Testing and detecting jumps based on a discretely observed process

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ABSTRACT

We propose a new nonparametric test for detecting the presence of jumps in asset prices using discretely observed data. Compared with the test in Aït-Sahalia and Jacod (2009), our new test enjoys the same asymptotic properties but has smaller variance. These results are justified both theoretically and numerically. We also propose a new procedure to locate the jumps. The jump identification problem reduces to a multiple comparison problem. We employ the false discovery rate approach to control the probability of type I error. Numerical studies further demonstrate the power of our new method.

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1. Introduction

The discontinuities of asset prices, so-called "jumps", play important roles in pricing and managing risks of many financial instruments such as asset returns, option prices, interest rates, and exchange rates. Recently, researchers have proved the existence of jumps and studied their financial implications in the literature both empirically and theoretically. See, for example, Merton (1976), Duffie et al. (2000), Pan (2002), Johannes (2004) and Andersen et al. (2007).

The wide availability of high-frequency data for a host array of financial instruments makes it feasible to accurately detect the locations of jumps with little time delay. The interest in testing and identifying jumps has surged recently. For example, Aït-Sahalia (2004) introduces methods to separate jumps from diffusion. Jiang and Oomen (2008) propose a test statistic that measures the impact of jumps on the third- and higher-order return moments and is directly related to the profit/loss function of a variance swap replication strategy. Barndorff-Nielsen and Shephard (2006) introduce a test statistic based on the bipower

variation of the asset price, which is consistent and asymptotically normal with mean zero under the null hypothesis of no jumps, and which converges in probability to some negative number depending on the jump sizes under the alternative hypothesis. A nonparametric test statistic was proposed by Aït-Sahalia and Jacod (2009), which converges to two different deterministic numbers that are independent of the dynamics of the diffusion process, depending on whether the sample path has or does not have jumps. Fan and Wang (2007) develop wavelet methods to estimate jump locations and jump sizes from a jump-diffusion process that is discretely observed with market microstructure noise. Lee and Mykland (2008) introduce and study a nonparametric test to detect jump arrival times up to the intra-day level. Their test statistic not only detects the presence of jumps but also gives estimates of the realized jump sizes in asset prices. Jacod and Todorov (2009) consider a bivariate asset price process and propose tests to decide whether these processes have common jumps or not using discretely observed data on a finite time interval. Other tests include Carr and Wu (2003), Mancini (2003) and Johannes et al. (2004a,b).

The asset price X_t is assumed to follow an Itô semimartingale process and is observed at discrete time points $t_i = i\Delta$, $i = 0, 1, \ldots, n$. In this paper, we consider high-frequency data—i.e., assuming that $\Delta \to 0$ as $n \to \infty$ and $T = n\Delta$ is a fixed positive number. To simplify the notation, we suppress the dependence of Δ on n when it causes no confusion. Aït-Sahalia and Jacod (2009)

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propose a nonparametric test statistic with the following form

$$S_n = \frac{\sum_{i=1}^{[n/K]} |X_{iK\Delta} - X_{(i-1)K\Delta}|^p}{\sum_{i=1}^{n} |X_{i\Delta} - X_{(i-1)\Delta}|^p},$$

where K is a positive integer, $p > 3$, and $[z]$ denotes the integer part of z. This is the ratio of power variations at two time scales (Zhang et al., 2005). The intuition of the test statistic S_n is that if there is a jump in the time interval $((i-1)\Delta, i\Delta]$, then the magnitude of the increment $\Delta X_i = X_{i\Delta} - X_{(i-1)\Delta}$ is large and independent of the sampling interval Δ, whereas the magnitude of ΔX_i is small and depends on Δ when there is no jump in the interval. A high power of the increment ΔX_i further separates the magnitudes of $|\Delta X_i|^p$ in the previous two cases. Since the increments containing jumps are much larger than those that do not, their contribution to the summation dominates all other terms. As a result, S_n behaves very differently when the sample path on the time interval $[0, T]$ has jumps from the case when there is no jump. In fact, S_n converges to 1 when a jump is present, and it converges to $K^{p/2-1}$ in the absence of jumps, as formally stated in (7). This limiting result holds for any Itô semimartingale X_t with no need to estimate the model parameters, and thus it is genuinely nonparametric. It serves as the basis for separating jumps from diffusion.

Aït-Sahalia and Jacod (2009) establish asymptotic distribution theorems for their test statistic. From their result, it can be easily seen that the asymptotic variance of S_n increases rapidly with both K and p. The inflation of the variance reduces the power of the test. Reducing the variance of the test statistic will undoubtedly increase the power of the test, which is particularly important when the sample size is not large. This is exactly the case when the test statistic is applied to a small window of data to detect whether there is any jump in that window.

We proceed with an idea of variance reduction and propose a new test statistic. Note that the numerator of the test statistic S_n uses only the subsequence $\{X_{iK\Delta} : i = 0, 1, \ldots, [n/K]\}$. For each $\ell = 1, \ldots, K$, we can construct a similar test statistic $S_{n,\ell}$ whose numerator uses data points $\{X_{(\ell-1+iK)\Delta} : i = 0, 1, \ldots, [n/K] - 1\}$, resulting in test statistics $S_{n,\ell}$, $\ell = 1, \ldots, K$, that have the same asymptotic distribution. Therefore, a proper linear combination of them can reduce the variance with mean unchanged and thus give a more powerful test statistic. This approach requires the study of the joint behavior of these test statistics under the assumption of jumps. Rooted in our analytical studies, a new test statistic, which is the average of those K test statistics, is proposed. We show that this new test statistic is the optimal one among all linear combinations of $S_{n,\ell}$ in terms of variance reduction.

In addition to detecting the presence of jumps, this paper also contributes to detecting the locations of jumps using the newly proposed test statistic. Our main idea is to first divide the whole time interval $[0, T]$ into many non-overlapping subintervals of equal length $2a_n\Delta$ with $a_n \to \infty$ as $n \to \infty$ and $a_n\Delta$ fixed, and then to apply the new test statistic to each subinterval. This reduces the problem of jump identification to a multiple comparison problem. Using the False Discovery Rate (FDR) approach, one can decide which subintervals contain jumps at a given level. Therefore, the jump locations can be identified within an accuracy of $2a_n\Delta$. In the identified jump intervals, we can further locate jumps by comparing the magnitude of increments. This FDR approach can be applied to many tests such as the ones proposed by Barndorff-Nielsen and Shephard (2006) and Jiang and Oomen (2008), and thus allows us to use these tests to locate the jumps. For this reason, our proposed method is very general.

The rest of the paper is organized as follows. Section 2 introduces the model and assumption. In Section 3, we construct the new test and present its asymptotic properties. Section 4 constructs the critical region and studies its asymptotic size and power. A new testing procedure to locate the jumps is proposed in Section 5. In Sections 6 and 7, we present the simulation studies and a real data application, respectively. Section 8 provides some discussions. All proofs are presented in the Appendix.

2. Model and assumption

Consider a one-dimensional asset price process X_t on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We are interested in testing jumps in the process X_t over the time interval $[0, T]$. In this paper, we assume that X_t is an Itô semimartingale that can be represented as

$$
\begin{aligned}
X_t = X_0 &+ \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s \\
&+ \int_0^t \int_E \kappa \circ \delta(s, x)(\mu - \nu)(\mathrm{d}s, \mathrm{d}x) \\
&+ \int_0^t \int_E \kappa' \circ \delta(s, x)\mu(\mathrm{d}s, \mathrm{d}x),
\end{aligned} \tag{1}
$$

where W_t is an $\mathcal{F}_t$-adapted standard Brownian motion and μ is a Poisson random measure on $[0, t] \times E$ with (E, ε) an auxiliary measurable space on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The intensity measure of μ is $\nu(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \otimes \lambda(\mathrm{d}x)$ with λ some finite or σ-finite measure on (E, ε). The function $\delta(\omega, t, x)$ is predictable, the function κ is continuous with compact support satisfying $\kappa(x) = x$ in a neighborhood of 0, and $\kappa'(x) = x - \kappa(x)$. We further assume that b_t and σ_t are $\mathcal{F}_t$-measurable, with σ_t being another Itô semimartingale

$$
\begin{aligned}
\sigma_t = \sigma_0 &+ \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + \int_0^t \widetilde{\sigma}'_s \mathrm{d}W'_s \\
&+ \int_0^t \int_E \kappa \circ \widetilde{\delta}(s, x)(\mu - \nu)(\mathrm{d}s, \mathrm{d}x) \\
&+ \int_0^t \int_E \kappa' \circ \widetilde{\delta}(s, x)\mu(\mathrm{d}s, \mathrm{d}x),
\end{aligned} \tag{2}
$$

where W'_t is another standard Brownian motion independent of W_t and μ, and $\widetilde{\delta}$ is a predictable function. This model was also considered in Aït-Sahalia and Jacod (2009).

Let $\Delta X_t = X_t - X_{t-}$ be the jump size of the process X at time t. Clearly, $\Delta X_t = 0$ when there is no jump at time t, and $|\Delta X_t| > 0$ whenever there is a jump at time t. Define $\tau = \inf\{s \in [0, T] : \Delta X_s \neq 0\}$ to be the first jump time on the time interval $[0, T]$. Then we usually but not necessarily have $\tau = 0$ almost surely when the jump activity is infinite, and we have $\tau > 0$ almost surely when the jump activity is finite.

We introduce some notation to facilitate the presentation. Let

$$B(p)_t = \sum_{s < t} |\Delta X_s|^p \quad \text{and} \quad A(p)_t = \int_0^t |\sigma_s|^p \mathrm{d}s.$$

Suppose on the time interval $[0, t]$ we have discrete observations at $t_i = i\Delta$ for $i = 0, 1, \ldots, [t/\Delta]$. We define

$$\widehat{B}(p, \Delta)_t = \sum_{j=1}^{[t/\Delta]} |X_{j\Delta} - X_{(j-1)\Delta}|^p. \tag{3}$$

For each $\ell = 1, \ldots, K$, let

$$\widehat{B}(p, K\Delta)_{\ell,t} = \sum_{j=1}^{[t/(K\Delta)]-1} |X_{(\ell-1+jK)\Delta} - X_{(\ell-1+(j-1)K)\Delta}|^p.$$

Then both $\widehat{B}(p, \Delta)_t$ and $\widehat{B}(p, K\Delta)_{\ell,t}$ estimate $B(p)_t$.

Jacod (2008) proves the following convergence results:

$$
\begin{cases}
p > 2 \implies \widehat{B}(p, \Delta)_t \xrightarrow{P} B(p)_t, \\
X \text{ is continuous} \implies \dfrac{\Delta^{1-p/2}}{m_p} \widehat{B}(p, \Delta)_t \xrightarrow{P} A(p)_t,
\end{cases} \tag{4}
$$

where $m_p = E|Z|^p$ and Z is a standard Gaussian random variable. The different behavior of the statistic $\widehat{B}(p, \Delta)_t$, depending on whether the process X has jumps or not, gives the theoretical foundation for jump detection.

We define

$$
\delta'_t(\omega) =
\begin{cases}
\displaystyle \int \kappa \circ \delta(\omega, t, x) \lambda(dx) & \text{if the integral exists} \\
\infty & \text{otherwise.}
\end{cases}
$$

Before presenting the main results, we make the following assumption, which is similar to that in Aït-Sahalia and Jacod (2009).

Assumption 1. (a) All paths $t \to \widetilde{b}_t$ are locally bounded.
(b) All paths $t \to b_t, t \to \widetilde{\sigma}_t$ and $t \to \widetilde{\sigma}'_t$ are right-continuous with left limits.
(c) All paths $t \to \delta(\omega, t, x)$ and $t \to \widetilde{\delta}(\omega, t, x)$ are left-continuous with right limits.
(d) All paths $t \to \sup_{x \in E} \frac{|\delta(\omega, t, x)|}{\gamma(x)}$ and $t \to \sup_{x \in E} \frac{|\widetilde{\delta}(\omega, t, x)|}{\gamma(x)}$ are locally bounded, where γ is a deterministic nonnegative function satisfying $\int_E [\gamma(x)^2 \vee 1] \lambda(dx) < \infty$.
(e) All paths $t \to \delta'_t(\omega)$ are left-continuous with right limits on $[0, \tau(\omega))$.
(f) $\int_0^t |\sigma_u| du > 0$ a.s. for any $t > 0$.

