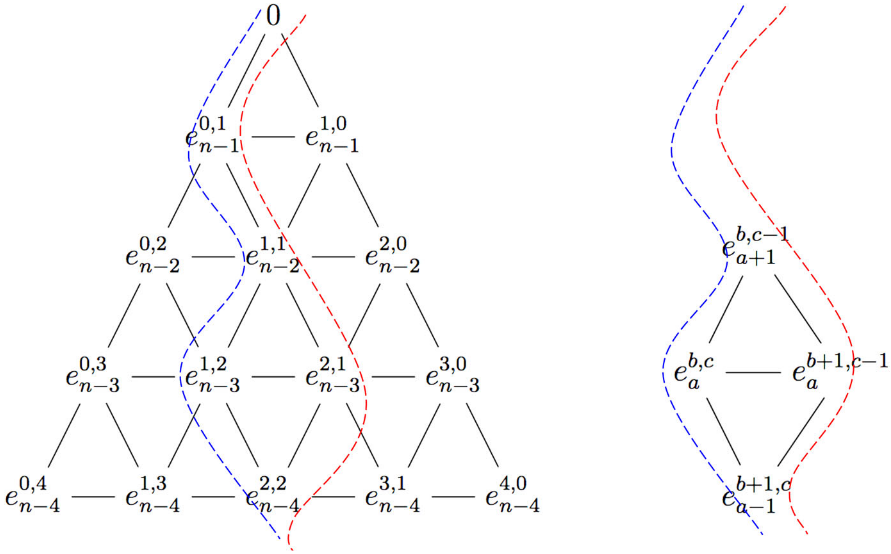


Fig. 27 Zig-zag paths and bases for the decorated flag F

If p, q are adjacent paths, see the right of Fig. 27, then by Lemma 3.1,

$$\chi_{\mathrm{F}}(u_{pq}) = \alpha_a^{b,c} = \frac{\Delta_{a-1,b+1,c}\Delta_{a+1,b,c-1}}{\Delta_{a,b,c}\Delta_{a,b+1,c-1}}.$$

One can transform the very left path to the very right by a sequence of adjacent paths. Let $u \in \mathrm{U_F}$ transform E_l to E_r. Then

$$\chi_{\mathrm{F}}(u) = \sum_{(a,b,c)\in\Gamma_m, a\neq 0, c\neq 0} \alpha_a^{b,c} = \sum_{(a,b,c)\in\Gamma_m, a\neq 0, c\neq 0} \frac{\Delta_{a-1,b+1,c}\Delta_{a+1,b,c-1}}{\Delta_{a,b,c}\Delta_{a,b+1,c-1}}.$$

Its tropicalization

$$\chi_{\mathrm{F}}^t = \min_{(a,b,c)\in\Gamma_n, a\neq 0, c\neq 0} \{\Delta^t_{a-1,b+1,c} + \Delta^t_{a+1,b,c-1} - \Delta^t_{a,b,c} - \Delta^t_{a,b+1,c-1}\}$$

delivers 1/3 of Knutson–Tao rhombus inequalities. Clearly, same holds for the other two directions. By definition,

$$\mathcal{W}(\mathrm{F}, \mathrm{G}, \mathrm{H}) = \chi_{\mathrm{F}} + \chi_{\mathrm{G}} + \chi_{\mathrm{H}}.$$

Our set $\mathrm{Conf}_3^+(\mathcal{A})(\mathbb{Z}^t)$ coincides with the set of hives in [54].

In Sects. 3.2–3.3 we show that the potential on the space $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ for GL_m, written in the special coordinates there, recovers Givental's potential and, after tropicalization, Gelfand–Tsetlin's patterns.

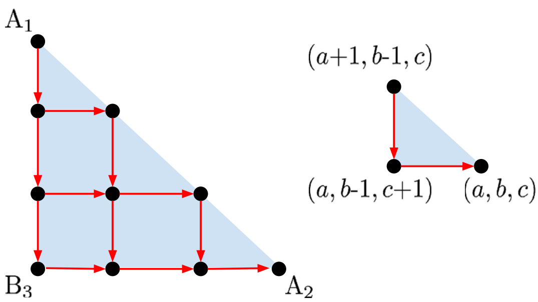


Fig. 28 Calculating the potential $\mathcal{W}$ on $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ in the special coordinates for GL_m

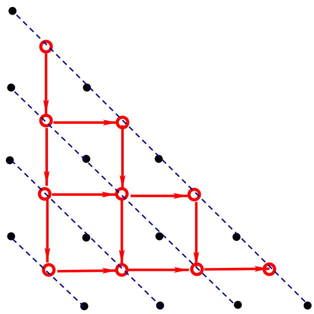


Fig. 29 The Givental quiver and special coordinates on $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ for GL_4

3.2 The potential for $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ and Givental's potential for GL_m

Let $\mathrm{G} = \mathrm{GL}_m$. Recall the set Γ_m, see (145). For each triple $(a, b, c) \in \Gamma_m$, there is a canonical function $\Delta_{a,b,c} : \mathrm{Conf}_3(\mathcal{A}) \to \mathbb{A}^1$. Consider the functions $\Delta_{a,b,c}$ with $(a, b, c) \in \Gamma_m - (0, 0, m)$, illustrated by the $\bullet$-vertices on Fig. 29. For each triple $(a, b, c) \in \Gamma_{m-1}$, let us set

$$\mathrm{R}_{a,b,c} := \frac{\Delta_{a,b+1,c}}{\Delta_{a+1,b,c}}. \tag{148}$$

The functions $\mathrm{R}_{a,b,c}$ are assigned naturally to the $\circ$-vertices on Fig. 29. Each of them is the ratio of the Δ-functions at the ends of the slant edge centered at a $\circ$-vertex. They are functions on $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ since $\mathrm{R}_{a,b,c}(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \cdot h) = \mathrm{R}_{a,b,c}(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$ for any $h \in \mathrm{H}$. The functions $\mathrm{R}_{a,b,c}$ form a coordinate system on $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$, referred to as the *special coordinate system*.

The functions $\{\mathrm{R}_{a,b,0}\}$ provide the canonical map

$$\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Conf}(\mathcal{A}, \mathcal{A}) = (\mathbb{G}_m)^{m-1}. \tag{149}$$

Consider now the *Givental quiver* Γ_{m-1}, whose vertices are the $\circ$-vertices, parametrised by the set Γ_{m-1}, with the arrows are going down and to the right, as shown on Fig. 29. For each arrow connecting two vertices, take the sourse/tail ratio of the corresponding functions. For example, see Fig. 28, the vertical arrow α connecting $(a+1, b-1, c)$ and $(a, b-1, c+1)$ provides

$$Q_\alpha = \frac{\mathrm{R}_{a,b-1,c+1}}{\mathrm{R}_{a+1,b-1,c}} = \frac{\Delta_{a,b,c+1}\Delta_{a+2,b-1,c}}{\Delta_{a+1,b-1,c+1}\Delta_{a+1,b,c}}. \tag{150}$$

Recall the function $\chi_{\mathrm{A}_1}, \chi_{\mathrm{A}_2}$ on $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$. Taking the sum of Q_α over the vertical arrows α, and a similar sum over the horizontal arrows β, and using (150), we get

$$\chi_{\mathrm{A}_1} = \sum_{\alpha \text{ vertical}} Q_\alpha, \quad \chi_{\mathrm{A}_2} = \sum_{\beta \text{ horizontal}} Q_\beta.$$

Relating to Givental's work. Givental [37, pages 3–4], introduced parameters $T_{i,j}$, $0 \leq i \leq j \leq m$, matching the vertices of the Givental quiver:

$$\begin{array}{llll} T_{0,0} & & & \\ T_{01} & T_{1,1} & & \\ T_{02} & T_{12} & T_{2,2} & \\ T_{03} & T_{13} & T_{2,3} & T_{3,3} \end{array}$$

He treats the entries on the main diagonal $\mathrm{a} = (T_{0,0}, T_{1,1}, \ldots, T_{m,m})$ as parameters, and defines the potential as a sum over the oriented edges of the quiver:

$$\mathcal{W}_{\mathrm{a}} = \sum_{0 \leq i < j \leq m} \big(\exp(T_{i,j} - T_{i,j-1}) + \exp(T_{i,j} - T_{i+1,j})\big).$$

Let Y_{a} be the subvariety with a given value of a. Then Givental's integral is

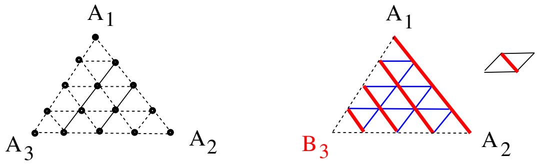
$$\mathcal{F}(\mathrm{a}, \hbar) = \int_{Y_{\mathrm{a}}} \exp(-\mathcal{W}_{\mathrm{a}}/\hbar) \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{i-1} dT_{i.j}.$$

Givental's variables $T_{i,j}$ match our coordinates $\mathrm{R}_{a,b,c}$ where $a+b+c = m-1$:

$$\mathrm{R}_{m-i-1,j,i-j} = \exp(T_{i,j}).$$

Observe that Y_{a} is the fiber of the map (149) over a point $\mathrm{a} = (\mathrm{R}_{m-1,0}, \mathrm{R}_{m-2,1}, \ldots, \mathrm{R}_{0,m-1})$. Givental's potential coincides with $\chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}$. Givental's volume form on Y_{a} coincides, up to a sign, with ours since

Fig. 30 Gelfand–Tsetlin patterns for GL_4 and the special coordinates for Conf$(\mathcal{A}, \mathcal{A}, \mathcal{B})$



$$\bigwedge_{i=1}^{n}\bigwedge_{j=0}^{i-1} dT_{i.j} = \pm \bigwedge_{a+b+c=m-1,c>0} d\log \mathrm{R}_{a,b,c}.$$

3.3 The potential for Conf$(\mathcal{A}, \mathcal{A}, \mathcal{B})$ and Gelfand–Tsetlin's patterns for GL_m

Gelfand–Tsetlin's patterns for GL_m [28] are arrays of integers $\{p_{i,j}\}$, $1 \leq i \leq j \leq m$, such that

$$p_{i,j+1} \leq p_{i,j} \leq p_{i+1,j+1}. \tag{151}$$

Theorem 3.2 *The special coordinate system on* $\mathrm{Conf}(\mathcal{A}_{\mathrm{GL}_m}, \mathcal{A}_{\mathrm{GL}_m}, \mathcal{B}_{\mathrm{GL}_m})$ *together with the potential* $\mathcal{W} = \chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}$ *provide a canonical isomorphism*

$$\{\textit{Gelfand– Tsetlin's patterns for } \mathrm{GL}_m\} = \mathrm{Conf}^+(\mathcal{A}_{\mathrm{GL}_m}, \mathcal{A}_{\mathrm{GL}_m}, \mathcal{B}_{\mathrm{GL}_m})(\mathbb{Z}^t).$$

Proof The space $\mathrm{Conf}(\mathcal{A}^3_{\mathrm{GL}_m}, \omega_m)$ of GL_m-orbits on $\mathcal{A}^3_{\mathrm{GL}_m} \times \det V_m^*$ has dimension $\frac{(m+1)(m+2)}{2}$. It has a coordinate system given by the functions $\Delta_{a,b,c}$, $a+b+c=m$, parametrized by the vertices of the graph Γ_m, shown on the left of Fig. 30. The coordinates on $\mathrm{Conf}(\mathcal{A}_{\mathrm{GL}_m}, \mathcal{A}_{\mathrm{GL}_m}, \mathcal{B}_{\mathrm{GL}_m})$ are parametrized by the edges E of the graph parallel to the edge $\mathrm{A}_1\mathrm{A}_2$ of the triangle. They are little red segments on the right of Fig. 30. They are given by the ratios of the coordinates at the ends of the edge E, recovering formula (148). Notice that the edges E are oriented by the orientation of the side $\mathrm{A}_1\mathrm{A}_2$. The monomials of the potential $\chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}$ are parametrized by the blue edges, that is by the edges of the graph inside of the triangle parallel to either side $\mathrm{B}_3\mathrm{A}_1$ or $\mathrm{B}_3\mathrm{A}_2$. We claim that the monomials of potential $\chi_{\mathrm{A}_1} + \chi_{\mathrm{A}_2}$ are in bijection with Gelfand–Tsetlin's inequalities. Indeed, a typical pair of inequalities (151) is encoded by a part of the graph shown on Fig. 31. The three coordinates (P_1, P_2, Q) on $\mathrm{Conf}(\mathcal{A}_{\mathrm{GL}_m}, \mathcal{A}_{\mathrm{GL}_m}, \mathcal{B}_{\mathrm{GL}_m})$ assigned to the red edges are expressed via the coordinates (A, B, C, D, F) at the vertices:

$$P_1 = \frac{B}{A}, \quad P_2 = \frac{C}{B}, \quad Q = \frac{E}{D}.$$

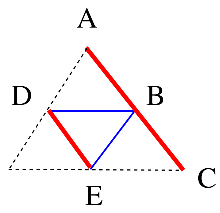


Fig. 31 Gelfand–Tsetlin patterns from the potential

The monomials of the potential at the two blue edges are $\frac{EA}{DB}$ and $\frac{DC}{EB}$. Their tropicalization delivers the inequalities $p_1 \leq q, q \leq p_2$. $\square$

4 Proof of Theorem 2.11

Let T be a split torus. Let $g := \sum_{\alpha \in X^*(\mathrm{T})} g_\alpha X^\alpha$ be a nonzero positive polynomial on T, i.e. its coefficients $g_\alpha \geq 0$ are non-negative. The integral tropical points $l \in \mathrm{T}(\mathbb{Z}^t) = X_*(\mathrm{T})$ are cocharacters of T. The tropicalization of g is a piecewise linear function on $\mathrm{T}(\mathbb{Z}^t)$:

$$g^t(l) = \min_{\alpha \mid g_\alpha > 0} \{\langle l, \alpha \rangle\}.$$

Fix an $l \in \mathrm{T}(\mathbb{Z}^t)$. Set

$$\Lambda_{g,l} := \{\alpha \in X^*(\mathrm{T}) \mid g_\alpha > 0, \ \langle l, \alpha \rangle = g^t(l)\}, \qquad g_l := \sum_{\alpha \in \Lambda_{g,l}} g_\alpha X^\alpha.$$

The set $\Lambda_{g,l}$ is non-empty. Therefore g_l is a nonzero positive polynomial. If f and g are two such polynomials, so is the product $f \cdot g$. We have $(f \cdot g)_l = f_l \cdot g_l$ for all $l \in \mathrm{T}(\mathbb{Z}^t)$.

Let h be a nonzero positive rational function on T. It can be expressed as a ratio f/g of two nonzero positive polynomials. Set $h_l := f_l/g_l$. Let $h = f'/g'$ be another expression. Then

$$f/g = f'/g' \implies f \cdot g' = f' \cdot g \implies f_l \cdot g'_l = f'_l \cdot g_l \implies f_l/g_l = f'_l/g'_l.$$

Hence h_l is well defined.

Lemma 4.1 *Let h, l be as above. For each $C \in \mathrm{T}_l$ such that*[12] $h_l(\mathrm{in}(C)) \in \mathbb{C}^*$, *we have*

$$\mathrm{val}(h(C)) = h^t(l), \quad \mathrm{in}(h(C)) = h_l(\mathrm{in}(C)). \tag{152}$$

[12] Every transcendental point $C \in \mathrm{T}_l^\circ$ automatically satisfies such conditions.

Proof Assume that h is a nonzero positive polynomial. By definition

$$\forall C \in \mathrm{T}_l, \quad h(C) = h_l(\mathrm{in}(C))t^{h^t(l)} + \text{ terms with higher valuation.}$$

If $h_l(\mathrm{in}(C)) \in \mathbb{C}^*$, then (152) follows. The argument for a positive rational function is similar. □

Let $f = (f_1, \ldots, f_k) : \mathrm{T} \to \mathrm{S}$ be a positive birational isomorphism of split tori. Let $l \in \mathrm{T}(\mathbb{Z}^t)$. We generalize the above construction by setting $f_l := (f_{1,l}, \ldots, f_{k,l}) : \mathrm{T} \longrightarrow \mathrm{S}$.

Lemma 4.2 *Let f, l be as above. Let $C \in \mathrm{T}_l^\circ$. Then*

$$\mathrm{val}(f(C)) = f^t(l), \quad \mathrm{in}(f(C)) = f_l(\mathrm{in}(C)). \tag{153}$$

Let h be a nonzero positive rational function on S. *Then*

$$\mathrm{in}\,(h \circ f(C)) = h_{f^t(l)}\,(\mathrm{in}(f(C)))\,. \tag{154}$$

Proof Here (153) follows directly from Lemma 4.1. Note that $h_{f^t(l)} \circ f_l$ is a nonzero positive rational function on T. Since C is transcendental, we get

$$h_{f^t(l)}\,(\mathrm{in}(f(C))) = h_{f^t(l)} \circ f_l(\mathrm{in}(C)) \in \mathbb{C}^*.$$

Thus (154) follows from Lemma 4.1. □

Proof of Theorem 2.11 It suffices to prove $f(\mathrm{T}_l^\circ) \subseteq \mathrm{S}^\circ_{f^t(l)}$. The other direction is the same.

Let $C = (C_1, \ldots, C_k) \in \mathrm{T}_l^\circ$. Let $f(C) := (C_1', \ldots, C_k')$. By (153), we get $f(C) \in \mathrm{S}_{f^t(l)}$ and the field extension $\mathbb{Q}(\mathrm{in}(C_1'), \ldots, \mathrm{in}(C_k')) \subseteq \mathbb{Q}(\mathrm{in}(C_1), \ldots, \mathrm{in}(C_k))$.

Let $g = (g_1, \ldots, g_k) : \mathrm{S} \to \mathrm{T}$ be the inverse morphism of f. Then $C_j = g_j \circ f(C)$ for $j \in [1, k]$. The functions g_j are nonzero positive rational functions on S. Therefore

$$\mathrm{in}(C_j) = \mathrm{in}(g_j \circ f(C)) \overset{(154)}{=} g_{j,f^t(l)}(\mathrm{in}(f(C))) \in \mathbb{Q}(\mathrm{in}(C_1'), \ldots, \mathrm{in}(C_k')).$$

Therefore $\mathbb{Q}(\mathrm{in}(C_1), \ldots, \mathrm{in}(C_k)) \subseteq \mathbb{Q}(\mathrm{in}(C_1'), \ldots, \mathrm{in}(C_k'))$. Summarizing, we get

$$\mathbb{Q}(\mathrm{in}(C_1), \ldots, \mathrm{in}(C_k)) = \mathbb{Q}(\mathrm{in}(C_1'), \ldots, \mathrm{in}(C_k')). \tag{155}$$

Therefore $f(C)$ is transcendental. Theorem 2.11 is proved. □

5 Positive structures on the unipotent subgroups U and U^-

5.1 Lusztig's data and MV cycles

Lusztig's data. Fix a reduced word $\mathbf{i} = (i_m, \ldots, i_1)$ for w_0. There are positive functions

$$F_{\mathbf{i},j} : \mathrm{U} \longrightarrow \mathbb{A}^1, \quad x_{i_m}(a_m)\ldots x_{i_1}(a_1) \longmapsto a_j. \tag{156}$$

Their tropicalizations induce an isomorphism $f_{\mathbf{i}} : \mathrm{U}(\mathbb{Z}^t) \stackrel{=}{\to} \mathbb{Z}^m$, $p \mapsto \{F^t_{\mathbf{i},j}(p)\}$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Lusztig proved [59] that the subset

$$f_{\mathbf{i}}^{-1}(\mathbb{N}^m) \subset \mathrm{U}(\mathbb{Z}^t) \tag{157}$$

does not depend on $\mathbf{i}$, and parametrizes the canonical basis in the quantum enveloping algebra of the Lie algebra of a maximal unipotent subgroup of the Langlands dual group G^L.

Lemma 5.1 *The subset* $\mathrm{U}^+_\chi(\mathbb{Z}^t) := \{l \in \mathrm{U}(\mathbb{Z}^t) \mid \chi^t(l) \geq 0\}$ *is identified with the set* (157).

Proof Note that $\chi = \sum_{j=1}^m F_{\mathbf{i},j}$. It tropicalization is $\min_{1\leq j\leq m}\{F^t_{\mathbf{i},j}\}$. Let $l \in \mathrm{U}(\mathbb{Z}^t)$. Then

$$\chi^t(l) \geq 0 \Longleftrightarrow F^t_{\mathbf{i},j}(l) \geq 0,\ \forall j \in [1, m] \Longleftrightarrow f_{\mathbf{i}}(l) \in \mathbb{N}^m.$$

□

Let $l \in \mathrm{U}(\mathbb{Z}^t)$. Recall the transcendental cell $\mathcal{C}^\circ_l \subset \mathrm{U}(\mathcal{K})$.

Lemma 5.2 *Let $u \in \mathcal{C}^\circ_l$. Then $u \in \mathrm{U}(\mathcal{O})$ if and only if $l \in \mathrm{U}^+_\chi(\mathbb{Z}^t)$.*

Proof Set $u = x_{i_m}(a_m)\ldots x_{i_1}(a_1) \in \mathcal{C}^\circ_l$. Note that u is transcendental. Using Lemma 2.13, we get

$$\chi^t(l) = \mathrm{val}(\chi(u)); \quad F^t_{\mathbf{i},j}(l) = \mathrm{val}(a_j),\ \forall j \in [1, m].$$

If $l \in \mathrm{U}^+_\chi(\mathbb{Z}^t)$, then $\mathrm{val}(a_j) = F^t_{\mathbf{i},j}(l) \geq 0$. Therefore $a_j \in \mathcal{O}$. Hence $u \in \mathrm{U}(\mathcal{O})$.

Note that χ is a regular function of U. If $u \in \mathrm{U}(\mathcal{O})$, then $\chi(u) \in \mathcal{O}$. Therefore $\chi^t(l) = \mathrm{val}(\chi(u)) \geq 0$. Hence $l \in \mathrm{U}^+_\chi(\mathbb{Z}^t)$. □

The positive morphism β. Let $[g]_0 := h$ if $g = u_+ h u_-$, where $u_\pm \in \mathrm{U}^\pm$, $h \in \mathrm{H}$. Define

$$\beta : \mathrm{U} \longrightarrow \mathrm{H}, \quad u \longmapsto [\overline{w}_0 u]_0. \tag{158}$$

Let $\mathbf{i} = (i_m, \ldots, i_1)$ as above. Let $w_k^{\mathbf{i}} := s_{i_1} \ldots s_{i_k} \in W$. Let $\beta_k^{\mathbf{i}} := w_{k-1}^{\mathbf{i}}(\alpha_{i_k}^\vee) \in \mathrm{P}$. The following Lemma shows that β is a positive map.

Lemma 5.3 [12, Lemma 6.4] *For each* $u = x_{i_m}(a_m) \ldots x_{i_1}(a_1) \in \mathrm{U}$, *we have* $[\overline{w}_0 u]_0 = \prod_{k=1}^m \beta_k^{\mathbf{i}}(a_k^{-1})$.

Let $l \in \mathrm{U}(\mathbb{Z}^t)$. The tropicalization β^t becomes $\beta^t(l) = -\sum_{k=1}^m F_{\mathbf{i},k}^t(l)\beta_k^{\mathbf{i}}$. Note that $\beta_k^{\mathbf{i}} \in \mathrm{P}$ are positive coroots. If $l \in \mathrm{U}_\chi^+(\mathbb{Z}^t)$, then $-\beta^t(l) \in \mathrm{R}^+$. Hence

$$\mathrm{U}_\chi^+(\mathbb{Z}^t) = \bigsqcup_{\lambda \in \mathrm{R}^+} \mathbf{A}_\lambda, \quad \mathbf{A}_\lambda := \{l \in \mathrm{U}_\chi^+(\mathbb{Z}^t) \mid -\beta^t(l) = \lambda\}. \tag{159}$$

The set $\mathbf{A}_\lambda$ is identified with Lusztig's set parametrizing the canonical basis of weight λ [59].

Kamnitzer's parametrization of MV cycles. Kamnitzer [45] constructs a canonical bijection between Lusztig's data (i.e. $\mathrm{U}_\chi^+(\mathbb{Z}^t)$ in our set-up) and the set of stable MV cycles. Let us briefly recall Kamnitzer's result for future use.

Let $\mathrm{U}_* := \mathrm{U} \cap \mathrm{B}^- w_0 \mathrm{B}^-$ and let $\mathrm{U}_*^- = \mathrm{U}^- \cap \mathrm{B} w_0 \mathrm{B}$. There is an well-defined isomorphism

$$\eta : \mathrm{U}_* \to \mathrm{U}_*^-, \quad u \longmapsto \eta(u). \tag{160}$$

such that $\eta(u)$ is the unique element in $\mathrm{U}^- \cap \mathrm{B} w_0 u$. The map η was used in [22]. Set

$$\kappa_{\mathrm{Kam}} : \mathrm{U}_*(\mathcal{K}) \longrightarrow \mathrm{Gr}, \quad u \longrightarrow [\eta(u)]. \tag{161}$$

Let $l \in \mathrm{U}(\mathbb{Z}^t)$. Then $\mathcal{C}_l^\circ \subset \mathrm{U}_*(\mathcal{K})$. Define

$$\mathrm{MV}_l := \overline{\kappa_{\mathrm{Kam}}(\mathcal{C}_l^\circ)} \subset \mathrm{Gr}. \tag{162}$$

The following Theorem is a reformulation of Kamnitzer's result.

Theorem 5.4 [45, Theorem 4.5] *Let* $l \in \mathbf{A}_\lambda$. *Then* MV_l *is an MV cycle of coweight* $(\lambda, 0)$. *It gives a bijection between* $\mathbf{A}_\lambda$ *and the set of such MV cycles.*

A stable MV cycle of coweight λ has a unique representative of coweight $(\lambda, 0)$. Therefore Theorem 5.4 tells that the set $\mathbf{A}_\lambda$ parametrizes the set of stable MV cycles of coweight λ.

5.2 Positive functions $\chi_i, \mathcal{L}_i, \mathcal{R}_i$ on U.

Let $i \in I$. We introduce positive rational functions χ_i, $\mathcal{L}_i$, $\mathcal{R}_i$ on U, and χ_i^-, $\mathcal{L}_i^-$, $\mathcal{R}_i^-$ on U^-.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a reduced word for w_0. Let

$$x = x_{i_1}(a_1) \ldots x_{i_m}(a_m) \in \mathrm{U}, \quad y = y_{i_1}(b_1) \ldots y_{i_m}(b_m) \in \mathrm{U}^-.$$

Using above decompositions of x and y, we set

$$\chi_i(x) := \sum_{p \mid i_p = i} a_p, \quad \chi_i^-(y) := \sum_{p \mid i_p = i} b_p.$$

By definition the characters χ and χ^- have decompositions $\chi = \sum_{i \in I} \chi_i$ and $\chi^- = \sum_{i \in I} \chi_i^-$.

