Kähler-Einstein metrics and K-stability

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Abstract

In this thesis, we study several problems related to the existence problem of Kähler-Einstein metric on Fano manifold. After introduction in the first chapter, in the second chapter, we review the basic theory both from PDE and variational point of view. Tian's program using finite dimensional approximation is then explained. Futaki invariant is discussed in detail for both its definition and calculation. K-stability is introduced following Tian and Donaldson. In the third chapter, we extend the basic theory to the twisted setting. As an important case, the analytic and algebraic theory are both extended to the conic setting. In the third chapter, we study the continuity method on toric Fano manifolds. We calculate the maximal value of parameter for solvability and study the limit behavior of the solution metrics. As a corollary, we prove Tian's partial C^0-estimate on toric Fano manifolds. The log-Futaki invariant is calculated on toric Fano manifolds too. In the fourth chapter, we discuss the recent joint work with Dr. Chenyang Xu. We use Minimal Model Program (MMP) to simplify the degeneration and prove Tian's conjecture which reduce the test for K-stability to special degenerations. In the final chapter, we construct examples of rotationally symmetric solitons. These solitons are local models of special singularities of Kähler-Ricci flow.

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Contents

Abstract . . . iii

Acknowledgements . . . iv

1 Introduction **1**

1.1 Main Theme: Kähler-Einstein metric as canonical metric . . . 1

1.2 Content of the thesis . . . 3

2 Preliminaries **8**

2.1 Basic Kähler geometry . . . 8

2.2 Kähler-Einstein problem and complex Monge-Ampère equation . . . 10

2.3 C^2-estimate and complex Krylov-Evans estimate . . . 12

2.3.1 C^2-estimate . . . 12

2.3.2 Complex Krylov-Evans estimate . . . 15

2.4 Energy functionals and Variational point of view . . . 16

2.5 Finite dimensional approximation . . . 18

2.5.1 Tian-Catlin-Zelditch-Lu expansion . . . 18

2.5.2 F^0_ω functional and Chow norm . . . 20

2.5.3 Application: CSCK as minimizer of K-energy . . . 21

2.6 Tian's Conjecture and Partial C^0-estimate . . . 27

2.7 Futaki invariant and K-stability . . . 34

2.7.1 Analytic and algebraic definition of Futaki Invariant . . . 34

2.7.2 Donaldson-Futaki invariant and asymptotic slope of K-energy along one parameter subgroup . . . 36

2.7.3 Test configuration and K-stability . . . 40

2.7.4 Calculation of Futaki invariant . . . 41

3 Some Extension of general theories **49**

3.1 Twisted Kähler-Einstein equation and Invariant $R_\eta(X)$ 49

3.2 Existence of conic Kähler-Einstein metric on Fano manifold 55

3.2.1 A calculation of bisectional curvature of a conic metric 59

3.3 Log-Futaki invariant and log-K-stability 65

3.3.1 Log-Futaki invariant 65

3.3.2 Integrating log-Futaki-invariant 68

3.3.3 log-K-stability 69

4 Continuity method in Kähler-Einstein problem on toric Fano manifold **74**

4.1 Introduction to results 74

4.2 Consequence of Wang-Zhu's theory 77

4.3 Calculate $R(X)$ on any toric Fano manifold 83

4.4 On behaviors of the limit metric 86

4.4.1 Equation for the limit metric 86

4.4.2 Change to Complex Monge-Ampère equation 88

4.4.3 Discussion on the conic behavior of limit metric 90

4.4.4 Proof of Theorem 20 91

4.5 Partial C^0-estimate and Multiplier ideal sheaf 99

4.6 Example 101

4.7 Log-Futaki invariant for 1psg on toric Fano variety 104

4.7.1 Example 107

5 Intersection formula of Donaldson-Futaki invariant and applications **108**

5.1 Introduction 108

5.2 Intersection formula for the Donaldson-Futaki invariant 109

5.2.1 Song-Weinkove's condition from intersection formula 113

5.3 Special degeneration and K-stability of Fano manifold 114

5.3.1 Preliminaries from algebraic geometry 115

5.3.2 Step 0: Normalization 116

5.3.3 Step 1: Equivariant semi-stable reduction 117

5.3.4 Step 2: Log canonical modification 118

5.3.5 Step 3: Running MMP with scaling 119

5.3.6 Decreasing of DF-intersection number 120

5.3.7 Step 4: $\mathbb{Q}$-Fano extension . . . 123
5.3.8 Completion of Proof of Theorem 23 . . . 125
5.3.9 Simplification in the unstable case and Discussions . . . 126

6 Rotationally symmetric Kähler-Ricci solitons on flips 128

6.1 Introduction and motivation . . . 128
6.2 General setup and the result . . . 131
6.3 Reduction to ODE . . . 132
6.3.1 Rotationally Symmetric Model of Tian-Yau metrics . . . 135
6.4 Boundary condition at zero section . . . 135
6.5 Complete noncompact case . . . 137
6.5.1 Condition at infinity . . . 137
6.5.2 Existence and asymptotics . . . 138
6.6 Compact shrinking soliton . . . 141

Chapter 1

Introduction

1.1 Main Theme: Kähler-Einstein metric as canonical metric

Complex geometry aims to study complex manifolds, which are manifolds with complex structures. Kähler manifolds form a large class of complex manifolds. A Kähler manifold is a complex manifold with a Riemannian metric compatible with the complex structure. One basic problem in Kähler geometry is to find the canonical metric on a given Kähler manifold. Canonical means that the metric depends on the complex structure and is unique up to biholomorphic automorphisms. This problem is similar to Thurston's Geometrization Conjecture for real 3-manifolds. They all intend to classify manifolds by extending the Uniformization Theorem for Riemann Surfaces to higher dimensions.

One class of canonical metric is Kähler-Einstein (KE) metric. In order for a Kähler manifold X to have a Kähler-Einstein metric, its first Chern class has to be definite, that is either $-c_1(X)$ is a Kähler class, or $c_1(X) = 0$, or $c_1(X)$ is a Kähler class. For the first case, Aubin [Aub] and Yau [Yau1] independently proved the existence of Kähler-Einstein metric in $-c_1(X)$. For the second case, the existence of Kähler-Einstein metric in any Kähler class follows from Yau's solution of Calabi conjecture [Yau1]. These Kähler-Einstein metrics with zero Ricci curvature are known as Calabi-Yau metrics and play a major role in the String Theory.

The remaining case is the Fano case, i.e. when the anti-canonical line bundle $-K_X$ is ample. In general, there are obstructions to the existence of Kähler-Einstein metric on X. For example, a theorem of Matsushima [Mat] says that, any Kähler-Einstein Fano manifold has a reductive automorphism group, which is the complexification of the isometry group of the Kähler-Einstein metric. The goal now is to find all obstructions and characterize all Fano manifolds with Kähler-Einstein metric. Now this is a big subject of intensive research.

On the one hand, finding Kähler-Einstein metric reduces to solving a fully non-linear complex Monge-Ampère equation. The standard tool from PDE is to use continuity method or flow method.

Since the equation in general can not be solved, in both of these methods, the difficulty lies in studying the singularities formed at certain threshold of the parameter and hopefully the information of the singularities will give us obstructions to the existence. These are tied with the compactness result from Riemannian geometry. More specifically, along continuity method, the metrics have a uniform lower Ricci bound. By Cheeger-Colding's theory, they converge in Gromov-Hausdorff sense to some metric space. But it's a weak limit and one does not know much things about the singularities. For the Kähler-Ricci flow case, there is the well-known Hamilton-Tian's conjecture, which predicts the limit to be Kähler-Ricci soliton on some normal Fano variety. In general there may be a jump to a different complex structure and even jump to a singular variety.

On the other hand, Kähler-Einstein problem is a variational problem. Futaki [Fut] found an important invariant as the obstruction to this problem. Then Mabuchi [Mab1] defined the K-energy functional by integrating this invariant. The minimizer of the K-energy is the Kähler-Einstein metric. Tian [Tia9] proved that there is a Kähler-Einstein metric if and only if the K-energy is proper on the space of all Kähler metrics in $c_1(X)$. So the problem is how to test the properness of the K-energy.

Tian developed a program to reduce this infinitely dimensional problem to finitely dimensional problems. More precisely, he proved in [Tia9] that the space of Kähler metrics in a fixed Kähler class can be approximated by a sequence of spaces consisting of Bergman metrics. The latter spaces are finitely dimensional symmetric spaces. Tian ([Tia9]) then introduced the K-stability condition using the generalized Futaki invariant ([DiTi]) for testing the properness of K-energy on these finitely dimensional spaces. Later Donaldson [Don4] reformulated it using algebraically defined Futaki invariant (see (2.27)), which is now called *the Donaldson-Futaki invariant.* The following folklore conjecture is the guiding question in this area.

Conjecture 1 (Yau-Tian-Donaldson conjecture)**.** *Let (X, L) be a polarized manifold. Then there is a constant scalar curvature Kähler metric in $c_1(L)$ if and only if (X, L) is K-polystable.*

1.2 Content of the thesis

Chapter 2 In 2.1, we will recall some standard notation in Kähler geometry. In 2.2, we will reduce the Kähler-Einstein problem to solving a complex Monge-Ampère equation. Then we introduce the continuity method and reduce the solvability to establishing a priori estimates. In 2.3, we explain how the C^2-estimate and higher order estimate follow from the C^0-estimate. This means the non-existence of KE is due to the failure of uniform C^0-estimate. In 2.4, we will view Kähler-Einstein metric as critical points of two functionals: K-energy and F-functional. We will also define various other energy functional which we will use frequently later. We will state Tian's important result which gives a analytic criterion of existence of Kähler-Einstein metric using properness of K-energy or F-functional. This raise the problem of how to test the properness of K-energy functional. In 2.5, we will explain the important results and idea of finite dimensional approximation. This is the link to the algebraic geometry side of the problem. As an application of finite dimensional approximation, we prove:

Theorem 1. *Constant scalar curvature Kähler (CSCK) metric obtains the minimum of K-energy.*

In 2.6, we will explain Tian's conjecture which reduces properness of K-energy on the infinite dimensional space of Kähler metrics to the properness of K-energy on finite dimensional spaces. We explain how Tian's another conjecture called partial C^0-estimate would complete the program if established. In 2.7, we will explain the important Futaki invariant, which is roughly the asymptotic slope of K-energy along a one parameter subgroup of projective transformations. This invariant is used to test the properness in finite dimensional spaces.

Chapter 3 In this chapter, we generalize some of the standard theory associated to classical Kähler-Einstein problem. In 3.1, we consider the twisted Kähler-Einstein equation. We define the corresponding twisted version of K-energy and F-functional. We define the invariant R_η which measures the extent to which we can untwist such equation. Continuity method is a special case of twisted Kähler-Einstein. The other important case is the conic Kähle-Einstein metric. In 3.2, we explain the recent work of Jeffrey-Mazzeo-Rubinstein-Li which proved existence of conic Kähler-Einstein metric on Fano manifold along a smooth anti-canonical divisor under the assumption that the log-K-energy is proper. The main strategy is the same as in the smooth case. But there are many technical difficulties to overcome. One technical point is

Proposition 1. *There exists a reference conic Kähler metric whose holomorphic bisectional curvature is bounded from above but not from below.*

In 3.3, we extend the algebraic part of story to the conic setting. In particular, we explain the log-Futaki invariant and log-K-stability, and extend conjectural K-stability picture to conic case.

Chapter 4 This chapter forms first main part of this thesis. Here we carry out a detailed study of continuity method in Kähler-Einstein problem on toric Fano manifolds. As explained in 4.2, this study is partly based on Wang-Zhu's [WaZh] a priori estimates of the convex solutions to real Monge-Ampère equations which are reductions of the complex Monge-Ampère equations by toric symmetry. In 4.3, we calculate the greatest lower bound of Ricci curvature in the first Chern class of toric Fano manifold. This invariant (called $R(X)$) is also the maximal parameter for which one can solve the equations in the continuity method. It is entirely determined by the geometry of the momentum polytope. Such a polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_c O}$ intersects the boundary $\partial\triangle$ at point Q.

Theorem 2. *If $P_c \neq O$,*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|}$$

Here $|\overline{OQ}|$, $|\overline{P_c Q}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_c Q}$. In other words,

$$Q = -\frac{R(X_\triangle)}{1 - R(X_\triangle)} P_c \in \partial\triangle$$

If $P_c = O$, then there is Kähler-Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

In 4.4, we study the limit behavior of the sequence of solutions to continuity family when the parameter approach the maximal value. We prove the convergence upon holomorphic transformation.

Theorem 3. *After some biholomorphic transformation $\sigma_t : X_\triangle \to X_\triangle$, there is a subsequence $t_i \to R(X)$, such that $\sigma_{t_i}^* \omega_{t_i}$ converge to a Kähler current $\omega_\infty = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty$, with $\psi_\infty \in L^\infty(X_\triangle) \cap C^\infty(X_\triangle \backslash Bs(\mathfrak{L}_{\mathcal{F}}))$, which satisfies a complex Monge-Ampère equation* *of the form*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty)^n = e^{-R(X)\psi_\infty}\left(\sum_\alpha{}' b_\alpha |s_\alpha|^2\right)^{-(1-R(X))}\Omega. \tag{1.1}$$

Here $\Omega = e^{h_\omega}\omega^n$ *is a smooth volume form. For each vertex lattice point* $p_\alpha^{\mathcal{F}}$ *of* $\mathcal{F}$*,* b_α *is a constant satisfying* $0 < b_\alpha \leq 1$*.* $\|\cdot\| = \|\cdot\|_{FS}$ *is (up to multiplication of a constant) the Fubini-Study metric on* $K_{X_\triangle}^{-1}$*. In particular*

$$Ric(\omega_{\psi_\infty}) = R(X)\omega_{\psi_\infty} + (1 - R(X))\sqrt{-1}\partial\bar{\partial}\log(\sum_\alpha{}' b_\alpha |s_\alpha|^2). \tag{1.2}$$

This convergence should be compared to the Cheeger-Gromov convergence of manifolds with lower Ricci bounds. In particular, the limit we get has conic type singularities whose information can be read out from the geometry of the moment polytope. As a corollary of the convergence result, in 4.5, we prove

Corollary 1. *Tian's partial* C^0*-estimate holds along the continuity method on toric Fano manifolds.*

For special toric Fano manfolds, we can describe the multipler ideal sheaf very explicitly. We give examples in 4.6. In 4.7, we do some calculation for the log-Futaki invariant on toric Fano manifolds.

Theorem 4. *Let* $X_\triangle$ *be a toric Fano variety with a* $(\mathbb{C}^*)^n$ *action. Let* Y *be a general hyperplane section of* $X_\triangle$*. When* $\beta < R(X_\triangle)$*,* $(X_\triangle, \beta Y)$ *is log-K-stable along any 1 parameter subgroup in* $(\mathbb{C}^*)^n$*. When* $\beta = R(X_\triangle)$*,* $(X_\triangle, \beta Y)$ *is semi-log-K-stable along any 1 parameter subgroup in* $(\mathbb{C}^*)^n$ *and there is a 1 parameter subgroup in* $(\mathbb{C}^*)^n$ *which has vanishing log-Futaki invariant. When* $\beta > R(X_\triangle)$*,* $(X_\triangle, \beta Y)$ *is not log-K-stable.*

Chapter 5 This chapter grow out of the joint work with Dr. Chenyang Xu. In 5.2, we derive the intersection formula for Donaldson-Futaki invariant.

Theorem 5. *[LiXu] Assume* $\mathcal{X}$ *is normal, then*

$$\frac{a_1 b_0 - a_0 b_1}{a_0^2} = \mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)! a_0}\left(\frac{a_1}{a_0}\bar{\mathcal{L}}^{n+1} + \frac{n+1}{2}K_{\overline{\mathcal{X}}/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^n\right). \tag{1.3}$$

In 5.3, we modify the test configuration using Minimal Model Program (MMP) step by step and calculate the derivative of the Donaldson-Futaki invariant. We prove that any test configuration can be modified to a special test configuration which means the central fibre is a $\mathbb{Q}$-Fano variety. Moreover, the Donaldson-Futaki invariant is decreasing along the process.

Theorem 6. *[LiXu]Let X be a $\mathbb{Q}$-Fano variety. Assume $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a test configuration of $(X, -rK_X)$. We can construct a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ and a positive integer m, such that $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ is a and*

$$m\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s}).$$

Furthermore, if we assume $\mathcal{X}$ is normal, then the equality holds only when $(\mathcal{X}, \mathcal{X}_0)$ itself is a special test configuration.

These important facts allows us to confirm one of Tian's conjecture: to test K-stability, one only needs to test on special test configurations.

Theorem 7 (Tian's conjecture)**.** *([LiXu])Assume X is a $\mathbb{Q}$-Fano variety. If X is destablized by a test configuration, then X is indeed destablized by a special test configuration. More precisely, the following two statements are true.*

1. *(K-semistability) If $(X, -rK_X)$ is not K-semi-stable, then there exists a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ with a negative Futaki invariant $\mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s}) < 0$.*
2. *(K-polystability) Let X be a K-semistable variety. If $(X, -rK_X)$ is not K-polystable, then there exists a special test configuration $(\mathcal{X}^{st}, -rK_{\mathcal{X}^s})$ with Donaldson-Futaki invariant 0 such that $\mathcal{X}^s$ is not isomorphic to $X \times \mathbb{A}^1$.*

Chapter 6 In the last chapter, we construct examples of Kähler-Ricci solitons on direct sum of KE line bundles on KE manifolds. The construction is straightforward using the rotational symmetry to reduce the equation to an ODE.

Theorem 8. *Assume $(L, h) \to (M, \omega_{KE})$ is a line bundle with Hermitian metric h over a Kähler-Einstein manifold, such that $-\sqrt{-1}\partial\bar{\partial}\log h = -\epsilon\omega_{KE}$ and $Ric(\omega_{KE}) = \tau\omega_{KE}$. Then on the total space of $L^{\oplus n}$, there exist complete rotationally symmetric solitons of types depending on the sign of $\lambda = \tau - n\epsilon$. If $\lambda > 0$, there exists a unique rotationally symmetric shrinking soliton. If $\lambda = 0$, there exists a family of rotationally symmetric steady solitons. If $\lambda < 0$, there exists a family of rotationally symmetric expanding solitons.*

When $n = 1$, this construction recovers the previous constructions by Cao [Cao], Koiso [Koi], Feldman-Ilmane-Knopf [FIK]. The singularities of Kähler-Ricci flow correspond to extremal contractions in MMP. While their examples can serve as the local model in the case of divisorial contraction, the example in the above theorem should correspond to the Kähler-Ricci soliton appeared when a special flipping contraction happens, because the contracting base is of high codimension.

In the compactified Fano manifold, there also exists shrinking Kähler-Ricci soliton.

Theorem 9. *[Li3] Using the notation as in Theorem 8, assume* $\lambda = \tau - n\epsilon > 0$*, then on the space* $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n}) = \mathbb{P}(L^{-1} \oplus \mathbb{C}^{\oplus n})$*, there exists a unique rotationally symmetric shrinking Kähler-Ricci soliton.*

Remark: Ideas in different chapters are inter-related. For example,

- The theory to the twisted KE problem, in particular for the conic KE, are essentially the same as in the smooth case except for several technical difficulties.
- The idea of using symmetry to simplify the problem is manifest in both Chapter 4 (toric symmetry) and Chapter 6 (rotational symmetry).
- Since the Kähler-Ricci flow is just the metric counter part of Minimal Model Program (MMP) with scaling. The solitons in Chapter 6 should be the scaling limit of the special extremal contractions in MMP with scaling. Actually, this is true if the singularity of Kähler-Ricci flow is of type I.
- The study of singularities are important for the existence problem. These are tied with the compactness result from Riemannian geometry. Chapter 4 and Chapter 6 can be seen as examples of the singularities developed along continuity method and Kähler-Ricci flow respectively.

Chapter 2

Preliminaries

2.1 Basic Kähler geometry

Let (X, J) be a complex manifold. g is a Riemannian metric on X. Assume g is compatible with respect to J, i.e.

$$g(JV, JW) = g(V, W), \text{ for any } V, W \in T_{\mathbb{R}}X$$

Define the form $\omega_g(X, Y) = g(JX, Y)$. Then $g(X, Y) = \omega_g(X, JY)$. Then

Lemma 1. *The following are equivalent: 1. $\nabla^{LC} J = 0$, 2. $d\omega_g = 0$.*

The Kähler condition is characterized by either of the condition. ω_g is called the Kähler form of the Kähler metric g.

In local complex coordinate, any Kähler metric can be represented by its Kähler form

$$\omega_g = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad \text{satisfying } g = (g_{i\bar{j}}) > 0 \text{ and } d\omega_g = 0$$

Since $d\omega_g = 0$ and $\omega_g^n \neq 0$, ω represents a nonzero cohomology class in $H^{1,1}(M, \mathbb{R})$. Since ω_g comes from a Kähler metric which is positive definite, we write $[\omega_g] > 0$. The Kähler cone on any Kähler manifold is characterized by Demailly-Paun [DePau]. If $[\omega] \in H^2(X, \mathbb{Z})$ then $[\omega]$ is the first Chern class of an ample line bundle L: $[\omega] = c_1(L)$.

The curvature tensor of a Kähler metric is easier to compute using the complex coordinates.

- $\nabla_{\partial_{z_i}} \partial_{z_j} = \Gamma_{ij}^k \partial_{z_k}$, with $\Gamma_{ij}^k = g^{k\bar{l}} \partial_{z_j} g_{i\bar{l}}$. Similarly, $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = g^{l\bar{k}} \partial_{\bar{z}_i} g_{l\bar{j}}$
- $\nabla_{\partial_{z_i}} \partial_{\bar{z}_j} = 0 = \nabla_{\partial_{\bar{z}_i}} \partial_{z_j}$

So the curvature is (use $\partial_i = \partial_{z_i}$ and $\partial_{\bar{j}} = \partial_{\bar{z}_j}$)

$$R(\partial_{z_i}, \partial_{\bar{z}_j})\partial_{z_k} = R^l_{i\bar{j}k}\partial_{z_l}, \text{ with } R^l_{i\bar{j}k} = -\partial_{\bar{z}_j}\Gamma^l_{ik} = g^{l\bar{s}}(\partial_{\bar{j}} g_{r\bar{s}})g^{r\bar{q}}(\partial_i g_{k\bar{q}}) - g^{l\bar{q}}\partial_i\partial_{\bar{j}} g_{k\bar{q}}$$

So we get:

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{r\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{r\bar{l}}}{\partial \bar{z}_j}$$

The Ricci curvature

$$R_{i\bar{j}} = R_{\bar{j}i} = R^k_{k\bar{j}i} = -\partial_{\bar{j}}\left(g^{k\bar{l}}\partial_i g_{k\bar{l}}\right) = -\partial_{\bar{j}}\partial_i \log\det(g_{k\bar{l}})$$

So we get the simple expression for the Ricci curvature of the

$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\omega^n = -\sqrt{-1}\sum_{i,j}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log\det(g_{k\bar{l}})dz^i \wedge d\bar{z}^j$$

We can give the Chern-Weil explanation of this formula. Any volume form Ω induces an Hermitian metric on K_X^{-1} by

$$|\partial_{z_1} \wedge \cdots \partial_{z_n}|^2_\Omega = \frac{2^n\Omega}{\sqrt{-1}^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}$$

By abuse of notation, we denote the Chern curvature of the $(K_X^{-1}, |\cdot|_\Omega)$ to be $Ric(\Omega)$ then we see that $Ric(\omega) = Ric(\omega^n)$. So we see that the Ricci form is a closed (1,1) form representing the class $2\pi c_1(X)$.

$$S(\omega_\phi) = g^{i\bar{j}}_\phi Ric(\omega_\phi)_{i\bar{j}}, \quad \underline{S} = \frac{\langle n(2\pi c_1(X))[\omega]^{n-1}, [X]\rangle}{\langle [\omega]^n, [X]\rangle}$$

are the scalar curvature of ω_ϕ and average of scalar curvature. Note that $\underline{S}$ is a topological constant.

Lemma 2 ($\partial\bar{\partial}$-Lemma). *If $[\omega_1] = [\omega_2]$, then there exists $\phi \in C^\infty(M)$ such that $\omega_2 - \omega_1 = \sqrt{-1}\partial\bar{\partial}\phi$.*

This Lemma is very useful because it reduces equations on Kähler metriccs to equations involving Kähler potentials.

Definition 1. *Fix a reference metric ω and define the space of smooth Kähler potentials as*

$$\mathcal{H} := \mathcal{H}_\omega = \{\phi \in C^\infty(M) | \omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\} \tag{2.1}$$

Remark 1. *The set $\mathcal{H}$ depends on reference Kähler metric ω. However in the following, we will omit writing down this dependence, because it's clear that $\mathcal{H}$ is also the set of Hermitian metrics h on L whose curvature form*

$$\omega_h := -\sqrt{-1}\partial\bar{\partial}\log h$$

is a positive (1,1) form on X. Since ω_ϕ determines ϕ up to the addition of a constant, $\mathcal{H}/\mathbb{C}$ is the space of smooth Kähler metric in the Kähler class $[\omega]$. By abuse of language, sometimes we will not distinguish $\mathcal{H}$ and $\mathcal{H}/\mathbb{C}$.

2.2 Kähler-Einstein problem and complex Monge-Ampère equation

We are interested in Kähler-Einstein metrics on Kähler manifolds, that is the Kähler metric $\omega_{KE} \in [\omega]$ satisfying

$$Ric(\omega_{KE}) = \lambda\omega_{KE}$$

There are three cases.

$c_1(X,J) < 0$: X is canonically polarized. Let $[\omega] = -c_1(X,J)$ and $\lambda = -1$. There exists a unique Kähler-Einstein metric ω_{KE} in $-c_1(X,J)$ such that $Ric(\omega_{KE}) = -\omega_{KE}$. This was proved independently by Aubin [Aub] and Yau [Yau1].

$c_1(X,J) = 0$: X is Calabi-Yau. Let $[\omega]$ be any Kähler class and $\lambda = 0$. There exists a unique Kähler metric in $[\omega]$ which is Ricci flat $Ric(\omega) = 0$. This is a consequence of Yau's solution of Calabi's conjecture in [Yau1].

$c_1(X,J) > 0$: X is Fano, i.e. anti-canonically polarized. Let $[\omega] = c_1(X)$ and $\lambda = 1$. In contrast with the previous 2 cases, in general there are obstructions to the existence of KE metric. Matsushima [Mat] proved Kähler-Einstein Fano manifold must have reductive automorphism group which is just the complexification of the isometric group $\mathrm{Isom}(X, g_{KE})$. Futaki [Fut] found an important invariant which is now called Futaki invariant as the obstruction to the existence. The Yau-Tian-Donaldson conjecture aims to characterize all the Fano manifolds which have Kähler-Einstein metrics.

From PDE point of view, the Kähler-Einstein equation is equivalent to the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{h_\omega - \lambda\phi}\omega^n$$

where h_ω is the function measuring the deviation of ω from ω_{KE} which is determined by the following Lemma.

Lemma 3. *There exists $h_\omega \in C^\infty(X)$ such that*

$$Ric(\omega) - \lambda\omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \quad \int_X e^{h_\omega}\omega^n/n! = V$$

Through out the paper, we will use the notation h_ω for this meaning and call it the Ricci potential of the smooth metric ω.

One classical method to solve the complex Monge-Ampère equation is continuity method. The main reason that we can solve the case when $\lambda = -1, 0$ is that along the continuity method we can get a priori C^0-estimate from which the C^2 and higher order estimate follows (as we will discuss in the next section). But for the $\lambda = 1$ case, the C^0-estimate in general does not hold. More precisely, for the Fano case, we fix a reference metric ω and consider a family of equations with parameter t:

$$Ric(\omega_\phi) = t\omega_\phi + (1-t)\omega \Longleftrightarrow (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{h_\omega - t\phi}\omega^n \qquad (*)_t$$

Define $\mathcal{S} = \{t \in [0,1]; ((*)_t) \text{ is solvable }\}$. Then by [Yau1], $0 \in \mathcal{S}$. By implicit function theorem, $\mathcal{S}$ is open. To show the closed-ness of $\mathcal{S}$ one needs to show uniform a priori estimate for ϕ_t. But when there is no Kähler-Einstein metric on X, then $\|\phi_t\|_{C^0}$ will blow up when t approaches some critical value. We will study this continuity method in detail on toric Fano manifolds in Chapter 4.

The other method to get a Kähler-Einstein metric is to use the Kähler-Ricci flow. Actually one can run Kähler-Ricci flow on any projective variety with mild singularties [SoTi]. This can be seen as the metric counterpart of Minimal Model Program in birational algebraic geometry. See Chapter 6 for more details.

2.3 C^2-estimate and complex Krylov-Evans estimate

2.3.1 C^2-estimate

Lemma 4. *[Yau1, Yau2]*

1. *Assume ω_ϕ satisfies $\omega_\phi^n = e^F\omega^n$. Let $\Lambda = \log(n+\Delta\phi) - \lambda\phi = \log tr_\omega\omega_\phi - \lambda\phi$, then*

$$\Delta'\Lambda \quad \geq \quad \left(\inf_{i\neq j} S_{i\bar{i}j\bar{j}} + \lambda\right)\sum_i \frac{1}{1+\phi_{i\bar{i}}} + \left(\Delta F - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}\right)\frac{1}{n+\Delta\phi} - \lambda n \qquad (2.2)$$

2. *Define $\Xi = \log tr_{\omega_\phi}\omega - \lambda\phi$, then*

$$\Delta'\Xi \geq (\inf R'_{i\bar{i}} - \lambda n) + (\lambda - \sup S_{i\bar{i}j\bar{j}})tr_{\omega_\phi}\omega \qquad (2.3)$$

Proof. Let $f = tr_{\omega_\phi}\omega$ and Δ' be the complex Laplacian associated with Kähler metric ω_ϕ. As in [Yau2], we have the Chern-Lu's identity:

$$\Delta' f = g'^{i\bar{l}}g'^{k\bar{j}}R'_{k\bar{l}}g_{i\bar{j}} + g'^{i\bar{j}}g'^{k\bar{l}}T^\alpha_{i,k}T^\beta_{\bar{j},\bar{l}}g_{\alpha\bar{\beta}} - g'^{i\bar{j}}g'^{k\bar{l}}S_{i\bar{j}k\bar{l}}.$$

Here the tensor $T^\alpha_{i,j} = \tilde{\Gamma}^\alpha_{ij} - \Gamma^\alpha_{ij}$ is the difference of Levi-Civita connections $\tilde{\Gamma}$ and Γ associated with g_ω and $g' = g_{\omega_\phi}$ respectively. $R'_{k\bar{j}}$ is the Ricci curvature of ω_ϕ and $S_{i\bar{j}k\bar{l}}$ is the curvature of reference metric ω. Let ∇' be the gradient operator associated with g_{ω_ϕ}, then

$$\begin{aligned}
\Delta'\log f \quad &= \quad \frac{\Delta' f}{f} - \frac{|\nabla' f|^2_{\omega_\psi}}{f^2} \\
&= \quad \frac{g'^{i\bar{l}}g'^{k\bar{j}}R'_{k\bar{l}}g_{i\bar{j}}}{f} - \frac{g'^{i\bar{j}}g'^{k\bar{l}}S_{i\bar{j}k\bar{l}}}{f} + \frac{g'^{i\bar{j}}g'^{k\bar{l}}T^\alpha_{i,k}T^\beta_{\bar{j},\bar{l}}g_{\alpha\bar{\beta}}}{f} - \frac{g'^{p\bar{q}}g'^{i\bar{j}}g'^{k\bar{l}}T^\alpha_{ip}T^{\bar{\beta}}_{\bar{l}\bar{q}}g_{\alpha\bar{j}}g_{k\bar{\beta}}}{f^2} \\
&= \quad \frac{\sum_i \mu_i^{-2}R'_{i\bar{i}}}{\sum_i \mu_i^{-1}} - \frac{\sum_{i,j}\mu_i^{-1}\mu_j^{-1}S_{i\bar{i}j\bar{j}}}{\sum_i \mu_i^{-1}} + \frac{\sum_{i,k,\alpha}\mu_i^{-1}\mu_k^{-1}|T^\alpha_{ik}|^2}{\sum_i \mu_i^{-1}} - \frac{\sum_p \mu_p^{-1}|\sum_i \mu_i^{-1}T^i_{ip}|^2}{(\sum_i \mu_i^{-1})^2} \\
&\geq \quad \inf_i R_{i\bar{i}} - (\sup_{i,j} S_{i\bar{i}j\bar{j}})\sum_i \mu_i^{-1} = \inf_i R_{i\bar{i}} - (\sup_{i,j} S_{i\bar{i}j\bar{j}})f
\end{aligned} \qquad (2.4)$$

In the 3rd equality in (2.4), for any fixed point $P \in X$, we chose a coordinate near P such that $g_{i\bar{j}} = \delta_{ij}$, $\partial_k g_{i\bar{j}} = 0$. We can assume $g' = g_{\omega_\phi}$ is also diagonalized so that

$$g'_{i\bar{j}} = \mu_i\delta_{ij}, \quad \text{with } \mu_i = 1 + \psi_{i\bar{i}}.$$

For the last inequality in (2.4), we the following inequality:

$$\begin{aligned}\sum_p \mu_p^{-1}|\sum_i \mu_i^{-1}T^i_{ip}|^2 &= \sum_p \mu_p^{-1}\left|\sum_i \mu_i^{-1/2}T^i_{ip}\mu_i^{-1/2}\right|^2\\ &\le (\sum_{p,i}\mu_p^{-1}\mu_i^{-1}|T^i_{ip}|^2)(\sum_i \mu_i^{-1})\\ &\le (\sum_{p,i,\alpha}\mu_p^{-1}\mu_i^{-1}|T^\alpha_{ip}|^2)(\sum_i \mu_i^{-1}).\end{aligned}$$

□

Proposition 2. *1. There exists $\lambda = \lambda(n, \inf \Delta F, \inf_{i\neq j} S_{i\bar{i}j\bar{j}})$ and $C = C(n, \inf \Delta F, \sup F, \inf_{i\neq j} S_{i\bar{i}j\bar{j}})$ such that*

$$tr_\omega \omega_\phi \le Ce^{\lambda(\mathrm{Osc}(\phi))}$$

2. There exists a constant $\lambda = \lambda(n, \sup S_{i\bar{i}j\bar{j}})$ and $C = C(\inf Ric(\omega_\phi), \sup S_{i\bar{i}j\bar{j}})$ such that

$$tr_{\omega_\phi}\omega \le Ce^{\lambda(\mathrm{Osc}(\phi))} \tag{2.5}$$

Remark 2. *This proposition implies that, under appropriate assumptions, the C^2 estimate is valid if there is a C^0 estimate for the potential. Yau [Yau2] used Chern-Lu's formula to deduce the Schwartz Lemma which generalize the classical Schwartz Lemma by Ahlfors.*

Proof. 1. Let $C_1 = -\min(0, \inf \Delta F - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}) \ge 0$. Since $\frac{1}{1+\phi_{i\bar{i}}} \ge \frac{1}{n+\Delta\phi}$, we get

$$\begin{aligned}\Delta'\Lambda &\ge (\lambda + \inf_{i\neq j} S_{i\bar{i}j\bar{j}} - C_1)\sum_i \frac{1}{1+\phi_{i\bar{i}}} - \lambda n = C_2\sum_i \frac{1}{1+\phi_{i\bar{i}}} - C_3\\ &\ge C_2(n+\Delta\phi)^{1/(n-1)}e^{\frac{-F}{n-1}} - C_3\end{aligned}$$

with $C_2 = (\lambda + \inf_{i\neq j} S_{i\bar{i}j\bar{j}} - C_1) > 0$ by choosing λ sufficiently large and $C_3 = \lambda n$. For the 2nd inequality, we used the following trick

$$\sum_i \frac{1}{1+\phi_{i\bar{i}}} \ge \left(\frac{\sum_i (1+\phi_{i\bar{i}})}{\prod_j (1+\phi_{j\bar{j}})}\right)^{1/(n-1)} = ((n+\Delta\phi)\omega^n/\omega_\phi^n)^{1/(n-1)}.$$

So at the maximal point P of Λ, we have

$$0 \ge \Delta'\Lambda \ge C_2(tr_\omega\omega_\phi)(P)^{\frac{1}{n-1}}e^{\frac{-F(P)}{n-1}} - C_3,$$

which implies $tr_\omega(\omega_\phi)(P) \leq (\frac{C_3}{C_2})^{n-1} e^{F(P)} \leq C_4 e^{\sup F} = C_5$. So at general point $x \in X$, we get the C^2-estimate:

$$n + \Delta\phi(x) \leq (tr_\omega \omega_\phi)(x) \leq tr_\omega \omega_\phi(P) e^{\lambda(\phi(x)-\phi(P))} \leq C_5 e^{\lambda \mathrm{Osc}(\phi)}$$

2. Assume $Ric(\omega_\phi) \geq -\delta\omega_\phi$. Let $f = tr_{\omega_\phi}\omega$, then by (2.3),

$$\Delta'(\log f - \lambda\phi) \geq -(\delta + \lambda n) + (\lambda - \sup S_{i\bar{i}j\bar{j}}) f =: C_1 f - C_2$$

for some constants $C_1 > 0$, $C_2 > 0$, if we choose λ to be sufficiently large. So at the maximum point P of the function $\log f - \lambda\phi$, we have

$$0 \geq \Delta'(\log f - \lambda\phi)(P) = C_1 f(P) - C_2.$$

with $C_1 = -(\delta + \lambda n)$ and $C_2 = \lambda - \sup S_{i\bar{i}j\bar{j}}$. So

$$f(P) = tr_{\omega_\phi}(\omega)(P) \leq C_3.$$

So for any point $x \in X$, we have

$$tr_{\omega_\phi}\omega(x) \leq C_3 e^{\lambda(\phi(x)-\phi(P))} \leq C_3 e^{\lambda(\mathrm{Osc}(\phi))}.$$

□

To apply the Chern-Lu's inequality method, we sometimes need to use the following observation [Yau2]:

Lemma 5. *Let (X, J, g) be a complex Kähler manifold. If $Y \subset X$ is a complex submanifold. Then the holomorphic bisectional curvature $R^Y_{i\bar{i}j\bar{j}}$ of Y is bounded from above by the holomorphic bisectional curvature $R^X_{i\bar{i}j\bar{j}}$ of X.*

Proof. By Gauss' formula:

$$R^X(\partial_i, \overline{\partial_i}, \partial_j, \overline{\partial_j}) = R^Y(\partial_i, \overline{\partial_i}, \partial_j, \overline{\partial_j}) + |\mathrm{II}(\partial_i, \partial_j)|^2 - |\mathrm{II}(\partial_i, \overline{\partial_j})|^2$$

For any $V, W \in TY$ and $N \in (TY)^{\perp}$, because $\nabla J = 0$ by Lemma 1, we have

$$\begin{aligned}
\langle \mathrm{II}(JV, JW), N\rangle &= \langle \nabla_{JV}(JW), N\rangle = \langle J\nabla_{JV}W, N\rangle = -\langle \nabla_{JV}W, JN\rangle \\
&= -\langle \nabla_W(JV), JN\rangle = -\langle J\nabla_W V, JN\rangle \\
&= -\langle \nabla_W V, N\rangle = -\langle \mathrm{II}(V, W), N\rangle
\end{aligned}$$

So $\mathrm{II}(JV, JW) = -\mathrm{II}(V, W)$ which implies $\mathrm{II}(\partial_i, \overline{\partial_j}) = 0$. □

In particular, if $X \subset \mathbb{P}^N$ is a projective manifold and $\omega = \omega_{FS}|_X$. Then where $\tilde{R}$ is the curvature of Fubini-Study metric of ambient $\mathbb{P}^N$. $\tilde{R}$ satisfies: $\tilde{R}_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$, So the holomorphic bisectional curvature of the restriction of Fubini-Study metric satisfies $R_{i\bar{i}j\bar{j}} \le \tilde{R}_{i\bar{i}j\bar{j}} = 2$. For any $\sigma \in PSL(N, \mathbb{C})$, let $\omega_\sigma = \sigma^*\omega_{FS}|_X = \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\phi_\sigma$. Then

$$\phi_\sigma = \log \frac{|\sigma \cdot Z|^2}{|Z|^2}$$

By the above discussion and by (2.5), we have the C^2-estimate of ϕ_σ in terms of oscillation of ϕ_σ

Corollary 2.

$$\omega_\sigma \le Ce^{\lambda\ osc(\phi_\sigma)}\omega$$

In particular, since $\mathrm{Osc}(\phi_\sigma)$ *has log polynomial growth,*

$$\log \frac{\omega_\sigma^n}{\omega^n} \le n \log C + n\lambda\ \mathrm{Osc}(\phi_\sigma)$$

has log polynomial growth.

2.3.2 Complex Krylov-Evans estimate

In the proof of Calabi conjecture in [Yau1], Yau proved 3rd order estimate. Define $S = \sum g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} \phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}}$. By complicated computation, Yau showed that

$$\Delta'(S + C_1\Delta\phi) \ge C_2 S - C_3$$

We have now a systematic way of getting higher order estimate thanks to the work of Krylov-Evans-(Safonov). This estimate is purely local in contrast with Yau's proof which is global and uses maximal principle. We record a version here from [Bło] (See also [Tia3]:

Theorem 10 ([Bło]). *Let u be a C^4 plurisubharmonic function in an open $\Omega \subset \mathbb{C}^n$ such that $f := \det(u_{i\bar{j}}) > 0$. Then for any $\Omega' \Subset \Omega$ there exist $\alpha \in (0,1)$ depending only on n and on upper bounds for $\|u\|_{C^{0,1}(\Omega)}$, $\sup_\Omega \Delta u$, $\|f\|_{C^{0,1}(\Omega)}$, $1/\inf_\Omega f$ and $C > 0$ depending in addition on a lower bound for $\operatorname{dist}(\Omega', \partial\Omega)$ such that*

$$\|u\|_{C^{2,\alpha}} \le C$$

2.4 Energy functionals and Variational point of view

The Kähler-Einstein problem is variational. Futaki [Fut] found an important invariant (now known as Futaki invariant) as the obstruction to its existence. Then Mabuchi [Mab1] defined K-energy functional by integrating this invariant:

$$\nu_\omega(\omega_\phi) = -\int_0^1 dt \int_X (S(\omega_{\phi_t}) - \underline{S})\dot{\phi}_t \omega_\phi^n / n!$$

where ϕ_t is any path connecting 0 and ϕ in the space of Kähler potentials $\mathcal{H}$. This is well-defined, i.e. it is independent of the path connecting ω and ω_ϕ. This follows from Stokes' Theorem because the one form defined by the variation:

$$\delta\nu_\omega(\omega_\phi) \cdot \delta\phi = -\int_X S(\omega_\phi) - \underline{S})\delta\phi\omega_\phi^n/n! \tag{2.6}$$

is a closed one form one $\mathcal{H}$ and $\mathcal{H}$ is contractible.

It's easy to see that the constant scalar curvature Kähler (CSCK) metric is the critical point of K-energy. Actually, CSCK metric obtains the absolute minimum of K-energy. (See section 2.5.3).

In the Fano case, it's easy to see that for $\omega \in 2\pi c_1(X)$,

$$\omega \text{ is CSCK } \Longleftrightarrow h_\omega = 0 \Longleftrightarrow \omega \text{ is KE}$$

Define the functional:

$$I_\omega(\omega_\phi) = \int_X \phi(\omega^n - \omega_\phi^n)/n!, \quad J_\omega(\omega_\phi) = \int_0^1 \frac{I(x\phi)}{x} dx, \quad F_\omega^0(\phi) = J_\omega(\phi) - \int_X \phi\omega^n/n!$$

We have the well known formula for K-energy [Tia10]:

$$\begin{aligned}
\nu_\omega(\omega_\phi) &= \int_X \log\frac{\omega_\phi^n}{e^{h_\omega}\omega^n}\frac{\omega_\phi^n}{n!} - (I_\omega - J_\omega)(\phi) + \int_X h_\omega \omega^n/n! \\
&= \int_X \log\frac{\omega_\phi^n}{e^{h_\omega-\phi}\omega^n}\frac{\omega_\phi^n}{n!} + F_\omega^0(\phi) + \int_X h_\omega \omega^n/n!
\end{aligned}$$

It's easy to verify that

$$(I_\omega - J_\omega)(\phi) = -\left(\int_X \phi\omega_\phi^n/n! + F_\omega^0(\phi)\right)$$

There is another energy functional associated with Monge-Ampère equation:

$$F_\omega(\phi) = F_\omega^0(\phi) - V\log\left(\frac{1}{V}\int_X e^{h_\omega-\phi}\omega^n/n!\right)$$

Lemma 6. *We have the following relations between these functionals:*

1.
$$F_\omega(\omega_\phi) = \nu_\omega(\omega_\phi) - \int_X h_\omega\frac{\omega^n}{n!} + \int_X h_{\omega_\phi}\frac{\omega_\phi^n}{n!} \le \nu_\omega(\omega_\phi) - \int_X h_\omega\frac{\omega^n}{n!}.$$

2.
$$J_\omega(\phi) = \sum_{i=0}^{n-1}\frac{n-i}{n+1}\int_X \partial\phi\wedge\bar\partial\phi\wedge\omega_\phi^k\wedge\omega^{n-1-k}/n!$$

3.
$$\frac{n+1}{n}J_\omega(\phi) \le I_\omega(\phi) \le (n+1)J_\omega(\phi),$$

4.
$$F_\omega^0(\phi) = -\frac{1}{n+1}\sum_{i=0}^{n}\int_X \phi\omega_\phi^i\wedge\omega^{n-i}/n!$$

5.
$$F_\omega^0(\phi_2) \ge F_\omega^0(\phi_1) - \int_X (\phi_2-\phi_1)\omega_{\phi_1}^n/n!$$

6.
$$\frac{d}{dt}J_\omega(\phi_t) = \int_X \dot\phi(\omega^n - \omega_\phi^n)/n!, \quad \frac{d}{dt}F_\omega^0(\phi_t) = -\int_X \dot\phi\omega_\phi^n/n!$$

7. *[Ding]*
$$J_\omega(t\phi) \le t^{1+1/n}J_\omega(\phi), \ \textit{for } 0 < t < 1.$$

Definition 2. *A functional $F : \mathcal{H} \to \mathbb{R}$ is called proper if there is an inequality of the type*

$$F(\omega_\phi) \geq f\left((I-J)_\omega(\omega_\phi)\right), \text{ for any } \omega_\phi \in \mathcal{H}$$

where $f(t) : \mathbb{R}_+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t\to+\infty} f(t) = +\infty$.

Theorem 11. *[Tia9] If $Aut(X,J)$ is discrete. There exists a Kähler-Einstein metric on X if and only if either $F_\omega(\omega_\phi)$ or $\nu_\omega(\omega_\phi)$ is proper.*

So, at least when there is no holomorphic vector field, the problem is how to test the properness of K-energy. The basic idea due to Tian ([Tia4], [Tia5]) is to use finite dimensional approximation.

2.5 Finite dimensional approximation

2.5.1 Tian-Catlin-Zelditch-Lu expansion

Let (X,L) be a fixed polarized manifold. h is any fixed Hermitian metric on L with positive Chern curvature $\omega_h = -\sqrt{-1}\partial\bar{\partial}\log h$. Let $N_k = \dim H^0(X, L^k)$, $V = \int_X \omega_h^n/n!$.

Definition 3.
1. *$\mathcal{B}_k := \left\{\frac{1}{k}\log\sum_{\alpha=1}^{N_k}|s_\alpha|^2; \{s_\alpha\} \text{ is a basis of } H^0(X,L^k)\right\} \subset \mathcal{H}$*
2. *$\mathcal{H}_k = ($ the space of inner products on the vector space $H^0(X, L^{\otimes k})) \cong GL(N_k,\mathbb{C})/U(N_k, C)$*
3. *Define two maps between $\mathcal{H}_k$ and $\mathcal{H}$.*

$$\begin{array}{rcl}
\mathrm{Hilb}_k : \mathcal{H} & \longrightarrow & \mathcal{H}_k \\
h & \mapsto & \langle s_1, s_2\rangle_{\mathrm{Hilb}_k(h)} = \displaystyle\int_X (s_1,s_2)_{h^{\otimes k}}\omega_h^n/n!, \quad \forall s_1, s_2 \in H^0(X, L^k) \\
\mathrm{FS}_k : \mathcal{H}_k & \longrightarrow & \mathcal{B}_k \subset \mathcal{H} \\
H_k & \mapsto & |s|^2_{\mathrm{FS}_k(H_k)} = \dfrac{|s|^2}{\left(\sum_{\alpha=1}^{N_k}\left|s_\alpha^{(k)}\right|^2\right)^{1/k}}, \quad \forall s \in L.
\end{array}$$

In the above definition, $\{s_\alpha^{(k)}; 1 \leq \alpha \leq N_k\}$ is an orthonormal basis of the Hermitian complex vector space $(H^0(X,L^k), H_k)$.

For any Hermitian metric h on L such that $\omega_h > 0$, the kth Bergman metric of h is

$$h_k = \mathrm{FS}_k(\mathrm{Hilb}_k(h)) \in \mathcal{H}.$$

Let $\{s_\alpha^{(k)}, 1 \leq \alpha \leq N_k\}$ be an orthonormal basis of $\mathrm{Hilb}_k(h)$. Define the kth (suitably normalized) Bergman kernel of ω

$$\rho_k(\omega) = \sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|^2_{h^{\otimes k}}.$$

Note that although h is determined by ω_h up to a multiplication by a positive constant, $\rho_k(\omega_h)$ doesn't depend on the choice of h.

The following proposition is now well known.

Proposition 3. *([Tia4],[Cat],[Zel],[Ruan],[Lu2])*

(i) For fixed ω, there is an asymptotic expansion as $k \to +\infty$

$$\rho_k(\omega) = A_0(\omega)k^n + A_1(\omega)k^{n-1} + \dots$$

where $A_i(\omega)$ are smooth functions on X defined locally by ω.

(ii) In particular

$$A_0(\omega) = 1, \quad A_1(\omega) = \frac{1}{2}S(\omega).$$

(iii) The expansion holds in C^∞ in that for any $r, N \geq 0$

$$\left\| \rho_k(\omega) - \sum_{i=0}^{N} A_i(\omega)k^{n-i} \right\|_{C^r(X)} \leq K_{r,N,\omega} k^{n-N-1}$$

for some constants $K_{r,N,\omega}$. Moreover the expansion is uniform in that for any r, N, there is an integer s such that if ω runs over a set of metrics, which are bounded in C^s, and with ω bounded below, the constants $K_{r,N,\omega}$ are bounded by some $K_{r,N}$ independent of ω.

The following approximation result is a corollary of Proposition 3.(i)–(ii).