Throughout the paper we consider a stochastic process X_t over a fixed time interval $[0, T]$. Thus, an application of the localization procedure shows that if any theorems to be presented later hold under the assumption

$$
\begin{aligned}
&|b_t| + |\sigma_t| + |\widetilde{b}_t| + |\widetilde{\sigma}_t| + |\widetilde{\sigma}'_t| \le M, \qquad \delta(t, x) \le \gamma(x), \\
&|\widetilde{\delta}(t, x)| \le \gamma(x), \quad \text{and} \quad \gamma(x) \le M
\end{aligned} \tag{5}
$$

for some positive constant M, then they hold under Assumption 1 as well. Therefore, (5) is implicitly assumed to be true by Assumption 1.

3. Test statistics

To simplify the notation, we will drop the dependence of the test statistic on t whenever there is no confusion. For instance, we write $\widehat{B}(p, K\Delta)_\ell$ for $\widehat{B}(p, K\Delta)_{\ell, t}$. Based on the convergence results in (4), Aït-Sahalia and Jacod (2009) propose the following statistic to test for jumps:

$$
\widehat{S}(p, K)_1 = \frac{\widehat{B}(p, K\Delta)_1}{\widehat{B}(p, \Delta)}. \tag{6}
$$

This test statistic behaves differently for sample paths without jumps from those with jumps. In fact, they proved that $\widehat{S}(p, K)_1$ converges in probability to the variable $S(p, K)$ given by

$$
S(p, K) =
\begin{cases}
1 & \text{on the event } \Omega_t^j, \\
K^{p/2-1} & \text{on the event } \Omega_t^c,
\end{cases} \tag{7}
$$

where $\Omega_t^j = \{\omega : X_s(\omega) \text{ is discontinuous on } [0, t]\}$ and $\Omega_t^c = \{\omega : X_s(\omega) \text{ is continuous on } [0, t]\}$. Note that the event Ω_t^j means that the model has jumps, whereas the event Ω_t^c does not mean that the model is continuous. In fact, Ω_t^c could also correspond to the case where the model has jumps but none are in the time interval $[0, t]$. The test statistic $\widehat{S}(p, K)_1$ enjoys nice properties. It is invariant under the scaling of X_t, and its limiting behavior is

independent of the dynamics of the process X_t. Aït-Sahalia and Jacod (2009) also derived the asymptotic distribution of $\widehat{S}(p, K)_1$. The asymptotic mean of $\widehat{S}(p, K)_1$ is 1 when jumps are present and is $K^{p/2-1}$ when there are no jumps. Its asymptotic variance is a complicated function of K and p that increases with both K and p. This indicates that although larger K and larger p can separate the asymptotic means further apart, the improvement would very likely be masked by the inflation of the asymptotic variance. Therefore, it is important to reduce the dependence of the asymptotic variance on K or p, which motivated our work.

For a given $K \ge 2$, there are K test statistics of the similar form:

$$
\widehat{S}(p, K)_\ell = \frac{\widehat{B}(p, K\Delta)_\ell}{\widehat{B}(p, \Delta)}, \quad \ell = 1, \ldots, K. \tag{8}
$$

Thanks to the similarity of their mathematical forms, they have the same asymptotic distribution. Intuitively, for finite K, $\{\widehat{S}(p, K)_\ell\}_{\ell=1}^K$ should have asymptotic correlation 1. Contrary to this intuition, the asymptotic correlation is less than 1, as formally presented in Theorems 1 and 2. This suggests that the test statistic defined in (6) can be improved by linearly combining the K test statistics defined in (8). We thus propose a new test statistic:

$$
\widehat{S}(p, K, \boldsymbol{\omega}) = \sum_{\ell=1}^K \omega_\ell \widehat{S}(p, K)_\ell,
$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)$ is the weight vector satisfying $\sum_{\ell=1}^K \omega_\ell = 1$. It is critical to derive the asymptotic joint distribution of these K test statistics $\widehat{S}(p, K)_\ell, \ell = 1, \ldots, K$. A similar technique was used by Aït-Sahalia et al. (2005) with $p = 2$, but it was mainly used for the reduction of measurement errors.

Define

$$
D(p) = \sum_{s \le t} |\Delta X_s|^p (\sigma_{s-}^2 + \sigma_s^2).
$$

The following theorem gives the asymptotic joint distributions of $\widehat{B}(p, \Delta)$ and $\sum_{\ell=1}^K \omega_\ell \widehat{B}(p, K\Delta)_\ell$ when jumps are present in the sample path of X_t.

Theorem 1. Assume that Assumption 1 holds, $\Delta \to 0$, and $p > 3$. Then the pair of variables

$$
\Delta^{-1/2} \left(\widehat{B}(p, \Delta) - B(p), \sum_{\ell=1}^K \omega_\ell \widehat{B}(p, K\Delta)_\ell - B(p) \right)
$$

converges stably in law to a bivariate random vector $(Z(p), Z(p) + \sum_{\ell=1}^K \omega_\ell Z'(p, K, \ell))$, where $Z(p)$ and $Z'(p, K, \ell)$ given by (29) in Appendix A.2 are defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \ge 0}, \widetilde{\mathbb{P}})$ of the original filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ and have mean zero conditional on $\mathcal{F}$. Conditional on $\mathcal{F}$, $Z(p)$ and $Z'(p, K, \ell)$ are independent, and $Z'(p, K, \ell)$ have the following conditional variance and covariance:

$$
\widetilde{\mathbb{E}} \left(Z'(p, K, \ell_1) Z'(p, K, \ell_2) | \mathcal{F} \right)
$$

$$
= p^2 \left\{ \frac{K-1}{2} - \frac{1}{K} (K - |\ell_2 - \ell_1|) |\ell_2 - \ell_1| \right\} D(p). \tag{9}
$$

Furthermore, if the processes X and σ have no common jumps, $Z(p)$ and $Z'(p, K, \ell)$ are $\mathcal{F}$-conditionally Gaussian.

Although Theorem 1 does not exclude the situation when X_t is continuous, both $B(p)$ and $D(p)$ are zero in the absence of jumps. Thus we only use Theorem 1 to derive the asymptotic distribution of $\widehat{S}(p, K, \boldsymbol{\omega})$ under the assumption of jumps. Since the asymptotic distributions are derived for arbitrary linear combinations, the above results also give the asymptotic joint distributions of the $K + 1$ random variables $\widehat{B}(p, \Delta), \widehat{B}(p, K\Delta)_1, \ldots, \widehat{B}(p, K\Delta)_K$. In view of (9), the conditional correlation between $Z'(p, K, \ell_1)$ and

$Z'(p, K, \ell_2)$ can be small if $|\ell_1 - \ell_2|$ is close to $K/2$. In particular, when $K = 2$, the conditional correlation between $Z'(p, K, 1)$ and $Z'(p, K, 2)$ is zero. Therefore, by choosing $\omega = (1/2, 1/2)$, compared with $\widehat{S}(p, K)_1$, we reduce the asymptotic variance by a factor of $1/2$. For general K, with a proper choice of ω, the reduction of the variance of the new test statistic $\widehat{S}(p, K, \omega)$ can also be significant.

Corollary 1. *Assume that the conditions of Theorem 1 hold. Then, conditional on Ω^j, $\Delta^{-1/2}(\widehat{S}(p, K, \omega) - 1)$ converges stably in law to a random variable $S(p, K, \omega)^j$ which, conditional on $\mathcal{F}$, has mean zero and variance*

$$\widetilde{\mathbb{E}}\left((S(p, K, \omega)^j)^2 | \mathcal{F}\right)$$

$$= \left[\frac{K-1}{2} - \sum_{i,j=1}^{K} \omega_i \omega_j (1 - K^{-1}|i-j|)|i-j|\right]$$

$$\times p^2 \frac{D(2p-2)}{B(p)^2}. \tag{10}$$

Moreover, if the processes X and σ have no common jumps, conditional on $\mathcal{F}$, $S(p, K, \omega)^j$ has Gaussian distribution.

We would like to remark that although in models (1) and (2), the processes X and σ are driven by the same Poisson random measure μ, the jump behaviors of X and σ can be very different and even independent if the functions δ and $\tilde{\delta}$ are chosen appropriately. This was also pointed out by Jacod (2007, Page 20). In fact, if δ and $\tilde{\delta}$ are chosen in a way such that X and σ have independent jump behaviors, with probability 1, X and σ have no common jumps. Thus, it is not restrictive to exclude the common jump case in Theorem 1 and Corollary 1.

Notice that under the assumption of jumps, the optimal weight vector ω_{opt} in terms of minimizing the variance of $S(p, K, \omega)^j$ is the solution to the following quadratic optimization problem

$$\arg\max_{\omega} \sum_{i,j} \omega_i \omega_j \left(1 - K^{-1}|i-j|\right)|i-j| \quad \text{subject to} \quad \sum_{i=1}^{K} \omega_i = 1.$$

It is not hard to show that its solution is of equal weight due to the exchangeability of the weights—i.e., $\omega_{\mathrm{opt}} = K^{-1}\mathbf{1}$. The corresponding variance in (10) is

$$\frac{(2K-1)(K-1)p^2}{6K} \frac{D(2p-2)}{B(p)^2}. \tag{11}$$

The optimal choice of K in terms of variance reduction is clearly attained at $K = 2$.

Deriving the asymptotic joint distribution of $\widehat{S}(p, K)_\ell$, $\ell = 1, \ldots, K$ when X_t is continuous is nontrivial. We can show that the optimal weight vector ω under the assumption of no jumps coincides with the optimal weight vector $\omega_{\mathrm{opt}} = K^{-1}\mathbf{1}$ under the assumption of jumps. This is clear once we observe that due to the similarity of the definitions of $\widehat{S}(p, K)_\ell$, $\ell = 1, \ldots, K$, the asymptotic covariance of $\widehat{S}(p, K)_{\ell_1}$ and $\widehat{S}(p, K)_{\ell_2}$ depends only on $|\ell_1 - \ell_2|$ for $1 \le \ell_1, \ell_2 \le K$. Therefore, minimizing the asymptotic variance of $\widehat{S}(p, K, \omega)$ yields $\omega_{\mathrm{opt}} = K^{-1}\mathbf{1}$. Hereinafter, we denote by $\widetilde{S}(p, K) = \widehat{S}(p, K, \omega_{\mathrm{opt}})$.

The following theorem characterizes the asymptotic joint distribution of $K^{-1}\sum_{\ell=1}^{K} \widehat{B}(p, K\Delta)_\ell$ and $\widehat{B}(p, \Delta)$ when X is continuous.

Theorem 2. *Under Assumption 1, if $\Delta \to 0$, $p \ge 2$, and X is continuous, then the pair of variables*

$$\Delta^{-1/2}\Bigg(\Delta^{1-p/2}\widehat{B}(p, \Delta) - m_p A(p),$$

$$\Delta^{1-p/2}K^{-1}\sum_{\ell=1}^{K} \widehat{B}(p, K\Delta)_\ell - K^{p/2-1}m_p A(p)\Bigg)$$

converges stably in law to a bivariate random vector $(Y(p), Y'(p))$, which is defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \ge 0}, \widetilde{\mathbb{P}})$ of the original filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$. Conditional on $\mathcal{F}$, $(Y(p), Y'(p))$ is Gaussian with mean zero and variance–covariance

$$\widetilde{\mathbb{E}}(Y(p)^2 | \mathcal{F}) = (m_{2p} - m_p^2)A(2p),$$

$$\widetilde{\mathbb{E}}((Y'(p))^2 | \mathcal{F}) = K^{-2}v(p, K)A(2p),$$

$$\widetilde{\mathbb{E}}(Y(p)Y'(p) | \mathcal{F}) = \tilde{v}(p, K)A(2p),$$

where $\widetilde{v}(p, K) = \mathrm{cov}(|U_1|^p, |U_1 + \sqrt{K-1}U_2|^p)$ and

$$v(p, K) = K^p(m_{2p} - m_p^2) + 2\sum_{\ell=1}^{K-1} \mathrm{cov}(|\sqrt{K-\ell}U_1$$

$$+ \sqrt{\ell}U_2|^p, |\sqrt{\ell}U_2 + \sqrt{K-\ell}U_3|^p)$$

with $m_p = E|U_1|^p$ and $U_1, U_2,$ and U_3 being independent standard Gaussian random variables.

Corollary 2 is a direct consequence of Theorem 2. It formally states the asymptotic distributions of $\widetilde{S}(p, K)$ when X is continuous.