We take $\mathbf{i}$ which starts from $i_1 = i$. Define the "left" functions:

$$\mathcal{L}_i(x) := a_1, \quad \mathcal{L}_i^-(y) := b_1.$$

We take $\mathbf{i}$ which ends by $i_m = i$. Define the "right" functions:

$$\mathcal{R}_i(x) := a_m, \quad \mathcal{R}_i^-(y) := b_m.$$

It is easy to see that the above functions are well-defined and independent of $\mathbf{i}$ chosen.

For each simple reflection $s_i \in W$, set s_{i^*} such that $w_0 s_{i^*} = s_i w_0$.

Set $\mathrm{Ad}_v(g) := vgv^{-1}$. For any $u \in \mathrm{U}$, set $\widetilde{u} := \mathrm{Ad}_{\overline{w}_0}(u^{-1}) \in \mathrm{U}^-$.

Lemma 5.5 *The map $u \mapsto \widetilde{u}$ is a positive birational isomorphism from* U *to* U^-. *Moreover,*

$$\chi_i(u) = \chi_{i^*}^-(\widetilde{u}), \quad \mathcal{L}_i(u) = \mathcal{R}_{i^*}(\widetilde{u}), \quad \mathcal{R}_i(u) = \mathcal{L}_{i^*}(\widetilde{u}) \quad \forall i \in I. \tag{163}$$

Proof Note that $\mathrm{Ad}_{\overline{w}_0}(x_i(-a)) = y_{i^*}(a)$. Let $u = x_{i_1}(a_1) \ldots x_{i_m}(a_m) \in \mathrm{U}$. Then

$$\widetilde{u} = \mathrm{Ad}_{\overline{w}_0}(u^{-1}) = y_{i_m^*}(a_m) \ldots y_{i_1^*}(a_1).$$

Clearly it is a positive birational isomorphism. Identities in (163) follow by definition. □

Lemma 5.6 *Let $h \in \mathrm{H}$, $x \in \mathrm{U}$ and $y \in \mathrm{U}^-$. For any $i \in I$, we have*

$$\begin{aligned} &\chi_i\left(\mathrm{Ad}_h(x)\right) = \chi_i(x) \cdot \alpha_i(h), \quad \mathcal{L}_i\left(\mathrm{Ad}_h(x)\right) = \mathcal{L}_i(x) \cdot \alpha_i(h), \\ &\mathcal{R}_i\left(\mathrm{Ad}_h(x)\right) = \mathcal{R}_i(x) \cdot \alpha_i(h). \end{aligned} \tag{164}$$

$$\chi_i^-(\mathrm{Ad}_h(y)) = \chi_i(y)/\alpha_i(h), \quad \mathcal{L}_i^-(\mathrm{Ad}_h(y)) = \mathcal{L}_i^-(y)/\alpha_i(h),$$
$$\mathcal{R}_i^-(\mathrm{Ad}_h(y)) = \mathcal{R}_i^-(y)/\alpha_i(h). \tag{165}$$

Proof Follows from the identities $\mathrm{Ad}_h(x_i(a)) = x_i(a\alpha_i(h))$ and $\mathrm{Ad}_h(y_i(a)) = y_i(a/\alpha_i(h))$. □

5.3 The positive morphisms Φ and η

We show that each χ_i is closely related to $\mathcal{L}_i^-$ by the following morphism.

Definition 5.7 There exists a unique morphism $\Phi : \mathrm{U}^- \longrightarrow \mathrm{U}$ such that

$$u_-\mathrm{B} = \Phi(u_-)w_0\mathrm{B}. \tag{166}$$

Lemma 5.8 *For each $i \in I$, one has*

$$1/\mathcal{L}_i^- = \chi_i \circ \Phi, \quad 1/\chi_i^- = \mathcal{L}_i \circ \Phi \tag{167}$$

Example. Let $\mathrm{G} = \mathrm{SL}_3$. We have

$$y = y_1(b_1)y_2(b_2)y_1(b_3) = y_2\left(\frac{b_2b_3}{b_1+b_3}\right)y_1(b_1+b_3)y_2\left(\frac{b_1b_2}{b_1+b_3}\right).$$

$$\Phi(y) = x_1\left(\frac{1}{b_1+b_3}\right)x_2\left(\frac{b_1+b_3}{b_2b_3}\right)x_1\left(\frac{b_3}{b_1(b_1+b_3)}\right)$$
$$= x_2\left(\frac{1}{b_2}\right)x_1\left(\frac{1}{b_1}\right)x_2\left(\frac{b_1}{b_2b_3}\right).$$

$$1/\mathcal{L}_1^-(y) = \chi_1(\Phi(y)) = \frac{1}{b_1}, \quad 1/\mathcal{L}_2^-(y) = \chi_2(\Phi(y)) = \frac{b_1+b_3}{b_2b_3}.$$

$$1/\chi_1^-(y) = \mathcal{L}_1(\Phi(y)) = b_1+b_3, \quad 1/\chi_2^-(y) = \mathcal{L}_2(\Phi(y)) = b_2.$$

The proof was suggested by the proof of Proposition 3.2 of [60].

Proof We prove the first formula. The second follows similarly by considering the inverse morphism $\Phi^{-1} : \mathrm{U} \to \mathrm{U}^-$ such that $u\mathrm{B}^- = \Phi^{-1}(u)w_0\mathrm{B}^-$.

Let $i \in I$. Let $w \in W$ such that its length $l(w) < l(s_iw)$. We use two basic identities:

$$y_i(b)x_i(a) = x_i\left(a/(1+ab)\right)y_i\left(b(1+ab)\right)\alpha_i^\vee\left(1/(1+ab)\right). \tag{168}$$

$$y_i(b)w\mathrm{B} = x_i(1/b)s_i w\mathrm{B}. \tag{169}$$

By (169), one can change $y_i(b)$ on the most right to $x_i(1/b)$. By (168), one can "move" $y_i(b)$ from left to the right. After finite steps, we get

$$\begin{aligned} &y_{i_1}(b_1)y_{i_2}(b_2)\dots y_{i_m}(b_m)\mathrm{B} \\ &\quad = y_{i_1}(b_1)x_{i_m}(a_m)x_{i_{m-1}}(a_{m-1})\dots x_{i_2}(a_2)s_{i_2}\dots s_{i_m}\mathrm{B}. \end{aligned} \tag{170}$$

The last step is to move the very left term $y_{i_1}(b_1)$ to the right. Let

$$f_s(c_1, c_2, \dots, c_m) = x_{i_m}(c_m)x_{i_{m-1}}(c_{m-1})\dots x_{i_{s+1}}(c_{s+1})y_{i_1}(c_1)x_{i_s}(c_s) \\ \dots x_{i_2}(c_2)s_{i_2}\dots s_{i_m}\mathrm{B}.$$

We will need the relations between $\{c_i\}$ and $\{c_i'\}$ such that

$$f_s(c_1, c_2, \dots, c_m) = f_{s-1}(c_1', c_2', \dots, c_m')$$

By (168)–(169), if $i_1 \neq i_s$, then $c_p = c_p'$ for all p. If $i_1 = i_s$, then

$$\begin{aligned} &c_p' = c_p \quad \text{for } p = s+1, \dots, m; \\ &c_s' = c_s/(1 + c_1 c_s), \quad c_1' = c_1(1 + c_1 c_s); \\ &c_p' = c_p(1 + c_1 c_s)^{-\langle \alpha_{i_1}^\vee, \alpha_{i_p}\rangle} \quad \text{for } p = 2, \dots, s-1. \end{aligned}$$

For each $q = f_s(c_1, c_2, \dots, c_m)$, we set

$$h(q) := \frac{1}{c_1} + \sum_{p \mid i_p = i_1,\ p > s} c_p. \tag{171}$$

If $i_s = i_1$, then

$$\begin{aligned} \frac{1}{c_1'} + \sum_{p \mid i_p = i_1,\ p > s-1} c_p' &= \frac{1}{c_1(1 + c_1 c_s)} + \frac{c_s}{1 + c_1 c_s} + \sum_{p \mid i_p = i_1,\ p > s} c_p \\ &= \frac{1}{c_1} + \sum_{p \mid i_p = i_1,\ p > s} c_p. \end{aligned}$$

Same is true for $i_s \neq i_1$. Therefore the function (171) does not depend on s.

Back to (170), we have

$$\begin{aligned} u\mathrm{B} &= y_{i_1}(b_1)y_{i_2}(b_2)\dots y_{i_m}(b_m)\mathrm{B} \\ &= y_{i_1}(b_1)x_{i_m}(a_m)\dots x_{i_2}(a_2)s_{i_2}\dots s_{i_n}\mathrm{B} \\ &= x_{i_m}(c_m)\dots x_{i_2}(c_2)y_{i_1}(c_1)s_{i_2}\dots s_{i_n}\mathrm{B} \\ &= x_{i_m}(c_m)\dots x_{i_2}(c_2)x_{i_1}(1/c_1)s_{i_1}\dots s_{i_n}\mathrm{B} \\ &= \Phi(u)w_0\mathrm{B} \end{aligned}$$

Hence $\Phi(u) = x_{i_m}(c_m)\dots x_{i_1}(c_2)x_{i_1}(1/c_1)$. Then

$$\chi_{i_1}(\Phi(u)) = \frac{1}{c_1} + \sum_{p \mid i_p = i_1,\ p>1} c_p = h(u\mathrm{B}) = \frac{1}{b_1} = \frac{1}{\mathcal{L}_{i_1}^{-}(u)}.$$

□

Lemma 5.9 *The morphism* $\Phi : \mathrm{U}^- \to \mathrm{U}$ *is a positive birational isomorphism with respect to Lusztig's positive atlases on* U^- *and* U.

Proof According to the algorithm in the proof of Lemma 5.8, clearly Φ is a positive morphism. By the same argument, one can show that Φ^{-1} is a positive morphism. The Lemma is proved. □

The morphism η in (160) is the right hand side version of Φ, i.e. $\mathrm{B}^- u = \mathrm{B}^- w_0 \eta(u)$. Similarly,

Lemma 5.10 *The morphism* $\eta : \mathrm{U} \to \mathrm{U}^-$ *is a positive birational isomorphism. Moreover,*

$$\forall i \in I, \quad 1/\mathcal{R}_i = \chi_j^- \circ \eta, \quad 1/\chi_i = \mathcal{R}_i^- \circ \eta. \tag{172}$$

5.4 Birational isomorphisms ϕ_i of U

Let $i \in I$. Define

$$z_i(a) := \alpha_i^\vee(a)y_i(-a), \quad z_i^*(a) := \alpha_i^\vee(1/a)y_i(1/a).$$

Clearly $z_i(a)z_i^*(a) = 1$.

Lemma-Construction 5.11 *There is a birational isomorphism*

$$\phi_i : \mathrm{U} \xrightarrow{\sim} \mathrm{U}, \quad u \longmapsto \overline{s}_i \cdot u \cdot z_i\left(\chi_i(u)\right). \tag{173}$$

Remark. The map ϕ_i is not a positive birational isomorphism.

Proof We need the following identities:

$$\overline{s}_i x_i(a) z_i(a) = x_i(-1/a). \tag{174}$$

$$z_i^*(a) x_i(b-a) z_i(b) = x_i(1/a - 1/b). \tag{175}$$

If $j \neq i$, then

$$z_i^*(a) x_j(b) z_i(a) = x_j\left(ba^{-\langle \alpha_i^\vee, \alpha_j \rangle}\right) \tag{176}$$

Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a reduced word for w_0 such that $i_1 = i$. For each $s \in [1, m]$, define

$$I_s^{\mathbf{i},i} := \{p \in [1, s] \mid i_p = i\}.$$

Let $u = x_{i_1}(a_1) \ldots x_{i_m}(a_m) \in \mathrm{U}$. Set $d_s := \sum_{k \in I_s^{\mathbf{i},i}} a_k$. In particular, $d_1 = a_1$, $d_m = \chi_i(u)$.

Let us assume that $u \in \mathrm{U}$ is generic, so that $d_s \neq 0$ for all $s \in [1, m]$. By (174)–(176), we get

$$\begin{aligned}\phi_i(u) &= \overline{s}_i \cdot x_{i_1}(a_1) x_{i_2}(a_2) \ldots x_{i_m}(a_m) \cdot z_i\left(\chi_i(u)\right)\\ &= \left(\overline{s}_i x_{i_1}(a_1) z_i(d_1)\right) \cdot \left(z_i^*(d_1) x_{i_2}(a_2) z_i(d_2)\right)\\ &\quad \cdot \ldots \cdot \left(z_i^*(d_{m-1}) x_{i_m}(a_m) z_i(d_m)\right)\\ &= x_{i_1}(a_1') x_{i_2}(a_2') \ldots x_{i_m}(a_m').\end{aligned} \tag{177}$$

Here $a_1' = -1/d_1$. For $s > 1$,

$$a_s' = \begin{cases} 1/d_{s-1} - 1/d_s, & \text{if } i_s = i,\\ a_s d_s^{-\langle \alpha_i^\vee, \alpha_{i_s} \rangle}, & \text{if } i_s \neq i. \end{cases} \tag{178}$$

Thus $\phi_i(u) \in \mathrm{U}$. The map ϕ_i is well-defined. By (178), we have $\chi_i(\phi_i(u)) = -1/\chi_i(u)$. Therefore

$$\phi_i \circ \phi_i(u) = \overline{s}_i \cdot \overline{s}_i \cdot u \cdot z_i(\chi_i(u)) \cdot z_i(-1/\chi_i(u)) = \overline{s}_i^2 \cdot u \cdot \overline{s}_i^2.$$

Since $\overline{s}_i^4 = 1$, we get $\phi_i^4 = \mathrm{id}$. Therefore ϕ_i is birational. □

Let $\lambda \in \mathrm{P}^+$. Recall $t^\lambda \in \mathrm{Gr}$. Recall the $\mathrm{G}(\mathcal{O})$-orbit Gr_λ of t^λ in Gr.

Lemma 5.12 *Let $l \in \mathrm{U}(\mathbb{Z}^t)$. For any $u \in \mathcal{C}_l^\circ$, the element $u \cdot t^\lambda \in \overline{\mathrm{Gr}_\lambda}$ if and only if $l \in \mathrm{U}_\chi^+(\mathbb{Z}^t)$.*

Proof If $l \in \mathrm{U}_\chi^+(\mathbb{Z}^t)$, by Lemma 5.2, we see that $u \in \mathrm{U}(\mathcal{O})$. Hence $u \cdot t^\lambda \in \overline{\mathrm{Gr}_\lambda}$.

If $\chi^t(l) = \min_{i \in I}\{\chi_i^t(l)\} < 0$, then pick i such that $\chi_i^t(l) < 0$. Set $\mu := \lambda - \chi_i^t(l) \cdot \alpha_i^\vee$. Since $y_i(t^{\langle \alpha_i, \lambda\rangle}/\chi_i(u)) \in \mathrm{G}(\mathcal{O})$, we get

$$z_i^*(\chi_i(u)) \cdot t^\lambda = \alpha_i^\vee(1/\chi_i(u)) \cdot t^\lambda \cdot y_i(t^{\langle \alpha_i, \lambda\rangle}/\chi_i(u)) = \alpha_i^\vee(1/\chi_i(u)) \cdot t^\lambda = t^\mu. \tag{179}$$

Recall the $\mathrm{U}_w(\mathcal{K})$-orbit S_w^ν of t^ν in Gr. We have

$$\begin{aligned} u \cdot t^\lambda = u z_i(\chi_i(u)) \cdot z_i^*(\chi_i(u)) t^\lambda \overset{(179)}{=} u z_i(\chi_i(u)) \cdot t^\mu \overset{(173)}{=} \overline{s}_i^{-1} \phi_i(u) \overline{s}_i \\ \cdot t^{s_i(\mu)} \in \mathrm{S}_{s_i}^{s_i(\mu)}. \end{aligned} \tag{180}$$

It is well-known that the intersection $\mathrm{S}_w^\nu \cap \overline{\mathrm{Gr}_\lambda}$ is nonempty if and only if $t^\nu \in \overline{\mathrm{Gr}_\lambda}$. In this case $t^{s_i(\mu)} \notin \overline{\mathrm{Gr}_\lambda}$. Therefore $\mathrm{S}_{s_i}^{s_i(\mu)} \cap \overline{\mathrm{Gr}_\lambda}$ is empty. Hence $u \cdot t^\lambda \notin \overline{\mathrm{Gr}_\lambda}$. □

6 A positive structure on the configuration space $\mathrm{Conf}_{\mathbf{I}}(\mathcal{A}; \mathcal{B})$

6.1 Left G-torsors

Let G be a group. Let X be a left principal homogeneous G-space, also known as a left G-torsor. Then for any $x, y \in X$ there exists a unique $g_{x,y} \in \mathrm{G}$ such that $x = g_{x,y} y$. Clearly,

$$g_{x,y} g_{y,z} = g_{x,z}, \quad g_{gx,y} = g g_{x,y}, \; g_{x,gy} = g_{x,y} g^{-1}, \quad g \in \mathrm{G}. \tag{181}$$

Given a reference point $p \in X$, one defines a "p-distance from x to y":

$$g_p(x, y) := g_{p,x} g_{y,p} \in \mathrm{G}. \tag{182}$$

If $i_p : X \to \mathrm{G}$ is a unique isomorphism of G-sets such that $i_p(p) = e$, then $g_p(x, y) = i_p(x)^{-1} i_p(y)$.

Lemma 6.1 *One has:*

$$g_p(x, y) g_p(y, z) = g_p(x, z). \tag{183}$$

$$g_p(gx, gy) = g_p(x, y), \quad g \in \mathrm{G}. \tag{184}$$

$$y = g_p(p, y) \cdot p. \tag{185}$$

Fig. 32 A frame $\{A_i, B_j\}$

Proof Indeed,

$$
\begin{aligned}
&g_p(x,y)g_p(y,z) = g_{p,x}g_{y,p}g_{p,y}g_{z,p} = g_{p,x}g_{z,p} = g_p(x,z),\\
&g_p(gx,gy) = g_{p,gx}g_{gy,p} \overset{(181)}{=} g_{p,x}g^{-1}gg_{y,p} = g_{p,x}g_{y,p} = g_p(x,y),\\
&y = g_{y,p}\cdot p = g_{p,p}g_{y,p}\cdot p = g_p(p,y)\cdot p.
\end{aligned}
$$

□

Recall $\mathcal{F}_{\mathrm{G}}$ in Definition 2.2. From now on, we apply the above construction in the set-up

$$X = \mathcal{F}_{\mathrm{G}}, \quad p = \{\mathrm{U}, \mathrm{B}^-\}.$$

Pick a collection $\{A_1, \ldots, A_n\}$ representing a configuration in $\mathrm{Conf}_n(\mathcal{A})$. We assign A_i to the vertices of a convex n-gon, so that they go clockwise around the polygon. Each oriented pair $\{A_i, A_j\}$ provides a frame $\{A_i, B_j\}$, shown on Fig. 32 by an arrow with a white dot.

6.2 Basic invariants associated to a generic configuration

We introduce several invariants that will be useful in the rest of this paper. We employ $\cdot$ to denote the action of G on (decorated) flags.
The invariant $u^{A_2}_{B_1,B_3} \in \mathrm{U}$. Let $(B_1, A_2, B_3) \in \mathrm{Conf}(\mathcal{B}, A, B)$ be a generic configuration. Set

$$u^{A_2}_{B_1,B_3} := g_{\{\mathrm{U},\mathrm{B}^-\}}(\{A_2, B_1\}, \{A_2, B_3\}). \tag{186}$$

By (184), the invariant $u^{A_2}_{B_1,B_3}$ is independent of the representative chosen. Clearly, $u^{A_2}_{B_1,B_3} \in \mathrm{U}$.
The invariant $h_{A_1,A_2} \in \mathrm{H}$. Let (A_1, A_2) be a generic configuration. There is a unique element $h_{A_1,A_2} \in \mathrm{H}$ such that

$$(A_1, A_2) = (\mathrm{U},\ h_{A_1,A_2}\overline{w}_0 \cdot \mathrm{U}). \tag{187}$$

Using the notation (182), we have

$$h_{A_1,A_2}\overline{w}_0 = g_{\{\mathrm{U},\mathrm{B}^-\}}(\{A_1, B_2\}, \{A_2, B_1\}). \tag{188}$$

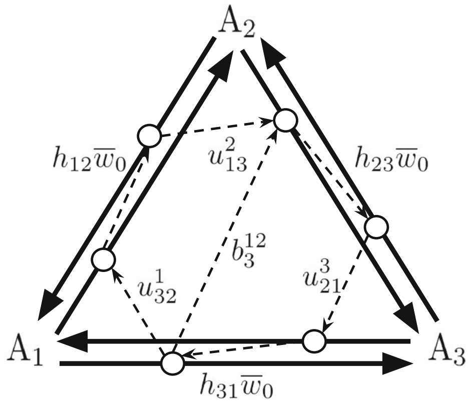


Fig. 33 Invariants of a configuration

The invariant $b^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3} \in \mathrm{B}^-$. Let $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3)$ be a generic configuration. Define

$$b^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3} := g_{\{\mathrm{U},\mathrm{B}^-\}}(\{\mathrm{A}_1, \mathrm{B}_3\}, \{\mathrm{A}_2, \mathrm{B}_3\}) \in \mathrm{B}^-.$$

Relations between basic invariants. Let $(\mathrm{A}_1, \ldots, \mathrm{A}_n) \in \mathrm{Conf}^*_n(\mathcal{A})$. Set

$$h_{ij} := h_{\mathrm{A}_i,\mathrm{A}_j} \in \mathrm{H}, \quad u^j_{ik} := u^{\mathrm{A}_j}_{\mathrm{B}_i,\mathrm{B}_k} \in \mathrm{U}, \quad b^{ij}_k := b^{\mathrm{A}_i,\mathrm{A}_j}_{\mathrm{B}_k} \in \mathrm{B}^-. \tag{189}$$

We denote these invariants by dashed arrows, see Fig. 33.

Lemma 6.2 *The data* (189) *satisfy the following relations:*

1. $h_{12}\overline{w}_0 h_{21}\overline{w}_0 = 1$.
2. $u^1_{23}u^1_{34} = u^1_{24}$, *in particular* $u^1_{23}u^1_{32} = 1$.
3. $b^{12}_4 b^{23}_4 = b^{13}_4$.
4. $b^{12}_3 = u^1_{32}h_{12}\overline{w}_0 u^2_{13} = h_{13}\overline{w}_0 u^3_{12}\overline{w}_0^{-1}h^{-1}_{23}$.
5. $u^1_{32}h_{12}\overline{w}_0 u^2_{13}h_{23}\overline{w}_0 u^3_{21}h_{31}\overline{w}_0 = 1$.

Proof We prove the first identity of 4. The others follow similarly. Let $p = \{\mathrm{U}, \mathrm{B}^-\}$. Let

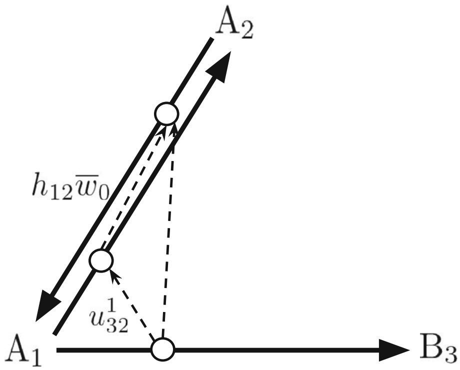
$$x_1 = \{\mathrm{A}_1, \mathrm{B}_3\},\ x_2 = \{\mathrm{A}_1, \mathrm{B}_2\},\ x_3 = \{\mathrm{A}_2, \mathrm{B}_1\},\ x_4 = \{\mathrm{A}_2, \mathrm{B}_3\}.$$

As illustrated by Fig. 33,

$$b^{12}_3 = g_p(x_1, x_4),\ u^1_{32} = g_p(x_1, x_2),\ h_{12}\overline{w}_0 = g_p(x_2, x_3),\ u^2_{13} = g_p(x_3, x_4).$$

By (183), we get $g_p(x_1, x_4) = g_p(x_1, x_2)g_p(x_2, x_3)g_p(x_3, x_4)$. ◻

Fig. 34 Invariants of a configuration (A_1, A_2, B_3)



Lemma 6.3 *Let* $x \in \mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})$ *be a generic configuration. Then it has a unique representative* $\{A_1, A_2, B_3\}$ *with* $\{A_1, B_3\} = \{U, B^-\}$. *Such a representative is*

$$\left\{U, u^1_{32} h_{12}\overline{w}_0 \cdot U, B^-\right\}. \tag{190}$$

Proof The existence and uniqueness are clear. It remains to show that it is (190). By Fig. 34,

$$g_{\{U,B^-\}}(\{A_1, B_3\}, \{A_2, B_1\}) = u^1_{32} h_{12}\overline{w}_0. \tag{191}$$

If $\{A_1, B_3\} = \{U, B^-\}$, then by (185), we get

$$\{A_2, B_1\} = g_{\{U,B^-\}}(\{A_1, B_3\}, \{A_2, B_1\}) \cdot \{U, B^-\} = \left\{u^1_{32} h_{12}\overline{w}_0 \cdot U, B\right\}.$$

□

Each $b \in B^-$ can be decomposed as $b = y_l \cdot h = h \cdot y_r$ where $h \in H$, $y_l, y_r \in U^-$. Thus B^- has a positive structure induced by positive structures on U^- and H. There are three positive maps

$$\pi_l, \pi_r : B^- \longrightarrow U^-, \ \pi_h : B^- \longrightarrow H, \quad \pi_l(b) = y_l, \ \pi_r(b) = y_r, \ \pi_h(b) = h. \tag{192}$$

These maps give rise to three more invariants.

