Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities

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1 Introduction

This is the first of a series of three papers which provide proofs of results announced in [8]. Let X be a Fano manifold of complex dimension n. Let $\lambda > 0$ be an integer and D be a smooth divisor in the linear system $|-\lambda K_X|$. For $\beta \in (0, 1]$ there is now a well-established notion of a Kähler-Einstein metric with a cone singularity of cone angle $2\pi\beta$ along D. (It is often called an "edge-cone" singularity). For brevity we will just say that ω has cone angle $2\pi\beta$ along D. The Ricci curvature of such a metric ω is $(1 - \lambda(1 - \beta))\omega$. Our primary concern is the case of positive Ricci curvature, so we suppose throughout most of the paper that $\beta \geq \beta_0 > 1 - \lambda^{-1}$. However our arguments also apply to the case of non-positive Ricci curvature: see Remark 3.9.

Theorem 1.1. *If ω is a Kähler-Einstein metric with cone angle $2\pi\beta$ along D with $\beta \geq \beta_0$, then (X, ω) is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics with positive Ricci curvature and with diameter bounded by a fixed number depending only on β_0, λ.*

This suffices for most of our applications but we also prove a sharper statement.

Theorem 1.2. *If ω is a Kähler-Einstein metric with cone angle $2\pi\beta$ along D then (X, ω) is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics ω_i with $Ric(\omega_i) \geq (1 - \lambda(1 - \beta))\omega_i$.*

One consequence of our approximation results is a uniform bound on the Sobolev constant; this bound has also been obtained by Jeffres, Mazzeo and Rubinstein in [13].

A Kähler-Einstein metric with cone angle $2\pi\beta > 0$ along the divisor D, satisfies the equation of currents:

$$Ric(\omega) = (1 - (1 - \beta)\lambda)\omega + 2\pi(1 - \beta)[D], \tag{1.1}$$

where $[D]$ is the current of integration along D. To prove our theorems, we first approximate $[D]$ by a sequence of smooth positive forms, and solve the corresponding complex Monge-Ampère equations; then we show that this sequence of solutions converges to the initial Kähler-Einstein metric as expected. We will make this more precise in Section 2.

We will treat the case when $\lambda = 1$. The general case can be done in an identical way.

In this article we fix ω_0 to be a smooth Kähler form in $2\pi c_1(X)$. Set the space of smooth Kähler potentials to be

$$\mathcal{H} = \{\varphi \in C^\infty(X;\mathbb{R}) : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ in } X\}.$$

Definition 1.3. *A Kähler metric ω' on X with cone angle $2\pi\beta$ along D is a current in $2\pi c_1(X)$ such that*

1. *ω' is a closed positive $(1,1)$ current on X, and is a smooth Kähler metric in $X \setminus D$;*
2. *for any point $p \in D$, there exists a chart $(\mathcal{U}, \{z_i\})$ so that z_1 is a local defining function for D and on this chart the metric is uniformly equivalent to the standard cone metric:*

$$\sqrt{-1}\sum_{j=2}^{n} dz_j \wedge d\bar{z}_j + \sqrt{-1}|z_1|^{2\beta-2} dz_1 \wedge d\bar{z}_1.$$

For any $\beta \in (0,1]$, let $\hat{\mathcal{H}}_\beta$ be the space of all potentials φ such that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler metric on X with cone angle $2\pi\beta$ along D. It is well-known that, for any $\varphi \in \mathcal{H}$, and for ϵ small enough (which may depend on φ), we have

$$\varphi + \epsilon|S|_h^{2\beta} \in \hat{\mathcal{H}}_\beta,$$

where h is a smooth Hermitian metric on $-K_X$ with Ricci curvature ω_0 and S is the defining section of D.

There are other definitions of metrics with cone singularities which, for Kähler-Einstein metrics, turn out to be equivalent. One definition is to require that a local Kähler potential lies in a version of Hölder space $\mathcal{C}^{2,\gamma,\beta}$ for $0 < \gamma < \frac{1}{\beta} - 1$ [9]. The equivalence with Definition 1.3, for Kähler-Einstein metrics, is proved in [13] (Theorem 2). Higher regularity is also established in [13], including the fact that such metrics have an asymptotic expansion about points on the divisor.

2 Proof of Theorem 1.1

Suppose ω_{φ_β} is a Kähler-Einstein metric on X with cone angle $2\pi\beta$ along a smooth anti-canonical divisor D. Let S be the defining section of D in $H^0(X, -K_X)$. By [9], [13], it satisfies the following Monge-Ampère equation

$$\omega_{\varphi_\beta}^n = e^{-\beta\varphi_\beta + h_{\omega_0}} \frac{\omega_0^n}{|S|_h^{2(1-\beta)}}, \qquad \text{on } X \setminus D. \tag{2.1}$$

where h_{ω_0} is the Ricci potential of ω_0 and we have chosen the normalization of φ_β so that

$$\int_X e^{-\beta\varphi_\beta + h_{\omega_0}} \frac{\omega_0^n}{|S|_h^{2(1-\beta)}} = \int_X \omega_0^n.$$

To prove Theorem 1.1, we need to achieve the following three goals simultaneously:

1. Approximate ω_{φ_β} by smooth Kähler metrics on X locally smoothly away from D;
2. The Ricci curvature of this sequence of metrics is positive and diameter is uniformly bounded from above;
3. The Gromov-Hausdorff limit of this sequence of metrics is precisely the metric $(X, \omega_{\varphi_\beta})$.