Corollary 3 ([Tia4])**.** *Using the notation at the beginning of this subsection, we have, as $k \to +\infty$, $h_k \to h$, and $\omega_k \to \omega$, the convergence in C^∞ sense. More precisely, for any $r > 0$, there exists a constant $C_{n,r,\omega}$ such that*

$$\left\| \log \frac{h_k}{h} \right\|_{C^r} \leq C_{n,r,\omega} \frac{\log k}{k}, \qquad \|\omega_k - \omega\|_{C^r} \leq C_{n,r,\omega} k^{-2}. \tag{2.7}$$

Proof. It's easy to see that

$$h_k = h \cdot \left(\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} \right)^{-\frac{1}{k}} =: he^{-\psi_k}.$$

Note that by the expansion in Proposition 3.(i)–(ii), we have

$$\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} = k^n \left(1 + \frac{1}{2} S(\omega) k^{-1} + O(k^{-2})\right) = k^n (1 + O(k^{-1})).$$

So

$$\psi_k = \frac{1}{k} \log \left(\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} \right) = n \frac{\log k}{k} + O(k^{-2}).$$

The error term is in C^∞ sense. So the first inequality in (2.7) holds. The second inequality in (2.7) follows because

$$\omega_k - \omega = \sqrt{-1}\partial\bar{\partial}\psi_k.$$

□

2.5.2 F^0_ω functional and Chow norm

Under the orthonormal basis $\{\tau_\alpha^{(k)}, 1 \le \alpha \le N_k\}$ of H_k^*, $H^0(X, L^k) \cong \mathbb{C}^{N_k}$ and $\mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{CP}^{N_k - 1}$.

For any $H_k \in \mathcal{H}_k$, take an orthonormal basis $\{s_\alpha, 1 \le \alpha \le N_k\}$ of H_k. Let $\det H_k$ denote the determinant of matrix $(H_k)_{\alpha\beta} = (H_k^*(s_\alpha, s_\beta))$. $\{s_\alpha\}$ determines a projective embedding into $\mathbb{CP}^{N_k-1}$. (Note that the fixed isomorphism $\mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{CP}^{N_k-1}$ is determined by the basis $\{\tau_\alpha^{(k)}\}$.) The image of this embedding is denoted by $X_k(H_k) \subset \mathbb{CP}^{N_k-1}$ and has degree $d_k = V n! k^n$. $X_k(H_k)$ has a Chow point ([Zha], [Paul]):

$$\hat{X}_k(H_k) \in \mathcal{W}_k := H^0(Gr(N_k - n - 2, \mathbb{P}^{N_k-1}), \mathcal{O}(d_k))$$

such that the corresponding divisor

$$Zero(\hat{X}_k(H_k)) = \{L \in Gr(N_k - n - 2, \mathbb{P}^{N_k-1}); L \cap X_k(H_k) \ne \emptyset\}.$$

Proposition 4 ([Zha], [Paul]). *$\mathcal{W}_k$ has a Chow norm $\|\cdot\|_{\mathrm{CH}(H_k^*)}$, such that for all $H_k \in \mathcal{H}_k$ we* *have*

$$\frac{1}{N_k}\log\det H_k - \frac{k}{V}F^0_\omega(\mathrm{FS}(H_k)) = \frac{1}{Vk^n}\log\|\hat{X}_k(H_k)\|^2_{\mathrm{CH}(H_k^*)}$$

$SL(N_k,\mathbb{C})$ acts on $\mathcal{H}_k$ and $\mathcal{W}_k$. Note that $X_k(\sigma\cdot H_k^*) = \sigma\cdot X_k(H_k^*)$. Define

$$f_k(\sigma) = \log\left(\|\hat{X}_k(\sigma\cdot H_k^*)\|^2_{\mathrm{CH}(H_k^*)}\right) \quad \forall\sigma\in SL(N_k,\mathbb{C})$$

It's easy to see that $f_k(\sigma\cdot\sigma_1) = f_k(\sigma)$ for any $\sigma_1\in SU(N_k)$, so f_k is a function on the symmetric space $SL(N_k,\mathbb{C})/SU(N_k)$. We have

Proposition 5. *([KeNe], [Zha], [Don2], [PhSt1]) $f_k(\sigma)$ is convex on $SL(N_k,\mathbb{C})/SU(N_k)$.*

To relate $\mathcal{H}$ and $\mathcal{H}_k$, following Donaldson [Don2], we change $FS(H_k)$ in the above formula into general $h_\phi\in\mathcal{H}$ and define:

Definition 4. *For all $h_\phi\in\mathcal{H}$ and $H_k\in\mathcal{H}_k$,*

$$\tilde{P}_k(h_\phi, H_k) = \frac{1}{N_k}\log\det H_k - \frac{k}{V}F^0_\omega(\phi).$$

Note that, for any $c\in\mathbb{R}$,

$$\tilde{P}_k(e^c h_\phi, e^{ck}H_k) = \tilde{P}_k(h_\phi, H_k). \tag{2.8}$$

Remark 3. *This definition differs from Donaldson's definition by omitting two extra terms, since we find no use for these terms in the following argument.*

2.5.3 Application: CSCK as minimizer of K-energy

The finite dimensional symmetric space $\mathcal{H}_k\cong GL(N_k)/U(N_k)$ in Definition section 5.3 can be identified as the space of Hermitian inner products on $H^0(X,L^k)$. There are natural convex functional on $\mathcal{H}_k$:

$$\frac{1}{N_k}\log\det H_k - \frac{k}{V}F^0_\omega(\mathrm{FS}(H_k)) =: \frac{1}{Vk^n}\log\|\widehat{X}_k(H_k)\|^2_{\mathrm{CH}(H_k^*)}$$

where I_k is called Aubin-Yau functional defined as integration of some Bott-Chern class. For any $H_k\in\mathcal{H}_k$, we can choose an orthonormal base of H_k and get a Kodaira's embedding into $X_k(H_k)\subset\mathbb{P}(H^0(X,L^k)^*)\cong\mathbb{P}^{N_k-1}$. $\widehat{X}_k(H_k)$ denotes the Chow form of $X_k(H_k)$. The functional defines some norm $\|\cdot\|_{\mathrm{CH}(H_k^*)}$ for any Chow form. These Chow-norm functionals approximate K-energy as $\mathcal{H}_k$ approximate $\mathcal{H}$ by the Tian-Catlin-Zelditch-Lu expansion. If there exists a constant scalar curvature Kähler (CSCK) metric ω_∞, then its Bergman metrics $\omega_\infty^{(k)}$ is the approximate minimum of the k-th Chow-norm. Using this, we can prove

Theorem 12. *[Li2] Assume there is CSCK metric ω_∞ on the polarized manifold (X, L), then it obtains the global minimum of K-energy.*

Remark 4. *In the Fano case, this was proved by Bando-Mabuchi [BaMa] using the continuity method. Donaldson [Don2] proved the above result assuming the automorphism group is discrete. Chen-Tian [ChTi] proved the same result for general Kähler class (not necessarily rational) using the convexity of K-energy along geodesics in $\mathcal{H}$.*

To prepare for the proof of the above theorem, assume we have a Kähler metric of constant scalar curvature ω_∞ in the Kähler class $c_1(L)$. Take a $h_\infty \in \mathcal{K}_1$ such that $\omega_\infty = \omega_{h_\infty}$. We will make extensive use of the kth Bergman metric of h_∞ and its associated objects, so for the rest of this note, we denote

$$H_k^* = \mathrm{Hilb}_k(h_\infty), \quad h_k^* = \mathrm{FS}_k(H_k^*) = \mathrm{FS}_k(\mathrm{Hilb}_k(h_\infty)), \quad \omega_k^* = \omega_{h_k^*} = \frac{1}{k}\sqrt{-1}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2\right). \tag{$*$}$$

Hereafter, we fix an orthonormal basis $\{\tau_\alpha^{(k)}, 1 \le \alpha \le N_k\}$ of $H_k^* = \mathrm{Hilb}_k(h_\infty)$.

The following is the direct corollary of Proposition 3.

Proposition 6. *For any $r > 0$, there exists some constant C_{n,r,ω_∞} such that*

$$\left\| \sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}} - \frac{N_k}{V} \right\|_{C^r} \le C_{n,r,\omega_\infty} k^{n-2} \tag{2.9}$$

So in particular,

$$\frac{1}{k}\sqrt{-1}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2\right) - \omega_\infty = O(k^{-3}) \tag{2.10}$$

Lemma 7. *For any h_{ϕ_1}, $h_{\phi_2} \in \mathcal{K}_k$, we have*

$$-\int_X (\phi_2 - \phi_1)\frac{\omega_{\phi_1}^n}{n!} \le F_\omega^0(\phi_2) - F_\omega^0(\phi_1) \le -\int_X (\phi_2 - \phi_1)\frac{\omega_{\phi_2}^n}{n!}.$$

This is the same as Lemma 6-5

Proof. This lemma just says F_ω^0 is a convex function on $\mathcal{H}$, regarded as an open subset of $C^\infty(X)$.

We only need to calculate its second derivative along the path $\phi_t = t\phi$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}F^0_\omega(t\phi) = -\int_X \phi\Delta_t\phi\frac{\omega_t^n}{n!} = \int_X |\nabla_t\phi|^2\frac{\omega_t^n}{n!} \geq 0$$

Δ_t and ∇_t are the Laplace and gradient operators of Kähler metric $c_1(L^k, h_k(t))$. $\square$

From now on, fix a $\mathcal{H}$ such that $\omega_\phi = \omega_{h_\phi} \in c_1(L)$.

Lemma 8.

$$\tilde{P}_k(h_\phi, \mathrm{Hilb}(h_\phi)) \geq \tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_\phi)), \mathrm{Hilb}(h_\phi)) - \log\frac{N_k}{V}.$$

This is a corollary of [Don2, Lemma 4]. Since the definition of $\tilde{P}$ is a little different from that in [Don2], we give a direct proof here.

Proof. Let $he^{-\phi_k} = \mathrm{FS}_k(\mathrm{Hilb}_k(h_\phi))$. Then

$$\tilde{P}_k(h_\phi, \mathrm{Hilb}(h_\phi)) - \tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_\phi)), \mathrm{Hilb}(h_\phi)) = \frac{k}{V}(F^0_\omega(\phi_k) - F^0_\omega(\phi)).$$

Let $\{s_\alpha^{(k)}, 1 \leq \alpha \leq N_k\}$ be an orthonormal basis of $\mathrm{Hilb}_k(h_\phi)$. Then $\phi - \phi_k = \phi - \frac{1}{k}\log\left(\sum_{\alpha=1}^{N_k}|s_\alpha^{(k)}|^2_{h^{\otimes k}}\right) = -\frac{1}{k}\log\rho_k(\omega_\phi)$. By Lemma 7 and concavity of the function log,

$$\begin{aligned}
\frac{k}{V}\left(F^0_\omega(\phi_k) - F^0_\omega(\phi)\right) &\geq -\frac{1}{V}\int_X \log\rho_k(\omega_\phi)\frac{\omega_\phi^n}{n!} \\
&\geq -\log\left(\frac{1}{V}\int_X\sum_\alpha |s_\alpha^{(k)}|^2_{h_k}\frac{\omega_\phi^n}{n!}\right) \\
&= -\log\frac{N_k}{V}
\end{aligned}$$

$\square$

Lemma 9. *There exists a constant $C > 0$, depending only on h_ϕ and h_∞, such that*

$$\tilde{P}_k(\mathrm{FS}_k(\mathrm{Hilb}_k(h_\phi)), \mathrm{Hilb}_k(h_\phi)) - \tilde{P}_k(\mathrm{FS}_k(H_k^*), H_k^*) \geq -Ck^{-1}.$$

Proof. Recall that $H_k^* = \mathrm{Hilb}_k(h_\infty)$ and $\{\tau_\alpha^{(k)}; 1 \leq \alpha \leq N_k\}$ is an orthonormal basis of H_k^* (see $(*)$). Let $H_k = \mathrm{Hilb}_k(h_\phi)$ and $\{s_\alpha^{(k)}; 1 \leq \alpha \leq N_k\}$ be an orthonormal basis of H_k. Transforming by a matrix in $SU(N_k)$, we can assume

$$s_\alpha^{(k)} = e^{\lambda_\alpha^{(k)}}\tau_\alpha^{(k)}$$

Evaluating the norm $\mathrm{Hilb}_k(h_\phi)$ on both sides, we see that

$$e^{-2\lambda_\alpha^{(k)}} = \int_X |\tau_\alpha^{(k)}|^2_{h_\phi^{\otimes k}} \frac{\omega_\phi^n}{n!}. \tag{2.11}$$

There exists a constant $C_1 > 0, C_2 > 0$, depending only on h_ϕ and h_∞, such that $C_1^{-1} \leq \frac{h_\phi}{h_\infty} \leq C_1$, $C_2^{-1}\omega_\infty \leq \omega_\phi \leq C_2\omega_\infty$, so we see from (2.11) that $|\lambda_\alpha^{(k)}| \leq Ck$.

Let $\underline{\lambda} = (1/N_k)\sum_{\beta=1}^{N_k} \lambda_\beta^{(k)}$, $H_k' = e^{2\underline{\lambda}}H_k$, $\hat{\lambda}_\alpha^{(k)} = \lambda_\alpha^{(k)} - \underline{\lambda}$. Then $\{\hat{s}_\alpha^{(k)} = e^{\hat{\lambda}_\alpha^{(k)}}\tau_\alpha^{(k)}\}$ is an orthonormal basis of H_k'. Note that $\hat{\lambda}_\alpha^{(k)}$ satisfies the same estimate as $\lambda_\alpha^{(k)}$:

$$|\hat{\lambda}_\alpha^{(k)}| \leq Ck. \tag{2.12}$$

$(e^{\hat{\Lambda}})_{\alpha\beta} = e^{\hat{\lambda}_\alpha^{(k)}}\delta_{\alpha\beta}$ is a diagonal matrix in $SL(N_k, \mathbb{C})$. By scaling invariance of $\tilde{P}_k$ (2.8) and Proposition 4, we have

$$\begin{aligned} &\tilde{P}_k(\mathrm{FS}(H_k), H_k) = \tilde{P}_k(\mathrm{FS}(H_k'), H_k') = -\frac{k}{V}F_\omega^0(FS_k(H_k')) \\ = &\frac{1}{Vk^n}\log\|\hat{X}_k(H_k')\|^2_{\mathrm{CH}(H_k^*)} = -\frac{1}{Vk^n}F^0_{k\omega}((FS_k(H_k'))^{\otimes k}) \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) = -\frac{k}{V}F_\omega^0(FS_k(H_k^*)) \\ = &\frac{1}{Vk^n}\log\|\hat{X}_k(H_k^*)\|^2_{\mathrm{CH}(H_k^*)} = -\frac{1}{Vk^n}F^0_{k\omega}((FS_k(H_k^*))^{\otimes k}) \end{aligned} \tag{2.14}$$

As in Section 2.5.2, let

$$X_k(s) = e^{s\hat{\Lambda}} \cdot X_k(H_k^*)$$

$$f_k(s) = \log\|\hat{X}_k(s)\|^2_{\mathrm{CH}(H_k^*)} = -F^0_{k\omega}((FS_k(e^{s\hat{\Lambda}} \cdot H_k^*))^{\otimes k}).$$

Then $X_k(0) = X_k(H_k^*)$ and $X_k(1) = X_k(H_k') = X_k(H_k)$. By Proposition 5, $f_k(s)$ is a convex function of s, so

$$f_k(1) - f_k(0) \geq f_k'(0).$$

We can estimate $f_k'(0)$ by the estimates in Proposition 6:

$$\begin{aligned}
f_k'(0) &= \int_X \frac{\sum_\alpha \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}{\sum_\alpha |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}} \left(\sqrt{-1}\partial\bar{\partial}\log\sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2\right)^n \\
&= \int_X \frac{\sum_\alpha \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}{N_k/V + O(k^{n-2})} (1+O(k^{-3})) \frac{(k\omega_\infty)^n}{n!} \\
&= \int_X O(k^{-2}) (\sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}) \frac{\omega_\infty^n}{n!}
\end{aligned}$$

where the last equality is because of

$$\int_X \sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}} \frac{\omega_\infty^n}{n!} = \sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} = 0$$

By the estimate for $\hat{\lambda}_\alpha^{(k)}$ (2.12), we get

$$|f_k'(0)| \leq Ck^{-2} k N_k \leq Ck^{n-1}.$$

So $f_k(1) - f_k(0) \geq f_k'(0) \geq -Ck^{n-1}$, and

$$\frac{1}{Vk^n}(\log \|\hat{X}_k(H_k')\|^2_{\mathrm{CH}} - \frac{1}{Vk^n}\log \|\hat{X}_k(H_k^*)\|^2_{\mathrm{CH}}) = \frac{1}{Vk^n}(f_k(1)-f_k(0)) \geq -C\frac{1}{Vk^n}k^{n-1} \geq -Ck^{-1}.$$

So the lemma follows from identities (2.13) and (2.14). □

Remark 5. *The proof of this lemma is similar to the argument in the beginning part of [Mab6, Section 5] where Mabuchi proved K-semistability of varieties with constant scalar curvature metrics. Roughly speaking, here we consider geodesic segment connecting H_k^* and H_k in $\mathcal{H}_k$, while Mabuchi [Mab6, Section 5] considered geodesic ray in $\mathcal{H}_k$ defined by a test configuration. The estimates in Proposition 6 show that, to prove the K-semistability as in Mabuchi's argument [Mab6, Section 5], we only need Bergman metrics of h_∞ instead of Mabuchi's T-balanced metrics.*

Remark 6. *Some similar argument also appears in the proof of Theorem 2 in [PhSt2].*

Remark 7. *In [Don2, Corollary 2], H_k^* is taken to be balance metric, that is, H_k^* is a fixed point of the mapping* $\mathrm{Hilb}(\mathrm{FS}(\cdot))$. *Then the difference in Lemma 9 is nonnegative, instead of bounded below by error term $-Ck^{-1}$.*

Lemma 10. *There exists a constant $C > 0$, which only depends on h_∞, such that*

$$\left|\tilde{P}_k(\mathrm{FS}_k(H_k^*), H_k^*) - \tilde{P}_k(h_\infty, \mathrm{Hilb}_k(h_\infty)) + \log\frac{N_k}{V}\right| \le Ck^{-2}$$

Proof. Recall from $(*)$ that: $\mathrm{Hilb}_k(h_\infty) = H_k^*$, $h_k^* = \mathrm{FS}_k(H_k^*) = \mathrm{FS}_k(\mathrm{Hilb}_k(h_\infty))$. Let $h_\infty = he^{-\phi_\infty}$, $h_k^* = he^{-\phi_k^*}$. Since $F_\omega^0(\phi + c) = F_\omega^0(\phi) - cV$. So defining $\tilde{\phi}_k^* = \phi_k^* - \frac{1}{k}\log(N_k/V)$, it's easy to see that,

$$\tilde{P}_k(\mathrm{FS}_k(H_k^*), H_k^*) - \tilde{P}_k(h_\infty, \mathrm{Hilb}_k(h_\infty)) = \frac{k}{V}(F_\omega^0(\phi_\infty) - F_\omega^0(\phi_k^*))$$
$$= \quad \frac{k}{V}\left(F_\omega^0(\phi_\infty) - F_\omega^0(\tilde{\phi}_k^*)\right) + \log(N_k/V)$$

For any section s of L, $|s|^2_{h_k^*} = \frac{|s|^2_{h_\infty}}{\left(\sum_\alpha |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}\right)^{1/k}}$. So by proposition 6.

$$\left|\tilde{\phi}_k^* - \phi_\infty\right| = \left|\log\frac{h_\infty}{h_k^*} - \frac{1}{k}\log\frac{N_k}{V}\right| = \left|\frac{1}{k}\log\left(\frac{V}{N_k}\sum_\alpha |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}\right)\right| = O(k^{-3}).$$

So by Lemma 7, we get

$$\left|\frac{k}{V}\left(F_\omega^0(\phi_\infty) - F_\omega^0(\tilde{\phi}_k^*)\right)\right| \le Ck^{-2}$$

□

Definition 5. *For any $h_\phi = he^{-\phi} \in \mathcal{H}$ and its corresponding Kähler form $\omega_\phi \in [\omega]$. Define*

$$\mathcal{L}_k(\omega_\phi) = \tilde{P}_k(h_\phi, \mathrm{Hilb}_k(h_\phi))$$

Lemma 11 ([Don2])**.** *There exist constants μ_k, such that*

$$\mathcal{L}_k(\omega_\phi) + \mu_k = \frac{1}{2}\nu_\omega(\omega_\phi) + O(k^{-1}).$$

Here $O(k^{-1})$ depends on ω and ω_ϕ.

Proof. Let $\psi(t) = t\phi \in \mathcal{H}$ connecting 0 and ϕ, $h_{t\phi} = he^{-t\phi}$, $\omega_t = \omega + t\sqrt{-1}\partial\bar{\partial}\phi$, Δ_t be the Laplace operator of metric ω_t. Plugging in expansions for Bergman kernels ρ_k in Proposition 3, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_k(h_{t\phi}, \mathrm{Hilb}_k(h_{t\phi})) &= \frac{1}{N_k}\int_X \sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} e^{-k\phi}(-k\phi + \triangle_t\phi)\frac{\omega_t^n}{n!} + \frac{k}{V}\int_X \phi\frac{\omega_t^n}{n!} \\
&= \frac{1}{V}\frac{1}{k^n + \frac{1}{2}\underline{S}\,k^{n-1} + \cdots}\int_X (-k\rho_k(\omega_t) + \triangle_t\rho_k(\omega_t))\phi\omega_t^n + \frac{k}{V}\int_X \phi\omega_t^n \\
&= -\frac{1}{2V}\int_X (S(\omega_t) - \underline{S})\phi\omega_t^n + O(k^{-1})
\end{aligned}$$

$\{\omega_t, 0 \leq t \leq 1\}$ have uniformly bounded geometry, so by Proposition 3.(3), the expansions above are uniform. So the lemma follows after integrating the above equation. □

Proof of Theorem 12. By Lemma 8, Lemma 9, Lemma 10

$$\begin{aligned}
\tilde{P}_k(h_\phi, \mathrm{Hilb}_k(h_\phi)) &\geq \tilde{P}_k(\mathrm{FS}_k(\mathrm{Hilb}_k(h_\phi)), \mathrm{Hilb}(h_\phi)) - \log\frac{N_k}{V} \\
&\geq \tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) - \log\frac{N_k}{V} + O(k^{-1}) \\
&= \tilde{P}_k(h_\infty, \mathrm{Hilb}_k(h_\infty)) + O(k^{-1})
\end{aligned}$$

So by Lemma 11

$$\begin{aligned}
\nu_\omega(\omega_\phi) &= 2\mathcal{L}_k(\omega_\phi) + 2\mu_k + O(k^{-1}) = 2\tilde{P}_k(h_\phi, \mathrm{Hilb}_k(h_\phi)) + 2\mu_k + O(k^{-1}) \\
&\geq 2\tilde{P}_k(h_\infty, \mathrm{Hilb}_k(h_\infty)) + 2\mu_k + O(k^{-1}) = 2\mathcal{L}_k(\omega_\infty) + 2\mu_k + O(k^{-1}) \\
&= \nu_\omega(\omega_\infty) + O(k^{-1})
\end{aligned}$$

The Theorem 12 follows by letting $k \to +\infty$. □

2.6 Tian's Conjecture and Partial C^0-estimate

The following conjecture of Tian is the analytic version of Yau-Tian-Donaldson conjecture specialized to the Fano case.

Conjecture 2 (Tian)**.** *There is a Kähler-Einstein metric on X if and only if for sufficiently large k, ν_ω is proper on $\mathcal{B}_k$.*

Lemma 12. *[PaTi2] For fixed k, there exists a constant $C_k > 0$ such that for any*

$$C_k^{-1}\cdot(I-J)_\omega(\omega_\phi) \leq \mathrm{Osc}(\phi) \leq C_k\cdot(I-J)_\omega(\omega_\phi)$$

Remark 8. *So the properness in the sense of Definition 2 is the same as saying that there exists some monotone increasing function f_k as in Definition 2 such that:*

$$\nu_\omega(\omega_\phi) \geq f_k(\mathrm{Osc}(\phi))$$

holds for any $\phi \in \mathcal{B}_k$. In particular, $\mathrm{Osc}(\phi) \to +\infty$ would imply $\nu_\omega(\omega_\phi) \to +\infty$.

Definition 6 (Partial C^0-estimate)**.** *Assume $\{\omega_{\phi_t}\} \subset c_1(L)$ is a family of Kähler metrics parametrized by t, we say the partial C^0-estimate holds for $\{\omega_{\phi_t}\}$ if there exists an integer k and positive constants C_k, both independent of t, such that $C_k^{-1} \leq \rho_k(\omega_{\phi_t}) \leq C_k$. Equivalently, if $\{s_\alpha^{(k)}(t)\}_{\alpha=1}^{N_k}$ are orthonormal basis of $\mathrm{Hilb}_k(h_\phi)$ then*

$$\left| \phi_t - \frac{1}{k} \log \sum_\alpha |s_\alpha^{(k)}(t)|^2_{h^{\otimes k}} \right| \leq \frac{\log C_k}{k}$$

To get the upper bound for Bergman kernel, we use the standard Moser iteration. We need to the estimate on L^2-Sobolev constant.

Definition 7 (L^2-Sobolev constant)**.** *For any fixed Kähler metric ω_g, there exists a positive constant $C_{\mathrm{Sob}} > 0$ such that for any $F \in C^\infty(X)$ we have*

$$C_{\mathrm{Sob}} \left(\int_X |F|^{\frac{2n}{n-1}} dV_\omega \right)^{\frac{n-1}{n}} \leq \int_X (|\nabla F|^2 + F^2) dV_\omega \tag{2.15}$$

Remark 9. *If $Ric(\omega_g) \geq \delta\omega_g$ with $\delta > 0$, then $C_{sob} \geq C(\delta, n) > 0$. The L^2-Sobolev constants is also uniformly bounded away from 0 along the normalized Kähler-Ricci flow [Zhu].*

Lemma 13. *Suppose $\phi = \phi(t)$ is a sequence of Kähler potentials. $k \geq 1$ is fixed integer. Then*

1. *If the Sobolev constant of ω_{ϕ_t} is uniformly bounded, then $\rho_k(\omega_{\phi_t}) \leq C(C_{sob}, k)$ or equivalently, if $\{s_\alpha^{(k)}\}$ is an orthonormal basis of $\mathrm{Hilb}_k(h_\phi)$ then*

$$\phi - \frac{1}{k} \log \left(\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} \right) \geq -\frac{\log C(C_{sob}, k)}{k}$$

2. *Assume the Sobolev constant of ω_{ϕ_t} is uniformly bounded. If exists a constant C_2 independent of t, such that for every t there exists a base $\{\tilde{s}_\alpha^{(k)}(t)\}$ of $H^0(X, -kK_X)$ such that the inequality*

$$\left| \phi - \frac{1}{k} \log \left(\sum_{\alpha=1}^{N_k} |\tilde{s}_\alpha^{(k)}(t)|^2_{h^{\otimes k}} \right) \right| \leq C_2$$

holds, then the the partial C^0-estimate holds.

3. *Assume ω_t converge to $\omega_\infty = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty$ with $\psi_\infty \in L^\infty(X)$ and the relative density function $\frac{\omega_\infty^n}{\omega^n}$ is integrable. Assume there is a closed subset $\mathcal{S} \subset X$ with zero Hausdorff measure, such that for any compact set $K \subset\subset X\backslash\mathcal{S}$, ϕ_t converge to ψ_∞ uniformly on K. Then the partial C^0-estimate holds.*

Proof. 1.

$$\Delta_{\omega_\phi}|s|^2_{h_\phi^{\otimes k}} = |\nabla s|^2_{h_\phi} - nk|s|^2_{h_\phi^{\otimes k}}$$

Let $\omega' = \omega_\phi$, $\Delta' = \Delta_{\omega_\phi}$ and $f = |s|_{h_\phi^{\otimes k}}$. We get $-\Delta' f^2 \leq nkf^2$ which implies

$$-\Delta' f \leq \frac{kn}{2} f$$

We want to use Moser iteration method, so we multiply both sides by f^p for $p \geq 1$ and integrate by parts with respect volume form $dV_{\omega'} = \omega'^n/n!$ to get

$$\frac{4p}{(p+1)^2}\int_X |\nabla f^{(p+1)/2}|^2_{\omega'} dV_{\omega'} \leq \frac{nk}{2}\int_X f^{p+1} dV_{\omega'}$$

Using this and rearranging the terms we get

$$\left(\int_X f^{(p+1)\frac{n}{n-1}} dV_{\omega'}\right)^{\frac{n-1}{n}} \leq \frac{1}{C(C_{sob}, n)}\left(\frac{(p+1)^2}{4p}\frac{nk}{2} + 1\right)\int_X f^{p+1} dV_{\omega'}$$

Taking $(p+1)$-th root on both sides, we get

$$\|f\|_{(p+1)\frac{n}{n-1}} \leq (\tilde{C}(C_{sob}, n)k(p+1))^{\frac{1}{p+1}}\|f\|_{p+1}$$

Define $p_i + 1 = (p_{i-1} + 1)\frac{n}{n-1}$ for $i \geq 1$ and $p_0 = 1$. We get $p_i + 1 = 2(\frac{n}{n-1})^i$. So

$$\|f\|_\infty = \lim_{i\to+\infty}\|f\|_{p_i+1} \leq C(k, n, C_{sob})\|f\|_2$$

where

$$
\begin{aligned}
C(k,n,C_{sob}) &= \prod_{i=0}^{\infty}(\tilde{C}k(p_i+1))^{\frac{1}{p_i+1}} \\
&= \exp\left[\frac{\log(\tilde{C}k)}{2}\sum_i\left(\frac{n-1}{n}\right)^i+\frac{1}{2}\sum_i\left(\frac{n-1}{n}\right)^i\left(\log 2+i\log\left(\frac{n}{n-1}\right)\right)\right] \\
&= Ck^{n/2}
\end{aligned}
$$

If we assume $\|s\|_{\mathrm{Hilb}(\mathrm{h}_\phi^{\otimes \mathrm{k}})}=\left\||s|_{h_\phi^{\otimes k}}\right\|_2=1$ then $|s|^2_{h_\phi^{\otimes k}}\leq Ck^n$ which implies

$$
\rho_k=\sup\{|s|^2_{h_\phi^{\otimes k}};\|s\|^2_{\mathrm{Hilb_k(h_\phi)}}=1\}\leq Ck^n
$$

2. By a unitary transformation, we can assume $s_\alpha^{(k)}=\sqrt{d_\alpha(t)}\tilde{s}_\alpha^{(k)}(t)$.

$$
\begin{aligned}
\|\sqrt{c_\alpha}\tilde{s}_\alpha\|^2_{\mathrm{Hilb_k}} &= \int_X c_\alpha|\tilde{s}_\alpha|^2_{h^{\otimes k}}e^{-k\phi}\frac{\omega_\phi^n}{n!} \\
&\leq e^{kC_2}\int_X\frac{c_\alpha|\tilde{s}_\alpha|^2}{\sum_\beta c_\beta|\tilde{s}_\beta|^2}\frac{\omega_\phi^n}{n!}\leq e^{kC_2}V
\end{aligned}
$$

where we used $\left(-\phi+\frac{1}{k}\log\sum_\alpha c_\alpha|s_\alpha|^2_{h^{\otimes k}}<C_2\right)$. So $c_\alpha\leq e^{kC_2}Vd_\alpha$, and

$$
\rho_k(\omega_t)=\sum_\alpha d_\alpha|\tilde{s}_\alpha|^2_{h_{\phi_t}^{\otimes k}}\geq e^{-kC_2}V^{-1}\sum_\alpha c_\alpha|\tilde{s}_\alpha|^2_{h_{\phi_t}^{\otimes k}}\geq e^{-2kC_2}V^{-1}>0
$$

The upper bound follows from part (1).

3. This follows from $\rho_k(\omega_t)\to\rho_k(\omega_\infty)\geq\inf\rho_k(\omega_\infty)>0$.

□

Remark 10. *In Section 4.5, we will show that the partial C^0-estimate holds along the classical continuity method on toric Fano manifolds. More precisely, on toric Fano manifolds, we will prove, along the continuity method, the condition (2) holds, and also (3) holds upon transformation by holomorphic automorphisms.*

But in general, one can not expect such strong convergence in the fixed complex manifold and one expects the jump of complex structure. This is where the notion of Gromov-Hausdorff convergence comes in. The partial C^0-estimate should comes from the understanding of Gromov-Hausdorff limit, in particular the structure of the singular set.

One such successful example is in Tian's solution of Kähler-Einstein problem on del Pezzo surfaces [Tia4] where the partial C^0-estimate for a sequence of Kähler-Einstein complex del Pezzo *surfaces played an important role. Tian proved the partial C^0-estimate by proving the Gromov-Hausdorff limit of a sequence of Kähler-Einstein del Pezzo surfaces is an complex orbifold.*

Partial C^0-estimate and Tian's Conjecture

The importance of partial C^0-estimate lies in its implication of Tian's conjecture 2. We will explain this now. For more discussion, see [Tia10].

Along the continuity method $(*)_t$, it's easy to see that the K-energy is decreasing. So in particular it's uniformly bounded from above: $\nu_\omega(\omega_t) \leq C$.

Lemma 14. *If the partial C^0-estimate holds, then $\nu_\omega(\omega_k(t)) \leq C'$, where $\omega_k(t)$ is the k-th Bergman metric of ω_{ϕ_t}.*

Proof. By the co-cycle property of ν-energy. $\nu_\omega(\omega_k(t)) = \nu_\omega(\omega_t) + \nu_{\omega_t}(\omega_k(t))$. So we only need to bound $\nu_{\omega_t}(\omega_k(t))$. By the explicit formula of ν-energy:

$$\nu_{\omega_t}(\omega_k(t)) = \int_X \log \frac{\omega_k(t)^n}{\omega_t^n} \frac{\omega_k(t)^n}{n!} - (I-J)_{\omega_t}(\omega_k(t)) \leq \int_X \log \frac{\omega_k(t)^n}{\omega_t^n} \frac{\omega_k(t)^n}{n!}$$

where $\psi_t = \frac{1}{k} \log \rho_k(\omega_t) = \frac{1}{k} \log \sum_\alpha |s_\alpha^{(k)}(t)|_h^2 - \phi$ is the relative potential between ω_t and $\omega_k(t)$: $\sqrt{-1}\partial\bar{\partial}\psi_t = \omega_k(t) - \omega_t$. By Corollary 2, we have the C^2-estimate: $\log \frac{\omega_k(t)^n}{\omega_t^n} \leq C \cdot \mathrm{Osc}(\psi)$ So $\nu_{\omega_t}(\omega_k(t)) \leq CV \cdot \mathrm{Osc}(\psi)$. The bound $\mathrm{Osc}(\psi)$ follows from partial C^0-estimate. □

Now we can prove Tian's Conjecture 2 assuming partial C^0-estimate as follows. If ν_ω is proper on $\mathcal{B}_k$, then by Remark 8 $\mathrm{Osc}(\phi_k(t))$ must be bounded. But since $|\phi_k(t) - \phi_t| = |\psi_t| \leq C$ by partial C^0-estimate, $\mathrm{Osc}(\phi_t)$ is uniformly bounded. By Harnack inequality, $\|\phi\|_{C^0}$ is uniformly bounded. So there exists Kähler-Einstein metric.

Partial C^0-estimate and effective finite generation

Following the idea of Siu [Siu], we explain how partial C^0-estimate implies some effective finite generation of the rings

$$\bigoplus_{m=1}^{\infty} H^0(X, kL)$$

Let's first recall the following generation theorem of Skoda.

Theorem 13. *([Sko]) Let Ω be a domain spread over $\mathbb{C}^n$ which is Stein. Let ψ be a plurisubharmonic function on Ω, $g_1, \ldots, g_p$ be holomorphic functions on Ω, $\alpha > 1$, $q = \min(n, p-1)$, and f*

be a holomorphic function on Ω. Assume that

$$\int_\Omega \frac{|f|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q+1}} < +\infty$$

Then there exist holomorphic functions $h_1, \cdots, h_p$ on Ω with $f = \sum_{j=1}^p h_j g_f$ on Ω such that

$$\int_\Omega \frac{|h_k|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q}} \leq \frac{\alpha}{\alpha-1} \int_\Omega \frac{|f|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q+1}}$$

for $1 \leq k \leq p$.

We also have the global version due to [Siu]. Note that we can choose $\alpha = \frac{n+l}{n}$ and $q = n$.

Theorem 14. *([Siu, 2.4]) Let X be a compact complex algebraic manifold of complex dimension n, G be a holomorphic line bundle over X, and E be a holomorphic line bunlde on X with $e^{-\psi}$ such that ψ is plurisubharmonic. Let $l \geq 1$ be an integer, $g_1, \ldots, g_p \in H^0(X, G)$, and $|g|^2 = \sum_{j=1}^p |g_j|^2$. If $f \in H^0(X, (n+l+1)G + E + K_X)$ satisfies*

$$\int_X \frac{|f|^2 e^{-\psi}}{|g|^{2(n+l+1)}} < C$$

Then $f = \sum_{j=1}^p h_j g_j$ with $h_j \in H^0(X, ((n+l)G + E + K_X))$ satisfying

$$\int_X \frac{|h_j|^2 e^{-\psi}}{|g|^{2(n+l)}} \leq \frac{n+l}{l} \int_X \frac{|f|^2 e^{-\psi}}{|g|^{2(n+l+1)}}$$

Write $mL = (n+l+1)(kL) + ((m-(n+l+1)k)L - K_X) + K_X =: (n+l+1)G + E + K_X$. Define the $\psi = (m-(n+l+1)k)\phi - \log \frac{\omega_\phi^n}{n!}$. Then

$$\sqrt{-1}\partial\bar{\partial}\psi = (m-(n+l+1)k)\omega_\phi + Ric(\omega_\phi) \geq (m-(n+l+1)k+\delta)\omega_\phi$$

Now assume the partial C^0-estimate in the definition 6 holds for some integer k. Let $s_1, \ldots, s_{N_k}$ be the orthonormal basis of $(H^0(X, kL), \mathrm{Hilb}_k(h_\phi))$. We want to prove

Proposition 7.

$$\bigoplus_{m=1}^{+\infty} H^0(X, mL)$$

is finite generated by

$$\bigoplus_{i=0}^{(n+2)k} H^0(X, iL)$$

with effective estimates.

Proof. • Fix $m \geq (n+2)k$, assume $m = (n+l+1)k + r$ with $0 \leq r < k$.

- For any $u^{(0)} := u \in H^0(mL)$, there exists $u_\alpha^{(1)} \in H^0(X, (n+l)G + E + K_X) = H^0(X, (m-k)L)$ with $u^{(0)} = \sum_{\alpha=1}^{N_k} u_\alpha^{(1)} s_\alpha$ such that

$$\int_X \frac{|u_\alpha^{(1)}|^2 e^{-\psi}}{|s|^{2(n+l)}} \leq \frac{n+l}{l} \int_X \frac{|u^{(0)}|^2 e^{-\psi}}{|s|^{2(n+l+1)}} \tag{2.16}$$

The left hand side

$$\begin{aligned}\int_X \frac{|u_\alpha^{(1)}|^2 e^{-\psi}}{|s|^{2(n+l)}} &= \int_X |u_\alpha^{(1)}|^2 e^{-(m-k)\phi} \frac{e^{-(n+l)k\phi}}{|s|^{2(n+l)}} \frac{\omega_\phi^n}{n!} \\ &= \int_X |u_\alpha^{(1)}|^2 e^{-(m-k)\phi} \rho_k^{-(n+l)} \frac{\omega_\phi^n}{n!}\end{aligned}$$

Similarly

$$\int_X \frac{|u^{(0)}|^2 e^{-\psi}}{|s|^{2(n+l+1)}} = \int_X |u^{(0)}|^2 e^{-m\phi} \frac{e^{-(n+l+1)k\phi}}{|s|^{2(n+l+1)}} \frac{\omega_\phi^n}{n!} = \int_X |u^{(0)}|^2 e^{-m\phi} \rho_k^{-(n+l+1)} \frac{\omega_\phi^n}{n!}$$

So (2.16) is equivalent to

$$\int_X |u_\alpha^{(1)}|^2 e^{-(m-k)\phi} \rho_k^{-(n+l)} \frac{\omega_\phi^n}{n!} \leq \frac{n+l}{l} \int_X |u^{(0)}|^2 e^{-m\phi} \rho_k^{-(n+l+1)} \frac{\omega_\phi^n}{n!} \tag{2.17}$$

- Repeat the above process for each $u_\alpha^{(1)} \in H^0(X, (m-k)L)$. We get $u_{\alpha_1,\alpha_2}^{(2)} \in H^0(X, (m-2k)L)$ with $u_{\alpha_1}^{(1)} = \sum_{\alpha_2=1}^{N_k} u_{\alpha_1,\alpha_2}^{(2)} s_\alpha$, such that

$$\int_X |u_{\alpha_1,\alpha_2}^{(2)}|^2 e^{-(m-2k)\phi} \rho_k^{-(n+l-1)} \frac{\omega_\phi^n}{n!} \leq \frac{n+l-1}{l-1} \int_X |u_\alpha^{(1)}|^2 e^{-(m-k)\phi} \rho_k^{-(n+l)} \frac{\omega_\phi^n}{n!}$$

- By induction, we get $u_{\alpha_1,\ldots,\alpha_{j+1}}^{(j+1)} \in H^0(X, (m-(j+1)k)L)$ with

$$u_{\alpha_1,\ldots,\alpha_j}^{(j)} = \sum_{\alpha_{j+1}=1}^{N_k} u_{\alpha_1,\ldots,\alpha_{j+1}}^{(j+1)} s_{\alpha_{j+1}}$$

such that

$$\int_X |u_{\alpha_1,\ldots,\alpha_{j+1}}^{(j+1)}|^2 e^{-(m-(j+1)k)\phi} \rho_k^{-(n+l-j)} \frac{\omega_\phi^n}{n!} \leq \frac{n+l-j}{l-j} \int_X |u_{\alpha_1,\ldots,\alpha_j}^{(j)}|^2 e^{-(m-jk)\phi} \rho_k^{-(n+l-j+1)} \frac{\omega_\phi^n}{n!}$$

- When $j+1=l$, we get $u^{(l)}_{\alpha_1,\ldots,\alpha_l}\in H^0(X,(m-lk)L)=H^0(X,((n+1)k+r)L)$ with

$$u^{(l-1)}_{\alpha_1,\ldots,\alpha_{l-1}}=\sum_{\alpha_l=1}^{N_k}u^{(l)}_{\alpha_1,\ldots,\alpha_l}s_{\alpha_l}$$

such that

$$\int_X |u^{(l)}_{\alpha_1,\ldots,\alpha_l}|^2e^{-(m-lk)\phi}\rho_k^{-(n+1)}\frac{\omega_\phi^n}{n!}\le(n+1)\int_X|u^{(l-1)}_{\alpha_1,\ldots,\alpha_{l-1}}|^2e^{-(m-(l-1)k)\phi}\rho_k^{-(n+2)}\frac{\omega_\phi^n}{n!}$$

- So for any $u\in H^0(X,mL)$ with $m\ge(n+2)k$, if we let $l=\lfloor\frac{m}{k}\rfloor-n-1$, then

$$u=\sum_{\alpha_1,\ldots,\alpha_l=1}^{N_k}u^{(l)}_{\alpha_1,\ldots,\alpha_l}s_{\alpha_1}\ldots s_{\alpha_l}$$

with $u^{(l)}_{\alpha_1,\ldots,\alpha_l}\in H^0(X,(m-lk)L)$ and

$$\|u^{(l)}_{\alpha_1,\ldots,\alpha_l}\|^2_{\mathrm{Hilb}_{m-lk}}\le\frac{(n+l)!}{l!n!}\frac{(\sup\rho_k)^{n+1}}{(\inf\rho_k)^{n+l+1}}\|u\|^2_{\mathrm{Hilb}_m}$$

□

2.7 Futaki invariant and K-stability

2.7.1 Analytic and algebraic definition of Futaki Invariant

Let X be an n dimensional normal variety. Assume it's Fano, i.e. its anticanonical line bundle K_X^{-1} is ample. If X is smooth, then for any Kähler form ω in $c_1(X)$, by $\partial\bar\partial$-lemma, we have a smooth function h_ω, such that $Ric(\omega)-\omega=\sqrt{-1}\partial\bar\partial h_\omega$. We call $h_\omega:=-\log\frac{\omega_h^n}{\eta_h}$ the Ricci potential of ω. Let v be a holomorphic vector field on X, i.e. v is of type (1,0) and $\bar\partial v=0$. Then the Futaki invariant is defined to be

$$F_{c_1(X)}(v)=\int_X v(h_\omega)\omega^n \tag{2.18}$$

It's a holomorphic invariant, as a character on the Lie algebra of holomorphic vector field, and independent of the choice of the Kähler form in $c_1(X)$. See [Fut]. The necessary condition of existence of Kähler-Einstein metric on X is that the Futaki invariant vanishes.

In [DiTi], the Futaki invariant is generalized to the singular case. When X is possibly singular normal, first use $|kK_X^{-1}|$ to embed X into projective spaces, $\phi_k = \phi_{|kK_X^{-1}|} : X \hookrightarrow \mathbb{CP}^{N_k}$. h_{FS} is the Fubini-Study metric determined by an inner product on $H^0(X, kK_X^{-1})$. $h = (\phi_k^* h_{FS})^{1/k}$ is an Hermitian metric on K_X^{-1}. Note that on the smooth part of X, Hermitian metrics on K_X^{-1} one-to-one corresponds to volume forms. If $\{z_i\}$ is a local holomorphic coordinate, denote $dz_1 \wedge \cdots \wedge dz_n$ by dz, and $d\bar{z}_1 \wedge \cdots d\bar{z}_n$ by $d\bar{z}$, the correspondence is given by

$$h \mapsto \sqrt{-1}^n \frac{dz_1 \wedge \cdots dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n}{|dz_1 \wedge \cdots \wedge dz_n|^2_{h^{-1}}} = \sqrt{-1}^n \frac{dz \wedge d\bar{z}}{|dz|^2_{h^{-1}}} =: \eta_h$$

$|dz|^{-2}_{h^{-1}} = |\partial_{z_1} \wedge \cdots \wedge \partial_{z_n}|^2_h$ is the induced Hermitian metric on K_X by the metric dual. On the smooth part of X,

$$\omega_h := \sqrt{-1}\bar{\partial}\partial \log h = -\sqrt{-1}\partial\bar{\partial} \log \frac{\eta_h}{\sqrt{-1}^n dz \wedge d\bar{z}} =: -\sqrt{-1}\partial\bar{\partial} \log \eta_h$$

is a Kähler form, its Ricci curvature is: $Ric(\omega_h) = -\sqrt{-1}\partial\bar{\partial} \log \det \omega_h^n$.

$$Ric(\omega_h) - \omega_h = -\sqrt{-1}\partial\bar{\partial} \log \frac{\omega_h^n}{\eta_h}$$

So the Ricci potential is $h_{\omega_h} = -\log \frac{\omega_h^n}{\eta_h}$.

$$\begin{aligned} -\int_{X_{sm}} v(\log \frac{\omega_h^n}{\eta_h})\omega_h^n &= -\int_{X_{sm}} v(\frac{\omega_h^n}{\eta_h})\eta_h = -\int_{X_{sm}} (L_v \omega_h^n - \frac{L_v \eta_h}{\eta_h}\omega_h^n) \\ &= \int_{X_{sm}} div_{\eta_h}(v)\omega_h^n = \frac{1}{n+1}\int_{X_{sm}} (div_{\eta_h}(v) + \omega_h)^{n+1} \end{aligned}$$

In [DiTi], it's proved this is still a well defined holomorphic invariant. Note that in local holomorphic coordinate, $L_v d\bar{z}_i = 0$, so

$$\frac{L_v(\eta_h)}{\eta_h} = \frac{L_v(dz)}{dz} + v(\log |dz|^{-2}_{h^{-1}})$$

Note that the first term on the right is holomorphic , so

$$\bar{\partial} div_{\eta_h}(v) = -i_v \bar{\partial}\partial \log |dz|^{-2}_{h^{-1}} = -\frac{2\pi}{\sqrt{-1}}\, i_v \omega_h \tag{2.19}$$

We can transform the expression of Futaki invariant (2.18) into another form:

$$F_{c_1(X)}(v) = -\int_X (S(\omega) - \omega)\theta_v \omega^n \tag{2.20}$$

where $S(\omega)$ is the scalar curvature of ω, and θ_v is the potential function of the vector field v satisfying

$$i_v\omega = \sqrt{-1}\bar{\partial}\theta_v$$

In this way, the Futaki invariant generalizes to any Kähler class. The vanishing of Futaki invariant is necessary for the existence of constant scalar Kähler metric in the fixed Kähler class.

Assume there is a $\mathbb{C}^*$ action on (X, L), there are induced actions on $H^0(X, L^k)$. Let w_k be the $k-th$ (Hilbert) weight of these actions. For k sufficiently large,

$$d_k = dimH^0(X, L^k) = a_0\frac{k^n}{n!} + a_1\frac{k^{n-1}}{2n!} + O(k^{n-2}) \quad , a_0 = \int_X \omega^n, \;\; a_1 = \int_X S(\omega)\omega^n$$

$$w_k = b_0\frac{k^{n+1}}{n!} + b_1\frac{k^n}{2n!} + O(k^{n-1}) \tag{2.21}$$

At least in the smooth (or normal) case, one can show that (See [Don4])

$$b_0 = \int_X \theta_v\omega^n, \;\; b_1 = \int_X S(\omega)\theta_v\omega^n \tag{2.22}$$

By this, Donaldson [Don4] gives an algebro-geometric definition of Futaki invariant:

$$F_{c_1(L)}(v) = \frac{a_1b_0 - a_0b_1}{a_0} \tag{2.23}$$

Remark 11. *Assume we can embed X into $\mathbb{P}(H^0(X, L)^*)$ using the complete linear system $|L|$ such that the $\mathbb{C}^*$ action is induced by a one parameter subgroup in $SL(d_1, \mathbb{C})$. Then we see that, at least in the smooth case, if we normalize θ_v, the (normalized) leading coefficient $((n + 1)b_0)$ in the expansion (2.21) is the Chow weight of this $\mathbb{C}^*$ action.*

2.7.2 Donaldson-Futaki invariant and asymptotic slope of K-energy along one parameter subgroup

Sean Paul's work

Assume $X \subset \mathbb{P}^N$ is embedded into the projective space and ω_{FS} is the standard Fubini-Study metric on $\mathbb{P}^N$. For any $\sigma \in SL(N + 1, \mathbb{C})$, denote $\omega_\sigma = \sigma^*\omega_{FS}|_X$.