The invariant $\mu^{A_1,A_2}_{B_3} \in H$. For each generic (A_1, A_2, B_3), we define

$$\mu^{A_1,A_2}_{B_3} := \pi_h\left(b^{A_1,A_2}_{B_3}\right). \tag{193}$$

The invariant $r_{\mathrm{B}_3}^{\mathrm{B}_1,\mathrm{A}_2} \in \mathrm{U}^-$. For any $h \in \mathrm{H}$, we have

$$b_{\mathrm{B}_3}^{\mathrm{A}_1\cdot h^{-1},\mathrm{A}_2} = h \cdot b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2}. \tag{194}$$

Thus we can define

$$r_{\mathrm{B}_3}^{\mathrm{B}_1,\mathrm{A}_2} := \pi_r\left(b_{\mathrm{B}_3}^{\mathrm{A}_1\cdot h^{-1},\mathrm{A}_2}\right) = \pi_r\left(b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2}\right) \in \mathrm{U}^-. \tag{195}$$

The invariant $l_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{B}_2} \in \mathrm{U}^-$. For any $h \in \mathrm{H}$, we have

$$b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2\cdot h} = b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2} \cdot h. \tag{196}$$

Define

$$l_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{B}_2} := \pi_l\left(b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2\cdot h}\right) = \pi_l\left(b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2}\right) \in \mathrm{U}^-. \tag{197}$$

For simplicity, we set

$$\mu_k^{ij} := \mu_{\mathrm{B}_k}^{\mathrm{A}_i,\mathrm{A}_j} \in \mathrm{H}, \quad r_k^{ij} := r_{\mathrm{B}_k}^{\mathrm{B}_i,\mathrm{A}_j} \in \mathrm{U}^-, \quad l_k^{ij} := l_{\mathrm{B}_k}^{\mathrm{A}_i,\mathrm{B}_j} \in \mathrm{U}^-. \tag{198}$$

Recall that $\widetilde{u} = \overline{w}_0 u^{-1} \overline{w}_0^{-1}$. By Relations 3, 4 of Lemma 6.2, we get

$$\mu_4^{12}\mu_4^{23} = \mu_4^{13}. \tag{199}$$

$$b_3^{12} = l_3^{12}\mu_3^{12} = \mu_3^{12}r_3^{12} = u_{32}^1 h_{12}\overline{w}_0 u_{13}^2 = h_{13}\widetilde{u_{21}^3}h_{23}^{-1}. \tag{200}$$

Recall the morphisms Φ, η and β in Sect. 5. By the definition of these morphisms, we get

Lemma 6.4 *We have*

1. $u_{32}^1 = \Phi(l_3^{12})$.
2. $\underset{\smile}{r_3^{12}} = \eta(u_{13}^2)$.
3. $\widetilde{u_{21}^3} = \mathrm{Ad}_{h_{13}^{-1}}(l_3^{12}) = \mathrm{Ad}_{h_{23}^{-1}}(r_3^{12})$.
4. $\mu_3^{12} = h_{12}\beta(u_{13}^2) = h_{13}h_{23}^{-1}, \quad beta(u_{13}^2) = h_{13}h_{23}^{-1}h_{12}^{-1}$.

Proof By (200), we have

$$l_3^{12}\mu_3^{12} = u_{32}^1\left(h_{13}\overline{w}_0 u_{13}^2\right).$$

The first identity follows. Similarly, the second identity follows from

$$\mu_3^{12} r_3^{12} = \left(u_{32}^1 h_{13} \overline{w}_0\right) u_{13}^2$$

The third identity follows from

$$l_3^{12}\mu_3^{12} = h_{13}\widetilde{u_{21}^3} h_{23}^{-1} = \mathrm{Ad}_{h_{13}}\left(\widetilde{u_{21}^3}\right) h_{13}h_{23}^{-1}, \quad \mu_3^{12} r_3^{12} = h_{13}h_{23}^{-1}\mathrm{Ad}_{h_{23}}\left(r_3^{12}\right).$$

The identity $\mu_3^{12} = h_{12}\beta(u_{13}^2)$ follows from

$$\mu_3^{12} r_3^{12} = u_{32}^1 h_{12} \cdot \left(\overline{w}_0 u_{13}^2\right).$$

The identity $\mu_3^{12} = h_{13}h_{23}^{-1}$ follows from

$$l_3^{12}\mu_3^{12} = \mathrm{Ad}_{h_{13}}\left(\widetilde{u_{21}^3}\right) h_{13}h_{23}^{-1}.$$

□

Lemma 6.5 *We have*

$$\chi(u_{21}^3) = \sum_{i\in I} \frac{\alpha_i(h_{13})}{\mathcal{L}_i(u_{32}^1)} = \sum_{i\in I} \frac{\alpha_i(h_{23})}{\mathcal{R}_i(u_{13}^2)}. \tag{201}$$

$$\alpha_i(h_{12}) = \alpha_{i^*}(h_{21}), \quad \forall i \in I. \tag{202}$$

Proof Use Lemmas 5.5, 6.4, 5.6 and 5.8 , we get

$$\begin{aligned}\chi\left(u_{21}^3\right) &= \chi^-\left(\widetilde{u_{21}^3}\right) = \chi^-\left(\mathrm{Ad}_{h_{13}^{-1}}(l_3^{12})\right)\\ &= \sum_{i\in I}\alpha_i(h_{13})\chi_i^-\left(l_3^{12}\right) = \sum_{i\in I}\frac{\alpha_i(h_{13})}{\mathcal{L}_i(u_{32}^1)}.\end{aligned}$$

By the same argument, we get the other identity in (201). By Relation 1 of Lemma 5.8, we get

$$h_{12} = \overline{w}_0 h_{21}^{-1}\overline{w}_0^{-1}\cdot s_{\mathrm{G}}.$$

Then (202) follows. □

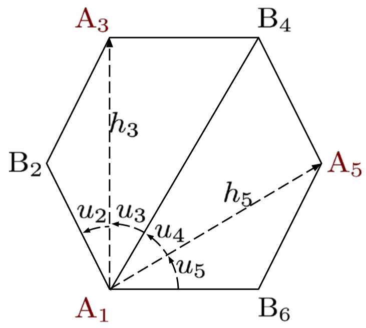


Fig. 35 The map α_1 for $\mathrm{I} = \{1, 3, 5\} \subset [1, 6]$

6.3 A positive structure on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$

Let $\mathrm{I} \subset [1, n]$ be a nonempty subset of cardinality m. Following [17, Section 8], there is a positive structure on the configuration space $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$. We briefly recall it below.

Let $x = (x_1, \ldots, x_n) \in \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ be a generic configuration such that

$$x_i = \mathrm{A}_i \in \mathcal{A} \text{ when } i \in \mathrm{I}, \text{ otherwise } x_i = \mathrm{B}_i \in \mathcal{B}. \tag{203}$$

Set $\mathrm{B}_j := \pi(\mathrm{A}_j)$ when $j \in \mathrm{I}$. Let $i \in \mathrm{I}$. For each $k \in [2, n]$, set

$$u_k^i(x) := u^{\mathrm{A}_i}_{\mathrm{B}_{i+k}, \mathrm{B}_{i+k-1}}, \quad \text{where the subscript is modulo } n. \tag{204}$$

For each pair $i, j \in \mathrm{I}$, recall

$$\pi_{ij}(x) := \begin{cases} h_{\mathrm{A}_i, \mathrm{A}_j}, & \text{if } i < j, \\ h_{s_{\mathrm{G}} \cdot \mathrm{A}_i, \mathrm{A}_j}, & \text{if } i > j. \end{cases} \tag{205}$$

Lemma 6.6 *Fix $i \in \mathrm{I}$. The following morphism is birational*

$$\alpha_i : \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}) \longrightarrow \mathrm{H}^{m-1} \times \mathrm{U}^{n-2}, \quad x \longmapsto (\{\pi_{ij}(x)\}, \{u_k^i(x)\}),$$
$$j \in \mathrm{I} - \{i\}, \; k \in [2, n-1].$$

Example. Figure 35 illustrates the map α_1 for $\mathrm{I} = \{1, 3, 5\} \subset [1, 6]$.

Proof Assume that $i = 1 \in \mathrm{I}$. Clearly α_1 is well defined on the subspace

$$\widetilde{\mathrm{Conf}}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}) := \{(x_1, \ldots, x_n) \mid (x_1, x_k) \text{ is generic for all } k \in [2, n]\}.$$

Note that $\widetilde{\mathrm{Conf}}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ is dense in $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$. We prove the Lemma by showing that α_1 is a bijection from $\widetilde{\mathrm{Conf}}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ to $\mathrm{H}^{m-1} \times \mathrm{U}^{n-2}$, Let $y = (\{h_j\}, \{u_k\}) \in \mathrm{H}^{m-1} \times \mathrm{U}^{n-2}$. Set $u'_n := 1$. Set $u'_k := u_{n-1} \ldots u_k$ for $k \in [2, n-1]$. Let $x = (x_1, \ldots, x_n) \in \widetilde{\mathrm{Conf}}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ such that

$$x_1 := \mathrm{U}; \quad x_j := u'_j h_j \overline{w}_0 \cdot \mathrm{U} \in \mathcal{A}, \ j \in \mathrm{I} - \{1\}; \quad x_k := u'_k \cdot \mathrm{B}^- \in \mathcal{B}, \ k \notin \mathrm{I}. \tag{206}$$

Clearly $\alpha_1(x) = y$. Hence α_1 is a surjection.

Let $x \in \widetilde{\mathrm{Conf}}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ such that $\alpha_1(x) = y$. Note that x has a unique representative $\{x_1, \ldots, x_n\}$ such that $\{x_1, x_n\} = \{\mathrm{U}, \mathrm{B}^-\}$ if $n \notin \mathrm{I}$, and $\{x_1, \pi(x_n)\} = \{\mathrm{U}, \mathrm{B}^-\}$ if $n \in \mathrm{I}$. By Lemma 6.3, each x_i is uniquely expressed by (206). The injectivity of α_1 follows. □

The product $\mathrm{H}^{m-1} \times \mathrm{U}^{n-2}$ has a positive structure induced by the ones on H and U.

When $\mathrm{I} = [1, n]$, we first introduce a positive structure on $\mathrm{Conf}_n(\mathcal{A})$ such that the map α_1 is a positive birational isomorphism. Such a positive structure is twisted cyclic invariant:

Theorem 6.7 [17, Section 8] *The following map is a positive birational isomorphism*

$$t : \mathrm{Conf}_n(\mathcal{A}) \stackrel{\sim}{\longrightarrow} \mathrm{Conf}_n(\mathcal{A}), \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longmapsto (\mathrm{A}_2, \ldots, \mathrm{A}_n, \mathrm{A}_1 \cdot s_{\mathrm{G}}).$$

Each α_i determines a positive structure on $\mathrm{Conf}_n(\mathcal{A})$. Theorem 6.7 tells us that these positive structures coincide. We prove the same result for $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$, using the following Lemmas.

Lemma 6.8 *Let $\mathcal{Y}$ be a space equipped with two positive structures denoted by $\mathcal{Y}^1$ and $\mathcal{Y}^2$. If for every rational function f on $\mathcal{Y}$, we have*

$$f \text{ is positive on } \mathcal{Y}^1 \Longleftrightarrow f \text{ is positive on } \mathcal{Y}^2,$$

then $\mathcal{Y}^1$ and $\mathcal{Y}^2$ share the same positive structure.

Proof It is clear. □

Lemma 6.9 *Let $\mathcal{Y}$, $\mathcal{Z}$ be a pair of positive spaces. If there are two positive maps $\gamma : \mathcal{Y} \to \mathcal{Z}$ and $\beta : \mathcal{Z} \to \mathcal{Y}$ such that $\beta \circ \gamma = \mathrm{id}_{\mathcal{Y}}$, then for every rational function f on $\mathcal{Y}$ we have*

$$f \text{ is positive on } \mathcal{Y} \Longleftrightarrow \beta^*(f) \text{ is positive on } \mathcal{Z}.$$

Proof If f is positive on $\mathcal{Y}$, since β is a positive morphism, then $\beta^*(f)$ is positive on $\mathcal{Z}$.

If $\beta^*(f)$ is positive on $\mathcal{Z}$, since γ is a positive morphism, then $\gamma^*(\beta^*(f)) = f$ is positive. □

Lemma 6.10 *Every* α_i $(i \in \mathrm{I})$ *determines the same positive structure on* $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$.

Remark. Lemma 6.10 is equivalent to say that for any pair $i, j \in \mathrm{I}$, the map $\phi_{i,j} := \alpha_i \circ \alpha_j^{-1}$ is a positive birational isomorphism of $\mathrm{H}^{m-1} \times \mathrm{U}^{n-2}$.

Proof Let us temporary denote the positive structure on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{X}; \mathcal{Y})$ by $\mathrm{Conf}_{\mathrm{I}}^i(\mathcal{A}; \mathcal{B})$ such that α_i is a positive birational isomorphism.

There is a projection $\beta : \mathrm{Conf}_n(\mathcal{A}) \to \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ which maps A_k to A_k if $k \in \mathrm{I}$ and maps A_k to $\pi(\mathrm{A}_k)$ otherwise. By Lemma 6.7, β is a positive morphism for all $\mathrm{Conf}_{\mathrm{I}}^i(\mathcal{A}; \mathcal{B})$.

Fix $i \in \mathrm{I}$. Each generic $x = (x_1, \ldots, x_n) \in \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ has a unique preimage $\gamma^i(x) := (\mathrm{A}_1, \ldots, \mathrm{A}_n) \in \mathrm{Conf}_n(\mathcal{A})$ such that

$$\mathrm{A}_j = x_j \text{ when } j \in \mathrm{I}, \text{ otherwise } \mathrm{A}_j \text{ is the preimage of } x_j$$
$$\text{such that } \pi_{ij}(\gamma^i(x)) = 1.$$

Clearly γ^i a positive morphism from $\mathrm{Conf}_{\mathrm{I}}^i(\mathcal{A}; \mathcal{B})$ to $\mathrm{Conf}_n(\mathcal{A})$. By definition $\beta \circ \gamma^i = \mathrm{id}$.

Let f be a rational function on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$. Let $i, j \in \mathrm{I}$. By Lemma 6.8,

$$f \text{ is positive on } \mathrm{Conf}_{\mathrm{I}}^i(\mathcal{A}; \mathcal{B}) \Longleftrightarrow \beta^*(f) \text{ is positive on } \mathrm{Conf}_n(\mathcal{A})$$
$$\Longleftrightarrow f \text{ is positive on } \mathrm{Conf}_{\mathrm{I}}^j(\mathcal{A}; \mathcal{B}).$$

This Lemma follows from Lemma 6.9. □

Thanks to Lemma 6.10, we introduce a canonical positive structure on $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$. From now on, we view $\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ as a positive space.

Given $k \in \mathbb{Z}/n$, we define the k-shift of the subset I by setting $\mathrm{I}(k) := \{i \in [1, n] \mid i + k \in \mathrm{I}\}$. The following Lemma is clear now.

Lemma 6.11 *The following map is a positive birational isomorphism*

$$t : \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}) \stackrel{\sim}{\longrightarrow} \mathrm{Conf}_{\mathrm{I}(1)}(\mathcal{A}; \mathcal{B}), \quad (x_1, \ldots, x_n) \longmapsto (x_2, \ldots, x_n, x_1 \cdot s_{\mathrm{G}}).$$

An invariant definition of positive structures. We have defined above positive structures on the configuration spaces using pinning in G, which allows to make calculations. Let us explain now how to define positive structures on the configurations spaces without choosing a pinning. When G is of type A_m, such a definition is given in [17, Section 9]. In general, given a decomposition of the longest Weyl group element $w_0 = s_{i_1} \ldots s_{i_n}$, for each generic pair $\{\mathrm{B}, \mathrm{B}'\}$ of flags, there exists a unique chain

$$\mathrm{B} = \mathrm{B}_0 \stackrel{i_1}{\longrightarrow} \mathrm{B}_1 \stackrel{i_2}{\longrightarrow} \ldots \stackrel{i_{n-1}}{\longrightarrow} \mathrm{B}_{n-1} \stackrel{i_n}{\longrightarrow} \mathrm{B}_n = \mathrm{B}'.$$

Here $\mathrm{B}_{k-1} \stackrel{i_k}{\to} \mathrm{B}_k$ indicates that $\{\mathrm{B}_{k-1}, \mathrm{B}_k\}$ is in the position s_{i_k}. The positive structure of $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B})$ can be defined via the birational map

$$\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) \longrightarrow (\mathbb{G}_m)^n, \quad (\mathrm{B}, \mathrm{A}, \mathrm{B}') \longmapsto (\chi^o(\mathrm{B}_0, \mathrm{A}, \mathrm{B}_1), \\ \chi^o(\mathrm{B}_1, \mathrm{A}, \mathrm{B}_2), \ldots, \chi^o(\mathrm{B}_{n-1}, \mathrm{A}, \mathrm{B}_n)).$$

Each generic pair $\{\mathrm{A}, \mathrm{A}'\} \in \mathcal{A}^2$ uniquely determines a pinning for G such that

$$x_i(a) \in \mathrm{U}_{\mathrm{A}}, \quad \chi_{\mathrm{A}}(x_i(a)) = a, \quad y_i(a) \in \mathrm{U}_{\mathrm{A}'}, \quad i \in I.$$

The pinning gives rise to a representative $\overline{w}_0 \in \mathrm{G}$ of w_0. There is a unique element $h \in \pi(\mathrm{A}) \cap \pi(\mathrm{A}')$ such that

$$\mathrm{A}' = h\overline{w}_0 \cdot \mathrm{A}.$$

Such an element h gives rise to a birational map from $\mathrm{Conf}_2(\mathcal{A})$ to the Cartan group of G, determining a positive structure of $\mathrm{Conf}_2(\mathcal{A})$. The positive structures of general configuration spaces are defined via the positive structures of $\mathrm{Conf}_2(\mathcal{A})$ and $\mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B})$.

6.4 Positivity of the potential $\mathcal{W}_{\mathrm{J}}$ and proof of Theorem 2.27

Let $\mathrm{J} \subset \mathrm{I} \subset [1, n]$. Consider the ordered triples $\{i, j, k\} \subset [1, n]$ such that

$$j \in \mathrm{J}, \text{ and } i, j, k \text{ seated clockwise.} \tag{207}$$

Let $x \in \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B})$ be presented by (203). Define $p_{j;i,k}(x) := u^{\mathrm{A}_j}_{\mathrm{B}_i, \mathrm{B}_k}$. In particular, we are interested in the triples $\{j-1, j, j+1\}$. Set

$$p_j(x) := p_{j;j-1,j+1} = u^{\mathrm{A}_j}_{\mathrm{B}_{j-1}, \mathrm{B}_{j+1}}, \quad \forall j \in \mathrm{J}. \tag{208}$$

Lemma 6.12 *The following morphisms are positive morphisms*

1. $\pi_{ij} : \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}) \longrightarrow \mathrm{H}, \ \forall\, i, j \in \mathrm{I}.$
2. $p_{j;i,k} : \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}) \longrightarrow \mathrm{U}, \ \forall\, \{i, j, k\} \in (207).$

Proof The positivity of π_{ij} is clear. By Relation 2 of Lemma 6.2, we get

$$u^{\mathrm{A}_j}_{\mathrm{B}_i, \mathrm{B}_k} = u^{\mathrm{A}_j}_{\mathrm{B}_i, \mathrm{B}_{i-1}} u^{\mathrm{A}_j}_{\mathrm{B}_{i-1}, \mathrm{B}_{i-2}} \cdots u^{\mathrm{A}_j}_{\mathrm{B}_{k+1}, \mathrm{B}_k}.$$

The product map $\mathrm{U} \times \mathrm{U} \to \mathrm{U}, \ (u_1, u_2) \mapsto u_1 u_2$ is positive. The positivity of $p_{j;i,k}$ follows. □

Positivity of the potential $\mathcal{W}_{\rm J}$. Recall the positive function χ on U. Let $x \in \mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})$ be a generic configuration presented by (203). By Lemma 6.3, each generic triple $(\mathrm{B}_{j-1}, \mathrm{A}_j, \mathrm{B}_{j+1})$ has a unique representative $\{\mathrm{B}^-, \mathrm{U}, u_{\mathrm{B}_{j-1},\mathrm{B}_{j+1}}^{\mathrm{A}_j} \cdot \mathrm{B}^-\}$. In this case u_j in (76) becomes $p_j(x)$. Therefore $\chi_{\mathrm{A}_j}(u_j) = \chi \circ p_j(x)$. The potential $\mathcal{W}_{\rm J}$ of $\mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})$ becomes

$$\mathcal{W}_{\rm J} = \sum_{j \in \rm J} \chi \circ p_j \tag{209}$$

Since p_j are positive morphisms, the positivity of $\mathcal{W}_{\rm J}$ follows.
By Relation 2 of Lemma 6.2, we get

$$\chi \circ p_j = \chi \circ p_{j;j-1,i} + \chi \circ p_{j;i,k} + \chi \circ p_{j;k,j+1} \tag{210}$$

All summands on right side are positive functions. By (209), the set $\mathrm{Conf}^+_{\rm J\subset I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$ of tropical points such that $\mathcal{W}^t_{\rm J} \geq 0$ is the set

$$\{l \in \mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t) \mid p^t_{j;i,k}(l) \in \mathrm{U}^+_\chi(\mathbb{Z}^t) \text{ for all } \{i, j, k\} \in (207)\}. \tag{211}$$

Proof of Theorem 2.27. Recall the moduli space $\mathrm{Conf}^{\mathcal{O}}_{\rm J\subset I}(\mathcal{A}; \mathcal{B})$ in Definition 2.26.

Lemma 6.13 *A generic configuration in* $\mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})(\mathcal{K})$ *is* $\mathcal{O}$*-integral relative to* J *if and only if* $u^{\mathrm{A}_j}_{\mathrm{B}_i,\mathrm{B}_k} \in \mathrm{U}(\mathcal{O})$ *for all* $\{i, j, k\} \in$(207).

Proof By definition $\mathrm{L}(\mathrm{A}_j, \mathrm{B}_k) = [g_{\{\mathrm{A}_j,\mathrm{B}_k\},\{\mathrm{U},\mathrm{B}^-\}}] \in \mathrm{Gr}$. Let $\{i, j, k\} \in (207)$. Then

$$\mathrm{L}(\mathrm{A}_j, \mathrm{B}_k) = \mathrm{L}(\mathrm{A}_j, \mathrm{B}_i) \Longleftrightarrow g^{-1}_{\{\mathrm{A}_j,\mathrm{B}_i\},\{\mathrm{U},\mathrm{B}^-\}} g_{\{\mathrm{A}_j,\mathrm{B}_k\},\{\mathrm{U},\mathrm{B}^-\}} = u^{\mathrm{A}_j}_{\mathrm{B}_i,\mathrm{B}_k} \in \mathrm{G}(\mathcal{O}).$$

The Lemma is proved. □

Let $l \in \mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})$. Let $x \in \mathcal{C}^\circ_l$ be presented by (203). By Lemma 5.2, $u^{\mathrm{A}_j}_{\mathrm{B}_i,\mathrm{B}_k} \in \mathrm{U}(\mathcal{O})$ if and only if $p^t_{j;i,k}(l) \in \mathrm{U}^+_\chi(\mathbb{Z}^t)$. Theorem 2.27 follows from Lemma 6.13 and (211). □

Tropicalizing the morphism (205), we get $\pi^t_{ij} : \mathrm{Conf}_{\rm I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t) \to \mathrm{H}(\mathbb{Z}^t) = \mathrm{P}$.

Lemma 6.14 *Let* $i, j \in \mathrm{J}$. *If* $l \in \mathrm{Conf}^+_{\rm J\subset I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$, *then* $\pi^t_{ij}(l) \in \mathrm{P}^+$.

Proof Since $\pi^t_{ij}(l) = -w_0(\pi^t_{ji}(l))$, we can assume that there exists k such that $\{i, j, k\} \in (207)$. Otherwise we switch i and j. Set $\lambda := \pi^t_{ij}(l)$, $u_1 := p^t_{i;k,j}(l)$,

$u_2 := p^t_{j;i,k}(l)$. We tropicalize (201):

$$\chi^t(u_2) = \min_{r \in I} \left\{ \langle \lambda, \alpha_r \rangle - \mathcal{R}^t_r(u_1) \right\}. \tag{212}$$

If $l \in$ (211), then $\chi^t(u_1) \geq 0$, $\chi^t(u_2) \geq 0$. By the definition of $\mathcal{R}_r$ and χ, we get $\mathcal{R}^t_r(u_1) \geq \chi^t(u_1)$. Therefore $\mathcal{R}^t(u_1) \geq 0$. Hence

$$\forall r \in I, \quad \langle \lambda, \alpha_r \rangle \geq \langle \lambda, \alpha_r \rangle - \mathcal{R}^t_r(u_1) \geq \chi^t(u_2) \geq 0 \implies \lambda \in \mathrm{P}^+.$$

$\square$

7 Main examples of configuration spaces

As discussed in Sect. 1, the pairs of configuration spaces especially important in representation theory are:

$$\{\mathrm{Conf}_n(\mathcal{A}), \quad \mathrm{Conf}_n(\mathrm{Gr})\}, \quad \left\{\mathrm{Conf}(\mathcal{A}^n, \mathcal{B}), \quad \mathrm{Conf}(\mathrm{Gr}^n, \mathcal{B})\right\},$$
$$\left\{\mathrm{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}), \quad \mathrm{Conf}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B})\right\}.$$

In Sect. 7 we express the potential $\mathcal{W}$ and the map κ in these cases under explicit coordinates.

7.1 The configuration spaces $\mathrm{Conf}_n(\mathcal{A})$ and $\mathrm{Conf}_n(\mathrm{Gr})$

Recall h_{ij}, u^k_{ij} in (189). Recall the positive birational isomorphism

$$\begin{aligned} \alpha_1 : \mathrm{Conf}_n(\mathcal{A}) &\stackrel{\sim}{\longrightarrow} \mathrm{H}^{n-1} \times \mathrm{U}^{n-2}, \\ (\mathrm{A}_1, \ldots, \mathrm{A}_n) &\longmapsto (h_{12}, \ldots, h_{1n}, u^1_{3,2}, \ldots, u^1_{n,n-1}). \end{aligned} \tag{213}$$

The potential $\mathcal{W}$ on $\mathrm{Conf}_n(\mathcal{A})$ induces a positive function $\mathcal{W}_{\alpha_1} := \mathcal{W} \circ \alpha_1{}^{-1}$ on $\mathrm{H}^{n-1} \times \mathrm{U}^{n-2}$.

Theorem 7.1 *The function*

$$\begin{aligned} &\mathcal{W}_{\alpha_1}(h_2, \ldots, h_n, u_2, \ldots, u_{n-1}) \\ &\quad = \sum_{j=2}^{n-1} \left(\chi(u_j) + \sum_{i \in I} \frac{\alpha_i(h_j)}{\mathcal{R}_i(u_j)} + \sum_{i \in I} \frac{\alpha_i(h_{j+1})}{\mathcal{L}_i(u_j)} \right). \end{aligned} \tag{214}$$

Proof By the scissor congruence invariance (89), we get $\mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n) = \sum_{j=2}^{n-1} \mathcal{W}(\mathrm{A}_1, \mathrm{A}_j, \mathrm{A}_{j+1})$. The rest follows from (209) and Lemma 6.5. $\square$

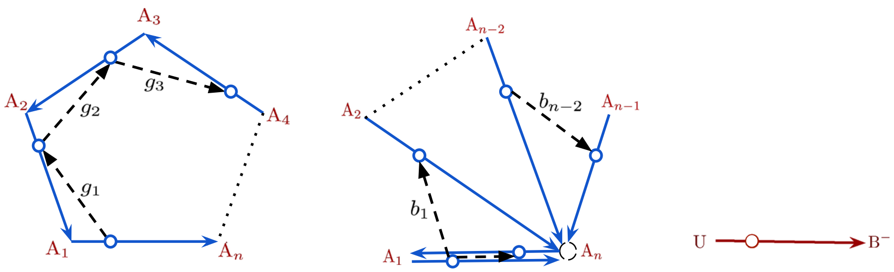


Fig. 36 The map ω expressed by two different choices of frames $\{A_i, B_{\alpha(i)}\}$

Let us choose a map without stable points which is not necessarily a bijection:

$$\alpha : [1, n] \longrightarrow [1, n], \quad \alpha(k) \neq k.$$

Let $x = (A_1, \ldots, A_n) \in \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$. Define

$$\omega_k(x) := [g_{\{U,B^-\}}(\{A_1, B_n\}, \{A_k, B_{\alpha(k)}\})] \in \mathrm{Gr}. \tag{215}$$

By the definition of $\mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$, the map ω_k is independent of the map α chosen. Define

$$\omega := (\omega_2, \ldots, \omega_n) : \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A}) \longrightarrow \mathrm{Gr}^{n-1}, \quad x \longmapsto (\omega_2(x), \ldots, \omega_n(x)). \tag{216}$$

Consider the projection

$$i_1 : \mathrm{Gr}^{n-1} \longrightarrow \mathrm{Conf}_n(\mathrm{Gr}), \quad \{L_2, \ldots, L_n\} \longmapsto ([1], L_2, \ldots, L_n)$$

Lemma 7.2 *The map κ in (97) is $i_1 \circ \omega$.*

Proof Here $\omega_k(x) = g_{\{U,B^-\},\{A_1,B_n\}} L(A_k, B_{\alpha(k)})$. In particular $\omega_1(x) = [1]$. The Lemma follows. □

Below we give two explicit expressions of ω based on different choices of the map α. We emphasize that although the expressions look entirely different from each other, they are the same map. As before, set $x = (A_1, \ldots, A_n) \in \mathrm{Conf}_n^{\mathcal{O}}(\mathcal{A})$.

