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Semiparametric estimation of Value at Risk

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Summary Value at Risk (VaR) is a fundamental tool for managing market risks. It measures the worst loss to be expected of a portfolio over a given time horizon under normal market conditions at a given confidence level. Calculation of VaR frequently involves estimating the volatility of return processes and quantiles of standardized returns. In this paper, several semiparametric techniques are introduced to estimate the volatilities of the market prices of a portfolio. In addition, both parametric and nonparametric techniques are proposed to estimate the quantiles of standardized return processes. The newly proposed techniques also have the flexibility to adapt automatically to the changes in the dynamics of market prices over time. Their statistical efficiencies are studied both theoretically and empirically. The combination of newly proposed techniques for estimating volatility and standardized quantiles yields several new techniques for forecasting multiple period VaR. The performance of the newly proposed VaR estimators is evaluated and compared with some of existing methods. Our simulation results and empirical studies endorse the newly proposed time-dependent semiparametric approach for estimating VaR.

Keywords: *Aggregate returns, Value at Risk, Volatility, Quantile, Semiparametric, Choice of decay factor.*

1. INTRODUCTION

Risk management has become an important topic for financial institutions, regulators, nonfinancial corporations and asset managers. Value at Risk (VaR) is a measure for gauging market risks of a particular portfolio; it shows the maximum loss over a given time horizon at a given confidence level . The review article by Duffie and Pan (1997) as well as the books edited by Alexander (1998) and written by Dowd (1998) and Jorion (2000) provide a nice introduction to the subject.

The field of risk management has evolved very rapidly, and many new techniques have since been developed. Aït-Sahalia and Lo (2000) introduced the concept of economic valuation of VaR and compared it with the statistical VaR. Other methods include historical simulation approaches and their modifications (Hendricks (1996), Mahoney (1996)); techniques based on parametric models (Wong and So, 2003), such as GARCH models (Engle (1995), Bollerslev (1986)) and their approximations; estimates based on extreme value theory (Embrechts *et al.*, 1997) and ideas based on variance–covariance matrices (Davé and Stahl, 1997). The problems on bank capital and

VaR were studied in Jackson *et al.* (1997). The accuracy of various VaR estimates was compared and studied by Beder (1995) and Davé and Stahl (1997). Engle and Manganelli (2000) introduced a family of VaR estimators, called CAViaR, using the idea of regression quantile.

An important contribution to the calculation of VaR is the RiskMetrics of J. P. Morgan (1996). The method can be regarded as a nonparametric estimation of volatility together with a normality assumption on the return process. The estimator of VaR consists of two steps. The first step is to estimate the volatility for holding a portfolio for 1 day before converting this into the volatility for multiple days. The second step is to compute the quantile of standardized return processes through the assumption that the processes follow a standard normal distribution. Following this important contribution by J. P. Morgan, many subsequent techniques developed share a similar principle.

Many techniques in use are local parametric methods. By using the historical data at a given time interval, parametric models such as GARCH(1,1) or even GARCH(0,1) were built. For example, the historical simulation method can be regarded as a local nonparametric estimation of quantiles. The techniques by Wong and So (2003) can be regarded as modeling a local stretch of data by using a GARCH model. In comparison, the volatility estimated by RiskMetrics is a kernel estimator of observed square returns, which is basically an average of the observed volatilities over the past 38 days (see Section 2.1). From the function approximation point of view (Fan and Gijbels, 1996), this method basically assumes that the volatilities over the last 38 days are nearly constant or that the return processes are locally modeled by a GARCH(0,1) model. The latter can be regarded as a discretized version of geometric Brownian over a short time period for the prices of a held portfolio.

An aim of this paper is to introduce a time-dependent semiparametric model to enhance the flexibility of local approximations. This model is an extension of the time-homogeneous parametric model for term structure dynamics used by Chan *et al.* (1992). The pseudo-likelihood technique of Fan *et al.* (2003) will be employed to estimate the local parameters. The volatility estimates of return processes are then formed.

The windows over which the local parametric models can be employed are frequently chosen subjectively. For example, in RiskMetrics, a decay factor of 0.94 and 0.97, respectively, is recommended by J. P. Morgan for computing daily volatilities and for calculating monthly volatilities (defined as a holding period of 25 days). It is clear that a large window size will reduce the variability of estimated local parameters. However, this will increase modeling biases (approximation errors). Therefore, a compromise between these two contradicting demands is the art required for smoothing parameter selection in nonparametric techniques (Fan and Gijbels, 1996). Another aim of this paper is to propose new techniques for automatically selecting the window size or, more precisely, the decay parameter. It allows us to use a different amount of smoothing for different portfolios to better estimate their volatilities.

With estimated volatilities, the standardized returns for a portfolio can be formed quantile of this return process is needed for estimating VaR. RiskMetrics uses the quantile of the standard normal distribution. This can be improved by estimating the quantiles from the standardized return process. In this paper, a new nonparametric technique based on the symmetric assumption on the distribution of the return process is proposed. This increases the statistical efficiency by more than a factor of two in comparison with usual sample quantiles. While it is known that the distribution of asset returns are asymmetric, the asymmetry of the percentiles at a moderate level of percentages α is not very severe. Our experience shows that for moderate α efficiency gains can still be made by using the symmetric quantile methods. In addition, the proposed technique is robust against mis-specification of parametric models and outliers created by large market movements. In contrast, parametric techniques for estimating quantiles have a higher statistical efficiency for estimated quantiles when the parametric models fit well with the return process. Therefore, in order to ascertain whether this gain can be materialized, we also fit parametric t-distributions with an unknown scale and unknown degree of freedom to the standardized return. The method of quantiles and the method of moments are proposed for estimating unknown parameters and, hence, the quantiles. The former approach is more robust, while the latter method is more efficient.

Economic and market conditions vary from time to time. It is reasonable to expect that the return process of a portfolio and its stochastic volatility depend in some way on time. Therefore, a viable VaR estimate should have the ability to self-revise the procedure in order to adapt to changes in market conditions. This includes modifications of the procedures for both volatility and quantile estimation. A time-dependent procedure is proposed for estimating VaR and has been empirically tested. It shows positive results.

The outline of the paper is as follows: Section 2 revisits the volatility estimation of J. P. Morgan's RiskMetrics before going on to introduce semiparametric models for return processes. Two methods for choosing time-independent and time-dependent decay factors are proposed. The effectiveness of the proposed volatility estimators is evaluated using several measures. Section 3 examines the problems of estimating quantiles of normalized return processes. A nonparametric technique and two parametric approaches are introduced. Their relative statistical efficiencies are studied. Their efficacies for VaR estimation are compared with J. P. Morgan's method. In Section 4, newly proposed volatility estimators and quantile estimators are combined to yield new estimators for VaR. Their performances are thoroughly tested by using simulated data as well as data from eight stock indices. Section 5 summarizes the conclusions of this paper.

2. ESTIMATION OF VOLATILITY

Let S_t be the price of a portfolio at time t. Let

$$r_t = \log(S_t/S_{t-1})$$

be the observed return at time t. The aggregate return at time t for a predetermined holding period τ is

$$R_{t,\tau} = \log(S_{t+\tau-1}/S_{t-1}) = r_t + \cdots + r_{t+\tau-1}.$$

Let Ω_t be the historical information generated by the process $\{S_t\}$, this is, Ω_t is the σ-field generated by $S_t, S_{t-1}, \ldots$. If S_t denotes the current market value of a portfolio, then the value of this portfolio at time $t+\tau$ will be $S_{t+\tau} = S_t \exp(R_{t+1,\tau})$. The VaR measures the extreme loss of a portfolio over a predetermined holding period τ with a prescribed confidence level $1-\alpha$. More precisely, letting $V_{t+1,\tau}$ be the α-quantile of the conditional distribution of $R_{t+1,\tau}$:

$$P(R_{t+1,\tau} > V_{t+1,\tau}|\Omega_t) = 1-\alpha,$$

the maximum loss of holding this portfolio for a period of τ is $S_t V_{t+1,\tau}$, that is, the VaR is $S_t V_{t+1,\tau}$. See the books by Jorion (2000) and Dowd (1998).

The current value S_t is known at time t. Thus, most efforts in the literature concentrate on estimating $V_{t+1,\tau}$. A popular approach to predict VaR is to determine, first, the conditional volatility

$$\sigma^2_{t+1,\tau} = \text{Var}(R_{t+1,\tau}|\Omega_t)$$

and then the conditional distribution of the scaled variable $R_{t+1,\tau}/\sigma_{t+1,\tau}$. This is also the approach that we follow.

2.1. Revisiting RiskMetrics

An important technique for estimating volatility is the RiskMetrics approach which estimates the volatility for a one-period return ($\tau = 1$), $\sigma_t^2 \equiv \sigma_{t,1}^2$, according to

$$\hat{\sigma}_t^2 = (1-\lambda) r_{t-1}^2 + \lambda \hat{\sigma}_{t-1}^2, \tag{2.1}$$

with $\lambda = 0.94$. For a τ-period return, the square-root rule is frequently used in practice:

$$\hat{\sigma}_{t,\tau} = \sqrt{\tau}\hat{\sigma}_t. \tag{2.2}$$

J. P. Morgan recommends using (2.2) with $\lambda = 0.97$ for forecasting the monthly ($\tau = 25$) volatility of aggregate return. The Bank of International Settlement suggests using (2.2) for the capital requirement for a holding period of 10 days. In fact, Wong and So (2003) showed that for the IGARCH(1,1) model defined similarly to (2.1), the square-root rule (2.2) holds. Beltratti and Morana (1999) employ the square-root rule with GARCH models to daily and half-hourly data. By iterating (2.1), it can be easily seen that

$$\hat{\sigma}_t^2 = (1-\lambda)\{r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2 + \cdots\}. \tag{2.3}$$

This is an example of exponential smoothing in the time domain (see Fan and Yao (2003)). Figure 1 depicts the weights for several choices of λ.

Exponential smoothing can be regarded as a kernel method that uses the one-sided kernel $K_1(x) = b^x I(x > 0)$ with $b < 1$. Assuming $E(r_t|\Omega_{t-1}) = 0$, then $\sigma_t^2 = E(r_t^2|\Omega_{t-1})$. The kernel estimator of $\sigma_t^2 = E(r_t^2|\Omega_{t-1})$ is given by

$$\hat{\sigma}_t^2 = \frac{\sum_{i=-\infty}^{t-1} K_1((t-i)/h_1) r_i^2}{\sum_{i=-\infty}^{t-1} K_1((t-i)/h_1)} = \frac{\sum_{i=-\infty}^{t-1} b^{\frac{t-i}{h_1}} r_i^2}{\sum_{i=-\infty}^{t-1} b^{\frac{t-i}{h_1}}},$$

where h_1 is the bandwidth (see Fan and Yao (2003)). It is clear that this is exactly the same as (2.3) with $\lambda = b^{1/h_1}$.