Corollary 2. *Assume that the conditions of Theorem 2 hold. Then, $\Delta^{-1/2}(\widetilde{S}(p, K) - K^{p/2-1})$ converges stably in law to a random variable $S(p, K)^c$ which, conditional on $\mathcal{F}$, is Gaussian with mean zero and variance*

$$\widetilde{\mathbb{E}}\left((S(p, K)^c)^2 | \mathcal{F}\right) = M(p, K)\frac{A(2p)}{A(p)^2}, \tag{12}$$

where $M(p, K) = \frac{1}{m_p^2 K}\left[K^{-1}v(p, K) - 2K^{p/2}\widetilde{v}(p, K) + K^{p-1}(m_{2p} - m_p^2)\right]$, and $v(p, K)$, $\widetilde{v}(p, K)$, and m_p are given in Theorem 2.

The test statistic $\widehat{S}(p, K)_1$ in Aït-Sahalia and Jacod (2009) has asymptotic variance

$$M^*(p, K)\frac{A(2p)}{A(p)} \tag{13}$$

with $M^*(p, K)$ being some constant depending only on p and K when X_t is continuous, and it has asymptotic variance

$$\frac{(K-1)p^2}{2}\frac{D(2p-2)}{B(p)^2} \tag{14}$$

when there are jumps. Comparing (11) with (14), we see that when the sample path contains jumps, the best linear combination reduces the variance by a factor of $\frac{K+1}{3K}$. In particular, when $K = 2$, the variance reduction is by a factor of $1/2$. Under the assumption of no jumps, the factor of variance reduction is given by $1 - M(p, K)/M^*(p, K)$. Fig. 1 depicts the ratio $M(p, K)/M^*(p, K)$ for $p = 4$ and 5. Hereinafter, we focus on the following test statistic:

$$\widetilde{S}(p, K) = \widehat{S}(p, K, \omega_{\mathrm{opt}}). \tag{15}$$

4. Testing for jumps

4.1. Estimation of asymptotic variance

The asymptotic variance of $\widetilde{S}(p, K)$ is a function of $D(p)$, $A(p)$, and $B(p)$, which are unknown and need to be estimated. The same estimators as those in Aït-Sahalia and Jacod (2009) are used in this paper. Specifically, the estimator for $D(p)$ is defined as

$$\widehat{D}(p) = \frac{1}{m_n\Delta}\sum_{i=1}^{n}|\Delta_i^n X|^p \sum_{j \in I_{n,t}(i)}(\Delta_j^n X)^2 1\{|\Delta_j^n X| \le \alpha\Delta^\gamma\},$$

where $\Delta_i^n X = X_{i\Delta} - X_{(i-1)\Delta}$, $\alpha > 0$, $\gamma \in (0, 1/2)$, m_n is the local
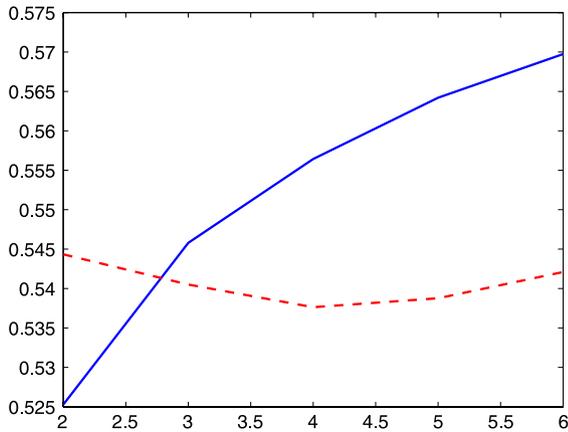
Fig. 1. Plot of $M(p, K)/M^*(p, K)$ against K for $p = 4$ (solid curve) and $p = 5$ (dashed curve).

window size for estimating σ_t^2, and $I_{n,t}(i) = \{j \in N : j \neq i, 1 \leq j \leq [t/\Delta], |i - j| \leq m_n\}$ is a local window of length $2m_n\Delta \to 0$ around $i\Delta$. A realized truncated p-th variation is used to estimate $A(p)$—that is, for $\alpha > 0$ and $\gamma \in (0, 1/2)$, the estimator is defined as

$$\widehat{A}(p) = \frac{\Delta^{1-p/2}}{m_p} \sum_{i=1}^{[t/\Delta]} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \leq \alpha \Delta^\gamma\}}.$$

The estimator $\widehat{B}(p)$ as defined in (3) is used to estimate $B(p)$. It has been proved by Jacod (2008) and Aït-Sahalia and Jacod (2009) that the above three estimators are consistent.

Let

$$\widehat{V}^j = \frac{(2K-1)(K-1)p^2\Delta}{6K} \frac{\widehat{D}(2p-2)}{\widehat{B}(p)^2} \quad \text{and}$$

$$\widehat{V}^c = \frac{\Delta M(p, K)\widehat{A}(2p)}{\widehat{A}(p)^2}.$$
(16)

The following corollary, which is a consequence of Theorems 1 and 2 and the property of stable convergence, gives the asymptotic distribution of the standardized test statistic.

Corollary 3. Assume that Assumption 1 holds and $\Delta \to 0$. We have:

(a) If $p > 3$, then restricted on the set Ω^j the random variable

$$(\widehat{V}^j)^{-1/2}(\widetilde{S}(p, K) - 1)$$
(17)

converges stably in law to a random variable which, conditional on $\mathcal{F}$, has mean 0 and variance 1, and which is standard Gaussian provided that the processes X and σ have no common jumps.

(b) If X is continuous and $p \geq 2$, then the random variable

$$(\widehat{V}^c)^{-1/2}(\widetilde{S}(p, K) - K^{p/2-1})$$
(18)

converges stably in law to a random variable that is standard Gaussian conditional on $\mathcal{F}$.

The beauty of the results in Corollary 3 is that they translate composite hypotheses asymptotically into two simple hypotheses. When there are jumps in the sample path, $\widetilde{S}(p, K)$ asymptotically has conditional distribution $N(1, \widetilde{V}^j)$, whereas when the model is continuous, $\widetilde{S}(p, K)$ asymptotically has conditional distribution $N(K^{p/2-1}, \widetilde{V}^c)$. Hence, the Neyman–Person lemma can be used to test for jumps.

4.2. Testing existence of jumps

Throughout this section, we consider $p > 3$. We aim at testing the existence of jumps in a fixed time interval $[0, T]$ using the

available observations $X_{i\Delta}$, $i = 0, 1, \ldots, n$ with $T = n\Delta$. Thus the null and alternative hypotheses are, respectively,

H_0 : There is no jump in the interval $[0, T]$,

H_1 : There are jumps in the interval $[0, T]$.

Note that the sets under which the null and alternative hypotheses hold are Ω_T^c and Ω_T^j, which are subsets of Ω instead of subsets of some parameter space.

For a critical value x of the above hypothesis testing problem, the type I error is given by

$$\alpha_n(x) = P(\widetilde{S}(p, K) \leq x|H_0),$$

and the power function is

$$\beta_n(x) = P(\widetilde{S}(p, K) \leq x|H_1).$$

We have the following asymptotic theorem for $\alpha_n(x)$ and $\beta_n(x)$.

Theorem 3. Assume that Assumption 1 holds and the critical value $x \in (1, K^{p/2-1})$. Then, we have:

(a) $\alpha_n(x) \to 0$; that is, the critical region $\{\widetilde{S}(p, K) \leq x\}$ has an asymptotic size 0.
(b) Let $\mathbb{P}(\Omega_T^j) > 0$. Then, the power function satisfies $\beta_n(x) \to 1$ as $n \to \infty$.

Similarly, we can show that the above asymptotic results hold for the type I error and power function if the roles of null and alternative hypotheses are switched.

Our new test $\widetilde{S}(p, K)$ is asymptotically more powerful than the test $\widehat{S}(p, K)_1$ by Aït-Sahalia and Jacod (2009). To understand this, recall that if we want to compare the power of tests, we fix their sizes at the same level. We add subscripts "FF" and "AJ" to $\widehat{V}^c$ and $\widehat{V}^j$ to denote the estimated asymptotic variances of $S(p, K)$ and $\widehat{S}(p, K)_1$ under H_0 and H_1, respectively. Then for any critical value x, the size of $\widetilde{S}(p, K)$ is approximately $\Phi(\frac{x-K^{p/2-1}}{\sqrt{\widehat{V}_{FF}^c}})$, and the size of the Aït-Sahalia and Jacod test is approximately $\Phi(\frac{x-K^{p/2-1}}{\sqrt{\widehat{V}_{AJ}^c}})$, where $\Phi(x)$ is the cumulative distribution function of standard Gaussian. If the critical value of our test $\widetilde{S}(p, K)$ is $x_{FF} \in (1, K^{p/2-1})$, then to make the Aït-Sahalia and Jacod test have approximately the same size, the critical value of $\widehat{S}(p, K)_1$ should be approximately

$$x_{AJ} = K^{p/2-1} + \left\{\frac{\widehat{V}_{AJ}^c}{\widehat{V}_{FF}^c}\right\}^{1/2} (x_{FF} - K^{p/2-1}).$$
(19)

Since we have proved in Corollary 1 that $\widehat{V}_{AJ}^c > \widehat{V}_{FF}^c$, it follows easily that $x_{AJ} < x_{FF}$. Let Z be a standard Gaussian random variable. Then the corresponding power of our new test is

$$P(\widetilde{S}(p, K) \leq x_{FF}|\Omega_t^j) \approx P\left(Z \leq \frac{x_{FF} - 1}{\sqrt{\widehat{V}_{FF}^j}}\right),$$

and the corresponding power of the Aït-Sahalia and Jacod test is

$$P(\widehat{S}(p, K)_1 \leq x_{AJ}|\Omega_t^j) \approx P\left(Z \leq \frac{x_{AJ} - 1}{\sqrt{\widehat{V}_{AJ}^j}}\right).$$

In view of Corollary 2, we have $\widehat{V}_{FF}^j < \widehat{V}_{AJ}^j$. This inequality together with $x_{FF} > x_{AJ}$ ensures that $\frac{x_{FF}-1}{\sqrt{\widehat{V}_{FF}^j}} > \frac{x_{AJ}-1}{\sqrt{\widehat{V}_{AJ}^j}}$, which indicates that our test is asymptotically more powerful than the Aït-Sahalia and Jacod test.

Although Theorem 3 shows that asymptotically any constant between 1 and $K^{p/2-1}$ can serve as a critical value and yield

asymptotic size 0 and asymptotic power 1, the finite sample performance of the test based on these critical values can be very different. Next, we discuss how to choose the critical value in practice.

By Corollary 3, conditional on $\mathcal{F}$, the null and alternative distributions of $\widetilde{S}(p, K)$ are both asymptotically Gaussian. Thus, asymptotically the likelihood ratio test rejects H_0 whenever

$$\Lambda = \frac{\varphi_0(\widetilde{S}(p, K))}{\varphi_1(\widetilde{S}(p, K))} \tag{20}$$

is small, where $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$ are the conditional asymptotic densities of $\widetilde{S}(p, K)$ under H_0 and H_1, respectively. So the critical value can be chosen using the classical likelihood ratio test theory. Another way to choose the critical value is to use the asymptotic normality of the test statistic $\widetilde{S}(p, K)$ under the null hypothesis, as in Aït-Sahalia and Jacod (2009).

An alternative approach is to minimize the sum of the probabilities of type I and type II errors. Unlike other scientific hypothesis testing problems, we do not have a strong preference here in distinguishing the null and alternative hypotheses. This is particularly the case when the test procedure is applied to detecting the location of possible jumps, as will be done in the next section. In this case, the critical value is the minimizer to $\alpha_n(x) + (1 - \beta_n(x))$. By Corollary 3, asymptotically the critical value minimizes the function

$$g(x) = \Phi\left(\frac{x - K^{p/2-1}}{\sqrt{\widehat{V}^c}}\right) + \Phi\left(\frac{1 - x}{\sqrt{\widehat{V}^j}}\right).$$

It can be shown that there exists a unique minimizer x_0 of $g(x)$, where $x_0 \in (1, K^{p/2-1})$ and solves the equation $\varphi_0(x) = \varphi_1(x)$ with φ_0 and φ_1 defined in (20). Denote the corresponding rejection region by

$$R = \{\widetilde{S}(p, K) \le x_0\}. \tag{21}$$

In the simulation study and real data analysis of this paper, we will use the rejection region defined in (21) when testing whether a fixed time interval has jumps or not.