1. Let $\alpha(k) = k - 1$. It provides frames $\{A_i, B_{i-1}\}$, see the first graph of Fig. 36. Set

$$g_k := g_{\{U,B^-\}}(\{A_k, B_{k-1}\}, \{A_{k+1}, B_k\}) \stackrel{*}{=} u^{A_k}_{B_{k-1},B_{k+1}} h_{A_k,A_{k+1}} \overline{w}_0. \tag{217}$$

See Fig. 34 for proof of $*$. By (183), we get

$$\omega_k(x) = [g_{\{\mathrm{U},\mathrm{B}^-\}}(\{\mathrm{A}_1, \mathrm{B}_n\}, \{\mathrm{A}_k, \mathrm{B}_{k-1}\})] = [g_1 \ldots g_{k-1}], \quad k \in [2, n] \tag{218}$$

Therefore

$$\omega(x) = ([g_1], \ldots, [g_1 \ldots g_{n-1}]) \in \mathrm{Gr}^{n-1}. \tag{219}$$

2. Let $\alpha(k) = n$ when $k \neq n$. Let $\alpha(n) = 1$. See the second graph of Fig. 36. Set

$$b_k := b_{\mathrm{B}_n}^{\mathrm{A}_k, \mathrm{A}_{k+1}}, \ k \in [1, n-2]; \quad h_n := h_{\mathrm{A}_1, \mathrm{A}_n}.$$

Then

$$\begin{aligned} \omega_k(x) = [g_{\{\mathrm{U},\mathrm{B}^-\}}(\{\mathrm{A}_1, \mathrm{B}_n\}, \{\mathrm{A}_k, \mathrm{B}_n\})] = [b_1 \ldots b_{k-1}], \\ k \in [2, n-1]; \quad \omega_n(x) = [h_n]. \end{aligned} \tag{220}$$

Therefore

$$\omega(x) = ([b_1], \ldots, [b_1 \ldots b_{n-2}], [h_n]) \in \mathrm{Gr}^{n-1}. \tag{221}$$

7.2 The configuration spaces $\mathrm{Conf}(\mathcal{A}^n, \mathcal{B})$ and $\mathrm{Conf}(\mathrm{Gr}^n, \mathcal{B})$

Consider the scissoring morphism

$$\begin{aligned} s : \mathrm{Conf}(\mathcal{A}^{m+n+1}, \mathcal{B}) &\longrightarrow \mathrm{Conf}(\mathcal{A}^{m+1}, \mathcal{B}) \times \mathrm{Conf}(\mathcal{A}^{n+1}, \mathcal{B}), \\ (\mathrm{A}_1, \ldots, \mathrm{A}_{m+n+1}, \mathrm{B}_0) &\longmapsto (\mathrm{A}_1, \ldots, \mathrm{A}_{m+1}, \mathrm{B}_0) \times (\mathrm{A}_{m+1}, \ldots, \mathrm{A}_{m+n+1}, \mathrm{B}_0). \end{aligned} \tag{222}$$

By Lemmas 6.6, 6.10, the morphism s is a positive birational isomorphism. In fact, the inverse map of s can be defined by "gluing" two configurations:

$$\begin{aligned} * : \mathrm{Conf}^*(\mathcal{A}^{m+1}, \mathcal{B}) \times \mathrm{Conf}^*(\mathcal{A}^{n+1}, \mathcal{B}) &\longrightarrow \mathrm{Conf}(\mathcal{A}^{m+n+1}, \mathcal{B}), \\ (a, b) &\longmapsto a * b. \end{aligned} \tag{223}$$

By Lemma 6.3, a has a unique representative $\{\mathrm{A}_1, \ldots, \mathrm{A}_m, \mathrm{U}, \mathrm{B}^-\}$, b has a unique representative $\{\mathrm{U}, \mathrm{A}'_1, \ldots, \mathrm{A}'_n, \mathrm{B}^-\}$. We define the *convolution product* $a * b := (\mathrm{A}_1, \ldots, \mathrm{A}_m, \mathrm{U}, \mathrm{A}'_1, \ldots, \mathrm{A}'_n, \mathrm{B}^-)$. The associativity of the convolution product is clear.

Recall b_k^{ij} in (189). Recall the morphisms π_r, π_l in (192).

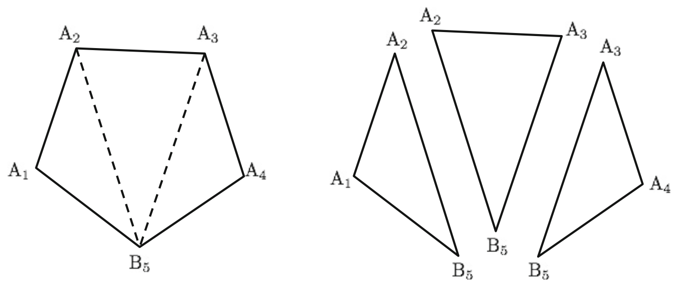


Fig. 37 A map given by scissoring a convex pentagon

Theorem 7.3 *The following morphism is a positive birational isomorphism*

$$
\begin{aligned}
c: \mathrm{Conf}(\mathcal{A}^n, \mathcal{B}) \longrightarrow (\mathrm{B}^-)^{n-1}, \quad & (\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B}_{n+1}) \\
& \longmapsto \left(b_{n+1}^{1,2}, \ldots, b_{n+1}^{i,i+1}, \ldots, b_{n+1}^{n-1,n}\right).
\end{aligned} \tag{224}
$$

Proof Scissoring the convex $(n{+}1)$-gon along diagonals emanating from $n{+}1$, see Fig. 37, we get a positive birational isomorphism $\mathrm{Conf}(\mathcal{A}^n, \mathcal{B}) \stackrel{\sim}{\rightarrow} (\mathrm{Conf}(\mathcal{A}^2, \mathcal{B}))^{n-1}$. The Theorem is therefore reduced to $n = 2$. Recall α_2 in Lemma 6.6. By Lemma 6.4, it is equivalent to prove that $\mathrm{H} \times \mathrm{U} \to \mathrm{H} \times \mathrm{U}^-$, $(h, u) \mapsto (\beta(u)h, \eta(u))$ is a positive birational isomorphism. Since η is a positive birational isomorphism, and β is a positive map, the Theorem follows. □

The potential $\mathcal{W}$ on $\mathrm{Conf}(\mathcal{A}^n, \mathcal{B})$ induces a positive function $\mathcal{W}_c = \mathcal{W} \circ c^{-1}$ on $(\mathrm{B}^-)^{n-1}$.

Lemma 7.4 *The function*

$$
\mathcal{W}_c(b_1, \ldots, b_{n-1}) = \sum_{j=1}^{n-1} \sum_{i \in I} \left(\frac{1}{\mathcal{L}_i^- \circ \pi_l(b_j)} + \frac{1}{\mathcal{R}_i^- \circ \pi_r(b_j)} \right) \tag{225}
$$

Proof Note that

$$
\begin{aligned}
\mathcal{W}(\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B}_{n+1}) &= \sum_{j=1}^{n-1} \mathcal{W}(\mathrm{A}_j, \mathrm{A}_{j+1}, \mathrm{B}_{n+1}) \\
&= \sum_{j=1}^{n-1} \left(\chi(u_{n+1,j+1}^{j}) + \chi(u_{j,n+1}^{j+1}) \right).
\end{aligned}
$$

The Lemma follows directly from Lemma 5.8, (172) and Lemma 6.4. □

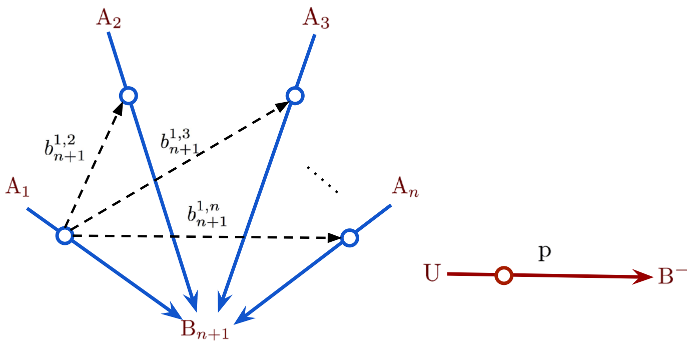


Fig. 38 Frames assigned to $(\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B}_{n+1})$

Define

$$\tau : \mathrm{Conf}^{\mathcal{O}}\left(\mathcal{A}^n, \mathcal{B}\right) \longrightarrow \mathrm{Gr}^{n-1}, \quad (\mathrm{A}_1, \ldots, \mathrm{A}_n) \longmapsto \{[b^{1,2}_{n+1}], \ldots, [b^{1,n}_{n+1}]\}. \tag{226}$$

Consider the projection

$$i_b : \mathrm{Gr}^{n-1} \longrightarrow \mathrm{Conf}(\mathrm{Gr}^n, \mathcal{B}), \quad \{\mathrm{L}_2, \ldots, \mathrm{L}_n\} \longmapsto ([1], \mathrm{L}_2, \ldots, \mathrm{L}_n, \mathrm{B}^-).$$

Recall the map κ in (109). As illustrated by Fig. 38, we get

Lemma 7.5 *When* $\mathrm{J} = \mathrm{I} = [1, n] \subset [1, n+1]$*, we have* $\kappa = i_b \circ \tau$.

7.3 The configuration spaces $\mathrm{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B})$ and $\mathrm{Conf}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B})$

Recall r^{ij}_k in (198). Similarly, there is a positive birational isomorphism

$$\begin{aligned} p : \mathrm{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}) &\longrightarrow \mathrm{U}^- \times (\mathrm{B}^-)^{n-1}, \quad (\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2}) \\ &\longmapsto \left(r^{1,2}_{n+2}, b^{2,3}_{n+2}, \ldots, b^{n,n+1}_{n+2}\right). \end{aligned} \tag{227}$$

The potential $\mathcal{W}$ on $\mathrm{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B})$ induces a positive function $\mathcal{W}_p := \mathcal{W} \circ p^{-1}$ on $\mathrm{U}^- \times (\mathrm{B}^-)^{n-1}$. We have

$$\begin{aligned} \mathcal{W}_p(r_1, b_2, \ldots, b_n) = &\sum_{i \in I} \frac{1}{\mathcal{R}^-_i(r_1)} \\ &+ \sum_{2 \leq j \leq n} \sum_{i \in I} \left(\frac{1}{\mathcal{L}^-_i \circ \pi_l(b_j)} + \frac{1}{\mathcal{R}^-_i \circ \pi_r(b_j)} \right). \end{aligned} \tag{228}$$

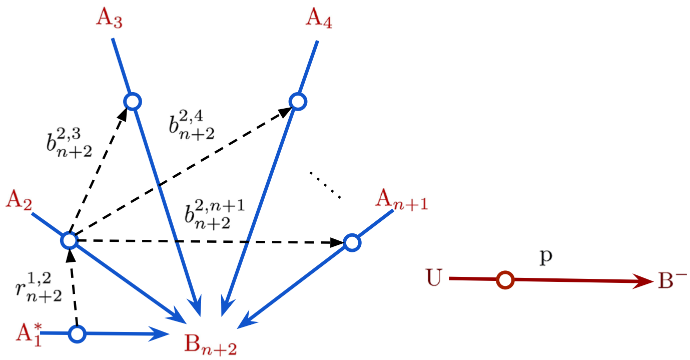


Fig. 39 Frames assigned to $(\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2})$. Here $\pi(\mathrm{A}_1^*) = \mathrm{B}_1$

Recall the map κ in (109). Define

$$\tau_s : \mathrm{Conf}_{w_0}^{\mathcal{O}}(\mathcal{A}, \mathcal{B}^n, \mathcal{A}) \longrightarrow \mathrm{Gr}^n, \quad (\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2}) \\ \longmapsto \left(\left[r_{n+2}^{1,2}\right], \left[r_{n+2}^{1,2} b_{n+2}^{2,3}\right], \ldots, \left[r_{n+2}^{1,2} b_{n+2}^{2,n+1}\right]\right). \tag{229}$$

Consider the projection

$$i_s : \mathrm{Gr}^n \longrightarrow \mathrm{Conf}_{w_0}(\mathcal{B}, \mathrm{Gr}^n, \mathcal{B}), \ \{\mathrm{L}_2, \ldots, \mathrm{L}_{n+1}\} \longmapsto (\mathrm{B}, \mathrm{L}_2, \ldots, \mathrm{L}_{n+1}, \mathrm{B}^-).$$

Let $x = (\mathrm{B}_1, \mathrm{A}_2, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2}) \in \mathrm{Conf}_{w_0}^{\mathcal{O}}(\mathcal{A}, \mathcal{B}^n, \mathcal{A})$. Let $\mathrm{A}_1^* \in \mathcal{A}$ be the preimage of B_1 such that $b_{\mathrm{B}_{n+2}}^{\mathrm{A}_1^*, \mathrm{A}_2} = r_{n+2}^{1,2}$. As illustrated by Fig. 39, we get

Lemma 7.6 *When* $\mathrm{J} = \mathrm{I} = [2, n+1] \subset [1, n+2]$, *we have* $\kappa = i_s \circ \tau_s$.

8 Proof of Theorems 2.24 and 2.36

8.1 Lemmas

Let $\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_k$ be a product of positive spaces. The positive structure on $\mathcal{Y}$ is induced by positive structures on $\mathcal{Y}_i$. Let $y_i \in \mathcal{Y}_i^{\circ}(\mathcal{K})$. Let $(y_{i,1}, \ldots, y_{i,n_i})$ be the coordinate of y_i in a positive coordinate system $\mathbf{c}_i$. Define the field extension

$$\mathbb{Q}(y_1, \ldots, y_k) := \mathbb{Q}\left(\mathrm{in}(y_{1,1}), \ldots, \mathrm{in}(y_{1,n_1}), \ldots, \mathrm{in}(y_{k,n_k})\right). \tag{230}$$

Thanks to (155), such an extension is independent of the positive coordinate systems chosen.

Recall the morphisms π_l, π_r in (192).

Lemma 8.1 *Fix $i \in I$. Let $(b, c) \in (\mathrm{B}^- \times \mathbb{G}_m)^\circ(\mathcal{K})$. Recall $y_i(c) \in \mathrm{U}^-(\mathcal{K})$. Then $b' := b \cdot y_i(c) \in (\mathrm{B}^-)^\circ(\mathcal{K})$.*

Moreover, if $\mathrm{val}(\mathcal{R}_i^- \circ \pi_r(b)) \leq \mathrm{val}(c)$*, then* $\mathrm{val}(b') = \mathrm{val}(b)$ *and* $\mathbb{Q}(b', c) = \mathbb{Q}(b, c)$.

Proof Let $b = h \cdot y$. Fix a reduced word for w_0 which ends with $i_m = i$. It provides a decomposition $y = y_{i_1}(c_1) \ldots y_{i_m}(c_m)$. Then $b' = h \cdot y_{i_1}(c_1) \ldots y_{i_m}(c_m + c)$. The rest is clear. □

Lemma 8.2 *Let $(b, h) \in (\mathrm{B}^- \times \mathrm{H})^\circ(\mathcal{K})$. Then $b' := b \cdot h \in (\mathrm{B}^-)^\circ(\mathcal{K})$.*

Moreover, if $h \in \mathrm{H}(\mathbb{C})$, then $\mathrm{val}(b') = \mathrm{val}(b)$ *and* $\mathbb{Q}(b', h) = \mathbb{Q}(b, h)$.

Proof Let $b = y \cdot h_b$. The rest is clear. □

Lemma 8.3 *Let $(b, p) \in (\mathrm{B}^- \times \mathrm{B}^-)^\circ(\mathcal{K})$. Assume $p \in \mathrm{B}^-(\mathbb{C})$.*

1. *If* $\mathrm{val}(\mathcal{R}_i^- \circ \pi_r(b)) \leq 0$ *for all $i \in I$, then $b \cdot p$ is a transcendental point. Moreover*

$$\mathrm{val}(b \cdot p) = \mathrm{val}(b), \quad \mathbb{Q}(b \cdot p, p) = \mathbb{Q}(b, p).$$

2. *If* $\mathrm{val}(\mathcal{L}_i^- \circ \pi_l(b)) \leq 0$ *for all $i \in I$, then $p^{-1} \cdot b$ is a transcendental point. Moreover*

$$\mathrm{val}(p^{-1} \cdot b) = \mathrm{val}(b), \quad \mathbb{Q}(p^{-1} \cdot b, p) = \mathbb{Q}(b, p).$$

Proof Combining Lemmas 8.1–8.2, we prove 1. Analogously 2 follows. □

8.2 Proof of Theorem 2.36.

Our first task is to prove Theorem 2.36 for the cases when $\mathrm{I} = [1, n] \subset [1, n+1]$.

Let $\mathrm{J} = \{j_1, \ldots, j_m\} \subset \mathrm{I}$. Recall $\mathcal{W}_{\mathrm{J}}$ in (110). Let $l \in \mathrm{Conf}(\mathcal{A}^n, \mathcal{B})(\mathbb{Z}^t)$ be such that $\mathcal{W}_{\mathrm{J}}^t(l) \geq 0$.

Let $\mathbf{x} \in \mathcal{C}_l^\circ$. Recall the map c in Theorem 7.3. Set $c(\mathbf{x}) := (b_1, \ldots, b_{n-1}) \in (\mathrm{B}^-)^{n-1}(\mathcal{K})$.

Lemma 8.4 *For every $i \in I$, we have*

1. $\mathrm{val}(\mathcal{L}_i^- \circ \pi_l(b_j)) \leq 0$ *if* $j \in [1, n-1] \cap \mathrm{J}$,
2. $\mathrm{val}(\mathcal{R}_i^- \circ \pi_r(b_{k-1})) \leq 0$ *if* $k \in [2, n] \cap \mathrm{J}$.

Proof Let $j \in [1, n-1] \cap \mathrm{J}$. By definition $b_j = b_{\mathrm{B}_{n+1}}^{\mathrm{A}_j, \mathrm{A}_{j+1}}$. By Lemmas 5.8, 6.4, we get

$$\mathrm{val}(\mathcal{L}_i^- \circ \pi_l(b_j)) = -\mathrm{val}\left(\chi_i\left(u_{\mathrm{B}_{n+1}, \mathrm{B}_{j+1}}^{\mathrm{A}_j}\right)\right) \leq -\chi_{\mathrm{A}_j}^t(l) \leq 0.$$

The second part follows similarly. □

As illustrated by Fig. 38, we see that

$$\mathbf{x}=(g_1\cdot \mathrm{U}, g_2\cdot \mathrm{U}, \ldots, g_n\cdot \mathrm{U}, \mathrm{B}^-), \quad g_1:=1,\ g_j := b_1\ldots b_{j-1},\ j\in[2,n].$$

If $j\in \mathrm{J}$, then $\mathrm{L}_j := \mathrm{L}(g_j\cdot \mathrm{U}, \mathrm{B}^-) = [g_j]\in \mathrm{Gr}$. Therefore

$$\kappa(\mathbf{x}) = (x_1,\ldots,x_n,\mathrm{B}^-), \quad x_j = \begin{cases} [g_j] & \text{if } j\in \mathrm{J},\\ g_j\cdot \mathrm{U} & \text{otherwise.}\end{cases}$$

Let $\{\mathrm{A}_{j_1},\ldots,\mathrm{A}_{j_m}\}\in \mathcal{A}^m(\mathbb{C})$ be a generic point in the sense of algebraic geometry. Define

$$\mathbf{y} := \left(\mathrm{A}'_1, \mathrm{A}'_2, \ldots, \mathrm{A}'_n, \mathrm{B}^-\right)\in \mathrm{Conf}(\mathcal{A}^n;\mathcal{B}),\ \ \mathrm{A}'_j = \begin{cases} g_j\cdot \mathrm{A}_j & \text{if } j\in \mathrm{J},\\ g_j\cdot \mathrm{U} & \text{otherwise.}\end{cases}$$

Let $F\in \mathbb{Q}_+(\mathrm{Conf}(\mathcal{A}^n;\mathcal{B}))$. By the very definition of D_F, we have $D_F(\kappa(\mathbf{x})) = \mathrm{val}(F(\mathbf{y}))$.

Since $\{\mathrm{A}_{j_1},\ldots,\mathrm{A}_{j_m}\}$ is generic, it can be presented by

$$\{\mathrm{A}_{j_1},\ldots,\mathrm{A}_{j_m}\}:=\{p_{j_1}\cdot \mathrm{U},\ldots,p_{j_m}\cdot \mathrm{U}\}, \quad \mathbf{p}=\{p_{j_1},\ldots,p_{j_m}\}\in (\mathrm{B}^-)^m(\mathbb{C}). \tag{231}$$

We can also assume that $(\mathbf{x},\mathbf{p})$ is a transcendental point, so that

$$(c(\mathbf{x}),\mathbf{p})\in \left((\mathrm{B}^-)^{m+n-1}\right)^\circ(\mathcal{K}). \tag{232}$$

Set $p_j=1$ for $j\notin \mathrm{J}$. Keep the same p_j for $j\in \mathrm{J}$. Then

$$\begin{aligned}\mathbf{y} &= (g_1p_1\cdot \mathrm{U},\ldots,g_np_n\cdot \mathrm{U},\mathrm{B}^-); \quad c(\mathbf{y}) = (\tilde{b}_1,\ldots,\tilde{b}_{n-1}),\\ \tilde{b}_j &:= p_j^{-1}b_jp_{j+1}\in \mathrm{B}^-(\mathcal{K}).\end{aligned}$$

By Lemmas 8.3–8.4, we get

$$\begin{aligned}\mathbb{Q}(c(\mathbf{x}),\mathbf{p}) &= \mathbb{Q}(b_1,\ldots,b_{n-1},p_{i_1},\ldots,p_{i_m})\\ &= \mathbb{Q}(\tilde{b}_1,\ldots,b_{n-1},p_{i_1},\ldots,p_{i_m}) = \ldots\\ &= \mathbb{Q}(\tilde{b}_1,\ldots,\tilde{b}_{n-1},p_{i_1},\ldots,p_{i_m}) = \mathbb{Q}(c(\mathbf{y}),\mathbf{p}). \end{aligned}\tag{233}$$

$$\mathrm{val}(b_j) = \mathrm{val}(\tilde{b}_j), \quad \forall j\in[1,n-1]. \tag{234}$$

Therefore $(c(\mathbf{y}),\mathbf{p})\in((\mathrm{B}^-)^{m+n-1})^\circ(\mathcal{K})$. Thus $c(\mathbf{y})$ is a transcendental point. Since $\mathrm{val}(c(\mathbf{y})) = \mathrm{val}(c(\mathbf{x})) = c^t(l)$, we get $\mathbf{y}\in \mathcal{C}_l^\circ$. By Lemma 2.13, $\mathrm{val}(F(\mathbf{y})) = F^t(l)$. Theorem 2.36 is proved.

Now consider the general cases when $\mathrm{J} \subset \mathrm{I} \subset [1, n]$. Consider the positive projection

$$d_{\mathrm{I}} = p_{\mathrm{I}} \circ d : \mathrm{Conf}(\mathcal{A}^n; \mathcal{B}) \xrightarrow{d} \mathrm{Conf}_n(\mathcal{A}) \xrightarrow{p_{\mathrm{I}}} \mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}).$$

Here the map d kills the last flag B_{n+1}. The map p_{I} keeps A_i intact when $i \in \mathrm{I}$, and takes A_i to $\pi(\mathrm{A}_i)$ otherwise.

Lemma 8.5 *Let $l \in \mathrm{Conf}^{+}_{\mathrm{J}\subset\mathrm{I}}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^t)$. There exists $l' \in \mathrm{Conf}(\mathcal{A}^n; \mathcal{B})(\mathbb{Z}^t)$ such that $\mathcal{W}^t_{\mathrm{J}}(l') \geq 0$ and $d^t_{\mathrm{I}}(l') = l$.*

Proof We prove the case when J contains $\{1, n\}$. In fact, the other cases are easier. Let $x = (\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{B}_{n+1})$. Consider a map $u : \mathrm{Conf}(\mathcal{A}^n; \mathcal{B}) \to \mathrm{U}$ given by $x \mapsto u^{\mathrm{A}_1}_{\mathrm{B}_{n+1},\mathrm{B}_n}$. Then

$$\begin{aligned}\mathcal{W}_{\mathrm{J}}(x) &= \mathcal{W}_{\mathrm{J}}(\mathrm{A}_1, \ldots, \mathrm{A}_n) + \mathcal{W}(\mathrm{A}_1, \mathrm{A}_n, \mathrm{B}_{n+1}) = \mathcal{W}_{\mathrm{J}}(d_{\mathrm{I}}(x)) \\ &\quad + \chi\left(u^{\mathrm{A}_1}_{\mathrm{B}_{n+1},\mathrm{B}_n}\right) + \chi\left(u^{\mathrm{A}_n}_{\mathrm{B}_1,\mathrm{B}_{n+1}}\right) \\ &= \mathcal{W}_{\mathrm{J}}(d_{\mathrm{I}}(x)) + \chi(u(x)) + \sum_{i\in I} \frac{\pi_{1,n}(d_{\mathrm{I}}(x))}{\mathcal{R}_i(u(x))}. \end{aligned} \tag{235}$$

By Lemma 6.14, we have $\lambda := \pi^t_{1,n}(l) \in \mathrm{P}^+$. Clearly there exists $l' \in \mathrm{Conf}(\mathcal{A}^n; \mathcal{B})(\mathbb{Z}^t)$ such that $d^t_{\mathrm{I}}(l') = l$ and $u^t(l') = 0 \in \mathrm{U}(\mathbb{Z}^t)$. We tropicalize (235):

$$\begin{aligned}\mathcal{W}^t_{\mathrm{J}}(l') &= \min\left\{\mathcal{W}^t_{\mathrm{J}}(l),\ \chi^t(0),\ \min_{i\in I}\{\langle\lambda, \alpha_i\rangle - \mathcal{R}^t_i(0)\}\right\} \\ &= \min\left\{\mathcal{W}^t_{\mathrm{J}}(l),\ 0,\ \min_{i\in I}\{\langle\lambda, \alpha_i\rangle\}\right\} = 0.\end{aligned}$$

□

Let l, l' be as above. Let $\mathbf{x} \in \mathcal{C}^{\circ}_{l}$. Clearly there exists $\mathbf{z} \in \mathcal{C}^{\circ}_{l'}$ such that $d_{\mathrm{I}}(\mathbf{z}) = \mathbf{x}$. For any $F \in \mathbb{Q}_+(\mathrm{Conf}_{\mathrm{I}}(\mathcal{A}; \mathcal{B}))$, we have

$$D_F(\kappa(\mathbf{x})) = D_{F\circ d_{\mathrm{I}}}(\kappa(\mathbf{z})) = (F \circ d_{\mathrm{I}})^t(l') = F^t \circ d^t_{\mathrm{I}}(l') = F^t(l).$$

The second identity is due to the special cases discussed before. The rest are by definition.