Exponential smoothing has the advantage of gradually, rather than radically, reducing the influence of remote data points. However, the effective number of points used in computing the local average is hard to quantify. If the one-sided uniform kernel $K_2(x) = I[0 < x \le 1]$ with bandwidth h_2 is used, then it is clear that there are h_2 data points used in computing the local average. According to the equivalent kernel theory (Section 5.4 of Fan and Yao (2003)), the kernel estimator with kernel function K_1 and bandwidth h_1 and the kernel estimator with kernel function K_2 and bandwidth h_2 conduct approximately the same amount of smoothing when

$$h_2 = \alpha(K_2) h_1 / \alpha(K_1),$$

where

$$\alpha(K) = \left\{\int_{-\infty}^{\infty} u^2 K(u) du\right\}^{-2/5} \left\{\int_{-\infty}^{\infty} K^2(u) du\right\}^{1/5}.$$

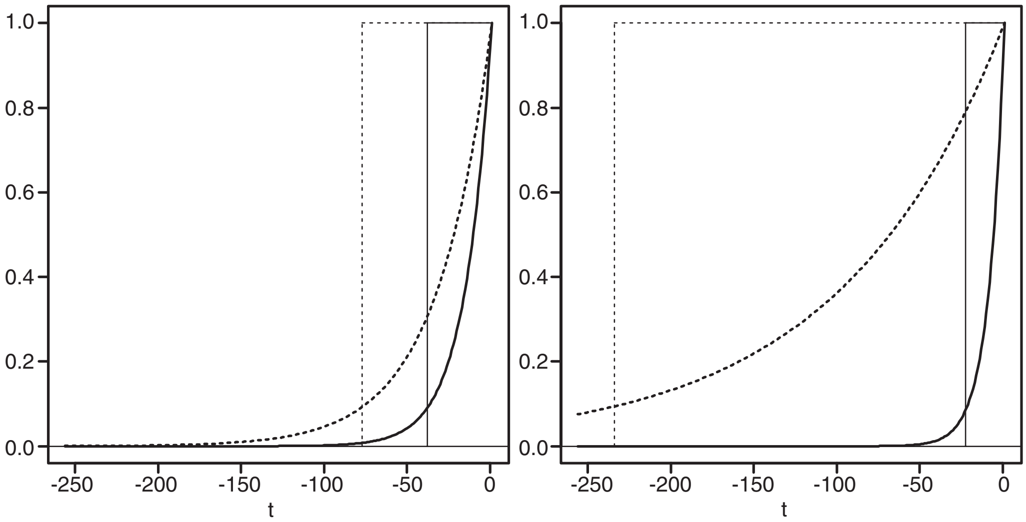

Figure 1. Weights for exponential smoothing with parameters $\lambda = 0.94$ (left panel, solid curve), $\lambda = 0.97$ (left panel, dashed curve), $\lambda = 0.90$ (right panel, solid curve) and $\lambda = 0.99$ (right panel, dashed curve) and the weights of their corresponding equivalent uniform kernels.

Table 1. Effective number of data points used in exponential smoothing.

Parameter λ	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
Effective number h_2	22.3	24.9	28.2	32.4	38.0	45.8	57.6	77.2	116.4	234.0

It is clear that $\alpha(K_2) = 3^{0.4} = 1.5518$ and that exponential smoothing corresponds to kernel smoothing with $K_1(x) = \lambda^x I(x > 0)$ and $h_1 = 1$. Hence, it effectively uses

$$h_2 = 1.5518/\alpha(K_1).$$

Table 1 records the effective number of data points used in exponential smoothing.

Assume now the model

$$r_t = \sigma_t \varepsilon_t, \tag{2.4}$$

where $\{\varepsilon_t\}$ is a sequence of independent random variables with mean zero and variance 1. It is well known that the kernel method can be derived from a local constant approximation (Fan and Gijbels, 1996). Assuming that $\sigma_u \approx \theta$ for u in a neighborhood of a point t, i.e.

$$r_u \approx \theta \varepsilon_u, \qquad \text{for } u \approx t \tag{2.5}$$

then the kernel estimator or, specifically the exponential smoothing estimator (2.1), can be regarded as a solution to the local least-squares problem

$$\sum_{i=-\infty}^{t-1} (r_i^2 - \theta)^2 \lambda^{(t-i-1)}, \tag{2.6}$$

where λ is a decay factor (smoothing parameter) that controls the size of the local neighborhood (Figure 1).

From the above function approximation point of view, the J. P. Morgan estimator of volatility assumes that locally the return process follows the model (2.5). The model can, therefore, be regarded as a discretized version of geometric Brownian motion with no drift, $d \log(S_u) = \theta d W_u$, for u around t or

$$d \log(S_u) = \theta(u) d W_u, \tag{2.7}$$

when the time unit is small, where W_u is the Wiener process.

2.2. Semiparametric models

The implicit assumption of J. P. Morgan's estimation of volatility is the local geometric Brownian motion stock price dynamics. To reduce possible modeling bias and to enhance the flexibility of the approximation, we enlarge the model (2.7) to the following semiparametric time-dependent model:

$$d \log(S_u) = \theta(u) S_u^{\beta(u)} d W_u, \tag{2.8}$$

allowing volatility to depend on the value of asset, where $\theta(u)$ and $\beta(u)$ are the coefficient functions. When $\beta(u) \equiv 0$, the model reduces to (2.7). This time-dependent diffusion model was used for interest rate dynamics by Fan *et al.* (2003). It is an extension of the time-dependent models previously considered by, among others, Hull and White (1990), Black *et al.* (1990), and Black and Karasinski (1991) and the time-independent model considered by Cox *et al.* (1985) and Chan *et al.* (1992). Unlike the yields of bonds, the scale of $\{S_u\}$ can be very different over a large time period. However, the model (2.8) is used locally, rather than globally.

Motivated by the continuous-time model (2.8), we model the return process discrete time as

$$r_u = \theta(u) S_{u-1}^{\beta(u)} \varepsilon_u, \tag{2.9}$$

where ε_u is a sequence of independent random variables with mean 0 and variance 1. To estimate the parameters $\theta(u)$ and $\beta(u)$, the local pseudo-likelihood technique is employed. For each given t and $u \leq t$ in a neighborhood of time t, the functions $\theta(u)$ and $\beta(u)$ are approximated by the constants

$$\theta(u) \approx \theta, \qquad \beta(u) \approx \beta.$$

Then, the conditional log-likelihood for r_u given S_{u-1} is

$$-\frac{1}{2} \log(2\pi\theta^2 S_{u-1}^{2\beta}) - \frac{r_u^2}{2\theta^2 S_{u-1}^{2\beta}},$$

when $\varepsilon_u \sim N(0, 1)$. In general, the above likelihood is a pseudo-likelihood. Dropping the constant factors and adding the pseudo-likelihood around the point t, we obtain the locally weighted pseudo-likelihood

$$\ell(\theta, \beta) = -\sum_{i=-\infty}^{t-1} \left\{ \log(\theta^2 S_{i-1}^{2\beta}) + \frac{r_i^2}{\theta^2 S_{i-1}^{2\beta}} \right\} \lambda^{t-1-i}, \tag{2.10}$$

where $\lambda < 1$ is the decay factor that makes this pseudo-likelihood use only the local data (see Figure 1 and (2.6)). Maximizing (2.10) with respect to the local parameters θ and β yields an estimate of the local parameters $\theta(t)$ and $\beta(t)$. Note that for given β, the maximum is achieved at

$$\hat{\theta}^2(t, \beta) = (1 - \lambda) \sum_{i=-\infty}^{t-1} \lambda^{t-1-i} r_i^2 S_{i-1}^{-2\beta}.$$

Substituting this into (2.10), the pseudo-likelihood $\ell(\hat{\theta}(t, \beta), \beta)$ is obtained. This is a one-dimensional maximization problem, and the maximization can easily be obtained by, for example, searching β over a grid of points or by using other more advanced numerical methods. Let $\hat{\beta}(t)$ be the maximizer. Then, the estimated volatility for the one-period return is

$$\hat{\sigma}_t^2 = \hat{\theta}^2(t) S_{t-1}^{2\hat{\beta}(t)}, \tag{2.11}$$

where $\hat{\theta}(t) = \hat{\theta}(t, \hat{\beta}(t))$. In particular, if we let $\beta(t) = 0$, the model (2.9) becomes the model (2.7) and the estimator (2.11) reduces to the J. P. Morgan estimator (2.3).

Our method corresponds to time-domain smoothing, which uses mainly the most recent data. There is also a large literature that postulates models on $\mathrm{Var}(r_t|\mathcal{F}_{t-1}) = g(r_{t-1}, \ldots, r_{t-p})$. This corresponds to the state-domain smoothing, using mainly the historical data to estimate the function g. See Engle and Manganelli (2000), Yang *et al.* (1999), Yatchew and Härdle (2003) and Fan and Yao (2003). Combination of both time-domain and state-domain smoothing for volatility estimation is an interesting direction for future research.

2.3. Choice of decay factor

The performance of volatility estimation depends on the choice of decay factor λ. In the RiskMetrics, $\lambda = 0.94$ is recommended for the estimation of 1-day volatility, while $\lambda = 0.97$ is recommended for the estimation of monthly volatility. In general, the choice of decay factor should depend on the portfolio and holding period, and should be determined from the data.

Our idea is related to minimizing the prediction error. In the current pseudo-likelihood estimation context, our aim is to maximize the pseudo-likelihood. For example, suppose that we have observed the price process $S_t, t = 1, \ldots, T$. Note that the pseudo-likelihood estimator $\hat{\sigma}_t^2$ depends on the data up to time $t - 1$. This estimated volatility can be used to predict the volatility at time t. The estimated volatility $\hat{\sigma}_t^2$ given by (2.11) can then be compared with the observed volatility r_t^2 to assess the effectiveness of the estimation. One way to validate the effectiveness of the prediction is to use square prediction errors

$$\mathrm{PE}(\lambda) = \sum_{t=T_0}^{T} (r_t^2 - \hat{\sigma}_t^2)^2, \tag{2.12}$$

where T_0 is an integer such that $\hat{\sigma}_{T_0}^2$ can be estimated with reasonable accuracy. This avoids the boundary problem caused by the exponential smoothing (2.1) or (2.3). The decay factor λ can be chosen to minimize (2.12). Using the model (2.9) and noting that $\hat{\sigma}_t$ is Ω_{t-1} measurable, the expected value can be decomposed as

$$E\{\mathrm{PE}(\lambda)\} = \sum_{t=T_0}^{T} E(\sigma_t^2 - \hat{\sigma}_t^2)^2 + \sum_{t=T_0}^{T} E(r_t^2 - \sigma_t^2)^2. \tag{2.13}$$

Note that the second term is independent of λ. Thus, minimizing $\text{PE}(\lambda)$ aims at finding an estimator that minimizes the mean-square error

$$\sum_{t=T_0}^{T} E(\sigma_t^2 - \hat{\sigma}_t^2)^2.$$

A question arises naturally why square errors, rather than, other types of errors, such as absolute deviation errors should be used in (2.12). In the current pseudo-likelihood context, a natural alternative is to maximize the pseudo-likelihood defined as

$$\text{PL}(\lambda) = -\sum_{t=T_0}^{T} (\log \hat{\sigma}_t^2 + r_t^2/\hat{\sigma}_t^2), \tag{2.14}$$

compared to (2.10). The likelihood function is a natural measure of the discrepancy between r_t and $\hat{\sigma}_t$ in the current context, and does not depend on an arbitrary choice of distance. The summand in (2.14) is the conditional likelihood, after dropping constant terms, of r_t given S_{t-1} with unknown parameters replaced by their estimated values. The decay factor λ can then be chosen to maximize (2.14). For simplicity in later discussion, we call this procedure semiparametric estimation of volatility (SEV).