5. Detecting jump locations

We have discussed how to test whether a fixed time interval contains jumps. Naturally, the next question is to locate these jumps if there are any. Throughout this section, we assume finite jump activity, that is, there are only finite jumps in $[0, T]$.

5.1. Test statistic as a process

Our idea of locating jumps is to apply our test to many small local time intervals in this period and then decide whether these time intervals contain jumps or not. Let the width of the local time interval be fixed at $2W$. For each $t \in [W, T - W]$, apply our new test statistic to the data in the time interval $[t - W, t + W]$, resulting in the value $\widetilde{S}(p, K)_t$. This defines a process $\widetilde{S}(p, K)_t$ over the time interval $[W, T - W]$. Suppose the number of jumps N_J in $[0, T]$ is finite. Denote the successive jump times by $J_1, J_2, \ldots, J_{N_J}$. Then for any $t \in (J_i - W, J_i + W)$, the interval $[t - W, t + W]$, over which $\widetilde{S}(p, K)_t$ is defined, contains the i-th jump. Define

$$\mathcal{A}^j = \bigcup_{i=1}^{N_J} (J_i - W, J_i + W), \tag{22}$$

which is the union of a finite number of time intervals with equal length $2W$. Let $\mathcal{E} = \{J_i - W, i = 1, \ldots, N_J\}$ be the union of left endpoints, and define $\mathcal{A}^c = [W, T - W] \setminus (\mathcal{A}^j \cup \mathcal{E})$.

Corollaries 1 and 2 give the results of our test statistic $\widetilde{S}(p, K)_t$ at a fixed time point t. For $t \in \mathcal{A}^j$, it should be approximately 1, and
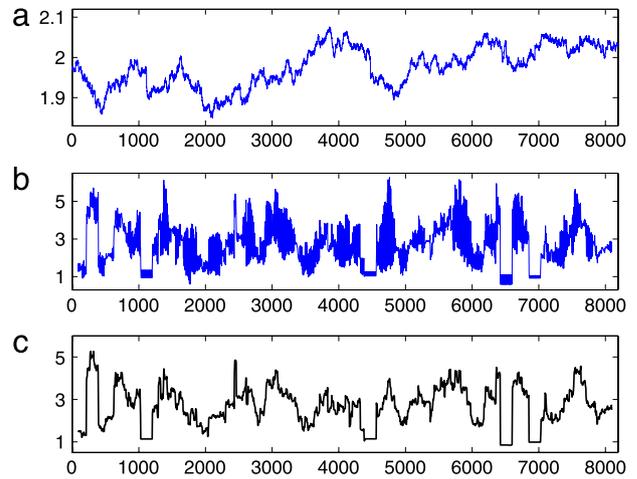

Fig. 2. Test statistics $\widetilde{S}(p, K)_t$ and $\widehat{S}(p, K)_{1,t}$, with $p = 4$ and $K = 2$ as processes of time. The price process X_t is generated from model (25), with $\Delta = 1$ min and $J_s = 0.05$. There are 4 jumps in total, which are located at time points 1115, 4474, 6512, and 6941. (a) X_t in a one-month period; (b) $\widehat{S}(p, K)_t$ with local window size $a_n = 90$; (c) $\widetilde{S}(p, K)_t$ with local window size $a_n = 90$.

for $t \in \mathcal{A}^c$, it should be approximately $K^{p/2-1}$. Fig. 2 shows the test statistic $\widetilde{S}(p, K)_t$ as a process of time for a simulated sample path, detailed in Section 7. In this simulation, we generate the asset price process X_t at time points $i\Delta$, $i = 0, 1, \ldots, n$, and calculate the test statistic $\widetilde{S}(p, K)_{i\Delta}$ for each $i \in \{[W/\Delta], \ldots, n - [W/\Delta]\}$ using data in the local window $[i\Delta - W, i\Delta + W]$. It is clear that the process $\widetilde{S}(p, K)_{i\Delta}$ is hovering around two values. The standard deviation of $\widetilde{S}(p, K)_{i\Delta}$ at the state of no jumps is much larger than that at the state with jumps. It is also clear that our new method is less volatile than the test statistic in Aït-Sahalia and Jacod (2009), which is consistent with our theory. From Fig. 2(c), it is clear that there are four jumps in the process, as there are four flat intervals with values around 1. The midpoints of these intervals are our estimated jump locations. Even though the process $\widetilde{S}(p, K)_t$ at several other locations has a value close to one, they are not jump points, as a jump point implies the process should be around one in an interval with width approximately $2W$.

5.2. Locating jumps using the FDR approach

Given observations $X_{i\Delta}$, $i = 0, 1, \ldots, n$ over a fixed time interval $[0, T]$, one can first construct the test statistic $\widetilde{S}(p, K)_T$ using all observations and then test whether there are jumps in $[0, T]$ using the rejection region (21). If the hypothesis of jumps cannot be rejected, one further uses the procedure described in the previous section to locate the jumps. In practical applications, one can apply the procedure using non-overlapping local intervals. More specifically, we divide the time interval $[0, T]$ into non-overlapping subintervals each with length $2a_n\Delta$ and construct test statistic $\widetilde{S}(p, K)_{(2i-1)a_n\Delta}$ using data in the interval $[(2i - 2)a_n\Delta, 2ia_n\Delta]$ for $i = 1, 2, \ldots, [n/(2a_n)]$. Then, the sequence of test statistics $\{\widetilde{S}(p, K)_{(2i-1)a_n\Delta}, i = 1, 2, \ldots, [n/(2a_n)]\}$ are expected to have normal distributions with different means and standard deviations at intervals with and without jumps. Thus, locating jumps is equivalent to a multiple comparison problem. Since $\widetilde{S}(p, K)_{(2i-1)a_n\Delta}$ has a smaller variance (see Fig. 1) when there are jumps in $[2(i - 1)a_n\Delta, 2ia_n\Delta]$, the null hypotheses for this multiple-comparison problem are chosen to be:

H_{0i} : There are jumps in the interval $\left(2(i - 1)a_n\Delta, 2ia_n\Delta\right]$,

$\quad i = 1, \ldots, [n/(2a_n)]$.

For each H_{0i}, the corresponding test statistic is $\widetilde{S}(p, K)_{(2i-1)a_n\Delta}$, whose null distribution is asymptotically Gaussian with mean 1 and variance

$$\frac{(2K-1)(K-1)p^2\Delta}{6K} \frac{D(2p-2)_{2ia_n\Delta} - D(2p-2)_{2(i-1)a_n\Delta}}{\left[B(p)_{2ia_n\Delta} - B(p)_{2(i-1)a_n\Delta}\right]^2}$$

by Theorem 1. It is well known that controlling the size of each individual null hypothesis will result in low power in the multiple-comparison problem, especially when there are many hypotheses. Thus, using the rejection point given in (21) is not realistic here. Therefore, we propose to use the adaptive control of the False Discovery Rate (FDR) procedure by Benjamini and Hochberg (1995) to control the type I error.

The false discovery rate is defined as $E(V/R)$, where R is the number of hypotheses rejected in total and V is the number of hypotheses rejected by mistake. The following procedure is proposed by Benjamini and Hochberg (1995) to control the FDR:

(1) Specify an allowable false discovery rate α.
(2) Estimate the total number of true null hypotheses, i.e., the number of intervals with jumps N_J^* and denote the estimated value as $\widehat{N}_J^*$.
(3) Calculate the p-value P_i corresponding to the null hypothesis H_{0i}. Rank the $[n/(2a_n)]p$-values from low to high: $P_{(1)} \le P_{(2)} \le \cdots \le P_{([n/(2a_n)])}$.
(4) Let k be the largest i for which $P_{(i)} \le i\alpha/\widehat{N}_J^*$. Reject all H_{0i}, $i = 1, \dots, k$.

In practice, N_J^* is unknown and needs to be estimated. We propose to estimate N_J^* as

$$\widehat{N}_J^* = \frac{\sum_{i=1}^{[n/(2a_n)]} 1(P_i > c)}{1-c}, \tag{23}$$

where $c \in (0, 1)$ is some constant, and $1(P_i > c)$ is a indicator function taking a value 1 if $P_i > c$ and taking a value 0 if $P_i \le c$. To understand why this is an estimator of N_J^* note that among $[n/(2a_n)]$ hypotheses to be tested, the P-values of true nulls are uniformly distributed over the interval [0, 1]. For large c, it is reasonable to assume that all P-values falling in $[c, 1]$ are contributed by the true null. Then theoretically, the P-value falling in $[c, 1]$ has density $(1 - c)N_J^*/[n/(2a_n)]$. On the other hand, empirically, the estimated density of P-values on the interval $[c, 1]$ is $\sum_{i=1}^{[n/(2a_n)]} 1(P_i > c)/[n/(2a_n)]$. By matching the theoretical density with the empirical density, we obtain the estimator proposed above. In practical implementations, we find that the results are not sensitive to c as long as c is not very close to 0.

6. Simulation studies

Throughout this section, the power p is fixed at 4 as it is the smallest integer greater than 3 (required by Theorem 1). To simulate the stochastic diffusion process, the Euler scheme is employed. We discard the burn-in period—i.e., the first 500 data points of the whole series—to avoid the starting value effect. More accurate simulations can be obtained by using the methods described in Fan (2005). To ease the presentation, we use "FF" to denote our new test, "AJ" to denote the Aït-Sahalia and Jacod test, "LM" to denote the Lee and Mykland (2008) test, "BNS" to denote the Barndorff-Nielsen and Shephard (2006) test, and "JO" to denote the Jiang and Oomen (2008) test.

6.1. Continuous diffusion process

The first model is the following continuous stochastic volatility process, which is taken from Aït-Sahalia and Jacod (2009):

$$dX_t/X_t = \sigma_t dW_t,$$
$$v_t = \sigma_t^2, \qquad dv_t = \kappa(\beta - v_t)dt + \gamma v_t^{1/2}dB_t, \tag{24}$$

Table 1
Means and standard deviations of FF test and AJ test when data are generated from model (24).

Frequency		$K = 2$	$K = 3$	$K = 4$
30 s	AJ	2.0006 (0.0800)	3.0068 (0.1633)	4.0188 (0.2608)
	FF	2.0020 (0.0568)	3.0042 (0.1214)	4.0059 (0.1960)
1 min	AJ	2.0094 (0.1132)	3.0029 (0.2364)	4.0111 (0.3591)
	FF	2.0047 (0.0842)	3.0023 (0.1703)	3.9965 (0.2739)
2 min	AJ	1.9973 (0.1554)	2.9826 (0.3225)	3.9880 (0.5111)
	FF	1.9942 (0.1127)	2.98884 (0.2357)	3.9821 (0.3827)
3 min	AJ	1.9964 (0.1917)	2.9971 (0.3928)	4.0203 (0.6109)
	FF	1.9965 (0.1397)	2.9946 (0.2930)	3.9973 (0.4836)

where W_t and B_t are both Brownian motions and $E[dW_t dB_t] = \rho dt$. We simulate 1000 sample paths of prices over a half-month $\left(T = \frac{1}{24}\right)$ period with parameters $\beta = 0.5^2$, $\gamma = 0.5$, $\kappa = 5$, and $\rho = -0.5$. The sampling frequencies are $\Delta = 30$ s, 1 min, 2 min, and 3 min.

In each of the 1000 simulations, for each $K = 2$, 3, and 4, the test statistic $\widehat{S}(p, K)_\ell$ and our new test statistic $\widetilde{S}(p, K)$ are computed to test whether there are any jumps in the half-month interval. We compare the sample means and standard deviations of $\widehat{S}(p, K)_\ell$ and $\widetilde{S}(p, K)$ across the 1000 simulations in Table 1. Although for each fixed K there are K test statistics $\widehat{S}(p, K)_\ell$, $\ell = 1, \dots, K$, the simulation results show that they all have very similar means and variances, as expected. Thus, only the mean and standard deviation of the first one of them are presented in Table 1. It is easy to see that the standard deviations of our new test statistic are smaller than those of $\widehat{S}(p, K)_\ell$, while the mean values are approximately the same. The results are in line with our asymptotic theory.