9 Configurations and generalized Mircović–Vilonen cycles

9.1 Proof of Theorem 2.35

In this section we use extensively the notation from Sect. 6.2, such as $u^{\mathrm{A}_2}_{\mathrm{B}_1,\mathrm{B}_3}$, $r^{\mathrm{B}_1,\mathrm{A}_2}_{\mathrm{B}_3} \in \mathrm{U}^-$. We identify the subset $\mathbf{A}_\nu$ in Theorem 2.35 with the subset $\mathbf{A}_\nu \subset \mathrm{U}^+_\chi(\mathbb{Z}^t)$ in (159) by tropicalizing

$$\alpha : \mathrm{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{U}, \quad (\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) \longmapsto u^{\mathrm{A}_2}_{\mathrm{B}_1,\mathrm{B}_3}. \tag{236}$$

Thanks to identity 4 of Lemma 6.4, the index ν for both definitions match.

Proof of Theorem 2.35 (2). Let $l \in \mathbf{A}_\nu$. Let $x = (\mathrm{B}_1, \mathrm{A}_2, \mathrm{B}_3) \in \mathcal{C}^\circ_l$. By Lemma 6.4, $r^{\mathrm{B}_1,\mathrm{A}_2}_{\mathrm{B}_3} = \eta(u^{\mathrm{A}_2}_{\mathrm{B}_1,\mathrm{B}_3})$. Recall κ_{Kam} in (161). Recall i_s in (126). By Lemma 7.6, we get

$$\kappa(x) = \left(\mathrm{B}, \left[r^{\mathrm{B}_1,\mathrm{A}_2}_{\mathrm{B}_3}\right], \mathrm{B}^-\right) = \left(\mathrm{B}, \kappa_{\mathrm{Kam}}\left(u^{\mathrm{A}_2}_{\mathrm{B}_1,\mathrm{B}_3}\right), \mathrm{B}^-\right) = i_s(\kappa_{\mathrm{Kam}}(\alpha(x))). \tag{237}$$

Recall MV_l in (162). Then $\mathcal{M}_l = i_s(\mathrm{MV}_l)$. Thus (2) is a reformulation of Theorem 5.4.

(1). Recall the map

$$p_i : \mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{U}, \quad (\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) \longmapsto u^{\mathrm{A}_i}_{\mathrm{B}_{i+2},\mathrm{B}_{i+1}}, \quad i = 1, 2. \tag{238}$$

Recall the map τ defined by (226)

$$\tau : \mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})(\mathcal{K}) \longrightarrow \mathrm{Gr}, \quad (\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) \longmapsto [b^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3}]. \tag{239}$$

Note that p_2^t induces a bijection from $\mathbf{P}^\mu_\lambda$ to $\mathbf{A}_{\lambda-\mu}$. The MV cycles of coweight $(\lambda - \mu, 0)$ are

$$\overline{\kappa_{\mathrm{Kam}} \circ p_2(\mathcal{C}^\circ_l)} = \overline{\kappa_{\mathrm{Kam}}(\mathcal{C}^\circ_{p_2^t(l)})} = \mathrm{MV}_{p_2^t(l)}, \quad l \in \mathbf{P}^\mu_\lambda.$$

Let $x = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) \in \mathcal{C}^\circ_l$. Note that

$$\tau(x) = \left[b^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3}\right] = \left[\mu^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3} r^{\mathrm{B}_2,\mathrm{A}_1}_{\mathrm{B}_3}\right] = \mu(x) \cdot \kappa_{\mathrm{Kam}}(p_2(x)),$$
$$\text{where } [\mu(x)] = \left[\mu^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_3}\right] = t^\mu.$$

We get $\overline{\tau(\mathcal{C}_l^\circ)} = t^\mu \cdot \mathrm{MV}_{p_2^t(l)}$. They are precisely MV cycles of coweight (λ, μ). Recall the isomorphism i in (121). Clearly $\mathcal{M}_l = i(\overline{\tau(\mathcal{C}_l^\circ)})$. Thus (1) is proved.

(3). The set $\mathbf{B}_\lambda^\mu$ is a subset of $\mathbf{P}_\lambda^\mu$ such that $p_1^t(\mathbf{B}_\lambda^\mu) \subset \mathrm{U}_\chi^+(\mathbb{Z}^t)$. By Lemma 6.14, $\mathbf{B}_\lambda^\mu$ is empty unless $\lambda \in \mathrm{P}^+$. So we assume $\lambda \in \mathrm{P}^+$. Let $l \in \mathbf{P}_\lambda^\mu$. Let $x = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3) \in \mathcal{C}_l^\circ$. By Lemma 6.2,

$$\tau(x) = \left[b_{\mathrm{B}_3}^{\mathrm{A}_1,\mathrm{A}_2}\right] = \left[u_{\mathrm{B}_3,\mathrm{B}_2}^{\mathrm{A}_1} h_{\mathrm{A}_1,\mathrm{A}_2} \overline{w}_0 u_{\mathrm{B}_1,\mathrm{B}_3}^{\mathrm{A}_2}\right] = p_1(x) \cdot t^\lambda. \tag{240}$$

The last identity is due to $p_2^t(l) \in \mathrm{U}_\chi^+(\mathbb{Z}^t)$ (hence $u_{\mathrm{B}_1,\mathrm{B}_3}^{\mathrm{A}_2} \in \mathrm{U}(\mathcal{O})$).

By Lemma 5.12, $\tau(x) \in \overline{\mathrm{Gr}_\lambda}$ if and only if $p_1^t(l) \in \mathrm{U}_\chi^+(\mathbb{Z}^t)$. Therefore

$$\overline{\tau(\mathcal{C}_l^\circ)} \subset \overline{\mathrm{Gr}_\lambda} \Longleftrightarrow p_1^t(l) \in \mathrm{U}_\chi^+(\mathbb{Z}^t) \Longleftrightarrow l \in \mathbf{B}_\lambda^\mu. \tag{241}$$

The rest follows from Lemma 7.5. $\square$

9.2 Proof of Theorems 2.31, 2.33, 2.34

By Theorem 2.35, we have

$$\mathrm{S}_{w_0}^\mu \cap \mathrm{S}_e^\lambda = \bigcup_{l \in \mathbf{P}_\lambda^\mu} \mathrm{N}_l, \quad \mathrm{S}_{w_0}^\mu \cap \mathrm{Gr}_\lambda = \bigcup_{l \in \mathbf{B}_\lambda^\mu} \mathrm{M}_l, \tag{242}$$

Here N_l (resp. M_l) are components containing $\tau(\mathcal{C}_l^\circ)$ as dense subsets. They are all of dimension $\langle \rho, \lambda - \mu \rangle$. The closures $\overline{\mathrm{N}_l} = \overline{\tau(\mathcal{C}_l^\circ)}$ are MV cycles.

Proof of Theorem 2.31. Scissoring the convex $(n+2)$-gon along diagonals emanating from the vertex labelled by $n+2$, see Fig. 37, we get a positive birational isomorphism between $\mathrm{Conf}(\mathcal{A}^{n+1}, \mathcal{B})$ and $(\mathrm{Conf}(\mathcal{A}^2, \mathcal{B}))^n$. Its tropicalization provides a decomposition

$$\mathbf{P}_{\lambda;\underline{\lambda}}^\mu = \bigsqcup_{\mu_1+\ldots+\mu_n=\mu} \mathbf{P}_\lambda^{\mu_1} \times \mathbf{B}_{\lambda_2}^{\mu_2} \ldots \times \mathbf{B}_{\lambda_n}^{\mu_n}, \quad \underline{\lambda} = (\lambda_2, \ldots, \lambda_n) \in (\mathrm{P}^+)^{n-1}. \tag{243}$$

Let $l = (l_1, \ldots, l_n) \in \mathbf{P}_{\lambda;\underline{\lambda}}^\mu$. We construct an irreducible subset

$$\begin{aligned} \mathrm{C}_l := \{&([b_1], [b_1 b_2], \ldots, [b_1 b_2 \ldots b_n]) \in \mathrm{Gr}^n \mid b_i \in \mathrm{B}^-(\mathcal{K}), \\ &[b_1] \in \mathrm{N}_{l_1},\ [b_i] \in \mathrm{M}_{l_i},\ i \in [2, n]\}. \end{aligned}$$

By induction, C_l is of dimension $\langle \rho, \lambda + \lambda_2 + \ldots + \lambda_n - \mu \rangle$.

Lemma 9.1 *Recall the subvariety* $\mathrm{Gr}^{\mu}_{\lambda,\underline{\lambda}}$ *in* (122)*. We have* $\mathrm{Gr}^{\mu}_{\lambda,\underline{\lambda}} = \cup \mathrm{C}_l$ *where* $l \in \mathbf{P}^{\mu}_{\lambda;\underline{\lambda}}$.

Proof Thanks to the isomorphism $\mathrm{B}^-(\mathcal{K})/\mathrm{B}^-(\mathcal{O}) \xrightarrow{\sim} \mathrm{Gr}$, each $x \in \mathrm{Gr}^{\mu}_{\lambda,\underline{\lambda}}$ can be presented as $([b_1], [b_1b_2]\ldots, [b_1\ldots b_n])$, where $b_i \in \mathrm{B}^-(\mathcal{K})$ for all $i \in [1, n]$. By the definition of $\mathrm{Gr}^{\mu}_{\lambda,\underline{\lambda}}$, we have

$$[b_i] \in \mathrm{Gr}_{\lambda_i},\ \forall i \in [2, n]; \quad [b_1] \in \mathrm{S}^{\lambda}_e,\ [b_1 \ldots b_n] \in \mathrm{S}^{\mu}_{w_0}.$$

Let $\mathrm{pr} : \mathrm{B}^-(\mathcal{K}) \to \mathrm{H}(\mathcal{K}) \to \mathrm{H}(\mathcal{K})/\mathrm{H}(\mathcal{O}) = \mathrm{P}$ be the composite of standard projections. Set $\mathrm{pr}(b_i) := \mu_i$. Then $[b_i] \in \mathrm{S}^{\mu_i}_{w_0}$.

When $i = 1$, $[b_1] \in \mathrm{S}^{\mu_1}_{w_0} \cap \mathrm{S}^{\lambda}_e$. Thus $[b_1] \in \mathrm{N}_{l_1}$ for some $l_1 \in \mathbf{P}^{\mu_1}_{\lambda}$.

When $i > 1$, $[b_i] \in \mathrm{S}^{\mu_i}_{w_0} \cap \mathrm{Gr}_{\lambda_i}$. Thus $[b_i] \in \mathrm{M}_{l_i}$ for some $l_i \in \mathbf{B}^{\mu_i}_{\lambda_i}$.

Note that $\mu_1 + \ldots + \mu_n = \mathrm{pr}(b_1) + \ldots + \mathrm{pr}(b_n) = \mathrm{pr}(b_1 \ldots b_n) = \mu$. Thus $l := (l_1, \ldots, l_n) \in \mathbf{P}^{\mu}_{\lambda,\underline{\lambda}}$. By definition $x \in \mathrm{C}_l$. Therefore $\mathrm{Gr}^{\mu}_{\lambda,\underline{\lambda}} \subseteq \cup_{l \in \mathbf{P}^{\mu}_{\lambda;\underline{\lambda}}} \mathrm{C}_l$. The other direction follows similarly. □

Let $l \in \mathbf{P}^{\mu}_{\lambda,\underline{\lambda}}$. Recall the map

$$\tau : \mathrm{Conf}(\mathcal{A}^{n+1}, \mathcal{B}) \longrightarrow \mathrm{Gr}^n, \quad (\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}, \mathrm{B}_{n+2}) \longmapsto \left(\left[b^{\mathrm{A}_1,\mathrm{A}_2}_{\mathrm{B}_{n+2}}\right], \ldots, \left[b^{\mathrm{A}_1,\mathrm{A}_{n+1}}_{\mathrm{B}_{n+2}}\right]\right).$$

Clearly $\tau(\mathcal{C}^{\circ}_l)$ is a dense subset of C_l. Recall the isomorphism i in (121). Following Lemma 7.5, the isomorphism i identifies $\tau(\mathcal{C}^{\circ}_l)$ with $\mathcal{M}^{\circ}_l$. By Theorem 2.36, the cells $\mathcal{M}^{\circ}_l$ are disjoint. Theorem 2.31 follows from Lemma 9.1. □

Proof of Theorem 2.33. The group $\mathrm{H}(\mathcal{K})$ acts diagonally on Gr^n. Let $h \in \mathrm{H}(\mathcal{K})$ be such that $[h] = t^{\nu}$. Then $h \cdot \mathrm{Gr}^{\mu}_{\lambda;\underline{\lambda}} = \mathrm{Gr}^{\mu+\nu}_{\lambda+\nu;\underline{\lambda}}$. One can choose h such that $[h] = t^{-\mu}$. The rest follows by the same argument in the proof of Theorem 2.31. □

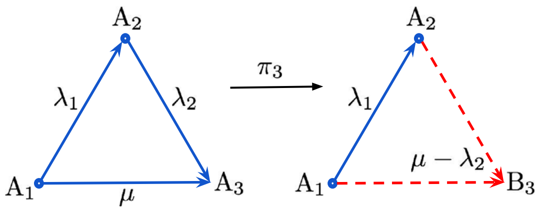
Proof of Theorem 2.34. By definition $\mathbf{B}^{\mu}_{\lambda_1,\lambda_2,\ldots,\lambda_n} \subset \mathbf{P}^{\mu}_{\lambda_1;\lambda_2,\ldots,\lambda_n}$. The Theorem follows by the same argument in the proof of Theorem 2.31. □

9.3 Components of the fibers of convolution morphisms

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathrm{P}^+)^n$. Recall the convolution variety $\mathrm{Gr}_{\underline{\lambda}}$ in (119). By the geometric Satake correspondence, $\mathrm{IH}(\overline{\mathrm{Gr}_{\underline{\lambda}}}) = V_{\underline{\lambda}} := V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$.

Set $|\underline{\lambda}| := \lambda_1 + \ldots + \lambda_n$. Set $\mathrm{ht}(\underline{\lambda}; \mu) := \langle \rho, |\underline{\lambda}| - \mu \rangle$. The *convolution morphism* $m_{\underline{\lambda}} : \overline{\mathrm{Gr}_{\underline{\lambda}}} \to \overline{\mathrm{Gr}_{|\underline{\lambda}|}}$ projects $(\mathrm{L}_1, \ldots, \mathrm{L}_n)$ to L_n. It is semismall,

Fig. 40 The projection π_3 induces a bijection $\pi_3^t : \widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2} \to \mathbf{B}^{\mu-\lambda_2}_{\lambda_1}$



i.e. for any $\mu \in \mathrm{P}^+$ such that $t^\mu \in \overline{\mathrm{Gr}_{|\underline{\lambda}|}}$, the fiber $m_{\underline{\lambda}}^{-1}(t^\mu)$ over t^μ is of top dimension $\mathrm{ht}(\underline{\lambda}; \mu)$. See [65] for proof.

By the decomposition theorem [8], we have

$$\mathrm{IH}(\overline{\mathrm{Gr}_{\underline{\lambda}}}) = \bigoplus_{\mu} F_\mu \otimes \mathrm{IH}(\overline{\mathrm{Gr}_\mu}).$$

Here the sum is over $\mu \in \mathrm{P}^+$ such that $t^\mu \subseteq \overline{\mathrm{Gr}_{|\underline{\lambda}|}}$, and F_μ is the vector space spanned by the fundamental classes of top dimensional components of $m_{\underline{\lambda}}^{-1}(t^\mu)$. As a consequence, the number of top components of $m_{\underline{\lambda}}^{-1}(t^\mu)$ equals the tensor product multiplicity $c_{\underline{\lambda}}^{\mu}$ of V_μ in $V_{\underline{\lambda}}$.

Recall the subsets $\mathbf{C}_{\underline{\lambda}}^{\mu}$ in (84). By Lemma 6.14, the set $\mathbf{C}_{\underline{\lambda}}^{\mu}$ is empty unless $(\mu, \underline{\lambda}) \in (\mathrm{P}^+)^{n+1}$. Recall the map ω in (216). In this subsection we prove

Theorem 9.2 *Let $\mathbf{T}_{\underline{\lambda}}^{\mu}$ be the set of top components of $m_{\underline{\lambda}}^{-1}(t^\mu)$. For each $l \in \mathbf{C}_{\underline{\lambda}}^{\mu}$, the closure $\overline{\omega(\mathcal{C}_l^\circ)} \in \mathbf{T}_{\underline{\lambda}}^{\mu}$. It gives a bijection between $\mathbf{C}_{\underline{\lambda}}^{\mu}$ and $\mathbf{T}_{\underline{\lambda}}^{\mu}$.*

First we prove the case when $n = 2$. In this case, the fiber $m_{\lambda_1,\lambda_2}^{-1}(t^\mu)$ is isomorphic to

$$\{\mathrm{L} \in \mathrm{Gr} \mid (\mathrm{L}, t^\mu) \in \overline{\mathrm{Gr}_{\lambda_1,\lambda_2}}\} = \overline{\mathrm{Gr}_{\lambda_1}} \cap t^\mu \overline{\mathrm{Gr}_{\lambda_2^\vee}}.$$

Here $\lambda_2^\vee := -w_0(\lambda_2) \in \mathrm{P}^+$. The following Theorem is due to Anderson.

Theorem 9.3 [3] *The top components of $\overline{\mathrm{Gr}_{\lambda_1}} \cap t^\mu \overline{\mathrm{Gr}_{\lambda_2^\vee}}$ are precisely the MV cycles of coweight $(\lambda_1, \mu - \lambda_2)$ contained in $\overline{\mathrm{Gr}_{\lambda_1}} \cap t^\mu \overline{\mathrm{Gr}_{\lambda_2^\vee}}$.*

Recall the positive morphisms

$$p_i : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow \mathrm{U}, \quad (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3) \longrightarrow u^{\mathrm{A}_i}_{\mathrm{B}_{i-1},\mathrm{B}_{i+1}}, \ i \in \mathbb{Z}/3$$

Let us put the potential condition on two vertices, see the left of Fig. 40, getting

$$\widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2} := \{l \in \mathrm{Conf}_3(\mathcal{A})(\mathbb{Z}^t) \mid (\pi_{12}, \pi_{23}, \pi_{13})^t(l) = (\lambda_1, \lambda_2, \mu),$$
$$p_1^t(l) \in \mathrm{U}_\chi^+(\mathbb{Z}^t),\ p_2^t(l) \in \mathrm{U}_\chi^+(\mathbb{Z}^t)\}.$$

Consider the projection $\pi_3 : \mathrm{Conf}_3(\mathcal{A}) \to \mathrm{Conf}(\mathcal{A}^2, \mathcal{B})$ which maps $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$ to $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_3)$. Its tropicalization π_3^t induces a bijection[13] from $\widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2}$ to $\mathbf{B}^{\mu-\lambda_2}_{\lambda_1}$. Recall ω_2 in (220). By (241), the cycles

$$\overline{\omega_2(\mathcal{C}_l^\circ)} = \overline{\tau(\mathcal{C}^\circ_{\pi_3^t(l)})}, \quad l \in \widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2}$$

are precisely MV cycles of coweight $(\lambda_1, \mu - \lambda_2)$ contained in $\overline{\mathrm{Gr}_{\lambda_1}}$.

Let $l \in \widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2}$. Let $x = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3) \in \mathcal{C}_l^\circ$. By identity 2 of Lemma 6.2,

$$\omega_2(x) = \left[\pi_{13}(x)\overline{w}_0 \cdot (p_3(x))^{-1}\pi_{32}(x)\right], \qquad \text{where } [\pi_{13}(x)] = t^{\mu}, \quad [\pi_{32}(x)] = t^{\lambda_2^\vee}.$$

Therefore

$$\begin{aligned}
\omega_2(x) \in t^{\mu}\overline{\mathrm{Gr}_{\lambda_2^\vee}} &\Longleftrightarrow t^{-\mu}\omega_2(x) \in \overline{\mathrm{Gr}_{\lambda_2^\vee}} \\
&\Longleftrightarrow t^{-\mu}\pi_{13}(x)\overline{w}_0 \cdot [(p_3(x))^{-1}\pi_{32}(x)] \in \overline{\mathrm{Gr}_{\lambda_2^\vee}} \\
&\Longleftrightarrow (p_3(x))^{-1} \cdot t^{\lambda_2^\vee} \in \overline{\mathrm{Gr}_{\lambda_2^\vee}}.
\end{aligned}$$

Here the last equivalence is due to the fact that $t^{-\mu}\pi_{13}\overline{w}_0 \in \mathrm{G}(\mathcal{O})$. Therefore for any $l \in \widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2}$,

$$\omega_2(\mathcal{C}_l^\circ) \subset t^{\mu}\overline{\mathrm{Gr}_{\lambda_2^\vee}} \Longleftrightarrow \left(p_3(\mathcal{C}_l^\circ)\right)^{-1} \cdot t^{\lambda_2^\vee} \subset \overline{\mathrm{Gr}_{\lambda_2^\vee}}$$

By Lemma 5.2, Lemma 5.12, and the definition of $\mathbf{C}^{\mu}_{\lambda_1,\lambda_2}$, we get

$$\begin{aligned}
&\left(p_3(\mathcal{C}_l^\circ)\right)^{-1} \cdot t^{\lambda_2^\vee} \in \overline{\mathrm{Gr}_{\lambda_2^\vee}} \Longleftrightarrow \left(p_3(\mathcal{C}_l^\circ)\right)^{-1} \in \mathrm{U}(\mathcal{O}) \Longleftrightarrow p_3(\mathcal{C}_l^\circ) \in \mathrm{U}(\mathcal{O}) \\
&\quad\Longleftrightarrow p_3^t(l) \in \mathrm{U}^+_{\chi}(\mathbb{Z}^t) \Longleftrightarrow l \in \mathbf{C}^{\mu}_{\lambda_1,\lambda_2}.
\end{aligned}$$

Let $l \in \mathbf{C}^{\mu}_{\lambda_1,\lambda_2}$. Let $x = (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3) \in \mathcal{C}_l^\circ$. Note that $\omega_3(x) = [h_{\mathrm{A}_1,\mathrm{A}_3}] = t^{\mu}$. Therefore $\omega(x) = (\omega_2(x), \omega_3(x)) \in m^{-1}_{\lambda_1,\lambda_2}(t^{\mu})$. The rest is due to Theorem 9.3.

[13] There is a positive Cartan group action on $\mathrm{Conf}_3(\mathcal{A})(\mathbb{Z}^t)$ defined via

$$\mathrm{H} \times \mathrm{Conf}_3(\mathcal{A}) \longrightarrow \mathrm{Conf}_3(\mathcal{A}), \qquad h \times (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3) \longmapsto (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \cdot h).$$

Its tropicalization determines a free $\mathrm{H}(\mathbb{Z}^t)$-action on $\mathrm{Conf}_3(\mathcal{A})(\mathbb{Z}^t)$. By definition, one can thus identify the $\mathrm{H}(\mathbb{Z}^t)$-orbits of $\mathrm{Conf}_3(\mathcal{A})(\mathbb{Z}^t)$ with points of $\mathrm{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})(\mathbb{Z}^t)$. Note that each element in $\mathbf{B}^{\mu-\lambda_2}_{\lambda_1}$ has a unique representative in $\widetilde{\mathbf{B}}^{\mu}_{\lambda_1,\lambda_2}$. Hence the map π_3^t is a bijection.

Now let us prove the general case. Consider the scissoring morphism

$$c = (c_1, c_2) : \mathrm{Conf}_{n+1}(\mathcal{A}) \longrightarrow \mathrm{Conf}_n(\mathcal{A}) \times \mathrm{Conf}_3(\mathcal{A}),$$
$$(\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}) \longmapsto (\mathrm{A}_1, \ldots, \mathrm{A}_{n-1}, \mathrm{A}_{n+1}) \times (\mathrm{A}_{n-1}, \mathrm{A}_n, \mathrm{A}_{n+1}) \tag{244}$$

Due to the scissoring congruence invariance, the map c^t induces a decomposition

$$\mathbf{C}^{\mu}_{\lambda_1,\ldots,\lambda_n} = \bigsqcup_{\nu\in \mathrm{P}^+} \mathbf{C}^{\mu}_{\lambda_1,\ldots,\lambda_{n-2},\nu} \times \mathbf{C}^{\nu}_{\lambda_{n-1},\lambda_n}. \tag{245}$$

Proposition 9.4 *The cardinality of* $\mathbf{C}^{\mu}_{\underline{\lambda}}$ *is the tensor product multiplicity* $c^{\mu}_{\underline{\lambda}}$ *of* V_μ *in* $V_{\underline{\lambda}}$.

Proof Decomposing the last tensor products in $V_{\lambda_1} \otimes \ldots \otimes (V_{\lambda_{n-1}} \otimes V_{\lambda_n})$ into a sum of irreducibles, and tensoring then each of them with $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-2}}$, we get

$$c^{\mu}_{\lambda_1,\ldots,\lambda_n} = \sum_{\nu\in \mathrm{P}^+} c^{\mu}_{\lambda_1,\ldots,\lambda_{n-2},\nu} c^{\nu}_{\lambda_{n-1},\lambda_n}.$$

As a consequence of $n = 2$ case, $|\mathbf{C}^{\nu}_{\lambda,\mu}| = c^{\nu}_{\lambda,\mu}$. The Lemma follows by induction and (245). □

Lemma 9.5 *For* $l \in \mathbf{C}^{\mu}_{\underline{\lambda}}$, *the cycles* $\omega(\mathcal{C}^{\circ}_l)$ *are disjoint.*

Proof By Lemma 7.2, $\kappa(\mathcal{C}^{\circ}_l) = i_1 \circ \omega(\mathcal{C}^{\circ}_l)$. The Lemma follows from Theorem 2.24. □

Lemma 9.6 *For any* $l \in \mathbf{C}^{\mu}_{\underline{\lambda}}$, *we have* $\omega(\mathcal{C}^{\circ}_l) \subset m^{-1}_{\underline{\lambda}}(t^{\mu})$.