2.4. Choice of adaptive smoothing parameter

The above choice of decay factor remains constant during post-sample forecasting. It relies heavily on the past history and has little flexibility to accommodate changes in stock dynamics over time. Therefore, in order to adapt automatically to changes in stock price dynamics, the decaying parameter λ should be allowed to depend on the time t. A solution to such problems has been explored by Mercurio and Spokoiny (2003) and Härdle *et al.* (2003).

To highlight possible changes of the dynamics of $\{S_t\}$, the validation should be localized around the current time t. Let g be a period for which we wish to validate the effectiveness of volatility estimation. Then, the pseudo-likelihood is defined as

$$\text{PL}(\lambda, t) = -\sum_{i=t-1-g}^{t-1} (\log \hat{\sigma}_i^2 + r_i^2/\hat{\sigma}_i^2). \tag{2.15}$$

Let $\hat{\lambda}_t$ maximize (2.15). In our implementation, we use $g = 20$, which validates the estimates in a period of about 1 month. The choice of $\hat{\lambda}_t$ is variable. To reduce this variability, the series $\{\hat{\lambda}_t\}$ can be smoothed further by using the exponential smoothing

$$\hat{\Lambda}_t = b\hat{\Lambda}_{t-1} + (1-b)\hat{\lambda}_t. \tag{2.16}$$

In our implementation, we use $b = 0.94$.

To sum up, in order to estimate the volatility $\hat{\sigma}_t$, we first compute $\{\hat{\sigma}_u\}$ and $\{\hat{\Lambda}_u\}$ up to time $t-1$ and obtain $\hat{\lambda}_t$ by minimizing (2.15) and then $\hat{\Lambda}_t$ from (2.16). The value of $\hat{\Lambda}_t$ is then used in (2.10) to estimate the local parameters $\hat{\theta}(t)$ and $\hat{\beta}(t)$, and hence the volatility $\hat{\sigma}_t^2$ using (2.11). The resulting estimator will be referred to as the adaptive volatility estimator (AVE).

The techniques in this section and Section 2.3 apply directly to the J. P. Morgan type of estimator (2.1). This allows different decay parameters for different portfolios.

2.5. Numerical results

In this section, the newly proposed procedures are compared by using three commonly used methods: RiskMetrics, historical simulation and a GARCH model using the quasi-maximum likelihood method (denoted by 'GARCH'). See Engle and Gonzalez-Rivera (1991) and Bollerslev and Wooldridge (1992). For the estimation of volatility, the historical simulation method is simply defined as the sample standard deviation of the return process for the past 250 days. For the newly proposed method, we employ the semiparametric estimator (2.11) with $\lambda = 0.94$ (denoted by 'semipara'); the estimator (2.11) with λ chosen by minimizing (2.12) (denoted by 'SEV'); and the estimator (2.11) (denoted by 'AVE') with the decay factor $\hat{\Lambda}_t$ chosen adaptively as in (2.16).

To compare the different procedures for estimating the volatility with a holding period of 1 day, eight stock indices and two simulated data sets were used together with the following five performance measures. For other related measures, see Davé and Stahl (1997). For a holding period of 1 day, the error distribution is not very far from normal.

Measure 1 (Exceedance ratio against confidence level). This measure counts number of events for which the loss of the asset exceeds the loss predicted by the normal model at a given confidence α. With estimated volatility, under the normal model, the 1-day VaR is estimated by $\Phi^{-1}(\alpha)\hat{\sigma}_t$, where $\Phi^{-1}(\alpha)$ is the α quantile of the standard normal distribution. For each estimated VaR, the exceedance ratio (ER) is computed as

$$\text{ER} = n^{-1} \sum_{t=T+1}^{T+n} I(r_t < \Phi^{-1}(\alpha)\hat{\sigma}_t),$$

for a post sample of size n. This gives an indication of how effectively volatility can be used for estimating one-period VaR. Note that the Monte Carlo error for this measure has an approximate size $\{\alpha(1-\alpha)/n\}^{1/2}$, even when the true σ_t is used. For example, with $\alpha = 5\%$ and $n = 1000$, the Monte Carlo error is around 0.68%. Thus, unless the post-sample size is large enough, this measure has difficulty in differentiating between various estimators due to the presence of large error margins.

Measure 2 (Mean absolute deviation error, MADE). To motivate this measure, let us first consider the mean square errors

$$\text{PE} = n^{-1} \sum_{t=T+1}^{T+n} (r_t^2 - \hat{\sigma}_t^2)^2.$$

Following (2.13), the expected value can be decomposed as

$$E(\text{PE}) = n^{-1} \sum_{t=T+1}^{T+n} E(\sigma_t^2 - \hat{\sigma}_t^2)^2 + n^{-1} \sum_{t=T+1}^{T+n} E(r_t^2 - \sigma_t^2)^2.$$

Note that the first term reflects the effectiveness of the estimated volatility while the second term is the size of the stochastic error, independent of estimators. As in all statistical prediction problems, the second term is usually of an order of magnitude larger than the first term. Thus, a small improvement on PE could mean substantial improvement over the estimated volatility. However, due to the well-known fact that financial time series contain outliers due to market crashes, the mean-square error is not a robust measure. Therefore, we will use the mean-absolute deviation error

$$\text{MADE} = n^{-1} \sum_{t=T+1}^{T+n} |r_t^2 - \hat{\sigma}_t^2|.$$

Measure 3 (Square-root absolute deviation error, RADE). An alternative variation to MADE is the RADE, which is defined as

$$\text{RADE} = n^{-1} \sum_{t=T+1}^{T+n} \left| |r_t| - \sqrt{\frac{2}{\pi}} \hat{\sigma}_t \right|.$$

The constant factor comes from the fact that $E|\varepsilon_t| = \sqrt{\frac{2}{\pi}}$ for $\varepsilon_t \sim N(0, 1)$.

Measure 4—Test of independence. A good VaR estimator should have the property that the sequence of the events exceeding VaR behaves like an *i.i.d.* Bernoulli distribution with probability of success α. Engle and Manganelli (1999) give an illuminating example showing that even a bad VaR estimator can have the right exceedance ratio α.

Let $I_t = I(r_t < \Phi^{-1}(\alpha)\hat{\sigma}_t)$ be the indicator of the event that the return exceeds VaR. Christoffersen (1998) introduced the likelihood ratio test for testing independence and for testing whether the probability $\Pr(I_t = 1) = \alpha$.

Assume $\{I_t\}$ is a first-order Markovian chain. Let $\pi_{ij} = \Pr(I_t = j | I_{t-1} = i)$ ($i = 0, 1$ and $j = 0, 1$) be the transition probability and n_{ij} be the number of events transferring from state i to state j in the post-sample period. The problem is to test

$$H_0 : \pi_{00} = \pi_{10} = \pi, \qquad \pi_{01} = \pi_{11} = 1 - \pi.$$

Then the maximum likelihood ratio test for independence is

$$\text{LR1} = 2 \log \left(\frac{\hat{\pi}_{00}^{n_{00}} \hat{\pi}_{01}^{n_{01}} \hat{\pi}_{10}^{n_{10}} \hat{\pi}_{11}^{n_{11}}}{\hat{\pi}^{n_0} (1 - \hat{\pi})^{n_1}} \right), \tag{2.17}$$

where $\hat{\pi}_{ij} = n_{ij}/(n_{i0} + n_{i,1})$, $n_j = n_{0j} + n_{1j}$, and $\hat{\pi} = n_0/(n_0 + n_1)$. The test statistic is a measure of deviation from independence. Under the null hypothesis, the test statistic LR1 is distributed approximately according to χ_1^2 when sample size is large. Thus, reporting the test statistic is equivalent to reporting the P-value.

Measure 5—Testing against a given confidence level. Christoffersen (1998) applied the maximum likelihood ratio test to the problem

$$H_0 : P(I_t = 1) = \alpha \qquad \text{vs.} \qquad H_1 : P(I_t = 1) \neq \alpha$$

under the assumption that $\{I_t\}$ is a sequence of *i.i.d.* Bernoulli random variables. The test statistic is given by

$$\text{LR2} = 2 \log \left(\frac{\hat{\pi}^{n_0} (1 - \hat{\pi})^{n_1}}{\alpha^{n_0} (1 - \alpha)^{n_1}} \right), \tag{2.18}$$

which follows the χ_1^2-distribution when the sample size is large. Again, the P-value is a measure of deviation from the null hypothesis, which is closely related to the ER.

Table 2. Comparisons of several volatility estimation methods.

Country	Index	In-sample period	Post-sample period
Australia	AORD	1988–1996	1997–2000
France	CAC 40	1990–1996	1997–2000
Germany	DAX	1990–1996	1997–2000
H.K.	HSI	1988–1996	1997–2000
Japan	Nikkei 225	1988–1996	1997–2000
UK	FTSE	1988–1996	1997–2000
USA	S&P 500	1988–1996	1997–2000
USA	Dow Joes	1988–1996	1997–2000

Example 1 (Stock indices). We first apply the five volatility estimators to the daily returns of eight stock indices (Table 2). For each stock index, the in-sample period terminated on December 30 1996 and the post-sample period starts from January 1 1997 to December 29 2000 ($n = 1014$). The results are summarized in Table 3. The initial period is set to $T_0 = 250$.

From Table 3, the smallest two MADE and RADE are always achieved by using semiparametric methods and GARCH methods. In fact, SEV, AVE and GARCH methods are the best three methods in terms of MADE and RADE. Of these, the semiparametric method with a decay parameter that is selected automatically by the data (SEV) performs the best. It achieved the two smallest MADE in eight out of eight times, and the two smallest RADE in four out eight times. This demonstrates that it is important to allow the algorithm to choose decay factors according to the dynamics of stock prices. The AVE and GARCH methods perform comparably with the SEV in terms of MADE and RADE. The GARCH method slightly outperforms the AVE according to MADE and RADE measures, but AVE outperforms the GARCH method for other measures such as ER and P-value from independence. This demonstrates the advantage of using a time-dependent decay parameter that adapts automatically to any changes in stock price dynamics. These results also indicate that our proposed methods for selecting decay parameters are effective. As shown in (2.13), both measures contain a large amount of stochastic error. A small improvement in MADE and RADE measures indicates a large improvement in terms of estimated volatilities.