We next check the probabilities of type I error of the AJ test and the FF test. To better compare these two tests, we first calculate the rejection point x_{FF} of the FF test using (21), and then we calculate the rejection point of the AJ test according to (19). Thus, the FF test and the AJ test have the same type I error. Both the AJ test and the FF test using all four values of sampling intervals Δ and different choices of K have a probability of type I error 0, which is consistent with our Theorem 3. As a comparison, we also list the probabilities of type I error of the LM test, the BNS test, and the JO test with three different significance levels: $\alpha = 0.1, 0.01$, and 0.001. The rejection region of the LM test is calculated by using the distribution derived in Lemma 1 of Lee and Mykland (2008), the rejection point of the BNS test is calculated based on the asymptotic normal distribution of the adjusted ratio jump test proposed in Section 2.2 of Barndorff-Nielsen and Shephard (2006), and the rejection point of the JO test is derived by using the asymptotic normal distribution of the ratio test given in Theorem 2.1 of their paper. For all three tests, if there are any parameters that need to be estimated, we follow the suggestions of the original papers to estimate them. In all cases, the LM test, BNS test, and JO test have higher probabilities of type I error, which, as expected, are close to the size of the test.

6.2. Diffusion process with Poisson jumps

The model we use to generate the data is

$$dX_t/X_t = \sigma_t dW_t + J_t dN_t,$$
$$v_t = \sigma_t^2, \qquad dv_t = \kappa(\beta - v_t)dt + \gamma v_t^{1/2}dB_t, \tag{25}$$

where N_t is a Poisson process with intensity $\lambda = 12 \times 4$, J_t measures the jump size, and W_t and B_t are both Brownian motions with $E[dW_t dB_t] = \rho dt$. This model is similar to that in Aït-Sahalia and Jacod (2009). In the simulations, the parameters are chosen to be the same as before; that is, $\beta^{1/2} = 0.5$, $\gamma = 0.5$, $\kappa = 5$, and $\rho = -0.5$. The jump size J_t is generated as $J_t = 0.01\beta^{1/2}U$, where U is a random variable uniformly distributed over the interval

Table 2
Probabilities of type I errors of FF test, AJ test, LM test, and BNS test when data are generated from model (24).

Frequency		$K = 2$	$K = 3$	$K = 4$		$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$
	AJ	0	0	0	BNS	0.090	0.014	0
30 s	FF	0	0	0	LM	0.121	0.015	0.001
					JO	0.219	0.024	0.002
	AJ	0	0	0	BNS	0.106	0.010	0.001
1 min	FF	0	0	0	LM	0.110	0.010	0
					JO	0.219	0.026	0.002
	AJ	0	0	0	BNS	0.111	0.005	0.001
2 min	FF	0	0	0	LM	0.144	0.011	0.002
					JO	0.210	0.020	0.003
	AJ	0	0	0	BNS	0.120	0.010	0
3 min	FF	0	0	0	LM	0.145	0.012	0.002
					JO	0.219	0.021	0.001

Table 3
Means and standard deviations of FF test and AJ test when data are generated from model (25).

Frequency		$K = 2$	$K = 3$	$K = 4$
30 s	AJ	1.1616 (0.1932)	1.3379 (0.3392)	1.5006 (0.4886)
	FF	1.1639 (0.1691)	1.3314 (0.3158)	1.4952 (0.4553)
1 min	AJ	1.2736 (0.2486)	1.5225 (0.4259)	1.8003 (0.6268)
	FF	1.2687 (0.2182)	1.5271 (0.3957)	1.7886 (0.5752)
2 min	AJ	1.3921 (0.2944)	1.7825 (0.5425)	2.191 (0.7852)
	FF	1.3872 (0.2496)	1.7832 (0.4764)	2.1747 (0.7047)
3 min	AJ	1.4896 (0.3431)	1.9796 (0.6188)	2.4758 (0.9101)
	FF	1.4846 (0.2815)	1.9718 (0.5391)	2.4617 (0.8057)

$[-2, 1] \cup [1, 2]$. Thus, the jump size is at most 0.02 times the average volatility level and at least 0.01 times the average volatility level. Since we consider a half-month period ($T = 1/24$) and $\lambda = 12 \times 4$, on average there are 2 jumps in this period.

With the simulated sample paths, we first compare the means and standard deviations of $\widehat{S}(p, K)$ to those of $\widehat{S}(p, K)_\ell$. Due to the similarities of $\widehat{S}(p, K)_\ell$ for $\ell = 1, \ldots, K$, only the results of the first one are presented. Table 2 summarizes the comparison results. The conclusions are the same as before: our new test statistic has the same mean values but smaller standard deviations.

To further compare the test statistics, we calculate the probabilities of type II errors of both methods with the type I error fixed at the same level. The rejection points of the FF test are calculated by using (21). The rejection point of the AJ test is calculated by using (19) to ensure that it has approximately the same type I error. The comparison results are shown in Table 4. We see that the probability of type II error increases when the data are sampled less frequently, as expected. Table 4 also shows that our new test statistic outperforms the AJ test statistic in all cases. The probabilities of type II error of the LM, BNS, and JO tests with significance level $\alpha = 0.1$, 0.01, and 0.001 are listed in the last three columns of Table 4. Recall that with critical values x_{AJ} and x_{FF}, the AJ and the FF tests both have zero probability of type I error (see Table 2). To fairly compare the type II error, all tests should be evaluated at a significance level smaller than $\alpha = 0.001$. In fact, as shown in Tables 2 and 3, although the LM test has the smallest probabilities of type II errors, it has much larger probabilities of type I errors (see Table 2) than the FF test. To better compare the type II error of the FF test with the LM test, we choose the critical value of the FF test in a way such that the FF test has the same type I error as that of the LM test when $\alpha = 0.1$. That is, we fix the type I error of the FF test at levels 0.121, 0.110, 0.144 and 0.145 for $\Delta = 30$ s, 1 min, 2 min, and 3 min, respectively. These corresponding type II errors are listed in the table and marked as FF*. So we only need to compare FF* with the LM test when $\alpha = 0.1$ for type II errors. It can be seen that with the same type I error, our FF test has the smallest type II errors when $\Delta = 30$ s and 1 min, and the LM test has the smallest type II errors when the sampling frequency is lower.

6.3. Diffusion process with Cauchy jumps

In this subsection we consider a diffusion process with Cauchy jumps. We generate data from the following model

$$dX_t/X_t = \sigma_t dW_t + J dY_t, \tag{26}$$

where σ_t is simulated in the same way as that in Section 6.2, $J > 0$ measures the size of the jumps relative to the volatility level, and Y_t is a Cauchy process with characteristic function $E \exp(iuY_t) = \exp\{-t|u|/2\}$. Note that the above model has infinite jump activity. We consider a half month period ($T = 1/24$) and set $J = 0.8$. To save space, we omit the means and standard deviations of the AJ and FF tests. The probabilities of type II errors for the five tests are summarized in Table 5. As discussed in the last section, in order to make the comparison of the probabilities of type II errors fair, the LM, BNS and JO tests should be evaluated at a significance level smaller than $\alpha = 0.001$. To make the comparison more precise, we also calculate the type II errors of the FF test when its type I errors are fixed at the same levels as those for the LM test with $\alpha = 0.1$ (see Table 2). These type II errors are marked as FF* in the table. Thus, we only need to compare FF* with the LM test with $\alpha = 0.1$. It can be seen that in this setting, the FF test outperforms the BNS and JO tests, and the LM test has the best performance.

We would like to note that the LM test is derived under the assumption that under the null hypothesis, the returns are normally distributed. Such an assumption is hard to validate for data even at daily frequency. All of our simulations fall in such a scenario, which gives advantages to the LM test. On the other hand, the AJ and FF tests do not require such an assumption.

6.4. Estimation of jump locations

In this section, the procedure proposed in Section 5 is used to identify the locations of jumps in a one-month ($T = 1/12$) period. The data are simulated from model (25) with parameter $\lambda = 12 \times 21$—i.e., there is one jump per day on average. The same as that in the previous section, the jump size J_t is generated as $J_t = 0.01\beta^{1/2}U$ with U being a random variable uniformly distributed over $[-2, -1] \cup [1, 2]$. We study the performance of our new test procedure with three different sampling frequencies $\Delta = 1$ min, 3 min, and 5 min.

We use a two-step method to identify the exact location of jumps. In the first step, we divide the one-month period into many non-overlapping intervals with equal length, and then identify the jump intervals using the FDR approach discussed in Section 6. In the second step, for each identified jump interval we locate the jumps by comparing the magnitude of the increment $X_{i\Delta} - X_{(i-1)\Delta}$. More specifically, in the first step we choose the window size $2W = 2a_n\Delta$ with a_n being a positive integer, and we divide the one-month period into $[n/(2a_n)]$ non-overlapping intervals with

Table 4
Probabilities of type II errors of FF test, AJ test, LM test, and BNS test when data are generated from model (25). FF* represents the FF test with type I errors fixed at levels 0.121, 0.110, 0.144, and 0.145 for $\Delta = 30$ s, 1 min, 2 min, and 3 min, respectively.

Frequency		$K = 2$	$K = 3$	$K = 4$		$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$
30 s	AJ	0.456	0.565	0.653	BNS	0.103	0.262	0.458
	FF	0.095	0.11	0.125	LM	0.015	0.02	0.021
	FF*	0	0	0	JO	0.038	0.063	0.084
1 min	AJ	0.669	0.751	0.825	BNS	0.282	0.6	0.792
	FF	0.265	0.302	0.333	LM	0.025	0.029	0.032
	FF*	0.010	0.005	0.003	JO	0.084	0.149	0.206
2 min	AJ	0.816	0.891	0.915	BNS	0.566	0.851	0.951
	FF	0.637	0.692	0.718	LM	0.019	0.054	0.069
	FF*	0.038	0.042	0.048	JO	0.15	0.301	0.396
3 min	AJ	0.874	0.926	0.956	BNS	0.671	0.909	0.981
	FF	0.637	0.692	0.718	LM	0.042	0.093	0.124
	FF*	0.115	0.110	0.111	JO	0.262	0.437	0.561

Table 5
Probabilities of type II errors of FF test, AJ test, LM test, and BNS test when data are generated from model (26). FF* represents the FF test with type I errors fixed at levels 0.121, 0.110, 0.144, and 0.145 for $\Delta = 30$ s, 1 min, 2 min, and 3 min, respectively.

Frequency		$K = 2$	$K = 3$	$K = 4$		$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$
30 s	AJ	0.280	0.312	0.359	BNS	0.041	0.073	0.106
	FF	0.148	0.159	0.167	LM	0	0.003	0.008
	FF*	0.015	0.020	0.017	JO	0.058	0.104	0.122
1 min	AJ	0.380	0.437	0.453	BNS	0.069	0.137	0.174
	FF	0.210	0.219	0.232	LM	0.009	0.015	0.028
	FF*	0.045	0.047	0.041	JO	0.089	0.147	0.174
2 min	AJ	0.506	0.533	0.550	BNS	0.117	0.200	0.250
	FF	0.281	0.284	0.293	LM	0.020	0.048	0.071
	FF*	0.085	0.081	0.076	JO	0.129	0.199	0.236
3 min	AJ	0.537	0.571	0.604	BNS	0.153	0.248	0.310
	FF	0.312	0.325	0.333	LM	0.033	0.082	0.118
	FF*	0.126	0.121	0.116	JO	0.148	0.243	0.283

equal length. In each of these intervals, our new test statistic FF and the test statistic AJ are applied using the data points in that interval. Section 6 shows that the variance of either test statistic is much smaller at the jump locations than at the continuous time points. Thus, the null hypothesis corresponding to the i-th time interval is chosen to be

$H_{0,i}$: There are jumps in i-th time interval.

If the i-th null hypothesis is rejected, we conclude that there is no jump in the i-th interval. The procedure introduced in Section 6 is used to control the FDR of this multiple-comparison problem. The FDR is controlled at 5% and the number of true hypotheses is estimated by using (23) with $c = 0.05$. Practical implementation suggests that the result is not sensitive to c as long as c is not very close to 0. Since the local window size a_n chosen in the first step is usually small, it is reasonable to assume that with high probability each identified jump interval has at most one jump. Thus in the second step we locate the jump in each interval by identifying the largest increment $|X_{i\Delta} - X_{(i-1)\Delta}|$ in that interval. The BNS and JO tests are applied in the same way to locate jumps.