To achieve the first goal, we want to smooth the volume form of ω_{φ_β} first, and then use the Calabi-Yau theorem on X to smooth the potential φ_β. Fix $p_0 \in (1, (1-\beta_0)^{-1})$. Note that the volume form of ω_{φ_β} is bounded in L^{p_0}. We can find a family of smooth volume forms $\eta_\epsilon (\epsilon \in (0,1])$ with $\int_X \eta_\epsilon = \int_X \omega_0^n$, which converges to $\omega_{\varphi_\beta}^n$ strongly in L^{p_0} and smoothly away from D. For each η_ϵ, by the Calabi-Yau theorem we can find a smooth Kähler potential $\varphi_\epsilon \in \mathcal{H}$ such that

$$\omega_{\varphi_\epsilon}^n = \eta_\epsilon.$$

Following [14], we obtain a uniform bound on $||\varphi_\epsilon||_{C^\gamma(X)}$ for some $\gamma \in (0,1)$. This bound and γ depend only on X, D, ω_0 and the L^{p_0} norm of $\frac{\omega_{\varphi_\beta}^n}{\omega_0^n}$. Furthermore $\{\varphi_\epsilon\}$ converges by sequence to φ_β in $C^{\gamma'}(X)$ for some γ' slightly smaller than γ.

To achieve our second goal, we need to modify the volume form to secure positive Ricci curvature. Following Yau [22], we can solve the following equation for $\epsilon \in (0,1]$:

$$\omega_{\psi_\epsilon}^n = e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}}. \tag{2.2}$$

Here we need to normalize φ_ϵ so that

$$\int_X e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}} = \int_X \omega_0^n.$$

In Theorem 8 of [22], Yau treated a more general case with meromorphic right hand side. Note our initial approximation φ_ϵ is smooth but does not have high regularity control outside D, and we will discuss a bit more after Proposition 2.3. There are some similarities in our approach and the work of Campana, Guenancia, and Paun [6]. For more recent work on complex Monge-Ampère equation on Kähler manifolds and generalizations, we refer to [11] for further references.

A direction calculation shows that

Proposition 2.1. *The Ricci form of* ω_{ψ_ϵ} *approximates* $\beta\omega_\beta + (1-\beta)[D]$ *as* $\epsilon \to 0$. *Moreover,*

$$Ric(\omega_{\psi_\epsilon}) \geq \beta\omega_{\varphi_\epsilon} > 0, \qquad \forall\ \epsilon \in (0,1].$$

Proof. For any smooth function $f > 0$, we have (c.f. [22])

$$\begin{aligned}
\sqrt{-1}\partial\bar\partial \log(f+\epsilon) &= \sqrt{-1}\partial\tfrac{\bar\partial f}{f+\epsilon} \\
&= \tfrac{\sqrt{-1}\partial\bar\partial f}{f+\epsilon} - \tfrac{\sqrt{-1}\partial f\wedge\bar\partial f}{(f+\epsilon)^2} \\
&= \tfrac{f}{f+\epsilon}\left(\tfrac{\sqrt{-1}\partial\bar\partial f}{f} - \tfrac{\sqrt{-1}\partial f\wedge\bar\partial f}{f^2} + \tfrac{\sqrt{-1}\partial f\wedge\bar\partial f}{f^2}\right) - \tfrac{\sqrt{-1}\partial f\wedge\bar\partial f}{(f+\epsilon)^2} \\
&= \tfrac{f}{f+\epsilon}\sqrt{-1}\partial\bar\partial\log f + \epsilon\tfrac{\sqrt{-1}\partial f\wedge\bar\partial f}{f(f+\epsilon)^2} \\
&\geq \tfrac{f}{f+\epsilon}\sqrt{-1}\partial\bar\partial \log f.
\end{aligned}$$

Using this, we can calculate the Ricci form on $X \setminus D$:

$$\begin{aligned}
Ric(\omega_{\psi_\epsilon}) &= -\sqrt{-1}\partial\bar\partial h_{\omega_0} + (1-\beta)\sqrt{-1}\partial\bar\partial\log(|S|_h^2+\epsilon) + Ric(\omega_0) + \beta\sqrt{-1}\partial\bar\partial\varphi_\epsilon \\
&= \omega_0 + (1-\beta)\sqrt{-1}\partial\bar\partial\log(|S|_h^2+\epsilon) + \beta\sqrt{-1}\partial\bar\partial\varphi_\epsilon \\
&= \beta\omega_{\varphi_\epsilon} + (1-\beta)(\omega_0 + \sqrt{-1}\partial\bar\partial\log(|S|_h^2+\epsilon)) \\
&= \beta\omega_{\varphi_\epsilon} + (1-\beta)\tfrac{\epsilon}{|S|_h^2+\epsilon}\omega_0 \\
&\geq \beta\omega_{\varphi_\epsilon}.
\end{aligned}$$

Since ψ_ϵ is smooth this also holds on the whole X. □

For later purpose we denote

$$(1-\beta)\chi_\epsilon = Ric(\omega_{\psi_\epsilon}) - \beta\omega_{\varphi_\epsilon}. \tag{2.3}$$

By the previous calculation, this converges to $2\pi(1-\beta)[D]$ in the sense of currents.

Now we derive estimates on ω_{ψ_ϵ}. We make the convention that unless otherwise emphasized all constants appearing below are positive and depend only on $X, D, \omega_0, \omega_{\varphi_\beta}$. Also the norms of the functions appearing in this article are always taken with respect to the background metric ω_0.