Presented in Table 3 are the P-values for testing independence and for testing whether the exceedance ratio is significantly different from 5%. Since the post sample size is more than 1000, we consider whether the deviations are significant at the 1% level. Most methods have a correct exceedance ratio except the GARCH method which tends to underestimate the risk. However, the GARCH method performs particularly well in terms of testing against independence. Its corresponding P-values tend to be large. Other method performs reasonably well in terms of independence.

As an illustration, Figure 2 presents the estimated volatilities for six stock indices in the post-sample period by using SEV and AVE. The β's parameters in model (2.11) depend on the stock prices and can vary substantially. Since they together predict the volatility, it is more meaningful to present the volatility plots. The volatility predicted by AVE is more variable than that by the SEV.

Example 2 (GARCH(1,1)-model). Next consider simulations from the GARCH model:

$$r_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = c + a\sigma_{t-1}^2 + br_{t-1}^2,$$

Table 3. Comparisons of several volatility estimation methods.

Index	Method	ER ($\times 10^{-2}$)	MADE ($\times 10^{-4}$)	RADE ($\times 10^{-3}$)	*P*-value (indep)	*P*-value (ER = 5%)
	Historical	5.23	0.85	4.33	0.01[a]	0.76
	RiskMetrics	4.93	0.848	4.25	0.01[a]	0.92
AORD	Semipara	4.93	0.84	4.231	0.01[a]	0.92
	SEV	5.23	**0.803**	**4.30**	0.02	0.76
	AEV	4.93	0.835	4.213	0.00[a]	0.92
	GARCH	5.52	**0.786**	**4.168**	0.12	0.46
	Historical	6.16	2.177	7.272	0.11	0.10
	RiskMetrics	6.36	2.157	7.102	0.33	0.06
CAC 40	Semipara	6.55	2.15	7.094	0.20	0.03
	SEV	6.45	**2.077**	**7.035**	0.17	0.04
	AEV	6.36	2.138	7.069	0.91	0.06
	GARCH	8.24	**1.931**	**6.883**	0.20	0.00
	Historical	6.55	2.507	7.814	0.18	0.03
	RiskMetrics	5.46	2.389	7.37	0.59	0.51
DAX	Semipara	5.56	2.389	7.342	0.90	0.43
	SEV	6.06	**2.368**	7.457	0.03	0.14
	AEV	5.96	2.377	**7.330**	0.90	0.18
	GARCH	7.94	**2.200**	**7.174**	0.78	0.00[a]
	Historical	6.08	5.71	11.167	0.00[a]	0.12
	RiskMetrics	5.99	5.686	10.818	0.00[a]	0.15
HSI	Semipara	5.89	5.567	**10.685**	0.00[a]	0.19
	SEV	5.61	**5.523**	10.743	0.00[a]	0.37
	AEV	6.55	5.578	10.686	0.00[a]	0.03
	GARCH	7.3	**5.293**	**10.565**	0.03	0.00[a]
	Historical	5.78	2.567	7.824	0.90	0.27
	RiskMetrics	5.78	2.526	7.656	0.35	0.27
Nikkei 225	Semipara	6.09	2.507	7.631	0.48	0.13
	SEV	5.68	**2.457**	**7.610**	0.68	0.34
	AEV	6.19	**2.479**	**7.565**	0.25	0.10
	GARCH	5.88	2.563	7.693	0.80	0.22
	Historical	6.83	1.369	5.761	0.05	0.01
	RiskMetrics	5.94	1.342	5.594	0.45	0.18
FTSE	Semipara	6.24	1.328	**5.567**	0.58	0.08
	SEV	6.93	**1.299**	5.598	0.02	0.01
	AEV	6.04	1.328	5.571	0.90	0.14
	GARCH	7.43	**1.256**	**5.497**	0.78	0.00[a]
	Historical	6.34	1.613	6.027	0.65	0.06
	RiskMetrics	5.55	1.647	6.056	0.62	0.44
S&P 500	Semipara	5.55	1.62	5.995	0.62	0.44
	SEV	5.85	**1.539**	**5.888**	0.46	0.23
	AEV	5.75	**1.611**	**5.984**	0.51	0.29
	GARCH	4.46	1.689	6.163	0.89	0.43
	Historical	6.15	1.493	5.84	0.25	0.11
	RiskMetrics	5.65	1.507	5.784	0.56	0.35
Dow Jones	Semipara	5.75	1.489	**5.739**	0.16	0.29
	SEV	5.75	**1.460**	5.743	0.51	0.29
	AEV	5.85	**1.480**	**5.731**	0.90	0.23
	GARCH	4.46	1.575	5.96	0.68	0.43

Note: GARCH refers to the GARCH(1,1) model. Numbers with bold face are the two smallest.
[a] Means statistically significant at the 1% level.

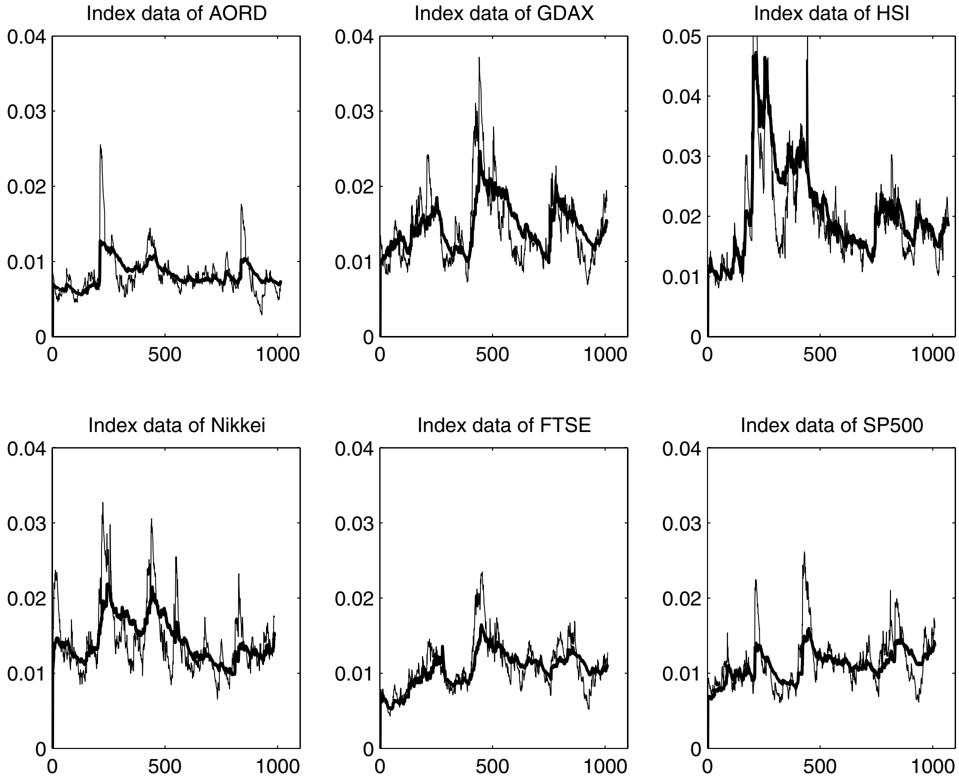

Figure 2. Predicted volatility in the out of sample period for several indices. Thin curves—AVE method; thick curves—SEV method.

where ε_t is the standard Gaussian noise. The first two hundred random series of length 3000 were simulated using the parameters $c = 0.000{,}000{,}38$, $a = 0.957{,}513$ and $b = 0.038{,}455$. These parameters are from the GARCH(1,1) fit to the SP500 index from January 4, 1988 to December 29, 2000. The parameter a is reasonably close to $\lambda = 0.94$ of the RiskMetrics. The second 200 time series of length 3000 were simulated using the parameters $c_0 = 0.000{,}009{,}025$, $a = 0.9$ and $b = 0.09$. The choice of c is to make resulting series have approximately the same standard deviation as the returns of the SP500. The first 2000 data points were used as the in-sample period, namely $T = 2000$, and the last 1000 data points were used as the post-sample, namely $n = 1000$. The performance of six volatility estimators of two models is shown in Tables 4 and 5 respectively.

The performance of each volatility estimator can be summarized by using the average and standard deviation of MADE and RADE over 200 simulations. However, MADE and RADE show quite large variability from one simulation to another. In order to avoid taking averages over different scales, for each simulated series, we first standardize the MADE using the median MADE in that series of the six methods and then average them across 200 simulations. The results are presented as the column ‘score’ in Tables 4 and 5. In addition, the frequency of each method that achieved the best MADE among 200 simulations was recorded, and is presented

Table 4. Comparisons of several volatility estimation methods [the first GARCH(1,1) model].

Method	ER ($\times 10^{-2}$)	Score ($\times 10^{-2}$)	Best	Best two	Reject times (indep)	Reject times (ER = 5%)
Historical	5.32 (0.92)	103.48 (4.87)	13.5	18	14	29
RiskMetrics	5.43 (0.56)	100.17 (0.49)	2	11.5	21	8
Semipara	5.67 (0.61)	99.81 (0.42)	11	35	18	20
SEV	5.44 (0.66)	99.73 (0.50)	16.5	51.5	15	17
AVE	5.94 (0.63)	99.30 (0.58)	51	75	12	53
GARCH(1,1)	4.41 (0.86)	103.67 (4.97)	6	9	23	42

Note: The values in the brackets are their corresponding standard deviations.

Table 5. Comparisons of several volatility estimation methods [the second GARCH(1,1) model].

Method	ER ($\times 10^{-2}$)	Score ($\times 10^{-2}$)	Best	Best two	Reject times (indep)	Reject times (ER = 5%)
Historical	5.58 (0.98)	109.64 (9.92)	3.5	4	65	49
RiskMetrics	5.54 (0.60)	100.30 (0.60)	1	10.5	14	14
Semipara	5.72 (0.62)	99.75 (0.37)	11.5	47	14	30
SEV	5.75 (0.62)	99.73 (0.42)	14	41	15	33
AVE	6.06 (0.61)	99.53 (0.69)	37	60.5	11	63
GARCH(1,1)	5.03 (0.78)	100.67 (2.65)	33	37	15	14

Note: The values in the brackets are their corresponding standard deviations.

in the column 'best'. Further, the frequency of each volatility estimator achieving the smallest two MADE in each simulation was also counted. More precisely, among 200 simulations, we computed the percentage of times a method performed the best as well as the percentage of times the methods ranked in the top two positions. The results are presented in the column 'best two' of Tables 4 and 5. For clarity, we omit similar presentations using the RADE measure—the results are nearly the same as the MADE. The numbers of rejections of null hypotheses are recorded in the column of 'reject times (indep)' and 'reject times (ER = 5%)'.