Theorem 15. *[Paul] Let $X^n \hookrightarrow \mathbb{P}^N$ be a smooth, linearly normal, complex algebraic variety of degree ≥ 2. Let R_X denote the **X-resultant** (the Cayley-Chow form of X). Let $\triangle_{X\times\mathbb{P}^{n-1}}$ denote* *the **X-hyperdiscriminant** of f format (n-1) (the defining polynomial for the dual of $X \times \mathbb{P}^{n-1}$ in the Segre embedding). Then there are norms such that the Mabuchi-energy restricted to the Bergman metrics is given as follows:*

$$\nu_\omega(\omega_\sigma) = \deg(R_X)\log\frac{\|\sigma\cdot\triangle_{X\times\mathbb{P}^{n-1}}\|^2}{\|\triangle_{X\times\mathbb{P}^{n-1}}\|^2} - \deg(\triangle_{X\times\mathbb{P}^{n-1}})\log\frac{\|\sigma\cdot R_X\|^2}{\|R_X\|^2}$$

Lemma 15 (Tian)**.**

$$\log\frac{\|\sigma\cdot R_X\|^2}{\|R_X\|^2} = (n+1)\int_0^1\int_X \dot{\phi}_\sigma\omega_\sigma^n$$

$$\log\frac{\|\sigma\cdot\triangle_{X\times\mathbb{P}^{n-1}}\|^2}{\|\triangle_{X\times\mathbb{P}^{n-1}}\|^2} = (N+n-1)\int_0^1 dt\int_{(X\times\mathbb{P}^{n-1})^\vee}\dot{\Phi}_\sigma\omega_{FS(\mathbb{P}^{N+n-1\vee})}^{N+n-1}$$

Lemma 16.

$$\deg(\triangle_{X\times\mathbb{P}^{n-1}}) = \deg((X\times\mathbb{P}^{n-1})^\vee) = \int_{X\times\mathbb{P}^{n-1}} c_{2n-1}(J(\mathcal{O}(1,1))) = (n(n+1)-n\mu)V$$

where $J(\mathcal{O}(1,1))$ is the jet bundle of $\mathcal{O}(1,1) = \pi_1^*\mathcal{O}_{\mathbb{P}^N}(1)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ while

$$\deg(R_X) = (n+1)d$$

The most important ingredient is the following identity

Theorem 16 (Hyper-discriminant part in the K-energy)**.**

$$(N+n-1)\int_0^1 dt\int_{(X\times\mathbb{P}^{N+n-1})^\vee}\dot{\Phi}_\sigma\omega_{FS(\mathbb{P}^\vee)}^{N+n-1} = \int_0^1 dt\int_X\dot{\phi}_\sigma(n(n+1)\omega_\sigma^n - nRic(\omega_\sigma)\wedge\omega_\sigma^{n-1}) \quad (2.24)$$

To get the above identity, the idea is to consider both sides as function of $G := SL(N+1,\mathbb{C})$. Then one takes $\partial\bar{\partial}$ of both sides and pair with any smooth test $(m-1,m-1)$-form η on G to conclude the above identity after taking $\partial\bar{\partial}$. To remove the transgression operator $\partial\bar{\partial}$ one uses the following trick by Tian. First compactify G to be $\overline{G}$ such that $\overline{G}\backslash G$ has an irreducible divisor. Then one verify the log polynomial growth of both sides. The following is an immediate corollary of Corollary 2 in Section 2.3.1.

Proposition 8. *The functional*

$$-\int_0^1 dt\int_X n\dot{\phi}_\sigma(Ric(\omega_\sigma)-Ric(\omega_0))\wedge\omega_\sigma^{n-1} = \int_X\log\frac{\omega_\sigma^n}{\omega_0^n}\omega_\sigma^n$$

has log polynomial growth as function on $SL(N+1,\mathbb{C})$. In particular, the K-energy $\nu(\omega_\sigma)$ has log polynomial growth as a function of $\sigma \in SL(N+1,\mathbb{C})$.

Futaki invariant as asymptotic slope

By Tian's Conjecture 2, we need to test if the K-energy functional is proper on $\mathcal{B}_k$. Following the Hilbert-Mumford Criterion for GIT stability, we consider any one parameter subgroup $\lambda(t) = t^A \in SL(N+1,\mathbb{C})$. Although the K-energy is not convex along $\lambda(t)$, the above theorem 15 by Sean Paul says that it is the difference of two convex functionals. As a corollary, we have the existence of asymptotic slope as the difference of Chow weight and the hyperdiscriminant weight. Define $\omega_{\lambda(t)} = \lambda(t)^*\omega_{FS}|_X$, and $\mathcal{X}_0$ to be the limit $\lim_{t\to 0}\lambda(t)\cdot X$ in the Hilbert scheme (which is the central fibre of the induced test configuration introduced in the next subsection). Then combined with [PaTi2], we also have the following expansion

Proposition 9. *[Tia9, PaTi2]*

$$\nu_\omega(\omega_{\lambda(t)}) = (F_1(\lambda)+a)\log\frac{1}{t} + O(1) \tag{2.25}$$

where F_1 is the Donaldson-Futaki invariant. $a \in \mathbb{Q}$ is negative if and only if the central fibre $\mathcal{X}_0$ has generically non-reduced fibre.

Remark 12. *In fact, if $\mathcal{X}_0$ is irreducible, then by ([Tia9], [PaTi2]) one can calculate that $-a = c\cdot(\mathrm{mult}(\mathcal{X}_0)-1)$ for $c > 0 \in \mathbb{Q}$.*

Without loss of generality, we assume each homogeneous coordinate Z_i are the eigenvector of $\lambda(t)$ on $H^0(X,\mathcal{O}(1))$ with eigenvalues $\lambda_0 = \cdots = \lambda_K < \lambda_{K+1} \le \cdots \le \lambda_N$. Let $\omega_{\lambda(t)} = \omega_{FS} + \sqrt{-1}\partial\bar\partial\phi_t$. Then

$$\phi_t = \log\frac{\sum_i t^{\lambda_i}|Z_i|^2}{\sum_i |Z_i|^2} \tag{2.26}$$

There are three possibilities for $\mathcal{X}_0$. Compare [Sto1].

1. (non-degenerate case)

   $\lim_{t\to 0}\mathrm{Osc}(\phi_t) \to +\infty$. By (2.26), this is equivalent to $\cap_{i=0}^K\{Z_i = 0\}\bigcap X \neq \emptyset$.

2. (degenerate case) $\mathrm{Osc}(\phi_t) \le C$ for C independent of t. This is equivalent to $\cap_{i=0}^K\{Z_i = 0\}\bigcap X = \emptyset$. In this case, $\mathcal{X}_0$ is the image of X under the projection $\mathbb{P}^N \to \mathbb{P}^K$ given by $[Z_0,\ldots,Z_N] \mapsto [Z_0,\ldots,Z_K,0,\ldots,0]$ and there is a morphism from $\Phi : X = \mathcal{X}_{t\neq 0} \to \mathcal{X}_0$ which is the restriction of the projection. There are two possibilities.

(a) $\deg(\Phi) > 1$. In this case, $\mathcal{X}_0$ is generically non-reduced. So $a < 0$ in (2.25).

Example: $X = \{Z_0^3+Z_1^3+Z_2^3+Z_3^3 = 0\}$. $\lambda(t) : (Z_0, Z_1, Z_2, Z_3) \mapsto (t^{-1}Z_0, t^{-1}Z_1, t^{-1}Z_2, Z_3)$. $\mathcal{X}_0 = 3\{Z_3 = 0\}$.

More generally, assume $X^n \subset \mathbb{P}^N$ is in general position. Then the generical linear subspace $\mathbb{L} \cong \mathbb{P}^{N-n-1}$ satisfies $\mathbb{L} \cap X = \emptyset$. Let $\mathbb{M} \cong \mathbb{P}^n$ be a complement of $\mathbb{L} \subset \mathbb{P}^N$. Then the projection of $\Phi : \mathbb{P}^N \backslash \mathbb{L} \to \mathbb{M}$ gives a projection $\Phi : X \to \Phi(X)$ with degree equal to the degree of X.

(b) $\deg(\Phi) = 1$. In this case, $\mathcal{X}_0$ is generically reduced. We have the following fact

Lemma 17. *If $\Phi(X)$ is normal, then $X \cong \Phi(X)$.*

Proof. Note that Φ is generically one to one, Φ is a birational morphism. Assume $y \in \Phi(X)$, such that $\Phi^{-1}(y)$ contains a positive dimensional subvariety C. The sub-linear system $\mathcal{L}$ of $|\mathcal{O}(1)|$ defining Φ has a basis $\{Z_0, \ldots, Z_K\}$ has no base point, so the form defined by

$$\omega = \sqrt{-1}\partial\bar{\partial} \log \sum_{i=1}^{K} |Z_i|^2$$

is a smooth form representing $c_1(\mathcal{O}(1))$. By assumption $\omega|_C = 0$. So

$$\deg C = \int_C \omega^{\dim C} = 0$$

This contradicts the ampleness of $\mathcal{O}(1)$. So the inverse image of Φ contains only finite many points. Since $\Phi(X)$ is normal, by Zariski's main theorem [Hart, 11.4], the fibre of Φ is connected, so Φ must be isomorphism. □

Example: Assume $X^n \subset \mathbb{P}^N$ is in general position. Assume $K \geq n+1$, then $N - K - 1 \leq N - n - 2$. So the generical linear subspace $\mathbb{L} \cong \mathbb{P}^{N-K-1}$ satisfies $\mathbb{L} \cap X = \emptyset$. Let $\mathbb{M} \cong \mathbb{P}^K$ be a complement of $\mathbb{L} \subset \mathbb{P}^N$. Then the projection of $\Phi : \mathbb{P}^N \backslash \mathbb{L} \to \mathbb{M}$ gives a projection $\Phi : X \to \Phi(X)$ with degree 1.

Now assume $\nu_\omega(\omega_\phi)$ is proper on $\mathcal{B}_k$ in the sense of Definition 2. Then by Lemma 12 and Remark 8 (See [PaTi2]), in case 1 or 2(a), $F_1 > 0$. However, in case 2(b), $F_1 = 0$. So there always exists non-product test configuration with vanishing Donaldson-Futaki invariant. This case was missing in most of previous works as pointed out in [LiXu] (See also [Sto2] and [Odak4]).

2.7.3 Test configuration and K-stability

Following [LiXu], we will state the definition for any $\mathbb{Q}$-Fano variety X which by definition is a normal, klt variety with $-K_X$ ample.

Definition 8. *1. Let X be a $\mathbb{Q}$-Fano variety. Assume $-rK_X$ is Cartier. A test configuration of $(X, -rK_X)$ consists of*

- *a variety $\mathcal{X}$ with a $\mathbb{G}_m$-action,*
- *a $\mathbb{G}_m$-equivariant ample line bundle $\mathcal{L} \to \mathcal{X}$,*
- *a flat $\mathbb{G}_m$-equivariant map $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication in the standard way $(t, a) \to ta$,*

such that for any $t \neq 0$, $(\mathcal{X}_t = \pi^{-1}(t), \mathcal{L}|_{\mathcal{X}_t})$ is isomorphic to $(X, -rK_X)$.
2. If $\mathcal{L}$ is only a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ such that for an integer $m \geq 1$, $(\mathcal{X}, m\mathcal{L})$ yields a test configuration of $(X, -mrK_X)$. We call $(\mathcal{X}, \mathcal{L})$ a $\mathbb{Q}$-test configuration of $(X, -rK_X)$.

Remark 13. *With this definition, in fact any test configuration comes from a $\mathbb{Q}$-test configuration of $(X, -K_X)$ by taking power of a $\mathbb{Q}$-polarization. In the following, by the abuse of notation, if we do not want to specify the exponent r, we will just call $(\mathcal{X}, \mathcal{L})$ a test configuration for both cases in the above definition.*

Similarly, we have the following definition.

Definition 9. *A $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$ is called a special test configuration if $\mathcal{L} = -rK_{\mathcal{X}}$ and $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety.*

For any test configuration, we can define *the Donaldson-Futaki invariant.* First by the Riemann-Roch theorem,

$$d_k = \dim H^0(X, \mathcal{O}_X(-rkK_X)) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

for some rational numbers a_0 and a_1. Let $(\mathcal{X}_0, \mathcal{L}_0)$ be the restriction of $(\mathcal{X}, \mathcal{L})$ over 0. Since $\mathbb{G}_m$ acts on $(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$, $\mathbb{G}_m$ also acts on $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$. We denote its total $\mathbb{G}_m$-weight by w_k. By the equivariant Riemann-Roch Theorem,

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

So we can expand

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

Definition 10 ([Don4])**.** *The (normalized) Donaldson-Futaki invariant (DF-invariant) of the test configuration* $(\mathcal{X}, \mathcal{L})$ *is defined to be*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = -\frac{F_1}{a_0} = \frac{a_1 b_0 - a_0 b_1}{a_0^2} \tag{2.27}$$

With the normalization in (2.27), we can define the Donaldson-Futaki invariant for any $\mathbb{Q}$-test configuration by $\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \mathrm{DF}(\mathcal{X}, m\mathcal{L})$. It is easy to see that this definition does not depend on the choice of m because of the normalization in the definition (2.27).

Definition 11. *Let X be a $\mathbb{Q}$-Fano variety.*

1. *X is called K-semistable if for any $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$ with $r > 0$, we have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$.*

2. *X is called K-stable (resp. K-polystable) if for any normal $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$ with $r > 0$, we have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, and the equality holds only if $(\mathcal{X}, \mathcal{L})$ is trivial (resp. $\mathcal{X} \cong X \times \mathbb{A}^1$).*

Remark 14. *The original definition of K-polystability and K-stability need to be amended as pointed out in [LiXu] or at the end of Subsection 2.7.2. Here for the triviality of the test configuration with Donaldson-Futaki invariant 0, we require the test configuration to be normal. See the case 2-(b) at the end of Section 2.7.2 and Remark 41 in Section 5.3.2. On the other hand, for K-semistability, it follows from [RoTh1, 5.2] that we only need to consider normal test configurations, too. See Subsection 5.3.2.*

2.7.4 Calculation of Futaki invariant

by Log Resolution

Assume $\tilde{X}$ is an equivariant log resolution of singularity of X such that

$$K_{\tilde{X}}^{-1} = \pi^* K_X^{-1} - \sum_i a_i E_i$$

E_i are exceptional divisors with normal crossings. v lifts to be a smooth holomorphic vector field $\tilde{v}$ on $\tilde{X}$, which is tangential to each exceptional divisor E_i. Let S_i be the defining section of

$[E_i]$, so $E_i = \{S_i = 0\}$. Let h_i be an Hermitian metric on $[E_i]$ and $R_{h_i} = \sqrt{-1}\bar{\partial}\partial \log h_i$ be the corresponding curvature form. By $\partial\bar{\partial}$ lemma (or Hodge theory), there is an Hermitian metric $\tilde{h}$ on $K_{\tilde{X}}^{-1}$ such that its curvature form $R_{\tilde{h}} = \sqrt{-1}\bar{\partial}\partial \log \tilde{h} = -\sqrt{-1}\partial\bar{\partial} \log \eta_{\tilde{h}}$ satisfies

$$R_{\tilde{h}} = \pi^* \omega_h - \sum_i a_i R_{h_i}$$

So

$$\pi^*(Ric(\omega_h) - \omega_h) = -\sqrt{-1}\partial\bar{\partial} \log \frac{\pi^* \omega_h^n}{\eta_{\tilde{h}}} + \sqrt{-1} \sum_i a_i \partial\bar{\partial} \log |S_i|^2_{h_i}$$

$$\pi^* h_{\omega_h} = -\log \frac{\pi^* \omega_h^n}{\eta_{\tilde{h}}} + \sum_i a_i \log |S_i|^2_{h_i} + C$$

$$\int_{X_{sm}} v(h_{\omega_h})\omega_h^n = \int_{X \setminus \cup_i E_i} \pi^*(v(h_{\omega_h}))\pi^*\omega_h^n = \int_{X \setminus \cup_i E_i} -\tilde{v}(\frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}})\eta_{\tilde{h}} + \sum_i a_i \tilde{v}(\log |S_i|^2_{h_i})\pi^*\omega_h^n$$

$\tilde{v}(\frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}})$ is a smooth function on $\tilde{X}$.

Lemma 18. $\theta_i = \tilde{v}(\log |S_i|^2_{h_i})$ *extends to a smooth function on* $\tilde{X}$ *such that*

$$\sqrt{-1}\bar{\partial}\theta_i = -i_{\tilde{v}} R_{h_i}$$

Proof. It's clearly true away from exceptional divisors. Let $p \in E_i$, in a neighborhood U of p, choose a local frame e_i of $[E_i]$, $S_i = f_i e_i$, and $E_i = \{f_i = 0\}$. We assume E_i is smooth at p, so we can take f_i to be a coordinate function, say z_1. Since $\tilde{v}$ is tangent to E_i, $\tilde{v}$ is of the form

$$\tilde{v}(z) = z_1 b_1(z)\partial_{z_1} + \sum_{i>1} c_i(z)\partial_{z_i}$$

$b_1(z)$, $c_i(z)$ are holomorphic functions near p. Now

$$\theta_i = \tilde{v}(\log |z_1|^2) + \tilde{v}(\log |e_i|^2_{h_i})$$

the second term is smooth near p, and

$$\tilde{v}(\log |z_1|^2) = \frac{\tilde{v}(z_1)}{z_1} = b_1(z)$$

is holomorphic near p. Also

$$\bar{\partial}\theta_i = \bar{\partial}(v(\log|e_i|^2_{h_i})) = -i_v\bar{\partial}\partial\log|e_i|^2_{h_i} = -\frac{2\pi}{\sqrt{-1}}i_v R_{h_i} \tag{2.28}$$

□

So the Futaki invariant can be written as

$$\begin{aligned} F_{c_1(X)}(v) &= \int_{\tilde{X}} (\frac{L_{\tilde{v}}\eta_{\tilde{h}}}{\eta_{\tilde{h}}} + \sum_i a_i\theta_i)(R_{\tilde{h}} + \sum_i a_i R_{h_i})^n \\ &= \frac{1}{n+1}\int_{\tilde{X}} (div_{\tilde{\eta}}(\tilde{v}) + \sum_i a_i\theta_i + R_{\tilde{h}} + \sum_i a_i R_{h_i})^{n+1} \end{aligned}$$

Now by (2.19) and (2.28), $(div_{\tilde{\eta}}(\tilde{v}) + \sum_i a_i\theta_i + R_{\tilde{h}} + \sum_i a_i R_{h_i})$ is an equivariantly closed form, so we can apply localization formula to this integral. See [BGV], [Tia10] for localization formula.

Remark 15. *Note that at any zero point p of $\tilde{v}$, the divergence $div_{\tilde{\eta}}(\tilde{v})$ is well defined independent of volume forms. Also by the proof of previous lemma, if $p \in E_i$, $\theta_i(p) = b_1(p)$ is the weight on the normal bundle of E_i at p, otherwise $\theta_i(p) = 0$. In any case, if $q = \pi(p) \in X$, then $div(\tilde{v})(p) + \sum_i a_i\theta_i(p)$ is the weight on $K_X^{-1}|_q$.*

An example of calculation

We calculate an example from [DiTi] using log resolution.

X is the hypersurface given by $F = Z_0Z_1^2 + Z_1Z_3^2 + Z_2^3$. v is given by $\lambda(t) = diag(1, e^{6t}, e^{4t}, e^{3t})$. The zero points of v are $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,0,1]$.

$[1,0,0,0]$ is an A-D-E singular point of type E_6. Locally, it's $\mathbb{C}^2/\Gamma$, Γ is the lifting to $SU(2)$ of the symmetric group of Tetrahedron in $SO(3)$. $|\Gamma| = 24$. After a (nonlinear) change of coordinate, we change it to the standard form $z_1^2 + z_2^3 + z_3^4$. The vector field is given by $v = 6z_1\partial_{z_1} + 4z_2\partial_{z_2} + 3z_3\partial_{z_3}$. By viewing the surface as a two-fold covering of $\mathbb{C}^2$, branched along a singular curve, we can equivariantly resolve the singularity by blowup and normalization (at the origin of each step). See [BPV].

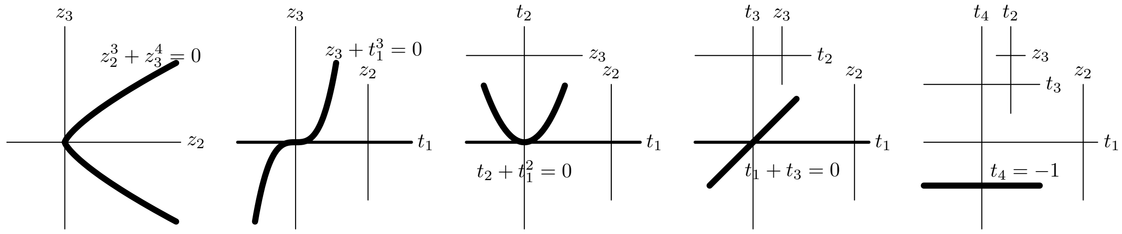
1. $z_1^2 + z_2^3 + z_3^4 = 0$. $z_1 \mapsto e^{6t}z_1$, $z_2 \mapsto e^{4t}z_2$, $z_3 \mapsto e^{3t}z_3$.

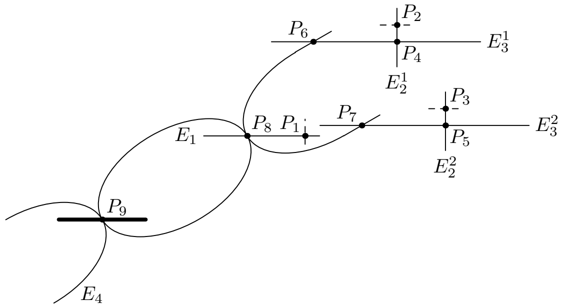
2. $s_1^2 + z_3(z_3 + t_1^3) = 0$. $t_1 = \frac{z_2}{z_3} \mapsto e^t t_1$, $s_1 = \frac{z_1}{z_3} \mapsto e^{3t}s_1$.

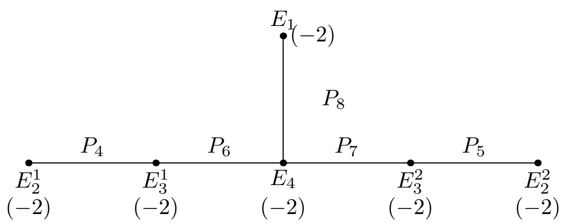
3. $s_2^2 + t_2(t_2 + t_1^2) = 0$. $t_2 = \frac{z_3}{t_1} \mapsto e^{2t}t_2$, $s_2 = \frac{s_1}{t_1} \mapsto e^{2t}s_2$.

4. $s_3^2 + t_3(t_3 + t_1) = 0$. $t_3 = \frac{t_2}{t_1} \mapsto e^t t_3$, $s_3 = \frac{s_2}{t_1} \mapsto e^t s_3$.

5. $s_4^2 + t_4(t_4+1) = 0$. $t_4 = \frac{t_3}{t_1} \mapsto t_4$, $s_4 = \frac{s_3}{t_1} \mapsto s_4$.







The intersection diagram of Exceptional divisors is of type E_6. Assume

$$K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i$$

Note that $\pi^* K_X \cdot E_i = 0$, then

$$K_{\tilde{X}} \cdot E_i = \sum_j a_j E_j \cdot E_i$$

By adjoint formula,

$$K_{\tilde{X}} \cdot E_i = K_{E_i} \cdot E_i - E_i^2 = 0$$

Because the intersection matrix $\{E_i \cdot E_j\}$ is negative definite, we have $a_i = 0$. So

$$K_{\tilde{X}} = \pi^* K_X$$

The zero points set of $\tilde{v}$ are: $\cup_{i=1}^5 \{P_i\} \cup E_4$.

1. equation near P_1 is: $u_1^2 + z_2(1 + t_1'^4 z_2) = 0$. $u_1 = \frac{z_1}{z_2} \mapsto e^{2t} u_1$, $t_1' = \frac{z_3}{z_2} \mapsto e^{-t} t_1'$.

2. equation near P_2, P_3 is: $u_2^2 + t_2'^3 z_3^2 + 1 = 0$. $t_2' = \frac{t_1}{z_3} \mapsto e^{-2t} t_2'$, $z_3 \mapsto e^{3t} z_3$.

3. equation near P_4, P_5 is: $u_3^2 + t_3'^2 t_2 + 1 = 0$. $t_3' = \frac{t_1}{t_2} \mapsto e^{-t} t_3'$, $t_2 \mapsto e^{2t} t_2$.

4. equation near E_4 (away from P_6, P_7) is: $s_4^2 + t_4(t_4+1) = 0$. (near P_6, P_7, the equation is $u_4^2 + t_4'^2 + 1 = 0$) $E_4 = \{t_1 = 0\}$. $t_1 \mapsto e^t t_1$.

So the contribution to the localization formula of Futaki invariant at point $[1,0,0,0]$ is:

$$\frac{1}{-2}+2\frac{1}{-6}+2\frac{1}{-2}+\int_E \frac{1}{1+c_1([E])}=\frac{1}{6}$$

the contributions from the other two fixed points are easily calculated, so the Futaki invariant is:

$$F_{c_1(X)}(v)=\frac{1}{3}\left(\frac{1}{6}+\frac{(-5)^3}{6}+\frac{(-2)^3}{-3}\right)=-6$$

Remark 16. *The contribution of the singular point can also be calculated using the localization formula for orbifolds given in [DiTi]. Note that the local uniformization is given by:*

$$\begin{aligned}
\pi_1:\mathbb{C}^2 &\longrightarrow \mathbb{C}^2/\Gamma\subset\mathbb{C}^3\\
(z_1,z_2) &\mapsto [1,(z_1^4+2\sqrt{-3}z_1^2z_2^2+z_2^4)^3,2(-3)^{\frac{3}{4}}z_1z_2(z_1^4-z_2^4),-(z_1^8+14z_1^4z_2^4+z_2^8)]
\end{aligned}$$

So $\pi_1^*v=\frac{1}{2}(z_1\partial_{z_1}+z_2\partial_{z_2})$, *and*

$$\frac{1}{|\Gamma|}\frac{(div(\pi_1^*v))^{n+1}}{det(\nabla(\pi_1^*v)|_{T_zX})}=\frac{1}{24}\frac{1^3}{1/4}=\frac{1}{6}$$

Futaki invariant of Complete Intersections

We will use the algebraic definition to calculate. Assume $X\in\mathbb{CP}^N$ is a complete intersection given by: $X=\cap_{\alpha=1}^r\{F_\alpha=0\}$. Assume $\deg F_\alpha=d_\alpha$, so $\deg X=\prod_\alpha d_\alpha$. Let $R=\mathbb{C}[Z_0,\cdots,Z_N]$. X has homogeneous coordinate ring

$$R(X)=\mathbb{C}[Z_0,\cdots,Z_N]/(I(X))=R/I(X)$$

$I(X)$ is the homogeneous ideal generated by homogeneous polynomial $\{F_\alpha\}$. It is well known that $R(X)$ has a minimal free resolution by Koszul complex:

$$0\to R(-\sum_{\alpha=0}^r d_\alpha)\otimes(\mathbb{C}\cdot\prod_\alpha F_\alpha)\to\cdots\to\bigoplus_{\alpha<\beta}^r R(-d_\alpha-d_\beta)\cdot(\mathbb{C}\cdot(F_\alpha F_\beta))\to\bigoplus_{\alpha=0}^r R(-d_\alpha)\otimes(\mathbb{C}\cdot F_\alpha)\to R\to R(X)\to 0$$

Let $\lambda(t)\in PSL(N+1,\mathbb{C})$ be a one-parameter subgroup generated by $A=diag(\lambda_0,\cdots,\lambda_N)$, and v be the corresponding holomorphic vector field. Assume that

$$\sum_{i=0}^N\lambda_iZ_i\frac{\partial}{\partial Z_i}F(Z)=\mu_\alpha F_\alpha$$

$(\mathbb{C}^*)^2$ acts on $S(X)$. Let $a_{k,l} = dimS(X)_{k,l}$ be the dimensions of weight spaces, then this action has character:

$$Ch(S(X)) = \sum_{(k,l)\in\mathbb{N}\times\mathbb{Z}} a_{k,l}t_1^k t_2^l = \frac{\prod_{\alpha=1}^r (1 - t_1^{d_\alpha} t_2^{\mu_\alpha})}{\prod_{i=0}^N (1 - t_1 t_2^{\lambda_i})} = f(t_1, t_2)$$

The $k-th$ Hilbert weight is (note it's a finite sum) $w_k = \sum_{l\in\mathbb{Z}} a_{k,l} \times l$ and

$$\begin{aligned}\sum_{k\in\mathbb{N}} w_k t_1^k &= \left.\frac{\partial f}{\partial t_2}\right|_{t_2=1} = -\frac{\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1-t_1^{d_\beta}))}{(1-t_1)^{N+1}} + (\sum_i \lambda_i)t_1 \frac{\prod_{\alpha=1}^r(1-t_1^{d_\alpha})}{(1-t_1)^{N+2}} \\ &= -\frac{\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1+\cdots+t_1^{d_\beta-1}))}{(1-t_1)^{N+2-r}} + \lambda t_1 \frac{\prod_{\alpha=1}^r(1+\cdots+t_1^{d_\alpha-1})}{(1-t_1)^{N+2-r}} \qquad (2.29)\end{aligned}$$

Lemma 19. *Let*

$$f(t) = \frac{g(t)}{(1-t)^{n+1}} = \frac{\sum_{i=0}^r a_i t^i}{(1-t)^{n+1}} = \sum_{k=0}^{+\infty} b_k t^k$$

then

$$b_k = \frac{k^n}{n!}g(1) + \frac{k^{n-1}}{2(n-1)!}((n+1)g(1) - 2g'(1)) + O(k^{n-2})$$

Proof.

$$f(t) = (\sum_{i=0}^r a_i t^i) \cdot \sum_{j=0}^{\infty} \binom{n+j}{n} t^j$$

So when $k \gg 1$,

$$\begin{aligned} b_k &= \sum_{i=0}^r a_i \binom{n+k-i}{n} = \sum_{i=0}^r a_i \frac{(n+k-i)\cdots(k-i+1)}{n!} \\ &= \frac{k^n}{n!}\sum_{i=0}^r a_i + \frac{k^{n-1}}{2(n-1)!}\sum_{i=0}^r a_i(n+1-2i) + O(k^{n-2}) \\ &= \frac{k^n}{n!}g(1) + \frac{k^{n-1}}{2(n-1)!}((n+1)g(1) - 2g'(1)) + O(k^{n-2}) \end{aligned}$$

□

Let $g(t) = -\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1+\cdots+t_1^{d_\beta-1})) + \lambda t_1 \prod_{\alpha=1}^r(1+\cdots+t_1^{d_\alpha-1})$, $n = N+1-r$, let $\tilde{\mu}_\alpha = \mu_\alpha - \frac{\lambda}{N+1}d_\alpha$, then $\tilde{\mu}$ is invariant when $\lambda(t)$ differs by a diagonal matrix. by the lemma, we can get

$$g(1) = -\sum_\alpha \mu_\alpha \prod_{\beta\neq\alpha} d_\beta + \lambda\prod_\alpha d_\alpha = -\prod_\alpha d_\alpha(\sum_\beta \frac{\mu_\beta}{d_\beta} - \lambda) = -\prod_\alpha d_\alpha\left(\sum_\beta \frac{\tilde{\mu}_\beta}{d_\beta} - \frac{\lambda}{N+1}(N+1-r)\right) \qquad (2.30)$$

$$
\begin{aligned}
(N-r+2)g(1)-2g'(1) &= -\prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\mu_\gamma}{d_\gamma}-\sum_\beta\mu_\beta-\lambda(N-\sum_\beta d_\beta)\right)\\
&= -\prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\tilde\mu_\gamma}{d_\gamma}-\sum_\beta\tilde\mu_\beta-\frac{\lambda}{N+1}(N-r)(N+1-\sum_\alpha d_\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
w_k &= -\prod_\alpha d_\alpha\sum_\beta\frac{\tilde\mu_\beta}{d_\beta}\frac{k^{N+1-r}}{(N+1-r)!}-\prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\tilde\mu_\gamma}{d_\gamma}-\sum_\beta\tilde\mu_\beta\right)\frac{k^{N-r}}{2(N-r)!}+O(k^{N-r-1})\\
&\quad+\frac{\lambda}{N+1}k\cdot dimH^0(X,\mathcal{O}(k)) \qquad (2.31)
\end{aligned}
$$

By (2.23), we can get the Futaki invariant

$$
F_{c_1(\mathcal{O}(1))}(v)=-\prod_\alpha d_\alpha\left(\sum_\beta\tilde\mu_\beta-\frac{N+1-\sum_\gamma d_\gamma}{N+1-r}\sum_\beta\frac{\tilde\mu_\beta}{d_\beta}\right)
$$

Remark 17. *In hypersurface case, the above formula becomes*

$$
F_{c_1(\mathcal{O}(1))}(v)=-\frac{(d-1)(N+1)}{N}(\mu-\frac{\lambda}{N+1}d)
$$

Apply this to the example in section 2.7.4, where $d=3$, $N=3$, $\lambda=6+3+4=13$, $\mu=12$, $\mathcal{O}(1)=K_X^{-1}$, then we get the same result as before.

$$
F_{c_1(X)}(v)=-\frac{2\cdot 4}{3}(12-\frac{13}{4}\cdot 3)=-6
$$

Remark 18. *We can calculate directly the leading coefficient of w_k in (2.31)using the Lelong-Poincáre equation. Also see [Lu1].*

Lemma 20 (Poincáre-Lelong equation)**.** *Assume L is a holomorphic line bundle on X, s is a nonzero holomorphic section of L, D is the zero divisor of s, i.e. $\{s=0\}$ counted with multiplicities. h is an Hermitian metric on L, $R_h=\sqrt{-1}\bar\partial\partial\log h$ is its curvature form. Then in the sense of distribution, we have the identity*

$$
\sqrt{-1}\partial\bar\partial\log|s|_h^2=\int_D -R_h
$$

i.e., for any smooth $(2n-2)$ form η on X, we have

$$\sqrt{-1}\int_X (\log|s|_h^2)\partial\bar{\partial}\eta = \int_D \eta - \int_X R_h \wedge \eta$$

Let $X_0 = \mathbb{CP}^N$, $X_{a+1} = X_a \cap \{F_a = 0\}$, then $X_0 \supset X_1 \cdots \supset X_r = X$. $\theta_v = \frac{\sum_i \lambda_i |Z_i|^2}{\sum_i |Z_i|^2}$, then $i_v\omega_{FS} = \sqrt{-1}\bar{\partial}\theta_v$. On X_{a-1}, by the lemma, we have

$$\sqrt{-1}\partial\bar{\partial}\log \left.\frac{|F_a|^2}{(\sum_i |Z_i|^2)^{d_a}}\right|_{X_{a-1}} = \int_{X_a} -d_a \cdot \omega_{FS}|_{X_{a-1}}$$

So

$$\int_{X_a} \theta_v \omega_{FS}^{N-a} = d_a \int_{X_{a-1}} \theta_v \omega_{FS}^{N-a+1} + \sqrt{-1}\int_{X_{a-1}} \theta_v \partial\bar{\partial}\log\frac{|F_a|^2}{(\sum_i |Z_i|^2)^d} \wedge \omega_{FS}^{N-a}$$

Using integration by parts, the second integral on the right equals

$$\begin{aligned}
\sqrt{-1}\int_{X_{a-1}} \bar{\partial}\theta_v \quad &\wedge \quad \partial \log\frac{|F_a|^2}{(\sum_i |Z_i|^2)^d} \wedge \omega_{FS}^{N-a} = \int_{X_{a-1}} i_v\omega_{FS} \wedge \partial\log\frac{|F_a|^2}{(\sum_i |Z_i|^2)^{d_a}} \wedge \omega_{FS}^{N-a} \\
&= \quad -\frac{1}{N-a+1}\int_{X_{a-1}} v(\log\frac{|F_a|^2}{(\sum_i |Z_i|^2)^{d_a}})\omega_{FS}^{N-a+1} \\
&= \quad -\frac{1}{N-a+1}\int_{X_{a-1}} (\mu_a - d_a\frac{\sum_i \lambda_i |Z_i|^2}{\sum_i |Z_i|^2})\omega_{FS}^{N-a+1} \\
&= \quad -\frac{1}{N-a+1}\mu_a \deg(X_{a-1}) + d_a\frac{1}{N-a+1}\int_{X_{a-1}} \theta_v \omega_{FS}^{N-a+1}
\end{aligned}$$

So

$$(N-a+1)\int_{X_a} \theta_v \omega_{FS}^{N-a} = -\mu_a \deg(X_{a-1}) + d_a(N-a+2)\int_{X_{a-1}} \theta_v \omega_{FS}^{N-a+1}$$

While

$$(N+1)\int_{X_0} \theta_v \omega_{FS}^N = (N+1)\int_{\mathbb{CP}^N} \frac{\sum_i \lambda_i |Z_i|^2}{\sum_i |Z_i|^2}\omega_{FS}^N = \sum_i \lambda_i = \lambda$$

By induction, we get

$$(N-r+1)\int_{X_r} \theta_v \omega_{FS}^{N-r} = -\prod_\alpha d_\alpha \sum_\beta \frac{\mu_\beta}{d_\beta} + \lambda\prod_\alpha d_\alpha = \prod_\alpha d_\alpha\left(-\sum_\beta \frac{\tilde{\mu}_\beta}{d_\beta} + (N+1-r)\frac{\lambda}{N+1}\right)$$

This is the same as $g(1)$, (2.30).

Chapter 3

Some Extension of general theories

3.1 Twisted Kähler-Einstein equation and Invariant $R_\eta(X)$

In the following, we will always use the notation: $\beta \in [0,1]$ and $\alpha = 1-\beta$.

Let's consider the twisted Kähler-Einstein equation

$$Ric(\omega_\phi) = \beta\omega_\phi + \alpha\eta \tag{3.1}$$

for some current $\eta \in 2\pi c_1(X)$ which is allowed to be non-positive and singular. This is equivalent to the following Monge-Ampère equation:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n$$

where $H_{\omega,\alpha\eta}$ satisfies:

$$\sqrt{-1}\partial\bar{\partial}H_{\omega,\alpha\eta} = Ric(\omega) - \beta\omega - \alpha\eta, \quad \int_X e^{H_{\omega,\alpha\eta}}\omega^n/n! = V$$

One can define the associated K-energy and F-energy:

- $$F_{\omega,\alpha\eta}(\omega_\phi) = F^0_\omega(\phi) - \frac{V}{\beta}\log\left(\frac{1}{V}\int_X e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n/n!\right)$$

- $$\nu_{\omega,\alpha\eta}(\omega_\phi) = \beta F^0_\omega(\phi) + \int_X \log \frac{\omega^n_\phi}{e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n}\frac{\omega^n_\phi}{n!} + \int_X H_{\omega,\alpha\eta}\omega^n/n!$$

It's easy to verify that these two functional are dual to each other under Legendre transform ([Berm]). Berman called $\nu_{\omega,\alpha\eta}$ the free energy associated with (3.24). The name comes from his statistical mechanical derivation of Kähler-Einstein equations. Note that in this notation, $\nu_\omega(\omega_\phi) = \nu_{\omega,0}(\omega_\phi)$.

Proposition 10. *We have the following formulas:*

1. $$\nu_{\omega,\alpha\eta}(\omega_\phi) = -\int_0^1 dt \int_X \dot{\phi}_t \cdot n(Ric(\omega_t) - \beta\omega_t - \alpha\eta) \wedge \omega_t^{n-1}$$

2. $$\begin{aligned} \nu_{\omega,\alpha\eta}(\omega_\phi) &= \beta F_{\omega,\alpha\eta}(\omega_\phi) + \int_X H_{\omega,\alpha\eta}\omega^n/n! - \int_X H_{\omega_\phi,\alpha\eta}\omega^n_\phi/n! \\ &\geq \beta F_{\omega,\alpha\eta} + \int_X H_{\omega,\alpha\eta}\omega^n/n! \end{aligned}$$

3. *Let ω_ψ solve the equation $\omega^n_\psi = e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n$. In other words, $Ric(\omega_\psi) = \beta\omega_\phi + \alpha\eta$. then*
$$\beta F_{\omega,\alpha\eta}(\omega_\phi) + \int_X H_{\omega,\alpha\eta}\frac{\omega^n}{n!} \geq \nu_{\omega,\alpha\eta}(\omega_\psi)$$

4. $$\nu_{\omega,\alpha\eta}(\omega_\phi) = \int_X \log \frac{\omega^n_\phi}{e^{H_{\omega,\alpha\eta}}\omega^n}\frac{\omega^n_\phi}{n!} - \beta(I_\omega - J_\omega)(\omega_\phi) + \int_X H_{\omega,\alpha\eta}\omega^n/n!$$

5. *Assume $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}\phi_1$, $\omega_2 = \omega + \sqrt{-1}\partial\bar{\partial}\phi_2 = \omega + \sqrt{-1}\partial\bar{\partial}(\phi_1 + \phi_2)$. Then*
$$\nu_{\omega,\alpha\eta}(\omega_1) + \nu_{\omega_1,\alpha\eta}(\omega_2) = \nu_{\omega,\alpha\eta}(\omega_2), \quad F_{\omega,\alpha\eta}(\omega_1) + F_{\omega_1,\alpha\eta}(\omega_2) = F_{\omega,\alpha\eta}(\omega_2)$$
In other words, $\nu_{\omega,\alpha\eta}$ and $F_{\omega,\alpha\eta}$ satisfy the cocycle condition.

Proof. (2),(3) and (5) follows from the formula relating twisted potentials of two Kähler metrics.

$$\begin{aligned} H_{\omega_\phi,\alpha\eta} &= H_{\omega,\alpha\eta} + \log\frac{\omega^n}{\omega^n_\phi} - \beta\phi - \log\left(\frac{1}{V}\int_X e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n/n!\right) \\ &= -\left(\log\frac{\omega^n_\phi}{e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n} + \log\left(\frac{1}{V}\int_X e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n/n!\right)\right) \end{aligned}$$

The inequality in (3.2) follows from concavity of log:

$$\frac{1}{V}\int_X H_{\omega_\phi,\alpha\eta}\frac{\omega_\phi^n}{n!} \le \log\left(\frac{1}{V}\int_X e^{H_{\omega_\phi,\alpha\eta}}\frac{\omega_\phi^n}{n!}\right) = 0$$

For (3), since we have normalized ϕ so that $\int_X e^{H_{\omega,\alpha\eta}-\beta\phi}\omega^n/n! = V$,

$$F_{\omega,\alpha\eta}(\phi) = F_\omega^0(\phi)$$

$$\nu_{\omega,\alpha\eta}(\omega_\psi) = \beta F_\omega^0(\psi) + \beta\int_X(\phi-\psi)\frac{\omega_\psi^n}{n!} + \int_X H_{\omega,\alpha\eta}\frac{\omega^n}{n!}$$

So (3) follows from

$$F_\omega^0(\phi) \ge F_\omega^0(\psi) - \int_X(\phi-\psi)\frac{\omega_\psi^n}{n!}$$

□

Corollary 4. *$F_{\omega,\alpha\eta}$ is bounded from below if and only if $\nu_{\omega,\alpha\eta}$ is bounded from below. In this case,*

$$\int_X H_{\omega,\alpha\eta}\frac{\omega^n}{n!} + \beta\inf F_{\omega,\alpha\eta} = \inf\nu_{\omega,\alpha\eta}$$

Proposition 11. *([Bern],[BaMa],[Ban]) If η is a* ***positive current****, and $\omega_\beta := \omega_{\phi_\beta}$ solves the equation* (3.1)*, then ω_β obtains the minimum of $F_{\omega,\alpha\eta}(\omega_\phi)$ and $\nu_{\omega,\alpha\eta}(\omega_\phi)$.*

Proof. If ω_β to the equation (3.1) is the critical point of $F_{\omega,\alpha\eta}(\omega_\phi)$. Berndtsson [Bern] proved $F_{\omega,\alpha\eta}(\omega_\phi)$ is convex on $\mathcal{H}$. So ω_β obtains the minimum of $F_{\omega,\alpha\eta}(\omega_\phi)$. The inequality (3.2) implies $\nu_{\omega,\alpha\eta}$ also obtains the minimum at ω_β. □

There are 2 cases we would like to consider in this Chapter.

- η is any smooth Kähler metric in $2\pi c_1(X)$.

  It was first showed by Tian [Tia6] that we may not be able to solve $(*)_t$ on certain Fano manifold for t sufficiently close to 1. Equivalently, for such a Fano manifold, there is some $t_0 < 1$, such that there is no Kähler metric ω in $c_1(X)$ which can have $Ric(\omega) \ge t_0\omega$. It is now made more precise. Define

  $$R(X) = \sup\{t:\ (*)_t \text{ is solvable}\}$$

  Székelyhidi proved

Proposition 12 ([Szé]).

$$R(X) = \sup\{t : Ric(\omega) > t\omega, \forall \text{ smooth Kähler metric } \omega \in c_1(X)\}$$

In particular, $R(X)$ is independent of $\eta \in c_1(X)$.

- $\eta = \{D = 0\}$ is the integration along a smooth anti-canonical divisor. One can extend the theory in smooth case to the conic case. We can define the logarithmic Futaki invariant after Donaldson (See Section 3.3, [Don6]) and integrate the log-Futaki invariant to get log-K-energy (See Section 3.3.2). If we assume the log-K-energy is proper, then there exists conic Kähler-Einstein metric. (Cf. [JMRL]). Note that, we need to relax the C^∞ condition for Kähler potentials to include the potential of Kähler metric with conic singularities. This conic type Höler space is studied by Donaldson [Don6] (also called wedge Hölder space in [JMRL]). See section 3.2 for sketched proof.

The two cases can be related as follows. Take $\eta_\epsilon = \omega + \sqrt{-1}\partial\bar{\partial}\log(|s|^2 + \epsilon)$, then it's easy to verify that $\lim_{\epsilon \to 0} \eta_\epsilon = \alpha\{s = 0\} =: D$, and

$$\lim_{\epsilon \to 0} \nu_{\omega, \alpha\eta_\epsilon}(\phi) = \nu_{\omega, D}(\phi)$$

$$\lim_{\epsilon \to 0} F_{\omega, \alpha\eta_\epsilon}(\phi) = F_{\omega, D}(\phi)$$

Proposition 13. *([Tia1],[Tia10],[Berm]) For the above two cases, when $0 < \beta \ll 1$, $\nu_{\omega,\alpha\eta}$ is proper.*

Proof. When η is smooth, this was proved by Tian ([Tia1], [Tia10]). The modification to the conic case was proved by [Berm]. First define the log-alpha-invariant.

$$\alpha(K_X^{-1}, (1-\beta)D) = \max\left\{u > 0; \exists 0 < C_u < +\infty \text{ such that } \frac{1}{V}\int_X e^{-u(\phi - \sup\phi)} \frac{e^{h_\omega}\omega^n}{|s_D|^{2(1-\beta)} n!} \leq C_u\right\}$$

Now for any $u < \alpha(K_X^{-1}, (1-\beta)D)$,

$$\begin{aligned}
\log C_u &\geq \log\left(\frac{1}{V}\int_X e^{-u(\phi-\sup\phi)} \frac{e^{h_\omega}\omega^n}{n!|s_D|^{2(1-\beta)}}\right) = \log\left(\frac{1}{V}\int_X e^{-u(\phi-\sup\phi) - \log\frac{|s_D|^{2(1-\beta)}\omega_\phi^n}{e^{h_\omega}\omega^n}} \frac{\omega_\phi^n}{n!}\right) \\
&\geq -\frac{1}{V}\int_X \log\left(\frac{|s_D|^{2(1-\beta)}\omega_\phi^n}{e^{h_\omega}\omega^n}\right)\frac{\omega_\phi^n}{n!} + u\left(\sup\phi - \frac{1}{V}\int_X \phi\frac{\omega_\phi^n}{n!}\right) \\
&\geq \frac{1}{V}\left(-\int_X \log\frac{|s_D|^{2(1-\beta)}\omega_\phi^n}{e^{h_\omega}\omega^n}\frac{\omega_\phi^n}{n!} + uI_\omega(\omega_\phi)\right)
\end{aligned}$$

$$\begin{aligned}\nu_{\omega,(1-\beta)D}(\omega_\phi) &\geq uI_\omega(\omega_\phi) - \beta(I-J)_\omega(\omega_\phi) - C(\beta) \\ &\geq \left(u - \beta\frac{n}{n+1}\right) I_\omega(\omega_\phi) - C\end{aligned}$$

So if

$$\beta < \frac{n+1}{n}\alpha(K_X^{-1}, (1-\beta)D),$$

then log-K-energy is proper for smooth reference metric. Berman estimated the log-alpha-invariant:

Proposition 14. *[Berm]*

$$\alpha(K_X^{-1}, (1-\beta)D) = \alpha(L_D, (1-\beta)D) \geq \min\{\beta, \alpha(L_D|_D), \alpha(L_D)\} > 0 \tag{3.2}$$

So when

$$0 < \beta < \frac{n+1}{n}\min\{\alpha(L_D|_D), \alpha(L_D)\}, \tag{3.3}$$

the log-K-energy is proper. In particular, when $0 < \beta \ll 1$, the log-K-energy is proper. $\square$

Remark 19. *Let D be a smooth divisor such that $D \sim_{\mathbb{Q}} -\lambda K_X$ for some $0 < \lambda \in \mathbb{Q}$. If we let $\eta = \lambda^{-1}\{D\} \in c_1(X)$, then as we will show in [LiSu], the conclusion in Lemma 13 in general is false if $\lambda < 1$.*

Proposition 15. *([Berm], [Rub]) $\nu_{\omega,\alpha\eta}$ is proper if and only if $F_{\omega,\alpha\eta}$ is proper.*

We give a proof due to Berman [Berm, Cor.3.5].

Proof. If $F_{\omega,\alpha\eta}$ is proper, then $\nu_{\omega,\alpha\eta}$ is proper by inequality in (3.2).

Now assume $\nu_{\omega,\alpha\eta}$ is proper. Then $\nu_{\omega,\alpha\eta} - \delta(I-J)_\omega$ is proper for δ small. Let $\eta' = (\alpha\eta - \delta\omega)/(\alpha-\delta)$. Note that $\alpha - \delta = 1 - (\beta+\delta)$. Then it's easy to verify that $H_{\omega,(\alpha-\delta)\eta'} = H_{\omega,\alpha\eta}$ and

$$\nu_{\omega,\alpha\eta} - \delta(I-J)_\omega = \nu_{\omega,(\alpha-\delta)\eta'}$$

Because it's bounded from below, by Corollary 4, $F_{\omega,(\alpha-\delta)\eta'}$ is bounded from below (even when

η' is not a positive current). So

$$
\begin{aligned}
F_{\omega,\alpha\eta}(\phi) &= F^0_\omega(\phi) - \frac{\beta+\delta}{\beta}\frac{V}{\beta+\delta}\log\left(\frac{1}{V}\int_X e^{H_{\omega,\alpha\eta}-(\beta+\delta)\frac{\beta\phi}{\beta+\delta}}\omega^n/n!\right) \\
&\geq F^0_\omega(\phi) - \frac{\beta+\delta}{\beta}F^0_\omega\left(\frac{\beta}{\beta+\delta}\phi\right) + \frac{\beta+\delta}{\beta}F_{\omega,(\alpha-\delta)\eta'}(\beta\phi/(\beta+\delta)) \\
&\geq F^0_\omega(\phi) - \frac{\beta+\delta}{\beta}F^0_\omega\left(\frac{\beta}{\beta+\delta}\phi\right) - C \\
&= J_\omega(\phi) - \frac{\beta+\delta}{\beta}J_\omega\left(\frac{\beta}{\beta+\delta}\phi\right) - C \\
&\geq (1-(\beta/(\beta+\delta))^{1/n})J_\omega(\phi) - C
\end{aligned}
$$

□

Now by Lemma 13,

Corollary 5. *In the two cases (smooth or conic), when $\beta \ll 1$, $F_{\omega,(1-\beta)\eta}$ is proper.*

The following lemma is observed together with Dr. Song Sun.