Theorem 2.2. *There exists a uniform constant* $C_1 > 0$ *such that for any* $\epsilon \in (0,1]$,

$$C_1^{-1}\omega_0 < \omega_{\psi_\epsilon} \leq \frac{C_1}{(\epsilon + |S|_h^2)^{1-\beta}} \cdot \omega_0.$$

Proof. First, by construction we have a constant c_1 and $p_0 > 1$ such that

$$||\frac{\omega_{\psi_\epsilon}^n}{\omega_0^n}||_{L^{p_0}} \leq c_1.$$

By a theorem of Kolodziej [14], this implies $||\psi_\epsilon||_{C^\gamma(X)} \leq c_2$ for some $\gamma \in (0,1)$. To derive C^2 estimate, we view the identity map $id : (X, \omega_{\psi_\epsilon}) \to (X, \omega_0)$ as a harmonic map with energy density

$$e(\psi_\epsilon) = \mathrm{tr}_{\omega_{\psi_\epsilon}} \omega_0$$

Then, using the fact that $Ric(\omega_{\psi_\epsilon}) > 0$ and $Rm(\omega_0) \leq c_3$, $e(\psi_\epsilon)$ satisfies the following Chern-Lu differential inequality [19] (c.f. also [13]):

$$\triangle_{\psi_\epsilon}(\log e(\psi_\epsilon) - c_4\psi_\epsilon) \geq c_5 e(\psi_\epsilon) - c_6.$$

Since ψ_ϵ is uniformly bounded by c_2, by maximum principle, we have

$$e(\psi_\epsilon) \leq c_6$$

or

$$c_6^{-1}\omega_0 \leq \omega_{\psi_\epsilon}.$$

Plugging this into the Monge-Ampère equation (2.2), we obtain

$$C_1^{-1}\omega_0 \leq \omega_{\psi_\epsilon} \leq C_1 \cdot (|S|_h^2 + \epsilon)^{-(1-\beta)}\omega_0. \tag{2.4}$$

□

It follows that we have a uniform bound on $\Delta_{\omega_0}\psi_\epsilon$ locally away from D, and a global uniform bound on $||\psi_\epsilon||_{C^\gamma(X)}$. So $\{\psi_\epsilon\}$ by sequence converges to a limit potential ψ_0 globally in $C^{\gamma'}(X)$ and in $C^{1,\alpha}$ locally away from D.

Proposition 2.3. *We have* $\psi_0 = \varphi_\beta + \text{constant}$.

Proof. This follows directly by the general uniqueness theorem for Monge-Ampère equation [1, 15, 4]. For the convenience of readers, we give a detailed account in our special case. Since

$$\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n$$

converges to 0 as $\epsilon \to 0$ in L^p topology for some fixed $p > 1$, we have

$$\int_X (\varphi_\epsilon - \psi_\epsilon)(\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n) \to 0.$$

It follows that

$$\int_X \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon)\wedge(\bar\partial\psi_\epsilon - \bar\partial\varphi_\epsilon)\wedge\sum_{k=0}^{n-1}\omega_{\psi_\epsilon}^k\omega_{\varphi_\epsilon}^{n-1-k}$$
$$= -\int_X (\psi_\epsilon - \varphi_\epsilon)(\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n) \to 0.$$

Positivity of the integrand means that for any $\delta > 0$, we have

$$\int_{X\setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon)\wedge(\bar\partial\psi_\epsilon - \bar\partial\varphi_\epsilon)\wedge\sum_{k=0}^{n-1}\omega_{\psi_\epsilon}^k\omega_{\varphi_\epsilon}^{n-1-k} \to 0,$$

where D_δ is the δ-tubular neighborhood of D, defined by the metric ω_0. Since every term is non-negative, we have

$$\int_{X\setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon)\wedge(\bar\partial\psi_\epsilon - \bar\partial\varphi_\epsilon)\wedge\omega_{\psi_\epsilon}^{n-1} \to 0,$$

By Theorem 2.2, $\omega_{\psi_\epsilon} \geq C_1^{-1}\omega_0$. It follows that

$$\int_{X\setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon)\wedge(\bar\partial\psi_\epsilon - \bar\partial\varphi_\epsilon)\wedge\omega_0^{n-1} \to 0,$$

In other words,

$$\partial\psi_0 - \partial\varphi_\beta = 0, \qquad \text{in } X\setminus D_\delta.$$

So

$$\psi_0 = \varphi_\beta + \text{constant}, \qquad \text{in } X\setminus D_\delta.$$

Since δ is arbitrary, this finishes the proof.

□

To obtain more regularity control, we can substitute φ_ϵ in Equation (2.2) by ψ_ϵ. Then, the right hand side of the equation will have a uniform $C^{1,1}$ bound (i.e. bound on its Laplacian Δ_{ω_0}) locally away from D. Then we can repeat the same procedure to obtain a new ψ'_ϵ for $\epsilon \in (0,1]$. As before, this new sequence ψ'_ϵ converges to φ_β globally in $C^{\gamma'}$ and they satisfy the same estimate as in Theorem 2.2. Now following standard theory of Evans-Krylov [10] [16] and bootstrapping, we can obtain an interior $C^{3,\gamma}$ estimate on $X\setminus D$ for some $\gamma\in(0,1)$. It follows that $\{\psi'_\epsilon\}$ by sequence converges in $C^{3,\gamma'}$ to φ_β locally away from D. For simplicity from now on we will denote by ψ_ϵ this new sequence. If we keep running this procedure, we get even higher derivative control away from D.