Using MADE or RADE as a measure, AVE and SEV consistently outperform other methods. GARCH performs quite reasonably in terms of MADE for the second GARCH(1,1) model, but not the first GARCH(1,1) model. Since the sum of the parameters a and b is close to one, the parameters in GARCH(1,1) cannot be estimated without large variability. This results in large variances in the computation of standardized MADE. In terms of ER or measure 5, which are closely related, RiskMetrics performs consistently well. Since the models used in the simulations are all stationary time-homogeneous models, AVE does not have much of its advantage while SEV performs better in terms of ER and measure 5. Except for the historical simulation method, all methods behave well in the independence tests (measure 4).

Example 3 (Continuous-time SV-model). Instead of simulating the data from GARCH(1,1) models, we simulate data from a continuous-time diffusion process with stochastic volatility

$$d \log(S_t) = \alpha dt + \sigma_t dW_t, \qquad d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \omega\sigma_t dB_t,$$

Table 6. Comparisons of several volatility estimation methods (SV model).

Method	ER ($\times 10^{-2}$)	Score ($\times 10^{-2}$)	Best	Best two	Reject times (indep)	Reject times (ER = 5%)
Historic	5.05 (0.74)	100.02 (1.94)	16.5	34	5	14
RiskMetrics	5.45 (0.59)	100.62 (0.59)	0	3	11	12
Semipara	5.90 (0.65)	100.38 (0.67)	0	5	10	45
SEV	5.64 (0.71)	98.69 (1.42)	34	66	11	30
AVE	6.06 (0.63)	99.02 (0.64)	32	72	8	60
GARCH(1,1)	4.10 (1.77)	108.73 (13.54)	17.5	20	9	116

Note: The corresponding standard deviations are in the brackets.

where W_t and B_t are two independent standard Brownian motions. See, for example, Barndorff-Nielsen and Shephard (2001, 2002). The parameters are chosen as $\alpha = 0$, $\kappa = 0.214{,}59$, $\theta = 0.085{,}71$, $\omega = 0.078{,}30$, following Chapman and Pearson (2000) and Fan and Zhang (2003). Two hundred series of 3000 daily data were simulated using the exact simulation method (see, e.g. Genon-Catalot *et al.* (1999), Fan and Zhang (2003)).

This simulation tests the performance of the six volatility estimators when the underlying dynamics differs from GARCH(1,1) and our semiparametric models. The same performance measures as those in Example 2 are used. Table 6 summarizes the results. Similar conclusions to those in Example 2 can be drawn. The AVE and SEV consistently outperform RiskMetrics using MADE as a measure even when the model is mis-specified. This is due, mainly, to the flexibility of the semiparametric model approximating the true dynamics, in addition to the data-driven smoothing parameter that enhanced the performance. The historical simulation method performs better in this example than those in the previous example. This is partially due to the fact that the stochastic volatility model produces more volatile returns and hence a larger smoothing parameter in the historical simulation method gives some advantages.

3. ESTIMATION OF QUANTILES

The conditional distribution of the multiple period return $R_{t,\tau}$ does not necessarily follow a normal distribution. Indeed, even under the IGARCH(1,1) model (2.1) with a normal error distribution in (2.4), Wong and So (2003) showed that the conditional distribution $R_{t,\tau}$ given Ω_t is not normal. This was also illustrated numerically by Lucas (2000). Thus, a direct application of $\Phi^{-1}(\alpha)\hat{\sigma}_{t+1,\tau}$ will provide an erroneous estimate of multiple period VaR, where $\hat{\sigma}_{t+1,\tau}$ is an estimated multiple period volatility of returns. In order to provide empirical evidence of non-normality of multiple period returns, the distributions $R_{t,\tau}/\hat{\sigma}_{t,\tau}$ for the S&P500, Hang-Seng, Nikkei 225, and FTSE 100 indices are shown in Figure 3. The multiple period volatility is computed by using the RiskMetrics $\hat{\sigma}_{t+1,\tau} = \sqrt{\tau}\hat{\sigma}_{t+1}$. The densities are estimated by the kernel density estimator with the rule of thumb bandwidth $h = 1.06n^{-1/5}s$, where n is the sample size and s is the sample standard deviation (taken as one to avoid outliers, since the data have already been normalized). See, for example, Chapter 2 of Fan and Gijbels (1996). It is evident that the one-period distributions are basically symmetric and have heavier tails than the standard

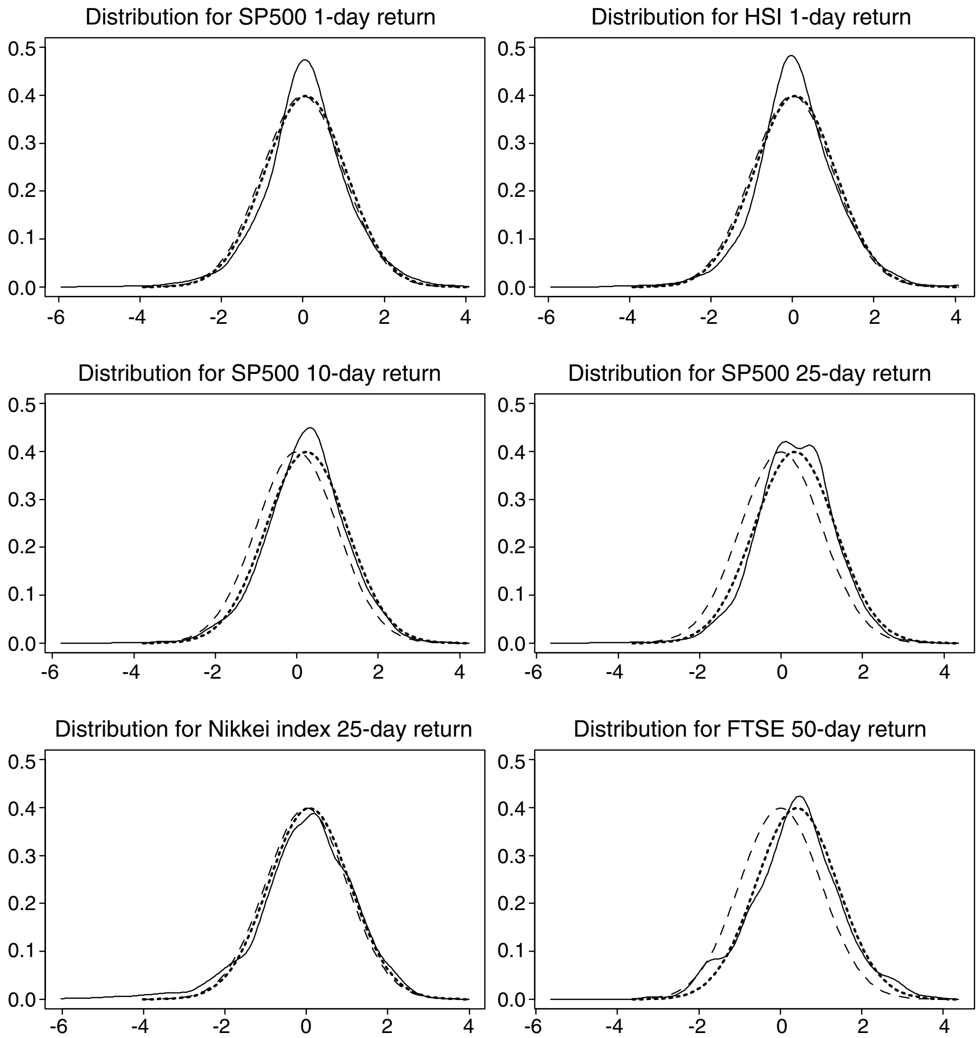

Figure 3. Estimated densities for the rescaled multiple period returns for several indices. Solid curves—estimated densities using the kernel density estimator; dashed curves—standard normal densities; thick dashed curves—normal densities centered at the median of data with standard deviation 1.

normal distribution. The deviations from normal are quite substantial for multiple period return processes. Indeed, the distribution is not centered around zero; the centered normal distributions (using medians of the data as the centers and one as the standard deviation) fit the data better.

3.1. Nonparametric estimation of quantiles

As discussed previously, the distributions of multiple period returns deviate from normality. Their distributions are generally unknown. In fact, Diebold *et al.* (1998) reported that converting 1-day volatility estimates to τ-day estimates by a scale factor $\sqrt{\tau}$ is inappropriate and produces overestimates of the variability of long time horizon volatility. Danielsson and de Vries (2000) suggested using the scaling factor $\tau^{1/\beta}$ with β being the tail index of extreme value distributions. Nonparametric methods can naturally be used to estimate the distributions of the residuals and correct the biases in the volatility estimation (the issue of whether the scale factor is correct becomes irrelevant when estimating the distribution of standardized return processes).

Let $\hat{\sigma}_{t,\tau}$ be an estimated τ-period volatility and $\hat{\varepsilon}_{t,\tau} = R_{t,\tau}/\hat{\sigma}_{t,\tau}$ be a residual. Denote by $\hat{q}(\alpha, \tau)$, the sample α-quantile of the residuals $\{\hat{\varepsilon}_{t,\tau}, t = T_0 + 1, \ldots, T - \tau\}$. This yields an estimated multiple period VaR as $\mathrm{VaR}_{t+1,\tau} = \hat{q}(\alpha, \tau)\hat{\sigma}_{t+1,\tau}$. Note that the choice of constant factor $\hat{q}(\alpha, \tau)$ is the same as selecting the constant factor c such that the difference between the exceedance ratio of the estimated VaR and confidence level is minimized in the in-sample period. More precisely, $\hat{q}(\alpha, \tau)$ minimizes the function

$$\mathrm{ER}(c) = \left| (T - \tau - T_0 + 1)^{-1} \sum_{t=T_0}^{T-\tau} I(R_{t+1,\tau} < c\hat{\sigma}_{t+1,\tau}) - \alpha \right|.$$

The nonparametric estimates of quantiles are robust against the possible mis-specification of parametric models and insensitive to a few large market movements for moderate α. Yet, they are not as efficient as parametric methods when parametric models are correctly given. To improve the efficiency of nonparametric estimates, we assume the distribution of $\{\hat{\varepsilon}_{t,\tau}\}$ is symmetric about the point 0. This implies that

$$q(\alpha, \tau) = -q(1 - \alpha, \tau),$$

where $q(\alpha, \tau)$ is the population quantile. Thus, an improved nonparametric estimator is

$$\hat{q}^{[1]}(\alpha, \tau) = 2^{-1}\{\hat{q}(\alpha, \tau) - \hat{q}(1 - \alpha, \tau)\}. \tag{3.1}$$

Denote by

$$\mathrm{VaR}^{[1]}_{t+1,\tau} = \hat{q}(\alpha, \tau)^{[1]}\hat{\sigma}_{t+1,\tau}$$

the corresponding estimated VaR. It is not difficult to show that the estimator $\hat{q}^{[1]}(\alpha, \tau)$ is a factor $(2 - 2\alpha)/(1 - 2\alpha)$ as efficient as the simple estimate $\hat{q}(\alpha, \tau)$ for $\alpha < 0.5$ (see Appendix A.1 for derivations).