Proof Let $x = (\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}) \in \mathcal{C}^{\circ}_l$. Recall the expression (219). We have

$$[g_i] := \left[u^{\mathrm{A}_i}_{\mathrm{B}_{i-1},\mathrm{B}_{i+1}} h_{\mathrm{A}_i,\mathrm{A}_{i+1}} \overline{w}_0\right] = u^{\mathrm{A}_i}_{\mathrm{B}_{i-1},\mathrm{B}_{i+1}} \cdot t^{\lambda_i} \in \mathrm{Gr}_{\lambda_i}, \quad i \in [1, n].$$

Thus $\omega(x) \in \mathrm{Gr}_{\underline{\lambda}}$. Meanwhile $m_{\underline{\lambda}} \circ \omega(x) = [h_{\mathrm{A}_1,\mathrm{A}_{n+1}}] = t^{\mu}$. The Lemma is proved. □

Lemma 9.7 *Let* $l \in \mathbf{C}^{\mu}_{\underline{\lambda}}$. *The closure* $\overline{\omega(\mathcal{C}^{\circ}_l)}$ *is an irreducible variety of dimension* $\mathrm{ht}(\underline{\lambda}; \mu)$.

Proof By construction, $\overline{\omega(\mathcal{C}_l^\circ)}$ is irreducible. Note that $m_{\underline{\lambda}}^{-1}(t^\mu)$ is of top dimension $\mathrm{ht}(\underline{\lambda};\mu)$. By Lemma 9.6, $\dim \overline{\omega(\mathcal{C}_l^\circ)} \leq \mathrm{ht}(\underline{\lambda};\mu)$. To show that $\dim \overline{\omega(\mathcal{C}_l^\circ)} \geq \mathrm{ht}(\underline{\lambda};\mu)$, we use induction.

Set $\pi^t_{n-1,n+1}(l) := \nu$. Recall $c = (c_1, c_2)$ in (244). Then $c_1^t(l) \in \mathbf{C}^\mu_{\lambda_1,\ldots,\lambda_{n-2},\nu}$, $c_2^t(l) \in \mathbf{C}^\nu_{\lambda_{n-1},\lambda_n}$. Consider the projection

$$\mathrm{pr} : \omega(\mathcal{C}_l^\circ) \longrightarrow \mathrm{Gr}^{n-1}, \quad (\mathrm{L}_1, \ldots, \mathrm{L}_{n-1}, \mathrm{L}_n) \longrightarrow (\mathrm{L}_1, \ldots, \mathrm{L}_{n-2}, \mathrm{L}_n)$$

Its image $\mathrm{pr}(\omega(\mathcal{C}_l^\circ)) = \omega(\mathcal{C}^\circ_{c_1^t(l)})$. Let $\mathbf{b} = (\mathrm{L}_1, \ldots, \mathrm{L}_{n-2}, \mathrm{L}_n) \in \omega(\mathcal{C}^\circ_{c_1^t(l)})$. The fiber over $\mathbf{b}$ is

$$\mathrm{pr}^{-1}(\mathbf{b}) := \{\mathrm{L} \in \mathrm{Gr} \mid (\mathrm{L}_1, \ldots, \mathrm{L}_{n-2}, \mathrm{L}, \mathrm{L}_n) \in \omega(\mathcal{C}_l^\circ)\}.$$

Let $y = (\mathrm{A}_1, \ldots, \mathrm{A}_{n-1}, \mathrm{A}_{n+1}) \in \mathcal{C}^\circ_{c_1^t(l)}$ such that $\omega(y) = \mathbf{b}$. Set $b_y := b^{\mathrm{A}_1,\mathrm{A}_{n-1}}_{\mathrm{B}_{n+1}}$. For any $x \in \mathcal{C}_l^\circ$ such that $c_1(x) = y$, we have $\mathrm{pr}(\omega(x)) = \omega(y) = \mathbf{b}$. By (220), we have

$$\omega_{n-1}(x) = \left[b^{\mathrm{A}_1,\mathrm{A}_n}_{\mathrm{B}_{n+1}}\right] = b_y \cdot \omega_2(c_2(x)) \in \mathrm{pr}^{-1}(b).$$

Then it is easy to see that $b_y \cdot \overline{\omega_2(\mathcal{C}^\circ_{c_2^t(l)})} \subset \overline{\mathrm{pr}^{-1}(b)}$. Therefore

$$\dim \overline{\omega(\mathcal{C}_l^\circ)} \geq \dim \overline{\omega(\mathcal{C}^\circ_{c_1^t(l)})} + \dim \overline{\omega(\mathcal{C}^\circ_{c_2^t(l)})}.$$

The case when $n = 2$ is proved above. The Lemma follows by induction. □

Proof of Theorem 9.2 By Lemmas 9.6, 9.7, the map $\mathbf{C}^\mu_{\underline{\lambda}} \longrightarrow \mathbf{T}^\mu_{\underline{\lambda}}, l \longmapsto \overline{\omega(\mathcal{C}_l^\circ)}$ is well-defined. By Lemma 9.5 and the very construction of the cell $\mathcal{C}_l^\circ$, it is injective. Since $|\mathbf{C}^\mu_{\underline{\lambda}}| = |\mathbf{T}^\mu_{\underline{\lambda}}| = c^\mu_{\underline{\lambda}}$, the map is a bijection. □

9.4 Proof of Theorem 2.20

We focus on the case when $\mu = 0$ for $\mathbf{C}^\mu_{\underline{\lambda}}$. Consider the scissoring morphism

$$\begin{aligned} c = (c_1, c_2) : \mathrm{Conf}_{n+1}(\mathcal{A}) &\longrightarrow \mathrm{Conf}_n(\mathcal{A}) \times \mathrm{Conf}_3(\mathcal{A}), \\ (\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{A}_{n+1}) &\longmapsto (\mathrm{A}_1, \ldots, \mathrm{A}_n) \times (\mathrm{A}_1, \mathrm{A}_n, \mathrm{A}_{n+1}). \end{aligned}$$

Due to the scissoring congruence invariance, the morphism (c_1^t, c_2^t) induces a decomposition

$$\mathbf{C}^0_{\underline{\lambda}} = \bigsqcup_{\nu} \mathbf{C}_{\lambda_1,\ldots,\lambda_{n-1},\nu} \times \mathbf{C}^0_{\nu^\vee,\lambda_n}.$$

Note that $\mathbf{C}^0_{\nu^\vee,\lambda_n}$ is empty if $\nu \neq \lambda_n$. Moreover $|\mathbf{C}^0_{\lambda_n^\vee,\lambda_n}| = 1$. Thus c_1^t : $\mathbf{C}^0_{\underline{\lambda}} \to \mathbf{C}_{\underline{\lambda}}$ is a bijection.

Consider the shifted projection

$$p_s : \mathrm{Gr}^n \longrightarrow \mathrm{Conf}_n(\mathrm{Gr}), \quad \{\mathrm{L}_1, \ldots, \mathrm{L}_n\} \longrightarrow (\mathrm{L}_n, \mathrm{L}_1, \ldots, \mathrm{L}_{n-1}).$$

Lemma 9.8 *Let $l \in \mathbf{C}^0_{\underline{\lambda}}$. Then $p_s \circ \omega(\mathcal{C}^\circ_l) = \kappa(\mathcal{C}^\circ_{c_1^t(l)})$.*

Proof Let $x = (\mathrm{A}_1, \ldots, \mathrm{A}_{n+1}) \in \mathcal{C}^\circ_l$. Then $u := u^{\mathrm{A}_1}_{\mathrm{B}_{n+1},\mathrm{B}_n} \in \mathrm{U}(\mathcal{O})$. Let $y := c_1(x) \in \mathcal{C}^\circ_{c_1^t(l)}$.

Recall ω_i in (215). Then $\omega_{n+1}(x) = [1]$. For $i \in [2, n]$, we have

$$\begin{aligned}\omega_i(x) &= [g_{\{\mathrm{U},\mathrm{B}^-\}}(\{\mathrm{A}_1, \mathrm{B}_{n+1}\}, \{\mathrm{A}_i, \mathrm{B}_1\})] = u \cdot [g_{\{\mathrm{U},\mathrm{B}^-\}}(\{\mathrm{A}_1, \mathrm{B}_n\}, \{\mathrm{A}_i, \mathrm{B}_1\})] \\ &= u \cdot \omega_i(y).\end{aligned}$$

Therefore

$$\begin{aligned}p_s \circ \omega(x) &= (\omega_{n+1}(x), u \cdot \omega_2(y), \ldots, u \cdot \omega_n(y)) \\ &= ([1], \omega_2(y), \ldots, \omega_n(y)) = \kappa(y).\end{aligned}$$

Here the last step is due to Lemma 7.2. Since $c_1(\mathcal{C}^\circ_l) = \mathcal{C}^\circ_{c_1^t(l)}$, the Lemma is proved. □

Recall $\mathrm{Gr}_{c(\underline{\lambda})}$ and the set $\mathbf{T}_{\underline{\lambda}}$ of its top components in Theorem 2.20. The connected group $\mathrm{G}(\mathcal{O})$ acts on $\mathrm{Gr}_{c(\underline{\lambda})}$. It preserves each component of $\mathrm{Gr}_{c(\underline{\lambda})}$. So these components live naturally on the stack $\mathrm{Conf}_n(\mathrm{Gr}) = \mathrm{G}(\mathcal{O})\backslash([1] \times \mathrm{Gr}^{n-1})$.

Recall the fiber $m^{-1}_{\underline{\lambda}}([1])$ and the set $\mathbf{T}^0_{\underline{\lambda}}$ in Theorem 9.2. Note that $p_s(m^{-1}_{\underline{\lambda}}([1])) = \mathrm{G}(\mathcal{O})\backslash\overline{\mathrm{Gr}_{c(\underline{\lambda})}} \subset \mathrm{Conf}_n(\mathrm{Gr})$. It induces a bijection $\mathbf{T}^0_{\underline{\lambda}} \stackrel{\sim}{\longrightarrow} \mathbf{T}_{\underline{\lambda}}$.

Proof of Theorem 2.20. By Theorem 9.2 and above discussions, there is a chain of bijections: $\mathbf{C}_{\underline{\lambda}} \stackrel{\sim}{\longrightarrow} \mathbf{C}^0_{\underline{\lambda}} \stackrel{\sim}{\longrightarrow} \mathrm{T}^0_{\underline{\lambda}} \stackrel{\sim}{\longrightarrow} \mathrm{T}_{\underline{\lambda}}$. By Lemma 9.8, this chain is achieved by the map κ. The Theorem is proved. □

10 Positive G-laminations and surface affine Grassmannians

A decorated surface S comes with an unordered collection $\{s_1, \ldots, s_n\}$ of special points, defined up to isotopy. Denote by ∂S the boundary of S. We assume that ∂S is not empty. We define *punctured boundary*

$$\widehat{\partial} S := \partial S - \{s_1, \ldots, s_n\}. \tag{246}$$

Its components are called *boundary circles* and *boundary intervals*.

Let us shrink all holes without special points on S into *punctures*, getting a homotopy equivalent surface. Abusing notation, we denote it again by S. We say that the punctures and special points on S form the set of *marked points* on S:

$$\{\text{marked points}\} := \{\text{special points } s_1, \ldots, s_n\} \cup \{\text{punctures}\}.$$

Pick a point $*s_i$ in each of the boundary intervals. The dual decorated surface $*S$ is given by the same surface S with the set of special points $\{*s_1, \ldots, *s_n\}$. We have a duality: $**S = S$.

Observe that the marked points are in bijection with the components of the punctured boundary $\widehat{\partial}(*S)$.

10.1 The space $\mathcal{A}_{\mathrm{G},S}$ with the potential $\mathcal{W}$

Twisted local systems and decorations. Let $\mathrm{T}'S$ be the complement to the zero section of the tangent bundle on a surface S. Its fiber T'_y at $y \in S$ is homotopy equivalent to a circle. Let $x \in \mathrm{T}'_yS$. The fundamental group $\pi_1(\mathrm{T}'S, x)$ is a central extension:

$$0 \longrightarrow \pi_1(\mathrm{T}'_yS, x) \longrightarrow \pi_1(\mathrm{T}'S, x) \longrightarrow \pi_1(S, y) \longrightarrow 0, \quad \pi_1(\mathrm{T}'_yS, x) = \mathbb{Z}. \tag{247}$$

Let $\mathcal{L}$ be a G-local system on $\mathrm{T}'S$ with the monodromy s_{G} around a generator of $\pi_1(\mathrm{T}'_yS, x)$. Let us assume that G acts on $\mathcal{L}$ on the right. We call $\mathcal{L}$ a *twisted* G-local system on S. It gives rise to the *associated decorated flag bundle* $\mathcal{L}_{\mathcal{A}} := \mathcal{L} \times_{\mathrm{G}} \mathcal{A}$.

Let C be a component of $\widehat{\partial}(*S)$. There is a canonical up to isotopy section $\sigma : \mathrm{C} \to \mathrm{T}'\mathrm{C}$ given by the tangent vectors to C directed according to the orientation of C. A *decoration on $\mathcal{L}$ over* C is a flat section of the restriction of $\mathcal{L}_{\mathcal{A}}$ to $\sigma(\mathrm{C})$.

Definition 10.1 [17] A twisted decorated G-local system on S is a pair $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a twisted G-local system on S, and α is given by a decoration on $\mathcal{L}$ over each component of $\widehat{\partial}(*S)$.

The moduli space $\mathcal{A}_{\mathrm{G},S}$ parametrizes twisted decorated G-local systems on S.

Abusing terminology, a decoration is given by decorated flags at the marked points.

Remark. Since the boundary ∂S of S is not empty, the extension (247) splits:

$$\pi_1(\mathrm{T}'S, x) \stackrel{\sim}{=} \pi_1\left(\mathrm{T}'_y S, x\right) \times \pi_1(S, y).$$

However the splitting is not unique. As a space, $\mathcal{A}_{\mathrm{G},S}$ is isomorphic, although non canonically if $s_{\mathrm{G}} \neq 1$, to its counterpart of usual unipotent G-local systems on S with decorations. The mapping class group Γ_S acts differently on the two spaces. For example, when S is a disk D_n with n special points on the boundary, then $\Gamma_{D_n} = \mathbb{Z}/n\mathbb{Z}$. Both moduli spaces are isomorphic to the configuration space $\mathrm{Conf}_n(\mathcal{A})$. The mapping class group $\mathbb{Z}/n\mathbb{Z}$ acts on the untwisted moduli space is by the cyclic rotation $(\mathrm{A}_1, \ldots, \mathrm{A}_n) \mapsto (\mathrm{A}_n, \mathrm{A}_1, \ldots, \mathrm{A}_{n-1})$, while its action on $\mathcal{A}_{\mathrm{G},D^n}$ is given by the "twisted" rotation

$$(\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n) \longmapsto (\mathrm{A}_n \cdot s_{\mathrm{G}}, \mathrm{A}_1, \ldots, \mathrm{A}_{n-1}).$$

Theorem 10.2 (loc.cit.) *The space $\mathcal{A}_{\mathrm{G},S}$ admits a natural positive structure such that the mapping class group Γ_S acts on $\mathcal{A}_{\mathrm{G},S}$ by positive birational isomorphisms.*

Below we give two equivalent definitions of the potential $\mathcal{W}$ on $\mathcal{A}_{\mathrm{G},S}$.

Potential via generalized monodromy. A decorated flag A provides an isomorphism

$$i_{\mathrm{A}} : \mathrm{U}_{\mathrm{A}}/[\mathrm{U}_{\mathrm{A}}, \mathrm{U}_{\mathrm{A}}] \stackrel{\sim}{\longrightarrow} \oplus_{\alpha\in\Pi}\mathbb{A}^1. \tag{248}$$

Let $\Sigma : \oplus_{\alpha\in\Pi}\mathbb{A}^1 \to \mathbb{A}^1$ be the sum map. Then $\chi_{\mathrm{A}} = \Sigma \circ i_{\mathrm{A}}$. This characterizes the map i_{A}.

Let us assign to each component C of $\widehat{\partial}(*S)$ a canonical rational map, called *generalized monodromy at* C: $\mu_{\mathrm{C}} : \mathcal{A}_{\mathrm{G},S} \longrightarrow \oplus_{\alpha\in\Pi}\mathbb{A}^1$. There are two possible cases.

(i) The component C is a boundary circle. The decoration over C is a decorated flag A_{C} in the fiber of $\mathcal{L}_{\mathcal{A}}$ on C, invariant under the monodromy around

C. It defines a conjugacy class in the unipotent subgroup $\mathrm{U}_{\mathrm{A_C}}$ preserving $\mathrm{A_C}$. So we get a regular map

$$\mu_{\mathrm{C}} : \mathcal{A}_{\mathrm{G},S} \longrightarrow \mathrm{U}_{\mathrm{A_C}}/[\mathrm{U}_{\mathrm{A_C}}, \mathrm{U}_{\mathrm{A_C}}] \stackrel{i_{\mathrm{A_C}}}{=} \oplus_{\alpha\in\Pi}\mathbb{A}^1.$$

(ii) The component C is a boundary interval on a hole h. The universal cover of h is a line. We get an infinite sequence of intervals on this line projecting to the boundary interval(s) on h. There are decorated flags assigned to these intervals. Take an interval C' on the cover projecting to C. Let C'_- and C'_+ be the intervals just before and after C'. We get a triple of decorated flags $(\mathrm{A}_-, \mathrm{A}, \mathrm{A}_+)$ sitting over these intervals. There is a unique $u \in \mathrm{U_A}$ such that $\mathrm{B}_+ = u\cdot\mathrm{B}_-$, where $\mathrm{B}_\pm = \pi(\mathrm{A}_\pm) \in \mathcal{B}$. Projecting u to $\mathrm{U_A}/[\mathrm{U_A}, \mathrm{U_A}]$, we get a map $\mu_{\mathrm{C}} : \mathcal{A}_{\mathrm{G},S} \to \oplus_{\alpha\in\Pi}\mathbb{A}^1$. It is clear that μ_{C} does not depend on the choice of C'.

Composing the generalized monodromy μ_{C} with the sum map $\oplus_{\alpha\in\Pi}\mathbb{A}^1 \to \mathbb{A}^1$, we get

$$\mathcal{W}_{\mathrm{C}} := \Sigma \circ \mu_{\mathrm{C}} : \mathcal{A}_{\mathrm{G},S} \longrightarrow \mathbb{A}^1, \tag{249}$$

called *the potential associated with* C.

Definition 10.3 The potential $\mathcal{W}$ on the space $\mathcal{A}_{\mathrm{G},S}$ is defined as

$$\mathcal{W} := \sum_{\text{components C of } \widehat{\partial}(*S)} \mathcal{W}_{\mathrm{C}}. \tag{250}$$

Potential via ideal triangulations.

Definition 10.4 An ideal triangulation of a decorated surface S is a triangulation of the surface whose vertices are the marked points of S.

Let T be an ideal triangulation of S. Pick a triangle t of T. The restriction to t provides a projection[14] π_t from $\mathcal{A}_{\mathrm{G},S}$ to $\mathrm{Conf}_3(\mathcal{A})$. Recall the potential $\mathcal{W}_3$ on the latter space.

Definition 10.5 The potential on the space $\mathcal{A}_{\mathrm{G},S}$ is defined as

$$\mathcal{W} := \sum_{\text{triangles } t \text{ of } T} \mathcal{W}_3 \circ \pi_t. \tag{251}$$

[14] If the vertices of t coincide, one can first pull back to a sufficient big cover $\widetilde{S}$ of S, and then consider the restriction to a triangle $\widetilde{t} \subset \widetilde{S}$ which projects onto t. Clearly the result is independent of the pair $\widetilde{t} \subset \widetilde{S}$ chosen.

Changing T by a flip we do not change the sum (251) since the potential on a quadrilateral is invariant under a flip (Sect. 2). Since any two ideal triangulations are related by a sequence of flips, the potential (251) is independent of the ideal triangulation T chosen.

The above definitions are equivalent. There is a natural bijection between the marked points, that is the vertices of T, and the components of $\widehat{\partial}(*S)$. Working with definition (251), the sum over all angles of the triangles shared by a puncture is the potential $\mathcal{W}_{\mathrm{C}}$ assigned to the corresponding boundary circle. A similar sum over all angles shared by a special point is the potential $\mathcal{W}_{\mathrm{C}}$ assigned to the corresponding boundary interval. Thus the potentials (250) and (251) coincide.

Positivity of the potential $\mathcal{W}$. In the positive structure of $\mathcal{A}_{\mathrm{G},S}$ introduced in [17], the projection $\pi_t : \mathcal{A}_{\mathrm{G},S} \to \mathrm{Conf}_3(\mathcal{A})$ is a positive morphism. By Theorem 2.5 and (251), we get

Theorem 10.6 *The potential $\mathcal{W}$ is a positive function on the space $\mathcal{A}_{\mathrm{G},S}$.*

Positive integral G**-laminations.** We define the set of *positive integral* G*-laminations on S*:

$$\mathcal{A}^{+}_{\mathrm{G},S}(\mathbb{Z}^t) = \{l \in \mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t) \mid \mathcal{W}^t(l) \geq 0\}. \tag{252}$$

By tropicalization, the mapping class group Γ_S acts on $\mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t)$. The potential $\mathcal{W}$ is Γ_S-invariant. Thus Γ_S acts on the subset $\mathcal{A}^{+}_{\mathrm{G},S}(\mathbb{Z}^t)$.

Partial potentials. Given any simple positive root α, there is a component $\chi_{A,\alpha}$ of the character χ_A so that $\chi_A = \sum_{\alpha\in\Pi} \chi_{A,\alpha}$. Let S be a decorated surface. Then to each boundary component $C \in \partial(*S)$ one associates a function $W_{C,\alpha}$. It is evidently invariant under the action of the mapping class group Γ_S of S.

Theorem 10.7 *Let S be a surface with n holes and no special points. Then the algebra of regular Γ_S-invariant functions on the space $\mathcal{A}_{G,S}$ is a polynomial algebra in* $n\mathrm{rk}(G)$ *variables freely generated by the partial potentials $W_{C,\alpha}$, where C run through all boundary circles on S, and α are simple positive roots.*

Proof It is well known that the action of the mapping class group Γ_S on the moduli space $\mathrm{Loc}^{\mathrm{un}}_{G,S}$ of unipotent G-local systems on a surface S with holes is ergodic. So there are no non-constant Γ_S-invariant regular functions on this space. On the other hand, there is a canonical Γ_S-invariant projection given by the generalised monodromy around the holes:

$$\mathcal{A}_{G,S} \longrightarrow \prod_{\text{holes of } S} (\mathbb{A}^1)^{\Pi}.$$

Its fiber over zero is the space $\mathrm{Loc}^{\mathrm{un}}_{G,S}$. $\square$

10.2 Duality conjectures for decorated surfaces

Definition 10.8 The moduli space $\mathrm{Loc}_{\mathrm{G},S}$ parametrizes pairs $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is a twisted G-local system on S, and γ assigns a decoration on $\mathcal{L}$ to each boundary interval of $\widehat{\partial}(*S)$.

It is important to consider several different types of twisted G-local system on S which differ by the data assigned to the boundary. Recall that components of the punctured boundary $\widehat{\partial}(*S)$ are in bijection with the marked points of S. There are three options for the data at a given marked point, which could be either a special point, or a puncture:

(1) No data.
(2) A decoration, that is a flat section of the associated decorated flag bundle $\mathcal{L}_{\mathcal{A}}$ near m.
(3) A framing, that is a flat section of the associated flag bundle $\mathcal{L}_{\mathcal{B}}$ near m.

In accordance to this, there are five different moduli spaces:

- $\mathcal{A}_{\mathrm{G},S}$: decorations at both special points and punctures.
- $\mathcal{L}oc_{\mathrm{G},S}$: no extra data.
- $\mathrm{Loc}_{\mathrm{G},S}$: decorations at the special points only. No extra data at the punctures.
- $\mathcal{P}_{\mathrm{G},S}$: decorations at the special points, framings at the punctures.
- $\mathcal{X}_{\mathrm{G},S}$: framings at the special points and punctures.

If S does have special points, it is silly to consider $\mathcal{L}oc_{\mathrm{G},S}$ since it ignores them.

If S has no punctures, then (besides $\mathcal{L}oc_{\mathrm{G},S}$) there are three different moduli spaces:

$$\mathcal{A}_{\mathrm{G},S} = \mathrm{Loc}_{\mathrm{G},S}, \quad \mathcal{P}_{\mathrm{G},S}, \quad \mathcal{X}_{\mathrm{G},S}.$$

If S has no special points, i.e. it is a punctured surface, there are three different moduli spaces:

$$\mathcal{A}_{\mathrm{G},S}, \quad \mathcal{L}oc_{\mathrm{G},S} = \mathrm{Loc}_{\mathrm{G},S}, \quad \mathcal{P}_{\mathrm{G},S} = \mathcal{X}_{\mathrm{G},S}.$$

Duality conjectures interchange a group G with the Langlands dual group G^L, and a decorated surface S with the dual decorated surface $*S$.[15] Here are some examples.

[15] Although the decorated surface $*S$ is isomorphic to S, the isomorphism is not quite canonical.

If S has no special points, the dual pairs look as follows:

$$\mathcal{A}_{\mathrm{G},S} \quad \text{is dual to} \quad \mathcal{P}_{\mathrm{G}^L,*S} = \mathcal{X}_{\mathrm{G}^L,*S}, \quad (\mathcal{A}_{\mathrm{G},S}, \mathcal{W}) \quad \text{is dual to} \quad \mathcal{L}oc_{\mathrm{G}^L,*S} = \mathrm{Loc}_{\mathrm{G}^L,*S}.$$

If S does have special points, the moduli space $\mathcal{X}_{\mathrm{G},S}$ plays a secondary role. The key dual pair is this:

$$(\mathcal{A}_{\mathrm{G},S}, \mathcal{W}) \quad \text{is dual to} \quad \mathrm{Loc}_{\mathrm{G}^L,*S}.$$

There are plenty of other dual pairs, obtained from this one by degenerating the potential, and simultaneously altering the dual space. Let us discuss some of them.

Generalisations. Let us assign to each marked point m of S a subset $I_m \subset I$, possibly empty.