To achieve the second goal, we need to prove the following proposition.

Proposition 2.4. *For any $\epsilon\in(0,1]$, the diameter of $(X,\omega_{\psi_\epsilon})$ is uniformly bounded above by a constant C_2.*

Proof. Since D is smooth, there exists a small constant $\delta > 0$ such that the restriction of the background metric ω_0 to the δ-*tubular* neighborhood of D, denoted by D_δ, is equivalent to the product metric on $D \times \mathbb{B}$, where $\mathbb{B}$ is the the standard disc of radius δ. Following the estimate in Theorem 2.2, for every point in D_δ, there is a curve connecting it to ∂D_δ with length bounded by $c_1 \delta^\beta$. On the other hand, the varying metric is bounded above by the metric $c_2 \delta^{\beta-1} \omega_0$ in $X \setminus D_\delta$. Therefore, the diameter of $(X, \omega_{\psi_\epsilon})$ is controlled above by $c_3(\delta^\beta + \delta^{\beta-1})$.

□

Finally, to achieve the third goal, we need to study the Gromov-Hausdorff limit of a sequence of Riemannian manifolds with positive Ricci curvature.

Proposition 2.5. *$(X, \omega_{\varphi_\beta})$ is the Gromov-Hausdorff limit of $(X, \omega_{\psi_\epsilon})$ as $\epsilon \to 0$.*

Proof. By Theorem 2.2 and Proposition 2.4, we have the following:

1. ω_{ψ_ϵ} converges as a current to ω_{φ_β};
2. $(X \setminus D, \omega_{\psi_\epsilon})$ converges locally in $C^{3,\gamma'}$ to $(X \setminus D, \omega_{\varphi_\beta})$;
3. Any fixed δ-tubular neighborhood D_δ of D with respect to the background metric ω_0 is contained in a $\eta(\delta)$-tubular neighborhood of D with respect to the varying metric ω_{ψ_ϵ}, where $\eta(\delta)$ tends to zero as δ tends to zero.

To prove the desired Gromov-Hausdorff convergence, we use the identity map from X to itself, for any $\delta > 0$ small, we want to show that if ϵ is small enough then

$$|d_{\psi_\epsilon}(x, y) - d_{\varphi_\beta}(x, y)| < \delta, \qquad \forall x, y \in X. \tag{2.5}$$

Fix a small constant

$$\delta_1 \ll \delta^{\frac{1}{\beta}}.$$

For any two points outside D_{δ_1}, for $\epsilon > 0$ sufficiently small, the preceding inequality (2.5) obviously holds with $\frac{\delta}{3}$ in the right hand side. Now for any two points $x, y \in D_{\delta_1}$, there exist two points $x_1, y_1 \in \partial D_{\delta_1}$ such that

$$d_{\varphi_\beta}(x_1, x) + d_{\varphi_\beta}(y_1, y) < \frac{\delta}{3} \qquad \text{and} \quad d_{\psi_\epsilon}(x_1, x) + d_{\psi_\epsilon}(y_1, y) < \frac{\delta}{3}.$$

A simple triangle inequality implies that

$$|d_{\psi_\epsilon}(x, y) - d_{\varphi_\beta}(x, y)| < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

□

To finish the proof of Theorem 1.1, what is left to prove is the uniform diameter bound (depending only on β_0). To see this, one notices that for a Kähler metric with cone singularities along D, the complement $X \setminus D$ is geodesically convex, so we can apply Myers' theorem to see that the diameter of ω_β is uniformly bounded by $\pi\sqrt{\frac{n-1}{\beta}} \le \pi\sqrt{\frac{n-1}{\beta_0}}$. Then for each $\beta \ge \beta_0$, we can apply Proposition 2.5 (choosing ϵ sufficiently small) to obtain a sequence of smooth Kähler metrics of positive Ricci curvature with Gromov-Hausdorff limit $(X, \omega_{\varphi_\beta})$ and the diameter of this sequence of metrics is uniformly bounded above by $2\pi\sqrt{\frac{n-1}{\beta_0}}$.

3 Proof of Theorem 1.2

To prove Theorem 1.2, we need to achieve the fourth goal: to approximate the Kähler-Einstein metric ω_{φ_β} by smooth Kähler metrics with Ricci curvature bounded from below by some uniform positive number. From the complex Monge-Ampère theory, this is very different from the first and second goal where we can obtain C^0 estimate via Kolodziej's theorem [14] directly. What we shall do is to use the metric constructed in the previous section as a starting point, and use continuity method to solve the twisted Kähler-Einstein equation up to $t = 1 - \beta$. To do this, we need to obtain a uniform C^0 estimate. The key observation is that for the family of $(1,1)$ forms χ_ϵ (see Equation (2.3)) which converges to $2\pi(1-\beta)[D]$, the twisted K-energy $E_{\epsilon,(1-\beta)D}$ dominates the K-energy $E_{(1-\beta)D}$ from above (See formulas (3.1), (3.2) and Lemma 3.5 below for precise statements).