Since we are interested in classifying between "jump" and "non-jump", our problem is essentially a two-class classification problem. Thus, we borrow measures from the two-class classification—that is, sensitivity and specificity—to evaluate our test procedure. These two measures are defined as

$$\text{Sensitivity} = \frac{\text{\# of correctly identified jumps by a test}}{\text{\# of true jumps in total}},$$

$$\text{Specificity} = \frac{\text{\# of correctly identified non-jumps by a test}}{\text{\# of true non-jumps in total}}.$$

Although larger values of these two measures indicate a better performance of a test, in practice there are tradeoffs between sensitivity and specificity. To understand this, just imagine that if a

test rejects all null hypotheses, then the sensitivity defined above will be 1, but the specificity defined above will be 0. On the other hand, if a test fails to reject any null hypothesis, then the sensitivity is 0 and the specificity is 1. So to compare different tests, we need to combine the results of sensitivity and specificity.

Specificity for the LM test is different from the specificity for other tests. To understand this, notice that the FF, AJ, BNS, and JO tests are all defined over local windows with $2a_n$ observations, while the LM test is defined over each sampling interval $[i\Delta, (i + 1)\Delta]$. If according to the FF, AJ, or BNS test, there is no jump in a local window, then none of the sampling intervals in that local window contain jumps. Thus, the specificities for these four tests are defined by considering each local window as a test unit, while the specificity for the LM test is defined by considering each sampling interval as a test unit. To make it more comparable, we redefine the specificity of the LM test using each local window as a unit as well. That is, if any time point in a local window is identified by the LM test as a jump time point, then the whole interval is identified by the LM test as a jump interval. Thus, the specificity we use in this paper takes the form

Specificity
$$= \frac{\text{\# of correctly identified non-jump intervals by a test}}{\text{\# of true non-jump intervals in total}}.$$

Since sensitivity and specificity usually trade off between each other, we report the weighted averages of sensitivities and specificities. For window sizes $2a_n = 30, 60, 90,$ and 120, the sensitivities and specificities of the test statistics are computed and the weighted averages of them are calculated. To save space, we only list the results when the weight for sensitivity is fixed at 0.5. See Tables 6–9. The significance level α for the LM test is chosen to be 0.001, as explained before.

It can be seen from Table 6 that the weighted averages of sensitivity and specificity of the FF test are larger than other tests in

Table 6
Weighted averages of means of sensitivities and specificities when $\Delta = 1$ min over 1000 simulations. The weight for sensitivity is 0.5, and the data are generated from model (25) with $\lambda = 12 \times 21$.

		$K = 2$	$K = 4$	$K = 6$	BNS	JO
$a_n = 15$	AJ	0.5069	0.559	0.5557	0.547	0.6075
	FF	0.5930	0.6179	0.6078		
$a_n = 30$	AJ	0.5383	0.5893	0.582	0.5595	0.5945
	FF	0.6286	0.6090	0.5884		
$a_n = 45$	AJ	0.5634	0.5952	0.5811	0.5595	0.5842
	FF	0.6209	0.5633	0.543		
$a_n = 60$	AJ	0.5872	0.5885	0.57	0.5562	0.5744
	FF	0.5981	0.535	0.5194		
	LM	0.5897				

Table 7
Weighted averages of means of sensitivities and specificities when $\Delta = 3$ min over 1000 simulations. The weight for sensitivity is 0.5, and the data are generated from model (25) with $\lambda = 12 \times 21$.

		$K = 2$	$K = 4$	$K = 6$	BNS	JO
$a_n = 15$	AJ	0.3308	0.3939	0.3958	0.5049	0.516
	FF	0.4196	0.4660	0.4658		
$a_n = 30$	AJ	0.3542	0.4282	0.4354	0.504	0.5093
	FF	0.4706	0.4976	0.4964		
$a_n = 45$	AJ	0.3892	0.4636	0.4669	0.5034	0.5062
	FF	0.4936	0.5025	0.5014		
$a_n = 60$	AJ	0.4154	0.4782	0.483	0.5032	0.5043
	FF	0.5002	0.5015	0.5008		
	LM	0.5043				

Table 8
Weighted averages of means of sensitivities and specificities when $\Delta = 5$ min over 1000 simulations. The weight for sensitivity is 0.5, and the data are generated from model (25) with $\lambda = 12 \times 21$.

		$K = 2$	$K = 4$	$K = 6$	BNS	JO
$a_n = 15$	AJ	0.2645	0.3392	0.3438	0.5017	0.502
	FF	0.3635	0.4221	0.4283		
$a_n = 30$	AJ	0.3002	0.3897	0.4009	0.5005	0.5001
	FF	0.4299	0.4767	0.4794		
$a_n = 45$	AJ	0.343	0.4375	0.4451	0.4996	0.4992
	FF	0.4669	0.4954	0.4971		
$a_n = 60$	AJ	0.3714	0.462	0.4681	0.4989	0.4987
	FF	0.4797	0.4988	0.4993		
	LM	0.5013				

Fig. 3. Mean of weighted averages of sensitivity and specificity for FF, AJ, and LM tests over 1000 simulations when $\Delta = 1$ min and $a_n = 30$. The x-axis represents the weight for sensitivity. The data are generated from model (25) with $\lambda = 12 \times 21$.

Fig. 4. Mean of weighted averages of sensitivity and specificity for FF, AJ, and LM tests over 1000 simulations when $\Delta = 3$ min and $a_n = 30$. The x-axis represents the weight for sensitivity. The data are generated from model (25) with $\lambda = 12 \times 21$.

inspecting the simulation results, we found that the LM, BNS, and JO tests classify almost all intervals as non-jump intervals, resulting in 1 specificity and 0 sensitivity. This can be confirmed by Fig. 4, which shows the weighted averages of the FF, AJ, and LM tests as a function of weights when $a_n = 30$. The same as before, the x-axis represents the weight for sensitivity. It can be seen that the FF and AJ tests have much higher sensitivities. And most of the time, the FF test has larger weighted averages than the other two tests. The BNS and JO tests perform very similarly to the LM test in this setting. When $\Delta = 5$ min, the results are very similar to the ones for $\Delta = 3$ min.

6.5. Impact of microstructure noise

In this subsection, we compare the performance of the AJ, FF, LM, BNS, and JO tests when market microstructure noise is present. The underlying asset price process is generated from (25) over a one-month period ($T = 1/12$), and the observed asset price process is generated as

$$X_t^* = X_t + \varepsilon_t, \tag{27}$$

where $\varepsilon_t \sim_{\text{i.i.d.}} N(0, \sigma_0^2 \beta)$ are the market microstructure noises and β is defined in (25) representing the mean volatility level. Three different noise levels are considered: small ($\sigma_0 = 0.01\%$), medium ($\sigma_0 = 0.1\%$), and large ($\sigma_0 = 1\%$). Due to the space limit, we only present the results when the local window size is $2a_n = 30$ and $\Delta = 1$ minute. The comparison results can be found in Table 9.

most cases when $K = 2$. Unreported simulation results show that the FF and AJ tests have much higher sensitivities than other tests, while the LM test usually has low sensitivity and high specificity. To better illustrate the idea, we plot the weighted averages of the FF, AJ, and LM tests when $a_n = 30$ in Fig. 3. The x-axis represents the weight for sensitivity, ranging from 0 to 1. It can be seen that most of the time, the FF test has larger weighted averages than either the AJ test or LM test. The LM test has almost 1 specificity and less than 0.2 sensitivity, indicating that the LM test classifies most intervals as non-jump intervals.

When the sampling frequency Δ becomes 3 min, all tests have worse performance. In fact, as shown in Table 7, the weighted averages become smaller than those in Table 6. Note that for LM, BNS, and JO tests, their weighted averages are close to 0.5. After
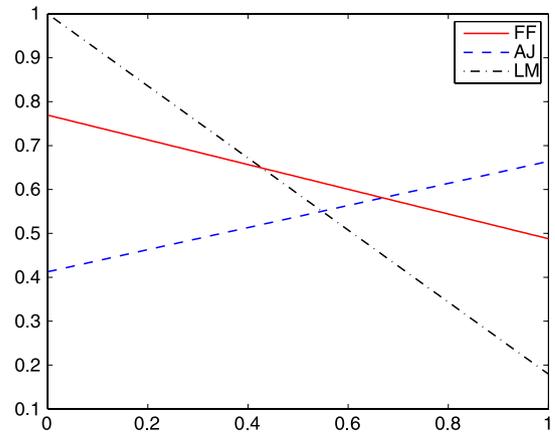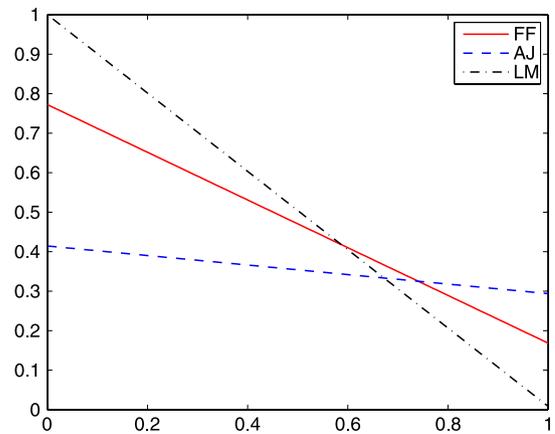
Table 9
Means of sensitivities and specificities of different methods when small, medium, and large microstructure noises are presented. The local window size is $2a_n = 30$, and the sampling frequency $\Delta = 1$ min.

| $a_n = 15$		Sensitivity			Specificity		
		$K = 2$	$K = 4$	$K = 6$	$K = 2$	$K = 4$	$K = 6$
Small	AJ	0.7404	0.628	0.5933	0.2568	0.4693	0.4946
	FF	0.6275	0.4854	0.4451	0.5282	0.7126	0.7304
	BNS		0.1044			0.9971	
	LM		0.1769			1	
	JO		0.2356			0.9749	
Medium	AJ	0.7389	0.6498	0.6147	0.152	0.3423	0.3845
	FF	0.6698	0.5473	0.5002	0.3175	0.5355	0.5842
	BNS		0.0004			0.996	
	LM		0.1218			1	
	JO		0.2			0.9776	
Large	AJ	0.0582	0.0173	0.011	0.5357	0.8685	0.9083
	FF	0.0121	0.0009	0.0006	0.9114	0.995	0.996
	BNS		0.0017			0.9964	
	LM		0			1	
	JO		0.0021			0.9968	
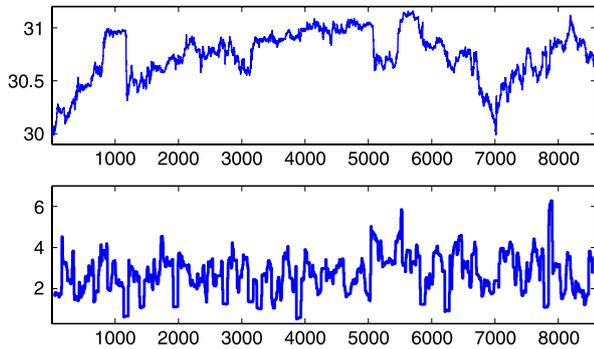
Fig. 5. Top panel: stock price data of Microsoft Corporation from May 1, 2007 to May 31, 2007 with 1-min frequency. Bottom panel: $\widetilde{S}(p, K)_t$ as a process of time with $2a_n = 78$.

With large noise, almost all tests fail to identify any jumps. For small and moderate noise levels, the LM, BNS, and JO tests tend to miss most of the true jumps and thus have low sensitivities, while the FF and AJ tests have much higher sensitivities.

7. Data analysis

We apply our new test to the high-frequency stock price data of Microsoft Corporation from May 1, 2007 to May 31, 2007. The sampling frequency is chosen to be 1 min. The total sample size is 8591. We first apply our new test with $K = 2$ to the entire data set, and the value of our new test statistic is 1.195. The rejection region of the null hypothesis that there are no jumps in this one-month period is obtained by using (21), and the value of x_0 is 1.244. Thus, the null hypothesis is rejected. We then locate the jumps. To this end, we calculate our new test $\widetilde{S}(p, K)_t$ at each data point with indices in $\{a_n, a_n + 1, \ldots, 8591 - a_n\}$. Jumps occurring when market closes and opens are not very interesting. To exclude these jumps, we choose the local window size $2a_n = 78$, that is, there are roughly 5 local windows per day for the FF test. As discussed in Section 5, if there is no microstructure noise the process $\widetilde{S}(p, K)_t$ should hover around two values 1 and 2 with some flat regions corresponding to the jumps. Due to the influence of the microstructure noise, the sample path of $\widetilde{S}(p, K)_t$ is a little bit wiggling. To avoid the influence of the noise, we smooth the process $\widetilde{S}(p, K)_t$ using the wavelet method. Fig. 5 shows the smoothed curve of $\widetilde{S}(p, K)_t$ as a process of time. We see that there are a few flat regions in the plot that may indicate jump intervals. However, due to the market microstructure noise in the real data, it

Table 10
Identified time periods with jumps using the stock price data of Microsoft Corporation from May 1, 2007 to May 31, 2007 with 1-min frequency.

