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Conical Kähler–Einstein Metrics Revisited

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Abstract: In this paper we introduce the “interpolation–degeneration” strategy to study Kähler–Einstein metrics on a smooth Fano manifold with cone singularities along a smooth divisor that is proportional to the anti-canonical divisor. By “interpolation” we show the angles in $(0, 2\pi]$ that admit a conical Kähler–Einstein metric form a connected interval, and by “degeneration” we determine the boundary of the interval in some important cases. As a first application, we show that there exists a Kähler–Einstein metric on $\mathbb{P}^2$ with cone singularity along a smooth conic (degree 2) curve if and only if the angle is in $(\pi/2, 2\pi]$. When the angle is $2\pi/3$ this proves the existence of a Sasaki–Einstein metric on the link of a three dimensional A_2 singularity, and thus answers a question posed by Gauntlett–Martelli–Sparks–Yau. As a second application we prove a version of Donaldson’s conjecture about conical Kähler–Einstein metrics in the toric case using Song–Wang’s recent existence result of toric invariant conical Kähler–Einstein metrics.

Contents

1. Introduction

The existence of Kähler–Einstein metrics on a smooth Kähler manifold X is a main problem in Kähler geometry. For the case when the first Chern class of X is negative, this problem was solved by Aubin [3] and Yau [74]. For the case when the first Chern class is zero, this problem was settled by Yau [74]. The main interest at present lies in the case of Fano manifolds, when the first Chern class is positive. There is the famous Yau–Tian–Donaldson program which relates the existence of Kähler–Einstein metrics to algebro-geometric stability.

More generally one could look at a pair (X, D) where D is a smooth divisor in a Kähler manifold X, and study the existence of Kähler–Einstein metrics on X with cone singularities along D and smooth away from D. This problem was classically studied on the Riemann surfaces [45,48,72] (see also a recent paper [20]) and was first considered in higher dimensions by Tian in [65]. Recently, there is much new interest on this generalized problem, mainly due to Donaldson's program (see [25]) on constructing smooth Kähler–Einstein metrics on X by varying the cone angle along an anti-canonical divisor. There are many subsequent works, see, for example, [6,34].

From now on in this paper, we assume X is a smooth Fano manifold, and D is a smooth divisor which is $\mathbb{Q}$-linearly equivalent to $-\lambda K_X$ with $0 < \lambda \in \mathbb{Q}$. β will always be a number in $(0, 1]$. We say (X, D) is *log canonical (resp. log Calabi–Yau, resp. log $\mathbb{Q}$-Fano) polarized* if $\lambda > 1$ (resp. $\lambda = 1$, resp. $\lambda < 1$). We will study Kähler–Einstein metrics in $2\pi c_1(X)$ with cone singularities along D. The equation is given by

$$Ric(\omega) = r(\beta)\omega + 2\pi(1-\beta)\{D\}, \tag{$*$}$$

where $2\pi\beta$ is the angle along D. For brevity we say ω is a *conical Kähler–Einstein metric* on $(X, (1-\beta)D)$. Note that when $\beta = 1$, conical Kähler–Einstein metrics become smooth Kähler–Einstein metrics.

Recall that the Ricci curvature form of a Kähler metric ω can be calculated as

$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\omega^n.$$

In other words, the volume form ω^n determines a Hermitian metric on K_X^{-1} whose Chern curvature is the Ricci curvature. So in particular, it represents the cohomology class $2\pi c_1(X)$. By taking cohomological class on both sides of the equation $(*)$, we obtain

$$r(\beta) = 1 - (1-\beta)\lambda. \tag{1}$$

We will use the above notation throughout this paper. Given a pair (X, D), we define the set

$$E(X, D) = \{\beta \in (0, 1] \mid \text{There is a conical Kähler–Einstein metric on } (X, (1-\beta)D)\}.$$

Theorem 1.1. *If $\lambda \geq 1$, then $E(X, D)$ is a connected relatively open interval in $(0, 1]$, which contains $(0, 1 - \lambda^{-1} + \epsilon)$ for some $\epsilon = \epsilon(\lambda) > 0$.*

The last property essentially follows from the work of [34] and [6], and we will review it in Sect. 2. Now suppose X admits a smooth Kähler–Einstein metric and $\lambda \geq 1$, then by Theorem 1.1 there exists a Kähler–Einstein metric ω_β on $(X, (1-\beta)D)$ for any $\beta \in (0, 1]$. By [11] and [6] we know ω_β is unique for $\beta \in (0, 1)$. Moreover, by the implicit function theorem in [25] ω_β varies continuously when β varies. When β goes to one, we have

Theorem 1.2. *If X admits a Kähler–Einstein metric and $Aut(X)$ is discrete, then the potential of ω_β converges to the potential of ω_{KE} in the C^0 norm, where ω_{KE} is the unique smooth Kähler–Einstein metric on X.*

Remark 1.3. This is in a similar flavor to Perelman's theorem [70], that the Kähler–Ricci flow converges on a Kähler–Einstein Fano manifold. In particular, when $\lambda = 1$ this provides evidence for Donaldson's program. An algebro-geometric counterpart about K-stability was shown in [49,59]. When β tends to zero, this is related to a conjecture of Tian [65] that the rescaled limit should be a complete Calabi–Yau metric on the complement of D.

When $Aut(X)$ is not discrete, we will prove the convergence of ω to a distinguished Kähler–Einstein metric ω_{KE}^D, modulo one technical issue, see Sect. 7. The issue is that, since we need to work in different function spaces corresponding to different cone angles, the application of implicit function theorem is more delicate as shown by Donaldson in [25], and Donaldson's linear theory does not provide uniform Schauder estimate when β is close to 1. However, in this case even though the Kähler–Einstein metrics on X are not unique, we can still identify the correct limit Kähler–Einstein metric in the moduli space. To do this, we use Bando–Mabuchi's bifurcation method. The result we find is that, the only obstruction for solving the conical Kähler–Einstein metric from $\beta = 1$ to $\beta = 1 - \epsilon$ (for $0 < \epsilon \ll 1$) comes from the holomorphic vector fields on X tangent to D, i.e., $LieAut(X, D)$. If we assume $\lambda \geq 1$, then $Aut(X, D)$ is discrete, and hence the obstruction vanishes. For more details, see the discussion in Sect. 7.

Another motivation for this paper comes from the study of conical Kähler–Einstein metrics on our favorite example $\mathbb{P}^2$. In this case when D is a smooth curve of degree bigger than two, we are in the setting of the above theorem and we know the conical Kähler–Einstein metrics exist on $(X, (1-\beta)D)$ for all $\beta \in (0, 1]$. When the degree is one or two, we are in the case $\lambda < 1$. We have an obstruction coming from log K-stability.

Theorem 1.4. *If $\lambda < 1$, then there is no conical Kähler–Einstein metric on $(X, (1-\beta)D)$ for $\beta < (\lambda^{-1} - 1)/n$, where n is the dimension of X.*

This immediately implies that there is no Kähler–Einstein metric on $\mathbb{P}^2$ which bends along a line, which could also be seen from the Futaki invariant obstruction. The most interesting case is when the degree is 2.

Theorem 1.5. *When D is a smooth conic in $\mathbb{P}^2$, i.e., a smooth degree two curve, then $E(X, D) = (1/4, 1]$.*

From the proof we also speculate the limit of the conical Kähler–Einstein metrics ω_β as β tends to $1/4$. As an application of the above theorem, we have

Corollary 1.6. *A three dimensional* A_2 *singularity* $x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$ *admits a Calabi-Yau cone metric with the natural Reeb vector field.*

This settles a question in [29]. As mentioned in [29], such a Calabi–Yau cone metric on A_2 might be dual to an exotic type of field theory since the corresponding Calabi–Yau cone does not admit a crepant resolution. Note this shows that the classification of cohomogeneity one Sasaki–Einstein manifolds given in [21] is incomplete, which is confirmed by the numerical result and calculations by the first author in [42]. See Remark 5.4.

Now we briefly discuss the strategy to prove the above results. The proof of Theorem 1.1 follows from the following "interpolation" result. Here one point in the statement is that the log-Mabuchi-energy is well defined on the space of *admissible functions* denoted by $\hat{\mathcal{H}}(\omega)$, which includes all the Kähler potentials of conical Kähler metrics for different angles. The definition of log-Mabuchi-energy and log-Ding-energy, as well as $\hat{\mathcal{H}}(\omega)$, will be given in Sect. 2.

Proposition 1. *As functionals on* $\hat{\mathcal{H}}(\omega)$*, the log-Mabuchi-energy* $\mathcal{M}_{\omega,(1-\beta)D}$ *is linear in* β*. The normalized log-Ding-energy* $r(\beta)F_{\omega,(1-\beta)D}$ *is concave downward in* β *up to a bounded constant. As a consequence, if the log-Mabuchi-energy (resp. log-Ding-energy) is proper for* $\beta_1 \in (0, 1]$ *and bounded from below for* $\beta_2 \in (0, 1]$*, then for any* β *between* β_1 *and* β_2*, the log-Mabuchi-energy (resp. log-Ding-energy) is proper, so there exists a conical Kähler–Einstein metric on* $(X, (1-\beta)D)$.

By combining Proposition 1 with the openness result of Donaldson [25], and the result of Berman [6] (see section 4.3) we easily see that

Corollary 1.7. *If* $\lambda \geq 1$ *and there is a conical Kähler–Einstein metric on* $(X, (1-\beta)D)$ *for* $0 < \beta < 1$*, then the log-Mabuchi energy* $\mathcal{M}_{\omega,(1-\beta)D}$ *is proper.*

Theorem 1.1 easily follows from the above proposition. In general, to apply Proposition 1, we often need to get the lower bound of log-Mabuchi-energy. For this we introduce the "degeneration" method. We have

Theorem 1.8. *If there exists a special degeneration* $(\mathcal{X}, (1-\beta)\mathcal{D}, \mathcal{L})$ *of* $(X, (1-\beta)D)$ *to a conical Kähler–Einstein variety* $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$*. Assume* $\mathcal{X}_0$ *has isolated* $\mathbb{Q}$*-Gorenstein singularities. Then the log-Ding-functional and log-Mabuchi-energy of* $(X, (1-\beta)D)$ *are bounded from below.*

Remark 1.9. Here the assumption that $\mathcal{X}_0$ has isolated singularities is purely technical, but it is satisfied for our main application here to prove Theorem 1.5. We now know a general statement to be true(see Remark 4.11 in Sect. 5).

In particular, we provide an alternative proof of a special case of a theorem of Chen [18]:

Corollary 1.10 (Chen's theorem in the Kähler–Einstein case). *If there exists a special degeneration of Fano manifold* (X, J) *to a Kähler–Einstein manifold* (X_0, J_0)*, then the Mabuchi energy on* X *in the class* $2\pi c_1(X)$ *is bounded from below.*

To rule out the existence for small angles stated in Theorem 1.4, we need to generalize the K-stability obstructions to the conical setting.

Theorem 1.11. *If the log-Ding-functional $F_{\omega,(1-\beta)D}$ or the log-Mabuchi-energy $\mathcal{M}_{\omega}$, $(1-\beta)D$ is bounded from below(resp. proper), then the polarized pair $((X,(1-\beta)D),-K_X)$ is log-K-semistable(resp. log-K-stable).*

Corollary 1.12. (1) *If there exists a conical Kähler–Einstein metric on $(X,(1-\beta)D)$, then $((X,(1-\beta)D);-K_X)$ is log-K-semistable. As a consequence, if $\lambda \geq 1$, then $((X,(1-\beta)D);-K_X)$ is log-K-semistable for $0 \leq \beta < 1-\lambda^{-1}+\epsilon$ for some $\epsilon > 0$*
(2) *Assume $\lambda \geq 1$ and $0 < \beta < 1$. If there exists a conical Kähler–Einstein metric on $(X,(1-\beta)D)$, then $((X,(1-\beta)D);-K_X)$ is log-K-stable.*

Theorem 1.5 is proved by the above "interpolation–degeneration" method. We first use Theorem 1.4 to show $E(X,D) \subset [1/4,1]$. Then we find an explicit special degeneration of $(X,3/4D)$ to $(\mathbb{P}(1,1,4),3/4D_0)$ where $D_0=\{z_3=0\}$ which admits the natural conical Kähler–Einstein metric. Since X itself admits a Kähler–Einstein metric, Theorem 1.5 will follow from the interpolation argument using Theorem 1.8. A technical point is that we do not get the full properness of Ding functional due to the presence of holomorphic vector fields. For details, see Sect. 5.

The organization of the paper is as follows. In Sect. 2, we review some preliminary materials, including the definition of Hölder norms with respect to conical Kähler metrics, various energy functionals, and existence theory for conical Kähler–Einstein metrics. We prove Proposition 1, Theorem 1.1 and Corollary 1.7 in Sect. 2.4. In Sect. 3, we explain the obstructions to the existence of conical Kähler–Einstein metrics. In particular, we prove Theorem 1.11 and its Corollary 1.12, and Theorem 1.4. In Sect. 4, we prove Theorem 1.8. In Sect. 5, we prove Theorem 1.5 and obtain Corollary 1.6. In Sect. 6, we discuss the construction of smooth Kähler–Einstein metrics using branch covers. In Sect. 7, we prove Corollary 1.2. We also prove (modulo one technical point) the convergence in the case when there are holomorphic vector fields on X.

After finishing the draft of this paper, we received the paper by Song and Wang [58]. In the last Sect. 8, we discuss the relation of their work to our paper. In addition to some overlaps with our results, they proved an existence result in the toric case. The conical Kähler–Einstein spaces they obtained can serve as the degeneration limits of toric Fano manifolds with some smooth pluri-anticanonical divisors. So combining their existence result in the toric case with the strategy in this paper, we show, in the toric case, a version of Donaldson's conjecture which relates the maximal cone angle and the greatest lower bound of Ricci curvature. To state this result, first define

$$R(X)=\sup\{t \mid \exists \text{ Kähler metric } \omega \in 2\pi c_1(X) \text{ such that } Ric(\omega) \geq t\omega\}. \tag{2}$$

Proposition 2. *Let X be a toric Fano manifold. For each λ sufficiently divisible, there exists a sub-linear system $\mathcal{L}_\lambda$ of $|-\lambda K_X|$ such that for any general member $D \in \mathcal{L}_\lambda$, if D is smooth, then there exists a conical Kähler–Einstein metric on $(X,(1-\gamma)\lambda^{-1}D)$ with positive Ricci curvature if and only if $\gamma \in (0,R(X))$.*

Remark 1.13. The smoothness assumption is easily satisfied when $\dim(X) \leq 2$. It seems to be guaranteed by choosing $\mathcal{L}_\lambda$ more carefully. See the discussion in Remark 8.6. In general, if D is not smooth, then there exists a weak solution (i.e., bounded solution) to the conical Kähler–Einstein equation.