Following Szekelyhidi [21], we define

$$R(X) = \sup\{t : \exists\ \omega' \in 2\pi c_1(X) \text{ such that } Ric(\omega') > t\omega'\}.$$

Theorem 1.2 is a consequence of the following

Theorem 3.1. *If there is a Kähler-Einstein metric ω_{φ_β} with cone angle $2\pi\beta$ along D, then $(X, \omega_{\varphi_\beta})$ is the Gromov-Hausdorff limit of a sequence of Kähler metrics with Ricci curvature bounded below by $\beta > 0$. In particular, $R(X) \ge \beta$.*

This verifies one aspect of a conjecture by the second named author earlier [9]. We need to do some preparation first. Set

$$\chi = \sqrt{-1}\partial\bar\partial \log |S|_h^2 + \omega_0.$$

By the Poincaré-Lelong equation, this is the same as the current $2\pi[D]$. For $\epsilon > 0$ sufficiently small, we define (c.f. Equation (2.3)):

$$\chi_\epsilon = \sqrt{-1}\partial\bar\partial \log(|S|_h^2 + \epsilon) + \omega_0 > 0$$

Clearly as currents

$$\chi_\epsilon \to \chi.$$

For any $\varphi \in \mathcal{H}$, we choose a smooth family of potentials $\varphi(t)(t \in [0,1])$ in $\mathcal{H}$ with $\varphi(0) = 0$ and $\varphi(1) = \varphi$. We denote $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$. For any smooth function $f(t,\cdot)$, we write $\dot{f}(t)$ for the time derivative $\frac{\partial f}{\partial t}(t,\cdot)$.

Definition 3.2. *Define a functional J_{χ_ϵ} by*

$$J_{\chi_\epsilon}(\varphi) = n\int_0^1 dt \int_X \dot{\varphi}(t)(\chi_\epsilon - \omega_t) \wedge \omega_t^{n-1},$$

and as ϵ tends to zero, we define a functional J_χ by

$$J_\chi(\varphi) = 2\pi n \int_0^1 dt \int_D \dot{\varphi}(t)\omega_t^{n-1} - n\int_0^1 dt \int_X \dot{\varphi}(t)\omega_t^n,$$

Definition 3.3. *Define the K-energy functional E by*

$$E(\varphi) = -n\int_0^1 dt \int_X \dot{\varphi}(t)(Ric(\omega_t) - \omega_t) \wedge \omega_t^{n-1}.$$

One can check these do not depend on the choice of the path and hence are well-defined functionals on $\mathcal{H}$. Of course they are only defined up to an additive constant. We have fixed this constant by imposing that they have value 0 at $\varphi = 0$. Then, we define the twisted K-energy

$$E_{\epsilon,(1-\beta)D}(\varphi) = E(\varphi) + (1-\beta)J_{\chi_\epsilon}(\varphi), \tag{3.1}$$

and

$$E_{(1-\beta)D}(\varphi) = E(\varphi) + (1-\beta)J_\chi(\varphi). \tag{3.2}$$

First, we give an explicit formula for J_{χ_ϵ} (c.f. [17]).

Proposition 3.4. *We have*

$$J_{\chi_\epsilon}(\varphi) = \int_X \log(|S|_h^2 + \epsilon)\cdot(\omega_\varphi^n - \omega_0^n) + n\int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t)\wedge\omega_t^{n-1}. \tag{3.3}$$

Proof. This is a direct calculation:

$$\begin{aligned}
& J_{\chi_\epsilon}(\varphi) \\
= \;& n\textstyle\int_0^1 dt \int_X \dot{\varphi}(t)\chi_\epsilon \wedge \omega_t^{n-1} - n\int_0^1 dt \int_X \dot{\varphi}(t)\,\omega_t^n \\
= \;& n\textstyle\int_0^1 dt \int_X \dot{\varphi}(t)\sqrt{-1}\partial\bar{\partial}\log(|S|_h^2+\epsilon)\wedge\omega_t^{n-1} + n\int_0^1 dt\int_X \dot{\varphi}(t)(\omega_0 - \omega_t)\wedge\omega_t^{n-1} \\
= \;& n\textstyle\int_0^1 dt \int_X \log(|S|_h^2+\epsilon)\cdot\triangle_{\omega_t}\dot{\varphi}(t)\omega_t^n + n\int_0^1 dt\int_X \dot{\varphi}(t)(\omega_0-\omega_t)\wedge\omega_t^{n-1} \\
= \;& \textstyle\int_0^1 dt\int_X \log(|S|_h^2+\epsilon)\cdot\frac{\partial}{\partial t}\omega_t^n + n\int_0^1 dt\int_X \dot{\varphi}(t)(\omega_0-\omega_t)\wedge\omega_t^{n-1} \\
= \;& \textstyle\int_X \log(|S|_h^2+\epsilon)\cdot(\omega_\varphi^n - \omega_0^n) + n\int_0^1 dt\int_X \dot{\varphi}(t)(\omega_0-\omega_t)\wedge\omega_t^{n-1}.
\end{aligned}$$

□

Similarly, we have

$$J_\chi(\varphi) = \int_X \log |S|_h^2 \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 d\,t \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1}.$$

Now we have the following observation

Lemma 3.5. *There exists a uniform constant $C_3 = C_3(X, D, \omega_0)$ such that for any $\epsilon \in (0,1]$ and for any smooth Kähler potential φ, we have*

$$J_{\chi_\epsilon}(\varphi) \geq J_\chi(\varphi) - C_3.$$

As a consequence, we also have

$$E_{\epsilon,(1-\beta)D}(\varphi) \geq E_{(1-\beta)D}(\varphi) - C_3.$$

Proof. This follows from an elementary calculation:

$$\begin{aligned}
J_{\chi_\epsilon}(\varphi) &= \int_X \log(|S|_h^2 + \epsilon) \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 d\,t \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\
&= \int_X \log(|S|_h^2 + \epsilon) \cdot \omega_\varphi^n - \int_X \log(|S|_h^2 + \epsilon) \cdot \omega_0^n + n \int_0^1 d\,t \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\
&\geq \int_X \log |S|_h^2 \cdot \omega_\varphi^n - \int_X \log |S|_h^2 \cdot \omega_0^n - \int_X \log \frac{|S|_h^2 + \epsilon}{|S|_h^2} \cdot \omega_0^n + n \int_0^1 d\,t \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\
&\geq \int_X \log |S|_h^2 \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 d\,t \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} - C_3 \\
&= J_\chi(\varphi) - C_3.
\end{aligned}$$