5/1/2007	10:52:55
5/2/2007	9:38:57
5/2/2007	14:37:57
5/4/2007	9:39:58
5/4/2007	13:41:57
5/11/2007	9:52:59
5/14/2007	10:50:58
5/15/2007	10:02:59
5/17/2007	15:34:57
5/30/2007	13:59:58
5/31/2007	13:30:57

is still somewhat hard to inspect which time intervals have jumps by solely looking at the plot. Thus, we further divide the one-month period into many small non-overlapping time intervals with equal length $2a_n = 78$. The FDR procedure is employed to identify the intervals with jumps with false discovery rate controlled at 5% level. Among the 110 time periods, 11 are detected with jumps, and the corresponding jump locations are listed in Table 10. We see that most of these identified intervals by the FDR approach correspond to a flat region in the plot of $\widetilde{S}(p, K)_t$, which is in line with our theory.

8. Discussions

We have observed that several nonparametric test statistics similar to the one in Aït-Sahalia and Jacod (2009) can be constructed to detect whether a continuous-time process has a continuous sample path or not. We have derived their asymptotic joint distribution, which shows that they are not highly correlated. As a consequence, we have proposed to linearly combine these test statistics to form a new one that has the same asymptotic properties as the original ones but with smaller variance. We have given explicitly the optimal weights. The critical region for testing the existence of jumps in a fixed time period is constructed by minimizing the summation of type I and type II errors. A new test procedure based on a test process, which is transformed from the original asset price process by applying the new test locally over the time, is proposed. This test process concentrates around two known constants indicating whether a jump occurs at a time point. Thus, the problem of jump identification becomes a multiple-comparison problem. The FDR procedure has been proposed to control the type I error. Simulation studies and a real data application have justified the theoretical results and further demonstrated the performance of our new test method.

Acknowledgments

Appendix. Proofs

A.1. A general result

Before proving Theorem 1, we first prove a more general result. Let n be the number of observations in the time interval $[0, t]$. Define

$$V^n(f)_t = \sum_{i=1}^{n} f(\Delta_i^n X),$$

$$\overline{V}_{\ell+1}^n(f)_t = \sum_{j=1}^{[n/K]} f(X_{(\ell+jK)\Delta} - X_{(\ell+(j-1)K)\Delta}),$$

for $\ell = 0, \ldots, K - 1$, and $V(f)_t = \sum_{s \leq t} f(\Delta X_s)$. Consider an auxiliary space $(\Omega', \mathcal{F}', \mathbb{P}')$ which supports the following variables and processes:

- a sequence of uniform random variables on $[0, 1]$ denoted by (κ_q);
- $3K$ sequences of i.i.d. standard Gaussian variables denoted by $(\overline{U}_{q,-K+1}), (\overline{U}_{q,-K+2}), \ldots, (\overline{U}_{q,2K-1})$; another two sequences of i.i.d. standard Gaussian variables denoted by (U_q) and (U_q');
- a sequence of uniform random variables on the finite set $\{0, 1, \ldots, K - 1\}$ denoted by (L_q);
- three standard Brownian motions $\overline{W}^1$, $\overline{W}^2$, and $\overline{W}^3$;

and all these processes or variables are mutually independent. Next define $\widetilde{\Omega} = \Omega \times \Omega'$, $\widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}'$, $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$. Extend the variables $X_t, b_t, \ldots$ defined on Ω and $\overline{W}^1, \overline{W}^2, \overline{W}^3, U_q, \ldots$ defined on Ω' to the product space $\widetilde{\Omega}$. For simplicity, we use the same notations to denote these variables. Hereinafter we write $\widetilde{\mathbb{E}}$ for the expectation with respect to $\widetilde{\mathbb{P}}$. Let $(S_q)_{q \geq 1}$ be the successive jump times of X, which are stopping times. Define $\widetilde{\mathcal{F}}_t$ as the smallest right continuous filtration of $\widetilde{\mathcal{F}}$ containing the filtration $(\mathcal{F}_t)$, and with respect to $\widetilde{\mathcal{F}}_t$, the processes $\overline{W}_t^1$, $\overline{W}_t^2$ and $\overline{W}_t^3$ are adapted, and all random variables and processes defined above are $\widetilde{\mathcal{F}}_{S_q}$-measurable for all q. Therefore, we have an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, \widetilde{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Then, the following results hold for $V^n(f)_t$, $\overline{V}_\ell^n(f)_t$, $\ell = 1, \ldots, K$, and $V(f)_t$.

Theorem 4. Let f be C^2 with $f(0) = f'(0) = f''(0) = 0$. Given Assumption 1, the following multivariate process

$$\left(\frac{1}{\sqrt{\Delta}}(V^n(f)_t - V(f)_{\Delta[t/\Delta]}), \frac{1}{\sqrt{\Delta}}(\overline{V}_1^n(f)_t - V(f)_{K\Delta[n/K]}), \right.$$

$$\left. \ldots, \frac{1}{\sqrt{\Delta}}(\overline{V}_K^n(f)_t - V(f)_{(K[n/K]+(K-1))\Delta}) \right)$$

converges stably in law to the process $(Z(f')_t, Z(f')_t + Z_1'(f')_t, \ldots, Z(f')_t + Z_K'(f')_t)$, where

$$Z(f)_t = \sum_{q: S_q \leq t} f(\Delta X_{S_q}) R_q \quad \text{and}$$

$$Z_\ell'(f)_t = \sum_{q: S_q \leq t} f(\Delta X_{S_q})(R_-(q, \ell - 1) + R_+(q, \ell - 1))$$

with R_q, $R_-(q, \ell - 1)$ and $R_+(q, \ell - 1)$ defined in the proof below.

Proof. This proof, which has two steps, is an extension of the proof of Theorem 8 in Aït-Sahalia and Jacod (2009). First, we prove the results for a special class of f, and then we extend the results to a more general function f.

Step 1. Assume that f vanishes on $[-2K\varepsilon, 2K\varepsilon]$ for some $\varepsilon > 0$. Let S_q be the successive jump times of the Poisson process $\mu([0, t] \times \{x : \gamma(x) > \varepsilon\})$. Let ΔX_{S_q} be the size of the q-th jump, and set

$$X(\varepsilon)_t = X_t - \sum_{\{q: S_q \leq t\}} \Delta X_{S_q}.$$

Define $\Omega_n(t, \varepsilon)$ to be the set of all ω such that each interval $[0, t] \cap (i\Delta, (i + 2K - 1)\Delta]$ contains at most one $S_q(\omega)$, $|X(\varepsilon)_{i\Delta} - X(\varepsilon)_{(i-1)\Delta}| \leq 2\varepsilon$ for all $i \leq t/\Delta$, the first jump time $S_1(\omega) > K\Delta$, and $|X(\varepsilon)_{i\Delta} - X(\varepsilon)_{(i-2K+1)\Delta}| \leq 2\varepsilon$ for all $2K - 1 \leq i \leq t/\Delta$. Next, for each q, on the set $\{(iK + j)\Delta < S_q \leq (iK + j + 1)\Delta\}$ for $i \geq 1$ and $0 \leq j < K$, define

$$L(n, q) = j \quad \text{and} \quad M(n, q) = S_q/\Delta - (iK + j).$$

We further define

$$\alpha_-(n, q) = \frac{1}{\sqrt{\Delta}}(W_{S_q} - W_{(iK+j)\Delta}),$$

$$\alpha_+(n, q) = \frac{1}{\sqrt{\Delta}}(W_{(iK+j+1)\Delta} - W_{S_q}),$$

$$\beta_-(n, q, \ell) = \frac{1}{\sqrt{\Delta}}(W_{(iK+j)\Delta} - W_{(iK+\ell-K1\{\ell>j\})\Delta}),$$

$$\beta_+(n, q, \ell) = \frac{1}{\sqrt{\Delta}}(W_{(iK+\ell+K1\{\ell\leq j\})\Delta} - W_{(iK+j+1)\Delta}),$$

where $0 \leq \ell \leq K - 1$. For the process $X(\varepsilon)_t$, define the increments

$$R_-(n, q, \ell) = X(\varepsilon)_{(iK+j)\Delta} - X(\varepsilon)_{(iK+\ell-K1\{\ell>j\})\Delta} \quad \text{and}$$

$$R_+(n, q, \ell) = X(\varepsilon)_{(iK+\ell+K1\{\ell\leq j\})\Delta} - X(\varepsilon)_{(iK+j+1)\Delta}$$

with $0 \leq \ell \leq K - 1$, and

$$R_q^n = \Delta_{iK+j+1}^n X(\varepsilon) = X(\varepsilon)_{(iK+j+1)\Delta} - X(\varepsilon)_{(iK+j)\Delta}.$$

Finally, for each $\ell = 0, \ldots, K - 1$ we define

$$R_q = \sqrt{\kappa_q} U_q \sigma_{S_{q-}} + \sqrt{1 - \kappa_q} U_q' \sigma_{S_q},$$

$$R_{q,\ell}'^n = R_q^n + R_-(n, q, \ell) + R_+(n, q, \ell),$$

$$R_-(q, \ell) = \sum_{l=\ell-K1\{\ell>L_q\}}^{L_q} \overline{U}_{q,l} \sigma_{S_{q-}},$$

$$R_+(q, \ell) = \sum_{l=L_q+1}^{\ell+K1\{\ell\leq L_q\}} \overline{U}_{q,l} \sigma_{S_q}.$$

Using a similar idea as that in Aït-Sahalia and Jacod (2009)—that is, extending the proof of Lemma 6.2 of Jacod and Protter (1998)—we obtain

$$\left(L(n, q), M(n, q), \alpha_-(n, q), \alpha_+(n, q), \beta_-(n, q, 0), \right.$$

$$\beta_+(n, q, 0), \ldots, \beta_-(n, q, \ell), \beta_+(n, q, \ell), \ldots, \beta_-(n, q, K - 1),$$

$$\left. \beta_+(n, q, K - 1) \right)_{q \geq 1}$$

$$\xrightarrow{\mathcal{L}_-(s)} \left(L_q, \kappa_q, \sqrt{\kappa_q} U_q, \sqrt{1 - \kappa_q} U_q', \sum_{l=0}^{L_q} \overline{U}_{q,l}, \right.$$

$$\left. \sum_{l=L_q+1}^{K} \overline{U}_{q,l}, \ldots, \sum_{l=\ell-K1\{L_q<\ell\}}^{L_q} \overline{U}_{q,l}, \sum_{l=L_q+1}^{\ell+K1\{L_q\geq\ell\}} \overline{U}_{q,l}, \ldots, \right.$$

$$\left(\sum_{l=K1\{L_q\geq K-1\}-1}^{L_q} \bar{U}_{q,l}, \quad \sum_{l=L_q+1}^{K-1+K\{L_q\geq K-1\}} \bar{U}_{q,l}\right)_{q\geq 1},$$

where $\xrightarrow{\mathcal{L}-(s)}$ denotes the stable convergence in law. The above result yields

$$\frac{1}{\sqrt{\Delta}}\Big(R_q^n, R_-(n, q, 1), R_+(n, q, 1), \ldots,$$

$$R_-(n, q, K), R_+(n, q, K)\Big)_{q\geq 1}$$

$$\xrightarrow{\mathcal{L}-(s)} \Big(R_q, R_-(q, 1), R_+(q, 1), \ldots, R_-(q, K), R_+(q, K)\Big)_{q\geq 1}. \quad (28)$$

Since $f(x) = 0$ for $|x| \leq 2K\varepsilon$, we obtain that on the set $\Omega_n(t, \varepsilon)$ and for all $s \leq t$,

$$V^n(f)_s - V(f)_{\Delta[s/\Delta]} = \sum_{q, S_q \leq \Delta[s/\Delta]} \big[f(\Delta X_{S_q} + R_q^n) - f(\Delta X_{S_q})\big]$$

$$= \sum_{q, S_q \leq \Delta[s/\Delta]} f'(\Delta X_{S_q} + \widetilde{R}_q^n)R_q^n,$$

where $\widetilde{R}_q^n$ is between ΔX_{S_q} and $\Delta X_{S_q} + R_q^n$. Similarly, it can be shown that

$$\overline{V}_\ell^n(f)_s - V(f)_{K\Delta[s/(K\Delta)]+(\ell-1)\Delta}$$

$$= \sum_{q, S_q \leq \Delta[s/K\Delta]+(\ell-1)\Delta} f'(\Delta X_{S_q} + \tilde{R}'^n_{q,(\ell-1)})R'^n_{q,(\ell-1)},$$

where $\tilde{R}'^n_{q,\ell}$ is between ΔX_{S_q} and $\Delta X_{S_q} + R'^n_{q,\ell}$. Since R_q^n, $R_-(n, q, \ell)$ and $R_+(n, q, \ell)$ converge to 0 for all $\ell = 0, \ldots, K-1$, we have $\tilde{R}_q^n$ and $\tilde{R}'^n_{q,\ell}$ also converging to 0 and $\Omega_n(t, \varepsilon) \to \Omega$. The continuity of f' together with (28) leads to the results in Theorem 4.