The idea of the proof is similar to the proof of Theorem 1.5, again using the "interpolation–degeneration" method. The sublinear system $\mathcal{L}_\lambda$ we construct has the property that for each general member D in $\mathcal{L}_\lambda$, (X,D) has a degeneration to the toric conical Kähler–Einstein pair (X,D_0) constructed by Song–Wang.

The interpolation properties of energy functionals obtained in this paper seem to be observed by some other experts in the field too. In particular, we were informed by Professor Arezzo that he also observed this.

2. Existence Theory on Conical Kähler–Einstein Metrics

2.1. Space of admissible potentials. In this paper, all Kähler metrics will be in the class $2\pi c_1(X)$.

Definition 2.1. (1) A conical Kähler metric on $(X, (1-\beta)D)$ is a Kähler current ω in the class $2\pi c_1(X)$ with locally bounded potential, smooth on $X \setminus D$, and for any point $p \in D$, there is a local coordinate $\{z_i\}$ in a neighborhood of p such that $D = \{z_1 = 0\}$ such that ω is quasi-isometric to the model metric:

$$\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i. \tag{3}$$

Geometrically, ω represents a Kähler metric with cone singularities along D of angle $2\pi\beta$.

(2) A conical Kähler–Einstein metric on $(X, (1-\beta)D)$ is a conical Kähler metric solving the equation

$$Ric(\omega) = r(\beta)\omega + 2\pi(1-\beta)\{D\}.$$

Here $\{D\}$ is the current of integration on D, and $r(\beta) = 1 - (1-\beta)\lambda$.

Now we follow Donaldson [25] to define the Hölder norm with respect to conical metric. let $(z_1, z_2, \ldots, z_n)$ be the coordinates near a point in D as chosen above. Let $z = re^{i\theta}$ and let $\rho = r^\beta$. The model metric in (3) becomes

$$(d\rho + \sqrt{-1}\beta\rho d\theta) \wedge (d\rho - \sqrt{-1}\beta\rho d\theta) + \sum_{j>1} dz_j \wedge d\bar{z}_j$$

Let $\epsilon = e^{\sqrt{-1}\beta\theta}(d\rho + \sqrt{-1}\beta\rho d\theta)$, we can write

$$\omega = \sqrt{-1}\left(f\epsilon \wedge \bar{\epsilon} + f_{\bar{j}}\epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j\right) \tag{4}$$

Definition 2.2. (1) A function f is in $C^{,\gamma,\beta}(X, D)$ if f is C^γ on $X \setminus D$, and locally near each point in D, f is C^γ in the coordinate $(\hat{\zeta} = \rho e^{i\theta} = z_1|z_1|^{\beta-1}, z_j)$.

(2) A (1,0)-form α is in $C^{,\gamma,\beta}(X, D)$ if α is C^γ on $X \setminus D$ and locally near each point in D, we have $\alpha = f_1\epsilon + \sum_{j>1} f_j dz_j$ with $f_i \in C^{,\gamma,\beta}$ for $1 \le i \le n$, and $f_1 \to 0$ as $z_1 \to 0$.

(3) A (1,1)-form ω is in $C^{,\gamma,\beta}(X, D)$ if ω is C^γ on $X \setminus D$ and near each point in D we can write ω as (4) such that $f, f_j, f_{\bar{j}}, f_{i\bar{j}} \in C^{,\gamma,\beta}$, and $f_j, f_{\bar{j}} \to 0$ as $z_1 \to 0$.

(4) A function f is in $C^{2,\gamma,\beta}(X, D)$ if $f, \partial f, \partial\bar{\partial} f$ are all in $C^{,\gamma,\beta}$.

It is easy to see that the above definitions do not depend on the particular choice of local complex chart. Donaldson set up the linear theory in [25].

Proposition 3 ([25]). *If $\gamma < \mu = \beta^{-1} - 1$, then the inclusion $C^{2,\gamma,\beta}(X, D) \to C^{,\gamma,\beta}(X, D)$ is compact. If ω is a $C^{,\gamma,\beta}$ Kähler metric on (X, D) then the Laplacian operator for ω defines a Fredholm map $\Delta_\omega : C^{2,\gamma,\beta}(X, D) \to C^{,\gamma,\beta}(X, D)$.*

In order to consider the conical Kähler–Einstein metrics for different cone angles at the same time, we define the following space

Definition 2.3. Fix a smooth metric ω_0 in $c_1(X)$, we define the space of admissible functions to be

$$\hat{\mathcal{C}}(X, D) = C^{2,\gamma}(X) \cup \bigcup_{0<\beta<1} \left(\bigcup_{0<\gamma<\beta^{-1}-1} C^{2,\gamma,\beta}(X, D) \right),$$

and the space of admissible Kähler potentials to be

$$\hat{\mathcal{H}}(\omega_0) = \{\phi \in \hat{\mathcal{C}}(X, D) | \omega_\phi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

Note that $\hat{\mathcal{H}}(\omega_0)$ includes the space of smooth Kähler potentials

$$\mathcal{H}(\omega_0) = \{\phi \in C^\infty(X) | \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\},$$

and is contained in the bigger space of bounded ω_0-plurisubharmonic functions $\mathcal{PSH}_\infty(\omega_0) = \mathcal{PSH}(\omega_0) \cap L^\infty(X)$ where

$$\mathcal{PSH}(\omega_0) = \{\phi \in L^1_{loc}(X); \phi \text{ is u.s.c. and } \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi \geq 0\}.$$

Modulo constants the space of admissible Kähler metrics corresponding to $\hat{\mathcal{H}}(\omega_0)$ consists exactly $C^{,\gamma,\beta}$ Kähler metrics on (X, D) for different γ and β.

We will need the following fundamental openness theorem proved by Donaldson.

Theorem 2.4 ([25]). *Let $\beta_0 \in (0, 1)$, $\alpha < \mu_0 = \beta_0^{-1} - 1$ and suppose there is a C^{2,α,β_0} conical Kähler–Einstein ω_β on $(X, (1-\beta_0)D)$. If there is no nonzero holomorphic vector fields on X tangent to D, then for β suifficiently close to β_0 there is a $C^{2,\alpha,\beta}$ conical Kähler–Einstein metric on $(X, (1-\beta)D)$.*

For later applications, we give a slight generalization. We use an idea from [54, Theorem 2]. Let $Aut(X, D)$ be the automorphism group of the pair (X, D), and let $G = Isom(X, D, \omega_{\beta_0})$ denote the isometry group of C^{2,α,β_0} conical Kähler–Einstein metric ω_{β_0} on $(X, (1-\beta_0)D)$, so that $Lie(G) = \{\text{Killing vector field of } (X, D, \omega_{\beta_0})\}$. Let

$$(\Lambda^{\mathbb{R}}_{r(\beta_0)})_0 = \left\{ \theta \in C^\infty(X, \mathbb{R}); (\Delta_{\omega_{\beta_0}} + r(\beta_0))\theta = 0, \int_X \theta \frac{\omega^n_{\beta_0}}{n!} = 0 \right\}.$$

Lemma 2.5 ([16,19,46]). *We have the isomorphism:*

$$\begin{gathered}(\Lambda^{\mathbb{R}}_{r(\beta_0)})_0 \cong Lie(G). \\ Lie(Aut(X, D)) = (\Lambda^{\mathbb{R}}_{r(\beta_0)})_0 \otimes_{\mathbb{R}} \mathbb{C} = (\Lambda^{\mathbb{C}}_{r(\beta_0)})_0.\end{gathered} \tag{5}$$

Note that the isomorphism (5) is given as follows. For any $\theta \in (\Lambda^{\mathbb{R}}_{r(\beta_0)})_0$, we associate $V_\theta := -J\nabla_{\omega_{\beta_0}}\theta \in Lie(G)$. Equivalently, we have the identity $\iota_V \omega_{\beta_0} = d\theta$. From this, it's easy to see that the isomorphism is equivariant under the action of $G = Isom(X, D, \omega_{\beta_0})$.

Proposition 4. *With the above notations, let $H < G$ be a closed subgroup. Assume that $Centr_G H$ is finite. Then for β sufficiently close to β_0, there is a H-invariant $C^{2,\alpha,\beta}$ conical Kähler–Einstein metric on $(X, (1-\beta)D)$.*

Proof. If we adapt Donaldson' proof to H-invariant $C^{2,\alpha,\beta}$ conical Kähler metrics, we just need to show that the H-invariant functions in $(\Lambda^{\mathbb{R}}_{r(\beta_0)})_0$ is 0. By the above isomorphism, any $\theta = \theta_V \in (\Lambda^{\mathbb{R}}_{r(\beta_0)})_0$ gives rise to a Killing vector field $V_\theta = -J\nabla_{\omega_{\beta_0}}\theta$. If θ is H-invariant, then by the G-equvariance of the isomorphism (5), V_θ is also H-invariant. By the assumption the only fixed point of H on $Lie(G)$ is 0. So $V_\theta = 0$ and hence $\theta = 0$. □

2.2. Energy functionals and analytic criterions. In the analytic study of Kähler–Einstein metrics, various functionals play important roles. We review them carefully in this subsection. Although they were originally defined on the space of smooth Kähler potentials $\mathcal{H}(\omega_0)$, they can be naturally extended for $\phi \in \hat{\mathcal{H}}(\omega_0)$, and some of them can even be defined on $\mathcal{PSH}(\omega_0) \cap L^\infty(X)$.

Definition 2.6. For any $\phi \in \mathcal{H}(\omega_0)$, we define the functionals

(1)

$$F^0_{\omega_0}(\phi) = -\frac{1}{(n+1)!}\sum_{i=0}^{n}\int_X \phi\omega^i_\phi \wedge \omega_0^{n-i}$$

(2)

$$J_{\omega_0}(\phi) = F^0_{\omega_0}(\phi) + \int_X \phi\omega_0^n/n!$$

(3)

$$I_{\omega_0}(\omega_\phi) = \int_X \phi(\omega_0^n - \omega_\phi^n)/n!,$$

By pluripotential theory the above functionals are also well-defined for $\phi \in \mathcal{PSH}(\omega_0) \cap L^\infty(X)$. The following facts are also well known.

Proposition 5 ([2,4]).

(1) If ϕ_t is a smooth path in $\mathcal{H}(\omega_0)$, then

$$\frac{d}{dt}F^0_\omega(\phi_t) = -\int_X \dot{\phi}\omega^n_\phi/n!, \tag{6}$$

(2)

$$\frac{n+1}{n}J_\omega(\phi) \le I_\omega(\phi) \le (n+1)J_\omega(\phi), \tag{7}$$

Remark 2.7. Equation (6) tells us that $F^0_\omega(\phi)$ is the integral of Bott–Chern form. (See [66]) If we let h be the Hermitian metric on K_X^{-1} such that $\omega_h := -\sqrt{-1}\partial\bar{\partial}\log h = \omega$. Denote $h_\phi = he^{-\phi}$. Connect h and h_ϕ by any path $h_t = he^{-\phi_t}$. The corresponding path of curvature forms $\omega_t = \omega_{h_t} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ connects ω and ω_ϕ. The Bott–Chern form is defined by

$$BC\left(c_1(K_X^{-1})^{n+1}; h, h_\phi\right) = -\int_0^1 dt(n+1)h_t^{-1}\dot{h}_t c_1(K_X^{-1}, h_t)^n = (n+1)\int_0^1 dt\dot{\phi}\omega_t^n.$$

So we have the following identities

$$F^0_\omega(\phi) = -\frac{1}{(n+1)!}\int_X BC\left(c_1(K_X^{-1})^{n+1}; h, h_\phi\right).$$

We now recall the generalization (see [6,34,40]) of the Mabuchi-energy ([47]) and Ding-energy ([23]) to the conical setting. In the next section we will show that log-Mabuchi-energy integrates log-Futaki invariant. First we introduce some more notations. For the smooth metric ω_0 in $c_1(X)$, define the twisted Ricci potential $H_{\omega_0,(1-\beta)D}$ by

$$Ric(\omega_0) - r(\beta)\omega_0 - 2\pi(1-\beta)\{D\} = \sqrt{-1}\partial\bar{\partial}H_{\omega_0,(1-\beta)D},$$
$$\int_X e^{H_{\omega_0,(1-\beta)D}}\frac{\omega_0^n}{n!} = \int_X \frac{\omega_0^n}{n!}.$$

It is easy to see that up to a constant $H_{\omega_0,(1-\beta)D} = h_{\omega_0} - (1-\beta)\log|s_D|^2$, where h_{ω_0} is the usual Ricci potential of ω_0, defined by the following identities:

$$Ric(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_{\omega_0}, \quad \int_X e^{h_{\omega_0}}\frac{\omega_0^n}{n!} = \int_X \frac{\omega_0^n}{n!}.$$

$|s_D|^2$ is the norm of the defining section of D under the Hermitian metric on $-K_X$ satisfying $-\sqrt{-1}\partial\bar{\partial}\log|s_D|^2 = \omega_0$. We will use the following definition of volumes in this paper:

$$Vol(X) = \int_X \frac{\omega^n}{n!} = (2\pi)^n\frac{\langle c_1(K_X^{-1})^n, [X]\rangle}{n!},$$
$$Vol(D) = \int_D \frac{\omega^{n-1}}{(n-1)!} = (2\pi)^{n-1}\frac{\langle c_1(K_X^{-1})^{n-1}, [D]\rangle}{(n-1)!}.$$

Definition 2.8. (1) (log-Mabuchi-energy) For any $\phi \in \hat{\mathcal{H}}_{\omega_0}$

$$\begin{aligned}\mathcal{M}_{\omega_0,(1-\beta)D}(\omega_\phi) &= \int_X \log\frac{\omega_\phi^n}{e^{H_{\omega_0,(1-\beta)D}}\omega_0^n}\frac{\omega_\phi^n}{n!} + r(\beta)\left(\int_X \phi\frac{\omega_\phi^n}{n!} + F^0_{\omega_0}(\phi)\right)\\ &\quad + \int_X H_{\omega_0,(1-\beta)D}\frac{\omega_0^n}{n!}\\ &= \int_X \log\frac{\omega_\phi^n}{e^{H_{\omega_0,(1-\beta)D}}\omega_0^n}\frac{\omega_\phi^n}{n!} - r(\beta)(I-J)_{\omega_0}(\omega_\phi)\\ &\quad + \int_X H_{\omega_0,(1-\beta)D}\frac{\omega_0^n}{n!}.\end{aligned}$$

(2) (log-Ding-energy)

$$F_{\omega_0,(1-\beta)D}(\omega_\phi)=F^0_{\omega_0}(\phi)-\frac{Vol(X)}{r(\beta)}\log\left(\frac{1}{Vol(X)}\int_X e^{H_{\omega_0,(1-\beta)D}-r(\beta)\phi}\frac{\omega_0^n}{n!}\right).$$

We will also call $r(\beta)F_{\omega_0,(1-\beta)}(\omega_\phi)$ the normalized log-Ding-energy.