Lemma 21. *If $F_{\omega,(1-\beta)\eta}$ is proper (resp. bounded) when $\beta = \beta_0$ and it's bounded (resp. proper) when $\beta = \beta_1$, then it's proper when $\beta = (1-t)\beta_0 + t\beta_1$ for $0 < t < 1$. The same conclusion holds for $\nu_{\omega,(1-\beta)\eta}$.*

Proof. Let $\beta_t = (1-t)\beta_0 + t\beta_1$. First note that $H_{\omega,(1-\beta_t)\eta} = (1-t)H_{\omega,(1-\beta_0)\eta} + tH_{\omega,(1-\beta_1)\eta}$. So by Hölder inequality we get

$$
\begin{aligned}
\int_X e^{H_{\omega,(1-\beta_t)\eta}-\beta_t\phi}\frac{\omega^n}{n!} &= \int_X \left(e^{H_{\omega,\alpha_0\eta}-\beta_0\phi}\right)^{1-t}\left(e^{H_{\omega,\alpha_1\eta}-\beta_1\phi}\right)^t\frac{\omega^n}{n!} \\
&\leq \left(\int_X e^{H_{\omega,(1-\beta_0)\eta}-\beta_0\phi}\frac{\omega^n}{n!}\right)^{1-t}\left(\int_X e^{H_{\omega,(1-\beta_1)\eta}-\beta_1\phi}\frac{\omega^n}{n!}\right)^t
\end{aligned}
$$

So the statement follows. The last statement follows by noting $\nu_{\omega,(1-\beta)\eta}$ is linear in β. □

Proposition 16. *In the above two cases (smooth or conic), if (3.1) is solvable for $\beta = \beta_1 \leq 1$, then it's solvable for any $0 < \beta < \beta_1$.*

Proof. Let ω_{β_1} be a solution of the equation (3.1). Because η is positive, ω_{β_1} obtains the minimum of $\nu_{\omega,\alpha_1\eta}(\omega_\phi)$ by Proposition 11. In particular it's bounded from below . By Proposition 13, $\nu_{\omega,\alpha_0\eta}$ is proper for any $0 < \beta_0 \ll 1$. So by Lemma 21, $\nu_{\omega,(1-\beta)\eta}$ is proper for any $0 < \beta < \beta_1$. By Proposition 15, $F_{\omega,(1-\beta)\eta}$ is proper for $0 < \beta < \beta_1$. Now the solvability in the smooth case is well known ([Tia9],[Tia10]). The conic case is solved in [JMRL]. See section 3.2 for sketched proof. □

For the above two cases (smooth or conic), we can define

$$R_\eta(X) = \sup\{s : (3.1) \text{ is solvable for } \beta \in (0, s]\}$$

Corollary 6. *The following are equivalent:*

1. $R_\eta(X) \geq t_0$*, i.e.* (3.1) *is solvable for* $0 < \beta < t_0$*;*

2. *The function* $\nu_{\omega,\alpha\eta}(\omega_\phi)$ *is proper for any* $0 < \beta < t_0$*;*

3. $F_{\omega,\alpha\eta}(\omega_\phi)$ *is proper for any* $0 < \beta < t_0$.

Proof. (1)⇒(2). Take any $\beta_1 < t_0$. The solution ω_{β_1} obtains the minimum of $\nu_{\omega,(1-\beta_1)\eta}$ by Proposition 11. By Proposition 13, $\nu_{\omega,(1-\beta_0)\eta}$ is proper for any $0 < \beta_0 \ll 1$. So by Lemma 21, $\nu_{\omega,(1-\beta)\eta}$ is proper for $\beta_0 < \beta < \beta_1$. (2)⇒(3) follows from Proposition 15. (3)⇒(1): This is well known in the smooth case (See e.g. [BaMa], [Tia10]). In the conic case, see Section 3.2. □

3.2 Existence of conic Kähler-Einstein metric on Fano manifold

There is another continuity method, which is via Kähler-Einstein metrics with conic singularities. This is equivalent to solving the following family of equations with parameter β:

$$Ric(\omega_\psi) = \beta\omega_\psi + (1-\beta)\{D\} \Longleftrightarrow (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{h_\omega - \beta\psi} \frac{\omega^n}{|s|^{2(1-\beta)}} \qquad (*)_\beta$$

where $D \in |-K_X|$ is a smooth divisor, s is the defining section of $[D] \cong -K_X$, and $|\cdot|^2$ is a Hermitian metric on $-K_X$ such that its curvature form $-\sqrt{-1}\partial\bar{\partial}\log|\cdot|^2 = \omega$.

Remark 20. *The weak solution was obtained by Berman [Berm] using pluripotential theory. For the geometric conic solution, in the early version of [JMRL], the authors need to assume the cone angle is in* $(0, \pi]$ *or the bisectional curvature of some reference conic metric has upper bound. Joint with Yanir Rubinstein, by carefully choosing adapted local coordinates, we showed in the appendix that the bisectional curvature of a natural reference conic metric is indeed bounded from above. So this allows any cone angle in* $(0, 2\pi]$.

We will use the log-K-energy associated with a reference conic metric. Let $\widehat{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}|s|^{2\beta}$.

So for any $\widehat{\omega} \in [\omega]$ with at most conic singularities, we can define the log-K-energy and log-F-energy to be

$$\widehat{\nu}_{\widehat{\omega}}(\psi) := \widehat{\nu}_{\widehat{\omega},D} = \int_X \log \frac{\widehat{\omega}_\psi^n}{e^{\widehat{h}_{\widehat{\omega}} - \beta\psi}\widehat{\omega}^n} \frac{\widehat{\omega}_\psi^n}{n!} + \beta F_{\widehat{\omega}}^0(\psi) + \int_X \widehat{h}_{\widehat{\omega}}\widehat{\omega}^n/n!.$$

$$\widehat{F}_{\widehat{\omega}}(\psi) = F_{\widehat{\omega}}^0(\psi) - \frac{V}{\beta}\log\left(\frac{1}{V}\int_X e^{\widehat{h}_{\widehat{\omega}} - \beta\psi}\widehat{\omega}^n/n!\right).$$

where $\widehat{h}_{\widehat{\omega}}$ satisfies

$$\sqrt{-1}\partial\bar{\partial}\widehat{h}_{\widehat{\omega}} = Ric(\widehat{\omega}) - \beta\widehat{\omega} - \alpha\{s=0\}, \text{ and } \int_X e^{\widehat{h}_{\widehat{\omega}}}\omega^n = V \tag{3.4}$$

Theorem 17. *[JMRL] Assume $\widehat{F}_{\widehat{\omega}}$ is proper or $\widehat{\nu}_{\widehat{\omega}}$ is proper, then there exists a conic metric along the divisor D of conic angle $2\pi\beta$.*

Sketch of the proof. The idea is using continuity method as in the proof in the smooth case. So we consider a family of equations:

$$(\widehat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{\widehat{h}_{\widehat{\omega}} - t\psi}\widehat{\omega}^n \tag{3.5}$$

This is equivalent to the equation

$$Ric(\widehat{\omega}_\psi) = t\widehat{\omega}_\psi + (\beta - t)\widehat{\omega} + \alpha\{s=0\}. \tag{3.6}$$

- Step 0: Set up good function space. This is essentially carried out by Donaldson in [Don6]. We first write the metric and differential operator in the conic coordinate. Let $z = re^{i\theta}$ be the ordinary complex coordinate. Define $\rho = r^\beta$. The model case tells us the right thing to do.

$$\begin{aligned}
\widehat{\omega} &= \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2\alpha}} + \sum_{j>1} dz_j \wedge d\bar{z}_j = \frac{(dr + ird\theta)\wedge(dr - ird\theta)}{r^{2\beta}} + \sum_{j>1} dz_j \wedge d\bar{z}_j \\
&= (d\rho + i\beta\rho d\theta)\wedge(d\rho - i\beta\rho d\theta) + \sum_{j>1} dz_j \wedge d\bar{z}_j
\end{aligned}$$

  Let $\epsilon = e^{i\beta\theta}(d\rho + i\beta\rho d\theta) = d\zeta$, where $\zeta = z^\beta = \rho e^{i\beta\theta}$ is defined by choosing a branch locally away from 0. Then the general (1,1) form, in particular the Käler form for a conic metric, also has cross terms:

$$\widehat{\omega} = \sqrt{-1}\left(f\epsilon \wedge \bar{\epsilon} + f_{\bar{j}}\epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j\right) \tag{3.7}$$

If $\widehat{\omega} = \sqrt{-1}\partial\bar{\partial}\phi$,

$$\begin{aligned} f &= \frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}} = \frac{1}{4}\left(e^{-i\beta\theta}(\frac{\partial}{\partial\rho} - \frac{i}{\beta\rho}\frac{\partial}{\partial\theta})\right)\left(e^{i\beta\theta}(\frac{\partial}{\partial\rho} + \frac{i}{\beta\rho}\frac{\partial}{\partial\theta})\right)\phi \\ &= \frac{1}{4}\Delta_{\mathbb{C}^2,\beta}\phi = \frac{1}{4}\frac{1}{\beta\rho}\left(\frac{\partial}{\partial\rho}(\beta\rho\frac{\partial}{\partial\rho}) + \frac{1}{\beta^2\rho^2}\frac{\partial}{\partial\theta}(\beta\rho\frac{\partial}{\partial\theta})\right)\phi \\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\beta^2\rho^2}\frac{\partial^2}{\partial\theta^2}\right)\phi \end{aligned}$$

where $\Delta_{\mathbb{C}^2,\beta}$ is the Laplacian associated with the standard conic metric on $\mathbb{C}^2$: $g_\beta = d\rho^2 + \beta^2\rho^2 d\theta^2$.

$$f_{\bar{j}} = \frac{\partial^2\phi}{\partial\zeta\partial\bar{z}_j} = \frac{1}{2}e^{-i\beta\theta}\left(\frac{\partial}{\partial\rho} - \frac{i}{\beta\rho}\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\bar{z}_j}\phi$$

Following Donaldson, we define

Definition 12. *1. f is in $C^{,\gamma,\beta}$ if f is Höler continuous in the coordinate*

$$(\hat{\zeta} = \rho e^{i\theta} = r^\beta e^{i\theta} = z|z|^{\beta-1}, z_j)$$

2. *A (1,0)-form $\alpha = f_1\epsilon + \sum_{j>1} f_j dz_j$ is in $C^{,\gamma,\beta}$ if $f_i \in C^{,\gamma,\beta}$ for $1 \le i \le n$, and $f_1 \to 0$ as $z_1 \to 0$.*

3. *A (1,1)-form ω is in $C^{,\gamma,\beta}$ if $f, f_j, f_{\bar{j}}, f_{i\bar{j}} \in C^{,\gamma,\beta}$, and $f_j, f_{\bar{j}} \to 0$ as $z_1 \to 0$.*

4. *A function f is in $C^{2,\gamma,\beta}$ if $f, \partial f, \partial\bar{\partial}f$ are all $C^{,\gamma,\beta}$.*

Remark 21. *The point is the derivatives involve only the following wedge differentials:*

$$\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z_j}, \frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}, \frac{\partial^2}{\partial\rho\partial\bar{z}_j}, \frac{1}{\rho}\frac{\partial^2}{\partial\theta\partial\bar{z}_j}, \frac{\partial^2}{\partial z_i\partial\bar{z}_j}$$

The linear theory is set up by Donaldson:

Proposition 17. *[Don6] If $\gamma < \mu = \beta^{-1} - 1$, the inclusion $C^{2,\alpha,\beta} \to C^{,\gamma,\beta}$ is compact. If $\widehat{\omega}$ is a $C^{,\gamma,\beta}$ Kähler metric on (X, D) then the Laplacian of $\widehat{\omega}$ defines a Fredholm map $\Delta_{\widehat{\omega}} : C^{2,\gamma,\beta} \to C^{,\gamma,\beta}$.*

- Step 1: Start the continuity method. Let

$$\mathcal{S} = \{t \in (-\infty, \beta); (3.5) \text{ is solvable}\}.$$

Then $\mathcal{S}$ is non-empty. This is achieved by solving the equation (3.5) when $t \ll 0$ using Newton-Moser iteration method.

- Step 2: Openness of solution set. This follows from implicit function theorem thanks to the Fredholm linear theory set up in Step 0. See [JMRL] for details.

- Step 3: C^0-estimate. This is the same as in the smooth case. By taking derivative with respect to t on both sides of (3.5)

$$\Delta_{\widehat{\omega}} \dot{\psi}_t = -t\dot{\psi} - \psi$$

By calculation, one can prove that

$$F^0_{\widehat{\omega}}(\psi_t) = -\frac{1}{t}\int_0^t (I-J)_{\widehat{\omega}_t}(\psi_t)\widehat{\omega}_t^n/n! \le 0$$

So

$$\begin{aligned} F_{\widehat{\omega}}(\psi_t) &\le -\frac{V}{\beta}\log\left(\frac{1}{V}\int_X e^{\widehat{h}_{\widehat{\omega}}-\beta\psi}\widehat{\omega}^n/n!\right) = -\frac{V}{\beta}\log\left(\frac{1}{V}\int_X e^{-(\beta-t)\psi}\widehat{\omega}_t^n/n!\right) \\ &\le \frac{\beta-t}{\beta}\int_X \psi_t\widehat{\omega}_t^n/n! \le \frac{\beta-t}{\beta^2}V\log\left(\frac{1}{V}\int_X e^{\beta\psi_t}\widehat{\omega}_t^n/n!\right) \\ &= \frac{\beta-t}{\beta^2}V\log\left(\frac{1}{V}\int_X e^{\widehat{h}_{\widehat{\omega}}}\widehat{\omega}^n/n!\right) = 0 \end{aligned}$$

So if $F_{\widehat{\omega}}(\psi)$ is proper, then $I_{\widehat{\omega}}(\widehat{\omega}_{\psi_t}) \le C$. Now the C^0-estimate follows from the following Proposition.

Proposition 18. *1.* $\mathrm{Osc}(\psi_t) \le I_{\widehat{\omega}}(\psi_t)$.

2. (Harnack estimate) $-\inf_X \psi_t \le n \sup_X \psi_t$.

- Step 3: C^2-estimate. To get the C^2-estimate, we can use the Chern-Lu's inequality and maximal principle as in Proposition 2. By the equation (3.6), $Ric(\widehat{\omega}_{\psi_t}) \ge t\widehat{\omega}_{\psi_t}$. The upper bound of bisectional curvature of $\widehat{\omega}$ is crucial, which will be shown in the following subsection.

Let $\Xi = \log tr_{\widehat{\omega}_\psi}\widehat{\omega} - \lambda\psi$ as in (2.3). To use the maximal principle in the conic setting, one can use Jeffrey's trick as in [Jef]. We add the barrier function $\epsilon\|s\|^{2\gamma'}$ for $0 < \gamma' < \gamma$ so that $\Xi + \epsilon\|s\|^{2\gamma'}$ obtains the maximum $x_0' \notin D$. We then apply the maximal principle to $\Xi + \epsilon\|s\|^{2\gamma'}$ and let $\epsilon \to 0$.

- Step 4: $C^{,2,\gamma,\beta}$-estimate. There is a Krylov-Evans' estimate in the conic setting as developed in [JMRL]. The proof is similar to the smooth case in Subsection 2.3.2 but adapted to the conic(wedge) Hölder space.

□

3.2.1 A calculation of bisectional curvature of a conic metric

We consider reference metric of the form

$$\omega = \omega' + \sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta}$$

ω' is a smooth Kähler form. Let $D = \{S = 0\}$. Assume we choose a local coordinate $\{z_i\}$, such that $D = \{z_n = 0\}$. Denote $\nabla z_n = dz_n + z_n a^{-1}\partial a$.

$$\partial\|S\|^{2\beta} = \beta\|S\|^{2(\beta-1)} a z_n \nabla z_n$$

$$\partial\bar{\partial}\|S\|^{2\beta} = -\beta\|S\|^{2\beta}\tilde{\omega} + \beta^2\|S\|^{2\beta-2} a\nabla z_n \wedge \overline{\nabla z_n}$$

where $\tilde{\omega} = c_1([D], \|\cdot\|) = -\partial\bar{\partial}\log a$. So

$$\omega = \omega' - \beta\|S\|^{2\beta}\tilde{\omega} + \beta^2\|S\|^{2\beta-2} a\nabla z_n \wedge \overline{\nabla z_n} \tag{3.8}$$

By scaling the Hermitian metric $\|\cdot\|$, we can assume this is positive definite.

To simplify the calculation of curvature of ω, we need lemma.

Lemma 22. *For any point $p \in X\backslash D$, there exists $\epsilon > 0$, such that if $dist_{g'}(p, D) \leq \epsilon$, then we can choose local holomorphic frame e of L_D and local coordinates $\{z_i\}_{i=1}^n$ such that $S = z_n e$, and $a = \|e\|^2$ satisfies $a(p) = 1$, $da(p) = 0$, $\frac{\partial^2 a}{\partial z_i \partial z_j} a(p) = 0$.*

Proof. Fix any point $q \in D$, we can choose local holomorphic frame e' and complex coordinates $\{w_i\}$ in $B_{g'}(q, \epsilon(q))$ for $\epsilon(q) \ll 1$. Let $S = f'e'$ with f' a holomorphic function and $\|e'\|^2 = c$. Let $e = he'$ for some holomorphic function h to be chosen. Then $a = \|he'\|^2 = |h|^2 c$. Now fix any point $p \in B_{g'}(q, \epsilon(q))$. In order for a to satisfy the vanishing property with respect to variables $\{w_i\}$ at point p, we can just choose h such that $h(p) = c(p)^{-1/2}$, $\tilde{\partial}_i h(p) = -c(p)^{-1}h(p)\tilde{\partial}_i c(p) = -c(p)^{-3/2}\tilde{\partial}_i c(p)$ and

$$\begin{aligned}\tilde{\partial}_i\tilde{\partial}_j h(p) &= -c(p)^{-1}(h(p)\tilde{\partial}_i\tilde{\partial}_j c(p) + \tilde{\partial}_j c(p)\tilde{\partial}_i h(p) + \tilde{\partial}_i c(p)\tilde{\partial}_j h(p)) \\ &= -c(p)^{-3/2}\tilde{\partial}_i\tilde{\partial}_j c(p) + 2c(p)^{-5/2}\tilde{\partial}_i c(p)\tilde{\partial}_j c(p)\end{aligned}$$

Here we used $\tilde{\partial}_i$ to mean partial derivatives with respect to variable w_i. Then we get $S = fe = f'e'$ with $f = f'h^{-1}$ a holomorphic function. Since $D = \{S = 0\}$ is a smooth divisor, we can assume $\frac{\partial f}{\partial w_n}(q) \neq 0$. Choose $\epsilon(q)$ sufficiently small, we can assume $\frac{\partial f}{\partial w_n} \neq 0$ in $B_{g'}(q, \epsilon(q))$. So by inverse function theorem, $z_1 = w_1, \cdots, z_{n-1} = w_{n-1}, z_n = f(w_1, \cdots, w_n)$ are complex coordinate in $B_{g'}(q, \epsilon(q)/2)$ and now $S = ef(w) = z_n e$. By chain rule, it's easy to verify that a satisfy the condition. $a(p) = 1$, $\partial_i a(p) = \partial_i\partial_j a(p) = 0$.

We can cover D by $B_{g'}(q, \epsilon(q)/2)$ for any $q \in D$. By compactness of D, the conclusion follows. □

Remark 22. *By the above proof, we can choose $z_n = f = f'h^{-1} = f'c^{1/2}$ for any point p in $B_{g'}(q, \epsilon(q))$. So z_n is uniformly equivalent to f', i.e. $C^{-1}|f'|(p) \leq |z_n|(p) \leq C|f'|(p)$ for any $p \in B_{g'}(q, \epsilon(q)/2)$.*

Proposition 19. *The holomorphic bisectional curvatures of ω are bounded from above.*

Proof. Using the above lemma, for any point $q \in D$, fix any point $p \in B_{g'}(q, \epsilon)$ and choose the adapted local holomorphic frame e and complex coordinate $\{z_i\}$ provided by the last lemma. Assume in the representation (3.8), $\omega' = \sum_{i,j} g'_{i\bar{j}} dz_i \wedge d\bar{z}_j$, $\tilde{\omega} = \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$ and $\omega = \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. We do some calculation. The calculation is straightforward although laborious. In particular, note that the formula we get has the right symmetry for the subindex.

$$\begin{aligned}g_{i\bar{j}} &= g'_{i\bar{j}} - \beta a^\beta |z_n|^{2\beta} h_{i\bar{j}} + \beta^2 a^\beta |z_n|^{2\beta}\left(\frac{(dz_n + a^{-1}\partial a z_n) \wedge (d\bar{z}_n + a^{-1}\bar{\partial} a \bar{z}_n)}{|z_n|^2}\right)_{i\bar{j}} \\ &= g'_{i\bar{j}} - \beta a^\beta h_{i\bar{j}} |z_n|^{2\beta} + \beta^2 a^{\beta-2}\partial_i a \partial_{\bar{j}} a |z_n|^{2\beta} + \beta^2 a^{\beta-1}\partial_i a |z_n|^{2(\beta-1)} z_n \delta_{jn} \\ &\quad + \beta^2 a^{\beta-1}\partial_{\bar{j}} a |z_n|^{2(\beta-1)} \bar{z}_n \delta_{in} + \beta^2 a^\beta |z_n|^{2(\beta-1)}\delta_{in}\delta_{jn}\end{aligned}$$

Using the vanishing property of a we get

$$g_{i\bar{j}}(p) = g'_{i\bar{j}} - \beta h_{i\bar{j}} |z_n|^{2\beta} + \beta^2 |z_n|^{2(\beta-1)}\delta_{in}\delta_{jn} \tag{3.9}$$

$$
\begin{aligned}
\frac{\partial g_{i\bar{j}}}{\partial z_k} &= \frac{\partial g'_{i\bar{j}}}{\partial z_k} - \beta\partial_k(a^\beta h_{i\bar{j}})|z_n|^{2\beta} + \beta^2\partial_k(a^{\beta-2}\partial_i a\partial_{\bar{j}}a)|z_n|^{2\beta} + \beta^2\partial_k(a^{\beta-1}\partial_i a)|z_n|^{2(\beta-1)}z_n\delta_{jn} \\
&\quad +\beta^2\partial_k(a^{\beta-1}\partial_{\bar{j}}a)|z_n|^{2(\beta-1)}\bar{z}_n\delta_{in} + \beta^2\partial_k(a^\beta)|z_n|^{2(\beta-1)}\delta_{in}\delta_{jn} \\
&\quad -\beta^2 a^\beta h_{i\bar{j}}|z_n|^{2(\beta-1)}\bar{z}_n\delta_{nk} + \beta^3 a^{\beta-2}\partial_i a\partial_{\bar{j}}a|z_n|^{2(\beta-1)}\bar{z}_n\delta_{nk} + \beta^3 a^{\beta-1}\partial_i a|z_n|^{2(\beta-1)}\delta_{nk}\delta_{nj} \\
&\quad \beta^2(\beta-1)a^{\beta-1}\partial_{\bar{j}}a|z_n|^{2(\beta-2)}\bar{z}_n^2\delta_{nk}\delta_{in} + \beta^2(\beta-1)a^\beta|z_n|^{2(\beta-2)}\bar{z}_n\delta_{nk}\delta_{in}\delta_{nj}
\end{aligned}
$$

Now we use the vanishing property of a and $\partial_i\partial_{\bar{j}}a(p) = \partial_i\partial_{\bar{j}}\log a(p) = -h_{i\bar{j}}$ to get

$$
\begin{aligned}
\partial_k g_{i\bar{j}}(p) &= \partial_k g'_{i\bar{j}} - \beta\partial_k h_{i\bar{j}}|z_n|^{2\beta} - \beta^2 h_{k\bar{j}}|z_n|^{2(\beta-1)}\bar{z}_n\delta_{in} - \beta^2 h_{i\bar{j}}|z_n|^{2(\beta-1)}\bar{z}_n\delta_{nk} \\
&\quad +\beta^2(\beta-1)|z_n|^{2(\beta-2)}\bar{z}_n\delta_{nk}\delta_{ni}\delta_{nj} \\
&= \partial_k g'_{i\bar{j}} - \beta\partial_k h_{i\bar{j}}|z_n|^{2\beta} - \beta^2(h_{k\bar{j}}\delta_{in} + h_{i\bar{j}}\delta_{nk})|z_n|^{2(\beta-1)}\bar{z}_n \\
&\quad +\beta^2(\beta-1)|z_n|^{2(\beta-2)}\bar{z}_n\delta_{nk}\delta_{ni}\delta_{nj}
\end{aligned}
$$

The last ingredient appearing in holomorphic bisectional curvature can also be calculated.

$$
\begin{aligned}
\partial_{\bar{l}}\partial_k g_{i\bar{j}}(p) &= \partial_{\bar{l}}\partial_k g'_{i\bar{j}} - \beta(-\beta h_{k\bar{l}}h_{i\bar{j}} + \partial_{\bar{l}}\partial_k h_{i\bar{j}})|z_n|^{2\beta} + \beta^2 h_{i\bar{l}}h_{k\bar{j}}|z_n|^{2\beta} - \beta^2\partial_k h_{i\bar{l}}|z_n|^{2(\beta-1)}z_n\delta_{jn} \\
&\quad -\beta^2\partial_{\bar{l}}h_{k\bar{j}}|z_n|^{2(\beta-1)}\bar{z}_n\delta_{in} - \beta^3 h_{k\bar{l}}|z_n|^{2(\beta-1)}\delta_{in}\delta_{jn} - \beta^2\partial_{\bar{l}}h_{i\bar{j}}|z_n|^{2(\beta-1)}\bar{z}_n\delta_{nk} \\
&\quad -\beta^3 h_{i\bar{l}}|z_n|^{2(\beta-1)}\delta_{nk}\delta_{nj} \\
&\quad -\beta^2(\partial_k h_{i\bar{j}})|z_n|^{2(\beta-1)}z_n\delta_{nl} - \beta^3 h_{k\bar{j}}|z_n|^{2(\beta-1)}\delta_{nl}\delta_{in} - \beta^3 h_{i\bar{j}}|z_n|^{2(\beta-1)}\delta_{nl}\delta_{nk} \\
&\quad +\beta^2(\beta-1)^2|z_n|^{2(\beta-2)}\delta_{nl}\delta_{nk}\delta_{ni}\delta_{nj} \\
&= \partial_{\bar{l}}\partial_k g'_{i\bar{j}} + \beta^2(h_{k\bar{l}}h_{i\bar{j}} + h_{i\bar{l}}h_{k\bar{j}})|z_n|^{2\beta} - \beta(\partial_{\bar{l}}\partial_k h_{i\bar{j}})|z_n|^{2\beta} \\
&\quad -\beta^2(\partial_k h_{i\bar{l}}\delta_{jn} + \partial_k h_{i\bar{j}}\delta_{nl})|z_n|^{2(\beta-1)}z_n - \beta^2(\partial_{\bar{l}}h_{i\bar{j}}\delta_{in} + \partial_{\bar{l}}h_{i\bar{j}}\delta_{nk})|z_n|^{2(\beta-1)}\bar{z}_n \\
&\quad -\beta^3(h_{k\bar{l}}\delta_{in}\delta_{jn} + h_{k\bar{j}}\delta_{nl}\delta_{in} + h_{i\bar{j}}\delta_{nl}\delta_{nk} + h_{i\bar{l}}\delta_{nk}\delta_{nj})|z_n|^{2(\beta-1)} \\
&\quad +\beta^2(\beta-1)^2|z_n|^{2(\beta-2)}\delta_{nl}\delta_{nk}\delta_{ni}\delta_{nj} \qquad (3.10)
\end{aligned}
$$

Now we can estimate the holomorphic bisectional curvature. Let $\{\xi_i\}_{i=1}^n$, $\{\eta_i\}_{i=1}^n$ be two unit vectors with respect to $g_{i\bar{j}}(p)$: $g_{i\bar{j}}(p)\xi_i\bar{\xi}_j = g_{i\bar{j}}(p)\eta_i\bar{\eta}_j = 1$. Using the formula for $g_{i\bar{j}}(p)$ in (3.9). This implies in particular

$$
\begin{aligned}
&|\xi_i| \le C, \quad |\eta_i| \le C, \quad 1 \le i \le n-1 \\
&|\xi_n| \le C|z_n|^{1-\beta}, \quad |\eta_n| \le C|z_n|^{1-\beta} \qquad (3.11)
\end{aligned}
$$

We want to estimate

$$R_{i\bar{j}k\bar{l}}(p)\xi_i\bar{\xi}_j\eta_k\bar{\eta}_l = \left(-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{\partial g_{i\bar{q}}}{\partial z_k} g^{p\bar{q}} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}\right) \xi_i\bar{\xi}_j\eta_k\bar{\eta}_l$$

There are several types of singular terms. In order to estimate them, we first estimate the inverse of $(g_{i\bar{j}})$. Let $f_{i\bar{j}} = g'_{i\bar{j}} - \beta h_{i\bar{j}}|z_n|^{2\beta}$, It's easy to get from (3.9) that

$$\det(g_{i\bar{j}}) = \frac{\beta^2}{|z_n|^{2(1-\beta)}} \det(f_{i\bar{j}})_{1\le i,j\le n-1} + \det(f_{i\bar{j}}) =: \frac{\beta^2}{|z_n|^{2(1-\beta)}}(a + b|z_n|^{2(1-\beta)})$$

with $a = \det(f_{i\bar{j}})_{1\le i,j\le n-1}$ and $b = \det(f_{i\bar{j}})/(\beta^2)$ and

$$\begin{cases} g^{p\bar{q}} & = & O(1) & p,q<n \\ g^{p\bar{n}}, g^{n\bar{q}} & = & O(|z_n|^{2(1-\beta)}) & p,q<n \\ g^{n\bar{n}} & = & \beta^{-2}|z_n|^{2-2\beta}\frac{a}{a+b|z_n|^{2(1-\beta)}} & \end{cases} \tag{3.12}$$

Remark 23. *Since when $|z_n|$ is small, $f_{i\bar{j}}$ is uniformly positive definite, a and b are uniformly positive when $|z_n|$ is small.*

1. Possible singular terms in $-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}\xi_i\bar{\xi}_j\eta_k\bar{\eta}_l$

   (a)
   $$\left|O(1)\delta_{jn}|z_n|^{2\beta-2}z_n\bar{\xi}_j\right| = \left||z_n|^{2\beta-1}\bar{\xi}_n\right| \le C|z_n|^\beta$$

   (b)
   $$\left|O(1)\delta_{in}\delta_{jn}|z_n|^{2\beta-2}\xi_i\bar{\xi}_j\right| = \left||z_n|^{2\beta-2}|\xi_n|^2\right| \le C$$
   $$\left|O(1)\delta_{in}\delta_{kn}|z_n|^{2\beta-2}\xi_i\eta_k\right| = \left||z_n|^{2\beta-2}\xi_n\eta_n\right| \le C$$

   (c) So here the singular term is the only last one
   $$-\beta^2(\beta-1)^2|z_n|^{2\beta-4}|\xi_n|^2|\eta_n|^2 \tag{3.13}$$

2. Possible singular terms in $\frac{\partial g_{i\bar{q}}}{\partial z_k} g^{p\bar{q}} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}\xi_i\bar{\xi}_j\eta_k\bar{\eta}_l$.

   We first define the bilinear form for local tensor $A = (A_{ik\bar{q}})$ satisfying $A_{ik\bar{q}} = A_{ki\bar{q}}$ as follows:
   $$A * B = A_{ik\bar{q}}g^{p\bar{q}}B_{\bar{j}\bar{l}p}\xi_i\bar{\xi}_j\eta_k\bar{\eta}_l \tag{3.14}$$

where $B_{\bar{j}\bar{l}p} = \overline{B_{jl\bar{p}}}$.

By diagonalization, it's easy to see that $*$ is a semipositive bilinear form.

Now we can write

$$g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l} \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l = (A+B+C+D) * (A+B+C+D)$$

where

$$A_{ik\bar{q}} = \partial_k g'_{i\bar{q}}, \quad B_{ik\bar{q}} = -\beta \partial_k h_{i\bar{q}} |z_n|^{2\beta}$$

$$C_{ik\bar{q}} = -\beta^2 (h_{k\bar{q}} \delta_{in} + h_{i\bar{q}} \delta_{nk}) |z_n|^{2(\beta-1)} \bar{z}_n, \quad D_{ik\bar{q}} = \beta^2 (\beta - 1) |z_n|^{2(\beta-2)} \bar{z}_n \delta_{nk} \delta_{ni} \delta_{nq}$$

Using (3.12) and (3.11), let's check all the possible singular terms:

(a) Terms in $C * C$:

$$\left| h_{k\bar{q}} \delta_{in} |z_n|^{2\beta-2} \bar{z}_n g^{p\bar{q}} h_{p\bar{l}} \delta_{jn} |z_n|^{2\beta-2} z_n \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l \right| = \left| O(1) |z_n|^{4\beta-2} |\xi_n|^2 \right| \le C |z_n|^{2\beta}$$

$$\left| h_{i\bar{q}} \delta_{kn} |z_n|^{2\beta-2} \bar{z}_n g^{p\bar{q}} h_{p\bar{l}} \delta_{jn} |z_n|^{2\beta-2} z_n \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l \right| = \left| O(1) |z_n|^{4\beta-2} \bar{\xi}_n \eta_n \right| \le C |z_n|^{2\beta}$$

(b) Terms in $A * C$:

$$\left| (\partial_k g'_{i\bar{q}}) g^{p\bar{q}} h_{p\bar{l}} \delta_{jn} |z_n|^{2\beta-2} z_n \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l \right| = \left| O(1) |z_n|^{2\beta-1} \bar{\xi}_n \right| \le C |z_n|^{\beta}$$

This also implies the terms in $B * C$ are bounded.

(c) Terms in $B * D$:

$$\left| (\partial_k h_{i\bar{q}}) |z_n|^{2\beta} g^{p\bar{q}} |z_n|^{2\beta-4} z_n \delta_{np} \delta_{nj} \delta_{nl} \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l \right| = \left| O(1) |z_n|^{4\beta-3} g^{n\bar{q}} \bar{\xi}_n \bar{\eta}_n \right| \le C |z_n|$$

(d) Terms in $C * D$:

$$\left| h_{k\bar{q}} \delta_{in} |z_n|^{2\beta-2} \bar{z}_n g^{p\bar{q}} |z_n|^{2\beta-4} z_n \delta_{np} \delta_{nj} \delta_{nl} \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l \right| = \left| g^{n\bar{q}} |z_n|^{4\beta-4} |\xi_n|^2 \bar{\eta}_n \right| \le C |z_n|^{1-\beta}$$

(e) So the only singular terms are contained in $(A+D) * (A+D)$.

Now to deal with this bad term, we first note the Cauchy-Schwartz inequality

$$(A+D) * (A+D) = |A|^2 + |D|^2 + A * D + D * A \le (1 + 1/\epsilon) A * A + (1+\epsilon) D * D$$

where ϵ is to determined. Indeed, this follows from $(\frac{1}{\sqrt{\epsilon}}A - \sqrt{\epsilon}D) * (\frac{1}{\sqrt{\epsilon}}A - \sqrt{\epsilon}D) \geq 0$. To find ϵ, replacing $g^{n\bar{n}}$ by it's explicit expression in (3.12) we get:

$$D * D = \beta^4(\beta-1)^2|z_n|^{4\beta-6}g^{n\bar{n}}|\xi_n|^2|\eta_n|^2 = \beta^2(\beta-1)^2|z_n|^{2\beta-4}|\xi_n|^2|\eta_n|^2\frac{a}{a+b|z_n|^{2-2\beta}} \tag{3.15}$$

We can add (3.13) and $(1+\epsilon)D * D$ to get

$$\beta^2(\beta-1)^2|z_n|^{2\beta-4}|\xi_n|^2|\eta_n|^2\left(-1+(1+\epsilon)\frac{a}{a+b|z_n|^{2-2\beta}}\right)$$

So if we choose ϵ:

$$1 = (1+\epsilon)\frac{a}{a+b|z_n|^{2-2\beta}} \Longrightarrow \epsilon = \frac{b}{a}|z_n|^{2-2\beta}$$

The only singular term left is

$$\mathbb{E} := (1+1/\epsilon)A * A = \frac{a+b|z_n|^{2-2\beta}}{b|z_n|^{2-2\beta}}g^{p\bar{q}}(\partial_k g'_{i\bar{q}})\ (\partial_{\bar{l}} g'_{p\bar{j}})\xi_i\bar{\xi}_j\eta_k\bar{\eta}_l \tag{3.16}$$

To bound this term, we first note that there were some ambiguity in choosing first $n-1$ coordinates $z_1, \cdots, z_{n-1}$ in Lemma 22. Now we can choose these coordinates more carefully.

Lemma 23. *At a fixed point P, We can modify the first $(n-1)$ coordinates $\{z_i\}_{i=1}^{n-1}$ and leave z_n unchanged such that $\partial_k g'_{i\bar{q}}(P) = 0$, for $1 \leq q \leq n-1$. Also under these coordinate, the condition for a is preserved: $a(p) = 1$, $\partial_i a(p) = 0$, $\partial_i\partial_j a(p) = 0$.*

Proof. Denote $\tilde{z}_i = z_i - z_i(P)$ for $1 \leq i \leq n$, we can write ω' as

$$\omega' = (c_{i\bar{j}} + a_{i\bar{j}k}\tilde{z}_k + \overline{a_{j\bar{i}k}}\overline{\tilde{z}_k})d\tilde{z}_i \wedge d\overline{\tilde{z}_j} + O(2) \tag{3.17}$$

We want to do coordinate change:

$$\tilde{z}_i = w_i + \frac{1}{2}b_{ilm}w_m w_l \tag{3.18}$$

with $b_{ilm} = b_{iml}$, and $b_{nlm} = 0$ since we want $\tilde{z}_n$ to stay unchanged.

When we substitute (3.18) into (3.17), we get

$$\omega' = \left(c_{i\bar{j}} + (a_{i\bar{j}k} + c_{r\bar{j}}b_{rki})w_k + (\overline{a_{j\bar{i}k}} + c_{i\bar{s}}\overline{b_{skj}})\bar{w}_k\right) dw_i \wedge d\bar{w}_j + O(2)$$

For any fixed $1 \leq i, k \leq n$, we want to solve (n-1) unknown $\{b_{rki}\}_{r=1}^{n-1}$ from (n-1) equations:

$$a_{i\bar{j}} + \sum_{r=1}^{n-1} c_{r\bar{j}}b_{rki} = 0, 1 \leq j \leq n-1$$

Note that the $1 \leq r \leq n-1$ since $b_{nki} = 0$.

If we denote the truncated square matrix $(\tilde{c}_{r\bar{j}} = c_{r\bar{j}})_{r,j=1}^{n-1}$, then $\tilde{c}_{r\bar{j}}$ is positive definite. Denote by $\tilde{c}^{r\bar{j}}$ its inverse matrix. Then we can choose

$$b_{rki} = -\sum_{r=1}^{n-1} \tilde{c}^{r\bar{j}} a_{i\bar{j}k} \text{ for } 1 \leq r, j \leq n-1, \quad \text{and} \quad b_{nki} = 0$$

to get

$$\frac{\partial}{\partial w_k} g' \left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j} \right)(P) = a_{i\bar{j}k} + c_{r\bar{j}}b_{rki} = 0 \text{ for } 1 \leq j \leq n-1$$

The last statement follows from chain rule. $\square$

Using this lemma and $g^{n\bar{n}}$ in (3.12), we see the last singular term in (3.16) becomes

$$\mathbb{E} = \frac{a + b|z_n|^{2-2\beta}}{b|z_n|^{2-2\beta}} g^{n\bar{n}} (\partial_k g'_{i\bar{n}})(\partial_{\bar{l}} g'_{n\bar{j}}) \xi_i \bar{\xi}_j \eta_k \bar{\eta}_l = \beta^{-2} \frac{a}{b} O(1) = O(1)$$

$\square$

3.3 Log-Futaki invariant and log-K-stability

3.3.1 Log-Futaki invariant

In this section, we recall Donaldson's definition of log-Futaki invariant (3.22). Let (X, L) be a polarized projective variety and D be a normal crossing divisor:

$$D = \sum_{i=1}^{r} \alpha_i D_i$$

with $\alpha_i \in (0, 1)$.

From now on, we fix a Hermitian metric $|\cdot|_i = h_i$ and defining section s_i of the line bundle

$[D_i]$. Assume $\omega \in c_1(L)$ is a smooth Kähler form. We define

$$\overline{\mathcal{P}}(\omega) = \left\{\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi;\ \phi \in L^\infty(X) \cap C^\infty(X\backslash D) \text{ such that } \omega + \sqrt{-1}\partial\bar{\partial}\phi \geq 0\right\}$$

Around any point $p \in X$, we can find local coordinate $\{z_i; i = 1, \cdots, n\}$, such that D is defined by

$$D = \cup_{i=1}^{r_p} \alpha_i \{z_i = 0\}$$

where $r_p = \sharp\{i; p \in D_i\}$.

Definition 13. *We say that $\widehat{\omega} \in \overline{\mathcal{P}}(\omega)$ is a conic Kähler metric on (X, D), if around p, ω is quasi-isometric to the metric*

$$\sum_{i=1}^{r_p} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2\alpha_i}} + \sum_{j=r_p+1}^{n} dz_j \wedge d\bar{z}_j$$

We will simply say that $\widehat{\omega}$ is a conic metric if it's clear what D is.

Geometrically, this means the Riemannian metric determined by ω has conic singularity along each D_i of conic angle $2\pi(1 - \alpha_i)$.

Remark 24. *Construction of Kähler-Einstein metrics with conic singularites was proposed long time ago by Tian, see [Tia8] in which he used such metrics to prove inequalities of Chern numbers in algebraic geometry.*

One consequence of this definition is that globally the volume form has the form

$$\widehat{\omega}^n = \frac{\Omega}{\prod_{i=1}^{r} |s_i|_i^{2\alpha_i}}$$

where Ω is a smooth volume form. For any volume form Ω, let $Ric(\Omega)$ denote the curvature of the Hermitian metric on K_X^{-1} determined by Ω. Then, by abuse of notation,

$$\begin{aligned} Ric(\widehat{\omega}) &= Ric(\widehat{\omega}^n) = Ric(\Omega) + \sqrt{-1}\sum_{i=1}^{r} \alpha_i \partial\bar{\partial} \log |s_i|_{h_i}^2 = Ric(\Omega) - \sum_{i=1}^{r} \alpha_i c_1([D_i], h_i) + \sum_{i=1}^{r} \alpha_i \{D_i\} \\ &= Ric(\Omega) - c_1([D], h) + \{D\} \end{aligned} \tag{3.19}$$

where $h = \otimes_{i=1}^{r} h_i^{\alpha_i}$ and $s = \otimes_{i=1}^{r} s_i^{\alpha_i}$ are Hermitian metric and defining section of the $\mathbb{R}$-line bundle $[D] = \otimes_{i=1}^{r} [D_i]^{\alpha_i}$.

Here we used the Poincáre-Lelong identity:

$$\sqrt{-1}\partial\bar{\partial}\log|s_i|^2_{h_i} = -c_1([D_i], h_i) + \{D_i\}$$

where $\{D_i\}$ is the current of integration along the divisor D_i.

The scalar curvature of $\widehat{\omega}$ on its smooth locus $X \backslash D$ is

$$S(\widehat{\omega}) = \widehat{g}^{i\bar{j}}\widehat{R}_{i\bar{j}} = \frac{nRic(\widehat{\omega}) \wedge \widehat{\omega}^{n-1}}{\widehat{\omega}^n} = \frac{n(Ric(\Omega) - c_1([D], h)) \wedge \widehat{\omega}^{n-1}}{\widehat{\omega}^n}$$

So if $S(\widehat{\omega})$ is constant, then the constant only depends on cohomological classes by the identity:

$$n\mu_1 := \frac{n(c_1(X) - c_1([D])) \wedge [c_1(L)]^{n-1}}{c_1(L)^n} = \frac{-n(K_X + D) \cdot L^{n-1}}{L^n} = n\mu - \frac{Vol(D)}{Vol(X)} \tag{3.20}$$

Here

$$n\mu = \frac{n\, c_1(X) \cdot c_1(L)^{n-1}}{c_1(L)^n} = \frac{-nK_X \cdot L^{n-1}}{L^n}$$

is the average scalar curvature for smooth Kähler form in $c_1(L)$. And

$$Vol(D) = \int_D \frac{c_1(L)^{n-1}}{(n-1)!} = \frac{D \cdot L^{n-1}}{(n-1)!}, \quad Vol(X) = \int_X \frac{c_1(L)^n}{n!} = \frac{L^n}{n!}$$

Now assume $\mathbb{C}^*$ acts on (X, L) and v is the generating holomorphic vector field. Recall that the ordinary Futaki-Calabi invariant ([Fut], [Cal3]) is defined by

$$F(c_1(L))(v) = -\int_X \theta_v (S(\omega) - n\mu)\frac{\omega^n}{n!}$$

where θ_v satisfies

$$\iota_v \omega = \bar{\partial}\theta_v$$

Now assume $\widehat{\omega}_\infty \in \overline{\mathcal{P}}(\omega)$ is a conic metric and satisfies

$$S(\widehat{\omega}_\infty) = n\mu_1 \tag{3.21}$$

Assume D is preserved by the $\mathbb{C}^*$ action. Let's calculate the ordinary Futaki invariant using the conic metric $\widehat{\omega}_\infty$. Let $\widehat{\theta}_v = \widehat{\theta}(\widehat{\omega}_\infty, v)$. Then near $p \in D$, $v \sim \sum_{i=1}^{r_p} c_i z_i \partial_{z_i} + \tilde{v}$ with $\tilde{v} = o(z_1 \cdots z_{r_p})$ holomorphic. $\widehat{\theta}_v \sim \sum_{i=1}^{r_p} |z_i|^{2(1-\alpha_i)}$.

We then make use of the distributional identity (3.19) to get

$$\begin{aligned}
F(c_1(L))(v) &= -\int_X \widehat{\theta}_v(nRic(\widehat{\omega}_\infty) - n\mu\widehat{\omega}_\infty) \wedge \frac{\widehat{\omega}_\infty^{n-1}}{n!} \\
&= -\int_X \widehat{\theta}_v \left[(nRic(\Omega) - nc_1([D],h) - n\mu_1\widehat{\omega}_\infty) + n\{D\} - (n\mu - n\mu_1)\widehat{\omega}_\infty\right] \wedge \frac{\widehat{\omega}_\infty^{n-1}}{n!} \\
&= -\int_X \widehat{\theta}_v(S(\widehat{\omega}_\infty) - n\mu_1)\frac{\widehat{\omega}_\infty^n}{n!} - \int_X \{D\}\widehat{\theta}_v \frac{\widehat{\omega}_\infty^{n-1}}{(n-1)!} + (n\mu - n\mu_1)\int_X \widehat{\theta}_v \frac{\widehat{\omega}_\infty^n}{n!} \\
&= -\left(\int_D \widehat{\theta}_v \frac{\widehat{\omega}_\infty^{n-1}}{(n-1)!} - \frac{Vol(D)}{Vol(X)}\int_X \widehat{\theta}_v \frac{\widehat{\omega}_\infty^n}{n!}\right)
\end{aligned}$$

So we get

$$0 = F(c_1(L))(v) + \left(\int_D \widehat{\theta}_v \frac{\widehat{\omega}_\infty^{n-1}}{(n-1)!} - \frac{Vol(D)}{Vol(X)}\int_X \widehat{\theta}_v \frac{\widehat{\omega}_\infty^n}{n!}\right)$$

Since the two integrals in the above formula is integration of (singular) equivariant forms, they are independent of the chosen Kähler metric in $\overline{\mathcal{P}}(\omega)$ with at worst conic singularities. In particular, we can choose the smooth Kähler metric ω, then we just discover the log-Futaki invariant defined by Donaldson:

Definition 14. *[Don6]*

$$F(c_1(L), D)(v) = F(c_1(L))(v) + \left(\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{Vol(D)}{Vol(X)}\int_X \theta_v \frac{\omega^n}{n!}\right) \tag{3.22}$$

Remark 25. *This differs from the formula in [Don6] by a sign. And we think of D as a cycle with real coefficients, so if we replace D by $(1-\beta)\triangle$, we have the same formua as that in [Don6].*

3.3.2 Integrating log-Futaki-invariant

We can integrate the log-Futaki-invariant to get log-K-energy

$$\begin{aligned}
\nu_{\omega,D}(\phi) &= -\int_0^1 dt \int_X (S(\omega_t) - \underline{S})\dot{\phi}\frac{\omega_t^n}{n!} + \int_0^1 dt \int_D \dot{\phi}\frac{\omega_t^{n-1}}{(n-1)!} - \frac{Vol(D)}{Vol(X)}\int_0^1 dt \int_X \dot{\phi}\frac{\omega_t^n}{n!} \\
&= \nu_\omega(\phi) + \int_0^1 \int_X \left(i\partial\bar{\partial}\log|s_D|^2 + 2\pi c_1([D],h)\right)\dot{\phi}\frac{\omega_t^{n-1}}{(n-1)!} + \frac{Vol(D)}{Vol(X)}F_\omega^0(\phi) \\
&= \nu_\omega(\phi) + \frac{Vol(D)}{Vol(X)}F_\omega^0(\phi) + \mathcal{J}_\omega^{\chi_D}(\phi) + \int_X \log|s_D|^2(\omega_\phi^n - \omega^n)/n!
\end{aligned} \tag{3.23}$$

where $\chi_D = c_1([D],h)$ is the Chern curvature form. The $\mathcal{J}_\omega^\chi(\phi)$ is defined by:

$$\mathcal{J}_\omega^\chi(\phi) = \int_0^1 dt \int_X \dot{\phi}\chi \wedge \frac{\omega_{\phi_t}^{n-1}}{(n-1)!}$$

Let's now focus on the Fano case as in the beginning of this paper. $((*)_\beta)$ is equivalent to the following singular complex Monge-Ampère equation:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-\beta\phi}\frac{\Omega_1}{|s|^{2(1-\beta)}} \tag{3.24}$$

with $\Omega_1 = e^{h_\omega}\omega^n$ and s is a defining section of $[Y]$. Note that the line bundle $[Y] = K_X^{-1}$ has the Hermitian metric $\|\cdot\|$ such that the curvature is ω.

We have $D = (1-\beta)Y$. Since $[Y] = K_X^{-1}$, we can assume $\chi_D = (1-\beta)\omega$, $Vol((1-\beta)D) = n(1-\beta)Vol(X)$. Then (3.23) becomes

$$\begin{aligned}
\nu_{\omega,D}(\omega_\phi) &= \nu_\omega(\omega_\phi) + (1-\beta)\left(nF_\omega^0(\phi) + \mathcal{J}_\omega^\omega(\phi)\right) + (1-\beta)\int_X \log|s|^2\frac{\omega_\phi^n}{n!} \\
&= \nu_\omega(\omega_\phi) + (1-\beta)(I_\omega - J_\omega)(\omega_\phi) + (1-\beta)\int_X \log|s|^2\frac{\omega_\phi^n}{n!} \\
&= \int_X \log\frac{\omega_\phi^n}{e^{h_\omega}\omega^n}\frac{\omega_\phi^n}{n!} - \beta(I_\omega - J_\omega)(\omega_\phi) + (1-\beta)\int_X \log|s|^2\frac{\omega_\phi^n}{n!} + \int_X h_\omega\omega^n/n! \\
&= \int_X \log\frac{\omega_\phi^n}{e^{h_\omega}\omega^n/|s|^{2(1-\beta)}}\frac{\omega_\phi^n}{n!} + \beta\left(\int_X \phi\omega_\phi^n/n! + F_\omega^0(\phi)\right) + \int_X h_\omega\omega^n/n! \\
&= \int_X \log\frac{\omega_\phi^n}{e^{h_\omega - \alpha\log|s|^2 - \beta\phi}\omega^n}\frac{\omega_\phi^n}{n!} + \beta F_\omega^0(\phi) + \int_X h_\omega\omega^n/n! \\
&= \int_X \log\frac{\omega_\phi^n}{e^{\widehat{h}_\omega - \beta\phi}\omega^n}\frac{\omega_\phi^n}{n!} + \beta F_\omega^0(\phi) + \int_X h_\omega\omega^n/n!
\end{aligned}$$

where $\widehat{h}_\omega = h_\omega - \alpha\log|s|^2$ satisfies

$$\sqrt{-1}\partial\bar{\partial}\widehat{h}_\omega = Ric(\omega) - \beta\omega - \alpha\{s=0\}, \text{ and } \int_X e^{\widehat{h}_\omega}\omega^n = V \tag{3.25}$$

3.3.3 log-K-stability

We imitate the definition of K-stability to define log-K-stability. First we recall the definition of test configuration [Don4] or special degeneration [Tia9] of a polarized projective variety (X, L).