□

It is well-known that the K-energy has an explicit expression (c.f. [7]):

$$E(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega_0^n} \omega_0^n + I(\varphi) + Q(\varphi),$$

where

$$I(\varphi) = n \int_0^1 dt \int_X \dot{\varphi}(t) \omega_t^n,$$

and

$$Q(\varphi) = -n \int_0^1 dt \int_X \dot{\varphi}(t) Ric(\omega_0) \wedge \omega_t^{n-1}.$$

Following [21] in the smooth case and [17] in the case with cone singularities, we have

Proposition 3.6. *If there exists a Kähler-Einstein metric ω_{φ_β} with cone angle $2\pi\beta > 0$ along D, then the twisted K-energy $E_{(1-\beta)D}$ is proper on $\mathcal{H}$. In other words, there are constants C_4, C_5 depending on X, D, ω_0 and β such that for any smooth Kähler potential φ, we have*

$$E_{(1-\beta)D}(\varphi) \geq C_4 \cdot J_0(\varphi) - C_5,$$

where

$$J_0(\varphi) = \int_X \varphi(\omega_0^n - \omega_\varphi^n).$$

Proof. By [20] there is no non-trivial holomorphic vector field on X that is tangential to D. So by the openness theorem [9] for a slightly larger $\beta' > \beta$ there exists a Kähler-Einstein metric on X with cone angle $2\pi\beta'$ along D. By [3] and [2], the twisted Ding functional is bounded from below on $\mathcal{H}$. By [18], the infimum of the Ding functional and the infimum of the K-energy are the same in the anti-canonical class. This is generalized in [2] to the case with cone singularities. It follows that the twisted K-energy $E_{(1-\beta')D}$ is bounded from below on $\mathcal{H}$. On the other hand, it is proved in [2] that for $\beta'' > 0$ sufficiently small $E_{(1-\beta'')D}$ is proper on $\mathcal{H}$. Since the twisted K-energy is linear in β [17], we see that $E_{(1-\beta)D}$ is proper on $\mathcal{H}$. □

To prove Theorem 3.1, we need to set up a continuity path. For any $\epsilon > 0$, we consider the following equation for $\phi_\epsilon(t) \in \mathcal{H}(t \in [0, \beta])$:

$$Ric(\omega_{\phi_\epsilon(t)}) = t\omega_{\phi_\epsilon(t)} + (\beta - t)\omega_{\varphi_\epsilon} + (1-\beta)\chi_\epsilon. \tag{3.4}$$

This is equivalent to the complex Monge-Ampère equation:

$$\begin{cases} \omega^n_{\phi_\epsilon(t,\cdot)} = e^{-t\phi_\epsilon(t,\cdot)-(\beta-t)\varphi_\epsilon + h_{\omega_0}} \frac{1}{(|S|_h^2+\epsilon)^{1-\beta}} \omega_0^n, \\ \phi_\epsilon(0,\cdot) = \psi_\epsilon. \end{cases} \tag{3.5}$$

Here ψ_ϵ is the solution in Equation (2.2); in other words, we have

$$\omega^n_{\psi_\epsilon} = e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{1}{(|S|_h^2+\epsilon)^{1-\beta}} \omega_0^n.$$

If we can solve Equation (3.5) up to $t = \beta$, then we have:

$$Ric(\omega_{\phi_\epsilon(\beta)}) = \beta\omega_{\phi_\epsilon(\beta)} + (1-\beta)\chi_\epsilon \geq \beta\omega_{\phi_\epsilon(\beta)}.$$

Lemma 3.7. *There is a constant $C_6 > 0$ such that*

$$|E(\epsilon, (1-\beta)D)(\psi_\epsilon)| \leq C_6.$$

Proof. By Theorem 2.2 this is a direct verification using the explicit expressions of energy functionals described above. □

Lemma 3.8. *Along the continuity path,* $E_{\epsilon,(1-\beta)D}(\phi_\epsilon(t))$ *decreases monotonically.*

Proof. Here we follow [21]. We take time derivative on both sides of Equation (3.5). Then,

$$\triangle_{\phi_\epsilon}\dot{\phi}_\epsilon = -t\dot{\phi}_\epsilon - (\phi_\epsilon - \varphi_\epsilon).$$