When $\beta=1$ these functionals go back to the original functionals on smooth manifolds, which we denote by $\mathcal{M}_{\omega_0}$ and F_{ω_0} for simplicity.

By studying the behavior of conical metrics near D, it is not hard to see that the above functionals $\mathcal{M}_{\omega_1}(\omega_2)$, etc. are all well defined for any a C^{γ_1,β_1} metric ω_1 and C^{γ_2,β_2} metric ω_2. We have the following proposition generalizing the facts in the smooth case. (see [6,69])

Proposition 6. *(1) The Euler–Lagrange equations of log-Mabuchi-energy and log-Ding-energy are the same:*

$$(\omega_0+\sqrt{-1}\partial\bar{\partial}\phi)^n=e^{-r(\beta)\phi}e^{H_{\omega_0,(1-\beta)D}}\omega_0^n$$

(2) The log-Mabuchi-energy and log-Ding-energy differ by a cocycle:

$$\mathcal{M}_{\omega_0,(1-\beta)D}(\omega_\phi)=r(\beta)F_{\omega_0,(1-\beta)D}(\omega_\phi)+\int_X H_{\omega_0,(1-\beta)D}\frac{\omega_0^n}{n!}-\int_X H_{\omega_\phi,(1-\beta)D}\frac{\omega_\phi^n}{n!}.$$

(3) log-Mabuchi-energy is bounded from below by log-Ding-energy:

$$\mathcal{M}_{\omega_0,(1-\beta)D}(\omega_\phi)\geq r(\beta)F_{\omega_0,(1-\beta)D}(\omega_\phi)+\int_X H_{\omega_0,(1-\beta)D}\frac{\omega_0^n}{n!}$$

The equality holds if and only if ω_ϕ is a conical Kähler–Einstein metric on $(X,(1-\beta)D)$.

(4) (co-cycle condition) Assume ω_i are $C^{,\gamma_i,\beta_i}$ Kähler metrics on (X,D), for $i=1,2,3$. Then

$$\mathcal{M}_{\omega_1,(1-\beta)D}(\omega_2)+\mathcal{M}_{\omega_2,(1-\beta)D}(\omega_3)=\mathcal{M}_{\omega_1,(1-\beta)D}(\omega_3)$$
$$F_{\omega_1,(1-\beta)D}(\omega_2)+F_{\omega_2,(1-\beta)D}(\omega_3)=F_{\omega_1,(1-\beta)D}(\omega_3)$$

Proof. Items (1), (2) and (4) easily follows from the formula relating twisted Ricci potentials of two Kähler metrics.

$$\begin{aligned}H_{\omega_\phi,(1-\beta)D}&=H_{\omega_0,(1-\beta)D}+\log\frac{\omega_0^n}{\omega_\phi^n}-r(\beta)\phi-\log\left(\frac{1}{V}\int_X e^{H_{\omega_0,(1-\beta)D}-r(\beta)\phi}\frac{\omega_0^n}{n!}\right)\\&=-\left(\log\frac{\omega_\phi^n}{e^{H_{\omega_0,(1-\beta)D}-r(\beta)\phi}\omega_0^n}+\log\left(\frac{1}{V}\int_X e^{H_{\omega_0,(1-\beta)D}-r(\beta)\phi}\frac{\omega_0^n}{n!}\right)\right)\end{aligned}$$

Item (3) follows from from concavity of logarithm. □

Theorem 2.9 ([11]). *If there exists a conical Kähler–Einstein metric ω_β on $(X,(1-\beta)D)$, then ω_β obtains the minimum of log-Ding-energy $F_{\omega_0,(1-\beta)D}(\omega_\phi)$.*

Berndtsson ([11]) proved the important property that the log-Ding-energy is convex along a bounded geodesic in $\mathcal{PSH}_\infty(\omega_0)$. By Proposition 6 (3) we get

Corollary 2.10. *ω_β also obtains the minimum of log-Mabuchi-energy $\mathcal{M}_{\omega_0,(1-\beta)D}$.*

Remark 2.11. One technical point here is that it is more difficult to use convexity of log-Mabuchi-energy than that of log-Ding-energy, as it requires more regularity.

The following properness of energy functions was introduced by Tian [68].

Definition 2.12. A functional $F : \mathcal{H}(\omega_0) \to \mathbb{R}$ is called proper if there is an inequality of the type

$$F(\omega_\phi) \geq f\left(I_{\omega_0}(\omega_\phi)\right), \quad \text{for any } \phi \in \mathcal{H}(\omega),$$

where $f(t) : \mathbb{R}_+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t\to+\infty} f(t) = +\infty$.

Note that, by the inequalities (7), we could replace $I_{\omega_0}(\omega_\phi)$ by equivalent norms $J_{\omega_0}(\omega_\phi)$ or $(I-J)_{\omega_0}(\omega_\phi)$ in the above definition. Now we state a fundamental theorem by Tian which gives an analytic criterion for the existence of Kähler–Einstein metric.

Theorem 2.13 ([68]). *If $Aut(X, J)$ is discrete. There exists a Kähler–Einstein metric on X if and only if either $F_{\omega_0}(\omega_\phi)$ or $\mathcal{M}_{\omega_0}(\omega_\phi)$ is proper on $\mathcal{H}(\omega_0)$.*

The case when $Aut(X, J)$ is not discrete is more subtle. (We thank Professor G. Tian, Professor J. Song and Professor R. J. Berman for pointing out this to us). The full general statement is a conjecture by Tian [68]. But for our application, we just need the following result obtained in [54]. Note that we used this idea of [54] in Proposition 4.

Theorem 2.14 ([54]). *Let ω_{KE} be a smooth Kähler–Einstein metric on a Fano manifold X and denote $G = Isom(X, \omega_{KE})$. If $K \subset G$ is a closed subgroup whose centralizer in G denoted by* $\mathrm{Centr_K G}$ *is finite, then F_{ω_0} is proper on K-invariant potentials.*

It is natural to extend the definition of properness to the conical case, where we simply replace $\mathcal{H}(\omega_0)$ by $\hat{\mathcal{H}}(\omega_0)$.

Lemma 2.15. *If log-Mabuchi-energy or log-Ding-energy is proper (resp. bounded from below) on the space of smooth Kähler potentials, then it's proper (resp. bounded from below) on the space of admissible Kähler potentials.*

Proof. By Proposition 6, we just need to prove for the log-Ding-energy. By [22] (see also [14]), we can approximate any admissible Kähler potential ϕ by a sequence of decreasing smooth potentials $\phi_j \in \mathcal{PSH}(\omega_0) \cap C^\infty(X)$. Moreover, we can assume $\|\phi_j - \phi\|_{C^0} \to 0$. So by co-cycle condition, we get that

$$F^0_{\omega_0}(\phi_j) - F^0_{\omega_0}(\phi) = F^0_{\omega_\phi}(\phi_j - \phi) = -\frac{1}{n+1}\sum_{j=0}^{n}\int_X (\phi_j - \phi)\omega_{\phi_j}^{n-j} \wedge \omega_\phi^j \longrightarrow 0.$$

The Lemma easily follows from this fact. □

The following Lemma is well known (see for example [69] and [61]). We record a proof for the reader's convenience. From the proof, we see that the conclusion holds for any **continuous** potentials.

Lemma 2.16. *Let ω_i be a $C^{,\gamma_i,\beta_i}$ metric. Then the norm defined by J_{ω_1} and J_{ω_2} are equivalent, that is, there is a constant $C(\omega_1, \omega_2)$ such that for any other metric $\omega_3 \in \hat{\mathcal{H}}(\omega)$,*

$$\left|J_{\omega_1}(\omega_3) - J_{\omega_2}(\omega_3)\right| \leq C(\omega_1, \omega_2).$$

Proof. Assume $\omega_2 = \omega_1 + \sqrt{-1}\partial\bar{\partial}\phi$ and $\omega_3 = \omega_2 + \sqrt{-1}\partial\bar{\partial}\psi$. Then by co-cycle condition of $F^0_{\omega_i}$, it's easy to verify that

$$J_{\omega_1}(\omega_3) - J_{\omega_2}(\omega_3) = J_{\omega_1}(\omega_2) + \int_X \psi(\omega_1^n - \omega_2^n) =: J_{\omega_1}(\omega_2) + \mathbf{E}.$$

To estimate the term **E** we do integration by part:

$$\begin{aligned}\mathbf{E} &= \frac{1}{n!}\int_X \psi(\omega_1 - \omega_2) \wedge \left(\sum_{i=0}^{n-1} \omega_1^{n-1-i} \wedge \omega_2^i\right) \\ &= \frac{1}{n!}\int_X -\phi(\omega_3 - \omega_2) \wedge \left(\sum_{i=0}^{n-1} \omega_1^{n-1-i} \wedge \omega_2^i\right) \\ |\mathbf{E}| &\leq \frac{1}{n!}\int_X |\phi|\,(\omega_2 + \omega_3) \wedge \left(\sum_{i=0}^{n-1} \omega_1^{n-1-i} \wedge \omega_2^i\right) \leq 2n\|\phi\|_{L^\infty} Vol(X).\end{aligned}$$

□

By the cocycle relations and the above lemmas, we obtain

Proposition 7. *Assume ω_i is a C^{γ_i,β_i} Kähler metric on (X, D). Then $\mathcal{M}_{\omega_1,(1-\beta)D}$(or $F_{\omega_1,(1-\beta)D}$) is proper if and only if $\mathcal{M}_{\omega_2,(1-\beta)D}$(or $F_{\omega_2,(1-\beta)D}$) is proper.*

Now we can state a theorem on the existence of conical Kähler–Einstein metric which is a generalization of Tian's sufficient criterion in the smooth case.

Theorem 2.17 ([34]). *If the log-Mabuchi-energy is proper on $C^{2,\gamma,\beta}(X, D)$, then there exists a conical Kähler–Einstein metric on $(X, (1-\beta)D)$.*

The idea in [34] is to use continuity method as in the proof in the smooth case. More precisely, fix a backgroubd conical Kähler metric on $(X, (1-\beta)D)$ and consider the following family of equations.

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{H_{\omega,(1-\beta)D} - t\psi}\omega^n \tag{8}$$

This is equivalent to the equation

$$Ric(\omega_\psi) = t\omega_\psi + (r(\beta) - t)\omega + (1-\beta)\{s = 0\}. \tag{9}$$

Note that the C^0-estimate and weak solution was first obtained by Berman [6]. The a priori uniform C^2-estimate for any $\beta \in (0, 1)$ was first obtained in [34] which depends heavily on deriving upper bound of bisectional curvature for reference conical Kähler metric (see [34, Appendix]). The higher order $C^{2,\alpha,\beta}$ estimate in the conical setting is more complicated than in the smooth case when we have Evans–Krylov theory. In [34], the edge calculus was used to attack this (alternatively in the Appendix B in [34], Tian's original argument in the smooth case was adapted to the conical setting).

2.3. Alpha-invariant and small cone angles. In [6] and [34], Tian's alpha invariant [63] was generalized to the conical setting. We will explain this modification.

Definition 2.18 *(log alpha-invariant).* Fix a smooth volume form Ω. For any Kähler class $[\omega]$, we define

$$\alpha([\omega], (1-\beta)D) = \max\left\{\alpha > 0; \exists 0 < C_\alpha < +\infty \text{ s.t. } \int_X e^{-\alpha(\phi-\sup\phi)} \frac{\Omega}{|s_D|^{2(1-\beta)}} \right.$$
$$\left. \leq C_\alpha \text{ for any } \phi \in \mathcal{PSH}_\infty(X, [\omega])\right\}.$$

When $\beta = 1$, we get Tian's alpha invariant $\alpha([\omega])$ in [63]. In the following, we will write $\alpha(L, (1-\beta)D) = \alpha(2\pi c_1(L), (1-\beta)D)$ for any line bundle L. For any $\alpha < \alpha(K_X^{-1}, (1-\beta)D)$, using concavity of log function, we can estimate, for any $\phi \in \hat{\mathcal{H}}(\omega_0) \subset \mathcal{PSH}_\infty(\omega_0)$,

$$\begin{aligned}
\log C_\alpha &\geq \log\left(\frac{1}{V}\int_X e^{-\alpha(\phi-\sup\phi)} \frac{e^{h_{\omega_0}}\omega_0^n}{n!|s_D|^{2(1-\beta)}}\right)\\
&= \log\left(\frac{1}{V}\int_X e^{-\alpha(\phi-\sup\phi)-\log\frac{|s_D|^{2(1-\beta)}\omega_\phi^n}{e^{h_{\omega_0}}\omega_0^n}} \frac{\omega_\phi^n}{n!}\right)\\
&\geq -\frac{1}{V}\int_X \log\left(\frac{|s_D|^{2(1-\beta)}\omega_\phi^n}{e^{h_{\omega_0}}\omega_0^n}\right)\frac{\omega_\phi^n}{n!} + \alpha\left(\sup\phi - \frac{1}{V}\int_X \phi\frac{\omega_\phi^n}{n!}\right)\\
&\geq \frac{1}{V}\left(-\int_X \log\frac{\omega_\phi^n}{e^{H_{\omega_0,(1-\beta)D}}\omega_0^n}\frac{\omega_\phi^n}{n!} + \alpha I_{\omega_0}(\omega_\phi)\right).
\end{aligned}$$

In the last inequality, we used the expression for $H_{\omega_0,(1-\beta)D} = h_{\omega_0} - (1-\beta)\log|s_D|^2$. Now using the expression for $\mathcal{M}_{\omega,(1-\beta)D}$ in Definition 2.8 and inequalities in (7), we get

$$\begin{aligned}
\mathcal{M}_{\omega_0,(1-\beta)D}(\omega_\phi) &\geq \alpha I_{\omega_0}(\omega_\phi) - r(\beta)(I-J)_{\omega_0}(\omega_\phi) - C'_\alpha\\
&\geq \left(\alpha - r(\beta)\frac{n}{n+1}\right) I_{\omega_0}(\omega_\phi) - C'_\alpha
\end{aligned}$$

So if

$$\alpha(K_X^{-1}, (1-\beta)D) > \frac{n}{n+1}r(\beta) = \frac{n}{n+1}\left(1-\lambda(1-\beta)\right), \tag{10}$$

then log-Mabuchi-energy is proper for smooth reference metric. To estimate the alpha-invariant, we can use Berman's estimate:

Proposition 8 ([6]). *If we let L_D denote the line bundle determined by the divisor D, we have the estimate for log-alpha-invariant:*

$$\begin{aligned}
\alpha(K_X^{-1}, (1-\beta)D) &= \lambda\alpha(L_D, (1-\beta)D) \geq \lambda\min\{\beta, \alpha(L_D|_D), \alpha(L_D)\}\\
&= \min\{\lambda\beta, \lambda\alpha(L_D|_D), \alpha(K_X^{-1})\} > 0.
\end{aligned} \tag{11}$$

Corollary 2.19. *When* $\lambda \geq 1$, *if*

$$0 < \beta < \min\left(1, (1 - 1/\lambda) + \frac{n+1}{n}\min\{\alpha(L_D|_D), \lambda^{-1}\alpha(K_X^{-1})\}\right), \tag{12}$$

then the log-Mabuchi-energy is proper. In particular, when $0 < \beta < 1 - \lambda^{-1} + \epsilon$ *for* $\epsilon = \epsilon(\lambda) \ll 1$, *the log-Mabuchi-energy is proper. When* $\lambda < 1$, *we need to assume in addition that* $\beta > n(\lambda^{-1} - 1)$.