First, let us define a new potential on the space $\mathcal{A}_{\mathrm{G},S}$. Observe that any non-degenerate additive character χ of U is naturally decomposed into a sum of characters parametrised by the set of positive simple roots: $\chi = \sum_{i \in I} \chi_i$. Then, replacing in the definition of the potential at a given marked point m the nondegenerate character χ by the character $\sum_{i \in I_m} \chi_i$, we get a new function $\mathcal{W}_{m,I_m}$ at m, and set

$$\mathcal{W}_{\{I_m\}} := \sum_{\text{marked points } m \text{ on } S} \mathcal{W}_{m,I_m}. \tag{253}$$

Next, let us define a modified moduli space $\mathcal{P}^{\{\mathrm{I_m}\}}_{\mathrm{G}^L,*S}$.

Recall that for each simple positive root α_i there is a G-invariant divisor in $\mathcal{B} \times \mathcal{B}$. Let D_i be its preimage in $\mathcal{A} \times \mathcal{A}$. We say that a pair $(A_1, A_2) \in \mathcal{A} \times \mathcal{A}$ is in position $I - I_m$ if $(A_1, A_2) \in \mathcal{A} \times \mathcal{A} - \cup_{i \in I - I_m} D_i$.

Recall that C_m is the boundary component of $*S$ matching a marked point m on S.

Definition 10.9 The moduli space $\mathcal{P}^{\{I_m\}}_{\mathrm{G}^L,*S}$ parameterizes twisted G^L-local systems on S plus

a) A reduction of the structure group G^L near each puncture m to the parabolic subgroup of type $I - I_m$.
b) A decoration at every boundary interval C_m of $*S$ such that

- The decorated flags at the ends of the boundary interval C_m are in the position $I - I_m$.

So if $I = I_m$, the data a) is empty, and the condition b) is vacuous.

Finally, we consider the largest subspace

$$\mathcal{A}_{\mathrm{G},S}^{\{I_m\}} \subset \mathcal{A}_{\mathrm{G},S}$$

on which the potential $\mathcal{W}_{\{I_m\}}$ is regular. This condition is vacuous at punctures, and boils down to the $\bullet$-condition from Definition 10.9 at boundary intervals of $*S$. So if $I_m = \emptyset$ at every special point m, then $\mathcal{A}_{\mathrm{G},S}^{\{I_m\}} = \mathcal{A}_{\mathrm{G},S}$.

Conjecture 10.10 $(\mathcal{A}_{\mathrm{G},S}^{\{I_m\}}, \mathcal{W}_{\{I_m\}})$ *is dual to* $\mathcal{P}_{\mathrm{G}^L,*S}^{\{I_m\}}$.

Let us now formulate what the Duality Conjecture tells about canonical bases for the most interesting moduli space $\mathrm{Loc}_{\mathrm{G}^L,S}$, leaving similar formulations in other cases as a straightforward exercise.
Duality Conjecture for the space $\mathrm{Loc}_{\mathrm{G}^L,*S}$. The group Γ_S acts on the set $\mathcal{A}_{\mathrm{G},S}^{+}(\mathbb{Z}^t)$, and on the space $\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^{\mathrm{L}},*S})$ of regular functions on $\mathrm{Loc}_{\mathrm{G}^L,*S}$.

Conjecture 10.11 *There is a canonical basis in the space* $\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^L,*S})$ *parametrized by the set* $\mathcal{A}_{\mathrm{G},S}^{+}(\mathbb{Z}^t)$. *This parametrization is* Γ_S*-equivariant.*

Example. If S is a disc D_n with n special points on the boundary, then $\Gamma_{D_n} = \mathbb{Z}/n\mathbb{Z}$. Theorem 2.6 provides a Γ_{D_n}-equivariant canonical basis. Thus Conjecture 10.11 is proved.

If $\mathrm{G} = \mathrm{SL}_2$ (or $\mathrm{G} = \mathrm{PGL}_2$), then [17] provides a concrete construction of the Γ_S-equivariant parametrization, using laminations.

The following Theorem tells that the set $\mathcal{A}_{\mathrm{G},S}^{+}(\mathbb{Z}^t)$ is of the right size.

Theorem 10.12 *Given an ideal triangulation T of a decorated surface S, there is a linear basis in* $\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^L,*S})$ *parametrized by the set* $\mathcal{A}_{\mathrm{G},S}^{+}(\mathbb{Z}^t)$.

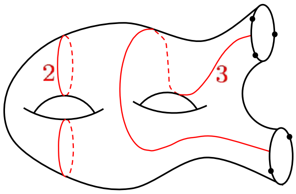
Remark. The parametrization depends on the choice of the ideal triangulations. In particular, it is not Γ_S-equivariant.

Proof The graph Γ dual to the triangulation T is a ribbon trivalent graph homotopy equivalent to S. An *end vertex* of Γ is a univalent vertex of the graph. It corresponds to a boundary interval of $\widehat{\partial}S$. Let $\mathrm{Loc}_{\mathrm{G}^L,\Gamma}$ be the moduli space of pairs $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is a G^L-local system on Γ, and γ is a flat section of the restriction of the local system $\mathcal{L}_{\mathcal{A}}$ to the end vertices of Γ.

Choose an orientation of the edges of Γ. Let $V(\Gamma)$ and $E(\Gamma)$ be the sets of vertices and edges of Γ. Pick an edge $E = (v_1, v_2)$ of Γ, oriented from v_1 to v_2. Given a function $\lambda : E(\Gamma) \longrightarrow \mathrm{P}^+$, we assign irreducible G^L-modules to the two flags of E, denoted $V_{v,E}$:

$$V_{(v_1,E)} := V_{\lambda(E)}, \quad V_{(v_2,E)} := V_{-w_0(\lambda(E))}.$$

Fig. 41 An integral lamination on a surface with two holes, with $2+3$ special points



According to [17, Section 12.5, (12.30)], there is a canonical isomorphism

$$\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^L,\Gamma}) = \bigoplus_{\{\lambda: E(\Gamma)\longrightarrow P^+\}} \bigotimes_{v\in V(\Gamma)} \left(\bigotimes_{(v,E)} V_{\lambda(v,E)}\right)^{\mathrm{G}^L} \qquad (254)$$

The second tensor product is over all flags incident to a given vertex v of Γ. By Applying Theorem 2.6 parametrizing a basis in the G^L-invariants of the tensor product for each vertex of Γ, it follows that $\mathcal{O}(\mathrm{Loc}_{\mathrm{G}^L,\Gamma})$ admits a linear basis parametrized by $\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$. Note that the central extension (247) is split. Following the remark after Definition 10.1, the moduli space $\mathrm{Loc}_{\mathrm{G}^L,S}$ is isomorphic to $\mathrm{Loc}_{\mathrm{G}^L,\Gamma}$. The Theorem is proved. □

10.3 Canonical basis in the space of functions on $\mathrm{Loc}_{\mathrm{SL}_2,S}$

Given any decorated surface S, there is a generalisation of integral laminations on S.

Definition 10.13 Let S be a decorated surface. An integral lamination l on S is a formal sum

$$l = \sum_i n_i[\alpha_i] + \sum_j m_j[\beta_j], \quad n_i, m_j \in \mathbb{Z}_{>0}. \qquad (255)$$

where $\{\alpha_i\}$ is a collection of simple nonisotopic loops, $\{\beta_j\}$ is a collection of simple nonisotopic intervals ending inside of boundary intervals on $\partial S - \{s_1, \ldots, s_n\}$, such that the curves do not intersect, considered modulo isotopy (Fig. 41). The set of integral laminations on S is denoted by $\mathcal{L}_{\mathbb{Z}}(S)$.

Let $\mathrm{Mon}_\alpha(\mathcal{L}, \alpha)$ be the monodromy of a twisted SL_2-local system $(\mathcal{L}, \alpha)$ over a loop α on S.

Let us show that a simple path β on S connecting two points x and y on $\widehat{\partial} S$ gives rise to a regular function Δ_β on $\mathrm{Loc}_{\mathrm{SL}_2,S}$.

Let $(\mathcal{L}, \alpha)$ be a decorated SL_2-local system on S. The associated flat bundle $\mathcal{L}_{\mathcal{A}}$ is a two dimensional flat vector bundle without zero section. Let v_x and v_y be the tangent vectors to ∂S at the points x, y. The decoration α at x and

y provides vectors l_x and l_y in the fibers of $\mathcal{L}_{\mathcal{A}}$ over v_x and v_y. The set S_β of non-zero tangent vectors to β is homotopy equivalent to a circle. Let us connect v_x and v_y by a path p in S_β, and transform the vector l_x at v_x to the fiber of $\mathcal{L}_{\mathcal{A}}$ over v_y, getting there a vector l'_x. We claim that $\Delta(l'_x, l_y)$ is independent of the choice of p. This uses crucially the fact that $\mathcal{L}$ is a twisted local system. So we arrive at a well defined number $\Delta(l'_x, l_y)$ assigned to $(\mathcal{L}, \alpha)$. We denote by Δ_β the obtained function on $\mathrm{Loc}_{\mathrm{SL}_2,S}$.

Given an integral lamination l on S as in (255), we a regular function M_l on $\mathrm{Loc}_{\mathrm{SL}_2,S}$ by

$$M_l(\mathcal{L}, \alpha) := \prod_i \mathrm{Tr}\left(\mathrm{Mon}_{\alpha_i}^{n_i}(\mathcal{L}, \alpha)\right) \prod_j \Delta_{\beta_j}^{m_j}(\mathcal{L}, \alpha).$$

Theorem 10.14 *The functions M_l, $l \in \mathcal{L}_{\mathbb{Z}}(S)$, form a linear basis in the space $\mathcal{O}(\mathrm{Loc}_{\mathrm{SL}_2,S})$.*

Theorem 10.15 *For any decorated surface S, there is a canonical isomorphism*

$$\mathcal{A}^+_{\mathrm{PGL}_2,S}(\mathbb{Z}^t) = \mathcal{L}_{\mathbb{Z}}(S).$$

Theorem 10.15 is proved similarly to Theorem 12.1 in [17]. Notice that $\mathcal{A}_{\mathrm{PGL}_2,S}$ is a positive space for the adjoint group PGL_2, the potential $\mathcal{W}$ lives on this space and is a positive function there. Theorem 10.14 is proved by using arguments similar to the proof of Theorem 10.12 and [17, Proposition 12.2].

Combining Theorem 10.14 and Theorem 10.15 we arrive at a construction of the canonical basis predicted by Conjecture 10.11 for $\mathrm{G} = \mathrm{PGL}_2$.

10.4 Surface affine Grassmannian and amalgamation

The surface affine Grassmannian $\mathrm{Gr}_{\mathrm{G},S}$. Given a twisted right $\mathrm{G}(\mathcal{K})$-local system $\mathcal{L}$ on S, there is the associated flat affine Grassmannian bundle $\mathcal{L}_{\mathrm{Gr}} := \mathcal{L} \times_{\mathrm{G}(\mathcal{K})} \mathrm{Gr}$. Similarly to Definition 10.1, we define

Definition 10.16 Let S be a decorated surface. The moduli space $\mathrm{Gr}_{\mathrm{G},S}$ parametrizes pairs $(\mathcal{L}, \nu)$ where $\mathcal{L}$ is a twisted right $\mathrm{G}(\mathcal{K})$-local system on S, and ν a flat section of the restriction of $\mathcal{L}_{\mathrm{Gr}}$ to the punctured boundary $\widehat{\partial}(*S)$.

Abusing terminology, the data ν is given by the lattices L_m at the marked points m on S.

The moduli space $\widetilde{\mathrm{Gr}}_{\mathrm{G},S}$ parametrizes similar data $(\widetilde{\mathcal{L}}, \nu)$, where $\widetilde{\mathcal{L}}$ is a twisted $\mathrm{G}(\mathcal{K})$-local system on S trivialized at a given point of S. So one has $\mathrm{Gr}_{\mathrm{G},S} = \mathrm{G}\backslash\widetilde{\mathrm{Gr}}_{\mathrm{G},S}$.

Example. Let D_n be a disc with n special points on the boundary. Then a choice of a special point provides isomorphisms

$$\mathrm{Gr}_{\mathrm{G},D_n} = \mathrm{Conf}_n(\mathrm{Gr}), \quad \widetilde{\mathrm{Gr}}_{\mathrm{G},D_n} = \mathrm{Gr}^n.$$

Cutting and amalgamating decorated surfaces. Let I be an ideal edge on a decorated surface S, i.e. a path connecting two marked points. Cutting S along the edge I we get a decorated surface S^*. Denote by I' and I'' the boundary intervals on S^* obtained by cutting along I.

Conversely, gluing boundary intervals I' and I'' on a decorated surface S^*, we get a new decorated surface S. We assume that the intervals I' and I'' on S^* are oriented by the orientation of the surface, and the gluing preserves the orientations.

More generally, let S be a decorated surface obtained from decorated surfaces $S_1, \ldots, S_n$ by gluing pairs $\{\mathrm{I}_1', \mathrm{I}_1''\}$, ..., $\{\mathrm{I}_m', \mathrm{I}_m''\}$ of oriented boundary intervals. We say that S is the *amalgamation* of decorated surfaces $S_1, \ldots, S_n$, and use the notation $S = S_1 * \cdots * S_n$. Abusing notation, we do not specify the pairs $\{\mathrm{I}_1', \mathrm{I}_1''\}, \ldots, \{\mathrm{I}_m', \mathrm{I}_m''\}$.

Amalgamating surface affine Grassmannians. There is a moduli space $\mathrm{Gr}_{\mathrm{G,I}}$ related to an oriented closed interval I, so that there is a canonical isomorphism of stacks

$$\mathrm{Gr}_{\mathrm{G,I}} = \mathrm{Conf}_2(\mathrm{Gr}).$$

Definition 10.17 Let I', I'' be boundary intervals on a decorated surface S^*, perhaps disconnected. The amalgamation stack $\mathrm{Gr}_{\mathrm{G},S^*}(\mathrm{I}' * \mathrm{I}'')$ parametrises triples $(\mathcal{L}, \gamma, g)$, where $(\mathcal{L}, \gamma)$ is the data parametrised by $\mathrm{Gr}_{\mathrm{G},S^*}$, and g is a gluing data, given by an equivalence of stacks

$$g : \mathrm{Gr}_{\mathrm{G,I'}} \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathrm{G,I''}}. \tag{256}$$

This immediately implies that there is a canonical equivalence of stacks:

$$\mathrm{Gr}_{\mathrm{G},S} \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathrm{G},S^*}(\mathrm{I}' * \mathrm{I}''). \tag{257}$$

Given decorated surfaces $S_1, \ldots, S_n$ and a collection $\{\mathrm{I}_1', \mathrm{I}_1''\}$, ..., $\{\mathrm{I}_m', \mathrm{I}_m''\}$ of pairs of boundary intervals, generalising the construction from Definition 10.17, we get the amalgamation stack

$$\mathrm{Gr}_{\mathrm{G},S_1*\cdots*S_n} = \mathrm{Gr}_{\mathrm{G},S_1*\cdots*S_n}(\mathrm{I}_1' * \mathrm{I}_1'', \ldots, \mathrm{I}_m' * \mathrm{I}_m'').$$

Applying equivalences (257) we get

Lemma 10.18 *There is a canonical equivalence of stacks:*

$$\mathrm{Gr}_{\mathrm{G},S} \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{G},S_1*\cdots*S_n}(\mathrm{I}'_1 * \mathrm{I}''_1, \ldots, \mathrm{I}'_m * \mathrm{I}''_m). \tag{258}$$

Let T be an ideal triangulation of a decorated surface S. Let $t_1, \ldots, t_n$ be the triangles of the triangulation. Abusing notation, denote by t_i the decorated surface given by the triangle t_i, with the special points given by the vertices. Denote by I'_i and I''_i the pair of edges obtained by cutting an edge I_i of the triangulation t, $i = 1, \ldots, m$. Then one has an isomorphism of stacks

$$\mathrm{Gr}_{\mathrm{G},S} = \mathrm{Gr}_{\mathrm{G},t_1*\cdots*t_n}(\mathrm{I}'_1 * \mathrm{I}''_1, \ldots, \mathrm{I}'_m * \mathrm{I}''_m). \tag{259}$$

10.5 Top components of the surface affine Grassmannian

10.5.1 Regularised dimensions

Recall that if a finite dimensional group A acts on a finite dimensional variety X, we define the dimension of the stack X/A by

$$\dim X/\mathrm{A} := \dim X - \dim \mathrm{A}.$$

Our goal is to generalise this definition to the case when X and A could be infinite dimensional.

Dimension torsors $\mathbf{t}^n$. Let us first define a rank one $\mathbb{Z}$-torsor $\mathbf{t}$. The kernel $\mathbb{N}$ of the evaluation map $\mathrm{G}(\mathcal{O}) \to \mathrm{G}(\mathbb{C})$ is a prounipotent algebraic group over $\mathbb{C}$. Let N be its finite codimension normal subgroup. We assign to each such an N a copy $\mathbb{Z}_{(N)}$ of $\mathbb{Z}$, and for each pair $N_1 \subset N_2$ such that N_2/N_1 is a finite dimensional, an isomorphism of $\mathbb{Z}$-torsors

$$i_{N_1,N_2} : \mathbb{Z}_{(N_1)} \longrightarrow \mathbb{Z}_{(N_2)}, \quad x \longmapsto x + \dim N_2/N_1. \tag{260}$$

Definition 10.19 A $\mathbb{Z}$-torsor $\mathbf{t}$ is given by the collection of $\mathbb{Z}$-torsors $\mathbb{Z}_{(N)}$ and isomorphisms i_{N_1,N_2}. We set $\mathbf{t}^n := \mathbf{t}^{\otimes n}$ for any $n \in \mathbb{Z}$.

In particular, $\mathbf{t}^0 = \mathbb{Z}$. To define an element of $\mathbf{t}^n$ means to exhibit a collection of integers d_N assigned to the finite codimension subgroups N of $\mathbb{N}$ related by isomorphisms (260).

Example. There is an element $\mathbf{dim}\, \mathrm{G}(\mathcal{O}) \in \mathbf{t}$, given by an assignment

$$\mathbf{dim}\, \mathrm{G}(\mathcal{O}) := \{N \longmapsto \dim \mathrm{G}(\mathcal{O})/N \in \mathbb{Z}_{(N)}\} \in \mathbf{t}.$$

More generally, there is an element

$$n\, \mathbf{dim}\, \mathrm{G}(\mathcal{O}) := \{N \longmapsto \dim (\mathrm{G}(\mathcal{O})/N)^n \in \mathbb{Z}_{(N)}\} \in \mathbf{t}^n.$$

For example, the stack $*/\mathrm{G}(\mathcal{O})^n$, where $* = \mathrm{Spec}(\mathbb{C})$ is the point, has dimension

$$\mathbf{dim}\ */\mathrm{G}(\mathcal{O})^n = -n\ \mathbf{dim}\ \mathrm{G}(\mathcal{O}) \in \mathbf{t}^{-n}.$$

If X and Y have dimensions $\mathbf{dim}\ X \in \mathbf{t}^n$ and $\mathbf{dim}\ Y \in \mathbf{t}^m$, then $\mathbf{dim}\ X \times Y \in \mathbf{t}^{n+m}$.

Dimension torsors $\mathbf{t}^n_{\mathrm{A}}$. We generalise this construction by replacing the group $\mathrm{G}(\mathcal{O})$ by a pro-algebraic group A, which has a finite codimension prounipotent normal subgroup.[16] Then there are the dimension torsor $\mathbf{t}_{\mathrm{A}}$, its tensor powers $\mathbf{t}^n_{\mathrm{A}}$, $n \in \mathbb{Z}$, and an element $\mathbf{dim}\ \mathrm{A} \in \mathbf{t}_{\mathrm{A}}$. One has $\mathbf{t}_{\mathrm{A}^n} = \mathbf{t}^n_{\mathrm{A}}$. Moreover,

$$n\ \mathbf{dim}\ \mathrm{A} \in \mathbf{t}^n_{\mathrm{A}}, \quad \mathbf{t}^n_{\mathrm{A}} = \{m + n\ \mathbf{dim}\ \mathrm{A}\}, m \in \mathbb{Z}.$$

Regularised dimension. Given such a group A, we can define the dimension of a stack $\mathcal{X}$ under the following assumptions.

1. There is a finite codimension prounipotent subgroup $\mathrm{N} \subset \mathrm{A}$ such that
$$\mathrm{N}^n \quad \text{acts freely on} \mathcal{X}.$$
2. There is a finite dimensional stack $\mathcal{Y}$ and an action of the group A^m on $\mathcal{Y}$ such that
$$\mathcal{Y}/\mathrm{A}^m = \mathcal{X}/\mathrm{N}^n. \tag{261}$$
3. There exists a finite codimension normal prounipotent subgroup $\mathrm{M} \subset \mathrm{A}$ such that the action of A^m on $\mathcal{Y}$ restricts to the trivial action of the subgroup M^m on $\mathcal{Y}$.

The last condition implies that we have a finite dimensional stack $\mathcal{Y}/(\mathrm{A}/\mathrm{M}^m)$. The stack $\mathcal{Y}/\mathrm{A}^m$ is the quotient of the stack $\mathcal{Y}/(\mathrm{A}/\mathrm{M}^m)$ by the trivial action of the group M^m.

In this case we define an element of the torsor $\mathbf{t}^{n-m}_{\mathrm{A}}$ by the assignment

$$\begin{aligned}(\mathrm{N}, \mathrm{M}) \longmapsto \dim(\mathcal{Y}/\mathrm{A}^m) + \mathbf{dim}\ (\mathrm{N}^n) &:= (n-m)\ \mathbf{dim}\ \mathrm{A} + \dim \mathcal{Y} \\ &-n \dim(\mathrm{A}/\mathrm{N}) \in \mathbf{t}^{n-m}_{\mathrm{A}}.\end{aligned} \tag{262}$$

Definition 10.20 Assuming (1)–(2), the assignment (262) defines the regularised dimension

$$\mathbf{dim}\ \mathcal{X} \in \mathbf{t}^{n-m}_{\mathrm{A}}.$$

[16] Taking the quotient by a unipotent group does not affect the category of equivariant sheaves. This is why we require the prounipotence condition here.

Remark. Often an infinite dimensional stack $\mathcal{X}$ does not have a canonical presentation (261), but rather a collection of such presentations. For instance such a presentation of the stack $\mathcal{M}_l^\circ$ defined below depends on a choice of an ideal triangulation T of S. Then we need to prove that the regularised dimension is independent of the choices.

10.5.2 Top components of the stack $\mathrm{Gr}_{\mathrm{G},S}$

Suppose that a decorated surface S is an amalgamation of decorated surfaces:

$$S = S_1 * \ldots * S_n. \tag{263}$$

Definition 10.21 Given an amalgamation pattern (263), define the amalgamation

$$\mathcal{A}_{\mathrm{G},S_1}(\mathbb{Z}^t) * \ldots * \mathcal{A}_{\mathrm{G},S_n}(\mathbb{Z}^t) := \{(l_1, \ldots, l_n) \in \mathcal{A}_{\mathrm{G},S_1}(\mathbb{Z}^t) \times \ldots \times \mathcal{A}_{\mathrm{G},S_n}(\mathbb{Z}^t) \ \mid \ (264) \text{ holds}\} :$$

$$\pi^t_{\mathrm{I}'_k}(l_i) = \pi^t_{\mathrm{I}''_k}(l_j) \text{ for any boundary intervals } \mathrm{I}'_k \subset S_i \text{ and } \mathrm{I}''_k \subset S_j \text{ glued in } \mathrm{S}. \tag{264}$$

Lemma 10.22 *Given an amalgamation pattern (263), there are canonical isomorphism of sets*

$$\mathcal{A}_{\mathrm{G},S}(\mathbb{Z}^t) = \mathcal{A}_{\mathrm{G},S_1}(\mathbb{Z}^t) * \ldots * \mathcal{A}_{\mathrm{G},S_n}(\mathbb{Z}^t).$$

$$\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t) = \mathcal{A}^+_{\mathrm{G},S_1}(\mathbb{Z}^t) * \ldots * \mathcal{A}^+_{\mathrm{G},S_n}(\mathbb{Z}^t).$$

In this case we say that l is presented as an amalgamation, and write $l = l_1 * \ldots * l_n$.

Let us pick an ideal triangulation T of S, and present S as an amalgamation of the triangles:

$$S = t_1 * \ldots * t_n. \tag{265}$$

By Lemma 10.22, any $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ is uniquely presented as an amalgamation

$$l = l_1 * \ldots * l_n, \quad l_i \in \mathcal{A}^+_{\mathrm{G},t_i}(\mathbb{Z}^t). \tag{266}$$

Recall that given a polygon D_n, there are cycles

$$\mathcal{M}_l^\circ := \kappa(\mathcal{C}_l^\circ) \subset \mathrm{Gr}_{G,D_n}, \quad l \in \mathcal{A}^+_{\mathrm{G},D_n}(\mathbb{Z}^t).$$

Definition 10.23 Given an ideal triangulation T of S and an $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ we set, using amalgamations (265) and (266),

$$\mathcal{M}^\circ_{T,l} = \mathcal{M}^\circ_{t_1,l_1} * \ldots * \mathcal{M}^\circ_{t_n,l_n}, \quad \mathcal{M}_{T,l} := \quad \text{Zariski closure of} \quad \mathcal{M}^\circ_{T,l}.$$

Thanks to Lemma 6.14, the restriction to the boundary intervals of S leads to a map of sets

$$\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t) \longrightarrow \mathrm{P}^{+\{\text{boundary intervals of} S\}}.$$

It assigns to a point $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$ a collection of dominant coweights $\lambda_{\mathrm{I}_1}, \ldots, \lambda_{\mathrm{I}_n} \in \mathrm{P}^+$ at the boundary intervals $\mathrm{I}_1, \ldots, \mathrm{I}_n$ of S.

For any decorated subsurface $i : S' \subset S$ there is a projection given by the restriction map for the surface affine Grassmannian: $r_{\mathrm{Gr}} : \mathrm{Gr}_{\mathrm{G},S} \longrightarrow \mathrm{Gr}_{\mathrm{G},S'}$. There are two canonical projections:

$$\begin{array}{ccc} \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t) & & \mathrm{Gr}_{\mathrm{G},S} \\ r^t_{\mathcal{A}} \downarrow & & \downarrow r_{\mathrm{Gr}} \\ \mathrm{Conf}^+_{\mathrm{G},S'}(\mathcal{A})(\mathbb{Z}^t) & & \mathrm{Gr}_{\mathrm{G},S'}) \end{array} \tag{267}$$

Theorem 10.24 *Let S be a decorated surface.*

(i) *The stack $\mathcal{M}_{T,l}$ does not depend on the triangulation T. We denote it by $\mathcal{M}_l$.*
(ii) *Let $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$. Let $\{\mathrm{I}_1, \ldots, \mathrm{I}_n\}$ be the set of boundary intervals of S, and $\lambda_{\mathrm{I}_1}, \ldots, \lambda_{\mathrm{I}_n}$ are the dominant coweights assigned to them by l. Then*

$$\mathbf{dim}\ \mathcal{M}_l = \langle \rho, \lambda_{\mathrm{I}_1} + \ldots + \lambda_{\mathrm{I}_n} \rangle - \chi(S)\ \mathbf{dim}\ \mathrm{G}(\mathcal{O}) \in \mathbf{t}^{-\chi(S)}. \tag{268}$$

(iii) *The stacks $\mathcal{M}_l$, $l \in \mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t)$, are top dimensional components of* $\mathrm{Gr}_{\mathrm{G},S}$.
(iv) *The map $l \longmapsto \mathcal{M}_l$ provides a bijection*

$$\mathcal{A}^+_{\mathrm{G},S}(\mathbb{Z}^t) \stackrel{\sim}{\longrightarrow} \{\textit{top dimensional components of the stack}\mathrm{Gr}_{\mathrm{G},S}\}.$$

This isomorphism commutes with the restriction to decorated subsurfaces of S.