When the distribution of the standardized return process is asymmetric, (3.1) will introduce some biases. For moderate α where $q(\alpha, \tau) \approx -q(1-\alpha, \tau)$, the biases are offset by the variance gain. As shown in Figure 3, the asymmetry for returns is not very severe for moderate α. Hence, the gain can still be materialized.

3.2. Adaptive estimation of quantiles

The above method assumes that the distribution of $\{\hat{\varepsilon}_{t,\tau}\}$ is stationary over time. To accommodate possible nonstationarity, for a given time t, we may use only the local data $\{\hat{\varepsilon}_{i,\tau}, i = t-\tau-h, t-h+1, \ldots, t-\tau\}$. This model was used by several authors, including Wong and So (2003) and Pant and Chang (2001). Let the resulting nonparametric estimator (3.1) be $\hat{q}_t^{[1]}(\alpha, \tau)$. To stabilize the estimated quantiles, we smooth further this quantile series to obtain the adaptive estimator of quantiles $\hat{q}_t^{[2]}(\alpha, \tau)$ via the exponential smoothing

$$\hat{q}_t^{[2]}(\alpha, \tau) = b\hat{q}_{t-1}^{[2]}(\alpha, \tau) + (1 - b)\hat{q}_{t-1}^{[1]}(\alpha, \tau). \tag{3.2}$$

In our implementation, we took $h = 250$ and $b = 0.94$.

3.3. Parametric estimation of quantiles

Based on empirical observations, one possible parametric model for the observed residuals $\{\hat{\varepsilon}_{t,\tau}, t = T_0 + 1, \ldots, T - \tau\}$ is to assume that the residuals follow a scaled t-distribution:

$$\hat{\varepsilon}_{t,\tau} = \lambda \varepsilon_t^*, \tag{3.3}$$

where $\varepsilon_t^* \sim t_\nu$, the Student's t-distribution with degree of freedom ν. The parameters λ and ν can be obtained by solving the following equations that are related to the sample quantiles:

$$\begin{cases} \hat{q}(\alpha_1, \tau) &= \lambda t(\alpha_1, \nu) \\ \hat{q}(\alpha_2, \tau) &= \lambda t(\alpha_2, \nu), \end{cases}$$

where $t(\alpha, \nu)$ is the α quantile of the t-distribution with degree of freedom ν. A better estimator to use is $\hat{q}^{[1]}(\alpha, \tau)$ in (3.1). Using the improved estimator and solving the above equations yields the estimates $\hat{\nu}$ and $\hat{\lambda}$ as follows:

$$\frac{t(\alpha_2, \hat{\nu})}{t(\alpha_1, \hat{\nu})} = \frac{\hat{q}^{[1]}(\alpha_2, \tau)}{\hat{q}^{[1]}(\alpha_1, \tau)}, \qquad \hat{\lambda} = \frac{\hat{q}^{[1]}(\alpha_1, \tau)}{t(\alpha_1, \hat{\nu})}. \tag{3.4}$$

Hence, the estimated quantile is given by

$$\hat{q}^{[3]}(\alpha, \tau) = \hat{\lambda} t(\alpha, \hat{\nu}) = \frac{t(\alpha, \hat{\nu})\hat{q}^{[1]}(\alpha_1, \tau)}{t(\alpha_1, \hat{\nu})}. \tag{3.5}$$

and the VaR of τ-period return is given by

$$\mathrm{VaR}_{t+1,\tau}^{[3]} = \hat{q}^{[3]}(\alpha, \tau)\hat{\sigma}_{t+1,\tau}. \tag{3.6}$$

In our implementation, we take $\alpha_1 = 0.15$ and $\alpha_2 = 0.35$. This choice is near optimal in terms of statistical efficiency (Figure 4).

The above method of estimating quantiles is robust against outliers. An alternative approach is to use the method of moments to estimate the parameters in (3.3). Note that if $\varepsilon \sim t_\nu$ with $\nu > 4$, then

$$E\varepsilon^2 = \frac{\nu}{\nu - 2} \qquad \text{and} \qquad E\varepsilon^4 = \frac{3\nu^2}{(\nu - 2)(\nu - 4)}.$$

The method of moments yields the following estimates:

$$\begin{cases} \hat{\nu} &= (4\hat{\mu}_4 - 6\hat{\mu}_2^2)/(\hat{\mu}_4 - 3\hat{\mu}_2^2) \\ \hat{\lambda} &= \{\hat{\mu}_2(\hat{\nu} - 2)/\hat{\nu}\}^{1/2}, \end{cases} \tag{3.7}$$

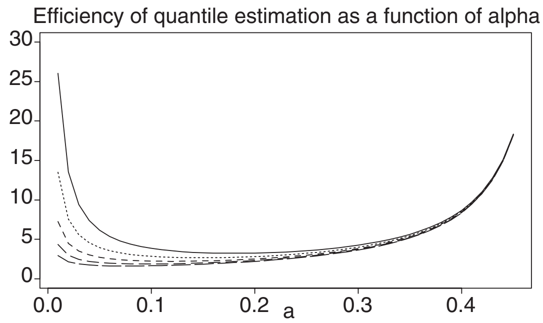

Figure 4. The efficiency function $g_\nu(\alpha)$ for degree of freedom $\nu = 2$, $\nu = 3$, $\nu = 5$, $\nu = 10$ and $\nu = 40$ (from solid, the shortest dash to the longest dash). For all ν, the minimum is almost attained at the interval [0.1, 0.2].

where $\hat{\mu}_j$ is the jth moment, defined as $\hat{\mu}_j = (T - \tau - T_0)^{-1} \sum_{t=T_0+1}^{T-\tau} \hat{\varepsilon}_{t,\tau}^j$. See Pant and Chang (2001) for similar expressions. Using these estimated parameters, we obtain the new estimated quantile and estimated VaR similarly to (3.5) and (3.6). The new estimates are denoted by $\hat{q}^{[4]}(\alpha, \tau)$ and $\mathrm{VaR}_{t+1,\tau}^{[4]}$, respectively. That is,

$$\hat{q}^{[4]}(\alpha, \tau) = \hat{\lambda} t(\alpha, \hat{\nu}), \qquad \mathrm{VaR}_{t+1,\tau}^{[4]} = \hat{q}^{[4]}(\alpha, \tau)\hat{\sigma}_{t+1,\tau}.$$

The method of moments is less robust than the method of quantiles. The former also requires the assumption that $\nu > 4$. We will compare their asymptotic efficiency in Section 3.4.

3.4. Theoretical comparisons of estimators for quantiles

Of the three methods for estimating the quantiles, the estimator $\hat{q}^{[1]}(\alpha, \tau)$ is the most robust method. It imposes very mild assumptions on the distribution of $\hat{\varepsilon}_{t,\tau}$ and, hence, is robust against model mis-specification. The two parametric methods rely on the model (3.3), which could lead to erroneous estimation if the model is mis-specified. The estimators $\hat{q}^{[1]}$, $\hat{q}^{[2]}$ and $\hat{q}^{[3]}$ are all robust against outliers, but $\hat{q}^{[4]}$ is not.

In order to give a theoretical study on the properties of the aforementioned three methods for estimation of quantiles, we assume that $\{\hat{\varepsilon}_{t,\tau}, t = T_0, \ldots, T - \tau\}$ is an independent random sample from the density f. Under this condition, for $0 < \alpha_1 < \cdots < \alpha_k < 1$,

$$\{\sqrt{m}[\hat{q}(\alpha_i, \tau) - q(\alpha_i, \tau)], 1 \le i \le k\} \xrightarrow{\mathcal{L}} N(0, \Sigma), \tag{3.8}$$

where $m = T - \tau - T_0 + 1$, $q(\alpha_i, \tau)$ is the population quantile of f, and $\Sigma = (\sigma_{ij})$ with

$$\sigma_{ij} = \alpha_i(1 - \alpha_j)/f(q(\alpha_i, \tau)) f(q(\alpha_j, \tau)), \qquad \text{for } i > j \text{ and } \sigma_{ji} = \sigma_{ij}.$$

See Prakasa Rao (1987).

To compare this with parametric methods, let us now assume for a moment that the model (3.3) is correct. Using the result in Appendix A.1, the nonparametric estimator $\hat{q}^{[1]}(\alpha, \tau)$ follows asymptotically a normal distribution with mean $\lambda t(\alpha, \nu)$ and variance (for $\alpha < 1/2$)

$$V_1(\alpha, \nu, \lambda) = \frac{\lambda^2 \alpha(1-2\alpha)}{2 f_\nu(t(\alpha, \nu))^2 m}, \tag{3.9}$$

where f_ν is the density of the t-distribution with degree of freedom ν given by

$$f_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}(1 + x^2/\nu)^{-(\nu+1)/2}.$$

Since ν is an integer, any consistent estimator of ν is equal to ν with probability tending to one. For this reason, ν can be treated as known in the asymptotic study. It follows directly from (3.8) that the estimator $\hat{q}^{[3]}(\alpha, \tau)$ has the asymptotic normal distribution with mean $\lambda t(\alpha, \nu)$ and variance

$$V_2(\alpha, \alpha_1, \nu, \lambda) = \frac{\lambda^2 t(\alpha, \nu)^2 \alpha_1(1-2\alpha_1)}{2 f_\nu(t(\alpha_1, \nu))^2 t(\alpha_1, \nu)^2 m}. \tag{3.10}$$

The efficiency of V_2 depends on the choice of α_1 through the function

$$g_\nu(\alpha) = \frac{\alpha(1-2\alpha)}{f_\nu(t(\alpha, \nu))^2 t(\alpha, \nu)^2}.$$

The function $g_\nu(\alpha)$ for several choices of ν is presented in Figure 3. It is clear that the choices of α_1 in the range $[0.1, 0.2]$ are nearly optimal for all values of ν. For this reason, $\alpha_1 = 0.15$ is chosen throughout this paper.