Proof. This follows from (10), (11) and the relation

$$\lambda\beta > \frac{n}{n+1}(1 - \lambda(1-\beta)) \Longleftrightarrow \beta > n(\lambda^{-1} - 1)$$

This is automatically true if $\lambda \geq 1$ and $\beta > 0$. □

Remark 2.20. If we use Hölder's inequality, we could get the estimate: $\alpha(K_X^{-1}, (1-\beta)D)) \geq \alpha(K_X^{-1})\beta > 0$. (Note that it's easy to get that $\lambda \geq \alpha(K_X^{-1})$ from the existence of smooth divisor $D \sim \lambda K_X^{-1}$) If we want to prove there always exists a conical Kähler–Einstein metric with small cone angle, this estimate only works when $\lambda > 1$ but not equal to 1. To see this, we study the inequality $\alpha(K_X^{-1})\beta > \frac{n}{n+1}r(\beta)$. When $\lambda > 1$, we get

$$\beta < (\lambda - 1)/(\lambda - \frac{n+1}{n}\alpha(K_X^{-1})) = (1 - \lambda^{-1})\left(1 + \frac{\frac{n+1}{n}\lambda^{-1}\alpha(K_X^{-1})}{1 - \frac{n+1}{n}\lambda^{-1}\alpha(K_X^{-1})}\right). \tag{13}$$

So again when $\beta < 1 - \lambda^{-1} + \epsilon$ for $\epsilon = \epsilon(\lambda) \ll 1$, the log-Mabuchi-energy is proper. When $\lambda = 1$, we get the condition $\alpha(K_X^{-1}) > \frac{n}{n+1}$. This condition is not always satisfied, and if it's true, then X has a smooth Kähler–Einstein metric by [63]. When $\lambda < 1$ we don't get useful condition on $\beta \in (0, 1)$. On the other hand, Berman's estimate works when $\lambda \geq 1$.

Corollary 2.21 (Berman, [6]). *When* $\lambda \geq 1$, *there is no holomorphic vector field on* X *tangent to* D.

Proof. If v is the holomorphic vector field tangent to D, then v generate a one-parameter subgroup $\lambda(t)$. Log-Mabuchi-energy is linear along $\sigma^*\omega$ with the slope given by the log-Futaki-invariant. This is in contradiction to Corollary 2.19. □

Remark 2.22. This corollary was speculated by Donaldson in [25]. This is also proved using pure algebraic geometry in Song–Wang's recent work [58].

2.4. Proof of Proposition 1, Theorem 1.1 and Corollary 1.7.

Proof of Proposition 1. Rewrite the log-Mabuchi-energy as:

$$\begin{aligned}\mathcal{M}_{\omega_0,(1-\beta)D}(\omega_\phi) = &\int_X \log\frac{\omega_\phi^n}{\omega_0^n}\omega_\phi^n + r(\beta)\left(\int_X \phi\omega_\phi^n + F_{\omega_0}^0(\phi)\right)\\ &+ \int_X (h_{\omega_0} - (1-\beta)\log|s|^2)\frac{\omega_0^n - \omega_\phi^n}{n!}.\end{aligned} \tag{14}$$

We see immediately that the linearity of log-Mabuchi-energy follows from the linearity of $r(\beta) = 1 - \lambda(1-\beta)$ in β and the relation $H_{\omega_0,(1-\beta_t)D} = (1-t)H_{\omega_0,(1-\beta_0)D} + tH_{\omega_0,(1-\beta_1)D} + C_t$. For the log-Ding-energy, let $\beta_t = (1-t)\beta_0 + t\beta_1$. So by Hölder inequality we get

$$\begin{aligned}&\int_X e^{H_{\omega_0,(1-\beta_t)D} - r(\beta_t)\phi} \frac{\omega_0^n}{n!}\\ &= e^{C_t} \int_X \left(e^{H_{\omega_0,(1-\beta_0)D} - r(\beta_0)\phi}\right)^{1-t} \left(e^{H_{\omega_0,(1-\beta_1)D} - r(\beta_1)\phi}\right)^t \frac{\omega_0^n}{n!}\\ &\le e^{C_t} \left(\int_X e^{H_{\omega_0,(1-\beta_0)D} - r(\beta_0)\phi} \frac{\omega_0^n}{n!}\right)^{1-t} \left(\int_X e^{H_{\omega_0,(1-\beta_1)D} - r(\beta_1)\phi} \frac{\omega_0^n}{n!}\right)^t\end{aligned}$$

By taking logarithm and using the definition of the log-Ding-energy we get, we get

$$r(\beta_t)F_{\omega_0,(1-\beta_t)D} \ge (1-t)r(\beta_0)F_{\omega_0,(1-\beta_0)D} + t \cdot r(\beta_1)F_{\omega_0+,(1-\beta_1)D} - \tilde{C}_t.$$

It's easy to see that $\tilde{C}_t$ is uniformly bounded with respect to $t \in [0, 1]$. □

Proof of Theorem 1.1. By the discussion above, when $\lambda \ge 1$, the $\mathcal{M}_{\omega,(1-\beta)D}$ is proper for $\beta \in (0, 1 - \lambda^{-1} + \epsilon)$ with some $\epsilon > 0$. On the other hand, when there is a conical Kähler–Einstein metric on $(X, (1-\beta_0)D)$, $\mathcal{M}_{\omega,(1-\beta_0)D}$ is bounded from below. So we can use Proposition 1 to get the properness of log-Mabuchi-energy for any $\beta \in (0, \beta_0)$. Now we use Theorem 2.17 to conclude. The openness follows from [25]. □

Proof of Corollary 1.7. Assume there exists a conical Kähler–Einstein metric for $0 < \beta = \beta_0 < 1$. Since we assume $\lambda \ge 1$, there is no holomorphic vector field on X fixing D by Corollary 2.21. By Donaldson's implicit function theorem 2.4 ([25]) for conical Kähler–Einstein metrics, there exists a conical Kähler–Einstein metric for $\beta = \beta_0 + \epsilon$ when $\epsilon \ll 1$. So the log-Mabuchi-energy is bounded for $\beta = \beta_0 + \epsilon$. Because log-Mabuchi-energy is proper for $0 < \beta \ll 1$, we can use interpolation result Proposition 1 to conclude the log-Mabuchi-energy is proper for $0 < \beta < \beta_0 + \epsilon$. □

3. Obstruction to Existence: Log-K-Stability

3.1. Log-Futaki invariant and log-K-(semi)stability. Fix a smooth Kähler metric $\omega \in 2\pi c_1(X)$. Assume D is a smooth divisor such that $D \sim_{\mathbb{Q}} -\lambda K_X$ for some $\lambda > 0 \in \mathbb{Q}$. Assume $\mathbb{C}^*$ acts on (X, D) with generating holomorphic vector field v. There exists a potential function $\theta_v \in C^\infty(X)$ satisfying $\sqrt{-1}\bar{\partial}\theta_v = \iota_v\omega$. The log-Futaki invariant, defined by Donaldson [25], is a generalization of the classical Futaki invariant (see [28]) to the conical setting.

Definition 3.1 ([25]). The log-Futaki invariant $F(X, (1-\beta)D) = F(X, (1-\beta)D; 2\pi c_1(X))$ of the pair $(X, (1-\beta)D)$ in the class $2\pi c_1(X)$ is a function on the Lie algebra of holomorphic vector fields, such that, for any holomorphic vector field v as above, its value is

$$F(X, (1-\beta)D; 2\pi c_1(X))(v) \tag{15}$$

$$\begin{aligned}&= F(2\pi c_1(X); v) + (1-\beta)\left(\int_{2\pi D} \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{Vol(2\pi D)}{Vol(X)} \int_X \theta_v \frac{\omega^n}{n!}\right)\\ &= -\int_X (S(\omega) - n)\theta_v \frac{\omega^n}{n!} + 2\pi(1-\beta)\left(\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - n\lambda \int_X \theta_v \frac{\omega^n}{n!}\right).\end{aligned} \tag{16}$$

Log-Futaki invariant is an obstruction to the existence of conical Kähler–Einstein metrics as explained in [40]. When $\lambda \geq 1$, one can show that there are no nontrivial $\mathbb{C}^*$ action for the pair (X, D). (See Corollary 2.21) To obtain the obstruction for the existence we define the log-K-stability by generalizing the original definition by Tian [68] and Donaldson [24].

Note that, although we state the following results by using general test configurations, by the results of [44], for the Fano manifolds or generally for the pairs considered in this paper, we only need to consider special degeneration first introduced by Tian [68]. We will construct special degeneration in our application later.

Definition 3.2 ([24,68]). A test configuration of a polarized pair $(X, D; L)$ consists of

(1) a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-action, and a subscheme $\mathcal{D} \subset \mathcal{X}$ which is invariant under the $\mathbb{C}^*$-action;
(2) a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L} \to \mathcal{X}$;
(3) a flat $\mathbb{C}^*$-equivariant map $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication in the standard way;

such that any fibre $(\mathcal{X}_t, \mathcal{D}_t) = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to (X, D) and $(X, D; L)$ is isomorphic to $(\mathcal{X}_t, \mathcal{D}_t; \mathcal{L}|_{X_t})$. The test configuration is called normal if the total space $\mathcal{X}$ is normal.

A test configuration is called a special test configuration or special degeneration, if the central fibre $(\mathcal{X}_0, \alpha\mathcal{D}_0)$ is a klt pair for some $\alpha \in (0, 1)$.

Note that any test configuration of X (without divisor) can be equivariantly embedded into $\mathbb{P}^N \times \mathbb{C}^*$ where the $\mathbb{C}^*$ action on $\mathbb{P}^N$ is given by a 1 parameter subgroup $\lambda(t)$ of $SL(N+1, \mathbb{C})$. If D is any reduced irreducible divisor of X, the one parameter subgroup $\lambda(t)$ associated with the test configuration of (X, L) induces a test configuration $(\mathcal{D}, \mathcal{L}|_{\mathcal{D}})$ of $(D, L|_D)$.

Let $d_k, \tilde{d}_k$ be the dimensions of $H^0(X, L^k)$, $H^0(Y, L|_Y^k)$, and $w_k, \tilde{w}_k$ be the weights of $\mathbb{C}^*$ action on $H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0}^k)$, $H^0(\mathcal{D}_0, \mathcal{L}|_{\mathcal{D}_0}^k)$, respectively. Then we have expansions:

$$d_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \quad w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1});$$

$$\tilde{d}_k = \tilde{a}_0 k^{n-1} + O(k^{n-2}), \quad \tilde{w}_k = \tilde{b}_0 k^n + O(k^{n-1})$$

If the central fibre $\mathcal{X}_0$ is smooth, we can use equivariant differential forms to calculate the coefficients as in [24]. Let ω be a smooth Kähler form in $2\pi c_1(L)$, and $\theta_v = 2\pi(\mathcal{L}_v - \nabla_v)$, then

$$(2\pi)^n a_0 = \int_X \frac{\omega^n}{n!} = Vol(X); \quad (2\pi)^n a_1 = \frac{1}{2}\int_X S(\omega)\frac{\omega^n}{n!}; \tag{17}$$

$$(2\pi)^n b_0 = -\int_X \theta_v \frac{\omega^n}{n!}; \quad (2\pi)^n b_1 = -\frac{1}{2}\int_X \theta_v S(\omega)\frac{\omega^n}{n!}; \tag{18}$$

$$(2\pi)^n \tilde{a}_0 = \int_{2\pi\mathcal{D}_0} \frac{\omega^{n-1}}{(n-1)!} = Vol(2\pi\mathcal{D}_0); \quad (2\pi)^n \tilde{b}_0 = -\int_{2\pi\mathcal{D}_0} \theta_v \frac{\omega^{n-1}}{(n-1)!}. \tag{19}$$

Comparing (17)–(19) with (15), we let $\alpha = 1 - \beta \in [0, 1)$ and define the algebraic log-Futaki invariant of $(\mathcal{X}, \alpha\mathcal{D}; \mathcal{L})$ to be

$$F((\mathcal{X},\alpha\mathcal{D});\mathcal{L}) = \frac{1}{(2\pi)^n}F((\mathcal{X},\alpha\mathcal{D});2\pi\mathcal{L}) = \frac{2(a_0b_1-a_1b_0)}{a_0}+\alpha\left(-\tilde{b}_0+\frac{\tilde{a}_0}{a_0}b_0\right)$$
$$= \frac{a_0(2b_1-\alpha\tilde{b}_0)-b_0(2a_1-\alpha\tilde{a}_0)}{a_0}. \tag{20}$$

Definition 3.3. $(X,Y;L)$ is log-K-semistable along $(\mathcal{X},\mathcal{L})$ if $F(\mathcal{X},\mathcal{Y};\mathcal{L})\le 0$. Otherwise, it's unstable.

$(X,Y;L)$ is log-K-polystable along test configuration $(\mathcal{X},\mathcal{L})$ if $F(\mathcal{X},\mathcal{Y};\mathcal{L})<0$, or $F(\mathcal{X},\mathcal{Y};\mathcal{L})=0$ and the normalization $(\mathcal{X}^\nu,\mathcal{Y}^\nu;\mathcal{L}^\nu)$ is a product configuration.

$(X,Y;L)$ is log-K-semistable (resp. log-K-polystable) if, for any integer $r>0$, $(X,Y;L^r)$ is log-K-semistable (log-K-polystable) along any test configuration of $(X,Y;L^r)$.