Definition 15. *A test configuration of (X, L), consists of*

1. *a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-action;*
2. *a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L} \to \mathcal{X}$*
3. *a flat $\mathbb{C}^*$-equivariant map $\pi : \mathcal{X} \to \mathcal{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication in the standard way;*

such that any fibre $X_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to X and (X, L) is isomorphic to $(X_t, \mathcal{L}|_{X_t})$.

Any test configuration can be equivariantly embedded into $\mathbb{P}^N \times \mathbb{C}^*$ where the $\mathbb{C}^*$ action on $\mathbb{P}^N$ is given by a 1 parameter subgroup of $SL(N+1, \mathbb{C})$. If Y is any subvariety of X, the test configuration of (X, L) also induces a test configuration $(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}})$ of $(Y, L|_Y)$.

Let d_k, $\tilde{d}_k$ be the dimensions of $H^0(X, L^k)$, $H^0(Y, L|_Y^k)$, and w_k, $\tilde{w}_k$ be the weights of $\mathbb{C}^*$ action on $H^0(X_0, \mathcal{L}|_{X_0}^k)$, $H^0(Y_0, \mathcal{L}|_{Y_0}^k)$, respectively. Then we have expansions:

$$w_k = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}), \quad d_k = b_0 k^n + b_1 k^{n-1} + O(k^{n-2})$$

$$\tilde{w}_k = \tilde{a}_0 k^n + O(k^{n-1}), \quad \tilde{d}_k = \tilde{b}_0 k^{n-1} + O(k^{n-2})$$

If the central fibre X_0 is smooth, we can use equivariant differential forms to calculate the coefficients by [Don4]. Let ω be a smooth Kähler form in $c_1(L)$, and $\theta_v = \mathcal{L}_v - \nabla_v$, then

$$a_0 = -\int_X \theta_v \frac{\omega^n}{n!}; \ a_1 = -\frac{1}{2}\int_X \theta_v S(\omega)\frac{\omega^n}{n!} \tag{3.26}$$

$$b_0 = \int_X \frac{\omega^n}{n!} = Vol(X); \ b_1 = \frac{1}{2}\int_X S(\omega)\frac{\omega^n}{n!} \tag{3.27}$$

$$\tilde{a}_0 = -\int_{Y_0} \theta_v \frac{\omega^{n-1}}{(n-1)!}; \ \tilde{b}_0 = \int_{Y_0} \frac{\omega^{n-1}}{(n-1)!} = Vol(Y_0) \tag{3.28}$$

Remark 26. *To see the signs of coefficients and give an example, we consider the case where $X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(k)$. $\mathbb{C}^*$ acts on $\mathbb{P}^1$ by multiplication: $t \cdot z = tz$. A general $D \in |L|$ consists of k points. As $t \to 0$, $t \cdot D \to k\{0\}$. D is the zero set of a general degree k homogeneous polynomial $P_k(z_0, z_1)$ and $k\{0\}$ is the zero set of z_1^k. $\mathbb{C}^*$ acts on $H^0(\mathbb{P}^1, \mathcal{O}(k))$ by $t \cdot z_0^i z_1^j = t^{-j} z_0^i z_1^j$ so that $\lim_{t\to 0}[t \cdot P_k(z_0, z_1)] = [z_1^k]$, where $[P_k] \in \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(k)))$. Take the Fubini-Study metric $\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log(1+|z|^2) = \sqrt{-1}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$, then $\theta_v = \frac{\partial \log(1+|z|^2)}{\partial \log |z|^2} = \frac{|z|^2}{1+|z|^2}$. So*

$$-a_0 = \int_{\mathbb{P}^1} \theta_v \omega_{FS} = \int_0^{+\infty} \frac{r^2}{(1+r^2)^3} 2r dr = \frac{1}{2}$$

$$-a_1 = \frac{1}{2}\int_{\mathbb{P}^1} S(\omega_{FS})\theta_v \omega_{FS} = \int_{\mathbb{P}^1} \theta_v \omega_{FS} = \frac{1}{2}$$

While

$$w_k = -(1 + \cdots + k) = -\frac{1}{2}k^2 - \frac{1}{2}k$$

which gives exactly $a_0 = a_1 = -\frac{1}{2}$.

Comparing (3.22), (3.26)-(3.28), we can define the algebraic log-Futaki invariant of the given test configuration to be

$$\begin{aligned} F(\mathcal{X}, \mathcal{Y}, \mathcal{L}) &= \frac{2(a_1 b_0 - a_0 b_1)}{b_0} + (-\tilde{a}_0 + \frac{\tilde{b}_0}{b_0} a_0) \\ &= \frac{(2a_1 - \tilde{a}_0) b_0 - a_0(2b_1 - \tilde{b}_0)}{b_0} \end{aligned} \tag{3.29}$$

Definition 16. *(X, Y, L) is log-K-stable along the test configuration $(\mathcal{X}, \mathcal{L})$ if $F(\mathcal{X}, \mathcal{Y}, \mathcal{L}) \leq 0$, and equality holds if and only if $(\mathcal{X}, \mathcal{Y}, \mathcal{L})$ is a product configuration.*

(X, Y, L) is semi-log-K-stable along $(\mathcal{X}, \mathcal{L})$ if $F(\mathcal{X}, \mathcal{Y}, \mathcal{L}) \leq 0$. Otherwise, it's unstable.

(X, Y, L) is log-K-stable (semi-log-K-stable) if, for any integer $r > 0$, (X, Y, L^r) is log-K-stable (semi-log-K-stable) along any test configuration of (X, Y, L^r).

Remark 27. *When Y is empty, then definition of log-K-stability becomes the definition of K-stability. ([Tia9], [Don4])*

Remark 28. *In applications, we sometimes meet the following situation. Let $\lambda(t) : \mathbb{C}^* \to SL(N+1, \mathbb{C})$ be a 1 parameter subgroup. As $t \to \infty$, $\lambda(t)$ will move $X, Y \subset \mathbb{P}^N$ to the limit scheme X_0, Y_0. Then stability condition is equivalent to the other opposite sign condition $F(X_0, Y_0, v) \geq 0$. This is of course related to the above definition by transformation $t \to t^{-1}$.*

Example 1 (Orbifold)**.** *Assume X is smooth. $Y = \sum_{i=1}^{r}(1 - \frac{1}{n_i})D_i$ is a normal crossing divisor, where $n_i > 0$ are integers. The conic Kähler metric on (X, Y) is just the orbifold Kähler metric on the orbifold (X, Y). Orbifold behaves similarly as smooth variety, but in the calculation, we need to use orbifold canonical bundle $K_{orb} = K_X + Y$. For example, think L as an orbifold line bundle on X, then the orbifold Riemann-Roch says that*

$$\begin{aligned} dim H^0_{orb}((X, Y), L) &= \frac{L^n}{n!} k^n + \frac{1}{2} \frac{-(K_X + Y) \cdot L^n}{(n-1)!} k^{n-1} + O(k^{n-2}) \\ &= b_0 k^n + \frac{1}{2}(2b_1 - \tilde{b}_0) k^{n-1} + O(k^{n-2}) \end{aligned}$$

For the $\mathbb{C}^*$-weight of $H^0_{orb}((X, Y), L)$, we have expansion:

$$w_k^{orb} = a_0^{orb} k^{n+1} + a_1^{orb} k^n + O(k^{n-1})$$

By orbifold equivariant Riemann-Roch, we have the formula:

$$a_0^{orb} = \int_X \widehat{\theta}_v \frac{\widehat{\omega}^n}{n!} = \int_X \theta_v \frac{\omega^n}{n!} = a_0$$

$$a_1^{orb} = \int_X \widehat{\theta}_v S(\widehat{\omega}) \frac{\widehat{\omega}^n}{n!}$$

To calculate the second coefficient a_1^{orb}, we choose an orbifold metric $\widehat{\omega}$, then by (3.26)*:*

$$\begin{aligned} a_1 &= -\frac{1}{2}\int_X \widehat{\theta}_v n \; Ric(\widehat{\omega}) \wedge \frac{\widehat{\omega}^{n-1}}{n!} \\ &= -\frac{1}{2}\int_X \widehat{\theta}_v n(Ric(\Omega) - c_1([D], h) + \{D\}) \wedge \frac{\widehat{\omega}^{n-1}}{n!} \\ &= -\frac{1}{2}\int_X \widehat{\theta}_v S(\widehat{\omega})\frac{\widehat{\omega}^n}{n!} - \frac{1}{2}\int_D \widehat{\theta}_v \frac{\widehat{\omega}^{n-1}}{(n-1)!} \\ &= a_1^{orb} - \frac{1}{2}\int_D \theta_v \frac{\omega^{n-1}}{(n-1!} = a_1^{orb} + \frac{1}{2}\tilde{a}_0 \end{aligned}$$

So

$$a_1^{orb} = \frac{1}{2}(2a_1 - \tilde{a}_0) \tag{3.30}$$

Comparing (3.29)*, we see that the log-Futaki invariant recovers the orbifold Futaki invariant, and similarly log-K-stability recovers orbifold K-stability. Orbifold Futaki and orbifold K-stability were studied by Ross-Thomas [RoTh2].*

Example 2. $X = \mathbb{P}^1$, $L = K_{\mathbb{P}^1}^{-1} = \mathcal{O}_{\mathbb{P}^1}(2)$, $Y = \sum_{i=1}^r \alpha_i p_i$. *For any $i \in \{1, \cdots, r\}$, we choose the coordinate z on $\mathbb{P}^1$, such that $z(p_i) = 0$. Then consider the holomorphic vector field $v = z\partial_z$. v generates the 1 parameter subgroup $\lambda(t) : \lambda(t) \cdot z = t \cdot z$. As $t \to \infty$, $\lambda(t)$ degenerate (X, Y) into the pair $(\mathbb{P}^1, \alpha_i\{0\} + \sum_{j\neq i} \alpha_j\{\infty\})$. We take $\theta_v = \frac{-|z|^{-2}+|z|^2}{|z|^{-2}+1+|z|^2}$. Then it's easy to get the log-Futaki invariant of the degeneration determined by λ:*

$$F(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i, \mathcal{O}_{\mathbb{P}^1}(2))(\lambda) = \sum_{j\neq i} \alpha_j - \alpha_i$$

If $(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i)$ is log-K-stable, by Remark 28, we have

$$\sum_{j\neq i} \alpha_j - \alpha_i > 0 \tag{3.31}$$

Equivalently, if we let $t \to 0$, we get $\alpha_i - \sum_{j\neq i} \alpha_j < 0$ from log-K-stability.

Let's consider the problem of constructing singular Riemannian metric g of constant scalar *curvature on $\mathbb{P}^1$ which has conic angle $2\pi(1-\alpha_i)$ at p_i and is smooth elsewhere. Assume $p_i \neq \infty$ for any $i = 1, \ldots, r$. Under conformal coordinate z of $\mathbb{C} \subset \mathbb{P}^1$, $g = e^{2u}|dz|^2$. u is a smooth function in the punctured complex plane $\mathbb{C} - \{p_1, \ldots, p_r\}$ so that near each p_i, $u(z) = -2\alpha_i \log|z - p_i| +$ a continuous function, where $\alpha_i \in (0,1)$ and $u = -2\log|z| +$ a continuous function near infinity. We call such function is of conic type. The condition of constant scalar curvature corresponds to the following Liouville equations.*

1. $\Delta u = -e^{2u}$
2. $\Delta u = 0$
3. $\Delta u = e^{2u}$

which correspond to scalar curvature=1, 0, -1 case respectively.

For such equations, we have the following nice theorem due to Troyanov, McOwen, Thurston, Luo-Tian.

Theorem 18 (See [LuTi] and the reference there)**.** *1. For equation 1, it has a solution of conic type if and only if*

(a) $\sum_{i=1}^r \alpha_i < 2$, and

(b) $\sum_{j\neq i} \alpha_j - \alpha_i > 0$, for all $i = 1, \ldots, n$.

2. For equation 2, it has a solution of conic type if and only if (a): $\sum_{i=1}^r \alpha_i = 2$.

In this case, (a) implies the condition: (b) $\sum_{j\neq i} \alpha_j - \alpha_i > 0$, for all $i = 1, \ldots, r$.

3. For equation 3, it has a solution of conic type if and only if (a): $\sum_{i=1}^r \alpha_i > 2$.

Again in this case, (a) implies the condition: (b) $\sum_{j\neq i} \alpha_j - \alpha_i > 0$, for all $i = 1, \ldots, r$.

Moreover, the above solutions are all unique.

Note that $\deg(-(K_{\mathbb{P}^1} + \sum_{i=1}^r \alpha_i p_i)) = 2 - \sum_{i=1}^r \alpha_i$, *so by* (3.20), *conditions (a) in above theorem correspond to the cohomological conditions for the scalar curvature to be positive, zero, negative respectively. While the condition (b) is the same as* (3.31). *So by the above theorem, if $(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i)$ is log-K-stable, then there is a conic metric on $(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i)$ with constant curvature whose sign is the same as that of $2 - \sum_i \alpha_i$.*

This example clearly suggests

Conjecture 3 (Logarithmic version of Tian-Yau-Donaldson conjecture)**.** *There is a constant scalar curvature conic Kähler metric on (X, Y) if and only if (X, Y) is log-K-stable.*

Chapter 4

Continuity method in Kähler-Einstein problem on toric Fano manifold

4.1 Introduction to results

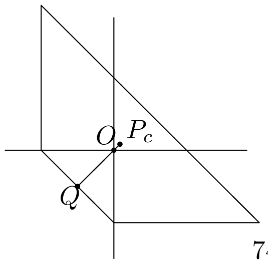
Recall that the continuity method on Fano manifold in $(*)_t$ is defined as the following family of equations parameterized by t

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n = e^{h_\omega - t\phi_t}\omega^n$$

where h_ω is defined by

$$Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \quad \int_X e^{h_\omega}\omega^n/n! = V \tag{4.1}$$

We get much information about the continuity method for toric Fano manifolds. A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$. For example, $Bl_p\mathbb{P}^2$ i.e. $\mathbb{P}^2$ blown up one point is represented by



Such a polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_c O}$ intersects the boundary $\partial\triangle$ at point Q.

Theorem 19. *([Li1]) If $P_c \neq O$,*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|}$$

Here $|\overline{OQ}|$, $|\overline{P_c Q}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_c Q}$. In other words,

$$Q = -\frac{R(X_\triangle)}{1 - R(X_\triangle)} P_c \in \partial\triangle$$

If $P_c = O$, then there is Kähler-Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

Remark 29. *The last statement was already proved by Wang-Zhu [WaZh].*

Remark 30. *Székelyhidi [Szé] proved that $R(Bl_p\mathbb{P}^2) = \frac{6}{7}$ and $R(Bl_{p,q}\mathbb{P}^2) \leq \frac{21}{25}$. My result gives the sharp value for $R(X_\triangle)$ of any toric Fano manifold.*

The next natural problem is how the limit metric looks like as $t \to R(X)$. For the special example $X = Bl_p\mathbb{P}^2$, which is also the projective compactification of the total space of line bundle $\mathcal{O}(-1) \to \mathbb{P}^2$. Székelyhidi [Szé] constructed a sequence of Kähler metric ω_t, with $Ric(\omega_t) \geq t\omega_t$ and ω_t converge to a metric with conic singularty along the divisor D_∞ of conic angle $2\pi \times 5/7$, where D_∞ is divisor at infinity added in projective compactification. Shi-Zhu [ShZh] proved that rotationally symmetric solutions to the continuity equations $(*)_t$ converge to a metric with conic singularity of conic angle $2\pi \times 5/7$ in Gromov-Hausdorff sense, which seems to be the first strict result on the limit behavior of solutions to $(*)_t$. Note that by the theory of Cheeger-Colding-Tian [CCT], the limit metric in Gromov-Hausdorff sense should have complex codimension 1 conic type singularities since we only have the positive lower Ricci bound.

For the more general toric case, if we use a special toric metric, which is just the Fubini-Study metric in the projective embedding given by the vertices of the polytope, then, after transforming by some biholomorphic automorphism, we prove that there is a sequence of Kähler metrics which solve the equation $(*)_t$, and converge to a limit metric satisfying a singular complex Monge-Ampère equation (Also see equivalent real version in Theorem 21). This generalizes the result of [ShZh] for the special reference Fubini-Study metric. Precisely, let $\{p_\alpha; \alpha = 1, \ldots, N\}$ be all the vertex lattice points of $\triangle$ and $\{s_\alpha; \alpha = 1, \ldots, N\}$ be the corresponding holomorphic sections of

$K_{X_\triangle}^{-1}$. Then we take the reference metric to be

$$\omega = \omega_{FS} := \sqrt{-1}\partial\bar{\partial}\log\sum_{\alpha=1}^{N}|s_\alpha|^2,$$

which is the pull-back of the Fubini-Study metric of $\mathbb{CP}^{N-1}$ under Kodaira embedding induced by $\{s_\alpha\}$. Now using the above notation, let $\mathcal{F}$ be the minimal face of $\triangle$ containing Q. Let $\{p_k^{\mathcal{F}}\}$ be the vertex lattice points of $\mathcal{F}$, then they correspond to a sub-linear system $\mathfrak{L}_{\mathcal{F}}$ of $|-K_{X_\triangle}|$. We let $Bs(\mathfrak{L}_{\mathcal{F}})$ denote the base locus of this sub-linear system. Also let $\sum_\alpha{}'$ denote the sum $\sum_{p_k^{\mathcal{F}}}$, then we have

Theorem 20. *([Li4]) After some biholomorphic transformation $\sigma_t : X_\triangle \to X_\triangle$, there is a subsequence $t_i \to R(X)$, such that $\sigma_{t_i}^*\omega_{t_i}$ converge to a Kähler current $\omega_\infty = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty$, with $\psi_\infty \in L^\infty(X_\triangle)\cap C^\infty(X_\triangle\backslash Bs(\mathfrak{L}_{\mathcal{F}}))$, which satisfies a complex Monge-Ampère equation of the form*

$$(\omega+\sqrt{-1}\partial\bar{\partial}\psi_\infty)^n = e^{-R(X)\psi_\infty}\left(\sum_\alpha{}'b_\alpha|s_\alpha|^2\right)^{-(1-R(X))}\Omega. \tag{4.2}$$

Here $\Omega = e^{h_\omega}\omega^n$ is a smooth volume form. For each vertex lattice point $p_\alpha^{\mathcal{F}}$ of $\mathcal{F}$, b_α is a constant satisfying $0 < b_\alpha \le 1$. $\|\cdot\| = \|\cdot\|_{FS}$ is (up to multiplication of a constant) the Fubini-Study metric on $K_{X_\triangle}^{-1}$. In particular

$$Ric(\omega_{\psi_\infty}) = R(X)\omega_{\psi_\infty} + (1-R(X))\sqrt{-1}\partial\bar{\partial}\log(\sum_\alpha{}'b_\alpha|s_\alpha|^2). \tag{4.3}$$

The above equation shows the conic type singularities for the limit metric. We can read out the place of conic singularities and conic angles from the geometry of the polytope. See Section 4.4.3 for the method and discussions. In particular, this can give a toric explanation of the special case $Bl_p\mathbb{P}^2$ just mentioned (See example 3).

Note that, although we can prove the limit metric is smooth outside the singular locus, to prove geometrically it's a conic metric along codimension one strata of singular set, we need to prove more delicate estimate that we wish to discuss in future. There are also difficulties for studying the behavior of limit metric around higher codimensional strata (See Remark 34 and Example 2).

Finally, we remark that, in view of the special case $Bl_p\mathbb{P}^2$ in [ShZh] and results in [LaSo], we expect the following statement is true: the Gromov-Hausdorff limit of $(X_\triangle, \omega_{t_i})$ is the metric completion of $(X_\triangle\backslash Bs(\mathfrak{L}_{\mathcal{F}}), \omega_\infty)$.

4.2 Consequence of Wang-Zhu's theory

For a reflexive lattice polytope $\triangle$ in $\mathbb{R}^n = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$, we have a Fano toric manifold $X_\triangle \supset (\mathbb{C}^*)^n$ with a $(\mathbb{C}^*)^n$ action. In the following, we will sometimes just write X for $X_\triangle$ for simplicity.

Let $(S^1)^n \subset (\mathbb{C}^*)^n$ be the standard real maximal torus. Let $\{z_i\}$ be the standard coordinates of the dense orbit $(\mathbb{C}^*)^n$, and $x_i = \log |z_i|^2$. We have a standard lemma about toric Kähler metric, which we omit the proof. See for example [WaZh].

Lemma 24. *Any $(S^1)^n$ invariant Kähler metric ω on X has a potential $u = u(x)$ on $(\mathbb{C}^*)^n$, i.e. $\omega = \sqrt{-1}\partial\bar{\partial}u$. u is a proper convex function on $\mathbb{R}^n$, and satisfies the momentum map condition:*

$$Du(\mathbb{R}^n) = \triangle.$$

Also,

$$\frac{(\sqrt{-1}\partial\bar{\partial}u)^n/n!}{\frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \cdots \wedge \frac{dz_n}{z_n} \wedge \frac{d\bar{z}_n}{\bar{z}_n}} = \det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) =: \det(u_{ij}). \tag{4.4}$$

Let $\{p_\alpha;\ \alpha = 1, \cdots, N\}$ be all the **vertex lattice points** of $\triangle$. Each p_α corresponds to a holomorphic section $s_\alpha \in H^0(X_\triangle, K^{-1}_{X_\triangle})$. We can embed $X_\triangle$ into $\mathbb{P}^N$ using $\{s_\alpha\}$. Let us first find the appropriate potential on $(\mathbb{C}^*)^n$ for the pull back of Fubini-Study metric. (Cf. [WaZh])

Recall that, for any section s of K_X^{-1}, the Fubini-Study metric as a Hermitian metric on K_X^{-1} is defined up to the multiplication by a positive constant:

$$|s|^2_{FS} = e^{-\tilde{C}} \frac{|s|^2}{\sum_\beta |s_\beta|^2}. \tag{4.5}$$

The righthand side is well defined by using local trivializations. $\tilde{C}$ is some normalizing constant which we choose now to simplify the computation later.

First, let $\tilde{s}_0$ be the section corresponding to the origin $0 \in \triangle$. On the open dense orbit $(\mathbb{C}^*)^n$, by standard toric geometry, we can assume

$$\frac{s_\alpha}{\tilde{s}_0} = \prod_{i=1}^n z_i^{p_{\alpha,i}}. \tag{4.6}$$

So the Fubini-Study norm of $\tilde{s}_0$ is

$$\|\tilde{s}_0\|^2_{FS} = e^{-\tilde{C}} \frac{|\tilde{s}_0|^2}{\sum_{\alpha=1}^N |s_\alpha|^2} = e^{-\tilde{C}} \left(\sum_{\alpha=1}^N \prod_{i=1}^n |z_i|^{2p_{\alpha,i}}\right)^{-1} = e^{-\tilde{C}} \left(\sum_{\alpha=1}^N e^{<p_\alpha, x>}\right)^{-1} =: e^{-\tilde{u}_0}.$$

In other words, we define

$$\tilde{u}_0 = \log\left(\sum_{\alpha=1}^{N} e^{<p_\alpha, x>}\right) + \tilde{C}. \tag{4.7}$$

Now we can choose $\tilde{C}$ by the normalization condition:

$$\int_{\mathbb{R}^n} e^{-\tilde{u}_0} dx = Vol(\triangle) = \frac{c_1(X_\triangle)^n}{n!} = \frac{1}{(2\pi)^n}\int_{X_\triangle} \frac{\omega^n}{n!}. \tag{4.8}$$

On the other hand, $Ric(\omega)$ is the curvature form of Hermitian line bundle K_M^{-1} with Hermitian metric determined by the volume form ω^n. Note that we can take $\tilde{s}_0 = z_1\frac{\partial}{\partial z_1}\wedge\cdots\wedge z_n\frac{\partial}{\partial z_n}$. Since $\frac{\partial}{\partial \log z_i} = \frac{1}{2}(\frac{\partial}{\partial \log|z_i|} - \sqrt{-1}\frac{\partial}{\partial\theta_i}) = \frac{\partial}{\partial \log|z_i|^2} = \frac{\partial}{\partial x_i}$ when acting on any $(S^1)^n$ invariant function on $(\mathbb{C}^*)^n$, we have

$$\begin{aligned} \|\tilde{s}_0\|^2_{\omega^n} &= \left\| z_1\frac{\partial}{\partial z_1}\wedge\cdots\wedge z_n\frac{\partial}{\partial z_n}\right\|^2_{\omega^n} = \det\left(\frac{\partial^2\tilde{u}_0}{\partial \log z_i\, \overline{\partial \log z_j}}\right) \\ &= \det\left(\frac{\partial^2 \tilde{u}_0}{\partial \log|z_i|^2\, \partial\log|z_j|^2}\right) = \det(\tilde{u}_{0,ij}). \end{aligned}$$

It's easy to see from definition of h_ω in (4.1) and normalization condition (4.8) that

$$\frac{e^{h_\omega}\omega^n/n!}{\frac{dz_1}{z_1}\wedge\frac{d\bar{z}_1}{\bar{z}_1}\cdots\wedge\frac{dz_n}{z_n}\wedge\frac{d\bar{z}_n}{\bar{z}_n}} = e^{h_\omega}\|\tilde{s}_0\|^2_{\omega^n} = \|\tilde{s}_0\|^2_{FS} = e^{-\tilde{u}_0}. \tag{4.9}$$

Remark 31. *We only use vertex lattice points because, roughly speaking, later in Lemma 23, vertex lattice points alone helps us to determine which sections become degenerate when doing biholomorphic transformation and taking limit. See remark 33. We expect results similar to Theorem 20 hold for general toric reference Kähler metric.*

So divide both sides of $(*)_t$ by meromorphic volume form $n!(\frac{dz_1}{z_1}\wedge\frac{d\bar{z}_1}{\bar{z}_1}\cdots\wedge\frac{dz_n}{z_n}\wedge\frac{d\bar{z}_n}{\bar{z}_n})$, We can rewrite the equations $(*)_t$ as a family of real Monge-Ampère equations on $\mathbb{R}^n$:

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}_0 - tu} \tag*{$(**)_t$}$$

where u is the potential for $\omega + \sqrt{-1}\partial\bar{\partial}\phi$ on $(\mathbb{C}^*)^n$, and is related to ϕ in $(*)_t$ by

$$\phi = u - \tilde{u}_0.$$

Every strictly convex function f appearing in $(**)_t$ ($f = \tilde{u}_0$, u, $w_t = (1-t)\tilde{u}_0 + tu_t$) must satisfy $Df(\mathbb{R}^n) = \triangle^\circ$ ($\triangle^\circ$ means the interior of $\triangle$). Since 0 is (the unique lattice point) contained in

$\triangle^\circ = Df(\mathbb{R}^n)$, the strictly convex function f is properly. For simplicity, let

$$w_t(x) = tu(x) + (1-t)\tilde{u}_0.$$

Then w_t is also a proper convex function on $\mathbb{R}^n$ satisfying $Dw_t(\mathbb{R}^n) = \triangle$. So it has a unique absolute minimum at point $x_t \in \mathbb{R}^n$. Let

$$m_t = \inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t).$$

Wang-Zhu's [WaZh] method for solving $(**)_t$ consists of two steps. The **first step** is to show some uniform a priori estimates for w_t. For $t < R(X_\triangle)$, the proper convex function w_t obtains its minimum value at a unique point $x_t \in \mathbb{R}^n$. Let

$$m_t = inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t)$$

Proposition 20 ([WaZh],See also [Don3]). *[1.]*

1. *there exists a constant C, independent of $t < R(X_\triangle)$, such that*

$$|m_t| < C$$

2. *There exists $\kappa > 0$ and a constant C, both independent of $t < R(X_\triangle)$, such that*

$$w_t \geq \kappa|x - x_t| - C \tag{4.10}$$

For the reader's convenience, we record the proof here.

Proof. Let $A = \{x \in \mathbb{R}^n; m_t \leq w(x) \leq m_t+1\}$. A is a convex set. By a well known lemma due to Fritz John, there is a unique ellipsoid E of minimum volume among all the ellipsoids containing A, and a constant α_n depending only on dimension, such that

$$\alpha_n E \subset A \subset E$$

$\alpha_n E$ means the α_n-dilation of E with respect to its center. Let T be an affine transformation with $\det(T) = 1$, which leaves x'=the center of E invariant, such that $T(E) = B(x', R)$, where

$B(x', R)$ is the Euclidean ball of radius R. Then

$$B(x', \alpha_n R) \subset T(A) \subset B(x', R)$$

We first need to bound R in terms of m_t. Since $D^2 w = tD^2 u + (1-t)D^2 \tilde{u}_0 \geq tD^2 u$, by $((**)_t)$, we see that

$$\det(w_{ij}) \geq t^n e^{-w}$$

Restrict to the subset A, it's easy to get

$$\det(w_{ij}) \geq C_1 e^{-m_t}$$

Let $\tilde{w}(x) = w(T^{-1}x)$, since $\det(T) = 1$, $\tilde{w}$ satisfies the same inequality

$$\det(\tilde{w}_{ij}) \geq C_1 e^{-m_t}$$

in $T(A)$.

Construct an auxiliary function

$$v(x) = C_1^{\frac{1}{n}} e^{-\frac{m_t}{n}} \frac{1}{2} \left(|x - x'|^2 - (\alpha_n R)^2 \right) + m_t + 1$$

Then in $B(x', \alpha_n R)$,

$$\det(v_{ij}) = C_1 e^{-m_t} \leq \det(\tilde{w}_{ij})$$

On the boundary $\partial B(x', \alpha_n R)$, $v(x) = m_t + 1 \geq \tilde{w}$. By the Comparison Principle for Monge-Amère operator, we have

$$\tilde{w}(x) \leq v(x) \ \text{ in } \ B(x', \alpha_n R)$$

In particular

$$m_t \leq \tilde{w}(x') \leq v(x') = C_1^{\frac{1}{n}} e^{-\frac{m_t}{n}} \frac{1}{2} (-\frac{R^2}{n^2}) + m_t + 1$$

So we get the bound for R:

$$R \leq C_2 e^{\frac{m_t}{2n}}$$

So we get the upper bound for the volume of A:

$$Vol(A) = Vol(T(A)) \leq CR^n \leq Ce^{\frac{m_t}{2}}$$

By the convexity of w, it's easy to see that $\{x; w(x) \leq m_t + s\} \subset s \cdot \{x; w(x) \leq m_t + 1\} = s \cdot A$, where $s \cdot A$ is the s-dilation of A with respect to point x_t. So

$$Vol(\{x; w(x) \leq m_t + s\}) \leq s^n Vol(A) \leq C s^n e^{\frac{m_t}{2}} \tag{4.11}$$

The lower bound for volume of sublevel sets is easier to get. Indeed, since $|Dw(x)| \leq L$, where $L = \max_{y \in \triangle} |y|$, we have $B(x_t, s \cdot L^{-1}) \subset \{x; w(x) \leq m_t + s\}$. So

$$Vol(\{x; w(x) \leq m_t + s\}) \geq C s^n \tag{4.12}$$

Now we can derive the estimate for m_t. First note the identity:

$$\int_{\mathbb{R}^n} e^{-w} dx = \int_{\mathbb{R}^n} \det(u_{ij}) dx = \int_{\triangle} d\sigma = Vol(\triangle) \tag{4.13}$$

Second, we use the coarea formula

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-w} dx &= \int_{\mathbb{R}^n} \int_{w}^{+\infty} e^{-s} ds dx = \int_{-\infty}^{+\infty} e^{-s} ds \int_{\mathbb{R}^n} 1_{\{w \leq s\}} dx \\
&= \int_{m_t}^{+\infty} e^{-s} Vol(\{w \leq s\}) ds \\
&= e^{-m_t} \int_{0}^{+\infty} e^{-s} Vol(\{w \leq m_t + s\}) ds
\end{aligned} \tag{4.14}$$

Using the bound for the volume of sublevel sets (4.11) and (4.12) in (4.14), and compare with (4.13), it's easy to get the bound for $|m_t|$.

Now we prove the estimate (4.10) following the argument of [Don3]. We have seen $B(x_t, L^{-1}) \subset \{w \leq m_t + 1\}$, and $Vol(\{w \leq m_t + 1\}) \leq C$ by (4.11) and uniform bound for m_t. Then we must have $\{w \leq m_t + 1\} \subset B(x_t, R(C, L))$ for some uniformly bounded radius $R(C, L)$. Otherwise, the convex set $\{w \leq m_t + 1\}$ would contain a convex subset of arbitrarily large volume. By the convexity of w, we have $w(x) \geq \frac{1}{R(C,L)} |x - x_t| + m_t - 1$ Since m_t is uniformly bounded, the estimate (4.10) follows. □

The **second step** is trying to bound $|x_t|$. In Wang-Zhu's [WaZh] paper, they proved the existence of Kähler-Ricci soliton on toric Fano manifold by solving the real Monge-Ampère equation corresponding to Kähler-Ricci solition equation. But now we only consider the Kähler-Einstein equation, which in general can't be solved because there is the obstruction of Futaki invariant.

Proposition 21 ([WaZh])**.** *the uniform bound of $|x_t|$ for any $0 \leq t \leq t_0$, is equivalent to that we can solve $(**)_t$, or equivalently solve $(*)_t$, for t up to t_0. More precisely, (by the discussion in introduction,) this condition is equivalent to the uniform C^0-estimates for the solution ϕ_t in $(*)_t$ for $t \in [0, t_0]$.*

Again we sketch the proof here.

Proof. If we can solve $(**)_t$ (or equivalently $(*)_t$) for $0 \leq t \leq t_0$. Then $\{w(t) = (1-t)\tilde{u_0} + tu; 0 \leq t \leq t_0\}$ is a smooth family of proper convex functions on $\mathbb{R}^n$. By implicit function theorem, the minimal point x_t depends smoothly on t. So $\{x_t\}$ are uniformly bounded in a compact set.

Conversely, assume $|x_t|$ is bounded. First note that $\phi_t = u - \tilde{u}_0 = \frac{1}{t}(w_t(x) - \tilde{u}_0)$.

As in Wang-Zhu [WaZh], we consider the enveloping function:

$$v(x) = \max_{p_\alpha \in \Lambda \cap \triangle} \langle p_\alpha, x \rangle$$

Then $0 \leq \tilde{u}_0(x) - v(x) \leq C$, and $Dw(\xi) \cdot x \leq v(x)$ for all $\xi, x \in \mathbb{R}^n$. We can assume $t \geq \delta > 0$. Then using uniform boundedness of $|x_t|$

$$\begin{aligned} \phi_t(x) &= \frac{1}{t}(w_t(x) - \tilde{u}_0) = \frac{1}{t}[(w_t(x) - w_t(x_t)) - v(x) + (v(x) - \tilde{u}_0(x)) + w_t(x_t)] \\ &\leq \delta^{-1}(Dw_t(\xi) \cdot x - v(x) - Dw_t(\xi) \cdot x_t) + C \leq C' \end{aligned}$$

Thus we get the estimate for $\sup_t \phi_t$. Then one can get the bound for $\inf_t \phi_t$ using the Harnack inequality in the theory of Monge-Ampère equations. For details see ([WaZh], Lemma 3.5) (see also [Tia1]). $\square$

By the above proposition, we have

Lemma 25. *If $R(X_\triangle) < 1$, then there exists a subsequence $\{x_{t_i}\}$ of $\{x_t\}$, such that*

$$\lim_{t_i \to R(X_\triangle)} |x_{t_i}| = +\infty$$

The observation now is that

Lemma 26. *If $R(X_\triangle) < 1$, then there exists a subsequence of $\{x_{t_i}\}$ which we still denote by $\{x_{t_i}\}$, and $y_\infty \in \partial\triangle$, such that*

$$\lim_{t_i \to R(X_\triangle)} D\tilde{u}_0(x_{t_i}) = y_\infty \tag{4.15}$$

This follows easily from the properness of $\tilde{u}_0$ and compactness of $\triangle$.

We now use the key relation (See [WaZh] Lemma 3.3, and also [Don3] page 29)

$$0 = \int_{\mathbb{R}^n} Dw(x)e^{-w}dx = \int_{\mathbb{R}^n} ((1-t)D\tilde{u}_0 + tDu)e^{-w}dx$$

Since

$$\int_{\mathbb{R}^n} Du\, e^{-w}dx = \int_{\mathbb{R}^n} Du\det(u_{ij})dx = \int_{\triangle} y d\sigma = Vol(\triangle)P_c$$

where P_c is the barycenter of $\triangle$, so

$$\frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w}dx = -\frac{t}{1-t}P_c. \tag{4.16}$$

Remark 32. *This identity is a toric form of a general formula for solutions of equations $(*)_t$:*

$$-\frac{1}{V}\int_X div_\Omega(v)\omega_t^n = \frac{t}{1-t}F_{2\pi c_1(X)}(v).$$

Here $\Omega = e^{h_\omega}\omega^n$. v is any holomorphic vector field, and $div_\Omega(v) = \frac{\mathcal{L}_v\Omega}{\Omega}$ is the divergence of v with respect to Ω.

$$F_{2\pi c_1(X)}(v) = \frac{1}{V}\int_X v(h_\omega)\omega^n$$

is the Futaki invariant in class $2\pi c_1(X)$ [Fut].

We will show this vector tend to a point on $\partial\triangle$ when t goes to $R(X_\triangle)$. To prove this we use the defining function of $\triangle$. Similar argument was given in the survey [Don3], page 30.

4.3 Calculate $R(X)$ on any toric Fano manifold

We now assume the reflexive polytope $\triangle$ is defined by inequalities:

$$\lambda_r(y) \geq -1,\ r = 1, \cdots, K \tag{4.17}$$

$\lambda_r(y) = \langle v_r, y\rangle$ are fixed linear functions. We also identify the minimal face of $\triangle$ where y_∞ lies:

$$\lambda_r(y_\infty) = -1,\ r = 1, \cdots, K_0 \tag{4.18}$$

$$\lambda_r(y_\infty) > -1,\ r = K_0 + 1, \cdots, K$$

Clearly, Theorem 19 follows from

Proposition 22. *If $P_c \neq O$,*

$$-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c \in \partial\triangle$$

Precisely,

$$\lambda_r\left(-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c\right) \geq -1 \tag{4.19}$$

Equality holds if and only if $r = 1, \cdots, K_0$. So $-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c$ and y_∞ lie on the same faces (4.18).

Proof. By (4.16) and defining function of $\triangle$, we have

$$\lambda_r\left(-\frac{t}{1-t}P_c\right)+1 = \frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w}dx+1 = \frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n}(\lambda_r(D\tilde{u}_0)+1)e^{-w}dx \tag{4.20}$$

The inequality (4.19) follows from (4.20) by letting $t \to R(X_\triangle)$. To prove the second statement, by (4.20) we need to show

$$\lim_{t_i\to R(X_\triangle)}\frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w_{t_i}}dx+1\begin{cases} =0 & : \quad r=1,\cdots,K_0 \\ >0 & : \quad r=K_0+1,\cdots,N\end{cases} \tag{4.21}$$

By the uniform estimate (4.10) and fixed volume (4.13), and since $D\tilde{u}_0(\mathbb{R}^n) = \triangle^\circ$ is a bounded set, there exists R_ϵ, independent of $t \in [0, R(X_\triangle))$, such that

$$\frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_t)}\lambda_r(D\tilde{u}_0)e^{-w_t}dx < \epsilon, \text{ and } \frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_t)}e^{-w_t}dx < \epsilon \tag{4.22}$$

Now (4.21) follows from the following claim.

Claim 1. *Let $R > 0$, there exists a constant $C > 0$, which only depends on the polytope $\triangle$, such that for all $\delta x \in B_R(0) \subset \mathbb{R}^n$,*

$$e^{-CR}(\lambda_r(D\tilde{u}_0(x_{t_i}))+1) \leq \lambda_r(D\tilde{u}_0(x_{t_i}+\delta x))+1 \leq e^{CR}(\lambda_r(D\tilde{u}_0(x_{t_i}))+1) \tag{4.23}$$

Assuming the claim, we can prove two cases of (4.21). First by (4.15) and (4.18), we have

$$\lim_{t_i\to R(X_\triangle)}\lambda_r(D\tilde{u}_0(x_{t_i}))+1 = \lambda_r(y_\infty)+1 = \begin{cases} 0 & : \quad r=1,\cdots,K_0 \\ a_r>0 & : \quad r=K_0+1,\cdots,N\end{cases} \tag{4.24}$$

1. $r = 1, \cdots, K_0$. $\forall\epsilon > 0$, first choose R_ϵ as in (4.22). By (4.23) and (4.24), there exists

$\rho_\epsilon > 0$, such that if $|t_i - R(X_\triangle)| < \rho_\epsilon$, then for all $\delta x \in B_{R_\epsilon}(0) \subset \mathbb{R}^n$,

$$0 \leq \lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 < e^{CR_\epsilon}(\lambda_r(D\tilde{u}_0)(x_{t_i}) + 1) < \epsilon$$

in other words, $\lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 \to 0$ uniformly for $\delta x \in B_{R_\epsilon}(0)$, as $t_i \to R(X_\triangle)$. So when $|t_i - R(X_\triangle)| < \rho_\epsilon$,

$$\begin{aligned} \frac{1}{Vol(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0) e^{-w} dx + 1 &= \frac{1}{Vol(\triangle)} \int_{\mathbb{R}^n \setminus B_{R_\epsilon}(x_{t_i})} \lambda_r(D\tilde{u}_0) e^{-w} dx \\ &+ \frac{1}{Vol(\triangle)} \int_{\mathbb{R}^n \setminus B_{R_\epsilon}(x_{t_i})} e^{-w} dx + \frac{1}{Vol(\triangle)} \int_{B_{R_\epsilon}(x_{t_i})} (\lambda_r(D\tilde{u}_0) + 1) e^{-w} dx \\ &\leq 2\epsilon + \epsilon \frac{1}{Vol(\triangle)} \int_{B_{R_\epsilon}(x_{t_i})} e^{-w} dx \leq 3\epsilon \end{aligned}$$

The first case in (4.21) follows by letting $\epsilon \to 0$.

2. $r = K_0 + 1, \cdots, N$. We fix $\epsilon = \frac{1}{2}$ and $R_{\frac{1}{2}}$ in (4.22). By (4.23) and (4.24), there exists $\rho > 0$, such that if $|t_i - R(X_\triangle)| < \rho$, then for all $\delta x \in B_{R_{\frac{1}{2}}}(0) \subset \mathbb{R}^n$,

$$\lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 > e^{-CR_{\frac{1}{2}}}(\lambda_r(D\tilde{u}_0(x_{t_i})) + 1) > e^{-CR_{\frac{1}{2}}} \frac{a_r}{2} > 0$$

$$\begin{aligned} \frac{1}{Vol(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0) e^{-w} dx + 1 &\geq \frac{1}{Vol(\triangle)} \int_{B_{R_{\frac{1}{2}}}(x_{t_i})} (\lambda_r(D\tilde{u}_0) + 1) e^{-w} dx \\ &\geq e^{-CR_{\frac{1}{2}}} \frac{a_r}{2} \frac{1}{Vol(\triangle)} \int_{B_{R_{\frac{1}{2}}}(x_{t_i})} e^{-w} dx \\ &\geq e^{-CR_{\frac{1}{2}}} \frac{a_r}{2} \frac{1}{2} > 0 \end{aligned}$$

Now we prove the claim. We can rewrite (4.23) using the special form of $\tilde{u}_0$ (4.53).

$$D\tilde{u}_0(x) = \sum_\alpha \frac{e^{<p_\alpha, x>}}{\sum_\beta e^{<p_\beta, x>}} p_\alpha = \sum_\alpha c_\alpha(x) p_\alpha$$

Here the coefficients

$$0 \leq c_\alpha(x) = \frac{e^{<p_\alpha, x>}}{\sum_\beta e^{<p_\beta, x>}}, \ \sum_{\alpha=1}^N c_\alpha(x) = 1$$

So

$$\lambda_r(D\tilde{u}_0(x)) + 1 = \sum_\alpha c_\alpha(x)(\lambda_r(p_\alpha) + 1) = \sum_{\{\alpha:\lambda_r(p_\alpha)+1>0\}} c_\alpha(x)(\lambda_r(p_\alpha) + 1)$$

Since $\lambda_r(p_\alpha) + 1 \geq 0$ is a fixed value, to prove the claim, we only need to show the same estimate for $c_\alpha(x)$.

But now

$$\begin{aligned} c_\alpha(x_{t_i} + \delta x) &= \frac{e^{<p_\alpha,x_{t_i}>} e^{<p_\alpha,\delta x>}}{\sum_\beta e^{<p_\beta,x_{t_i}>} e^{<p_\beta,\delta x>}} \leq e^{|p_\alpha|R} \cdot e^{max_\beta |p_\beta| \cdot R} \frac{e^{<p_\alpha,x_{t_i}>}}{\sum_\beta e^{<p_\beta,x_{t_i}>}} \\ &\leq e^{CR} \frac{e^{<p_\alpha,x_{t_i}>}}{\sum_\beta e^{<p_\beta,x_{t_i}>}} = e^{CR} c_\alpha(x_{t_i}) \end{aligned}$$

And similarly

$$c_\alpha(x_{t_i} + \delta x) \geq e^{-CR} c_\alpha(x_{t_i})$$

So the claim holds and the proof is completed. □

4.4 On behaviors of the limit metric

4.4.1 Equation for the limit metric

We first fix the reference metric to be the Fubini-Study metric.

$$\omega = \sqrt{-1}\partial\bar{\partial}\tilde{u}_0 = \sqrt{-1}\partial\bar{\partial}\log(\sum_\alpha |s_\alpha|^2)$$

We want to see what's the limit of ω_t as $t \to R(X)$ under suitable transformation, where

$$\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$$

is solution of continuity equation $(*)_t$. We now use notations from previous section. So in toric coordinates,

$$\omega_t = \frac{\partial^2 u}{\partial \log z_i \partial \log z_j} d\log z_i \wedge d\log z_j = -\sqrt{-1} u_{ij} dx_i d\theta_j,$$

where $u = u_t$ is the solution of real Monge-Ampère equation $(**)_t$.

Let $\sigma = \sigma_t$ be the holomorphic transformation given by

$$\sigma_t(x) = x + x_t.$$

Assume $x_t = (x_t^1, \cdots, x_t^n)$, then under complex coordinate, we have

$$\sigma_t(\{z_i\}) = \{e^{x_t^i/2} z_i\}.$$

By the analysis of previous section, we do the following transformation.

$$U(x) = \sigma_t^* u(x) - u(x_t) = u(x + x_t) - u(x_t), \quad \tilde{U}_t(x) = \sigma_t^* \tilde{u}_0(x) - \tilde{u}_0(x_t) = \tilde{u}_0(x + x_t) - \tilde{u}_0(x_t). \tag{4.25}$$

Note that $w_t(x) = tu + (1-t)\tilde{u}_0$. Then $U = U_t(x)$ satisfies the following Monge-Ampère equation

$$\det(U_{ij}) = e^{-tU-(1-t)\tilde{U}-w(x_t)} \tag*{$(**)'_t$}$$

By Proposition 22, we know that $Q = -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c$ lies on the boundary of $\triangle$. Let $\mathcal{F}$ be the minimal face of $\triangle$ which contains Q. Now the observation is

Proposition 23. *There is a subsequence $t_i \to R(X)$, $\tilde{U}_{t_i}$ converge locally uniformly to a convex function of the form:*

$$\tilde{U}_\infty := \log\left(\sum_{p_\alpha \in \mathcal{F}} b_\alpha e^{\langle p_\alpha, x\rangle}\right), \tag{4.26}$$

where $0 < b_\alpha \leq 1$ are some constants. For simplicity, we will use $\sum_\alpha{}' = \sum_{p_\alpha \in \mathcal{F}}$ to denote the sum over all the ***vertex lattice points*** *contained in $\mathcal{F}$.*

Proof. By (4.53) and (4.25), we have

$$\tilde{U}(x) = \log(\sum_\alpha e^{\langle p_\alpha, x + x_t\rangle}) - \log(\sum_\alpha e^{\langle p_\alpha, x_t}) = \log(\sum b(p_\alpha, t) e^{\langle p_\alpha, x\rangle}), \tag{4.27}$$

where

$$b(p_\alpha, t) = \frac{e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}}.$$

Since $0 < b(p_\alpha, t) < 1$, we can assume there is a subsequence $t_i \to R(X)$, such that for any vertex lattice point p_α,

$$\lim_{t \to R(X)} b(p_\alpha, t) = b_\alpha. \tag{4.28}$$

We need to prove $b_\alpha \neq 0$ if and only if $p_\alpha \in \mathcal{F}$. To prove this, we first note that

$$D\tilde{u}_0(x_t) = \frac{\sum_\alpha p_\alpha e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}} = \sum_\alpha b(p_\alpha, t) p_\alpha. \tag{4.29}$$

By Lemma 25, $D\tilde{u}_0(x_t) \to y_\infty \in \partial\triangle$. So by letting $t \to R(X)$ in (4.29) and using (4.28), we get

$$y_\infty = \sum_\alpha b_\alpha p_\alpha.$$

By Proposition 22, $y_\infty \in \partial\triangle$ lies on the same faces as Q does, i.e. $\mathcal{F}$ is also the minimal face containing y_∞, so we must have $b_\alpha = 0$ if $p_\alpha \notin \mathcal{F}$. We only need to show if $p_\alpha \in \mathcal{F}$, then $b_\alpha \neq 0$.

If dim $\mathcal{F}$=k, then there exists k+1 vertex lattice points $\{p_1, \cdots, p_{k+1}\}$ of $\mathcal{F}$, such that the corresponding coefficient $b_i \neq 0$, $i = 1, \cdots, k+1$, i.e. $\lim_{t\to R(X)} b(p_i, t) = b_i > 0$.

Remark 33. *Here is why we need to assume p_α are all vertex lattice points.*

Let p be any vertex point of $\mathcal{F}$, then

$$p = \sum_{i=1}^{k+1} c_i p_i, \quad \text{where} \quad \sum_{i=1}^{k+1} c_i = 1.$$

Then

$$b(p,t) = \frac{e^{\langle \sum_{i=1}^{k+1} c_i p_i, x_t \rangle}}{\sum_\beta e^{\langle p_\beta, x_t \rangle}} = \prod_{i=1}^{k+1} \left(\frac{e^{\langle p_i, x_t \rangle}}{\sum_\beta e^{\langle p_\beta, x_t \rangle}} \right)^{c_i} = \prod_{i=1}^{k+1} b(p_i, t)^{c_i} \xrightarrow{t \to R(X)} \prod_{i=1}^{k+1} b_i^{c_i} > 0.$$

$\square$

We can now state a real version of Theorem 20

Theorem 21. *There is a subsequence $t_i \to R(X)$, $U_{t_i}(x)$ converge to a smooth entire solution of the following equation on $\mathbb{R}^n$*

$$\det(U_{ij}) = e^{-R(X)U(x) - (1-R(X))\tilde{U}_\infty(x) - c} \qquad (**)'_\infty$$

$c = \lim_{t_i \to R(X)} w(x_{t_i})$ *is some constant.*

4.4.2 Change to Complex Monge-Ampère equation

The proof of Theorem 21 might be done by the theory of real Monge-Ampère equation. But here, we will change our view and rewrite $(**)'_t$ as a family of complex Monge-Ampère equations. This will alow us to apply some standard estimates in the theory of complex Monge-Ampère equations.