As explained previously, ν can be treated as known. Under this assumption, as shown in Appendix A.2, the method of moment estimator $\hat{q}^{[4]}(\alpha, \tau)$ is asymptotically normal with mean $\lambda t(\alpha, \nu)$ and variance

$$V_3(\alpha, \nu, \lambda) = \frac{\lambda^2(\nu-1)t(\alpha, \nu)^2}{2(\nu-4)m}. \tag{3.11}$$

Table 7 depicts the relative efficiency among the three estimators. The nonparametric estimator $\hat{q}^{[1]}$ always outperforms the parametric quantile estimator $\hat{q}^{[2]}$, unless the α and ν are small. The former is more robust against any mis-specification of the model (3.3). The nonparametric estimator $\hat{q}^{[1]}$ is more efficient than the method of moment $\hat{q}^{[3]}$ when the degree of freedom is small, and has reasonable efficiency when ν is large. This, together with the robustness of the nonparametric estimator $\hat{q}^{[1]}$ to mis-specification of models and outliers, indicates that our newly proposed nonparametric estimator is generally preferable to the method of moment estimator. This finding is consistent with our empirical studies.

In summary, in consideration of the fact that the return series have heavily tails and contain outliers due to large market movements, and in light of its high statistical efficiency even in the parametric models, the symmetric nonparametric estimator $\hat{q}^{[1]}$ is the most preferred for $\alpha = 5\%$. Among the two parametric methods, the method of moments may be preferred because of its high efficiency when the degree of freedom is large.

3.5. Empirical comparisons of quantile estimators

We compared, empirically, the performances of three different methods for quantile estimation by using the eight stock indices in Example 1. To make the comparison easier, the same volatility

Table 7. Relative efficiency for three estimators of quantiles.

		$\alpha = 5\%$		$\alpha = 1\%$		$\alpha = 10\%$	
ν	V_1/λ^2	V_2/V_1	V_3/V_1	V_2/V_1	V_3/V_1	V_2/V_1	V_3/V_1
2	26.243	1.071	∞	0.253	∞	1.708	∞
3	10.928	1.348	∞	0.394	∞	1.873	∞
4	7.117	1.532	∞	0.518	∞	1.965	∞
5	5.528	1.659	1.469	0.621	0.550	2.024	1.792
6	4.682	1.750	1.008	0.706	0.406	2.064	1.189
7	4.164	1.820	0.862	0.775	0.367	2.093	0.992
10	3.381	1.952	0.729	0.922	0.344	2.146	0.801
20	2.666	2.119	0.663	1.138	0.356	2.210	0.691
40	2.373	2.208	0.647	1.266	0.371	2.242	0.657
100	2.214	2.263	0.642	1.351	0.383	2.261	0.641

estimator was applied to all indices and holding periods. The multi-period estimation of volatility in the RiskMetrics is employed for $\tau = 10$, $\tau = 25$ and $\tau = 50$. The estimated quantiles $\hat{q}^{[j]}(\alpha, \tau)$ $(j = 1, 2, 3, 4)$ are used to obtain the estimated VaR. The effectiveness of the estimated VaR is measured by the exceedance ratio

$$\mathrm{ER} = (n - \tau)^{-1} \sum_{t=T+1}^{T+n-\tau} I(R_{t,\tau} < \mathrm{VaR}_{t,\tau})$$

in the post-sample period (January 1, 1997 to December 31, 2000 with $n = 1014$). Note that due to the insufficient number of non-overlapping τ-day intervals in the post-sample period, overlapping intervals are used. This has two advantages: (1) increasing the number of intervals by a factor of approximately τ and (2) making the results insensitive to the starting date of the post-sample period (not the case for non-overlapping intervals). The confidence level $1 - \alpha = 95\%$ is used. To compare with the performance of RiskMetrics, we also form $\mathrm{VaR}^{[0]}_{t+1,\tau} = \Phi^{-1}(\alpha)\hat{\sigma}_{t+1,\tau}$. This follows exactly the recommendations of J. P. Morgan's RiskMetrics. The results are presented in Table 8.

To summarize the performance presented in Table 8, we computed the average and standard deviation of exceedance ratios for holding period $\tau = 10, 25, 50$ for each of the eight stock indices. In addition, the mean absolute deviation errors from the nominal level $\alpha = 5\%$ are also computed. Table 9 depicts the results of these computations.

For a holding period of 10 days, the adaptive nonparametric estimator $\mathrm{VaR}^{[2]}$ has the smallest bias as well as the second smallest standard deviation of the five competing methods. RiskMetrics has a comparable amount of bias, yet its variability is larger. The nonparametric estimator $\mathrm{VaR}^{[1]}$ has the smallest standard deviation, though its bias is the third largest. The adaptive nonparametric method has, also, the smallest mean absolute deviation error from 5%. For a holding period of 25 days, RiskMetrics has the smallest biases, but its reliability is quite poor. Its variability is ranked third out of the five competing methods. The overall deviation (measured by MADE) from the nominal level is, again, achieved by the two nonparametric methods of estimation of quantiles. For a holding period of 50 days, the variability of the RiskMetrics are, again, very large. The adaptive nonparametric method is the best in terms of bias, variance and overall

Table 8. Comparisons of exceedance ratios of several VaR estimators ($\times 10^{-2}$).

Index	Holding period (τ)	$VaR^{[0]}$	$VaR^{[1]}$	$VaR^{[2]}$	$VaR^{[3]}$	$VaR^{[4]}$
	10	5.42	3.55	5.52	2.37	1.28
AORD	25	5.52	2.56	5.23	1.08	0.99
	50	1.28	0.10	3.75	0.10	0.00
	10	3.67	3.67	3.48	3.48	2.88
CAC 40	25	4.47	4.57	4.07	3.57	3.48
	50	2.78	3.38	2.48	2.78	2.38
	10	4.87	4.27	5.06	5.06	4.67
DAX	25	5.66	4.97	5.06	6.95	5.36
	50	4.07	3.97	3.77	4.27	3.97
	10	6.55	4.49	4.96	5.52	4.40
HS	25	9.17	6.08	8.04	5.89	5.71
	50	7.11	2.99	5.14	3.46	3.55
	10	4.97	4.26	5.98	3.65	4.06
Nikkei 225	25	6.59	4.56	7.20	4.46	4.67
	50	11.76	2.54	11.56	8.22	5.17
	10	4.75	4.06	5.05	3.47	4.06
FTSE	25	3.96	2.48	5.35	3.17	2.77
	50	2.67	2.48	4.95	2.08	2.67
	10	3.77	3.67	5.05	2.38	4.36
S&P500	25	3.17	3.57	5.05	3.96	3.67
	50	2.78	3.77	5.95	3.87	3.77
	10	5.06	4.96	5.95	3.77	5.16
DJ	25	4.86	5.16	7.24	6.65	5.75
	50	3.47	3.87	4.66	3.67	4.07

deviation from the target level of 5%. It is also worthwhile to note that the variability increases as the holding period gets longer. This is understandable since the prediction involves a longer time horizon.

The above performance comparisons show convincingly that the newly proposed nonparametric methods for estimation of quantiles outperform the two parametric methods and the RiskMetrics.

4. ESTIMATION OF VALUE AT RISK

We have proposed three new volatility estimators based on semiparametric model (2.8). These, together with the RiskMetrics and the GARCH model estimator, give rise to five volatility

Table 9. Summary of performance of several VaR estimators ($\times 10^{-2}$).

Measure	Holding period (τ)	VaR$^{[0]}$	VaR$^{[1]}$	VaR$^{[2]}$	VaR$^{[3]}$	VaR$^{[4]}$
AVE	10	4.88	4.12	5.13	3.71	3.86
STD	10	0.91	0.48	0.79	1.12	1.23
MADE	10	0.63	0.88	0.52	1.43	1.18
AVE	25	5.42	4.24	5.90	4.47	4.05
STD	25	1.85	1.27	1.39	1.97	1.66
MADE	25	1.31	1.07	1.18	1.66	1.41
AVE	50	4.49	2.89	5.28	3.56	3.20
STD	50	3.39	1.27	2.75	2.30	1.55
MADE	50	2.73	2.11	1.63	2.25	1.85

estimators. Further, we have introduced four new quantile estimators, based on parametric and nonparametric models. These and the normal quantile give five quantile estimators. Combinations of these volatility estimators and quantile estimators yield 25 methods for estimating VaR. To make comparisons easier, we eliminate a few unpromising combinations. For example, our previous studies indicate that the semiparametric volatility estimator with $\lambda = 0.94$ does not work very well and that the parametric methods for quantile estimation do not perform as well as their nonparametric counterparts. Therefore, these methods are not considered. Instead, a few promising methods are considered to highlight the points that we advocate. Namely, that the decay parameter should be determined by data and that the time-dependent decay parameter should have a better ability to adapt to changes in market conditions. In particular, we select the following procedures:

RiskMetrics:	Normal quantile and volatility estimator (2.1)
Nonparametric RiskMetrics (NRM):	Nonparametric quantile $q^{[1]}$ and (2.1)
Semiparametric Risk Estimator (SRE):	Nonparametric quantile $q^{[1]}$ and SVE
Adaptive Risk Estimator (ARE):	Adaptive nonparametric quantile $q^{[2]}$ and AVE
GARCH estimator (GARCH):	Normal quantile and GARCH(1,1) model.

The SRE and ARE are included in the study because they are promising. The former has time-independent decay parameters and quantiles, while the latter has time-dependent parameters and quantiles. NRM is also included in our study because of its simplicity. It possesses a very similar spirit to RiskMetrics. GARCH is included because of its popularity in analyzing financial data.

To compare these five methods, we use simulated data sets and the eight stock indices. We begin with the simulated data. Two hundred series of length 3000 were simulated from the continuous-time SV model in Example 3. As in Example 2, the first 2000 data points were regarded as the in-sample period and the last 1000 data points were treated as the post-sample period. The exceedance ratios were computed for each series for the holding periods $\tau = 1, 10, 25$ and 50. The results are summarized in Table 10. Using the mean absolute deviation error from the nominal confidence level 5% as an overall measure, the RiskMetrics and ARE perform the best. One reason for the RiskMetrics to perform well is that the stochastic noises are generated from a normal distribution. It is easy to understand that if the noise does not follow a normal distribution, the RiskMetrics will not perform well.

Table 10. Summary of the performance of five VaR estimators.

Measure	Holding period (τ)	RiskMetrics	NRM	SRE	ARE	GARCH
AVE	1	5.45	4.95	4.99	4.99	4.29
STD	1	0.55	0.58	0.66	0.62	2.16
MADE[a]	1	0.61	0.47	0.54	0.49	1.81
AVE	10	5.19	5.12	5.16	5.12	4.29
STD	10	1.60	1.82	1.83	1.75	2.52
MADE	10	1.24	1.45	1.41	1.37	2.12
AVE	25	5.00	5.18	5.18	5.18	4.17
STD	25	2.53	2.76	2.80	2.71	3.16
MADE	25	1.96	2.14	2.16	2.04	2.73
AVE	50	5.41	5.41	5.37	5.33	4.21
STD	50	3.89	3.89	3.94	3.77	3.70
MADE	50	3.07	3.07	3.10	2.98	3.16

[a] Mean absolute deviation error from the nominal confidence level 5%.