Remark 3.4. When Y is empty, then the definition of log-K-stability becomes the definition of K-stability. (See [24,68])

3.2. Log-Mabuchi-energy and log-Futaki-invariant.

3.2.1. Integrate log-Futaki-invariant. We now integrate the log-Futaki invariant to get log-Mabuchi-energy, which was already defined in the previous section. Fix a smooth Kähler metric $\omega\in 2\pi c_1(X)$. Define the functional on $\mathcal{H}(\omega)$ as

$$F^0_{\omega,D}(\phi) = -\int_0^1 dt\int_D \dot{\phi}_t\frac{\omega_{\phi_t}^{n-1}}{(n-1)!},$$

where ϕ_t is a family of Kähler potentials connecting 0 and ϕ. We can define the log-Mabuchi-energy as

$$\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) = \mathcal{M}_\omega(\omega_\phi)+2\pi(1-\beta)\left(-F^0_{\omega,D}(\phi)+\frac{Vol(D)}{Vol(X)}F^0_\omega(\phi)\right), \tag{21}$$

so that if $\omega_t=\omega+\sqrt{-1}\partial\bar{\partial}\phi_t$ is a sequence of smooth Kähler metrics in $2\pi c_1(X)$, then

$$\frac{d}{dt}\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) = -\int_X (S(\omega_t)-n)\dot{\phi}_t\frac{\omega_\phi^n}{n!}+2\pi(1-\beta)\left(\int_D\dot{\phi}\frac{\omega_\phi^{n-1}}{(n-1)!}-n\lambda\int_X\dot{\phi}\frac{\omega_\phi^n}{n!}\right).$$

Proposition 9 ([40]). *The log-Mabuchi-energy can be written as*

$$\begin{aligned}\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) &= \int_X \log\frac{\omega_\phi^n}{\omega^n}\frac{\omega_\phi^n}{n!}-r(\beta)(I_\omega-J_\omega)(\omega_\phi)\\ &\quad+\int_X\left(h_\omega-(1-\beta)\log|s|_h^2\right)\frac{\omega^n-\omega_\phi^n}{n!}\\ &=\int_X\log\frac{\omega_\phi^n}{e^{H_{\omega,(1-\beta)D}}\omega^n}\frac{\omega_\phi^n}{n!}+r(\beta)\left(\int_X\phi\frac{\omega_\phi^n}{n!}+F^0_\omega(\phi)\right)\\ &\quad+\int_X H_{\omega,(1-\beta)D}\frac{\omega^n}{n!},\end{aligned}$$

so it agrees the definition in the previous section.

Proof. Recall the Poincaré–Lelong equation $\sqrt{-1}\partial\bar{\partial}\log|s_D|_h^2 = -\lambda\omega + 2\pi\{D\}$. Then

$$\begin{aligned}F^0_{\omega,2\pi D}(\phi) = 2\pi F^0_{\omega,D}(\phi) &= -\int_0^1 dt\int_{2\pi D}\dot{\phi}_t\omega_{\phi_t}^{n-1}/(n-1)!\\ &= -\int_0^1 dt\int_X \dot{\phi}_t(\sqrt{-1}\partial\bar{\partial}\log|s_D|_h^2+\lambda\omega)\omega_{\phi_t}^{n-1}/(n-1)!\\ &= -\int_0^1 dt\int_X \log|s_D|_h^2\frac{d}{dt}\frac{\omega_{\phi_t}^n}{n!}+\lambda\mathcal{J}_\omega^\omega(\omega_\phi)\\ &= -\int_X\log|s_D|_h^2\frac{\omega_\phi^n-\omega^n}{n!}+\lambda\mathcal{J}_\omega^\omega(\omega_\phi)\end{aligned}$$

Here, for any smooth closed (1, 1)-form χ, we define

$$\mathcal{J}_\omega^\chi(\phi) = -\int_0^1 dt\int_X\dot{\phi}_t\chi\wedge\omega_{\phi_t}^{n-1}/(n-1)!$$

By taking derivatives, it's easy to verify that $nF_\omega^0(\phi)-\mathcal{J}_\omega^\omega = (I-J)_\omega(\omega_\phi)$. So

$$\begin{aligned}\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) &= \mathcal{M}_\omega(\omega_\phi)+\frac{(1-\beta)Vol(2\pi D)}{Vol(X)}F_\omega^0(\phi)-(1-\beta)F^0_{\omega,2\pi D}(\phi)\\ &= \mathcal{M}_\omega(\omega_\phi)+(n\lambda)(1-\beta)F_\omega^0(\phi)-(1-\beta)\lambda\mathcal{J}_\omega^\omega(\phi)+(1-\beta)\int_X\log|s_D|^2\frac{\omega_\phi^n-\omega^n}{n!}\\ &= \mathcal{M}_\omega(\omega_\phi)+\lambda(1-\beta)(I-J)_\omega(\omega_\phi)+(1-\beta)\int_X\log|s_D|^2\frac{\omega_\phi^n-\omega^n}{n!}.\end{aligned}$$

Then the statement follows from the expression for $\mathcal{M}_\omega$ and that $H_{\omega_0,(1-\beta)D} = h_{\omega_0}-(1-\beta)\log|s_D|^2$. □

3.2.2. Log-Futaki invariant and asymptotic slope of log-Mabuchi-energy. In this section, we adapt S. Paul's work in [51] to the conical setting and prove Theorem 1.11 using the argument from [68] and [52]. Assume $X\subset\mathbb{P}^N$ is embedded into the projective space and $\omega_{FS}\in 2\pi c_1(\mathbb{P}^N)$ is the standard Fubini-Study metric on $\mathbb{P}^N$. For any $\sigma\in SL(N+1,\mathbb{C})$, denote $\omega_\sigma = \sigma^*\omega_{FS}|_X$. We first recall S. Paul's formula for Mabuchi-energy $\mathcal{M}_\omega = \mathcal{M}_{\omega,0}$ on the space of Bergman metrics.

Theorem 3.5 ([51]). *Let $Emb_k : X^n\hookrightarrow\mathbb{P}^N = \mathbb{P}^{N_k}$ be the embedding by the complete linear system $|-kK_X|$ for k sufficiently large. Let $R_X^{(k)}$ denote the **X-resultant** (the Cayley–Chow form of X). Let $\triangle^{(k)}_{X\times\mathbb{P}^{n-1}}$ denote the **X-hyperdiscriminant** of format (n-1) (the defining polynomial for the dual of $X\times\mathbb{P}^{n-1}$ in the Segre embedding). Then there are continuous norms such that the Mabuchi-energy restricted to the Bergman metrics is given as follows:*

$$\frac{n!k^n}{(2\pi)^n}\cdot\mathcal{M}_\omega(\omega_\sigma/k) = \log\frac{\|\sigma\cdot\triangle^{(k)}_{X\times\mathbb{P}^{n-1}}\|^2}{\|\triangle^{(k)}_{X\times\mathbb{P}^{n-1}}\|^2}-\frac{\deg(\triangle^{(k)}_{X\times\mathbb{P}^{n-1}})}{\deg(R_X^{(k)})}\log\frac{\|\sigma\cdot R_X^{(k)}\|^2}{\|R_X^{(k)}\|^2}. \quad (22)$$

For some notes on Paul's proof, see [37]. One ingredient in S. Paul's formula is

Lemma 3.6 ([50,67,75]). *There is a continuous norm on Chow forms, which satisfies, for any projective variety $X^n \subset \mathbb{P}^N$,*

$$(2\pi)^n \cdot \log \frac{\|\sigma \cdot R_X^{(k)}\|^2}{\|R_X^{(k)}\|^2} = (n+1)\int_0^1 dt \int_X \dot{\phi}_\sigma \omega_\sigma^n = -(n+1)! k^{n+1} F_\omega^0(\phi_\sigma / k). \quad (23)$$

In particular, this holds when X^n is replaced by D^{n-1} and F_ω^0 is replaced by $F_{\omega,D}^0$.

We also know the degree of Cayley–Chow forms:

Lemma 3.7. *The degree of Cayley–Chow form $R_X^{(k)}$ and $R_D^{(k)}$ are given by*

$$\deg(R_X^{(k)}) = (n+1)k^n \deg(X, K_X^{-1}) = \frac{(n+1)!k^n}{(2\pi)^n} \cdot Vol(X),$$
$$\deg(R_D^{(k)}) = nk^{n-1} \deg(D, K_X^{-1}) = \frac{n!k^{n-1}}{(2\pi)^{n-1}} \cdot Vol(D).$$

Combining the formulas (21), (22), and (23), we get

Corollary 3.8. *We have the following formula for log-Mabuchi-energy:*

$$\frac{n!k^n}{(2\pi)^n} \cdot \mathcal{M}_{\omega,(1-\beta)D}(\omega_\sigma / k) = \log \frac{\|\sigma \cdot \triangle_{X\times\mathbb{P}^{n-1}}^{(k)}\|^2}{\|\triangle_{X\times\mathbb{P}^{n-1}}^{(k)}\|^2} - \frac{\deg(\triangle_{X\times\mathbb{P}^{n-1}}^{(k)})}{\deg(R_X^{(k)})} \log \frac{\|\sigma \cdot R_X^{(k)}\|^2}{\|R_X^{(k)}\|^2}$$
$$+(1-\beta)\left(\log \frac{\|\sigma \cdot R_D^{(k)}\|^2}{\|R_D^{(k)}\|^2} - \frac{\deg(R_D^{(k)})}{\deg(R_X^{(k)})} \log \frac{\|\sigma \cdot R_X^{(k)}\|^2}{\|R_X^{(k)}\|^2}\right)$$

For any one parameter subgroup $\lambda(t) = t^A \in SL(N_k + 1, \mathbb{C})$. Although the log-Mabuchi-energy is not convex along $\lambda(t)$, the above Corollary says that it is the linear combination of convex functionals. As a consequence, we have the existence of asymptotic slope. Define $\omega_{\lambda(t)} = \lambda(t)^* \omega_{FS}|_X$, and $(\mathcal{X}_0, \mathcal{D}_0) = \lim_{t\to 0} \lambda(t) \cdot (X, D)$ in the Hilbert scheme (which is the central fibre of the induced test configuration introduced in Sect. 3.1). Then by combining Corollary 3.8 with the argument in [52], we have the following expansion

Proposition 10.

$$\mathcal{M}_{\omega,(1-\beta)D}(\omega_{\lambda(t)}/k) = (F + a)\log t + O(1) \quad (24)$$

where $F = F(X, (1-\beta)D; 2\pi c_1(X))(\lambda)$ is the log-Futaki invariant. $a \geq 0 \in \mathbb{Q}$ is nonnegative and is positive if and only if the central fibre $\mathcal{X}_0$ has generically non-reduced fibre.

Remark 3.9. In fact, if $\mathcal{X}_0$ is irreducible, then by ([52,68]) one can calculate that $a = c \cdot (\mathrm{mult}(\mathcal{X}_0) - 1)$ for $c > 0 \in \mathbb{Q}$.

Without loss of generality, we assume the homogeneous coordinates Z_i are the eigenvectors of $\lambda(t)$ on $H^0(\mathbb{P}^N, \mathcal{O}(1)) = \mathbb{C}^{N+1}$ with eigenvalues $\lambda_0 = \cdots = \lambda_K < \lambda_{K+1} \leq \cdots \leq \lambda_N$. Let $\omega_{\lambda(t)} = \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\phi_t$. Then

$$\phi_t = \log \frac{\sum_i t^{\lambda_i}|Z_i|^2}{\sum_i |Z_i|^2} \quad (25)$$

There are three possibilities for $\mathcal{X}_0$.

(1) (non-degenerate case) $\lim_{t\to 0} \mathrm{Osc}(\phi_t) \to +\infty$. By (25), this is equivalent to $\cap_{i=0}^{K}\{Z_i = 0\} \bigcap X \neq \emptyset$.
(2) (degenerate case) $\mathrm{Osc}(\phi_t) \leq C$ for C independent of t. This is equivalent to $\cap_{i=0}^{K}\{Z_i = 0\} \bigcap X = \emptyset$. In this case, $\mathcal{X}_0$ is the image of X under the projection $\mathbb{P}^N \to \mathbb{P}^K$ given by $[Z_0, \ldots, Z_N] \mapsto [Z_0, \ldots, Z_K, 0, \ldots, 0]$ and there is a morphism from $\Phi : X = \mathcal{X}_{t\neq 0} \to \mathcal{X}_0$ which is the restriction of the projection. There are two possibilities.
   (a) $\deg(\Phi) > 1$. In this case, $\mathcal{X}_0$ is generically non-reduced. So $a > 0$ in (24).
   Example: Assume $X^n \subset \mathbb{P}^N$ is in general position. Then the generical linear subspace $\mathbb{L} \cong \mathbb{P}^{N-n-1}$ satisfies $\mathbb{L} \cap X = \emptyset$. Let $\mathbb{M} \cong \mathbb{P}^n$ be a complement of $\mathbb{L} \subset \mathbb{P}^N$. Then the projection of $\Phi : \mathbb{P}^N \backslash \mathbb{L} \to \mathbb{M}$ gives a projection $\Phi : X \to \Phi(X)$ whose mapping degree equals the algebraic degree of X.
   (b) $\deg(\Phi) = 1$. In this case, $\mathcal{X}_0$ is generically reduced and $a = 0$. This case was pointed out in [44]
   Example: Assume $X^n \subset \mathbb{P}^N$ is in general position. Assume $K \geq n + 1$, then $N - K - 1 \leq N - n - 2$. So the generical linear subspace $\mathbb{L} \cong \mathbb{P}^{N-K-1}$ satisfies $\mathbb{L} \cap X = \emptyset$. Let $\mathbb{M} \cong \mathbb{P}^K$ be a complement of $\mathbb{L} \subset \mathbb{P}^N$. Then the projection of $\Phi : \mathbb{P}^N \backslash \mathbb{L} \to \mathbb{M}$ gives a projection $\Phi : X \to \Phi(X)$ with degree 1.

Proposition 11. *As a functional on the space $\mathcal{H}(\omega)$ of smooth Kähler potentials, if $\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi)$ is bounded from below (resp. proper), then $(X, -K_X, (1-\beta)D)$ is log-K-semistable (resp. log-K-stable).*

Proof. If log-Mabuchi-energy is bounded from below, then $F \leq 0$ by the expansion (24) since $a \geq 0$.

Assume $\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi)$ is proper on $\mathcal{H}(\omega)$ in the sense of Definition 2.12, then in particular it's proper on the space of Bergman potentials, so by [52], in case 1 or 2(a), $F < 0$. In case 2(b), $(\mathcal{X}, \mathcal{Y}, \mathcal{L})$ has vanishing log-Futaki invariant and its normalization is a product test configuration. See [44] and [41] for more details (Actually, using [44], we can always assume that we have a special degeneration). □

Remark 3.10. Tian [68] introduced K-stability and proved K-stability for smooth Kähler–Einstein Fano manifolds with discrete automorphism groups. Recently, Berman [7] showed Kähler–Einstein (log) $\mathbb{Q}$-Fano variety is (log) K-polystable without assumption on the automorphism group. His approach is based on the expansion of Ding-functional along any (special) test configuration.

3.3. Log-slope stability and log-Fano manifold . Recall that when $\lambda < 1$, $r(\beta) = 1 - \lambda(1-\beta)$. So when $\beta = 0$, $r(\beta) = 1 - \lambda > 0$. The conical metric in this case would correspond to complete metric with infinite diameter and with $Ric = 1 - \lambda > 0$. This contradicts Myers theorem. So we expect when β is very small, there does not exist such conical Kähler–Einstein metrics.