Proof Let us calculate first dimensions of the stacks $\mathcal{M}^{\circ}_{T,l}$, and show that they are given by formula (268). We present first a heuristic dimension count, and then fill the necessary details.

Heuristic dimension count. Let us present a decorated surface S as an amalgamation of a (possible disconnected) decorated surface along a pair of boundary intervals I', I'', as in Definition 10.17. The space of isomorphisms g from (256) is a disjoint union $\mathrm{G}(\mathcal{K})$-torsors parametrised by dominant coweights λ, since the latter parametrise $\mathrm{G}(\mathcal{K})$-orbits on $\mathrm{Gr} \times \mathrm{Gr}$. Pick one of them.

Let $\mathrm{L}'_0 \xrightarrow{\lambda} \mathrm{L}'_1$ (respectively $\mathrm{L}''_0 \xrightarrow{\lambda} \mathrm{L}''_1$) be a pair of lattices assigned to the vertices of the interval I' (respectively I''). Then the gluing data is a map $g : (\mathrm{L}'_0, \mathrm{L}'_1) \longrightarrow (\mathrm{L}''_0, \mathrm{L}''_1)$. Let G_λ be the subgroup stabilising the pair $\mathrm{L}'_0 \xrightarrow{\lambda} \mathrm{L}'_1$. The space of gluings is a G_λ-torsor. The group G_λ is a subgroup of codimension $2\langle \rho, \lambda \rangle$ in $\mathrm{Aut}\, \mathrm{L}_0 \stackrel{\sim}{=} \mathrm{G}(\mathcal{O})$. So

$$\textbf{dim}\, \mathrm{G}_\lambda = \textbf{dim}\, \mathrm{G}(\mathcal{O}) - 2\langle \rho, \lambda \rangle = \textbf{dim}\, \mathrm{G}(\mathcal{O}) - \textbf{dim}\, \mathrm{Gr}_{\lambda, \lambda^\vee}.$$

Take the stack $\mathcal{M}^{\circ}_{t,l}$ assigned to a triangle t and a point $l \in \mathrm{Conf}^+_3(\mathcal{A})(\mathbb{Z}^t)$. Let $\lambda_1, \lambda_2, \lambda_3$ be the dominant coweights assigned to the sides of the triangle by l. Then $\mathcal{M}^{\circ}_{t,l}$ is an open part of a component of the stack $\mathrm{Gr}_{\lambda_1, \lambda_2, \lambda_3}/\mathrm{G}(\mathcal{O})$. Thus

$$\textbf{dim}\, \mathcal{M}^{\circ}_{t,l} = \langle \rho, \lambda_1 + \lambda_2 + \lambda_3 \rangle - \textbf{dim}\, \mathrm{G}(\mathcal{O}) \in \mathbf{t}^{-1}. \tag{269}$$

Let us calculate now the dimension of the stack $\mathcal{M}^{\circ}_{T,l}$. Let $|\mathcal{T}|$ be the number of triangles, and $\mathcal{E}_{\mathrm{int}}$ (respectively $\mathcal{E}_{\mathrm{ext}}$) the set of the internal (respectively external) edges of the triangulation T. Then the dimension of the product of stacks assigned to the triangles is

$$\sum_{E \in \mathcal{E}_{\mathrm{ext}}} \langle \rho, \lambda_E \rangle + 2 \sum_{E \in \mathcal{E}_{\mathrm{int}}} \langle \rho, \lambda_E \rangle - |\mathcal{T}|\, \textbf{dim}\, \mathrm{G}(\mathcal{O}) \in \mathbf{t}^{-|\mathcal{T}|}.$$

Gluing two boundary intervals into an internal edge E, with the dominant weights λ_E associated to it, we have to add the dimension of the corresponding gluing data torsor, that is

$$\textbf{dim}\, \mathrm{G}(\mathcal{O}) - 2\langle \rho, \lambda_E \rangle \in \mathbf{t}.$$

So, gluing all the intervals, we get

$$\sum_{E\in\mathcal{E}_{\mathrm{ext}}} \langle \rho, \lambda_E\rangle + (|\mathcal{E}_{\mathrm{int}}| - |\mathcal{T}|)\ \mathbf{dim}\ \mathrm{G}(\mathcal{O}) = \sum_{E\in\mathcal{E}_{\mathrm{ext}}} \langle \rho, \lambda_E\rangle - \chi(S)\ \mathbf{dim}\ \mathrm{G}(\mathcal{O}) = (268).$$

Notice that $|\mathcal{E}_{\mathrm{int}}| - |\mathcal{T}| = -\chi(S)$. Indeed, the triangles t with external sides removed cover the surface S minus the boundary, which has the same Euler characteristic as S.

Rigorous dimension count. For each of the triangles t of the triangulation T there are three dominant coweights $\underline{\lambda}(t) := \lambda_1(t), \lambda_2(t), \lambda_3(t)$ assigned by l to the sides of t. Pick a vertex $v(t)$ of the triangle t. We present the stack $\mathrm{Gr}_{\mathrm{G},t}$ as a quotient of the convolution variety

$$\mathrm{Gr}_{\mathrm{G},t} = \mathrm{Gr}_{\underline{\lambda}(t)}/\mathrm{G}(\mathcal{O}). \tag{270}$$

Namely, choose the lattice $\mathrm{L}_{v(t)}$ at the vertex $v(t)$ to be the standard lattice $\mathrm{L}_{v(t)} = \mathrm{G}(\mathcal{O})$.

There exists a finite codimension normal prounipotent subgroup $N_{t,l} \subset \mathrm{G}(\mathcal{O})$ acting trivially on $\mathrm{Gr}_{\underline{\lambda}(t)}$. It depends on the choice of coweights $\underline{\lambda}(t)$, and, via them, on the choice of the t and l. We assign to each finite codimension normal subgroup $N'_{t,l} \subset N_{t,l}$ a finite dimensional stack

$$\frac{\mathrm{Gr}_{\underline{\lambda}(t)}}{\mathrm{G}(\mathcal{O})/N'_{t,l}}.$$

Its dimension is $\langle \rho, \lambda_1 + \lambda_2 + \lambda_3\rangle - \dim \mathrm{G}(\mathcal{O})/N'_{t,l}$. This just means that we have formula (269).

There is a canonical surjective map of stacks

$$\mathrm{Gr}_{G,S} \longrightarrow \prod_{t\in T} \mathrm{Gr}_{G,t} = \prod_{t\in T} \mathrm{Gr}_{\underline{\lambda}(t)}/\mathrm{G}(\mathcal{O}). \tag{271}$$

Its fibers are torsors over the product over the set $\mathcal{E}_{\mathrm{int}}$ of internal edges E of T of certain groups $G_{\lambda(E)}$ defined as follows. Let $\lambda(E)$ be the dominant coweight assigned to E by l. Consider the pair E', E'' of edges of triangles glued into the edge E. For each of them, there is a pair of the lattices assigned to its vertices. We get two pairs of lattices:

$$\left(\mathrm{L}^-_{E'} \stackrel{\lambda(E)}{\longrightarrow} \mathrm{L}^+_{E'}\right) \quad \text{and} \quad \left(\mathrm{L}^-_{E''} \stackrel{\lambda(E)}{\longrightarrow} \mathrm{L}^+_{E''}\right).$$

Choose one of the edges, say E'. Set $\mathrm{G}_{\lambda(E)} := \mathrm{Aut}\,(\mathrm{L}^-_{E'} \stackrel{\lambda(E)}{\longrightarrow} \mathrm{L}^+_{E'})$. Therefore we conclude that

$$\text{The fibers of the map (271) are torsors over the group} \quad \prod_{E\in\mathcal{E}_{\rm int}} \mathrm{G}_{\lambda(E)}.$$

For each E, choose a finite codimension subgroup $N_{\lambda(E)} \subset \mathrm{G}_{\lambda(E)}$. Then we are in the situation discussed right before Definition 10.20, where

$$\mathcal{X} = \mathcal{M}^\circ_l, \quad \mathrm{A} = \mathrm{G}(\mathcal{O}), \quad N := \cap_{E\in\mathcal{E}_{\rm int}} N_{\lambda(E)}, \quad M = \cap_t N'_{t,l},$$
$$n = |\mathcal{E}_{\rm int}|, \quad m = |\mathcal{T}|.$$

So we get the expected formula for the regularised dimension of $\mathcal{M}^\circ_{T,l}$.

The resulting regularised dimension does not depend on the choice of ideal triangulation T—the triangulation does not enter to the answer.

Alternatively, one can see this as follows. Any two ideal triangulations of S are related by a sequence of flips. Let $T \longrightarrow T'$ be a flip at an edge E. Let R_E be the unique rectangle of the triangulation T with the diagonal E. Consider the restriction map $\pi : \mathrm{Gr}_{G,S} \longrightarrow \mathrm{Gr}_{R_E,S}$. So one can fiber $\mathcal{M}^\circ_l$ over the component $\mathcal{M}^\circ_{\pi^t(l)}$. The dimension of the latter does not depend on the choice of the triangulation of the rectangle.

A similar argument with a flip of triangulation proves (i). Combining with the formula for the regularised dimension of $\mathcal{M}^\circ_{T,l}$ we get (ii).

(iii), (iv). Present S as an amalgamation of the triangles of an ideal triangulation. It is known that the cycles $\mathcal{M}_l$ are the top dimensional components of the convolution variety, and thus the stack $\mathrm{Gr}_{G,t}$, assigned to the triangle. It remains to use Lemma 10.18. □

11 Cluster varieties, frozen variables and potentials

11.1 Basics of cluster varieties

Definition 11.1 A quiver $\mathbf{q}$ is described by a data $(\Lambda, \Lambda_0, \{e_i\}, (*,*))$, where

1. Λ is a lattice, Λ_0 is a sublattice of Λ, and $\{e_i\}$ is a basis of Λ such that Λ_0 is generated by a subset of *frozen basis vectors*;
2. $(*,*)$ is a skewsymmetric $\frac{1}{2}\mathbb{Z}$-valued bilinear form on Λ with $(e_i, e_j) \in \mathbb{Z}$ unless $e_i, e_j \in \Lambda_0$.

Any non-frozen basis element e_k provides a *mutated in the direction* e_k quiver $\mathbf{q}'$. The quiver $\mathbf{q}'$ is defined by changing the basis $\{e_i\}$ only. The new basis $\{e'_i\}$ is defined via halfreflection of the $\{e_i\}$ along the hyperplane $(e_k, \cdot) = 0$:

$$e_i' := \begin{cases} e_i + [\varepsilon_{ik}]_+ e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k. \end{cases} \tag{272}$$

Here $[\alpha]_+ := \alpha$ if $\alpha \geq 0$ and $[\alpha]_+ := 0$ otherwise. The frozen/non-frozen basis vectors of the mutated quiver are the images of the ones of the original quiver. The composition of two mutations in the same direction k is an isomorphism of quivers.

Set $\varepsilon_{ij} := (e_i, e_j)$. A quiver can be described by a data $\mathbf{q} = (\mathrm{I}, \mathrm{I}_0, \varepsilon)$, where I (respectively I_0) is the set parametrising the basis vectors (respectively frozen vectors). Formula (272) amounts then to the Fomin–Zelevinsky formula telling how the ε-matrix changes under mutations.

$$\varepsilon_{ij}' := \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \quad k \notin \{i, j\} \\ \varepsilon_{ij} + |\varepsilon_{ik}| \cdot \varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \quad k \notin \{i, j\}. \end{cases} \tag{273}$$

We assign to every quiver $\mathbf{q}$ two sets of coordinates, each parametrised by the set I: the $\mathcal{X}$-coordinates $\{X_i\}$, and the $\mathcal{A}$-coordinates $\{A_i\}$. Given a mutation of quivers $\mu_k : \mathbf{q} \longmapsto \mathbf{q}'$, the cluster coordinates assigned to these quivers are related as follows. Denote the cluster coordinates related to the quiver $\mathbf{q}'$ by $\{X_i'\}$ and $\{A_i'\}$. Then

$$A_k A_k' := \prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}; \qquad A_i' = A_i, \quad i \neq k. \tag{274}$$

If any of the sets $\{j|\varepsilon_{kj} > 0\}$ or $\{j|\varepsilon_{kj} < 0\}$ is empty, the corresponding monomial is 1.

$$X_i' := \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases} \tag{275}$$

The tropicalizations of these transformations are

$$a_k' := -a_k + \min\left\{ \sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj} a_j, \sum_{j|\varepsilon_{kj}<0} -\varepsilon_{kj} a_j \right\}; \qquad a_i' = a_i, \quad i \neq k. \tag{276}$$

$$x_i' := \begin{cases} -x_k & \text{if } i = k \\ x_i - \varepsilon_{ik}\min\{0, -\mathrm{sgn}(\varepsilon_{ik})x_k\} & \text{if } i \neq k, \end{cases} \tag{277}$$

Cluster transformations are transformations of cluster coordinates obtained by composing mutations. Cluster $\mathcal{A}$-coordinates and mutation formulas (272) and (274) are main ingredients of the definition of cluster algebras [23]. Cluster $\mathcal{X}$-coordinates and mutation formulas (275) describe a dual object, introduced in [18] under the name *cluster $\mathcal{X}$-variety*.

The cluster volume forms [21]. Given a quiver $\mathbf{q}$, consider the volume forms

$$\mathrm{Vol}^{\mathbf{q}}_{\mathcal{A}} := d\log A_1 \wedge \ldots \wedge d\log A_n, \quad \mathrm{Vol}^{\mathbf{q}}_{\mathcal{X}} := d\log X_1 \wedge \ldots \wedge d\log X_n.$$

Cluster transformations preserve them up to a sign: given a mutation $\mathbf{q} \longmapsto \mathbf{q}'$, we have

$$\mathrm{Vol}^{\mathbf{q}'}_{\mathcal{A}} = -\mathrm{Vol}^{\mathbf{q}}_{\mathcal{A}}, \qquad \mathrm{Vol}^{\mathbf{q}'}_{\mathcal{X}} = -\mathrm{Vol}^{\mathbf{q}}_{\mathcal{X}}.$$

Denote by Or_Λ the two element set of orientations of a rank n lattice Λ, given by expressions $l_1 \wedge \cdots \wedge l_n$ where $\{l_i\}$ form a basis of Λ. *An orientation* or_Λ of Λ is a choice of one of its elements. Given a basis $\{e_i\}$ of Λ, we define its sign $\mathrm{sign}(e_1, \ldots, e_n)$ by $e_1 \wedge \cdots \wedge e_n = \mathrm{sign}(e_1, \ldots, e_n)\mathrm{or}_\Lambda$. A quiver mutation changes the sign of the basis, and the sign of each of the cluster volume forms. So there is a definition of the cluster volume forms invariant under cluster transformations.

Definition 11.2 Choose an orientation or_Λ for a quiver $\mathbf{q}$. Then in any quiver obtained by from $\mathbf{q}$ by mutations, the cluster volume forms are given by

$$\mathrm{Vol}_{\mathcal{A}} = \mathrm{sign}(e_1, \ldots, e_n) d\log A_1 \wedge \ldots \wedge d\log A_n,$$
$$\mathrm{Vol}_{\mathcal{X}} = \mathrm{sign}(e_1, \ldots, e_n) d\log X_1 \wedge \ldots \wedge d\log X_n.$$

Residues of the cluster volume form $\mathrm{Vol}_{\mathcal{A}}$ **and frozen variables.** Take a space M equipped with a cluster $\mathcal{A}$-coordinate system $\{A_i\}$.

Lemma 11.3 *Let us assume that $k \in \mathrm{I} - \mathrm{I}_0$ is nonfrozen, and $\varepsilon_{kj} \neq 0$ for some j. Then*

$$\mathrm{Res}_{A_k=0}(\mathrm{Vol}_{\mathcal{A}}) = 0. \tag{278}$$

Proof We have $\mathrm{Res}_{A_k=0}(\mathrm{Vol}_{\mathcal{A}}) = \pm \bigwedge_{i\neq k} d\log A_i$. Since k is nonfrozen, there is an exchange relation (274). It implies a monomial relation on the locus $A_k = 0$: $\prod_j A_j^{\varepsilon_{kj}} = -1$. Since ε_{kj} is not identically zero, this monomial is nontrivial. Thus $\bigwedge_{i\neq k} d\log A_i = 0$ at the $A_k = 0$ locus. □

Corollary 11.4 *A coordinate A_k, with $\varepsilon_{kj} \neq 0$ for some j, can be nonfrozen only if we have (278), i.e. the functions $A_1, \ldots, \widehat{A}_k, \ldots, A_n$ become dependent on every component of the $A_k = 0$ locus.*

If we define a cluster algebra axiomatically, without referring to a particular space on which it is realised, then any subset of an initial quiver can be declared to be the frozen subset. However if a cluster algebra is realised geometrically, we do not have much freedom in the definition of frozen variables, as Corollary 11.4 shows. This leads to the following geometric definition of the frozen coordinates.

Definition 11.5 Let M be a space equipped with a cluster $\mathcal{A}$-coordinate system. Then a cluster variable A is a frozen variable if and only if the residue form $\mathrm{Res}_{\mathrm{A}}(\mathrm{Vol}_{\mathcal{A}})$ is not zero.

Non-negative real points for a cluster algebra. The space of positive real points of any positive space is well defined. Let us define the space of non-negative real points for a cluster algebra.

Let $\{A_i^{\mathbf{q}}\}$, $i \in \mathrm{I}$, be the set of all cluster coordinates in a given quiver $\mathbf{q}$. The cluster algebra $\mathcal{O}_{\mathrm{aff}}(\mathcal{A})$ is the algebra generated by the formal variables $\{A_i^{\mathbf{q}}\}$, for all quivers $\mathbf{q}$ related by mutations to a given one, modulo the ideal generated by exchange relations (274):

$$\mathcal{O}_{\mathrm{aff}}(\mathcal{A}) := \frac{\mathbb{Z}[A_i^{\mathbf{q}}]}{(\text{ exchange relations })}. \tag{279}$$

This ring is not necessarily finitely generated. Let $\mathcal{A}_{\mathrm{aff}}$ be its spectrum. Then the points of $\mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0})$ are just the collections of positive real numbers $\{a_i^{\mathbf{q}} \in \mathbb{R}_{\geq 0}\}$ satisfying the exchange relations. The *positive boundary* is defined as the complement to the set of positive real points:

$$\partial\mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0}) := \mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0}) - \mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{>0}).$$

Let A_f be a frozen variable. Then $\{A_f = 0\} \cap \partial\mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0})$ is of real codimension one in $\mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0})$. Indeed, the frozen $\mathcal{A}$-cluster coordinates do not mutate, and so the codimension one domain given by the points with the coordinates $A_{f_t} = 0$, $A_j^{\mathbf{q}} > 0$ where j is different then f_t is a part of the intersection.

Let $A_k^{\mathbf{q}}$ be a non-frozen variable. It is likely, although we did not prove this, that in many cases

$$\{A_k^{\mathbf{q}} = 0\} \cap \partial\mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0}) \quad \text{is of real codimension } \geq 2 in \quad \mathcal{A}_{\mathrm{aff}}(\mathbb{R}_{\geq 0}). \tag{280}$$

Indeed, the exchange relation for the $A_k^{\mathbf{q}}$, restricted to the $A_k^{\mathbf{q}} = 0$ hyperplane, reads

$$0 \cdot A_k^{\mathbf{q}'} = \prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}.$$

So both monomials on the right, being non-negative, are zero, and each of them is non-empty: the empty one contributes 1, violating 0 on the left. So we get at least two different cluster coordinates equal to zero. It is easy to see that then in any cluster coordinate system at least two of cluster coordinates are zero.

11.2 Frozen variables, partial compactification $\hat{\mathcal{A}}$, and potential on the $\mathcal{X}$-space

Potential on the $\mathcal{X}$-space

Lemma 11.6 *Any frozen $f \in \mathrm{I}_0$ gives rise to a tropical point $l_f \in \mathcal{A}(\mathbb{Z}^t)$ such that in any cluster $\mathcal{A}$-coordinate system all tropical $\mathcal{A}$-coordinates except a_f are zero, and $a_f = 1$.*

Proof Pick a cluster $\mathcal{A}$-coordinate system $\alpha = \{\mathrm{A}_f, \dots\}$ starting from a coordinate A_f. Consider a tropical point in $\mathcal{A}(\mathbb{Z}^t)$ with the coordinates $(1, 0, \dots, 0)$. It is clear from (276) that the coordinates of this point are invariant under mutations at non-frozen vertices. Indeed, at least one of the two quantities we minimize in (276) is zero, and the other must be non-negative. $\square$

The potential. Let us assume that there are canonical maps, implied by the cluster Duality Conjectures for the dual pair $(\mathcal{A}, \mathcal{X}^\vee)$ of cluster varieties:

$$\mathbb{I}_\mathcal{A}: \quad \mathcal{A}(\mathbb{Z}^t) \longrightarrow \mathbb{L}_+(\mathcal{X}^\vee), \quad \mathbb{I}_\mathcal{X}: \quad \mathcal{X}^\vee(\mathbb{Z}^t) \longrightarrow \mathbb{L}_+(\mathcal{A}).$$

Here $\mathbb{L}_+(\mathcal{X}^\vee)$ and $\mathbb{L}_+(\mathcal{A})$ are the sets of universally Laurent functions.

Definition 11.7 Let us assume that for each frozen $f \in \mathrm{I}_0$ there is a function

$$\mathcal{W}_{\mathcal{X}^\vee, f} := \mathbb{I}_\mathcal{A}(l_f) \in \mathbb{L}_+(\mathcal{X}^\vee)$$

predicted by the Duality Conjectures. Then the potential on the space $\mathcal{X}$ is given by the sum

$$\mathcal{W}_{\mathcal{X}^\vee} := \sum_{f \in \mathrm{I}_0} \mathcal{W}_{\mathcal{X}^\vee, f}.$$

Partial compactifications of the $\mathcal{A}$-space. Given any subset $\mathrm{I}_0' \in \mathrm{I}_0$, we can define a partial completion $\mathcal{A} \bigsqcup_{f \in \mathrm{I}_0'} \mathrm{D}_f$ of $\mathcal{A}$ by attaching to $\mathcal{A}$ the divisor D_f corresponding to the equation $\mathrm{A}_f = 0$ for each $f \in \mathrm{I}_0'$. The duality should look like

$$\left(\mathcal{A}\bigsqcup_{f\in \mathrm{I}_0'}\mathrm{D}_f\right) <=> \left(\mathcal{X}^\vee, \sum_{f\in \mathrm{I}_0'}\mathcal{W}_f\right).$$

The order of pole of $\mathbb{I}_{\mathcal{X}}(l)$ at the divisor D_f should be equal to $\mathcal{W}^t_f(l)$. In particular, $\mathbb{I}_{\mathcal{X}}(l)$ extends to $\mathcal{A}\bigsqcup\mathrm{D}_f$ if and only if it is in the subset $\{l \in \mathcal{X}^\vee(\mathbb{Z}^t) \mid \mathcal{W}^t_f(l)\geq 0\} \subset \mathcal{X}^\vee(\mathbb{Z}^t)$.

Canonical tropical points of the $\mathcal{X}$-space. Let $i \in \mathrm{I}$. Given a cluster $\mathcal{X}$-coordinate system, consider a point $t_i \in \mathcal{X}(\mathbb{Z}^t)$ with the coordinates ε_{ji}, $j\in \mathrm{I}$.

Lemma 11.8 *The point t_i is invariant under mutations of cluster $\mathcal{X}$-coordinate systems. So there is a point $t_i \in \mathcal{X}(\mathbb{Z}^t)$ which in any cluster $\mathcal{X}$-coordinate system has coordinates ε_{ji}, $j\in \mathrm{I}$.*

Proof Given a mutation in the direction of k, let us compare, using (277), the rule how the $\mathcal{X}$-coordinates $\{\varepsilon_{ji}\}$, $j \in \mathrm{I}$ change with the mutation formulas (273) for the matrix ε_{ij}.

Let us assume that $k\notin\{i,j\}$. Then, due to formula (277) for mutation of tropical $\mathcal{X}$-points, we have to prove that

$$\varepsilon'_{ji} \stackrel{?}{=} \varepsilon_{ji} - \varepsilon_{jk}\min\{0, -\mathrm{sgn}(\varepsilon_{jk})\varepsilon_{ki}\}. \tag{281}$$

Let us assume now that $\varepsilon_{jk}\varepsilon_{ki} < 0$. Then $\mathrm{sgn}(-\varepsilon_{jk})\varepsilon_{ki} > 0$. So $\min\{0, \mathrm{sgn}(-\varepsilon_{jk})\varepsilon_{ki}\} = 0$, and the right hand side is ε_{ji}. This agrees with $\varepsilon'_{ij} = \varepsilon_{ij}$, see (273), in this case.

If $\varepsilon_{jk}\varepsilon_{ki} > 0$, then $\mathrm{sgn}(-\varepsilon_{jk})\varepsilon_{ki} < 0$. So the right hand side is

$$\varepsilon_{ji} - \varepsilon_{jk}\min\{0, \mathrm{sgn}(-\varepsilon_{jk})\varepsilon_{ki}\} = \varepsilon_{ji} - \varepsilon_{jk}\mathrm{sgn}(-\varepsilon_{jk})\varepsilon_{ki} = \varepsilon_{ji} + |\varepsilon_{jk}|\varepsilon_{ki}.$$

Comparing with (273), we see that in both cases we get the expected formula (281).

Finally, if $k\in\{i,j\}$, then $\varepsilon'_{ij} = -\varepsilon_{ij}$, and by formula (277), we also get $-\varepsilon_{ij}$. □

Let us assume that, for each frozen $f \in \mathrm{I}_0$, there is a function $\mathbb{I}_{\mathcal{X}}(t_f) \in \mathbb{L}_+(\mathcal{A}^\vee)$. predicted by the duality conjectures. Then we conjecture that in many situations there exist monomials M_f of frozen $\mathcal{A}$-coordinates such that the potential on the space $\mathcal{A}$ is given by

$$\mathcal{W}_{\mathcal{A}^\vee} := \sum_{f\in \mathrm{I}_0} M_f\cdot \mathbb{I}_{\mathcal{X}}(t_f).$$

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