We now apply the five VaR estimators to the eight stock indices depicted in Example 1. As in Example 1 and shown in Table 2, the post-sample was from January 1, 1997 to December 31, 2000. The exceedance ratios are computed for each method. The results are shown in Table 11. To make the comparison easier, Table 12 shows the summary statistics of Table 11.

Table 12 shows that the ARE is the best procedure among the five VaR estimators for all holding periods. For one-period, SRE outperforms RiskMetrics, but for multi-period, RiskMetrics outperforms the SRE. NRM improves somewhat the performance of RiskMetrics. For the real data sets, it is clear that it is worthwhile to use the time-dependent methods such as ARE. Indeed, the gain is more than the price that we have to pay for the adaptation to the changes of market conditions. The results also provide stark evidence that the quantile of standardized return should be estimated and the decay parameters should be determined from data.

Table 13 shows the results for hypothesis testing, as in the performance measures 4 and 5. The results are shown for a 1-day holding period. As indicated before, for multiple-day VaR prediction, overlapping intervals are used. Hence, hypothesis testing cannot be applied. Except for the HSI, there is little evidence against the hypothesis of independence and the hypothesis that ER = 5%. The P-values for AORD also tend to be small.

5. CONCLUSIONS

We have proposed semiparametric methods for estimating volatility, as well as nonparametric and parametric methods for estimating the quantiles of scaled residuals. The performance comparisons are studied both empirically and theoretically. We have shown that the proposed semiparametric model is flexible in approximating stock price dynamics.

For volatility estimation, it is evident from our study that the decay parameter should be chosen from data. Our proposed method of choosing the decay parameter has been demonstrated

Table 11. Comparisons of exceedance ratios of five VaR estimators.

Index	Holding period (τ)	RiskMetrics	NRM	SRE	ARE	GARCH
	1	4.93	4.93	4.73	5.23	5.42
AORD	10	5.42	3.85	4.34	4.73	5.13
	25	5.52	3.25	3.55	5.33	4.93
	50	1.28	0.39	1.28	4.34	1.38
	1	6.36	6.45	6.06	5.56	8.24
CAC 40	10	3.67	3.67	2.68	3.28	4.57
	25	4.47	4.57	3.87	3.67	5.46
	50	2.78	3.38	2.78	2.28	3.28
	1	5.46	5.46	5.56	5.76	7.85
DAX	10	4.87	4.27	4.47	4.47	6.26
	25	5.66	4.97	4.47	5.06	6.95
	50	4.07	3.97	3.48	3.58	4.27
	1	5.99	5.71	6.27	4.96	7.30
HSI	10	6.55	4.49	5.43	5.05	9.17
	25	9.17	6.08	7.95	8.98	12.25
	50	7.11	2.99	4.40	5.15	11.23
	1	5.78	5.68	5.68	5.27	5.88
Nikkei 225	10	4.97	4.26	3.96	5.58	4.97
	25	6.59	4.56	3.96	6.90	6.69
	50	11.76	2.54	1.22	10.75	10.85
	1	5.94	6.04	6.04	6.14	7.43
FTSE	10	4.75	4.06	3.96	5.05	4.95
	25	3.96	2.48	2.97	6.63	4.65
	50	2.67	2.48	2.38	5.15	2.87
	1	5.55	5.25	5.15	4.96	4.46
S&P500	10	3.77	3.67	3.87	5.35	2.78
	25	3.17	3.57	3.47	4.96	1.98
	50	2.78	3.77	4.36	6.64	2.28
	1	5.65	5.65	5.65	5.65	4.96
Dow Jones	10	5.06	4.96	4.96	5.85	4.17
	25	4.86	5.16	4.46	6.15	4.37
	50	3.47	3.87	4.27	5.16	3.37

to be quite effective. An adaptive procedure has also been proposed, which allows automatic adaptation to periodic changes in market conditions. It is shown that the AVE outperforms the other procedures, while the SVE also performs competitively.

Table 12. Summary of the performance of five VaR estimators.

Measure	Holding period (τ)	RiskMetrics	NRM	SRE	ARE	GARCH
AVE	1	5.71	5.65	5.64	5.44	6.44
STD	1	0.42	0.46	0.51	0.41	1.43
MADE	1	0.73	0.66	0.71	0.46	1.59
AVE	10	4.88	4.15	4.21	4.92	5.25
STD	10	0.91	0.44	0.82	0.80	1.86
MADE	10	0.63	0.85	0.90	0.55	1.14
AVE	25	5.43	4.33	4.34	5.96	5.91
STD	25	1.85	1.16	1.54	1.60	2.99
MADE	25	1.31	0.98	1.40	1.30	1.93
AVE	50	4.49	2.92	3.02	5.38	4.94
STD	50	3.39	1.17	1.32	2.52	3.86
MADE	50	2.73	2.08	1.98	1.58	3.08

For quantile estimation, our study shows that the nonparametric method has a very high efficiency compared to its parametric counterparts. Furthermore, it is robust against model misspecifications. An adaptive procedure was introduced to accommodate the changes in market conditions over time and allows it to outperform other competing approaches.

For VaR estimation, it is natural to combine the adaptive volatility estimator with the adaptive quantile estimator (ARE), and to combine the semiparametric estimator of volatility with the nonparametric estimator of quantiles (SRE), to yield effective estimators for VaR. The former is designed to accommodate changes in market conditions over time while the latter is introduced for situations where the market conditions do not change abruptly. Both methods perform outstandingly although preference is given to the ARE method, due partially to its ability to adapt to the changes in market conditions over time.

Some parameters in the the adaptive volatility and adaptive nonparametric quantile estimators were chosen arbitrarily. The performance of our proposed procedure can further be ameliorated if these parameters are optimized. An advantage of our procedure is that it can be combined with other volatility estimators and quantile estimators to yield new and more powerful procedures for prediction of VaR.

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Table 13. Comparisons of exceedance ratios and the test results of five VaR estimators.

Index	Holding period	RiskMetrics	NRM	SRE	ARE	GARCH
	ER	4.93	4.93	4.73	5.23	5.42
AORD	P-value(indep)	0.01	0.01	0.01	0.02	0.12
	P-value(ER = 5%)	0.92	0.92	0.71	0.76	0.46
	ER	6.36	6.45	6.06	5.56	8.24
CAC 40	P-value(indep)	0.33	0.37	0.90	0.12	0.36
	P-value(ER = 5%)	0.06	0.04	0.14	0.43	0.00[a]
	ER	5.46	5.46	5.56	5.76	7.85
DAX	P-value(indep)	0.59	0.59	0.90	0.16	0.78
	P-value(ER = 5%)	0.51	0.51	0.43	0.28	0.00[a]
	ER	5.99	5.71	6.27	4.96	7.30
HSI	P-value(indep)	0.00[a]	0.00[a]	0.00[a]	0.00[a]	0.03
	P-value(ER = 5%)	0.15	0.30	0.07	0.96	0.00[a]
	ER	5.78	5.68	5.68	5.27	5.88
Nikkei 225	P-value(indep)	0.35	0.32	0.32	0.19	0.80
	P-value(ER = 5%)	0.27	0.34	0.34	0.71	0.22
	ER	5.94	6.04	6.04	6.14	7.43
FTSE	P-value(indep)	0.45	0.49	0.82	0.11	0.78
	P-value(ER = 5%)	0.18	0.14	0.14	0.11	0.00[a]
	ER	5.55	5.25	5.15	4.96	4.46
S&P500	P-value(indep)	0.62	0.88	0.32	0.89	0.89
	P-value(ER = 5%)	0.44	0.73	0.85	0.96	0.43
	ER	5.65	5.65	5.65	5.65	4.96
Dow Jones	P-value(indep)	0.56	0.56	0.18	0.56	0.68
	P-value(ER = 5%)	0.35	0.35	0.35	0.35	0.43

Note: Means statistically significant at the 1% level.

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APPENDIX

In this Appendix, we give some theoretical derivations on the relative efficiencies of several nonparametric and parametric estimators of quantiles. The basic assumption is that $\{\hat{\varepsilon}_{t,\tau}, t = T_0, \ldots, T-\tau\}$ is an independent random sample from a population with probability density f.

A.1. Relative efficiency of estimator (3.1)

By the symmetry assumption, $f(q(\alpha,\tau)) = f(q(1-\alpha,\tau))$. By using (3.8), we obtain that the estimators $\hat{q}(\alpha,\tau)$ and $\hat{q}(1-\alpha,\tau)$ are joint asymptotically normal with mean $(q(\alpha,\tau), -q(\alpha,\tau))^T$ and covariance matrix

$$f(q(\alpha,\tau))^{-2}\begin{pmatrix}\alpha(1-\alpha) & \alpha^2\\ \alpha^2 & \alpha(1-\alpha)\end{pmatrix}.$$

It follows that $\hat{q}^{[3]}(\alpha,\tau)$ has the asymptotic normal distribution with mean $q(\alpha,\tau)$ and variance

$$4^{-1}f(q(\alpha,\tau))^{-2}[\alpha(1-\alpha)-2\alpha^2+\alpha(1-\alpha)] = 2^{-1}f(q(\alpha,\tau))^{-2}\alpha(1-2\alpha),$$

and that $\hat{q}(\alpha,\tau)$ is asymptotically normal with mean $q(\alpha,\tau)$ and variance

$$f(q(\alpha,\tau))^{-2}\alpha(1-\alpha).$$

Consequently, the estimator $\hat{q}^{[3]}(\alpha,\tau)$ is a factor of $2(1-\alpha)/(1-2\alpha)$ as efficient as $\hat{q}(\alpha,\tau)$.

A.2. Asymptotic normality for the method of moments estimator

Recall that the variance of the squared t-random variable with degree of freedom ν is given by $2\nu^2(\nu-1)/(\nu-2)^2(\nu-4)$. By the central limit theorem,

$$\sqrt{m}\left(\hat{\mu}_2 - \frac{\nu}{\nu-2}\lambda^2\right) \xrightarrow{\mathcal{L}} N\left(0, \frac{2\nu^2(\nu-1)}{(\nu-2)^2(\nu-4)}\lambda^4\right).$$

Using the delta method, we deduce that

$$\sqrt{m}\left(\sqrt{\hat{\mu}_2} - \sqrt{\frac{\nu}{\nu-2}}\lambda\right) \xrightarrow{\mathcal{L}} N\left(0, \frac{\nu(\nu-1)}{2(\nu-2)(\nu-4)}\lambda^2\right).$$

Hence, the estimated quantile

$$\hat{q}^{[4]}(\alpha, \tau) = \sqrt{\hat{\mu}_2}\sqrt{\frac{\nu-2}{\nu}}t(\alpha, \tau)$$

has the asymptotic normal distribution with mean $\lambda t(\alpha, \tau)$ and variance

$$\frac{\lambda(\nu-1)t(\alpha, \tau)^2}{2(\nu-4)}.$$