This is indeed the case. To see this, we first generalize Ross-Thomas' slope stability [55] to the log setting (see [59]). For any subscheme $Z \subset X$, we blow up the ideal sheaf $\mathcal{I}_Z + (t)$ on $X \times \mathbb{C}$ to get the degeneration of X to the deformation to the normal cone $T_Z X$. For the polarization, we denote $\mathcal{L}_c = \pi^* L - cE$, where $E = P_Z(\mathbb{C} \oplus N_Z X)$ is the exceptional divisor, and $0 < c <$ Seshadri constant of Z with respect to X. By Ross–Thomas [55], we have the identity:

$$\begin{aligned}H^0(\mathcal{X}, \mathcal{L}_c^k) &= H^0(X \times \mathbb{C}, L^k \otimes ((t) + \mathcal{I}_Z)^{ck}) \\ &= \bigoplus_{j=0}^{ck-1} t^j H^0(X, L^k \otimes \mathcal{I}_Z^{ck-j}) \oplus t^{ck}\mathbb{C}[t] H^0(X, L^k).\end{aligned}$$

So for k sufficiently large,

$$\begin{aligned}H^0(\mathcal{X}_0, \mathcal{L}_c^k) &= H^0(X, L^k \otimes \mathcal{I}_Z^{ck}) \oplus \bigoplus_{i=0}^{ck} t^j H^0(X, L^k \otimes \mathcal{I}_Z^{ck-j}/\mathcal{I}_Z^{ck-j+1}) \\ &= H^0(X, L^k \otimes \mathcal{I}_Z^{ck}) \oplus \bigoplus_{i=0}^{ck} t^j \frac{H^0(X, L^k \otimes \mathcal{I}_Z^{ck-j})}{H^0(X, L^k \otimes \mathcal{I}_Z^{ck-j+1})}.\end{aligned}$$

By Riemann–Roch, we have the expansion:

$$\chi(X, L^k \otimes \mathcal{I}_Z^{xk}) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-1}).$$

By the calculation in by Ross–Thomas in [55], we know that

$$b_0 = \int_0^c a_0(x)dx - ca_0, \quad b_1 = \int_0^c (a_1(x) + \frac{1}{2}a_0'(x))dx - ca_1.$$

Similarly, if we restrict to $\mathcal{D}$, we have

$$\begin{aligned}H^0(\mathcal{D}, \mathcal{L}_c^k) &= H^0(X \times \mathbb{C}, L^k \otimes ((t) + \mathcal{I}_Z)^{ck} \otimes \mathcal{O}_X/\mathcal{I}_D) \\ &= H^0(D \times \mathbb{C}, L^k \otimes ((t) + \mathcal{I}_Z \cdot \mathcal{O}_D)^{ck})\end{aligned}$$

and

$$H^0(\mathcal{D}_0, \mathcal{L}_c^k) = H^0(D, L^k \otimes (\mathcal{I}_Z \cdot \mathcal{O}_D)^{ck}) \oplus \bigoplus_{i=0}^{ck} t^j \frac{H^0(D, L^k \otimes (\mathcal{I}_Z \cdot \mathcal{O}_D)^{ck-j})}{H^0(X, L^k \otimes (\mathcal{I}_Z \cdot \mathcal{O}_D)^{ck-j+1})}.$$

So, by [55] again, if

$$\chi(D, L^k \otimes (\mathcal{I}_Z \cdot \mathcal{O}_D)^{xk}) = \tilde{a}_0(x)k^{n-1} + O(k^{n-2}),$$

then

$$\tilde{b}_0 = \int_0^c \tilde{a}_0(x)dx - c\tilde{a}_0.$$

So we can calculate the log-Futaki invariant as

$$\begin{aligned}a_0 F(\mathcal{X}, \alpha\mathcal{D}; \mathcal{L}) &= 2(a_0 b_1 - a_1 b_0) + \alpha(\tilde{a}_0 b_0 - a_0 \tilde{b}_0) \\ &= a_0(2b_1 - \tilde{b}_0) - b_0(2a_1 - \tilde{a}_0) \\ &= 2a_0 \left(\int_0^c (a_1(x) - \frac{\alpha}{2}\tilde{a}_0(x) + \frac{1}{2}a_0'(x))dx \right) \\ &\quad -2(a_1 - \alpha\tilde{a}_0/2) \int_0^c a_0(x)dx.\end{aligned} \tag{26}$$

In other words, we can define the log-slope invariant:

$$\mu_c^{\log}((X,\alpha D);\mathcal{I}_Z)=\frac{\int_0^c(a_1(x)-\frac{\alpha}{2}\tilde a_0(x)+\frac{1}{2}a_0'(x))dx}{\int_0^c a_0(x)dx}$$
$$=\frac{\int_0^c(a_1(x)-\frac{\alpha}{2}\tilde a_0(x))dx+\frac{1}{2}(a_0(c)-a_0)}{\int_0^c a_0(x)dx}$$
$$=\mu_c(X;\mathcal{I}_Z)-\alpha\frac{\int_0^c\tilde a_0(x)dx}{2\int_0^c a_0(x)dx}.$$
$$\mu^{\log}((X,\alpha D))=\frac{a_1-\alpha\tilde a_0/2}{a_0}=-\frac{n}{2}\cdot\frac{(K_X+\alpha Y)\cdot L^{n-1}}{L^n}=\mu_c(X)-\frac{n\alpha D\cdot L^{n-1}}{2L^n}.$$

Definition 3.11. We call $(X,\alpha D)$ is log-slope-stable, if for any subscheme $Z\subset X$, we have

$$\mu_c^{\log}((X,\alpha D);\mathcal{I}_Z)<\mu^{\log}((X,\alpha D)).$$

Proposition 12. *Let X be a Fano manifold, and D a Cartier divisor which is numerically equivalent to $-\lambda K_X$. Then if $\lambda<1$, the pair $(X,(1-\beta)D)$ is not log-slope-stable for $\beta<(\lambda^{-1}-1)/n$. As a consequence, in the log-Fano case, the log-Mabuchi-energy is not bounded from below for very small angle.*

Proof. The idea is to look at the test configuration $\mathcal{X}$ given by deformation to the normal cone to D, as in [59]. By Lemma 3.13, the Seshadri constant of $(-K_X,D)$ is equal to $c=1/\lambda$. We will calculate the Futaki invariant for the semi test configuration polarized by $\mathcal{L}=L(-\frac{1}{\lambda}D)$ with $L=-K_X$ and show it is negative for $\beta<(\lambda^{-1}-1)/n$.

In our case, if we choose $Z=D\sim-\lambda K_X$, then the calculation simplifies to

$$a_0(x)=\frac{(L-xD)^n}{n!}=(1-x\lambda)^n\frac{(-K_X)^n}{n!}=(1-x\lambda)^n a_0,\quad a_0'(x)=-n\lambda(1-x\lambda)^{n-1}a_0.$$
$$a_1(x)=\frac{-K_X\cdot(L-xD)^{n-1}}{2(n-1)!}=(1-x\lambda)^{n-1}\frac{(-K_X)^n}{2(n-1)!}=\frac{n}{2}(1-x\lambda)^{n-1}a_0.$$

Recall that when $x>0$, $\tilde a_0(x)$ is defined as follows:

$$\chi(D,L^k\otimes(\mathcal{I}_Z\cdot\mathcal{O}_D)^{xk})=\tilde a_0(x)k^{n-1}+O(k^{n-2}).$$

Because $Z=D$, $\mathcal{I}_Z\cdot\mathcal{O}_D=0$. So

$$\tilde a_0(x)=\begin{cases}0, & \text{when } x>0\\ \frac{L^{n-1}\cdot D}{(n-1)!}=n\lambda a_0, & \text{when } x=0\end{cases}$$

To calculate the log-Futaki-invariant, we first calculate:

$$\int_0^c a_0(x)dx = \frac{a_0}{\lambda(n+1)}(1-(1-c\lambda)^{n+1});$$
$$\int_0^c a_0'(x)dx = a_0(c) - a_0 = a_0((1-c\lambda)^n - 1);$$
$$\int_0^c a_1(x)dx = \frac{a_0}{2\lambda}(1-(1-c\lambda)^n), \quad \int_0^c \tilde{a}_0(x)dx = 0.$$

Using (26) we can calculate that the log-Futaki invariant is equal to

$$a_0 F(\mathcal{X}, (1-\beta)\mathcal{D}; \mathcal{L}) = a_0^2(1-(1-c\lambda)^{n+1})\frac{n}{n+1} \times \left[(\lambda^{-1}-1)\left(\frac{n+1}{n}\cdot\frac{1-(1-c\lambda)^n}{1-(1-c\lambda)^{n+1}} - 1\right) - \beta\right].$$

So we get $F(\mathcal{X}, (1-\beta)\mathcal{D}; \mathcal{L}) \le 0 \Longleftrightarrow \beta \ge \beta(\lambda, c)$, where

$$\beta(\lambda, c) = (\lambda^{-1}-1)\left(\frac{n+1}{n}\frac{1-(1-c\lambda)^n}{1-(1-c\lambda)^{n+1}} - 1\right) = \frac{\lambda^{-1}-1}{n}\left(1 - \frac{n+1}{\sum_{i=0}^n (1-c\lambda)^{-i}}\right)$$

From the above formula for $\beta(\lambda, c)$ we easily get that

$$\sup_{0<c<\lambda^{-1}} \beta(\lambda, c) = \frac{\lambda^{-1}-1}{n}.$$

So when $\beta < (\lambda^{-1}-1)/n$ there exists $c \in (0, \lambda^{-1})$ such that $(X, (1-\beta)D)$ is destabilized by the subscheme cD. □

Example 3.12. On $\mathbb{P}^2$, when D is a line, then $(X, (1-\beta)D)$ is unstable for all $\beta \in [0, 1)$; when D is a conic, then $(X, (1-\beta)D)$ is unstable for $\beta \in (0, 1/4)$, and it will be proved below that it is semi-stable for $\beta = 1/4$, and hence poly-stable for $\beta \in (1/4, 1)$. On $\mathbb{P}^1 \times \mathbb{P}^1$, when D is a diagonal line, $(X, (1-\beta)D)$ is unstable for $\beta \in (0, 1/2)$. By viewing $\mathbb{P}^1 \times \mathbb{P}^1$ as a double cover of $\mathbb{P}^2$ along a conic curve (See Remark 5.1 in Sect. 5.1 for details) we see these observations match. It is an interesting question whether the bounds of β given by the above proposition is sharp for a smooth hypersurface of degree d in $\mathbb{P}^n$ with $d < n+1$.

Lemma 3.13. *The Seshadri constant of* $(-K_X, D)$ *is equal to* λ^{-1}.

Proof. Note that $\mathcal{X}_0 = X \cup_{D_\infty} E$. Here $E \cong P(N_D \oplus \mathbb{C})$ is the exceptional divisor and $D \cong D_\infty \subset P(N_D \oplus \mathbb{C})$ is the divisor at infinity. $\mathcal{L}_c|_X = K_X^{-1} - cD = (1-c\lambda)K_X^{-1}$. This is ample if and only if $c < \lambda^{-1}$.

On the other hand, $\mathcal{L}_c|_{P(N_D\oplus\mathbb{C})} = \pi^* K_X^{-1} + c\mathcal{O}_E(1) = \pi^* K_X^{-1} + cD_\infty$. Let h be a Hermitian metric on $\mathcal{O}(D)$ such that $\omega_h := -\sqrt{-1}\partial\bar{\partial}\log h$ is a Kähler form. Then if we define $\Omega = \lambda^{-1}\pi^*\omega_h + c\sqrt{-1}\partial\bar{\partial}\log(1+h)$, this gives a smooth rotationally symmetric (1,1)-form on E. To write Ω in local coordinate, we choose two kinds of coordinate charts on E which covers the neighborhood of zero section D_0 and infinity section D_∞

of $P(N_D \oplus \mathbb{C})$ respectively. To do this, just choose local trivialization of $N_D|_D$ to get holomorphic coordinate along the fibre, which is denoted by ξ. Then $h = a|\xi|^2$ for some smooth positive definite function a. Note that $\omega_h = -\sqrt{-1}\partial\bar{\partial}\log a$. In this local coordinate one can easily calculate that

$$\Omega = (\lambda^{-1} - c\frac{a|\xi|^2}{1+a|\xi|^2})\omega_h + c\sqrt{-1}\frac{a}{(1+a|\xi|^2)^2}\nabla\xi \wedge \overline{\nabla\xi}.$$

where for simplicity we denote $\nabla\xi = d\xi + \xi a^{-1}\partial a$. For the coordinate at infinity, we use coordinate change $\eta = \xi^{-1}$, then

$$\Omega = (\lambda^{-1} - c\frac{a}{a+|\eta|^2})\omega_h + c\sqrt{-1}\frac{a}{(|\eta|^2+a)^2}\nabla'\eta \wedge \overline{\nabla'\eta}$$

with $\nabla'\eta = d\eta - \eta a^{-1}\partial a$. So we easily sees that Ω is positive definite if an only if $c < \lambda^{-1}$. The lemma clearly follows from the combination of above discussions. □

The following example is in the log-Calabi-Yau case ($\lambda = 1$).

Example 3.14. Let $X = Bl_p\mathbb{P}^2$, $D \in |-K_X|$ be a general smooth divisor. Choose $Z = E$ to be the exceptional divisor. If we perform the operation of deformation to the normal cone, the central fibre is given by $\tilde{\mathcal{X}}_0 = X \cup_{E=D_\infty} \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(-1))$. The Seshadri constant equals 2 and the line bundle $\mathcal{L}_2$ contracts X along its fibration direction and the resulting test configuration has central fibre $\mathcal{X}_0 = \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(-1)) \cong X$. The boundary divisor on $\mathcal{X}_0$ is given by $F + 2D_\infty$ where F is the fibre over the intersection point $F \cap E \in E = D_\infty$. Denote $L = K_X^{-1}$ and $Y = (1-\beta)D$. Then the calculation specializes to

$$a_0(x) = \frac{(L-xZ)^2}{2} = 4 - x - \frac{x^2}{2}, \quad a_1(x) = \frac{-K_X \cdot (L-xZ)}{2} = 4 - \frac{x}{2}.$$
$$\tilde{a}_0(x) = \deg(L-xZ)|_D = (8-x).$$

Using formula (26), it's easy to calculate the log-Futaki invariant as

$$F(\mathcal{X}, (1-\beta)\mathcal{D}, \mathcal{L}_2) = 8\left(\frac{7\beta}{3} - 2\right).$$

This is negative if and only if $\beta < 6/7$. This is compatible with the calculation in [25] (See also [40]), where, instead of taking deformation to normal cone, the same test configuration is generated by one parameter subgroup in the torus action.