In this calculation we have omitted the parameter t for simplicity. A straightforward calculation then follows,

$$\begin{array}{rl}
& \frac{d}{dt}(E + (1-\beta)J_{\chi_\epsilon})(\phi_\epsilon(t)) \\
= & n\int_X \dot{\phi}_\epsilon(-Ric(\omega_{\phi_\epsilon}) + \omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon - (1-\beta)\omega_{\phi_\epsilon}) \wedge \omega_{\phi_\epsilon}^{n-1} \\
= & n\int_X \dot{\phi}_\epsilon(-t\omega_{\phi_\epsilon} - (\beta - t)\omega_{\varphi_\epsilon} - (1-\beta)\chi_\epsilon + \beta\omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon) \wedge \omega_{\phi_\epsilon}^{n-1} \\
= & n(\beta - t)\int_X \dot{\phi}_\epsilon(\omega_{\phi_\epsilon} - \omega_{\varphi_\epsilon}) \wedge \omega_{\phi_\epsilon}^{n-1} \\
= & n(\beta - t)\int_X (\phi_\epsilon - \varphi_\epsilon)\cdot \triangle_{\phi_\epsilon}\dot{\phi}_\epsilon\, \omega_{\phi_\epsilon}^n \\
= & -n(\beta - t)\int_X (\phi_\epsilon - \varphi_\epsilon)\cdot (t\dot{\phi}_\epsilon + (\phi_\epsilon - \varphi_\epsilon))\omega_{\phi_\epsilon}^n \\
= & -n(\beta - t)\int_X (\phi_\epsilon - \varphi_\epsilon)^2\omega_{\phi_\epsilon}^n - nt(\beta - t)\int_X (\phi_\epsilon - \varphi_\epsilon)\cdot \dot{\phi}_\epsilon\, \omega_{\phi_\epsilon}^n \\
\leq & nt(\beta - t)\int_X (\triangle_{\phi_\epsilon}\dot{\phi}_\epsilon + t\dot{\phi}_\epsilon)\cdot \dot{\phi}_\epsilon\, \omega_{\phi_\epsilon}^n \\
\leq & 0.
\end{array}$$

The last inequality holds because $Ric(\omega_{\phi_\epsilon(t,\cdot)}) > t\omega_{\phi_\epsilon(t,\cdot)}$ and $\triangle_{\phi_\epsilon} + t$ is a negative operator. □

Now we are ready to prove Theorem 3.1.

Proof. According to Proposition 3.6, the twisted K-energy $E_{(1-\beta)D}(\varphi)$ is proper on $\mathcal{H}$. Following Lemma 3.5,

$$E_{\epsilon,(1-\beta)D}(\varphi) \geq E_{(1-\beta)D}(\varphi) - C_3$$

is also proper on $\mathcal{H}$. By monotonicity, we have

$$E_{\epsilon,(1-\beta)D}(\phi_\epsilon(t)) \leq E_{\epsilon,(1-\beta)D}(\phi_\epsilon(0)) = E_{\epsilon,(1-\beta)D}(\psi_\epsilon), \qquad \forall t \in [0,\beta].$$

So by Lemma 3.7,

$$E_{\epsilon,(1-\beta)D}(\phi_\epsilon(t)) \leq C_6, \qquad \forall\, t \in [0,\beta].$$

By definition of properness there is a constant C_7 with

$$J_0(\phi_\epsilon(t)) \leq C_7.$$

It follows from the standard argument that we can solve Equation (3.5) up to $t = \beta$, and there is a constant C_8 such that

$$\sup_{\epsilon\in(0,1]}\max_{t\in[0,\beta]} ||\phi_\epsilon(t)||_{L^\infty} \leq C_8.$$

As in the proof of Theorem 2.2, there is a constant C_9 such that

$$C_9\omega_0 < \omega_{\phi_\epsilon(t)} \le \frac{C_9}{(\epsilon+|S|_h^2)^{1-\beta}}\cdot\omega_0, \forall\, t\in[0,\beta]\ \epsilon\in(0,1).$$

As before following [14] and Evans-Krylov theory to bootstrap regularity away from divisor, one can prove that $\phi_\epsilon(t,\cdot)$ converges to $\phi_0(t,\cdot)$ globally in $C^{\gamma'}(X)$ and locally in $C^{3,\gamma'}$ away from D. Moreover, in $X\setminus D$, it satisfies the equation

$$\omega^n_{\phi_0(t,\cdot)} = e^{-t\phi_0(t,\cdot)-(\beta-t)\varphi_\beta+h_{\omega_0}}\frac{1}{|S|_h^{2-2\beta}}\omega_0^n, \qquad \forall t\in[0,\beta].$$

with

$$\phi_0(0,\cdot)=\varphi_\beta.$$

This can be written in a more concise form as

$$\omega^n_{\phi_0} = e^{-t(\phi_0-\varphi_\beta)}\omega^n_{\varphi_\beta}, \qquad t\in[0,\beta]. \tag{3.6}$$

Since $\phi_\epsilon(t,\cdot)$ is uniformly bounded, we see $\phi_0(\beta,\cdot)$ is in the weak sense (c.f. [2]) a Kähler-Einstein metric on X with cone angle $2\pi\beta$ along D. By [20], there is no non-trivial holomorphic vector field on X which is tangential to D. So we can use the uniqueness theorem of Berndtsson [3] to obtain

$$\phi_0(\beta,\cdot)=\varphi_\beta(\cdot), \forall t\in[0,\beta].$$

Then Proposition 2.5 implies that $(X,\omega_{\phi_\epsilon(\beta)})$ converges in the Gromov-Hausdorff topology to $(X,\omega_{\varphi_\beta})$ as $\epsilon\to 0$.

□

Here we give an alternative proof which makes use of the openness theorem proved by the second named author, bypassing Berndtsson's theorem.