We rewrite the formula for $\tilde{U}(x)$ in (4.27) as

$$e^{\tilde{U}} = \frac{\sum_\alpha b(p_\alpha,t)e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}}\sum_\beta e^{\langle p_\beta,x\rangle} = \frac{\sum_\alpha b(p_\alpha,t)|s_\alpha|^2}{\sum_\beta |s_\beta|^2}e^{-\tilde{C}+\tilde{u}_0} = (\sum_\alpha b(p_\alpha,t)|s_\alpha|^2)e^{\tilde{u}_0}. \quad (4.30)$$

s_α is the holomorphic section of K_X^{-1} corresponding to lattice point p_α. Here and in the following $\|\cdot\| := \|\cdot\|_{FS}$ is the Fubini-Study metric on K_X^{-1}. Recall that, by (4.5), for any section s,

$$|s|^2_{FS} = e^{-\tilde{C}}\frac{|s|^2}{\sum_\beta |s_\beta|^2}.$$

The second equality in (4.30) holds because $e^{\langle p_\alpha,x\rangle} = \left|\frac{s_\alpha}{\tilde{s}_0}\right|^2$ by (4.6). We also used the definition of $\tilde{u}_0$ in (4.53).

$(**)'_t$ can then be rewritten as

$$\det(U_{ij}) = e^{-t\psi}e^{-\tilde{u}_0}\left(\sum_\alpha b(p_\alpha,t)|s_\alpha|^2\right)^{-(1-t)} e^{-w(x_t)}.$$

where

$$\psi = \psi_t = U - \tilde{u}_0. \quad (4.31)$$

By (4.4) and (4.54), $(**)'_t$ can finally be written as the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-t\psi}\left(\sum_\alpha b(p_\alpha,t)|s_\alpha|^2\right)^{-(1-t)} e^{h_\omega - w(x_t)}\omega^n \qquad (***)_t$$

Similarly for $\tilde{U}_\infty$ in (4.26), we write

$$e^{\tilde{U}_\infty} = \frac{\sum_\alpha{}' b_\alpha e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}}\sum_\beta e^{\langle p_\beta,x\rangle} = (\sum_\alpha{}' b_\alpha|s_\alpha|^2)e^{\tilde{u}_0}.$$

And the limit equation $(**)'_\infty$ becomes:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-R(X)\psi}\left(\sum_\alpha{}' b_\alpha|s_\alpha|^2\right)^{-(1-R(X))} e^{h_\omega - c}\omega^n \qquad (***)_\infty$$

So we reformulate Theorem 21 as the main Theorem 20 in the introduction.

4.4.3 Discussion on the conic behavior of limit metric

For any lattice point $p_\alpha \in \triangle$, let $D_{p_\alpha} = \{s_\alpha = 0\}$ be the zero divisor of the corresponding holomorphic section s_α. By toric geometry , we have

$$D_{p_\alpha} = \{s_\alpha = 0\} = \sum_{i=1}^{K} (\langle p_\alpha, v_i \rangle + 1) D_i.$$

Here v_i is the primitive inward normal vector to the i-th codimension one face, and D_i is the toric divisor corresponding to this face.

Recall that $\mathcal{F}$ is the minimal face containing Q. Let $\{p_k^{\mathcal{F}}\}$ be all the vertex lattice points of $\mathcal{F}$. They correspond to a sublinear system $\mathfrak{L}_{\mathcal{F}}$ of $|K_X^{-1}|$. The base locus of $\mathfrak{L}_{\mathcal{F}}$ is given by the schematic intersection

$$Bs(\mathfrak{L}_{\mathcal{F}}) = \bigcap_k D_{p_k^{\mathcal{F}}}$$

The fixed components in $Bs(\mathfrak{L}_{\mathcal{F}})$ are

$$D^{\mathcal{F}} = \sum_{i=1}^{r} a_i D_i, \tag{4.32}$$

where

$$\mathbb{N} \ni a_i = 1 + \min_k \langle p_k^{\mathcal{F}}, v_i \rangle > 0, i = 1, \ldots, r.$$

For $i = 1, \ldots, K$, we always have $a_i = 1 + \min_k \langle p_k^{\mathcal{F}}, v_i \rangle \geq 0$. In (4.32), the coefficients a_i are those with $a_i \neq 0$.

Pick any generic point p on $D^{\mathcal{F}}$. p lies on only one component of $D^{\mathcal{F}}$. Without loss of generality, assume $p \in D_1$, and in a neighborhood $\mathcal{U}_p$ of p, choose local coordinate $\{z_i\}$ such that D_1 is defined by $z_1 = 0$, then the singular Monge-Ampère equation (4.2) locally becomes:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = |z_1|^{-2a_1(1-R(X))}\Omega \tag{4.33}$$

with Ω a nonvanishing smooth volume form in $\mathcal{U}_p$.

So locally around generic point on p, we have

$$Ric(\omega_\psi) = 2\pi(1 - R(X))a_1(\{z_1 = 0\}) + Ric(\Omega) \tag{4.34}$$

where $\{z_1 = 0\}$ is the current of integration along divisor $\{z_1 = 0\}$.

Note that we have the following singular conic metric in $\mathcal{U}_p$

$$\eta = \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2\alpha}} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i.$$

η has conic singularity along $\{z_1 = 0\}$ with conic angle $2\pi(1-\alpha)$, and satisfies

$$\eta^n = \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{|z_1|^{2\alpha}}, \text{ and } \quad Ric(\eta) = 2\pi\alpha(\{z_1 = 0\}).$$

Comparing this with (4.33) and (4.34), we expect that the limit Kähler metric around p has conic singularity along D_1 with conic angle equal to $2\pi(1-(1-R(X))a_1)$ and the same hold for generic points on D_i, i.e. the limit metric should have conic singularity along D_i of conic angle equal to $2\pi(1-(1-R(X))a_i)$.

Remark 34. *At present, it seems difficult to speculate the behavior of limit metric around higher codimensional strata of $D^{\mathcal{F}}$. See the discussion in example 2. We hope to return to this issue in future.*

4.4.4 Proof of Theorem 20

We are now in the general setting of complex Monge-Ampère equations. $(***)_\infty$ is a complex Monge-Ampère equation with poles at righthand side. $(***)_t$ can be seen as regularizations of $(***)_\infty$. We ask if the solutions of $(***)_t$ converge to a solution of $(***)_\infty$. Starting from Yau's work [Yau1], similar problems have been considered by many people. Due to the large progress made by Kołodziej[Kolo], complex Monge-Ampère equation can be solved with very general, usually singular, righthand side. Kołodziej's result was also proved by first regularizing the singular Monge-Ampère equation, and then taking limit back to get solution of original equation.

We will derive several apriori estimate to prove Theorem 20. For the C^0-estimate, the upper bound follows from how we transform the potential function in (4.25). The lower bound follows from a Harnack estimate for the transformed potential function which we will prove using Tian's argument in [Tia3]. For the proof of partial C^2-estimate, higher order estimates and convergence of solutions, we use some argument similar to that used by Ruan-Zhang [RuZh], and Demailly-Pali [DePai].

C^0-estimate

We first derive the C^0-estimate for $\psi = U - \tilde{u}_0$. Let $\bar{v} = \bar{v}(x)$ be a piecewise linear function defined to be

$$\bar{v}(x) = \max_{p_\alpha} \langle p_\alpha, x \rangle.$$

Then u_0 is asymptotic to $\bar{v}$ and it's easy to see that $|\bar{v} - \tilde{u}_0| \le C$. So we only need to show that $|U(x) - \bar{v}(x)| \le C$. Here and in the following, C is some constant independent of $t \in [0, R(X))$.

One side is easy. Since $DU(\mathbb{R}^n) = \triangle$ and $U(0) = 0$, we have for any $x \in \mathbb{R}^n$, $U(x) = U(x) - U(0) = DU(\xi) \cdot x \le \bar{v}(x)$. ξ is some point between 0 and x. So

$$\psi = (U - \bar{v}) + (\bar{v} - \tilde{u}_0) \le C.$$

To prove the lower bound for ψ, we only need to prove a Harnack inequality

Proposition 24.

$$\sup_X (-\psi) \le n \sup_X \psi + C(n) t^{-1}. \tag{4.35}$$

For this we use the same idea of proof in [Tia3]. First we rewrite the $(***)_t$ as

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-t\psi + F - B_t}\omega^n, \tag{4.36}$$

where

$$B_t = (1-t)\log\left(\sum_\alpha b(p_\alpha, t)|s_\alpha|^2\right), \quad F = h_\omega - w(x_t).$$

Now consider a new continuous family of equations

$$(\omega + \sqrt{-1}\partial\bar{\partial}\theta_s)^n = e^{-s\theta_s + F - B_t}\omega^n. \tag*{$(4.36)_s$}$$

Define $S = \{s' \in [0,t] | (4.36)_s \text{ is solvable for } s \in [s', t]\}$. We want to prove $S = [0,t]$. Since (4.36) has a solution ψ, $t \in S$ and S is nonempty. It is sufficient to show that S is both open and closed.

For openness, we first estimate the first eigenvalue of the metric g_θ associated with the Kähler form $\omega_\theta = \omega + \sqrt{-1}\partial\bar{\partial}\theta$ for the solution θ of $(4.36)_s$.

$$\begin{aligned} Ric(\omega_\theta) &= s\sqrt{-1}\partial\bar{\partial}\theta - \sqrt{-1}\partial\bar{\partial}F + \sqrt{-1}\partial\bar{\partial}B_t + Ric(\omega) \\ &= s\sqrt{-1}\partial\bar{\partial}\theta + \omega + (1-t)(\sigma^*\omega - \omega) = s(\sqrt{-1}\partial\bar{\partial}\theta + \omega) + (t-s)\omega + (1-t)\sigma^*\omega \\ &= s\omega_\theta + (t-s)\omega + (1-t)\sigma^*\omega. \end{aligned} \tag{4.37}$$

In particular, $Ric(\omega_\theta) > s\omega_\theta$. So by Bochner's formula, the first nonzero eigenvalue $\lambda_1(g_{\theta_s}) > s$. This gives the invertibility of linearization operator $(-\Delta_s) - s$ of equation $(4.36)_s$, so the openness of solution set S follows.

Recall the functional I_ω, J_ω defined in Section 2.4.

Lemma 27 ([BaMa],[Tia3]). *(i)* $\frac{n+1}{n} J_\omega(\theta_s) \le I_\omega(\theta_s) \le (n+1) J_\omega(\theta_s)$,

(ii) $\frac{d}{ds}(I(\theta_s) - J(\theta_s)) = -\int_X \theta_s(\Delta_s\dot{\theta}_s)\omega_{\theta_s}^n$.

Using $\lambda_1(g_{\theta_s}) > s$, Lemma 27.(ii) gives

Lemma 28 ([BaMa],[Tia3]). *$I(\theta_s) - J(\theta_s)$ is monotonically increasing.*

Let's recall Bando-Mabuchi's estimate for Green function.

Proposition 25 ([BaMa]). *For every m-dimensional compact Riemannian manifold (X,g) with $diam(X,g)^2 Ric(g) \ge -(m-1)\alpha^2$, there exists a positive constant $\gamma = \gamma(m,\alpha)$ such that*

$$G_g(x,y) \ge -\gamma(m,\alpha) diam(X,g)^2 / V_g. \tag{4.38}$$

Here the Green function $G_g(x,y)$ is normalized to satisfy

$$\int_M G_g(x,y) dV_g(x) = 0.$$

Bando-Mabuchi used this estimate to prove the key estimate:

Proposition 26 ([BaMa]). *Let*

$$\mathcal{H}^s = \{\theta \in C^\infty(X); \omega_\theta = \omega + \sqrt{-1}\partial\bar{\partial}\theta > 0, Ric(\omega_\theta) \ge s\omega_\theta\},$$

then for any $\theta \in \mathcal{H}^s$, we have

(1)

$$\sup_X(-\theta) \le \frac{1}{V}\int_X (-\theta)\omega_\theta^n + C(n)s^{-1}, \tag{4.39}$$

(2)

$$Osc(\theta) \le I(\theta) + C(n)s^{-1}. \tag{4.40}$$

Proposition 27. $(4.36)_s$ *is solvable for* $0 \le s \le t$.

Proof. From $(4.36)_s$, there exists $x_s \in X$ such that $-s\theta_s(x_s) + F(x_s) - B_t(x_s) = 0$, so $|\theta_s(x_s)| = \frac{1}{s}|F - B_t|(x_s) \le C_t s^{-1}$. By (4.40) and $I \le (n+1)(I-J)$ (by Lemma 27-(i)), we get

$$\sup_X \theta_s \le Osc(\theta) + \theta(x_s) \le (n+1)(I-J)(\theta) + C(n)s^{-1} + C_t s^{-1}.$$

By Lemma 28, for any $\delta > 0$, we get uniform estimate for $\sup_X \theta_s$ and hence also $\inf_X \theta_s$ for $s \in [\delta, t]$. So $\|\theta_s\|_{C^0} \le C\delta^{-1}$. We can use Yau's estimate to get C^2 and higher order estimate. So we can solve $(4.36)_s$ for $s \in [\delta, t]$, for any $\delta > 0$.

On the otherhand, by Yau's theorem, we can solve $(4.36)_s$ for $s = 0$. And by implicit function theorem, we can solve $(4.36)_s$ for $s \in [0, \tau)$ for τ sufficiently small. We can pick δ such that $\delta < \tau$, so we get solution of $(4.36)_s$ for $s \in [\delta, \tau)$ in two ways. They must coincide by the recent work of Berndtsson [Bern] on the uniqueness of solutions for the twisted Kähler-Einstein equation (4.37). So we complete the proof. ☐

Then one can use the same argument as in [Tia3] to prove

Proposition 28 ([Tia3])**.**

$$-\frac{1}{V}\int_X \theta\omega_\theta^n \le \frac{n}{V}\int_X \theta\omega^n \le n\sup_X \theta. \tag{4.41}$$

Proof. First by taking derivatives to equation $(4.36)_s$, we get

$$\Delta_s\dot{\theta} = -\theta - s\dot{\theta}.$$

So

$$\begin{aligned}
\frac{d}{ds}(I-J)(\theta_s) &= -\int_X \theta\frac{d}{ds}\omega_\theta^n = -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) + \int_X \dot{\theta}\omega_\theta^n \\
&= -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) - \frac{1}{s}\int_X \theta\omega_\theta^n = -\frac{1}{s}\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right).
\end{aligned}$$

So

$$\frac{d}{ds}(s(I-J)(\theta_s)) - (I-J)(\theta_s) = -\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right). \tag{4.42}$$

By Proposition 27, θ_s can be solved for $s \in [0,t]$, and $\theta_t = \psi = \psi_t$, we can integrate to get

$$t(I-J)(\psi) - \int_0^t (I-J)(\theta_s)ds = -t\int_X \psi\omega_\psi^n.$$

Divide both sides by t to get

$$(I-J)(\psi) - \frac{1}{t}\int_0^t (I-J)(\theta_s)ds = -\int_X \psi\omega_\psi^n.$$

By Lemma 27.(i), we can get

$$\frac{n}{n+1}\int_X \psi(\omega^n - \omega_\psi^n) = \frac{n}{n+1}I(\psi) \geq -\int_X \psi\omega_\psi^n.$$

(4.41) follows from this inequality imediately. □

Combine (4.41) with Bando-Mabuchi's estimate (4.39) when $s = t$, we then prove the Harnack estimate (4.35). So we can derive the lower bound of ψ from the upper bound of ψ and C^0-estimate is obtained.

Remark 35. *Professor Jian Song showed me that by modifying the above argument one can prove Harnack inequality using the weaker statement instead of that in Proposition 27:* $(4.36)_s$ *can be solved for* $s \in (0,t]$*. In this way, one can avoid using Berndtsson's uniqueness result. Here we give his nice argument from [Son] for comparison. First by the concavity of log function and using* $(4.36)_s$*, we have*

$$\frac{1}{V}\left(-s\int_X \theta_s\omega^n + \int_X (F-B_t)\omega^n\right) \leq \log\left(\frac{1}{V}\int_X e^{-s\theta_s+F-B_t}\omega^n\right) = \log\left(\frac{1}{V}\int_X \omega_{\theta_S}^n\right) = 0.$$

So

$$-s\int_X \theta_s\omega^n \leq \int_X (B_t - F)\omega^n \leq C. \tag{4.43}$$

C is a constant independent of both s and t. Now we integrate (4.42) *from any s to t, then*

$$t(I-J)(\psi) - s(I-J)(\theta_s) - \int_s^t (I-J)(\theta_s)ds = -t\int_X \psi\omega_\psi^n + s\int_X \theta_s(\omega_{\theta_s}^n - \omega^n) + s\int_X \theta_s\omega^n.$$

Using positivity of $I-J$, (4.43), Lemma 27 and Lemma 28, we get

$$\begin{aligned} t(I-J)(\psi) &\geq -t\int_X \psi\omega_\psi^n - sI(\theta_s) - C \\ &\geq -t\int_X \psi\omega_\psi^n - s(n+1)(I-J)(\theta_s) - C \\ &\geq -t\int_X \psi\omega_\psi^n - s(n+1)(I-J)(\psi) - C. \end{aligned}$$

Now letting $s \to 0$, we get

$$(I-J)(\psi) \geq -\int_X \psi\omega_\psi^n - \frac{C}{t}.$$

Now we can argue as before to get Harnack.

Partial C^2-estimate

$(*)_t$ is equivalent to

$$Ric(\omega_\phi) = t\omega_\phi + (1-t)\omega.$$

From our transformation (4.25), we get

$$Ric(\omega_\psi) = t\omega_\psi + (1-t)\sigma^*\omega. \tag{4.44}$$

In particular, $Ric(\omega_\psi) > t\omega_\psi$.

By C^0-estimate of ψ and Proposition 2, we get the estimate $tr_{\omega_\psi}\omega \leq C_4$. So $\omega_\psi \geq C_4\omega$. If we choose local coordinate such that $\omega(\partial_i, \partial_{\bar{j}}) = \delta_{ij}$ and $\omega_\psi(\partial_i, \partial_{\bar{j}}) = \mu_i\delta_{ij}$, then $\mu_i \geq C_4$.

Now by (4.36),

$$\prod_j \mu_j = \frac{\omega_\psi^n}{\omega^n} = e^{-t\psi+F-B}$$

with $F = h - w(x_t)$ and $B = (1-t)\log\left(\sum_\alpha b(p_\alpha,t)|s_\alpha|^2\right)$. So by the C^0-estimate of ψ, we get

$$\mu_i = \frac{\prod_j \mu_j}{\prod_{j\neq i}\mu_j} \leq \frac{e^{-t\psi+F-B}}{C_4^{n-1}} \leq C_5 e^{-B}.$$

In conclusion, we get the partial C^2-estimate

$$C_4\omega \leq \omega_\psi \leq C_5 e^{-B}\omega. \tag{4.45}$$

Remark 36. *The partial C^2-upper bound $\omega_\psi \leq C_5 e^{-B}\omega$ can also be proved by maximal principle.*

In fact, let

$$\Lambda = \log(n + \Delta\psi) - \lambda\psi + B, \tag{4.46}$$

where $\Delta = \Delta_\omega$ *is the complex Laplacian with respect to reference metric* ω. *Then by standard calculation as in Yau [Yau1], we have*

$$\begin{aligned}
\Delta'\Lambda &\geq \left(\inf_{i\neq j} S_{i\bar{i}j\bar{j}} + \lambda\right)\sum_i \frac{1}{1+\psi_{i\bar{i}}} + \left(\Delta F - \Delta B - t\Delta\psi - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}\right)\frac{1}{n+\Delta\psi} - \lambda n + \Delta' B \\
&= \left(\inf_{i\neq j} S_{i\bar{i}j\bar{j}} + \lambda\right)\sum_i \frac{1}{1+\psi_{i\bar{i}}} + \left(\Delta F + nt - n^2 \inf_{i\neq j} S_{i\bar{i}j\bar{j}}\right)\frac{1}{n+\Delta\psi} + \\
&\quad + \sum_i B_{i\bar{i}}\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) - (\lambda n + t).
\end{aligned} \tag{4.47}$$

Since for each i, $\frac{1}{n+\Delta\psi} \leq \frac{1}{1+\psi_{i\bar{i}}}$, *so* $\frac{1}{n+\Delta\psi} \leq \frac{1}{n}\sum_i \frac{1}{1+\psi_{i\bar{i}}}$. *So the second term on the right of* (4.47) *is bounded below by* $-C_0 \sum_i \frac{1}{1+\psi_{i\bar{i}}}$ *for some positive constant* $C_0 > 0$

For the 3rd term, we observe from (4.25) *and* (4.30) *that*

$$\sqrt{-1}\partial\bar{\partial}B = (1-t)(\sigma^*\omega - \omega) \geq -(1-t)\omega.$$

So, since again $\frac{1}{n+\Delta\psi} \leq \frac{1}{1+\psi_{i\bar{i}}}$, *we have*

$$B_{i\bar{i}}\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geq -(1-t)\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geq -(1-t)\frac{1}{1+\psi_{i\bar{i}}}.$$

By the above discussion, at the maximal point P_t *of* Λ, *we have*

$$0 \geq \Delta'\Lambda \geq (\lambda + \inf_{i\neq j} S_{i\bar{i}j\bar{j}} - C_0 - (1-t))\sum_i \frac{1}{1+\psi_{i\bar{i}}} - (\lambda n + t) = C_2 \sum_i \frac{1}{1+\psi_{i\bar{i}}} - C_3 \tag{4.48}$$

for some constants $C_2 > 0$, $C_3 > 0$, *by choosing* λ *sufficiently large.*

Now we use the following inequality from [Yau1]

$$\begin{aligned}
\sum_i \frac{1}{1+\psi_{i\bar{i}}} &\geq \left(\frac{\sum_i (1+\psi_{i\bar{i}})}{\prod_j (1+\psi_{j\bar{j}})}\right)^{1/(n-1)} = (n+\Delta\psi)^{1/(n-1)} e^{\frac{B-F+t\psi}{n-1}} \\
&= e^{\frac{\Lambda}{n-1}} e^{\frac{-F+(t+\lambda)\psi}{n-1}}.
\end{aligned} \tag{4.49}$$

By (4.48) *and* (4.49), *we get the bound*

$$e^{\Lambda(P_t)} \leq C_4 e^{-(t+\lambda)\psi(P_t)}.$$

So we get estimate that for any $x \in X = X_\triangle$,

$$(n+\Delta\psi)e^{-\lambda\psi}e^{B} \le e^{\Lambda(P_t)} \le C_4 e^{-(t+\lambda)\psi(P_t)}.$$

Since we have C^0-estimate for ψ, we get partial C^2-upper estimate:

$$(n+\Delta\psi)(x) \le C_4 e^{-(t+\lambda)\psi(P_t)} e^{\lambda\psi(x)} e^{-B} \le C_5 \left(\sum_\alpha b(p_\alpha,t)|s_\alpha|^2\right)^{-(1-t)}. \tag{4.50}$$

In particular,

$$1+\psi_{i\bar{i}} \le C_5 e^{-B},$$

which is same as $\omega_\psi \le C_5 e^{-B}$.

Higher order estimate and completion of the proof of Theorem 20

For any compact set $K \subset X\backslash D$, we first get the gradient estimate by interpolation inequality:

$$\max_K |\nabla\psi| \le C_K(\max_K \Delta\psi + \max_K |\psi|). \tag{4.51}$$

Next, by the complex version of Evans-Krylov theory (Section 2.3.2, [Tia2], [Bło]), we have a uniform $C_{2,K} > 0$, such that $\|\psi\|_{C^{2,\alpha}(K)} \le C_{2,K}$ sor some $\alpha \in (0,1)$. Now take derivative to the equation:

$$\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = \log\det(g_{i\bar{j}}) - t\psi + F - B$$

to get

$$g'^{i\bar{j}}\psi_{i\bar{j},k} = -t\psi_k + F_k - B_k + g^{i\bar{j}}g_{i\bar{j},k} - g'^{i\bar{j}}g_{i\bar{j},k}. \tag{4.52}$$

By (4.45), (4.51) and $\|\psi\|_{C^{2,\alpha}(K)} \le C_{2,K}$, (4.52) is a linear elliptic equation with C^α coefficients. By Schauder's estimate, we get $\|\psi_k\|_{C^{2,\alpha}} \le C$, i.e. $\|\psi\|_{C^{3,\alpha}} \le C$. Then we can iterate in (4.52) to get $\|\psi\|_{C^{r,\alpha}} \le C$ for any $r \in \mathbb{N}$. So we see that $(\psi = \psi(t))_{t<R(X)} \subset C^\infty(X\backslash D)$ is precompact in the smooth topology.

Now we can finish the proof of Theorem 20 using argument from [DePai]

Proof of Theorem 20. The uniform estimate $\|\psi\|_{L^\infty}$ implies the existence of a L^1-convergent sequence $(\psi_j = \psi_{t_j})_j$, $t_j \uparrow R(X)$ with limit $\psi_\infty \in \mathcal{PSH}(\omega) \cap L^\infty(X)$. We can assume that a.e.-convergence holds too. The precompactness of the family $(\psi_j) \subset C^\infty(X\backslash D)$ in the smooth topology implies the convergence of the limits over $X\backslash D$:

$$\begin{aligned}(\omega+\sqrt{-1}\partial\bar{\partial}\psi_\infty)^n &= \lim_{t_j\to R(X)}(\omega+\sqrt{-1}\partial\bar{\partial}\psi_j)^n\\ &= \lim_{t_j\to R(X)} e^{-t_j\psi_{t_j}}\left(\sum_\alpha b(p_\alpha,t_j)|s_\alpha|^2\right)^{-(1-t_j)} e^{h_\omega-w(x_{t_j})}\omega^n\\ &= e^{-R(X)\psi_\infty}\left(\sum_\alpha{}' b_\alpha|s_\alpha|^2\right)^{-(1-R(X))} e^{h_\omega-c}\omega^n.\end{aligned}$$

The fact that ψ_∞ is a bounded potential implies that the global complex Monge-Ampère measure $(\omega+\sqrt{-1}\partial\bar{\partial}\psi_\infty)^n$ does not carry any mass on complex analytic sets. This follows from pluripotential theory ([Klim]) because complex analytic sets are pluripolar. We conclude that ψ_∞ is a global bounded solution of the complex Monge-Ampère equation $(***)_\infty$ which belongs to the class $\mathcal{PSH}(\omega)\cap L^\infty(X)\cap C^\infty(X\backslash D)$. □

4.5 Partial C^0-estimate and Multiplier ideal sheaf

Proposition 29. *The partial C^0-estimate holds along the continuity method $(**)_t$, i.e. the k-th Bergman kernel of $\sqrt{-1}\partial\bar{\partial}u_t$ is uniformly bounded away from zero for any $k\geq 1$.*

This follows from immediately from Lemma 13-(3). Alternatively, define

$$c(p_\alpha,t)=\frac{e^{-\langle p_\alpha,x_t\rangle}}{\sum_\beta e^{-\langle p_\beta,x_t\rangle}}.$$

Note that $\sum_\alpha c(p_\alpha,t)p_\alpha = D\tilde{u}(-x_t)$. By Lemma 13, the above Proposition follows from the following Lemma-(2).

Lemma 29. *There exists a constant C independent of t such that*

$$\left|\phi-\sup\phi-\log\sum_\alpha c(p_\alpha,t)|s_\alpha|_h^2\right|\leq C$$

Proof. Let $\xi_t=\phi_t-\sup\phi_t$, then

$$\sqrt{-1}\partial\bar{\partial}\xi_t=\omega_t-\tilde{\omega}=(\sigma_t^{-1})^*(\sigma_t^*\omega_t-\tilde{\omega})+((\sigma_t^{-1})^*\tilde{\omega}-\tilde{\omega}).$$

For the first bracket on the right

$$(\sigma_t^{-1})^*(\sigma_t^*\omega_t - \tilde{\omega}) = \sqrt{-1}\partial\bar{\partial}((\sigma_t^{-1})^*\psi_t)$$

For the second bracket on the right, we can take the potential of $(\sigma_t^{-1})^*\tilde{\omega}$ to be

$$(\sigma_t^{-1})^*\tilde{u} - \tilde{u}(-x_t) \quad = \quad \log\sum_\alpha e^{\langle p_\alpha, x - x_t\rangle} - \log\sum_\alpha e^{\langle p_\alpha, -x_t\rangle} = \log\sum_\alpha c(p_\alpha, t)e^{\langle p_\alpha, x\rangle}$$

So

$$(\sigma_t^{-1})^*\tilde{\omega} - \tilde{\omega} = \sqrt{-1}\partial\bar{\partial}\log\sum_\alpha c(p_\alpha, t)|s_\alpha|^2$$

$$\xi_t = (\sigma_t^{-1})^*\psi_t + \log\sum_\alpha c(p_\alpha, t)|s_\alpha|^2 + C(t)$$

For the first term on the right $\|(\sigma_t^{-1})^*\psi_t\|_{C^0} = \|\psi_t\|_{C^0} \leq C_1$ with C_1 independent of t. So it's easy to see that

$$\left|\sup\xi_t - \sup\log\sum_\alpha c(p_\alpha, t)|s_\alpha|^2 - C(t)\right| \leq C_1$$

To estimate $C(t)$, note that $\sup\xi_t = 0$, and

$$\sup\log\sum_\alpha c(p_\alpha, t)|s_\alpha|^2 \geq \log\left(\frac{1}{N+1}\min_\alpha\{\sup|s_\alpha|^2\}\right) = C_2'$$

$$\sup\log\sum_\alpha c(p_\alpha, t)|s_\alpha|^2 \leq \log\sum_\alpha |s_\alpha|^2 = C_2''$$

So $|C(t)| \leq C_1 + \max(|C_2'|, |C_2''|)$. $\square$

By the similar argument as in [DeKo], the space of closed positive currents contained in a given class is compact for the weak topology. So there exists a subsequence $\omega_{\phi_{t_i}}$ converging weakly to a limit $\omega_\infty = \omega_{\phi_\infty}$. The potential ϕ_∞ can be recovered from $tr_\omega\omega_\infty$ by means of the Green kernel, and ϕ_{t_i} converges to ϕ_∞ in $L^1(X)$. We define the multiplier ideal sheaf of ϕ_∞ with exponent γ to be

$$\mathcal{I}(\gamma\phi_\infty)(U) = \{f \in \mathcal{O}_X(U); \int_U |f|^2 e^{-\gamma\phi_\infty}dV < +\infty\}$$

where $U \subset X$ is any open set and dV is any smooth volume form on U. Define

$$Aut_{\mathbb{Z}^n}(\triangle) = \{g \in GL(n, \mathbb{Z}); g(\triangle) = \triangle\}$$

Corollary 7. *Assume the toric Fano manifold has very large symmetry in the sense that the fixed points of the action of $Aut_{\mathbb{Z}^n}(\triangle)$ on $\mathbb{R}^n$ is a one dimensional space generated by the barycenter $P_c \in \mathbb{R}^n$. Assume the metric ω is $Aut_{\mathbb{Z}^n}(\triangle)$ invariant. Let $\hat{Q}$ be the intersection of the ray $\overrightarrow{O(-P_c)}$ with $\partial\triangle$. Let $\hat{\mathcal{F}}$ be the minimal face of $\triangle$ containing $\hat{Q}$. Then the multiplier ideal sheaf $\mathcal{I}(\gamma\phi_\infty)$ is the same as the multiplier ideal sheaf $\mathcal{I}(\gamma\hat{\phi})$ where*

$$\hat{\phi} = \log \sum_{\hat{p}_\alpha \in \hat{\mathcal{F}}} |\hat{s}_\alpha|^2.$$

Proof. By symmetry, it's easy to see that $c_t = Du(-x_t)$ converge to the intersection of $\overrightarrow{O(-P_c)}$ and $\partial\triangle$. □

Remark 37. *The Kähler-Ricci flow version of the above corollary was obtained by Sano [Sano]. His argument depends on the convergence results obtained in [Zhu]. Here in the continuity method, our convergence result Theorem 20 plays the role of Zhu's convergence result in Kähler-Ricci flow setting.*

4.6 Example

Example 3. *$X_\triangle = Bl_p\mathbb{P}^n$. The polytope $\triangle$ is defined by*

$$x_i \geq -1, i = 1, \cdots, n; \quad \sum_i x_i \geq -1; \quad \textit{and} \quad -\sum_i x_i \geq -1.$$

Using the symmetry of the polytope, we can calculate that

$$Vol(\triangle) = \frac{1}{n!}((n+1)^n - (n-1)^n),$$

$$P_c = \left(x_i = \frac{2(n-1)^n}{(n+1)((n+1)^n - (n-1)^n)} \right), \quad \textit{and} \quad Q = \left(x_i = -\frac{1}{n} \right).$$

So

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|} = \left(1 + \frac{|\overline{OP_c}|}{|\overline{OQ}|}\right)^{-1} = \frac{(n+1)((n+1)^n - (n-1)^n)}{(n+1)^{(n+1)} + (n-1)^{(n+1)}}.$$

$\mathcal{F}$ is the (n-1)-dimensional simplex with vertices

$$P_i = (-1, \cdots, \overset{i-th\ place}{n-2}, \cdots, -1) \quad i = 1, \cdots, n.$$

Let e_j be the j-th coordinate unit vector, then $\langle P_i, e_j\rangle = -1$ for $i \neq j$. $\langle P_i, e_i\rangle = n-2$. $\langle P_i, \pm(1,\cdots,1)\rangle = \mp 1$. So P_i corresponds to a holomorphic section s_i with $\{s_i = 0\} = (n-1)D_i + 2D_\infty$, where D_i is the toric divisor corresponding to the codimension one face with inward normal e_i, and D_∞ is the toric divisor corresponding to the simplex face with vertices $Q_i = (-1,\cdots,n,\cdots,-1)$.

It's easy to see that $Bs(\mathfrak{L}_{\mathcal{F}}) = 2D_\infty$. If we view $X_\triangle$ as the projective compactification of $\mathcal{O}(-1)\to\mathbb{P}^{n-1}$, then D_∞ is just the divisor added at infinity. So the limit metric should have conic singularity along D_∞ with conic angle

$$\theta = 2\pi\times(1-(1-R(X))\times 2) = 2\pi\frac{(n+1)^{n+1}-(3n+1)(n-1)^n}{(n+1)^{n+1}+(n-1)^{n+1}}.$$

In particular, if $n=2$, i.e. $X_\triangle = Bl_p\mathbb{P}^2$ which is the case of the figure in the Introduction, then

$$R(X_\triangle) = \frac{6}{7}, \quad \theta = 2\pi\times\frac{5}{7}.$$

This agrees with the results of [Szé] and [ShZh]. In fact, the results in [Szé] and [ShZh] can be easily generalized to $Bl_p\mathbb{P}^n$ which give the same results as here.

For the multiplier ideal sheaf, by symmetry, we see that $\hat{\mathcal{F}}$ is the facet corresponding to D_∞. $\hat{\mathcal{F}}$ has vertex given by

$$\hat{P}_i = (-1,\cdots,\overset{i-th\ place}{n},\cdots,-1) \quad i=1,\cdots,n.$$

Let $\hat{s}_i$ be the holomorphic section corresponding to $\hat{P}_i$. Then it's easy to see that

$$\{\hat{s}_i = 0\} = 2E + (n+1)D_i$$

where E is the exceptional divisor corresponding to the facet $\mathcal{F}$ above. So

$$\mathcal{I}(\gamma\phi_\infty) = \mathcal{I}(2\gamma E)$$

So when $\gamma < 1/2$, the multiplier ideal scheme is empty. When $\gamma \geq 1/2$, the support of multiplier ideal scheme is the exceptional divisor E.

Example 4. *$X_\triangle = Bl_{p,q}\mathbb{P}^2$, $P_c = \frac{2}{7}(-\frac{1}{3},-\frac{1}{3})$, $-\frac{21}{4}P_c \in \partial\triangle$, so $R(X_\triangle) = \frac{21}{25}$.*

$\mathcal{F} = \overline{Q_1Q_2}$. Q_1 corresponds to holomorphic section s_1 with $\{s_1 = 0\} = 2D_1 + D_2 + D_5$. Q_2

corresponds to s_2 *with* $\{s_2 = 0\} = D_1 + 2D_2 + D_3$. *The fixed components in* $Bs(\mathfrak{L}_{\mathcal{F}})$ *are* $D_1 + D_2$. *Here as in the picture,* D_1 *to* D_5 *are the divisors corresponding to the facets. So at generic point of* D_1 *(or* D_2*), the conic angle along* D_1 *(or* D_2*) should be*

$$2\pi \times (1 - (1 - \frac{21}{25}) \times 1) = 2\pi \times \frac{21}{25}.$$

While around the point $p = D_1 \cap D_2$, *if we choose local coordinate around* p *such that* $D_1 = \{z_1 = 0\}$ *and* $D_2 = \{z_2 = 0\}$, *the ideal defining the base locus is* $(z_1^2 z_2, z_1 z_2^2) = (z_1)(z_2)(z_1, z_2)$. *the limit singular Monge-Ampère equation locally looks like*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = \frac{\Omega}{|z_1|^{2\alpha}|z_2|^{2\alpha}(|z_1|^2 + |z_2|^2)^\alpha},$$

where Ω *is a nonvanishing smooth volume form near* p *and* $\alpha = 1 - R(X) = \frac{4}{25}$. *The author does not know a candidate singular Kähler metric as local model yet. See Remark 34.*

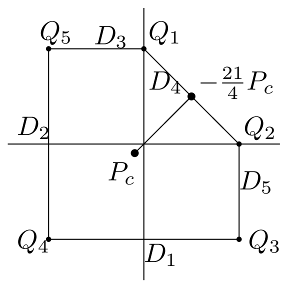
For the multiplier ideal sheaf, it's easy to see that $\hat{\mathcal{F}} = Q_4$. *The corresponding section* $\hat{s}$ *has divisor given by*

$$\{\hat{s} = 0\} = 2(D_3 + D_5) + 3D_4$$

So

$$\mathcal{I}(\gamma\phi_\infty) = \mathcal{I}(\gamma(2(D_3 + D_5) + 3D_4))$$

So when $\gamma < 1/3$, *then multiplier ideal scheme is empty. When* $1/2 \leq \gamma < 1/3$, *the support of multiplier ideal scheme is* D_4. *When* $\gamma \geq 1/2$, *the support of multiplier ideal scheme is* $D_3 \cup D_4 \cup D_5$.



4.7 Log-Futaki invariant for 1psg on toric Fano variety

Donaldson made a conjecture relating the two continuity methods.

Conjecture 1. *[Don6] There is a cone-singularity solution ω_β to $(*)_\beta$ for any parameter $\beta \in (0, R(X))$. If $R(X) < 1$, there is no solution for parameter $\beta \in (R(X), 1)$.*

The case of conic Riemann surface was known by the work of Troyanov, McOwen, Thurston, Luo-Tian, etc. we can provide more evidence on toric Fano manifolds:

Theorem 22. *[Li5] Let $X_\triangle$ be a toric Fano variety with a $(\mathbb{C}^*)^n$ action. Let Y be a general hyperplane section of $X_\triangle$. When $\beta < R(X_\triangle)$, $(X_\triangle, \beta Y)$ is log-K-stable along any 1 parameter subgroup in $(\mathbb{C}^*)^n$. When $\beta = R(X_\triangle)$, $(X_\triangle, \beta Y)$ is semi-log-K-stable along any 1 parameter subgroup in $(\mathbb{C}^*)^n$ and there is a 1 parameter subgroup in $(\mathbb{C}^*)^n$ which has vanishing log-Futaki invariant. When $\beta > R(X_\triangle)$, $(X_\triangle, \beta Y)$ is not log-K-stable.*

Let $\{p_\alpha;\ \alpha = 1, \cdots, N\}$ be all the lattice points of $\triangle$. Each p_α corresponds to a holomorphic section $s_\alpha \in H^0(X_\triangle, K^{-1}_{X_\triangle})$. We can embed $X_\triangle$ into $\mathbb{P}^N$ using $\{s_\alpha\}$. Define u to be the potential on $(\mathbb{C}^*)^n$ for the pull back of Fubini-Study metric (i.e. $\sqrt{-1}\partial\bar{\partial}u = \omega_{FS}$):

$$u = \log\left(\sum_{\alpha=1}^{N} e^{<p_\alpha, x>}\right) + C \tag{4.53}$$

C is some constant determined by normalization condition:

$$\int_{\mathbb{R}^n} e^{-u}dx = Vol(\triangle) = \frac{1}{n!}\int_{X_\triangle} \omega^n = \frac{c_1(X_\triangle)^n}{n!}$$

By the above normalization of u, it's easy to see that

$$e^{h_\omega} = \frac{|\cdot|^2_{FS}}{|\cdot|^2_{\omega^n}} = \frac{e^{-u}}{\omega^n/(\frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \cdots \wedge \frac{dz_n}{z_n} \wedge \frac{d\bar{z}_n)}{\bar{z}_n}}$$

So

$$h_\omega = -\log\det(u_{ij}) - u \tag{4.54}$$

Now let's calculate the log-Futaki invariant for any 1-parameter subgroup in $(\mathbb{C}^*)^n$. Each 1-parameter subgroup in $(\mathbb{C}^*)^n$ is determined by some $\lambda \in \mathbb{R}^n$ such that the generating holomorphic vector field is

$$v_\lambda = \sum_{i=1}^{n} \lambda_i z_i \frac{\partial}{\partial z_i}$$

A general Calabi-Yau hypersurface $Y \in |-K_X|$ is a hyperplane section given by the equation:

$$s := \sum_{\alpha=1}^{N} b(p_\alpha) z^{p_\alpha} = 0$$

By abuse of notation, we denote $\lambda(t)$ to be the 1 parameter subgroup generated by v_λ, then

$$\lambda(t) \cdot s = \sum_{\alpha=1}^{N} b(p_\alpha) t^{-\langle p_\alpha, \lambda \rangle} z^{p_\alpha} \tag{4.55}$$

Let

$$W(\lambda) = max_{p \in \triangle} \langle p, \lambda \rangle$$

Then $H_\lambda = \{p \in \mathbb{R}^n, \langle p, \lambda \rangle = W(\lambda)\}$ is a supporting plane of $\triangle$, and

$$\mathcal{F}_\lambda := \{p \in \triangle; \langle p, \lambda \rangle = W(\lambda)\} = H_\lambda \cap \triangle$$

is a face of $\triangle$.

We have $\lim_{t \to 0}[s] = \left[s_0 := \sum_{p_\alpha \in \mathcal{F}_\lambda} b(p_\alpha) z^{p_\alpha} \right]$, and by (4.55), the C^*-weight of s_0 is $-W(\lambda)$.

Proposition 30. *Let $F(K_X^{-1}, \beta Y)(\lambda)$ denote the Futaki invariant of the test configuration associated with the 1 parameter subgroup generated by v_λ. We have*

$$F(K_X^{-1}, \beta Y)(\lambda) = -\left(\beta \langle P_c, \lambda \rangle + (1-\beta) W(\lambda)\right) Vol(\triangle) \tag{4.56}$$

Proof. We will use the algebraic definition of log-Futaki invariant (3.29) to do the calculation.

Note that (X, Y, K_X^{-1}) degenerates to (X, Y_0, K_X^{-1}) under λ.

Y_0 is a hyperplane section of X, and $s_0 \in H^0(X, K_X^{-1})$ is the defining section, i.e. $Y_0 = \{s_0 = 0\}$. Then

$$H^0(Y_0, K_X^{-1}|_{Y_0}^k) \cong H^0(X, K_X^{-k}) / (s_0 \otimes H^0(X, K_X^{-(k-1)}))$$

So

$$\tilde{w}_k = w_k - (w_{k-1} - W(\lambda) d_{k-1})$$

Plugging the expansions, we get

$$\tilde{a}_0 = (n+1) a_0 + W(\lambda) b_0$$

Note that $\tilde{b}_0 = nb_0 = nVol(\triangle)$, we have

$$-\tilde{a}_0 + \frac{\tilde{b}_0}{b_0} a_0 = -a_0 - W(\lambda) b_0$$

where

$$-a_0 = \int_X \theta_v \frac{\omega^n}{n!} = \int_{\mathbb{R}^n} \sum_i \lambda_i u_i \det(u_{ij}) dx = \int_\triangle \sum_i \lambda_i y_i dy = Vol(\triangle)\langle P_c, \lambda\rangle$$

By (4.54), the ordinary Futaki invariant is given by

$$\begin{aligned} F(c_1(X))(v_\lambda) &= \int_X v(h_\omega) \frac{\omega^n}{n!} = -\int_{\mathbb{R}^n} \sum_{i=1}^n \lambda_i \frac{\partial u}{\partial x_i} \det(u_{ij}) dx \\ &= -\int_\triangle \sum_i \lambda_i y_i dy = -Vol(\triangle)\langle P_c, \lambda\rangle \end{aligned}$$

Substituting these into (3.29), we get

$$\begin{aligned} F(K_X^{-1}, \beta Y)(\lambda) &= -Vol(\triangle)\langle P_c, \lambda\rangle + (1-\beta)(Vol(\triangle)\langle P_c, \lambda\rangle - W(\lambda)Vol(\triangle)) \\ &= -(\beta\langle P_c, \lambda\rangle + (1-\beta)W(\lambda))Vol(\triangle) \end{aligned}$$

□

Proof of Theorem 22. Note that for any $P_\lambda \in \mathcal{F}_\lambda \subset \partial\triangle$, $W(\lambda) = \langle P_\lambda, \lambda\rangle$. By Theorem 19, we have

$$\begin{aligned} F(K_X^{-1}, \beta Y)(\lambda) &= \left(\frac{\beta}{1-\beta} \frac{1-R(X)}{R(X)} \langle Q, \lambda\rangle - W(\lambda)\right)(1-\beta)Vol(\triangle) \\ &= \langle Q_\beta - P_\lambda, \lambda\rangle \end{aligned}$$

where $Q_\beta = \frac{\beta}{1-\beta}\frac{1-R(X)}{R(X)} Q$.

Note that λ is a outward normal vector of H_λ. By convexity of $\triangle$, it's easy to see that (see the picture after Example 2)

- $\beta < R(X)$: $Q_\beta \in \triangle^\circ$. For any $\lambda \in \mathbb{R}^n$, $\langle Q_\beta - P_\lambda, \lambda\rangle < 0$.
- $\beta = R(X)$: $Q_\beta = Q \in \partial\triangle$. For any $\lambda \in \mathbb{R}^n$, $\langle Q_\beta - P_\lambda, \lambda\rangle \le 0$. Equality holds if and only if $\langle Q, \lambda\rangle = W(\lambda)$, i.e. H_λ is a supporting plane of $\triangle$ at point Q.
- $\beta > R(X)$: $Q_\beta \notin \overline{\triangle}$. There exists $\lambda \in \mathbb{R}^n$ such that $\langle Q_\beta - P_\lambda, \lambda\rangle > 0$

$\square$

4.7.1 Example

1. $X_\triangle = Bl_p\mathbb{P}^2$. See the picture in Introduction. $P_c = \frac{1}{4}(\frac{1}{3}, \frac{1}{3})$, $Q = -6P_c \in \partial\triangle$, so $R(X) = \frac{6}{7}$.

   If we take $\lambda = \langle -1, -1\rangle$, then $W(\lambda) = 1$. So by (4.56)

   $$F(K_X^{-1}, \beta Y)(\lambda) = \frac{2}{3}\beta - 4(1-\beta)$$

   So $F(K_X^{-1}, \beta Y)(\lambda) \leq 0$ if and only if $\beta \leq \frac{6}{7}$, and equality holds exactly when $\beta = \frac{6}{7}$.

2. $X_\triangle = Bl_{p,q}\mathbb{P}^2$, $P_c = \frac{2}{7}(-\frac{1}{3}, -\frac{1}{3})$, $Q = -\frac{21}{4}P_c \in \partial\triangle$, so $R(X_\triangle) = \frac{21}{25}$.

   If we take $\lambda_1 = \langle 1, 1\rangle$, then $W(\lambda_1) = 1$. By (4.56),

   $$F(K_X^{-1}, \beta Y)(\lambda_1) = \frac{2}{3}\beta - \frac{7}{2}(1-\beta)$$

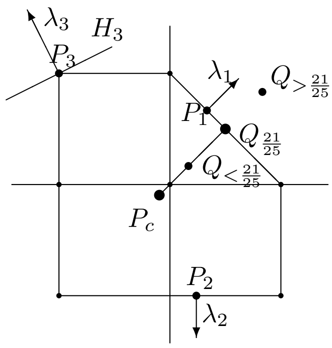
   $F(K_X^{-1}, \beta Y)(\lambda_1) \leq 0$ if and only if $\beta \leq \frac{21}{25}$.

   This is essentially the same as Donaldson's calculation in [Don6].

   If we take $\lambda_3 = \langle -1, 2\rangle$, then $W(\lambda_3) = \langle -1, 2\rangle \cdot \langle -1, 1\rangle = 3$. By (4.56)

   $$F(K_X^{-1}, \beta Y)(\lambda_3) = \frac{1}{3}\beta - \frac{21}{2}(1-\beta)$$

   So $F(K_X^{-1}, \beta Y)(\lambda_3) \leq 0$ if and only if $\beta \leq \frac{63}{65}$ which means that $(X, \beta Y)$ is log-K-stable along λ_3 when $\beta \leq \frac{21}{25} < \frac{63}{65}$.



Chapter 5

Intersection formula of Donaldson-Futaki invariant and applications

5.1 Introduction

K-stability is defined via the notion of test configurations. They are just $\mathbb{C}^*$-equivariant degenerations of the polarized manifold (X, L), usually denoted by $(\mathcal{X}, \mathcal{L})$. When the central fibre $\mathcal{X}_0$ is normal, Ding-Tian [DiTi] defined generalized Futaki invariant by extending the differential geometric defining formula from smooth case to normal case. Later Donaldson [Don4] defined Futaki invariant algebraically so that one can define it for any test configuration as scheme which may be very singular. However, motivated by compactness result for Kähler-Einstein metrics (cf. [CCT]), Tian conjectured that

Conjecture 2 (Tian's conjecture)**.** *When X is Fano, one only needs to consider those test configurations with normal $\mathbb{Q}$-Fano central fibers.*

As in [Tia9], we will call such a test configuration to be a **Special Test Configuration**. Recall that, in birational geometry, a $\mathbb{Q}$-Fano variety is a Kawamata Log Terminal (klt) Fano variety.

In this Chapter, we prove Theorem 24 which verify Tian's conjecture. This will be an immediate consequence of the following theorem proved jointly with Dr. Chenyang Xu.