4. Special Degeneration to Kähler–Einstein Svarieties

4.1. Kähler metrics on singular varieties. We will first establish some standard notations following [27].

Definition 4.1 ($\mathbb{Q}$-*Fano variety*). A normal variety X is $\mathbb{Q}$-Fano if X is klt and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Assume X is an n-dimensional $\mathbb{Q}$-Fano variety. D is a smooth divisor such that $D \cap X^{sing} = \emptyset$. Define the space of Kähler metrics on $\mathbb{Q}$-Fano varieties following [27]. So a plurisubharmonic(psh) function ϕ on X is an upper semi-continuous function on X with values in $\mathbb{R} \cup \{-\infty\}$, which is not locally $-\infty$, and extends to a psh function in some local embedding $X \to \mathbb{C}^N$. ϕ is said to be smooth(locally bounded) if there exists a smooth(locally bounded) local extension is smooth(bounded). Similarly a smooth Kähler metric on X is locally $\sqrt{-1}\partial\bar{\partial}$ of a smooth plurisubharmonic function. We are only interested in the class of bounded plurisubharmonic functions. Fix a smooth Kähler metric ω on X, we define

$$\mathcal{PSH}_\infty(\omega) := \{\phi \in L^\infty(X); \omega + \sqrt{-1}\partial\bar{\partial}\phi \geq 0 \text{ and } \phi \text{ is u.s.c.}\}.$$

Remark 4.2. Any function $\phi \in \mathcal{PSH}_\infty(X, \omega)$ is of finite self-energy in the sense of Definition 1.1 in [27].

Remark 4.3 (Orbifold metric induces L^∞ Hermitian metrics). When the Cartier index of K_X divides r, $K_X^{\otimes r}$ is a line bundle. Any orbifold metric induces a Hermitian metric on $K_X^{\otimes r}$ and hence on $K_X^{-\otimes r}$. In fact, for any point x, we can choose local uniformization chart $\tilde{U} \to U \ni x$ such that $U = \tilde{U}/G$ for some finite group G and we choose local coordinates $\{\tilde{z}_i\}$ on $\tilde{U}$. Define r=order of G. Then the Cartier index of K_X at x divides r. The r-pluri-anticanonical form $\tilde{\tau} = (\partial_{\tilde{z}_1} \wedge \cdots \wedge \partial_{\tilde{z}_n})^{\otimes r}$ on $\tilde{U}$ is G-invariant, so it induces a local generator τ of $K_X^{-\otimes r}$ downstairs. If we have an orbifold metric which is locally induced by a smooth G-invariant metric $\tilde{g}$ on $\tilde{U}$. We just define the Hermitian metric on $K_X^{-\otimes r}$ by requiring $|\tau|^2 = |\partial_{\tilde{z}}|_{\tilde{g}}^{2r} = \det(\tilde{g})^r$.

Example 4.4 . Let $\mathbb{Z}_4$ acts on $\mathbb{C}^2$ by $\xi : (\tilde{z}_1, \tilde{z}_2) \mapsto (\xi\tilde{z}_1, \xi\tilde{z}_2)$ where $\xi = \exp(2\pi\sqrt{-1}/4)$. Let $X = \mathbb{C}^2/\mathbb{Z}_4$, then X has an isolated singularity of index 2, which is usually denoted by $\frac{1}{4}(1, 1)$. We can embed X into $\mathbb{C}^5$ by defining $u_i = \tilde{z}_1^{4-i}\tilde{z}_2^i$ for $i = 0, \ldots, 4$.

We can choose the orbifold metric induced by the following smooth metric on $\tilde{U} = \mathbb{C}^2$:

$$\tilde{\omega} = \sqrt{-1}\partial\bar{\partial}(|z_1|^2 + |z_1|^4 + |z_2|^2) = (1 + 4|z_1|^2)dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$$

Then $\tilde{\tau} = (\partial_{\tilde{z}_1} \wedge \partial_{\tilde{z}_2})^{\otimes 2}$ induces a generator τ of $K_X^{-\otimes 2}$ with $|\tau|_{\tilde{g}}^2 = (1 + 4|\tilde{z}_1|^2)^2 = (1 + 4|u_1|^{1/2})^2$.

By the above discussion, we see that the Hermitian metric determined by an orbifold metric does not give rise to a smooth plurisubharmonic function. However, it is locally bounded, so we can use pluripotential theory to deal with them.

4.2. Degenerate Complex Monge–Ampère equation on Kähler manifolds with boundary. Let M be a Kähler manifold of dimension $n + 1$ with smooth boundary ∂M. We will be interested in solving degenerate Dirichlet problem of complex Monge–Ampère equation on M. We recall some important results for this problem. First, there is existence of weak solution

Theorem 4.5 ([11]). *Let ω be a nonnegative, smooth (1,1)-form on X. Assume $\phi_i \in \mathcal{PSH}(\omega) \cap C^0(X)$, $i = 0, 1$. Then there exists a bounded geodesic Φ_t connecting ϕ_0 and ϕ_1. In other words, there exists a bounded solution of the Dirichlet problem to the following homogeneous complex Monge–Ampère equation on $X \times [0, 1] \times S^1$.*

$$\begin{cases} \pi^*\omega + \sqrt{-1}\partial\bar{\partial}\Phi \geq 0, \\ (\pi^*\omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0, \\ \Phi|_{X\times\{0\}\times S^1} = \phi_0, \Phi|_{X\times\{1\}\times S^1} = \phi_1. \end{cases} \tag{27}$$

Remark 4.6. The existence of weak $C^{1,1}$-geodesics ("weak" means that $\Delta\Phi$ is bounded) connecting smooth Kähler metrics was first proved by Chen in [17]. Since we want to deal with mildly singular varieties, we choose to work with just bounded solutions. There are many other important related works to this result. See for example [5,13,15,32,35].

We also record a result by Phong–Sturm.

Theorem 4.7 ([53]). *Assume $\Omega \geq 0$ and there exists a smooth divisor E in the interior of M such that $\Omega > 0$ on $M \backslash E$. Also assume the line bundle $\mathcal{O}(E)$ has a Hermitian metric H, such that $\Omega_\epsilon = \Omega + \epsilon\sqrt{-1}\partial\bar{\partial}\log H > 0$ for $0 < \epsilon \ll 1$ sufficiently small. Consider the following homogeneous complex Monge–Ampère equation*

$$(\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0, \quad \Phi|_{\partial M} = \phi. \tag{28}$$

If there exists a subsolution $\Psi \in C^\infty(M)$ such that $\Omega + \sqrt{-1}\partial\bar{\partial}\Psi \geq 0$ and $\Psi|_{\partial M} = \phi$, then (28) has a bounded solution $\Phi \in L^\infty(M)$. Moreover, $\Phi \in C^{1,\alpha}(M \backslash E)$ for any $0 < \alpha < 1$.

4.3. Proof of Theorem 1.8. Assume $\pi : (\mathcal{X}, -K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ is a special degeneration. Assume for simplicity, $\mathcal{X}$ has only finite many isolated singularities $\{p_i\}$. Let $\triangle = \{w \in \mathbb{C}; |w| \leq 1\}$ be the unit disk and $\mathcal{X}_\triangle = \pi^{-1}(\triangle)$. We embed the special test configuration equivariantly into $\mathbb{P}^N \times \mathbb{C}$:

$$\phi_{\mathcal{X}} : (\mathcal{X}, -K_{\mathcal{X}/\mathbb{C}}) \hookrightarrow \mathbb{C} \times (\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)).$$

We get a smooth S^1-invariant Kähler metric on $\mathcal{X}_\triangle$ by pulling back $\Omega = \phi_{\mathcal{X}}^*(\omega_{FS} + \sqrt{-1}dw \wedge d\bar{w})$. We define the reference metric X to be $\omega = \Omega|_{\mathcal{X}_1}$, where $\mathcal{X}_1 \cong X$ is the fibre above $\{w = 1\}$. For any $\phi \in C^\infty(X)$, such that $\omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, we are going to solve the homogeneous Monge–Ampère equation

$$(\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0, \Phi|_{S^1 \times X} = \phi. \tag{29}$$

Proposition 13. *There exists bounded solution Φ for (29). $\Phi \in C^{1,\alpha}(\mathcal{X} \backslash \{p_i\})$.*

Proof. We choose a equivariant resolution $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$. Then we solve the equation on $\tilde{\mathcal{X}}$:

$$(\tilde{\Omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\Phi})^{n+1} = 0, \tilde{\Phi}|_{S^1 \times X} = \phi \tag{30}$$

with $\tilde{\Omega} = \pi^*\Omega$ being a smooth, closed, non-negative form. By the following Proposition, we have smooth subsolution for (30). So by Phong–Sturm's result (Theorem 4.7), we can get bounded solution $\tilde{\Phi}$ of (30) and, moreover, $\tilde{\Phi}$ is $C^{1,\alpha}$ on $\tilde{X} \backslash E$, where E is exceptional divisor. Because $\tilde{\Phi}$ is plurisubharmonic along the fibres of the resolution which are compact subvarieties, so $\tilde{\Phi}$ is constant on the fibre of the resolution and hence $\tilde{\Phi}$ descends to a solution Φ of (29). □

As pointed out in the above proof, to apply Theorem 4.7, we need to know the existence of subsolutions. Let $\mathcal{X}^* = \mathcal{X}\backslash\mathcal{X}_0$. To construct such subsolution, we first note that there is an equivariant isomorphism

$$\rho : \mathbb{C}^* \times X \cong \mathcal{X}^* \hookrightarrow \mathcal{X}. \tag{31}$$

Proposition 14. *For any smooth Kähler potential ϕ, there exists a smooth S^1-invariant smooth Kähler metric Ω_Ψ on $\mathcal{X}_\triangle$ such that $\rho^*\Omega_\Psi|_{S^1\times X} = \pi_2^*\omega_\phi$. As a consequence, Ψ is a subsolution of the homogeneous Monge–Ampère equation* (29).

Remark 4.8. Similar result was proved in [53]. For the reader's convenience, we give a proof here.

Proof. Under the isomorphism (31), we can write

$$\pi_2^*\omega_\phi + \sqrt{-1}dw \wedge d\bar{w} = \Omega + \sqrt{-1}\partial\bar{\partial}\Psi_0$$

by taking $\Psi_0 = -\log(h_\phi/\rho^*\phi_{\mathcal{X}}^* h_{FS})$. Note that this only holds on $\mathbb{C}^* \times X$. Now let $\eta(w)$ be a smooth cut-off function such that $\eta(w) = 1$ for $|w| \le 1/3$ and $\eta(w) = 0$ for $|w| \ge 2/3$. Now we define a new metric on $\mathbb{C}^* \times X$:

$$\begin{aligned}\Omega + \sqrt{-1}\partial\bar{\partial}\Psi &:= \pi_2^*\omega_\phi + \sqrt{-1}dw \wedge d\bar{w} - \sqrt{-1}\partial\bar{\partial}(\eta(w)\Psi_0) + a\sqrt{-1}dw \wedge d\bar{w}\\ &= \Omega + \sqrt{-1}\partial\bar{\partial}(\Psi_0 - \eta(w)\Psi_0 + a|w|^2).\end{aligned}$$

In other words we let $\Psi = (1 - \eta(w))\Psi_0 + a|w|^2 + c$ for some constant c.

We will show when $\mathbb{R} \ni a \gg 1$ is chosen to be big enough, then we get a smooth Kähler metric on $\mathcal{X}_\triangle$ with the required condition.

For $|w| \ge 2/3$, $\Omega_\Psi = \pi^*\omega_\phi + a\sqrt{-1}dw \wedge d\bar{w}$. When $|w| \le 1/3$, $\Omega_\Psi = \Omega + a\sqrt{-1}dw \wedge d\bar{w}$. we can use the glue map ρ to get a smooth S^1-invariant Kähler metric on $\pi^{-1}(\{|w| \le 1/3\})$. So Ω_Ψ is a smooth S^1-invariant Kähler metric for $|w| \le 1/3$ or $|w| \ge 2/3$. We now need to consider the behavior of Ω_Ψ at any point $p \in \mathbb{C}^* \times X$ such that $1/3 < |w(p)| < 2/3$.

$$\begin{aligned}\Omega_\Psi &= \pi^*\omega_\phi - \eta\sqrt{-1}\partial\bar{\partial}\Psi_0 - \Psi_0\sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}\left(\partial\eta \wedge \bar{\partial}\Psi_0 + \partial\Psi_0 \wedge \bar{\partial}\eta\right)\\ &\quad +(a+1)\sqrt{-1}dw \wedge d\bar{w} \ge (1-\eta)(\omega_\phi + \sqrt{-1}dw \wedge d\bar{w}) + \eta\Omega\\ &\quad -\epsilon\sqrt{-1}\partial\Psi_0 \wedge \bar{\partial}\Psi_0 + (a - \epsilon^{-1}|\eta_w|^2 - \Psi_0\eta_{w\bar{w}})\sqrt{-1}dw \wedge d\bar{w}.\end{aligned}$$

Note that the first two terms together are strictly positive definite. Because on $\mathcal{X}_{|w|\ge 1/3} = \pi^{-1}(\{|w| \ge 1/3\})$, Ψ_0 is a well defined smooth function there. So we can choose ϵ sufficiently small and a sufficiently big such that this is a positive form on $\mathcal{X}_{|w|\ge 1/3}$. □

Proof of Theorem 1.8. There exists a metric h_Ω on $K^{-1}_{\mathcal{X}/\mathbb{C}}$ such that $\Omega = -\sqrt{-1}\partial\bar{\partial}\log h_\Omega$. h_Ω defines a volume form on each fibre. If we choose local coordinate $\{z_i\}$ on $\mathcal{X}_t$ and denote $\partial_z = \partial_{z_1} \wedge \cdots \wedge \partial z_n$ and $dz = dz_1 \wedge \cdots \wedge dz_n$. Then the volume form is given by

$$dV(h_\Omega|_{\mathcal{X}_t}) = |\partial_z|^2_{h_\Omega|_{\mathcal{X}_t}}\, dz \wedge \overline{dz}.$$

Let $\mathcal{S}$ be the defining section of the divisor $\mathcal{D}$. Fix the Hermitian metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ such that $-\sqrt{-1}\partial\bar{\partial}\log|\cdot|^2 = \lambda\Omega$.

Let $\omega_t = \Omega|_{\mathcal{X}_t}$. To prove the lower boundedness of log-Ding-functional $F_{\omega,(1-\beta)D}$, by Lemma 2.15, we only need to consider smooth Kähler potentials. For any smooth potential $\phi \in C^\infty(X)$, we solve the homogeneous complex Monge–Ampère equation (29) to get the geodesic ray Φ. Then consider the function on the base defined by

$$f(t) = F^0_{\omega_t}(\Phi|_{\mathcal{X}_t}) - \frac{V}{r(\beta)} \log\left(\frac{1}{V}\int_{\mathcal{X}_t} e^{-r(\beta)\Phi} \frac{dV(h_\Omega|_{\mathcal{X}_t})}{|\mathcal{S}|^{2(1-\beta)}}\right).$$

Claim 4.9. $f(t)$ satisfies $\Delta f \geq 0$.