Step 2. For the general case, the idea of the proof is the same as that in Aït-Sahalia and Jacod (2009).

Combining Steps 1 an 2 above yields the stable convergence results in Theorem 4. □

A.2. Proof of Theorem 1

The proof in Theorem 4 applied with the function $f(x) = |x|^p$ yields the stable convergence in law of the process

$$\left(\frac{1}{\sqrt{\Delta}}\big[\widehat{B}(p, \Delta)_t - B(p)_{\Delta[t/\Delta]}\big],\right.$$

$$\left.\frac{1}{\sqrt{\Delta}}\sum_{\ell=1}^{K} a_\ell\big[\widehat{B}(p, K\Delta)_{\ell,t} - B(p)_{K\Delta[t/K\Delta]+(\ell-1)\Delta}\big]\right)$$

to $\Big(Z(f')_t, Z(f')_t + \sum_{j=1}^{K} a_j Z'_j(f')_t\Big)$ with $f'(x) = p|x|^{p-1}\text{sgn}(x)$. Hence, if we let

$$Z(p)_t = Z(f')_t \quad \text{and} \quad Z'(p, K, \ell)_t = Z'_\ell(f')_t \quad (29)$$

with $Z(f')_t$ and $Z'(p, K, \ell)_t$ defined in Theorem 4, then

$$\widetilde{\mathbb{E}}(Z(p)_t^2|\mathcal{F}) = \frac{1}{2}p^2 \sum_{s\leq t} |\Delta X_s|^{2p-2}(\sigma_{s-}^2 + \sigma_s^2).$$

It has been proved in Aït-Sahalia and Jacod (2009) that for any positive integer K,

$$\big[B(p)_t - B(p)_{K\Delta[t/K\Delta]+(\ell-1)\Delta}\big]/\sqrt{\Delta} \xrightarrow{P} 0.$$

Then, the convergence results in Theorem 1 are proved.

We next calculate the variance and covariance of $Z'(p, K, \ell)_t$. Note that for $q_1 < q_2$, any pairs of random vectors $(R_-(q_1, \cdot), R_+(q_1, \cdot))$ and $(R_-(q_2, \cdot), R_+(q_2, \cdot))$ are independent, regardless of

the value of ℓ. Thus, we only need to consider the correlation between $R_-(q, \cdot)$ and $R_+(q, \cdot)$ for a fixed q. For $\ell_1 \leq \ell_2$,

$$\widetilde{\mathbb{E}}(R_-(q, \ell_1)R_-(q, \ell_2)|\mathcal{F})$$

$$= \widetilde{\mathbb{E}}\big[L_q - (\ell_1 - K1\{L_q < \ell_1\}) \vee (\ell_2 - K1\{L_q < \ell_2\})|\mathcal{F}\big]\sigma_{S_q-}^2$$

$$= \left\{\frac{K-1}{2} - \frac{1}{K}[K - (\ell_2 - \ell_1)](\ell_2 - \ell_1)\right\}\sigma_{S_q-}^2.$$

Similarly, we have

$$\widetilde{\mathbb{E}}(R_+(q, \ell_1)R_+(q, \ell_2)|\mathcal{F})$$

$$= \left\{\frac{K-1}{2} - \frac{1}{K}[K - (\ell_2 - \ell_1)](\ell_2 - \ell_1)\right\}\sigma_{S_q}^2.$$

Moreover, for any $0 \leq \ell_1, \ell_2 \leq K-1$,

$$\widetilde{\mathbb{E}}(R_-(q, \ell_1)R_+(q, \ell_2)|\mathcal{F}) = 0.$$

From the above three equations, we deduce that for any $\ell_1 \leq \ell_2$,

$$\widetilde{\mathbb{E}}(Z'(p, K, \ell_1)_t Z'(p, K, \ell_2)_t|\mathcal{F})$$

$$= \sum_{q:S_q\leq t} f(\Delta X_{S_q})^2 \widetilde{\mathbb{E}}(R_-(q, \ell_1 - 1)R_-(q, \ell_2 - 1)|\mathcal{F})$$

$$+ \sum_{q:S_q\leq t} f(\Delta X_{S_q})^2 \widetilde{\mathbb{E}}(R_+(q, \ell_1 - 1)R_+(q, \ell_2 - 1)|\mathcal{F})$$

$$= p^2\left\{\frac{K-1}{2} - \frac{1}{K}(K - \ell_2 + \ell_1)(\ell_2 - \ell_1)\right\}D(p)_t.$$

This completes the proof. □

A.3. Proof of Corollary 1

This is similar to the proof of Theorem 3 in Aït-Sahalia and Jacod (2009). □

A.4. Proof of Theorem 2

Consider the 2-dimensional function $f = (f_1, f_2)$ with $f_1(x_1, \ldots, x_K) = |x_1 + \cdots + x_K|^p$ and $f_2(x_1, \ldots, x_K) = |x_1|^p$. Write $\rho_\sigma^{\otimes K}(f) = \int f(x)\rho_\sigma^{\otimes K}(dx)$ with $\rho_\sigma^{\otimes K}$ the K-fold tensor product of the law $N(0, \sigma^2)$. Define

$$V'(f, K, \Delta)_t = \sum_{i=1}^{n-K+1} f(\Delta_i^n X/\sqrt{\Delta}, \ldots, \Delta_{i+K-1}^n X/\sqrt{\Delta}), \quad (30)$$

where $\Delta_i^n X = X_{i\Delta} - X_{(i-1)\Delta}$. By Theorem 7.1 of Jacod (2007), the 2-dimensional processes

$$\frac{1}{\sqrt{\Delta}}\left(\Delta V'(f, K, \Delta)_t - \int_0^t \rho_{\sigma_u}^{\otimes K}(f)du\right) \quad (31)$$

converge stably in law to a continuous process $V'(f, K)$ defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, \mathbb{P})$, which conditionally on the σ-field $\mathcal{F}$ is a centered Gaussian $\mathbf{R}^2$-valued process with independent increments, satisfying

$$\widetilde{\mathbb{E}}[V'(f_i, K)_t V'(f_j, K)|\mathcal{F}] = \int_0^t R_{\sigma_u}^{ij}(f, K)du. \quad (32)$$

Here, $R_\sigma^{ij}(f, K)$ is defined as

$$R_\sigma^{ij}(f, K) = \sum_{\ell=-K+1}^{K-1} \widetilde{\mathbb{E}}\big[f_i(\sigma U_K, \ldots, \sigma U_{2K-1})$$

$$\times f_j(\sigma U_{\ell+K}, \ldots, \sigma U_{l+2K-1})\big]$$

$$- (2K-1)\widetilde{\mathbb{E}}\big[f_i(\sigma U_1, \ldots, \sigma U_K)\big]\widetilde{\mathbb{E}}$$

$$\times \big[f_j(\sigma U_1, \ldots, \sigma U_K)\big],$$

where $(U_i)_{i\geq 1}$ independent standard Gaussian random variables. By the definition of f, we can derive that

$$
\begin{aligned}
R_\sigma^{11}(f, K) &= \sigma^{2p} \sum_{l=0}^{K-1} \Big\{ \mathrm{cov}(|\sqrt{l}U_1 + \sqrt{K-l}U_2|^p, \\
&\quad |\sqrt{K-l}U_2 + \sqrt{l}U_3|^p) + \mathrm{cov}(|\sqrt{K-l}U_1 + \sqrt{l}U_2|^p, \\
&\quad |\sqrt{l}U_2 + \sqrt{K-l}U_3|^p) \Big\} \\
&= \sigma^{2p}K^p(m_{2p} - m_p^2) + 2\sigma^{2p} \\
&\quad \times \sum_{\ell=1}^{K-1} \mathrm{cov}(|\sqrt{K-\ell}U_1 + \sqrt{\ell}U_2|^p, \\
&\quad |\sqrt{\ell}U_2 + \sqrt{K-\ell}U_3|^p),
\end{aligned}
$$

$$
R_\sigma^{12}(f, K) = \sigma^{2p}K\,\mathrm{cov}(|U_1 + \sqrt{K-1}U_2|^p, |U_1|^p),
$$

$$
R_\sigma^{22}(f, K) = \sigma^{2p}(m_{2p} - m_p^2).
$$

Next note that $V'(f_2, K, \Delta)_t = \Delta^{-p/2}\widehat{B}(p, \Delta)$ and

$$
\Delta^{-p/2}K^{-1} \sum_{l=1}^{K} \widehat{B}(p, K\Delta)_l
$$

$$
= K^{-1} \sum_{i=1}^{[n/K]} f_1(\Delta_{(i-1)K+1}^n X/\sqrt{\Delta}, \ldots, \Delta_{(i+1)K-1}^n X/\sqrt{\Delta}).
$$

Define $Y(p)_t = K^{-1}V'(f_1, K)_t$ and $Y'(p, \mathbf{a})_t = V'(f_2, K)_t$ with $V'(f_1, K)_t$ and $V'(f_2, K)_t$ defined in (32). In view of (30), the bivariate process

$$
\Delta^{-1/2} \Big(\Delta^{1-p/2}\widehat{B}(p, \Delta) - m_p A(p),
$$

$$
\Delta^{1-p/2}K^{-1} \sum_{\ell=1}^{K} \widehat{B}(p, K\Delta)_\ell - K^{p/2-1}m_p A(p) \Big)
$$

converges stably to the 2-dimensional process $(Y(p)_t, Y'(p, \mathbf{a})_t)$. This follows the results in Theorem 2. $\quad\square$

A.5. Proof of Corollary 2

Similar to the proof of Theorem 3 in Aït-Sahalia and Jacod (2009). $\quad\square$

A.6. Proof of Theorem 3

(a) By Corollary 3, when X_t is continuous, $(\widehat{V}^c)^{-1/2}(\widetilde{S}(p, K) - K^{p/2-1})$ converges stably in law to $N(0, 1)$. Since $x \in (1, K)$,

$$
\frac{K^{p/2-1} - x}{\sqrt{\widehat{V}^j}} \xrightarrow{\text{a.s.}} +\infty.
$$

Thus,

$$
\alpha_{n,t} = P\left(\frac{\widetilde{S}(p, K) - K^{p/2-1}}{\sqrt{\widehat{V}^c}} < \frac{x - K^{p/2-1}}{\sqrt{\widehat{V}^c}} \,\Big|\, H_0 \right) \longrightarrow 0.
$$

In the case when X_t is not continuous but the sample path is continuous over $[0, T]$, using a similar technique to that in Theorem 6 of Aït-Sahalia and Jacod (2009) completes the proof.

(b) When the sample path of X_t exhibits jumps, it follows from Corollary 3 that $(\widehat{V}^j)^{-1/2}(\widetilde{S}(p, K) - 1)$ converges stably in law to $N(0, 1)$. Since $1 < x < K$, we know that $\frac{x-1}{\sqrt{\widehat{V}^c}} \xrightarrow{\text{a.s.}} +\infty$ as $\Delta \to 0$. Thus, the following result holds:

$$
P(\widetilde{S}(p, K) < x|H_1) = P\left(\frac{\widetilde{S}(p, K) - 1}{\sqrt{\widehat{V}^j}} < \frac{x - 1}{\sqrt{\widehat{V}^j}} \,\Big|\, H_1 \right) \longrightarrow 1.
$$

This completes the proof. $\quad\square$

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