Theorem 23. *[LiXu] Let X be a $\mathbb{Q}$-Fano variety. Assume $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a test configuration of $(X, -rK_X)$. We can construct a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ and a positive integer m, such that $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ is a and*

$$m\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s}).$$

Furthermore, if we assume $\mathcal{X}$ is normal, then the equality holds only when $(\mathcal{X}, \mathcal{X}_0)$ itself is a special test configuration.

As a immediate corollary, we can verify Tian's conjecture:

Theorem 24 (Tian's conjecture)**.** *([LiXu]) Assume X is a $\mathbb{Q}$-Fano variety. If X is destablized by a test configuration, then X is indeed destablized by a special test configuration. More precisely, the following two statements are true.*

1. *(unstable case) If $(X, -rK_X)$ is not K-semi-stable, then there exists a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ with a negative Futaki invariant $\mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s}) < 0$.*

2. *(semistable\ polystable case) Let X be a K-semistable variety. If $(X, -rK_X)$ is not K-polystable, then there exists a special test configuration $(\mathcal{X}^{st}, -rK_{\mathcal{X}^s})$ with Donaldson-Futaki invariant 0 such that $\mathcal{X}^s$ is not isomorphic to $X \times \mathbb{A}^1$.*

To prove the above result, we first derive an intersection formula for Donaldson-Futaki invariant. Then we use various birational transformations in MMP to modify the test configuration and prove the Donaldson-Futaki invariant is decreasing along the process. The end product will be $\mathcal{X}^s$.

5.2 Intersection formula for the Donaldson-Futaki invariant

Given any test configuration $(\mathcal{X}, \mathcal{L})$, we first compactify it. So we want to glue $(\mathcal{X}, \mathcal{L})$ with $(X \times \mathbb{A}^1, p_1^* L)$. In this section, we will compute the Donaldson-Futaki invariants by the intersection formula on this compactified space. The same formula appeared before (see [Wang] and [Odak2]). We include a proof here using Donaldson's argument.

Example 5. *$\mathbb{G}_m$ acts on $(X, L^{-1}) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$ by*

$$t \circ ([Z_0, Z_1], \lambda(Z_0, Z_1)) = ([Z_0, tZ_1], \lambda(Z_0, tZ_1)).$$

In particular, the $\mathbb{G}_m$-weights on

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_0, \mathcal{O}_{\mathbb{P}^1}(1)|_0, \mathcal{O}_{\mathbb{P}^1}(-1)|_\infty \text{ and } \mathcal{O}_{\mathbb{P}^1}(1)|_\infty$$

are 0,0,1 and -1. Let $\tau_0 = Z_1$, $\tau_\infty = Z_0$ be the holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Then the $\mathbb{G}_m$-weights of τ_0 and τ_∞ are -1 and 0.

Take $\bar{\mathcal{X}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ and $\bar{\mathcal{L}} = \mathcal{O}_{\bar{\mathcal{X}}}(1) = \mathcal{O}_{D_\infty}$, where D_∞ is the divisor at infinity. We see that $(\mathcal{X} := \bar{\mathcal{X}} \setminus \mathbb{P}^1_\infty, \mathcal{L} := \bar{\mathcal{L}}|_{\mathcal{X}})$ yields a test configuration of (X, L). Then $H^0(\mathbb{P}^1, L^{\otimes k})$ is of dimension $d_k = k+1$ and by the calculation in the first paragraph the total $\mathbb{G}_m$-weight of $H^0(\mathbb{P}^1, L^{\otimes k})$ is $w_k = -\frac{1}{2}(k^2+k)$. We know $D_\infty^2 = -1$ and $K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty = 1$. So

$$w_k = \frac{D_\infty^2}{2}k^2 + \left(\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1\right)k \qquad \textit{and} \qquad \mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{D_\infty^2}{2} - \left(\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1\right) (= 0).$$

This example can be generalized to more general cases (see (5.2), (5.3)) by using Donaldson's argument in the following way. Also see the proof of Proposition 4.2.1 in [Don4].

First note that, after identifying the fiber $\mathcal{X}_1$ over $\{1\}$ and X, we have an equivariant isomorphism:

$$(\mathcal{X} \backslash \mathcal{X}_0, \mathcal{L}) \cong (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^* L)$$

by $(p, a, s) \to (a^{-1} \circ p, a, a^{-1} \circ s)$. Therefore, $\mathbb{G}_m$ acts on the right hand side by

$$t \circ (\{p\} \times \{a\}, s) = (\{p\} \times \{ta\}, s)$$

for any $p \in X$, $a \in \mathbb{A}^1$ and $s \in \mathcal{L}_p$. The gluing map is given by

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{L}) & & (X \times \mathbb{P}^1 \setminus \{0\}, p_1^* L) \\ \bigcup & & \bigcup \\ (\mathcal{X} \backslash \mathcal{X}_0, \mathcal{L}) & \longrightarrow & (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^* L) \\ & & \\ (p, a, s) & \longmapsto & (\{a^{-1} \circ p\} \times \{a\}, a^{-1} \circ s), \end{array}$$

where $\mathbb{G}_m$ only acts by multiplication on the factor $\mathbb{P}^1 \setminus \{0\}$ of $(X \times \mathbb{P}^1 \setminus \{0\}, p_1^* L)$.

Using the above gluing map, we get a compact complex manifold projective over $\mathbb{P}^1$: $\bar{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$. In the following, we will denote $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ by $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ for simplicity. Note that there exists an integer N, such that $\bar{\mathcal{M}} = \bar{\mathcal{L}} \otimes \bar{\pi}^*(\mathcal{O}_{\mathbb{P}^1}(N \cdot \{\infty\}))$ is ample on $\bar{\mathcal{X}}$ (cf. [KoMo, 1.45]).

We need the following weak form of the Riemann-Roch formula whose proof is well known.

Lemma 30. *Let X be a normal projective variety and L an ample divisor on X then*

$$\dim H^0(X, L^{\otimes k}) = \frac{L^n}{n!}k^n + \frac{1}{2}\frac{K_X^{-1}\cdot L^{n-1}}{(n-1)!}k^{n-1} + O(k^{n-2}).$$

We define

$$d_k = \dim H^0(X, L^{\otimes k}) =: a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

Theorem 25. *Assume $\mathcal{X}$ is normal, then*

$$\frac{a_1 b_0 - a_0 b_1}{a_0^2} = \mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)!a_0}\left(\frac{a_1}{a_0}\bar{\mathcal{L}}^{n+1} + \frac{n+1}{2}K_{\overline{\mathcal{X}}/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^n\right). \tag{5.1}$$

Proof. For $k \gg 0$, by Kodaira Vanishing Theorem, we have two exact sequences:

$$\begin{array}{ccccccc}
 & A & & B & & C & \\
 & \| & & \| & & \| & \\
0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\mathcal{X}_0)) & \xrightarrow{\otimes\sigma_0} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\mathcal{X}_0, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0}) & \longrightarrow 0 \\
 & & & & & & \\
0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\mathcal{X}_\infty)) & \xrightarrow{\otimes\sigma_\infty} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\mathcal{X}_\infty, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_\infty}) & \longrightarrow 0, \\
 & \| & & \| & & \| & \\
 & A & & B & & D &
\end{array}$$

where σ_0, σ_∞ are sections of $\bar{\pi}^*\mathcal{O}_{\mathbb{P}^1}(1)$ which are pull back of the divisors $\{0\}, \{\infty\}$ on $\mathbb{P}^1$.

We can assume the $\mathbb{G}_m$-weights of σ_0 and σ_∞ are -1 and 0. Note the first terms in the two exact sequences are the same as $A := H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k} \otimes \bar{\pi}^*\mathcal{O}_{\mathbb{P}^1}(-1))$. We have the equation: $w_B = w_A - d_A + w_C = w_A + w_D$, where we denote d_A and w_A to be the dimension and the $\mathbb{G}_m$-weight of the vector space A and similarly for d_B etc. Since the $\mathbb{G}_m$-weight of $\mathcal{O}_{\mathbb{P}^1}(1)|_\infty$ is -1 and $\mathbb{G}_m$ acts on $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ trivially, we have $w_D = -kN\mathrm{dim}H^0(\bar{\mathcal{X}}_\infty, \bar{\mathcal{L}}^k|_{\bar{\mathcal{X}}_\infty})$. So we get

$$w_C = d_A + w_D = d_B - d_C - kNd_D = d_B - (kN+1)d_C,$$

In other words, we get the $\mathbb{G}_m$-weight on $H^0(\mathcal{X}_0, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0}) = H^0(\mathcal{X}_0, \bar{\mathcal{L}}^{\otimes k}|_{\mathcal{X}_0})$:

$$w_k = \dim H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) - (kN+1)\dim H^0(\mathcal{X}_0, \bar{\mathcal{L}}^{\otimes k}|_{\mathcal{X}_0}).$$

Expanding w_k, we get:

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$$

with

$$b_0 = \frac{\bar{\mathcal{M}}^{n+1}}{(n+1)!} - Na_0 = \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)!}, \text{ and} \tag{5.2}$$

$$b_1 = \frac{1}{2}\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot \bar{\mathcal{M}}^n}{n!} - Na_1 - a_0 = \frac{1}{2}\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot \bar{\mathcal{L}}^n}{n!} - a_0. \tag{5.3}$$

We get (5.1) by substituting the coefficients into (2.27). □

Remark 38. *This intersection formula is related to the interpretation of Donaldson-Futaki invariant as the CM-weight as in [PaTi2]. It was extensively used in [ALV], [Odak1], [RoTh1], etc.*

The intersection formula gives us a way to define the Donaldson-Futaki invariants when $\mathcal{L}$ is a π-big and semiample $\mathbb{R}$-divisor.

Definition 17. *(Donaldson-Futaki invariant for a semi-test configuation) Let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ be a $\mathbb{G}_m$-equivariant flat family with $\mathcal{L}$ being π-big and π-semiample, such that for $t \in \mathbb{A}^1 \backslash \{0\}$, $(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t}) \cong (X, -rK_X)$. We call $(\mathcal{X}, \mathcal{L})$ a semi-test configuration and $\mathcal{L}$ a semi-polarization. Assume $\mathcal{X}$ is normal. We define the Donaldson-Futaki invariant by the intersection formula:*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)!a_0}\left(\frac{a_1}{a_0}\bar{\mathcal{L}}^{n+1} + \frac{n+1}{2}K_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\right) \tag{5.4}$$

A frequent example is as follows: Suppose there is a $\mathbb{G}_m$-equivariant birational morphism between normal varieties $\rho : \mathcal{X}'^{tc} \to \mathcal{X}$ over $\mathbb{A}^1$ and $\mathcal{L}' = \rho^*(\mathcal{L})$ where $(\mathcal{X}, \mathcal{L})$ is a test configuration. If ρ is an isomorphism over the preimage images of $\mathbb{A}^1 \setminus \{0\}$, then $\mathcal{L}'$ is big and semi-ample but not ample in general. So $(\mathcal{X}', \mathcal{L}')$ is a semi-test configuration of $(X, -rK_X)$. The projection formula and (5.1) imply that

$$\mathrm{DF}(\mathcal{X}', \mathcal{L}') = \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Remark 39. *For any relative big and semi-ample line bundle, this definition of Futaki invariant coincides with the definition via computing the $\mathbb{G}_m$-weights of cohomological groups as in [ALV]. For more details, see [RoTh1] and [ALV].*

Remark 40. *Using the above argument, we can also get the intersection formula in the log case. Any test configuration can be equivariantly embedded into $\mathbb{P}^N \times \mathbb{C}^*$ where the $\mathbb{C}^*$ action on*

$\mathbb{P}^N$ is given by a 1 parameter subgroup of $SL(N+1,\mathbb{C})$. If Y is any subvariety of X, the test configuration of (X,L) also induces a test configuration $(\mathcal{Y},\mathcal{L}|_{\mathcal{Y}})$ of $(Y,L|_Y)$. Let d_k, $\tilde{d}_k$ be the dimensions of $H^0(X,L^k)$, $H^0(Y,L|_Y^k)$, and w_k, $\tilde{w}_k$ be the weights of $\mathbb{C}^*$ action on $H^0(X_0,\mathcal{L}|_{X_0}^k)$, $H^0(Y_0,\mathcal{L}|_{Y_0}^k)$, respectively. Then we have expansions:

$$d_k = a_0k^n + a_1k^{n-1} + O(k^{n-2}), \quad a_0 = \frac{L^n}{n!}, \quad a_1 = -\frac{K_X\cdot L^{n-1}}{2(n-1)!}$$

$$w_k = b_0k^{n+1} + b_1k^n + O(k^{n-1}), \quad b_0 = \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)!}, \quad b_1 = -\frac{K_{\bar{\mathcal{X}}}\cdot \bar{\mathcal{L}}^n}{2n!} - a_0$$

$$\tilde{d}_k = \tilde{a}_0k^{n-1} + O(k^{n-2}), \quad \tilde{a}_0 = \frac{L^{n-1}\cdot Y}{(n-1)!}$$

$$\tilde{w}_k = \tilde{b}_0k^n + O(k^{n-1}), \quad \tilde{b}_0 = \frac{\bar{\mathcal{L}}^n\cdot\bar{\mathcal{Y}}}{n!}$$

$$a_1^{\log} = a_1 - \frac{1}{2}\tilde{a}_0, \quad b_1^{\log} = b_1 - \frac{1}{2}\tilde{b}_0$$

$$K_{\bar{\mathcal{X}}}^{\log} = K_{\bar{\mathcal{X}}} + \bar{\mathcal{Y}} + \bar{\mathcal{X}}_0$$

So we can calculate the intersection formula for the log-Futaki invariant:

$$\begin{aligned}
\mathrm{DF}(\mathcal{X},\mathcal{Y},\mathcal{L}) &= \frac{2(a_1b_0 - a_1b_0) + (a_0\tilde{b}_0 - \tilde{a}_0b_0)}{a_0^2}\\
&= \frac{(2a_1-\tilde{a}_0)b_0 - a_0(2b_1-\tilde{b}_0)}{a_0^2}\\
&= \frac{1}{a_0(n+1)!}\left[c_1\bar{\mathcal{L}}^{n+1} + (n+1)(K_{\bar{\mathcal{X}}}+\bar{\mathcal{Y}})\cdot\bar{\mathcal{L}}^n\right] + 2\\
&= \frac{1}{a_0(n+1)!}\left[\frac{2a_1^{\log}}{a_0}\bar{\mathcal{L}}^{n+1} + (n+1)K^{\log}_{(\bar{\mathcal{X}},\bar{\mathcal{Y}})/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^n\right]
\end{aligned}$$

$$c_1 = \frac{a_1^{\log}}{a_0} = \frac{2a_1-\tilde{a}_0}{a_0} = \frac{n(K_X+Y)\cdot L^{n-1}}{L^n}$$

5.2.1 Song-Weinkove's condition from intersection formula

As a preliminary application of the intersection formula, we prove (X,L) is K-stable in the special case when $c_1(X)<0$ and L satisfies the inequality d. Following [Odak1], we only need to consider the blow up of $X\times\mathbb{P}^1$ along a flag ideal

$$\mathcal{J} = J_0 + tJ_1 + \cdots + t^kJ_k$$

Let $\mathcal{X} := \mathrm{Bl}_{\mathcal{J}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1$ be the blow up of along $\mathcal{J}$. Let $\mathcal{L}_t = \pi^* L - tE$ we need to show that

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}_t) \geq 0, \text{ for } 0 < t < \text{Seshadri constant of } \mathcal{J} \text{ with respect to } L$$

Following [Odak1], we first rewrite the intersection formula for Donaldson-Futaki invariant

$$\begin{aligned}
\mathrm{DF}(\mathcal{X}, \mathcal{L}_t) &= \frac{2a_1}{a_0}(L(-tE))^{n+1} + (n+1)K_X(L(-tE))^n + (n+1)K_{\mathcal{X}/(X\times\mathbb{P}^1)}(L(-tE))^n \\
&= \mathrm{I}(t) + \mathrm{II}(t)
\end{aligned}$$

Since $X \times \mathbb{P}^1$ is smooth, so $K_{\mathcal{X}/(X\times\mathbb{P}^1)}$ is effective so we have $\mathrm{II}(t) \geq 0$. Let $n\mu = 2a_1/a_0$, then

$$\frac{d}{dt}\mathrm{I}(t) = -(n+1)n\mathcal{L}_t^{n-1} \cdot (\mu\mathcal{L}_t + K_X) \cdot E$$

Note that $\frac{d}{dt}\mathrm{I}(0) = 0$. We can differentiate again:

$$\begin{aligned}
\frac{d^2}{dt^2}\mathrm{I}(t) &= n(n+1)\left(n\mu\mathcal{L}_t^{n-1} + (n-1)K_X \cdot \mathcal{L}_t^{n-2}\right) \cdot E^2 \\
&= -n(n+1)\left(-n\frac{c_1(X)\cdot[L]^{n-1}}{[L]^n}\mathcal{L}_t - (n-1)K_X\right) \cdot \mathcal{L}_t^{n-2} \cdot E^2 \qquad (5.5)
\end{aligned}$$

One should compare this intersection product with [Theorem 1.2 in [SoWe]]

Theorem 26 (Song-Weinkove)**.** *Suppose that X satisfies $c_1(X) < 0$. Let V be the cone of all Kähler class $[\chi_0]$ with the property that there exist metrics ω in $-c_1(X)$ and χ' in $[\chi_0]$ with*

$$\left(-n\frac{c_1(X)\cdot[\chi_0]^{n-1}}{[\chi_0]^n}\chi' - (n-1)\omega\right) \wedge \chi'^{n-2} > 0$$

Then the Mabuchi energy is proper on every class $[\chi_0]$ in V.

So by discussion in section 2.7.2, (X, L) is K-stability. This should be able to follow from the positivity of (5.5) for $0 < t <$ Seshadri constant of $\mathcal{J}$ with respect to L.

5.3 Special degeneration and K-stability of Fano manifold

In this section, we will prove Theorem 23 by simplifying the test configuration through various birational transformations from MMP.

5.3.1 Preliminaries from algebraic geometry

Definition and Theorem 1 (Relative MMP with scaling). *([BCHM], [KoMo])*

Let $(X^0, \Delta^0) := (X, \Delta)$ be a klt pair, projective over a smooth curve C. Let H be an ample line bundle such that $\Delta + H$ is nef. Assume $K_X + \Delta$ is not pseudo-effective. Let

$$\mu = \min\{t; K_X + \Delta + tH \text{ pseudo-effective } \}.$$

We can define a sequence rational numbers $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_K > \lambda_{K+1} = \mu$ and a sequence of models over C related by birational transformations:

$$(X^0, \Delta^0) \dashrightarrow (X^i, \Delta^i) \dashrightarrow (X^K, \Delta^K)$$

by the following inductive process. Assume (X^i, Δ^i) is defined. Let H^i be the birational transform of H on X^i. Define

$$\lambda_{i+1} = \min\{t \in [\mu, 1]; K_{X^i} + \Delta^i + tH^i \text{ is nef } \}.$$

So there exists an extremal ray $R \in \overline{NE}(X^i/C)$ such that $(K_{X^i} + \Delta^i) \cdot R = -H^i \cdot R < 0$. We can contract R by a contraction map $\pi_R : X^i \to Y$. There are two possibilities:

1. *π_R is divisorial contraction. Let $X^{i+1} = Y$ and $\Delta^{i+1} = (\pi_R)_* \Delta^i$.*

2. *π_R is a flipping contraction. Let $f : X^i \dashrightarrow X^{i+}$ be the $(K_{X^i} + \Delta^i)$-flip. Define $X^{i+1} = X^{i+}$ and $\Delta^{i+1} = f_* \Delta^i$.*

On the end product $(\mathcal{X}^k, \Delta^k)$ we have $K_{X^k} + \Delta^k + \mu H^k \sim_{\mathbb{Q},C} 0$.

The following two theorem are very important in the theory of MMP. We will frequently use in the following argument too.

Theorem 27 (Relative base point free theorem). *Let $(X, \triangle)$ be a klt pair, $\triangle$ effective, and $f : X \to Y$ a proper morphism of projective varieties over $\mathbb{C}$. Let D be an f-nef Cartier divisor such that $aD - K_X - \triangle$ is f-nef and f-big for some $a > 0$. Then bD is f-free for all $b \gg 0$.*

Lemma 31 (Negativity Lemma). *([KoMo, 3.39]) Let $h : Z \to Y$ be a proper birational morphism between normal varieties. Let $-B$ is an h-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z. Then*

1. *B is effective iff $h_* B$ is.*

2. *Assume B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \mathrm{Supp}B$ or $h^{-1}(y) \cap \mathrm{Supp}B = \emptyset$.*

We also need the higher dimensional version of the Zariski Lemma in the following calculation of Donaldson-Futaki invariant. See [LiXu].

Lemma 32 (Zariski's Lemma). *Let $\mathcal{X} \to C$ be a projective dominant morphism from a n-dimensional normal variety to a proper smooth curve. Let E be a $\mathbb{Q}$-divisor which supports on some fiber $\mathcal{X}_0$. Let $\mathcal{L}_1, .., \mathcal{L}_{n-2}$ be $n-2$ nef divisors on $\mathcal{X}$. Then*

$$E^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-2} \le 0.$$

If all $\mathcal{L}_i$'s are ample, then the equality holds if and only if $E = t\mathcal{X}_0$ for some $t \in \mathbb{Q}$.

5.3.2 Step 0: Normalization

Assume $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a test configuration. Let $\pi^\nu : \mathcal{X}^\nu \to \mathcal{X}$ be the normalization of $\mathcal{X}$. By [RoTh1, Proposition 5.1] and [ALV, Corollary 3.9],

$$DF(\mathcal{X}^\nu, (\pi^\nu)^*\mathcal{L}) \le DF(\mathcal{X}, \mathcal{L})$$

with equality holds if and only if $\mathcal{X}_{non-normal}$ (i.e. the set of non-normal points of $\mathcal{X}$ has codimension at least two.

Remark 41. *Any test configuration can be equivariantly embedded into $\mathbb{P}^N \times \mathbb{A}^1$. So it's induced by a one parameter subgroup $\lambda(t)$ of $SL(N+1, \mathbb{C})$. As we have seen in the end of Section 2.7.2, there are three cases for $\mathcal{X}_0$. For the 3rd case, in general, the test configuration $\mathcal{X}$ is non-normal (hence non-product), isomorphic to $X \times \mathbb{A}^1$ in codimension 1, with Donaldson-Futaki invariant 0. This case was missing in most of precious work, e.g. [Sto1], as pointed out in [LiXu]. See [Sto2], [Odak4] for related issues and corrections.*

From now on, we will assume the test configuration $\mathcal{X}$ is normal. Since we can naturally compactify any test configuration as in Section 5.2, we will consider more generally a polarized family over a proper smooth curve whose total space is normal.

Definition 18. *A polarized generically $\mathbb{Q}$-Fano family is a projective morphism $\pi : (\mathcal{X}, \mathcal{L}) \to C$ from a normal polarized variety $\mathcal{X}$ to a smooth curve, whose generic fibre $(\mathcal{X}_\eta, \mathcal{L}_\eta) \cong (\mathcal{X}_\eta, -rK_{\mathcal{X}_\eta})$*

is a $\mathbb{Q}$-Fano variety for some $r \in \mathbb{Q}$. If each fibre is a $\mathbb{Q}$-Fano variety, then $(\mathcal{X}, -rK_{\mathcal{X}/C})$ called a $\mathbb{Q}$-Fano family.

Definition 19. *Assume $(\mathcal{X}, \mathcal{L}) \to C$ is a polarized generically $\mathbb{Q}$-Fano family over a proper smooth curve. Assume for generic fibre $\mathcal{L}|_{\mathcal{X}_t} \sim_{\mathbb{Q}} -rK_{\mathcal{X}_t}$. We can define the Donaldson-Futaki intersection number as*

$$\mathrm{DF}(\mathcal{X}/C, \mathcal{L}) = \frac{1}{2(n+1)r^n(-K_{\mathcal{X}_t})^n}\left(\frac{n}{r}\mathcal{L}^{n+1} + (n+1)K_{\mathcal{X}/C}\cdot\mathcal{L}^n\right),$$

If $C = \mathbb{P}^1$ and $(\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ is the compactification of a $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ as in Section 5.2 by simply adding a 'trivial fiber' over the point $\infty \in \mathbb{P}^1$, then we have the equality

$$\mathrm{DF}(\bar{\mathcal{X}}/\mathbb{P}^1, \bar{\mathcal{L}}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Remark 42. *In the following, without loss of generality, we assume there is only one degenerate fibre over $0 \in C$, or $0' \in C'$. We also note that although the following step of modification was originally obtained without group action, they all can be carried out equivariantly. For explanations, see [LiXu, Section 2.3], [And], [KoMo].*

5.3.3 Step 1: Equivariant semi-stable reduction

Theorem 28. *[LiXu] There exists a finite morphism $\phi : C' \to C$ such that, if we denote $\tilde{\mathcal{X}}$ to be the main component of the normalization of $\mathcal{X} \times_C C'$, then*

1. *$\tilde{\mathcal{X}}_0$ is reduced.*
2. *$\mathrm{DF}(\tilde{\mathcal{X}}, \phi^*_{\tilde{\mathcal{X}}/\mathcal{X}}\mathcal{L}) \leq \deg(\phi)\mathrm{DF}(\mathcal{X}, \mathcal{L})$, where $\phi_{\tilde{\mathcal{X}}/\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is the natural finite morphism. The identity holds if and only if $\mathcal{X}_0$ is reduced.*
3. *then there exists a birational morphism $f : \mathcal{Y} \to \tilde{\mathcal{X}}$ such that $\mathcal{Y}$ is smooth and the degenerate fibre $\mathcal{Y}_0$ is simple normal crossing.*

Proof. The existence of semi-stable reduction (1. and 3.) was obtained in ([KKMS] and [KoMo]). We have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{Y} \arrow[dr] \arrow[drr, bend left, "\phi_{\mathcal{Y}/\mathcal{X}}"] \arrow[ddr, bend right] & & \\
& \tilde{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
& C' \arrow[r] & C
\end{tikzcd}$$

such that

- $\mathcal{Y}$ is smooth
- $\mathcal{Y}_0 = \sum_{i=1}^{N_1} \mathcal{Y}_{0,i}$ is simple normal crossing.

Because $K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0 = \phi_{\mathcal{X}}^*(K_{\mathcal{X}} + \mathrm{red}(\mathcal{X}_0))$ and $K_{C'} + \{0'\} = \phi^*(K_C + \{0\})$, we see that

$$K_{\tilde{\mathcal{X}}/C'} = \phi_{\mathcal{X}}^*(K_{\mathcal{X}/C} + \mathrm{red}(\mathcal{X}_0) - \mathcal{X}_0)$$

So 2. follows from projection formula. Equality holds if and only if $\mathcal{X}_0 = \mathrm{red}(\mathcal{X}_0)$ is reduced. $\square$

5.3.4 Step 2: Log canonical modification

Theorem 29. *[LiXu] Let $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \to C$ be the polarized generic $\mathbb{Q}$-Fano family such that $\tilde{\mathcal{X}}_0$ is reduced and there exists a birational morphism $\mathcal{Y} \to \tilde{\mathcal{X}}_0$ such that $\mathcal{Y}$ is smooth and $\mathcal{Y}_0$ is simple normal crossing. Then there exists a log canonical modification $\pi^{lc} : \mathcal{X}^{lc} \to \tilde{\mathcal{X}}$ and polarization $\mathcal{L}^{lc}$ on $\mathcal{X}^{lc}$ such that*

(1) $(\mathcal{X}^{lc}, \mathcal{X}_0^{lc})$ is log canonical

(2) $K_{\mathcal{X}^{lc}}$ is relatively ample over $\tilde{\mathcal{X}}$.

(3) $\mathrm{DF}(\mathcal{X}^{lc}, \mathcal{L}^{lc}) \le \mathrm{DF}(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$. The equality holds if and only if $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$ is log canonical.

Proof. $\mathcal{X}^{lc}$ is obtained by running $(K_{\mathcal{Y}} + \mathcal{Y}_0)$-minimal model program on $\mathcal{Y}$ over $\tilde{\mathcal{X}}$. So $\mathcal{X}^{lc} = \mathrm{Proj}\, R(\mathcal{Y}/\tilde{\mathcal{X}}, K_{\mathcal{Y}} + \mathcal{Y}_0)$. For details, see ([LiXu],[OdXu]). Let

$$\mathcal{L}_t = \frac{r}{r-t}\left(\pi^{lc*}\mathcal{L} + tK_{\mathcal{X}^{lc}}\right)$$

Since $K_{\mathcal{X}^{lc}}$ is relatively ample over $\tilde{\mathcal{X}}$, $\mathcal{L}_t$ is relatively ample over C if t is sufficiently small. Then $\frac{1}{r}\mathcal{L}_t + K_{\mathcal{X}^{lc}} = \frac{r}{r-t}\left(\frac{1}{r}\pi^{lc*}\mathcal{L} + K_{\mathcal{X}^{lc}}\right)$ and $\mathcal{L}_t' = \frac{r^2}{(r-t)^2}\left(\frac{1}{r}\pi^*\mathcal{L} + K_{\mathcal{X}^{lc}}\right)$. Let $E = \frac{1}{r}\pi^{lc*}\mathcal{L} + K_{\mathcal{X}^{lc}}$, then E is supported on $\mathcal{X}_0^{lc}$.

$$\begin{aligned}\frac{d}{dt}\mathrm{DF}(\mathcal{X}^{lc},\mathcal{L}_t) &= \frac{1}{(n+1)r^n(-K_{\mathcal{X}_t})^n}n(n+1)\mathcal{L}_t^{n-1}\left(\frac{1}{r}\mathcal{L}_t+K_{\mathcal{X}^{lc}}\right)\cdot\mathcal{L}_t' \\ &= \frac{nr^3}{r^n(r-t)^3(-K_{\mathcal{X}_t})^n}\cdot\mathcal{L}_t^{n-1}\cdot\left(\frac{1}{r}\pi^{lc*}\mathcal{L}+K_{\mathcal{X}^{lc}}\right)^2\leq 0\end{aligned}$$

for t sufficiently small. So we choose $\mathcal{L}^{lc}=\mathcal{L}_\epsilon$ for sufficiently small rational number ϵ. Then $\mathrm{DF}(\mathcal{X}^{lc},\mathcal{L}^{lc})\leq\mathrm{DF}(\tilde{\mathcal{X}},\tilde{\mathcal{L}})$. The equality holds only if $E\sim_{\mathbb{Q},C}\mathcal{X}_0^{lc}$, but then $K_{\mathcal{X}^{lc}}\sim_{\mathbb{Q},\mathcal{X}}0$, so $\mathcal{X}^{lc}$ is isomorphic to $\tilde{\mathcal{X}}$. □

5.3.5 Step 3: Running MMP with scaling

Now we let $\mathcal{X}^0=\mathcal{X}^{lc}$, $\mathcal{L}^0=\mathcal{L}^{lc}$. Given an exceptional divisor E, if its center dominates C then $a(E,\mathcal{X}^0)>-1$ because $\mathcal{X}^*$ is klt; if its center is vertical over C, then $a(E,\mathcal{X}^0)\geq 0$, since $(\mathcal{X}^0,\mathcal{X}_t^0)$ is log canonical for any t in C. In particular, $\mathcal{X}^0$ is klt.

Since $\mathcal{L}^0$ is ample, we can choose λ_0 sufficiently large such that $K_X+\lambda_0\mathcal{L}$ is ample. To simplify the family, we run a sequence of $K_{\mathcal{X}^0}$-MMP over C with scaling of $\mathcal{L}^0$ as in Subsection 5.3.1. (This is equivalent to running $(K_{\mathcal{X}^0}+\mathcal{L}^0)$-MMP with scaling $\mathcal{L}^0$. Compare with Section 5.3.9). So we obtain a sequence of models

$$\mathcal{X}^0 \dashrightarrow \mathcal{X}^1 \dashrightarrow \cdots \dashrightarrow \mathcal{X}^k.$$

Recall that, as in 5.3.1, we have a sequence of critical value of scaling factors

$$\lambda_{i+1}=\min\{\lambda \mid K_{\mathcal{X}^i}+\lambda\mathcal{L}^i \text{ is nef over } C\}$$

with $1=\lambda_0\geq\lambda_1\geq ...\geq\lambda_k>\lambda_{k+1}=\frac{1}{r}$. Note that $\frac{1}{r}$ is the pseudo-effective threshold of $K_{\mathcal{X}^0}$ with respect to $\mathcal{L}^0$ over C, since it is the pseudo-effective threshold for the generic fiber. Any $\mathcal{X}^i$ appearing in this sequence of $K_{\mathcal{X}^0}$-MMP with scaling of $\mathcal{L}^0$ is a relative weak log canonical model of $(\mathcal{X}^0,t\mathcal{L}^0)$ for any $t\in[\lambda_i,\lambda_{i+1}]$ (see [BCHM, 3.6.7] for the definition of weak log canonical model).

For $\lambda\in[\lambda_{i+1},\lambda_i]$, we denote by

$$\mathcal{L}_\lambda^i=\frac{r}{\lambda r-1}(K_{\mathcal{X}^i}+\lambda\mathcal{L}^i). \tag{5.6}$$

where $\mathcal{L}^i$ is the push forward of $\mathcal{L}^0$ to $\mathcal{X}^i$. As is clear from the context, this should not be confused with the i-th power or intersection product of $\mathcal{L}_\lambda$.

Lemma 33. *$-rK_{\mathcal{X}^k} \sim_{\mathbb{Q},C} \mathcal{L}^k$ is big and semi-ample over C.*

Proof. Since $\lambda_{k+1} = \frac{1}{r}$, the line bundle $K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}^k$ is relatively nef over C and its restriction to the generic fiber is trivial, so it is $\mathbb{Q}$-linearly equivalent to a linear sum of components of $\mathcal{X}^k_0$. By its nefness, we can apply Lemma 32 to get

$$K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}^k \sim_{\mathbb{Q},C} 0.$$

Now for any $\lambda \in [1/r, \lambda_{k+1}]$, $\lambda^k_\lambda \sim \mathcal{L}^k$ is nef. $\mathcal{L}^k$ is big because $\lambda_k > \frac{1}{r}$, and from the relative base-point free theorem (cf. Theorem 3.3 in [KoMo]), it is semi-ample over C. $\square$

By the above Lemma, we can define

$$\mathcal{X}^{an} = \mathrm{Proj}R(\mathcal{X}^k/C, \mathcal{L}^k) = \mathrm{Proj}R(\mathcal{X}^k/C, -rK_{\mathcal{X}^k/C}).$$

Since $(\mathcal{X}^0, \mathcal{X}^0_0)$ is log canonical and and $\mathcal{X}^0_0 = (f \circ \pi^{lc})^*(\{0\})$, this is a also a sequence of $(K_{\mathcal{X}^0} + \mathcal{X}^0_0)$-MMP and thus $(\mathcal{X}^k, \mathcal{X}^k_0)$ is log canonical which implies that $(\mathcal{X}^{an}, \mathcal{X}^{an}_0)$ is log canonical as well.

5.3.6 Decreasing of DF-intersection number

For any $\lambda > \frac{1}{r}$, the restriction of $K_{\mathcal{X}^0} + \lambda\mathcal{L}^0$ over C^* is ample. So the MMP with scaling does not change $\mathcal{X}^0 \times_C C^*$, i.e., $\mathcal{X}^0 \times_C C^* \cong \mathcal{X}^i \times_C C^*$ for any $i \le k$. Let's calculate the variation of DF-intersection number.

Proposition 31. *([LiXu]) With the notation above, we have*

$$\mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0) \ge \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}^k) = \mathrm{DF}(\mathcal{X}^k/C, -rK_{\mathcal{X}^k/C}) = \mathrm{DF}(\mathcal{X}^{an}/C, -rK_{\mathcal{X}^{an}/C})$$

The firt equality holds if and only if $h : \mathcal{X}^0 \dashrightarrow \mathcal{X}^k$ is an isomorphism.

Decreasing of DF on a fixed model

Assume $\mathcal{X}^i_0 = \sum_\alpha E^i_\alpha$, where E^i_α's are the prime divisors. Since $(\mathcal{X}^0, \mathcal{L}^0) \times_C C^*$ is isomorphic to $(\mathcal{X}^0 \times_C C^*, -rK_{\mathcal{X}^0 \times_C C^*})$, there exist $a^i_\alpha \in \mathbb{R}$ such that

$$K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}^i \sim_{\mathbb{R},C} \sum_{\alpha \in I} a^i_\alpha E_\alpha.$$

Let $\mathcal{Z}_\lambda$ be the relative log canonical model of $(\mathcal{X}^0, \lambda\mathcal{L}^0)$ over C. Then there is a morphism $\pi_\lambda : \mathcal{X}^i \to \mathcal{Z}_\lambda$ and an relatively ample $\mathbb{Q}$-divisor $\mathcal{M}_\lambda$ on $\mathcal{Z}_\lambda$ whose pull back is $\mathcal{L}^i_\lambda$.

Lemma 34. *If $\lambda_i \geq a > b \geq \lambda_{i+1}$ and $b > \frac{1}{r}$, then $\mathrm{DF}(\mathcal{X}^i, \mathcal{L}^i_a) \geq \mathrm{DF}(\mathcal{X}^i, \mathcal{L}^i_b)$. The inequality is strict if there is a rational number $\lambda \in [a, b]$, such that the push forward of $\sum_\alpha a^i_\alpha E_\alpha$ to $\mathcal{Z}_\lambda$ is not a multiple of the pull back of $0 \in C$ on $\mathcal{Z}_\lambda$.*

Proof. Note that

$$(\mathcal{L}^i_\lambda)' = \frac{d}{d\lambda}\mathcal{L}^i_\lambda = -\frac{r^2}{(\lambda r - 1)^2}(K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}^i), \quad K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}^i_\lambda = \frac{\lambda r}{\lambda r - 1}(K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}^i)$$

Now we compute the derivative of the Donaldson-Futaki invariants:

$$\begin{aligned}
\frac{d}{d\lambda}\mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}^i_\lambda) &= C_0\left((\mathcal{L}^i_\lambda)^{n-1} \cdot (\mathcal{L}^i_\lambda)' \cdot (\frac{1}{r}\mathcal{L}^i_\lambda + K_{\mathcal{X}^i})\right) \\
&= -C_0'(\mathcal{L}^i_\lambda)^{n-1} \cdot \left(K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}^i\right)^2 \\
&= -C_0'(\mathcal{M}^i_\lambda)^{n-1} \cdot \left(\sum_\beta a^i_\beta \tilde{E}_\beta\right)^2,
\end{aligned}$$

where C_0, C_0' are positive constants. Then the lemma follows from Lemma 32. $\square$

Lemma 35.

$$\lim_{\lambda \to +\infty} \mathrm{DF}(\mathcal{X}^0, \mathcal{L}^0_\lambda) = \mathrm{DF}(\mathcal{X}^0, \mathcal{L}^0)$$

Proof. This follows from $\lim_{\lambda \to +\infty} \mathcal{L}^0_\lambda = \mathcal{L}^0$ and the intersection formula. $\square$

The following corollary is a consequence of above two Lemmas.

Corollary 8.

$$\mathrm{DF}(\mathcal{X}^0, \mathcal{L}^0) \geq \mathrm{DF}(\mathcal{X}^0, \mathcal{L}^0_{\lambda_0})$$

Invariance of DF at contraction or flip points

If $\lambda_{i+1} > \frac{1}{r}$, then by the definition of MMP with scaling, we pick up a $K_{\mathcal{X}^i}$-negative extremal ray $[R]$ in $\mathrm{NE}(\mathcal{X}^i/C)$ such that $R \cdot (K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i) = 0$. we perform a birational transformation:

$$\begin{tikzcd} \mathcal{X}^i \arrow[rr, "f^i"] \arrow[dr] & & \mathcal{Y}^i \arrow[dl] \\ & \mathbb{P}^1, & \end{tikzcd}$$

which contracts all curves R' whose classes $[R']$ are in the ray $\mathbb{R}_{>0}[R]$. There are two cases:

1. (Divisorial Contraction) If f^i is a divisorial contraction. Then $\mathcal{X}^{i+1} = \mathcal{Y}^i$. Since f^i is a $(K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i)$-trivial morphism by the definition of the MMP with scaling, we have

$$K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i = (f^i)^*(K_{\mathcal{Y}^i} + \lambda_{i+1}\mathcal{L}^{i+1}),$$

which implies

$$\mathcal{L}^i_{\lambda_{i+1}} = (f^i)^*\mathcal{L}^{i+1}_{\lambda_{i+1}}.$$

Then it follows from projection formula that

$$\mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}^i_{\lambda_{i+1}}) = \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}^{i+1}_{\lambda_{i+1}}).$$

2. (Flipping Contraction) If f^i is a flipping contraction, let $\phi^i : \mathcal{X}^i \dashrightarrow \mathcal{X}^{i+1}$ be the flip.

$$\begin{tikzcd} \mathcal{X}^i \arrow[rr, dashed, "\phi^i"] \arrow[dr, "f^i"', "-K_{\mathcal{X}^i} \text{ is } f^i\text{-ample}"'] & & \mathcal{X}^{i+1} \arrow[dl, "f^{i+}", "K_{\mathcal{X}^{i+1}} \text{ is } f^{i+}\text{-ample}"'] \\ & \mathcal{Y}^i & \end{tikzcd}$$

As f^i is a $(K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i)$-trivial morphism, $(K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i) = (f^i)^*D_{\mathcal{Y}^i}$ for some divisor $D_{\mathcal{Y}^i}$. Since f^i, f^{i+}, ϕ^i are isomorphisms in codimension one, we also have $K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{L}^{i+1} = (f^{i+})^*D_{\mathcal{L}^i}$. Therefore, using the intersection formula, we see that

$$\mathrm{DF}(\mathcal{X}^i/C, K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{L}^i) = \mathrm{DF}(\mathcal{Y}^i/C, D_{\mathcal{Y}^i}) = \mathrm{DF}(\mathcal{X}^{i+1}/C, K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{L}^{i+1}).$$

Now we can finish the proof of Proposition 31:

Proof.

$$\begin{aligned}
\mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0) &\geq \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0_{\lambda_0}) \geq \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0_{\lambda_1}) \\
&= \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}^1_{\lambda_1}) \geq \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}^1_{\lambda_2}) \\
&\quad \cdots \quad \cdots \quad \cdots \\
&= \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}^i_{\lambda_i}) \geq \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}^i_{\lambda_{i+1}}) \\
&= \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}^{i+1}_{\lambda_{i+1}}) \geq \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}^{i+1}_{\lambda_{i+2}}) \\
&\quad \cdots \quad \cdots \quad \cdots \\
&= \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}^k_{\lambda_k}) = \mathrm{DF}(\mathcal{X}^k/C, -rK_{\mathcal{X}^k}).
\end{aligned}$$

Now we characterize the equality case. Since $-K_{\mathcal{X}^k} \sim_{\mathbb{Q},C} \frac{1}{r}\mathcal{L}^k$ is relatively nef over C, we conclude that $f^{k-1} : \mathcal{X}^{k-1} \to \mathcal{X}^k$ is a divisorial contraction contracting a divisor E. So if we let

$$aE = K_{\mathcal{X}^{k-1}} + \frac{1}{r}\mathcal{L}^{k-1}_{\lambda_k} - (f^{k-1})^*(K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}^k)$$

then because f^{k-1} is $(K_{\mathcal{X}^{k-1}} + \lambda_k \mathcal{L}^{k-1})$- trivial, for any curve $C \in E$,

$$aE \cdot C = \left(\frac{1}{r} - \lambda_k\right) \mathcal{L}^{k-1} \cdot C < 0$$

because $\lambda_k > \frac{1}{r}$. Since $E{\cdot}C < 0$, we have $a > 0$. Because $K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}^k \sim_{\mathbb{Q},C} 0$, $K_{\mathcal{X}^{k-1}} + \frac{1}{r}\mathcal{L}^{k-1}_{\lambda_k} \sim_{\mathbb{Q},C} aE$ whose support E is a proper subset of $\mathcal{X}^{k-1}_0$. So the equality condition of Lemma 34 can not hold on $\mathcal{X}^{k-1}$. Thus

$$\begin{aligned}
\mathrm{DF}(\mathcal{X}^{lc}/C, \mathcal{L}^{lc}) &\geq \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0_{\lambda_0}) \geq \mathrm{DF}(\mathcal{X}^{k-1}/C, \mathcal{L}^{k-1}_{\lambda_{k-1}}) \\
&> \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}^k_{\lambda_k}) = \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}^k) \\
&= \mathrm{DF}(\mathcal{X}^{an}/C, -rK_{\mathcal{X}^{an}}).
\end{aligned}$$

□

5.3.7 Step 4: $\mathbb{Q}$-Fano extension

From last section, we get some family $\mathcal{X}^k$ and its anti-canonical model $\mathcal{X}^{an}$. Now we need to use results from MMP to carry on. We collect the result we need in the following theorem.

Theorem 30. *[LiXu]*

1. *There exists a finite morphism $\phi : C' \to C$ and a birational morphism $\pi : \mathcal{X}' \to \tilde{\mathcal{X}} := \mathcal{X}^{an} \times_C C'$ such that, if $\mathcal{X}'_0 = \sum_i E_i$, then*

   - *We can find some polarization $\mathcal{L}'$ such that*

   $$\mathcal{L}' + K_{\mathcal{X}'/C'} = \sum_{E_i \neq E^s} a_i E_i$$

   with $a_i > 0$.

   - *π is a divisorial contraction contracting divisor E^s with $a(E^s, \tilde{\mathcal{X}}) = 0$.*

2. *We can run $(K_{\mathcal{X}'} + \mathcal{L}')$-MMP over C' and get a klt model $\mathcal{X}^s$ such that $\mathcal{X}^s_0$ is an irreducible $\mathbb{Q}$-Fano variety which is the strict transform of E^s.*

Theorem 31. *[LiXu]*

$$\mathrm{DF}(\tilde{\mathcal{X}}/C', -K_{\tilde{\mathcal{X}}/C'}) \geq \mathrm{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s})$$

with equality holds if and only if $\tilde{\mathcal{X}} \cong \mathcal{X}^s$.

Proof. The Donaldson-Futaki intersection number becomes very simple for relative anti-canonical polarization:

$$\begin{aligned}\mathrm{DF}(\tilde{\mathcal{X}}/C', -K_{\tilde{\mathcal{X}}/C'}) &= -\frac{1}{2(n+1)(-K_{\mathcal{X}_t})^n}(-K_{\tilde{\mathcal{X}}/C'})^{n+1} \\ &= -\frac{1}{2(n+1)(-K_{\mathcal{X}_t})^n}(-K_{\mathcal{X}'/C'})^{n+1}\end{aligned}$$

The second equality comes from $\pi^* K_{\tilde{\mathcal{X}}} = K_{\mathcal{X}'}$ because of $a(E^s, \tilde{\mathcal{X}}) = 0$. Similarly,

$$\mathrm{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s/C'}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}_t})^n}(-K_{\mathcal{X}^s/C'})^{n+1}$$

Let $p : \hat{\mathcal{X}} \to \mathcal{X}'$ and $q : \hat{\mathcal{X}} \to \mathcal{X}^s$ be common resolution. Define $E = p^* K_{\mathcal{X}'} - q^* K_{\mathcal{X}^s}$ so that

$$q^*(-K_{\mathcal{X}^s/C'}) = p^*(-K_{\mathcal{X}'/C'}) + E$$

Then $-E$ is q-nef and E is supported on $\hat{\mathcal{X}}_0$.

Let $\tilde{p} = \pi \circ p : \hat{\mathcal{X}} \to \tilde{\mathcal{X}}$. We can write $K_{\hat{\mathcal{X}}} = \tilde{p}^* K_{\tilde{\mathcal{X}}} + B = p^* K_{\mathcal{X}'} + B$ with B being exceptional over $\mathcal{X}'$. Because $\mathcal{X}' \dashrightarrow \mathcal{X}^s$ is a birational contraction, B is also exceptional over $\mathcal{X}^s$. Similarly,

$K_{\hat{\mathcal{X}}} = q^* K_{\mathcal{X}^s} + F$ with F exceptional over $\mathcal{X}^s$. So $E = -B + F$ is exceptional over $\mathcal{X}^s$. By the Negativity Lemma 31, $E \geq 0$.

If $E = 0$ or, $\mathcal{X}'$ and $\mathcal{X}^s$ are not isomorphic in codimension 1, then

$$\tilde{\mathcal{X}} = \mathrm{Proj}R(\mathcal{X}', -K_{\mathcal{X}'/C'}) = \mathrm{Proj}R(\mathcal{X}^s, -K_{\mathcal{X}^s/C'}) = \mathcal{X}^s$$

So we can assume $E > 0$ and $\mathcal{X}'$ is not isomorphic to $\mathcal{X}^s$ in codimension 1. Then $E > 0$ contains some divisor E_1 which is the strict transform of $\tilde{E}_1 \subset \tilde{\mathcal{X}}_0$ and is contracted via the birational map $\mathcal{X}' \dashrightarrow \mathcal{X}^s$.

Let $\mathcal{L}_t = p^*(-K_{\mathcal{X}'/C'}) + tE = (1-t)p^*(-K_{\mathcal{X}'/C'}) + tq^*(-K_{\mathcal{X}^s/C'})$. Then $\mathcal{L}_t$ is nef for $0 \leq t \leq 1$. We can differentiate again:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\mathcal{L}_t^{n+1}}{n+1}\right) &= \mathcal{L}_t^n \cdot E \\
&\geq (1-t)^n p^*(-K_{\mathcal{X}'/C'})^n \cdot E_1 \\
&= (1-t)^n (-K_{\mathcal{X}'/C'})^n \cdot p_* E_1 \\
&= (1-t)^n (-K_{\tilde{\mathcal{X}}/C'})^n \cdot \tilde{E}_1 > 0
\end{aligned}$$

□

5.3.8 Completion of Proof of Theorem 23

Proof. For any test configuration $\mathcal{X}$, we modify it by the above steps.

$$\begin{aligned}
\mathcal{X}/C &\rightsquigarrow \mathcal{X}^\nu/C && (\text{ Step 0: normalization })\\
&\rightsquigarrow \tilde{\mathcal{X}}/C' && (\text{ Step 1: base change and normalization})\\
&\rightsquigarrow \mathcal{X}^{lc}/C' && (\text{ Step 2: log canonical modification })\\
&\rightsquigarrow \mathcal{X}^{an}/C' && (\text{ Step 3: Run MMP with scaling })\\
&\rightsquigarrow \mathcal{X}'/C'' && (\text{ Step 4a: Base change and crepant blow up})\\
&\rightsquigarrow \mathcal{X}^s/C'' && (\text{ Step 4b: Contracting extra components})
\end{aligned}$$

For each step, the Donaldson-Futaki invariant decreases up to a factor due to base change $\deg(C'/C)$ and $\deg(C''/C)$. So

$$\mathrm{DF}(\mathcal{X}/C, \mathcal{L}) \leq \deg(C''/C')\mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s})$$

The equality holds if and only if $\mathcal{X}_{non-normal}$ has codimension at least two and $\mathcal{X}^\nu \times_C C'' = \mathcal{X}^s$ is a special test configuration. So if $\mathcal{X}$ is normal, then the equality holds if and only if $\mathcal{X}$ is itself a special test configuration. $\square$

5.3.9 Simplification in the unstable case and Discussions

If we want to prove the following weaker statement of the theorem.

Theorem 32. *Given any test configuration $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}^1$, for any $\epsilon \ll 1$, we can construct a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ and a positive integer m, such that*

$$m(\epsilon + \mathrm{DF}(\mathcal{X}, \mathcal{L})) \geq \mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s})$$

Then we can simplify the argument. Note that the weaker statement implies Tian's conjecture in the un-stable case.