Proof. Note that the expression $e^{-t(\phi_0-\varphi_\beta)}$ in Equation (3.6) is Hölder continuous on X, so it lies in the Hölder space $\mathcal{C}^{,\gamma,\beta}$ for some $\gamma<\frac{1}{\beta}-1$. Following [9], the Laplacian operator $\triangle_{\varphi_\beta}$ defines a continuous and invertible map

$$\triangle_{\varphi_\beta}:\ \mathcal{C}_0^{2,\gamma,\beta}(X,D)\to\mathcal{C}^{,\gamma,\beta}(X,D).$$

Here $\mathcal{C}_0^{2,\gamma,\beta}(X,D)$ consists of functions in $\mathcal{C}^{2,\gamma,\beta}(X,D)$ with zero average. It follows that, there exists a continuous family $\psi(t,\cdot)\in\mathcal{C}_0^{2,\gamma,\beta}(X,D)$ for small t, say $t\in[0,\epsilon_0]$, which solves

$$\omega^n_\psi = e^{-t(\phi_0-\varphi_\beta)}\omega^n_{\varphi_\beta}$$

and $\psi(0,\cdot) = \varphi_\beta$. Then, either following the uniqueness in [15] or Proposition 2.3, we have

$$\psi(t,\cdot) = \phi_0(t,\cdot) + \text{constant}$$

for $t \in [0, \epsilon_0]$. It follows that $\phi_0(t,\cdot) \in \mathcal{C}^{2,\gamma,\beta}(X, D)$ for $t \in [0, \epsilon_0]$.

Now we define the constant family $\varphi_\beta(t,\cdot) = \varphi_{\beta,\cdot}$, then it satisfies the same Equation (3.6) for $t \in [0, \epsilon_0]$.

$$\omega^n_{\varphi_\beta(t)} = e^{-t(\varphi_\beta(t)-\varphi_\beta)}\omega^n_{\varphi_\beta}, \qquad t \in [0, \beta]. \tag{3.7}$$

By [20], there is no non-trivial holomorphic vector fields on X which is tangential to D. It follows that, the first eigenvalue of $\omega_{\varphi_\beta} = \omega_{\phi_0(0,\cdot)}$ is strictly bigger than $\beta > 0$. Consequently for t sufficiently small, $\omega_{\phi_0(t,\cdot)}$ has eigenvalue strictly bigger than $\frac{\beta}{2}$. Now compare the two families of solutions to Equation (3.6), by implicit function theorem again we see the uniqueness holds. In other words, by making ϵ_0 even smaller we have

$$\phi_0(t,\cdot) = \varphi_\beta, \forall\ t \in [0, \epsilon_0].$$

Repeating the same procedure as we increase $t \leq \beta$, we see the same holds for all $t \in [0, \beta]$. Our theorem is then proved. $\square$

Remark 3.9. *Finally we remark that in the case when $\lambda > 1$ and $1 - (1-\beta)\lambda \leq 0$ there is a complete existence theory [13], but the argument in this article also applies to prove the following*

Theorem 3.10. *Let $\lambda > 1$ and $\beta_0 \in (0, 1-\lambda^{-1}]$. If ω is a Kähler-Einstein metric with cone angle $2\pi\beta$ along $D \in |-\lambda K_X|$ with $\beta \in [\beta_0, 1-\lambda^{-1}]$, then (X, ω) is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics ω_i with $Ric(\omega_i) \geq c_\beta \omega_i$ where $c_\beta = (1 - \lambda(1-\beta)) \leq 0$, and diameter bounded above by a uniform constant depending only on X, D and β_0.*

Proof. The main issue is that in our previous argument the diameter bound depends on the particular β, and in the case of positive Ricci curvature (as assumed before that $\beta_0 > 1-\lambda^{-1}$) we can apply Myers' theorem to show that the bound only depends on β_0. Under our assumptions, we are in the case of nonpositive Ricci curvature. In general the diameter can not have a uniform upper bound, if one varies the complex structure on (X, D), even when X has complex dimension one. However for a fixed (X, D), a closer look at the argument in the proof of Theorem 2.2 and Proposition 2.4 shows that the diameter bound really depends only on the lower bound on the Ricci curvature of ω_{φ_β} and the L^∞ bound on φ_β (which in turn depends on the L^{p_0} bound on the volume form of ω_{φ_β}). Notice we assume $c_\beta \leq 0$, then the metric ω_{φ_β} satisfies the equation

$$\omega^n_{\varphi_\beta} \quad = \quad e^{-c_\beta\varphi_\beta + h_{\omega_0}} \frac{1}{|S|_h^{2(1-\beta)}} \omega_0^n, \tag{3.8}$$

Similar to the arguments in Section 2, by Yau's theorem [22] for $\epsilon \in (0,1]$ one can solve the equation for ψ_ϵ:

$$\omega_{\psi_\epsilon}^n = e^{-c_\beta \psi_\epsilon + h_{\omega_0}} \frac{1}{(|S|_h^2 + \epsilon)^{(1-\beta)}} \omega_0^n.$$

Direct calculation as before shows that $Ric(\omega_{\psi_\epsilon}) \geq c_\beta \omega_{\psi_\epsilon}$. Moreover by the maximum principle we see that there are constants $p_0 \in (1, \frac{1}{1-\beta_0})$, and $A > 0$ depending only on X, D, ω_0 and β_0 such that for any $\epsilon \in (0,1]$,

$$\sup_X \psi_\epsilon + || \frac{\omega_{\psi_\epsilon}^n}{\omega_0^n} ||_{L^{p_0}} \leq A.$$

Following the arguments in Section 2, one can show that as $\epsilon \to 0$ the Gromov-Hausdorff limit of $(X, \omega_{\psi_\epsilon})$ is exactly $(X, \omega_{\varphi_\beta})$. Moreover, there is a uniform diameter bound independent of $\beta \in [\beta_0, 1 - \lambda^{-1}]$. $\square$

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