Proof.

1. Step 1: Equivariant semistable reduction. $\mathcal{Y} \to \tilde{\mathcal{X}} \to \mathcal{X}$.

2. Step 2: perturb the pull back polarization.

$$\mathcal{L}_{\mathcal{Y}} = \epsilon A + \phi^*_{\mathcal{Y}/\mathcal{X}}(\mathcal{L})$$

   by an ample divisor A ($\epsilon \ll 1$) such that

   - $\mathcal{L}_{\mathcal{Y}}$ is still ample
   - For some $a \in \mathbb{Q}$
$$\mathcal{L}_{\mathcal{Y}} + K_{\mathcal{Y}} + a\mathcal{Y}_0 = \sum_{i=\mathbf{2}}^{N} a_\alpha \mathcal{Y}_{0,\alpha}$$
     with $a_\alpha > 0$ for any $\alpha \geq 2$.

3. step 3: Run $(K_X + \mathcal{L}_{\mathcal{Y}})$-MMP with scaling $\mathcal{L}_{\mathcal{Y}}$ over C. Define

$$\bar{\mathcal{M}}_\lambda = \frac{(K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}}) + \lambda \bar{\mathcal{L}}}{\lambda}, \quad \bar{\mathcal{M}}_1 = \bar{\mathcal{L}}.$$

   so that $\mathcal{M}_{+\infty} = \mathcal{L}$. As λ decreases from $+\infty$ to 0, we get a sequence of critical points of λ and a sequence of models:

$$\begin{array}{ccccccccc} +\infty & \geq & \lambda_1 & \geq & \dots & \geq & \lambda_k & > & \lambda_{k+1} = 0 \\ \mathcal{X}^0 & \dashrightarrow & \mathcal{X}^1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{X}^k & \to & \mathbb{C} \end{array}$$

The $\mathcal{X}^k$ in the above sequence has very good properties:

- E^k is linearly trivial over C. $\mathcal{L}^k \sim_C -K_{\mathcal{X}^k}$ is semiample.
- Assume $\mathcal{X}_0 = \sum_{\alpha=1}^N a_\alpha \mathcal{X}_{0,\alpha} = E \geq 0$, then $\mathcal{X} \dashrightarrow \mathcal{X}^0$ contracts precisely Supp(E). This can easily seen as follows. Using hyperplane section, we can assume $\dim(\mathcal{X}) = 2$. Denote the strict transform of E on $\mathcal{X}^i$ by E^i. Because $E^i \cdot \mathcal{X}^i_{0,1} > 0$, $\mathcal{X}^i_{0,1}$ is never contracted. If E^k still contained $\mathcal{X}^k_{0,\beta}$ for some $\beta \geq 2$, then $E^k \cdot \mathcal{X}^k_{0,1} > 0$. This contradicts the $E^k \sim_C 0$.

By property 1 above, we can define: $\mathcal{X}^s = \mathcal{X}^{an} = Proj(\mathcal{X}^k, -K_{\mathcal{X}^k/C})$ so that $-K_{\mathcal{X}^s}$ is ample and $\mathcal{X}^s_0$ is $\mathbb{Q}$-Fano.

Again, Donaldson-Futaki invariant decreases along $\mathcal{M}_\lambda$. Note that

$$K_{\mathcal{X}} + \mathcal{M}_\lambda = \frac{\lambda+1}{\lambda}(K_{\mathcal{X}} + \mathcal{L}), \quad \frac{d}{d\lambda}\mathcal{M}_\lambda = -\frac{1}{\lambda^2}(K_{\mathcal{X}} + \mathcal{L}).$$

As before, we can calculate

$$\frac{d}{d\lambda}\mathrm{DF}(\mathcal{X}, \mathcal{M}_\lambda) \quad = \quad -C(n,\lambda,r)\mathcal{M}_\lambda^{n-1} \cdot (\mathcal{L} + K_{\mathcal{X}/\mathbb{P}^1})^2 \geq 0$$

by the Hodge index theorem, because $\mathcal{L} + K_{\mathcal{X}/\mathbb{P}^1} = \sum_i a_i \mathcal{X}_{0,i}$ only supports on $\mathcal{X}_0$. This means that DF invariant decreases as λ decreases. $\square$

Remark 43. *Let's explain the formal similarity between MMP and (normalized) Kähler Ricci flow. Assume $\mathcal{L}|_{\mathcal{X}_t} \sim_{\mathbb{C}} -K_X$. In MMP, we vary the polarization in the direction of K_X:*

$$\mathcal{M}_s := \frac{\mathcal{L} + sK_{\mathcal{X}}}{1-s}, \quad \mathcal{M}_s|_{\mathcal{X}_t} \sim -K_X.$$

The derivative of $\mathcal{M}_s$ is

$$\frac{d}{ds}\mathcal{M}_s = \frac{1}{(1-s)^2}(\mathcal{L} + K_{\mathcal{X}}) \overset{\tilde{s}=1/(1-s)}{\Longleftrightarrow} \frac{d}{d\tilde{s}}\mathcal{M}_{\tilde{s}} = K_{\mathcal{X}} + \mathcal{L}$$

This variation corresponds to the normalized Kähler-Ricci flow.

$$\frac{\partial \omega}{\partial \tilde{s}} = -Ric(\omega_{\tilde{s}}) + \omega_{\tilde{s}}$$

Chapter 6

Rotationally symmetric Kähler-Ricci solitons on flips

6.1 Introduction and motivation

Song-Tian [SoTi] used Kähler-Ricci flow to study general algebraic varieties. The goal is to construct canonical metrics on algebraic varieties obtained after surgeries and this can be seen as the metric counterpart of Minimal Model Program(MMP). Tian-Zhang [TiZha] proved singularity-occurring time is the same as nef threshold. It is proved in the theory of Ricci flow that, the type-I singularity will produce Ricci soliton after rescaling. In general, we don't know whether the singularity is type-I or not. In Fano case, this is related to the famous Hamilton-Tian conjecture on the limit behavior of Kähler-Ricci flow on any Fano manifold. If we assume the singularity is type-I, it's interesting to see examples of Kähler-Ricci solitons.

In [FIK], the authors constructed some examples of gradient Kähler-Ricci soliton. Among them is the shrinking soliton on the $Bl_0\mathbb{C}^m$. They also glue this to an expanding soliton on $\mathbb{C}^m$ to extend the Ricci flow across singular time.

Recently, La Nave and Tian [LaTi] studied the formation of singularity along Kähler-Ricci flow by symplectic quotient. The idea is explained by the following example.

Let $\mathbb{C}^*$ act on $\mathbb{C}^{m+n}$ by

$$t\cdot(x_1,\cdots,x_m,y_1,\cdots,y_n)=(t\,x_1,\cdots,t\,x_m,t^{-1}\,y_1,\cdots,t^{-1}\,y_n)$$

$S^1 \subset \mathbb{C}^*$ preserves the standard Kähler structure on $\mathbb{C}^{m+n}$:

$$\omega = \sqrt{-1}(\sum_{i=1}^{m} dx_i \wedge d\bar{x}_i + \sum_{\alpha=1}^{n} dy_\alpha \wedge d\bar{y}_\alpha)$$

Let $z = (x, y) = (x_1, \cdots, x_m, y_1, \cdots, y_n)$. The momentum map of this Hamiltonian action is

$$m(z) = \sum_{i=1}^{m} |x_i|^2 - \sum_{\alpha=1}^{n} |y_\alpha|^2 = |x|^2 - |y|^2$$

The topology of symplectic quotient $X_a = m^{-1}(a)/S^1$ changes as a across 0.

Let $\mathcal{O}_{\mathbb{P}^N}(-1)$ be the tautological line bundle on the complex projective space $\mathbb{P}^N$. We will use $Y_{N,R}$ to represent the total space of holomorphic vector bundle $(\mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus R} \to \mathbb{P}^N)$.

1. (a>0) $\forall z = (x, y) \in X_a$, $m(z) = |x|^2 - |y|^2 = a > 0$, so $x \neq 0$.

   $X_a \simeq Y_{m-1,n} \simeq \{\mathbb{C}^{m+n} - \{x = 0\}\}/\mathbb{C}^*$. The isomorphism is given by

   $$(x_1, \cdots, x_m, y_1, \cdots, y_n) \mapsto ([x_1, \cdots, x_m], y_1 \cdot x, \cdots, y_n \cdot x)$$

   There is an induced Kähler metric on X_a. Choose a coordinate chart $u_1 = \frac{x_2}{x_1}, \cdots, u_{m-1} = \frac{x_m}{x_1}, \xi_1 = x_1 y_1, \cdots, \xi_n = x_1 y_n$. The C^* action is then trivialized to: $(x_1, u, \xi) \mapsto (tx_1, u, \xi)$. The Kähler potential can be obtained by some Legendre transformation (see [BuGu]). Specifically, the potential for the standard flat Kähler metric on $\{\mathbb{C}^{m+n} - \{x = 0\}\}$ is

   $$\phi = |x|^2 + |y|^2 = |x_1|^2(1 + |u|^2) + \frac{|\xi|^2}{|x_1|^2} = e^{r_1}(1 + |u|^2) + e^{-r_1}|\xi|^2$$

   where $r_1 = \log |x_1|^2$. ϕ is a convex function of r_1. $a = \frac{\partial \phi}{\partial r_1}$ is the momentum map of the S^1 action. In the induced coordinate chart (u, ξ), the Kähler potential of the induced metric on the symplectic quotient is the Legendre transform of ϕ with respect to r_1:

   $$\Phi_a = a \log(1 + |u|^2) + \sqrt{a^2 + 4(1 + |u|^2)|\xi|^2} - a \log(a + \sqrt{a^2 + 4(1 + |u|^2|)|\xi|^2}) + (\log 2)\, a \tag{6.1}$$

2. (a<0) By symmetry, $X_a \simeq Y_{n-1,m} \simeq \{\mathbb{C}^{m+n} - \{y = 0\}\}/\mathbb{C}^*$. Choose a coordinate chart $v_1 = \frac{y_2}{y_1}, \cdots, v_{n-1} = \frac{y_n}{y_1}, \eta_1 = y_1 x_1, \cdots, \eta_m = y_1 x_m$. The Kähler potential has the same expression as (6.1) but replacing a by $-a$, u by v, and ξ by η.

3. (a=0) $X_a \cong$ affine cone over the Segre embedding of $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{mn-1}$:

$$(x_1, \cdots, x_m, y_1, \cdots, y_n) \mapsto \{x_i y_\alpha\}$$

   Away from the vertex of the affine cone, choose a coordinate chart: $u_1 = \frac{x_2}{x_1}, \cdots, u_{m-1} = \frac{x_m}{x_1}, v_1 = \frac{y_2}{y_1}, \cdots, v_{n-1} = \frac{y_n}{y_1}, \zeta = x_1 y_1$. The Kähler potential is given by

$$\Phi_0 = 2\sqrt{(1+|u|^2)(1+|v|^2)|\zeta|^2}$$

   Note that Φ_0 is obtained from Φ_a by coordinate change $\xi_1 = \zeta, \xi_2 = v_1\zeta, \cdots, \xi_n = v_{n-1}\zeta$, and let a tend to 0.

This is a simple example of flip when $m \neq n$, or flop when $m = n$, in the setting of symplectic geometry. $X_{<0}$ is obtained from $X_{>0}$ by first blowing up the zero section $\mathbb{P}^{m-1}$, and then blowing down the exceptional divisor $E \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$. Note that when $n = 1$, this process is just blow-down of exceptional divisor in $Bl_0\mathbb{C}^m$.

One hopes to have a Kähler metric on a larger manifold $\mathcal{M}$ such that induced Kähler metrics on symplectic quotient would satisfy the Kähler-Ricci flow equation as the image of momentum varies. See [LaTi] for details.

Our goal to construct a Kähler-Ricci soliton on $Y = Y_{m-1,n}$ and its projective compactification, and this generalizes constructions of [Koi], [Cao] [FIK]. See also [DaWa]. The construction follows these previous constructions closely, but we need to modify them to fit our setting. The higher dimensional analogs have the new phenomenon of contracting higher codimension subvariety to highly singular point. To continue the flow, surgery are needed. The surgeries in these cases should be the naturally appearing flips.

The organization of this note is as follows. In section 2, we put the construction in a more general setting where the base manifold is Kähler-Einstein, and state the main results: Theorem 33 and Theorem 34. In section 3, by the rotational symmetry, we reduce the Kähler-Ricci soliton equation to an ODE. In section 4, we analyze the condition in order for the general solution of the ODE to give a smooth Kähler metric near zero section. In section 5.1, we get the condition for the metric to be complete near infinity. In section 5.2, we prove theorem 33, i.e. construct Kähler-Ricci solitons in the noncompact complete Kähler manifold and study its behavior as time approaches the singular time. Finally in section 6, we prove theorem 34 by constructing the compact shrinking soliton on projective compactification.

6.2 General setup and the result

Let M be a Kähler manifold of **dimension d**. Kähler-Ricci soliton on M is a Kähler metric ω satisfying the equation

$$Ric(\omega) = \lambda\omega + L_V\omega \tag{6.2}$$

where V is a holomorphic vector field. The Kähler-Ricci soliton is called gradient if $V = \nabla f$ for some potential function f. If $\sigma(t)$ is the 1-parameter family of automorphisms generated by V, then

$$\omega(t) = (1-\lambda t)\sigma\left(-\frac{1}{\lambda}\log(1-\lambda t)\right)^* \omega \tag{6.3}$$

is a solution of Kähler-Ricci flow equation:

$$\frac{\partial\omega(t)}{\partial t} = -Ric(\omega(t))$$

We will construct gradient Kähler-Ricci solitons on the total space of special vector bundle $L^{\oplus n} \to M$ and its projective compactification $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n}) = \mathbb{P}(L^{-1} \oplus \mathbb{C}^{\oplus n})$. Here M is a Kähler-Einstein manifold:

$$Ric(\omega_M) = \tau\omega_M$$

L has an Hermitian metric h, such that

$$c_1(L,h) = -\sqrt{-1}\partial\bar{\partial}\log h = -\epsilon\omega_M$$

In the following, we always consider the case $\epsilon \geq 0$.

We consider the Kähler metric of the form considered by Calabi [Cal1]:

$$\omega = \pi^*\omega_M + \partial\bar{\partial}P(s) \tag{6.4}$$

Here s is the norm square of vectors in L. Under local trivialization of holomorphic local section e_L,

$$s(\xi e_L) = a(z)|\xi|^2,\ \xi = (\xi^1, \cdots, \xi^n)$$

P is a smooth function of s we are seeking for.

Using the form (6.4), we can determine λ immediately. Let M be the zero section. By adjoint formula,

$$-K_Y|_M = -K_M + \wedge^n N_M|_M = -K_M + nL$$

Note that $\omega|_M = \omega_M$, so by restricting both sides of (6.2) to M, and then taking cohomology, we see that

$$\tau[\omega_M] - n\epsilon[\omega_M] = c_1(Y)|_M = \lambda[\omega_M] \tag{6.5}$$

So $\lambda = \tau - n\epsilon$.

Remark 44. *If we rescale the Kähler-Einstein metric: $\omega_M \to \kappa\omega_M$, then $\tau \to \tau/\kappa$, $\epsilon \to \epsilon/\kappa$, $\lambda \to \lambda/\kappa$.*

The main theorem is

Theorem 33. *On the total space of $L^{\oplus n}$, there exist rotationally symmetric solitons of types depending on the sign of $\lambda = \tau - n\epsilon$. If $\lambda > 0$, there exists a unique shrinking soliton. If $\lambda = 0$, there exists a family of steady solitons. If $\lambda < 0$, there exists a family of expanding solitons. (The solitons are rotationally symmetric in the sense that it's of the form of* (6.4)*)*

Remark 45. *If we take $M = \mathbb{P}^{m-1}$, $L = \mathcal{O}(-1)$, $\omega_M = \omega_{FS}$, then $\tau = m$, $\epsilon = 1$. Then we get to the situation in section 1. So depending on the sign of $\lambda = m - n$, there exist either a unique rotationally symmetric shrinking KR soliton when $m > n$, or a family of rotationally symmetric steady KR solitons when m=n, or a family of rotationally symmetric expanding KR solitons when $m < n$.*

We also have the compact shrinking soliton:

Theorem 34. *Using the above notation, assume $\lambda = \tau - n\epsilon > 0$, then on the space $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n}) = \mathbb{P}(L^{-1} \oplus \mathbb{C}^{\oplus n})$, there exists a unique shrinking Kähler-Ricci soliton.*

6.3 Reduction to ODE

The construction of solitons is straightforward by reducing the soliton equation to an ODE.

First, in local coordinates, (6.4) is expressed as

$$\omega = (1 + \epsilon P_s s)\omega_M + a(P_s\delta_{\alpha\beta} + P_{ss}a\overline{\xi^\alpha}\xi^\beta)\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta} \tag{6.6}$$

Here

$$\nabla\xi^\alpha = d\xi^\alpha + a^{-1}\partial a\,\xi^\alpha$$

Note that $\{dz^i, \nabla\xi^\alpha\}$ are dual to the basis consisting of horizontal and vertical vectors:

$$\nabla_{z^i} = \frac{\partial}{\partial z^i} - a^{-1}\frac{\partial a}{\partial z^i}\sum_\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha}, \quad \frac{\partial}{\partial \xi^\alpha}$$

ω is positive if and only if

$$1 + \epsilon P_s s > 0,\ P_s > 0,\ \text{and } P_s + P_{ss}s > 0 \tag{6.7}$$

$$\omega^{d+n} = (1+\epsilon P_s s)^d \omega_M^d a^n P_s^{n-1}(P_s + P_{ss}s)\prod_{\alpha=1}^n d\xi^\alpha \wedge d\bar{\xi}^\alpha$$

Since we assume $Ric(\omega_M) = \tau\omega_M = (\lambda + n\epsilon)\omega_M$,

$$\partial\bar{\partial}\log\det\omega^{d+n} + \lambda(\omega_M + \partial\bar{\partial}P) = \partial\bar{\partial}\left[d\cdot\log(1+\epsilon P_s s) + (n-1)\log P_s + \log((P_s s)_s) + (\tau - n\epsilon)P\right]$$

Let $r = \log s$, then $\partial_r = s\partial_s$. Define

$$\begin{aligned} Q &:= d\cdot\log(1+\epsilon P_s s) + (n-1)\log P_s + \log((P_s s)_s) + (\tau - n\epsilon)P \\ &= d\cdot\log(1+\epsilon P_r) + (n-1)\log P_r + \log P_{rr} - nr + (\tau - n\epsilon)P \end{aligned} \tag{6.8}$$

To construct a gradient Kähler-Ricci soliton (6.2), it is sufficient to require that $Q(t)$ is a potential function for the holomorphic vector field $-V$. Notice that, for the radial holomorphic vector field:

$$V_{rad} = \sum_{\alpha=1}^n \xi^\alpha \frac{\partial}{\partial \xi^\alpha} \tag{6.9}$$

$$i_{V_{rad}}\omega = (P_s + P_{ss}s)a\sum_\beta \xi^\beta \overline{\nabla\xi^\beta} = (P_s s)_s \bar{\partial}s$$

Now

$$-i_V\omega = \bar{\partial}Q(s) = Q_s\bar{\partial}s = \frac{Q_s}{(P_s s)_s} i_{V_{rad}}\omega$$

which means $-V = \frac{Q_s}{(P_s s)_s}V_{rad}$, so $\frac{Q_s}{(P_s s)_s}$ is a holomorphic function. Since $s = a(z)|\xi|^2$ is not holomorphic, $\frac{Q_s}{(P_s s)_s}$ has to be a constant μ. We assume $\mu \neq 0$, since $V \neq 0$. So we get the equation: $Q_s = \mu(P_s s)_s$. Multiplying by s on both sides, this is equivalent to

$$Q_r = \mu P_{rr} \tag{6.10}$$

Remark 46. *Note that, if $\mu = 0$, then we go back to Calabi's construction of Kähler-Einstein metrics in [Cal1].*

Define $\phi(r) = P_r(r)$ and substitute (6.8) into (6.10), then we get

$$d\frac{\epsilon\phi_r}{1+\epsilon\phi} + (n-1)\frac{\phi_r}{\phi} + \frac{\phi_{rr}}{\phi_r} + (\tau - n\epsilon)\phi - n = \mu\phi_r \tag{6.11}$$

Since $\phi_r = P_{rr} = (P_s s)_s s = (P_s + P_{ss}s)s > 0$ by (6.7), we can solve r as a function of ϕ: $r = r(\phi)$. Define $F(\phi) = \phi_r(r(\phi))$, then $F'(\phi) = \phi_{rr}r'(\phi) = \frac{\phi_{rr}}{\phi_r}$. So the above equation change into an ODE

$$F'(\phi) + d\frac{\epsilon F(\phi)}{1+\epsilon\phi} + (n-1)\frac{F(\phi)}{\phi} - \mu F(\phi) = n - (\tau - n\epsilon)\phi = n(1+\epsilon\phi) - \tau\phi \tag{$*$}$$

Remark 47. *We will explain how this equation is related to the ODE in [FIK],(25). with our notation, in [FIK], $M = \mathbb{P}^d$, $L = O_{\mathbb{P}^d}(-k)$, $n = 1$. For the shrinking soliton case, $d+1-k > 0$, $\omega_M = (d+1-k)\omega_{FS}$, $\tau = \frac{d+1}{d+1-k}$, $\epsilon = \frac{k}{d+1-k}$, $\lambda = \tau - \epsilon = 1$. Let $r = k\tilde{r}$, $P(r) = \tilde{P}(\tilde{r}) - (d+1-k)\tilde{r} = \tilde{P}(\frac{r}{k}) - \frac{d+1-k}{k}r$, $\phi(r) = P_r(r) = \tilde{P}_{\tilde{r}}(\tilde{r})\frac{1}{k} - \frac{d+1-k}{k} = \frac{1}{k}(\tilde{\phi}(\tilde{r}) - (d+1-k))$, $F(\phi) = \phi_r(r(\phi)) = \frac{1}{k^2}\tilde{\phi}_{\tilde{r}}(\tilde{r}(\tilde{\phi})) = \frac{1}{k^2}\tilde{F}(\tilde{\phi})$, $F'_\phi(\phi) = \frac{1}{k}\tilde{F}'_{\tilde{\phi}}(\tilde{\phi})$. Substitute these expressions into $(*)$, then we get the ODE*

$$\tilde{F}'_{\tilde{\phi}} + \left(\frac{d}{\tilde{\phi}} - \frac{\mu}{k}\right)\tilde{F} - ((d+1) - \tilde{\phi}) = 0$$

So we see this is exactly the ODE in [FIK], (25). The expanding soliton case is the similar.

We can solve $(*)$ by multiplying the integral factor: $(1+\epsilon\phi)^d\phi^{n-1}e^{-\mu\phi}$:

$$\phi_r = F(\phi) = \nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} - (1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi}\int h(\phi)e^{-\mu\phi}d\phi \tag{6.12}$$

where

$$h(\phi) = \tau(1+\epsilon\phi)^d\phi^n - n(1+\epsilon\phi)^{d+1}\phi^{n-1} \tag{6.13}$$

is a polynomial of ϕ with degree $d+n$. Note the identity:

$$\int h(\phi)e^{-\mu\phi}d\phi = -\sum_{k=0}^{+\infty}\frac{1}{\mu^{k+1}}h^{(k)}(\phi)e^{-\mu\phi}$$

Since $h(\phi)$ is a polynomial of degree d+n, the above sum is a finite sum. So

$$F(\phi) = (1+\epsilon\phi)^{-d}\phi^{1-n}\left(\nu e^{\mu\phi} + \sum_{k=0}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(\phi)\right) \tag{6.14}$$

6.3.1 Rotationally Symmetric Model of Tian-Yau metrics

Using similar calculation, we can also give an interpretation of the leading term in Tian-Yau's [TiYa1, TiYa2] construction of complete Ricci flat Kähler metrics on $X\backslash D$ where X is a Fano manifold and D is a smooth divisor such that $-K_X \sim_{\mathbb{Q}} \beta D$. By adjunction formula, $K_D^{-1} = K_X^{-1}|_D - D \sim (\beta-1)D$. Let $M = D$, $L = N_D$, $n = 1$, $\epsilon = -1$, $d = dimD$, $F(r) = -r + P$ in the above. Then

$$\omega^{d+1} = (-F_r)^d F_{rr}\omega_D^d \wedge d\log\xi \wedge d\overline{\log\xi}$$

So $Ric(\omega) = -\sqrt{-1}\partial\bar\partial\log((-F_r)^dF_{rr}) + \tau\omega_D = -\sqrt{-1}\partial\bar\partial\left(\log\left((-F_r)^dF_{rr}\right) + \tau r\right)$. In order for $Ric(\omega) = 0$, it suffices for

$$\log\left((-F_r)^dF_{rr}\right) + \tau r = \text{constant} \tag{6.15}$$

1. $(\beta = 1)$ $\tau = 0$. We can solve $F = C\cdot(-r)^{(d+2)/(d+1)}$which is same as $F = C\cdot(-\log\|s\|^2)^{(n+1)/n}$ using Tian-Yau's notation in [TiYa1, equation 4.1].

2. $(\beta > 1)$ $\tau = \beta - 1$. We can solve $F = C\cdot e^{-\tau r/(d+1)}$ which is $C\cdot\|s\|^{-2(\beta-1)/n}$ in Tian-Yau's notation in [TiYa2, equation 2.2].

6.4 Boundary condition at zero section

Since $\lim_{r\to-\infty}\phi(r) = \lim_{s\to 0}P_s s = 0$, we have the boundary condition

$$\lim_{\phi\to 0}F(\phi) = \lim_{r\to-\infty}\phi_r = \lim_{s\to 0}(P_s s)_s s = 0 \tag{6.16}$$

So $\phi^{n-1}(1+\epsilon\phi)^dF(\phi) = O(\phi^n)$. Now the l-th term of Taylor expansion of $\phi^{n-1}(1+\epsilon\phi)^dF(\phi)$ at $\phi = 0$ is

$$(\phi^{n-1}(1+\epsilon\phi)^dF(\phi))^{(l)}\Big|_{\phi=0} = \nu\mu^l + \sum_{k=0}^{+\infty}\frac{1}{\mu^{k+1}}h^{(k+l)}(0) = \mu^l\left(\nu + \sum_{k=l}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(0)\right) \tag{6.17}$$

Note that by (6.13) $h^{(k)}(\phi) = 0$ for $k > d+n$, and $h^{(k)}(0) = 0$ for $k < n-1$. The vanishing of the 0-th(constant) term in expansion gives the equation:

$$\nu + \sum_{k=n-1}^{d+n} \frac{1}{\mu^{k+1}} h^{(k)}(0) = 0 \tag{6.18}$$

Using relation (6.18) we see that, when $l < n$, $(\phi^{n-1}(1+\epsilon\phi)^d F(\phi))^{(l)}\big|_{\phi=0} = 0$, and

$$(\phi^{n-1}(1+\epsilon\phi)^d F(\phi))^{(n)}\Big|_{\phi=0} = \mu^n \left(\nu + \sum_{k=n}^{d+n} \frac{1}{\mu^{k+1}} h^{(k)}(0) \right) = -h^{(n-1)}(0) = n! > 0$$

So we see that (6.16) and (6.18) are equivalent, and if they are satisfied,

$$\phi^{n-1}(1+\epsilon\phi)^d F(\phi) = \phi^n + O(\phi^{n+1}), \text{ or } F(\phi) = \phi + O(\phi^2)$$

So $F(\phi) > 0$ for ϕ near 0.

We can rewrite the relation (6.18) more explicitly:

$$\begin{aligned} \nu &= \sum_{k=0}^{+\infty} \frac{1}{\mu^{k+1}} \left[n((1+\epsilon\phi)^{d+1}\phi^{n-1})^{(k)} - \tau((1+\epsilon\phi)^d \phi^n)^{(k)} \right] \Bigg|_{\phi=0} \\ &= \sum_{k=n-1}^{d+n} \frac{1}{\mu^{k+1}} \left(n \binom{k}{n-1} (n-1)! \frac{(d+1)!}{(d+n-k)!} \epsilon^{k-n+1} - \tau \binom{k}{n} n! \frac{d!}{(d+n-k)!} \epsilon^{k-n} \right) \\ &= \sum_{k=n-1}^{d+n} C_k \frac{1}{\mu^{k+1}} \epsilon^{k-n} \end{aligned} \tag{6.19}$$

Here

$$C_k = \frac{k!d!}{(k-n+1)!(d+n-k)!} (n\epsilon(d+1) - \tau(k-n+1)) \tag{6.20}$$

$$C_{n-1} = n!\epsilon, \quad C_{d+n} = -(d+n)!(\tau - n\epsilon)$$

So, when k starts from $n-1$ to $d+n$, C_k change signs from positive to negative if and only if $\lambda = \tau - n\epsilon > 0$. We need the following simple lemma later.

Lemma 36. *Let $P(x) = \sum_{i=0}^{l} a_i x^i - \sum_{j=l+1}^{N} a_j x^j$ be a polynomial function. Assume $a_i > 0$ for $0 \le a_i \le N$. Then there exists a unique root for $P(x)$ on $[0, \infty)$.*

Proof. First $P(0) = a_0 > 0$. Since $a_N < 0$, when x is large enough $P(x) < 0$. So there exists at least one root on $[0, \infty)$. Assume there are more than one root, than it's easy to see that $P'(x)$ has at least two roots on $[0, \infty)$. Note that $P'(x)$ has the same form as $P(x)$, so $P''(x)$ has at least two roots on $[0, \infty)$. By induction, $P^{(l)}(x)$ has at least two roots on $[0, \infty)$, but $P^{(l)}(x)$ has only negative coefficients, so it has no root at all. This contradiction proves the lemma. $\square$

6.5 Complete noncompact case

We prove theorem 33 in this section.

6.5.1 Condition at infinity

As $\phi \to +\infty$,

$$F(\phi) = \nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} + \frac{\tau - n\epsilon}{\mu}\phi + O(1) \tag{6.21}$$

Let $\phi = b_1 > 0$ be the first positive root for $F(\phi) = 0$, then $F'(b_1) \leq 0$. By $(*)$, $F'(b_1) = n - (\tau - n\epsilon)b_1$. So if $\lambda = \tau - n\epsilon \leq 0$, there exits no such b_1. If $\lambda = \tau - n\epsilon > 0$, we integrate (6.12) to get

$$r = r(\phi) = \int_{\phi_0}^{\phi} \frac{1}{F(u)}du + r(\phi_0) \tag{6.22}$$

then the metric is defined for $-\infty < r < r(b_1)$. We require that

$$r_{max} = r(b_1) = +\infty \tag{6.23}$$

We can also calculate the length of radial curve extending to infinity. In a fixed fibre, the radial vector

$$\frac{\partial}{\partial r} = \frac{1}{2}|\xi|\frac{\partial}{\partial|\xi|} = \frac{1}{2}\sum_{\alpha=1}^{n}\xi^\alpha\frac{\partial}{\partial\xi^\alpha} = \frac{1}{2}V_{rad}$$

$$\left|\frac{\partial}{\partial r}\right|^2 = \frac{1}{4}g_\omega(V_{rad}, V_{rad}) = C(P_s s + P_{ss}s^2) = C\phi_r$$

The completeness implies that the length of the radial curve extending to infinity is infinity:

$$\int_{-\infty}^{r(b_1)} \sqrt{\phi_r}dr = \int_0^{b_1} \sqrt{\phi_r}\phi_r^{-1}d\phi = \int_0^{b_1} \phi_r^{-\frac{1}{2}}d\phi = \int_0^{b_1} F(\phi)^{-\frac{1}{2}}d\phi = +\infty \tag{6.24}$$

If $0 < b_1 < +\infty$,(6.24) means $F(\phi) = c(\phi - b_1)^2 + O((\phi - b_1)^3)$, i.e. $F'(b_1) = F(b_1) = 0$.

But this can't happen: $b_1 = \frac{n}{\tau - n\epsilon}$, and $c = -(\tau - n\epsilon)$. First we have $b_1 > 0$. Second, $c \geq 0$ since $F(\phi) > 0$ when $\phi < b_1$. But they contradict with each other.

In conclusion, there can't be any finite value positive root for $F(\phi)$.

6.5.2 Existence and asymptotics

1. ($\lambda = \tau - n\epsilon > 0$) The solution is a shrinking Kähler-Ricci soliton. If $\mu < 0$, then when ϕ becomes large, the dominant term in $F(\phi)$(6.21) is $\frac{\lambda}{\mu}\phi < 0$, so there exists $0 < b < +\infty$ such that $F(b) = 0$. But this is excluded by former discusions. So we must have $\mu > 0$.

   If $\nu < 0$ the dominant term is $\nu\phi^{1-n}(1+\epsilon\phi)^{-m}e^{\mu\phi} < 0$, so there is $0 < b < +\infty$, such that $F(b) = 0$. Again, this is impossible. If $\nu > 0$, when ϕ becomes large, the dominant term is $\nu\phi^{1-d-n}e^{\mu\phi}$,

   $$\int_{\phi_0}^{+\infty} \frac{1}{F(s)} ds \le C \int_{\phi_0}^{+\infty} \frac{1}{\nu}\phi^{d+n-1}e^{-\mu\phi} d\phi < +\infty$$

   This contradicts (6.23). So we must have $\nu = 0$. This gives us an equation for μ via (6.19). Since when $\lambda = \tau - n\epsilon > 0$, C_k change signs exactly once, by lemma 36, there exists a unique μ such that $\nu(\mu) = 0$ in (6.19). We now verify this μ guarantees the positivity of ϕ_r. Since the dominant term in (6.21) is $\frac{\lambda}{\mu}\phi > 0$, $F(\phi) \overset{\phi\to+\infty}{\longrightarrow} +\infty$. We have also $F(\phi) > 0$ for ϕ near 0. If $\phi = b_1 > 0$ is the first root and $\phi = b_2 > 0$ is the last root of $F(\phi)$, then $b_1 \le b_2$, and

   $$F'(b_1) = -\lambda b_1 + n \le 0, \quad F(b_2) = -\lambda b_2 + n \ge 0$$

   So $b_1 \ge \frac{n}{\lambda} \ge b_2$, this implies $b_1 = b_2$ and $F'(b_1) = 0$. We have ruled out this possibility before. In conclusion, $F(\phi) > 0$ for all $\phi > 0$, or equivalently $\phi_r > 0$ for all r.

   So we already get the soliton. In the following, we study the limit of flow as time approaches singularity time.

   Define $p = \frac{\lambda}{\mu} = \frac{\tau - n\epsilon}{\mu}$,

   $$r(\phi) - r(\phi_0) = \int_{\phi_0}^{\phi} \frac{du}{F(u)} = \int_{\phi_0}^{\phi} \frac{du}{pu} + \int_{\phi_0}^{\phi} \frac{pu - F(u)}{puF(u)} ds = \frac{1}{p}(\log\phi - \log\phi_0) + G(\phi_0, \phi)$$

   $$\phi(r) = \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0,\phi)} e^{pr} = \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0,\phi(r))} s^p \tag{6.25}$$

   The holomorphic vector field $-\frac{V}{2} = \frac{\mu}{2}\sum_\alpha \xi^\alpha \frac{\partial}{\partial\xi^\alpha}$ generates the 1-parameter family of automorphisms: $\sigma(\tilde{t}) \cdot (u, \xi) = (u, e^{\frac{\tilde{t}\mu}{2}}\xi)$. Let

   $$\tilde{t}(t) = -\frac{1}{\lambda}\log(1 - \lambda t) \tag{6.26}$$

$$\begin{aligned}\lim_{t\to\frac{1}{\lambda}}(1-\lambda t)\sigma(\tilde{t})^*\phi &= \phi_0 e^{-pr(\phi_0)}e^{-\lim_{\phi\to+\infty}G(\phi_0,\phi)}\lim_{t\to\frac{1}{\lambda}}(1-\lambda t)((1-\lambda t)^{-\frac{\mu}{\lambda}}s)^p \\ &= \phi_0 e^{-pr(\phi_0)}e^{-G(\phi_0,+\infty)}s^p = D_0 s^p\end{aligned}$$

So

$$\lim_{t\to\frac{1}{\lambda}}(1-\lambda t)\sigma(\tilde{t})^*\phi_r = \lim_{t\to\frac{1}{\lambda}}(1-\lambda t)\sigma(\tilde{t})^*F(\phi) = pD_0 s^p$$

(6.6) can be rewritten as

$$\omega = (1+\epsilon\phi)\pi^*\omega_{FS} + (\phi|\xi|^{-2}\delta_{\alpha\beta} + (\phi_r-\phi)|\xi|^{-4}\overline{\xi^\alpha}\xi^\beta)\nabla\xi^\alpha\wedge\overline{\nabla\xi^\beta} \tag{6.27}$$

$$\begin{aligned}\lim_{t\to\frac{1}{\lambda}}(1-\lambda t)\sigma(\tilde{t})^*\omega &= D_0\left[s^p\epsilon\omega_M + s^p(|\xi|^{-2}\delta_{\alpha\beta}+(p-1)|\xi|^{-4}\overline{\xi^\alpha}\xi^\beta)\nabla\xi^\alpha\wedge\overline{\nabla\xi^\beta}\right] \\ &= D_0\partial\bar{\partial}\left(\frac{1}{p}s^p\right)\end{aligned}$$

Remark 48. *One sees that as $t\to\frac{1}{\lambda}$, the flow shrinks the base (zero section of the vector bundle). In the model case, $M=\mathbb{P}^{m-1}$, $L=\mathcal{O}(-1)$, the flow contracts the manifold to the affine cone of the Segre embedding $\mathbb{P}^{m-1}\times\mathbb{P}^{n-1}\hookrightarrow\mathbb{P}^{mn-1}$. This is the same phenomenon as that appears for the symplectic quotients at the beginning of this note.*

2. ($\lambda=\tau-n\epsilon=0$) the solution is a steady Kähler-Ricci soliton.

$$F(\phi) = \nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} - n(1+\epsilon\phi)^{-d}\phi^{1-n}\sum_{k=0}^{d+n-1}\frac{1}{\mu^{k+1}}\left((1+\epsilon\phi)^d\phi^{n-1}\right)^{(k)}$$

If $\mu>0$, then $v(\mu)>0$ by (6.19). So the dominant term in (6.21) is $\nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi}$, so $r_{max}<+\infty$.

So $\mu<0$ and the dominant term in (6.14) is the constant term $-\frac{n}{\mu}>0$. As $\phi\to+\infty$,

$$F(\phi) = -\frac{n}{\mu} - \frac{n(d+n-1)}{\mu^2}\frac{1}{\phi} + O\left(\frac{1}{\phi^2}\right) = c_1 - c_2\frac{1}{\phi} + O\left(\frac{1}{\phi^2}\right)$$

$$\int\frac{du}{c_1-\frac{c_2}{u}} = \frac{1}{c_1}u + \frac{c_2}{c_1^2}\log(c_1u-c_2) = R(u)$$

$$\begin{aligned}r(\phi)-r(\phi_0) &= \int_{\phi_0}^{\phi}\frac{du}{F(u)} = \int_{\phi_0}^{\phi}\frac{du}{c_1-\frac{c_2}{u}} + \int_{\phi_0}^{\phi}\left(\frac{1}{F(u)}-\frac{1}{c_1-\frac{c_2}{u}}\right)du \\ &= R(\phi)-R(\phi_0)+G(\phi_0,\phi)\end{aligned}$$

Since $c_1 > 0$ and $c_2 > 0$ ($\mu < 0$, $d > 0$, $n \geq 1$), $R(u)$ is an increasing function for $u \gg 0$, and has an inverse function denoted by R^{-1}. Let $\tilde{G}(r) = -G(\phi_0, \phi(r)) + R(\phi_0) - r(\phi_0)$, then $\tilde{G}$ is a bounded smooth function of r. We have

$$\phi(r) = R^{-1}(r + \tilde{G}(r))$$

The condition (6.24) is always satisfied. There is a family of steady Kähler-Ricci solitons.

Remark 49. *If we let $d = 0$, then we get expanding solitons on $\mathbb{C}^n$. 6.18 becomes $\nu\mu^n = n!$. The equation becomes*

$$(|\mu|\phi)_r = (-1)^n n!(|\mu|\phi)^{1-n} e^{-|\mu|\phi} + n! \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)!(|\mu|\phi)^k}$$

In particular, if $n = 1$, the equation becomes

$$\phi_r = F(\phi) = \nu e^{\mu\phi} - \frac{1}{\mu}$$

$$\phi(r) = -\frac{1}{\mu}\log(\mu\nu + Ce^r) = -\frac{1}{\mu}\log(1 + C|z|^2), \quad \phi_r = -\frac{C|z|^2}{\mu(1 + C|z|^2)}$$

$$\omega = \frac{\phi_r}{|z|^2} dz \wedge d\bar{z} = -\frac{C dz \wedge d\bar{z}}{\mu(1 + C|z|^2)} \overset{w=\sqrt{C}z}{=} \frac{1}{-\mu}\frac{dw \wedge d\bar{w}}{1 + |w|^2}$$

This is cigar steady soliton.

3. ($\lambda = \tau - n\epsilon < 0$) the solution is an expanding Kähler-Ricci soliton. By similar argument, we see that $\mu < 0$. The situation is similar to the shrinking soliton case. Now $t \to \frac{1}{\lambda} < 0$, or equivalently $\tilde{t} \to -\infty$ (6.26),

$$\phi(r) = \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0, \phi(r))} s^p$$

$$\lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^* \phi_r = \lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^* F(\phi) = pD_0 s^p$$

$$\lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^* \omega = D_0 \partial\bar{\partial}\left(\frac{1}{p}s^p\right)$$

The condition (6.24) is always satisfied. So there is a family of expanding Kähler-Ricci solitons.

Remark 50. *One could apply the same argument in [FIK] to get the Gromov-Hausdorff conver-*

gence and continuation of flow through singularity time.

6.6 Compact shrinking soliton

We prove theorem 34 in this section. First we show the considered manifold is Fano. For some results on Fano manifolds with a structure of projective space bundle, see [SzWi].

Lemma 37. *If $\lambda = \tau - n\epsilon > 0$, then $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n})$ is Fano.*

Proof. Let $E = \mathbb{C} \oplus L^{\oplus n}$ and $X = \mathbb{P}(E)$. We have the formula for anti-canonical bundle:

$$K_X^{-1} = (n+1)\mathcal{O}(1) + \pi^*(K_M^{-1} + L^{\otimes n})$$

$\mathcal{O}(1)$ is the relative hyperplane bundle. Since $c_1(L^{-1}) = \epsilon[\omega_M] \geq 0$, one can prove $\mathcal{O}(1)$ is nef on X [Laz]. $c_1(K_M^{-1} + L^{\otimes n}) = (\tau - n\epsilon)[\omega_M] > 0$, so $K_M^{-1} + L^{\otimes n}$ is an ample line bundle on M. So $\mathcal{O}(1)$ and $\pi^*(K_M^{-1} + L^{\otimes n})$ are different rays of the cone of numerically effective divisors in $Pic(\mathbb{P}(E)) = \mathbb{Z}Pic(M) + \mathbb{Z}\mathcal{O}(1)$. So K_X^{-1} is ample, i.e. X is Fano. □

The construction of shrinking soliton was developed for the n=1 case, see [Cal2], [Koi], [Cao], [FIK]. We will give a simple direct argument under our setting. Note here we will use Tian-Zhu's theory [TiZhu] to get the uniqueness of Kähler-Ricci soliton.

First we need to know the expression for the metric near infinity. By change of coordinate

$$[1, \xi_1, \xi_2, \cdots, \xi_n] = [\eta, 1, u_2, \cdots, u_n]$$

So the coordinate change is given by

$$\xi_1 = \frac{1}{\eta}, \xi_2 = \frac{u_2}{\eta}, \cdots, \xi_n = \frac{u_n}{\eta} \quad \Longleftrightarrow \quad \eta = \frac{1}{\xi_1}, u_2 = \frac{\xi_2}{\xi_1}, \cdots, u_n = \frac{\xi_n}{\xi_1}$$

Since

$$\xi_1 \frac{\partial}{\partial \xi_1} = -\eta \frac{\partial}{\partial \eta} - \sum_{i=2}^{n} u_\alpha \frac{\partial}{\partial u_\alpha}, \quad \xi_\alpha \frac{\partial}{\partial \xi_\alpha} = u_\alpha \frac{\partial}{\partial u_\alpha}$$

So the radial vector $\sum_{i=\alpha}^n \xi_\alpha \frac{\partial}{\partial \xi_\alpha} = -\eta \frac{\partial}{\partial \eta}$ is a holomorphic vector field on $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n})$. The dual 1-forms transform into

$$\nabla \xi^1 = -\frac{1}{\eta^2}(d\eta - \eta a^{-1}\partial a) = -\frac{1}{\eta^2}\omega_0, \quad \nabla \xi^\alpha = \frac{du_\alpha}{\eta} - \frac{u_\alpha}{\eta^2}(d\eta - \eta a^{-1}\partial a) = \frac{du_\alpha}{\eta} - \frac{u_\alpha}{\eta^2}\omega_0$$

Note that the dual basis for the basis $\{dz_i, \omega_0, du_\alpha\}$ is

$$\nabla_{z_i} = \frac{\partial}{\partial z_i} + a^{-1}\frac{\partial a}{\partial z_i}\eta\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial u_\alpha}$$

So making coordinate change,

$$\begin{aligned}
\omega &= (1+P_s s)\omega_M + \sum_{\alpha=2}^{n}\sum_{\beta=2}^{n} a\left(P_s\delta_{\alpha\beta} + P_{ss}a\frac{\bar{u}_\alpha u_\beta}{|\eta|^2}\right)\left(\frac{1}{\eta}du_\alpha - \frac{u_\alpha}{\eta^2}\omega_0\right)\wedge\left(\frac{1}{\bar{\eta}}d\bar{u}_\beta - \frac{\bar{u}_\beta}{\bar{\eta}^2}\overline{\omega_0}\right) \\
&\quad -\sum_{\beta=2}^{n} P_{ss}a^2\frac{u_\beta}{|\eta|^2}\frac{\omega_0}{\eta^2}\wedge\left(\frac{1}{\bar{\eta}}d\bar{u}_\beta - \frac{\bar{u}_\beta}{\bar{\eta}^2}\overline{\omega_0}\right) - \sum_{\alpha=2}^{n} P_{ss}a^2\frac{\bar{u}_\alpha}{|\eta|^2}\left(\frac{1}{\eta}du_\alpha - \frac{u_\alpha}{\eta^2}\omega_0\right)\wedge\frac{\overline{\omega_0}}{\bar{\eta}^2} \\
&\quad + a(P_s + P_{ss}a\frac{1}{|\eta|^2})\frac{\omega_0\wedge\overline{\omega_0}}{|\eta|^4} \\
&= (1+P_s s)\omega_M + \sum_{\alpha=2}^{n}\sum_{\beta=2}^{n}(P_s\delta_{\alpha\beta} + P_{ss}s\frac{\bar{u}_\alpha u_\beta}{1+|u|^2})\frac{s}{1+|u|^2}du_\alpha\wedge d\bar{u}_\beta \\
&\quad -\sum_{\alpha=2}^{n}\frac{\bar{u}_\alpha\eta}{a(1+|u|^2)^2}(P_s + P_{ss}s)s^2 du_\alpha\wedge\overline{\omega_0} - \sum_{\beta=2}^{n}\frac{u_\beta\bar{\eta}}{a(1+|u|^2)^2}(P_s + P_{ss}s)s^2\omega_0\wedge d\bar{u}_\beta \\
&\quad + \frac{1}{a(1+|u|^2)}(P_s + P_{ss}s)s^2\omega_0\wedge\overline{\omega_0}
\end{aligned} \tag{6.28}$$

In the above calculation, we used many times the relation $s = a|\xi|^2 = \frac{a(1+|u|^2)}{|\eta|^2}$.

Lemma 38. *The closing condition for compact shrinking soliton is: there exists a $b_1 > 0$, such that*

$$F(b_1) = 0, F'(b_1) = -1 \tag{6.29}$$

Proof. Define $\tilde{s} = s^{-1} = \frac{|\eta|^2}{a(1+|u|^2)}$. Under the condition (6.29), then near b_1, $\phi_r = F(\phi) = -(\phi - b_1) + O((\phi - b_1)^2$. So up to the main term, $\phi - b_1 \sim -C_0 e^{-r} = -C_0\frac{1}{s} = -C_0\tilde{s}$ for some $C_0 > 0$, $(P_s s)_s s^2 = \phi_s s^2 \sim C_0$, $P_{ss}s^2 = (P_s s)_s s - P_s s = \phi_s s - \phi \sim -b_1 + \frac{2C_0}{s} = -b_1 + 2C_0\tilde{s}$, . So we first see that the coefficients in (6.28) are smooth near infinity divisor defined by $\eta = 0$ (or equivalently $\tilde{s} = 0$). We only need to show ω is positive definite everywhere. In fact, we only need to check when $\tilde{s} = 0$. When $\tilde{s} = 0$, we have

$$\omega = (1+b_1)\omega_M + \sum_{\alpha=2}^{n}\sum_{\beta=2}^{n}(b_1\delta_{\alpha\beta} - b_1\frac{\bar{u}_\alpha u_\beta}{1+|u|^2})\frac{1}{1+|u|^2}du_\alpha\wedge d\bar{u}_\beta + \frac{C_0}{a(1+|u|^2)}\omega_0\wedge\overline{\omega_0}$$

So ω is positive definite. So it defines a smooth Kähler metric on the projective compactification.

□

By $(*)$, this condition determines

$$b_1 = \frac{n+1}{\tau - n\epsilon} = \frac{n+1}{\lambda}$$

Since $F(0) = 0$, this condition is equivalent to

$$0 = F(b_1) - F(0) = \int_0^{b_1} h(\phi) e^{-\mu\phi} d\phi = T(\mu) \tag{6.30}$$

$$\begin{aligned}
T(0) &= \int_0^{b_1} h(\phi) d\phi = \int_0^{b_1} (1+\epsilon\phi)^d \phi^{n-1}((\tau-\epsilon)\phi - 1) \\
&= \int_0^{b_1} \sum_{k=0}^{d} \binom{d}{k} \epsilon^k (\lambda \phi^{k+n} - \phi^{k+n-1}) d\phi \\
&= \sum_{k=0}^{d} \binom{d}{k} \epsilon^k b_1^{k+n} \left(\frac{\lambda b_1}{k+n+1} - \frac{n}{k+n} \right) \\
&= \sum_{k=0}^{d} \binom{d}{k} \epsilon^k b_1^{k+n} \frac{k}{(k+n+1)(k+n)} > 0
\end{aligned}$$

On the other hand,

$$T(\mu) = \frac{1}{\mu^{d+n+1}} \sum_{k=0}^{d+n} \mu^{d+n-k} (h^{(k)}(0) - h^{(k)}(b_1) e^{-\mu b_1})$$

Since $h^{(0)}(0) < 0$ (6.20), and $\lim_{\mu \to +\infty} e^{-\mu b_1} = 0$. It's easy to see that $T(\mu) < 0$ for μ sufficiently large. So there is a zero point for $T(\mu)$ on $(0, \infty)$. The uniqueness is difficult to see directly, but because different solutions of (6.30) would give proportional vector fields and hence proportional potential functions, by using Tian-Zhu's invariant [TiZhu], we indeed have the uniqueness.

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