Assuming the claim, let's finish the proof of Theorem 1.8. By maximal principle of subharmonic function, we have

$$F^X_{\omega_1,(1-\beta)D}(\phi) = f(1) = \max_{t\in\partial\triangle} f(t) \geq f(0) = F^{\mathcal{X}_0}_{\omega_0,(1-\beta)\mathcal{D}_0}(\Phi|_{\mathcal{X}_0}).$$

Now since by assumption, there exists a conical Kähler–Einstein metric $\widehat{\omega}_{KE} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\widehat{\phi}_{KE}$ on $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$. By Berndtsson's Theorem 2.9 and its generalization to the $\mathbb{Q}$-Fano case [8], we have

$$F_{\omega_0,(1-\beta)\mathcal{D}_0}(\Phi|_{\mathcal{X}_0}) \geq F_{\omega_0,(1-\beta)\mathcal{D}_0}(\widehat{\phi}_{KE}).$$

So combining the above two inequality, we indeed get the lower bound of log-Ding-energy:

$$F^X_{\omega_1,(1-\beta)D}(\phi) \geq F^{\mathcal{X}_0}_{\omega_0,(1-\beta)\mathcal{D}_0}(\widehat{\phi}_{KE}).$$

Now, to prove the claim, we write $f(t)$ as parts: $f(t) = \mathrm{I} + \mathrm{II}$:

$$\mathrm{I} = F^0_{\omega_t}(\Phi|_{\mathcal{X}_t}) = -\frac{1}{(n+1)!}\int_X BC(\Omega^{n+1}, (\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1}),$$
$$\mathrm{II} = -\frac{V}{r(\beta)} \log\left(\frac{1}{V}\int_{\mathcal{X}_t} e^{-r(\beta)\Phi} \frac{dV(h_\Omega|_{\mathcal{X}_t})}{|\mathcal{S}|^{2(1-\beta)}}\right).$$

For part I, we use the property of Bott–Chern form and the geodesic equation to get that (see Remark 2.7):

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\mathrm{I} &= -\frac{1}{(n+1)!}\int_{\mathcal{X}_t} \sqrt{-1}\partial\bar{\partial} BC(\Omega^{n+1}, (\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1})\\ &= -\frac{1}{(n+1)!}\int_{\mathcal{X}_t} (\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} - \Omega^{n+1}\\ &= \frac{1}{(n+1)!}\int_{\mathcal{X}_t} \Omega^{n+1} \geq 0.\end{aligned}$$

For part II, we can write locally $1 = \partial_z \otimes dz$ in the decomposition $\mathcal{O}_{\mathcal{X}} = -K_{\mathcal{X}/\mathbb{C}} + K_{\mathcal{X}/\mathbb{C}}$. Then we think $1 \in \mathcal{O}_{\mathbb{C}}$ is a holomorphic section in $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathbb{C}}$.

$$\mathrm{II} = -\frac{V}{r(\beta)} \log \|1\|^2_{L^2}.$$

where $\|\cdot\|^2_{L^2}$ is the L^2-metric induced by the singular metric $H = h_\Omega e^{-r(\beta)\Phi}/|\mathcal{S}|^{2(1-\beta)}$ on $-K_{\mathcal{X}/\mathbb{C}}$. Then the subharmonicity is given by the next proposition. □

Proposition 15. II *is a subharmonic function of t.*

Proof. First note that

$$\begin{aligned} -\sqrt{-1}\partial\bar{\partial}\log H &= \Omega + (1-\lambda(1-\beta))\sqrt{-1}\partial\bar{\partial}\Phi + (1-\beta)(-\lambda\Omega + \{\mathcal{D}=0\}) \\ &= (1-\lambda(1-\beta))\Omega_\Phi + (1-\beta)\{\mathcal{D}\} \end{aligned}$$

is a positive current. If $\mathcal{X}$ is smooth, then the subharmonicity follows immediately from Berndtsson's important result in [10]. In our case, $\mathcal{X}$ has isolated singularities. We can use divisors to cut out this singularity and reduce the problem to smooth fibrations of Stein manifolds. We apply Berndtsson–Paun's argument in [12]. They construct a sequence of smooth fibrations $\pi_j : \mathcal{X}_j \to \mathbb{C}$, such that

(1) π_j is a smooth fibration. Each fibre is a Stein manifold.
(2) As $j \to +\infty$, $\{\mathcal{X}_j\}$ form an exhaustion of $\mathcal{X}$.

Note that in our equivariant setting, we can also require the $\mathcal{X}_j$ is $\mathbb{C}^*$-invariant.

In [12] Berndtsson–Paun proved that the relative Bergman kernel metric h_j of the bundle $\mathcal{O}_{\mathcal{X}_j} = K_{\mathcal{X}_j/\mathbb{C}} + (-K_{\mathcal{X}_j/\mathbb{C}})$ has semipositive curvature current. See also [9]. In other words, $-\log |1|^2_{h_j}$ is plurisubharmonic on $\mathcal{X}_k$. If we use $\|\cdot\|_j$ to denote the L^2-metric on $(\pi_j)_*\mathcal{O}_{\mathcal{X}_j}$ induced by $H = h_\Omega e^{-r(\beta)\Phi}/|\mathcal{S}|^{2(1-\beta)}$ on $K^{-1}_{\mathcal{X}_j/\mathbb{C}}$ and $\mathcal{K}_j(z,z)$ to denote the relative Bergman kernel of $\mathcal{O}_{\mathcal{X}_j} = K_{\mathcal{X}_j/\mathbb{C}} + (-K_{\mathcal{X}_j/\mathbb{C}})$, then

$$\mathcal{K}_j(z,z) = \max\{|f|^2; \|f\|_j \le 1\}, \quad |1|^2_{h_j} = \frac{1}{\mathcal{K}_j(z,z)}.$$

Now, as showed by Berndtsson–Paun, the relative Bergman kernel $\mathcal{K}$ of $\mathcal{O}_\mathcal{X} = K_{\mathcal{X}/\mathbb{C}} + (-K_{\mathcal{X}/\mathbb{C}})$ is the decreasing limit of the Bergman kernel $\mathcal{K}_j$, and hence the relative Bergman kernel metric on $\mathcal{O}_\mathcal{X}$ also has semipositive curvature current. Since, for any $t \in \mathbb{C}$, $H^0(\mathcal{X}_t, \mathcal{O}_\mathcal{X}|_{\mathcal{X}_t}) = \mathbb{C}$ which is generated by constant function 1, using the extremal characterization of the relative Bergman kernel, it's straight forward to verify that the relative Bergman kernel metric (BK) on $\mathcal{O}_\mathcal{X} = K_{\mathcal{X}/\mathbb{C}} + (-K_{\mathcal{X}/\mathbb{C}})$ is given by $|1|^2_{BK} = \frac{1}{\mathcal{K}(z,z)} = \|1\|^2_{L^2}$ which is the pull-back of a function from the base $\mathbb{C}$. So we get that $\text{II} = -\log \|1\|^2_{L^2}$ is plurisubharmonic on the disk $\{|w| \le 1\}$. □

Remark 4.10. When the central fiber is smooth, Theorem 1.8 is a special case of a theorem of Chen [18], where a more general statement concerning constant scalar curvature Kähler metrics is proved, using the weak convexity of Mabuchi functional on the space of Kähler metrics. It seems difficult to adapt Chen's argument to the singular setting. The advantage here(in the log setting) is to use (log)-Ding's functional, which requires much weaker regularity of the geodesics. A fundamental result of Berndtsson [11] says that the Ding functional is genuinely geodesically convex. This technique has been demonstrated in [11], [8] and [7].

Remark 4.11. During the writing of this paper, the paper by Berman [7] appeared in which some more results about subharmonicity of Ding-functional in the singular setting was proved. Recently, the following result is proved by the first author [43] by proving the continuity (log)-Ding energy at $t = 0$.

Theorem 4.12. *Let $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ be a special degeneration for (X, D). Suppose the central fiber $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$ admits a singular Kähler–Einstein metric of cone angle $2\pi\beta$ along $\mathcal{D}_0$. Then the log-Ding functional $F_{\omega,(1-\beta)D}$ is bounded below. As a consequence, the log-Mabuchi functional $\mathcal{M}_{\omega,(1-\beta)D}$ is also bounded below.*

This covers Theorem 1.8. Note that by the work [8], the pair $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$ must be klt in order for it to admit a conical Kähler–Einstein metric.

5. Kähler–Einstein Metrics on $X = \mathbb{P}^2$ Singular Along a Conic

5.1. Proof of Theorem 1.5. We first apply Phong–Song–Sturm–Weinkove's properness result in Theorem 2.14 to show that the Ding-energy $F_\omega^{\mathbb{P}^2}$ is proper on the space of $SO(3)$-invariant Kähler metrics. For this, we need to show that the centralizer of $SO(3, \mathbb{R})$ in $SU(3)$ is finite. Indeed, if $\gamma \in \mathrm{Centr}_{\mathrm{SO}(3,\mathbb{R})}\mathrm{SU}(3)$, then $\gamma \cdot C$ is a degree 2 curve invariant under $SO(3, \mathbb{R})$. But there is a unique curve invariant under $SO(3, \mathbb{R})$ which is just C itself. So $\gamma \cdot C = C$ and we conclude $\gamma \in SO(3, \mathbb{R})$. Since the center of $SO(3, \mathbb{R})$ is finite, so the conclusion follows.

By the calculation in Example 3.12, we see that $(\mathbb{P}^2, (1-\beta)D)$ is unstable when $0 < \beta < 1/4$, so there is no conical Kähler–Einstein metric for $\beta \in (0, 1/4)$, by Corollary 2.10 and Proposition 11.

When $\beta = 1/4$, we can show there is no conical Kähler–Einstein metric. Indeed, if there is such a Kähler–Einstein metric, then by Proposition 4, there exists $SO(3, \mathbb{R})$ invariant conical Kähler–Einstein metric on $(X, (1-\beta)D)$ for some $\beta < 1/4$. This would imply $(X, (1-\beta)D)$ is semi-stable for some $\beta < 1/4$ which we know is not true by above discussion. (Alternatively, we can also use the fact that the deformation to the normal cone considered in Proposition 12 shows that $(\mathbb{P}^2, 3/4D)$ is not log-K-polystable since it has log-Futaki-invariant zero and is not a product degeneration, and conclude the nonexistence of conical Kähler–Einstein metrics for $\beta = 1/4$ by appealing to the more general result of Berman [7]. See Remark 3.10).

To prove the existence for all $\beta \in (1/4, 1]$, by Proposition 1, we only need to show the lower boundedness of log-Mabuchi-energy when $\beta = 1/4$. To do this, we construct a special degeneration to conical Kähler–Einstein variety and apply Theorem 1.8. The special degeneration comes from deformation to the normal cone. Let $\tilde{\mathcal{X}}$ be the blow up of $\mathbb{P}^2 \times \mathbb{C}$ along $D \times \{0\}$. Choose the line bundle $\mathcal{L}_{3/2} := \pi^* K_{\mathbb{P}^2}^{-1} - 3/2E$ where E is the exceptional divisor. Then $\mathcal{L}_{3/2}$ is semi-ample and the map given by the complete linear system $|k\mathcal{L}_{3/2}|$ for k sufficiently big contracts the $\mathbb{P}^2$ in the central fibre and we get a special test configuration $\mathcal{X}$ with central fibre being the weighted projective space $\mathbb{P}(1, 1, 4)$. It inherits an orbifold Kähler–Einstein metric from the standard Fubini-Study metric on $\mathbb{P}^2$ by the quotient map $\mathbb{P}^2 = \mathbb{P}(1, 1, 1) \to \mathbb{P}(1, 1, 4)$ given by $(Z_0, Z_1, Z_2) \to (Z_0, Z_1, Z_2^4) =: [W_0, W_1, W_2]$. (See Example 8.5 in Sect. 8 for a toric explanation) The induced orbifold Kähler–Einstein metric is the same as the conical Kähler–Einstein metric on $\mathbb{P}(1, 1, 4)$ singular along the divisor $[W_2 = 0]$ with cone angle $2\pi/4$. There is one orbifold singularity on $\mathbb{P}(1, 1, 4)$ of type $\frac{1}{4}(1, 1)$ as explained in Example 4.4. But this does not cause any difficulty by the discussion in Sect. 4.1. So by Theorem 1.8, we get that the log-Ding-energy $F_{\omega,3/4D}^{\mathbb{P}^2}$ is bounded from below. So by Proposition 1, $F_{\omega,(1-\beta)D}(\phi)$ is proper for $\beta \in (1/4, 1]$ on the space of $SO(3, \mathbb{R})$ invariant conical metrics. So by the existence Theorem 2.17, we get the existence of conical Kähler–Einstein metric on $(\mathbb{P}^2, (1-\beta)D)$ for any $\beta \in (1/4, 1]$.

Remark 5.1. When $\beta = 1/2$, there exists an orbifold metric on $(\mathbb{P}^2, 1/2D)$ coming from the branched covering map given by

$$p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$$
$$[U_0, U_1], [V_0, V_1] \mapsto [U_0V_0 + U_1V_1, i(U_0V_1 + U_1V_0), i(U_0V_0 - U_1V_1)]$$

This is a degree 2 cover branching along the diagonal $\triangle = \{[U_0, U_1], [U_0, U_1]\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. Note that

$$p(\triangle) = D = \{Z_0^2 + Z_1^2 + Z_2^2 = 0\} \subset \mathbb{P}^2.$$

From the above proof it is tempting to expect that

Conjecture 5.2. *The conical Kähler–Einstein metric ω_β on $\mathbb{P}^2$ with cone angle $2\pi\beta$ along a smooth degree 2 curve converge in the Gromov–Hausdorff sense to the standard orbifold Kähler–Einstein metric on $\mathbb{P}(1, 1, 4)$ as β tends to $1/4$.*

Actually, more generally, assume there is a special degeneration $(\mathcal{X}, \mathcal{Y})$ of the pair (X, Y), such that $(\mathcal{X}_0, \mathcal{Y}_0)$ is a conical Kähler–Einstein pair. Then we expect (X, Y) converges to $(\mathcal{X}_0, \mathcal{Y}_0)$ in Gromov–Hausdorff sense along certain continuity method (either by the classical continuity method by increasing Ricci curvature (cf. [4,39]), or by changing cone angles(cf. [25]) , or even by log-Kähler–Ricci flow(cf. in [60]). This philosophy is certainly well known to the experts in the field. In particular, this is related to [71] and [26].

Remark 5.3. In [33], $\mathbb{Q}$-Gorenstein smoothable degenerations of $\mathbb{P}^2$ are classified. They are given by partial smoothings of weighted projective planes $\mathbb{P}^2(a^2, b^2, c^2)$ where (a, b, c) satisfies the Markov equation: $a^2 + b^2 + c^2 = 3abc$. Different solutions are related by an operation called mutation: $(a, b, c) \to (a, b, 3ab - c)$. The first several solutions are $(1, 1, 1)$, $(1, 1, 2)$, $(1, 5, 2)$, $(1, 5, 13)$, $(29, 5, 2)$. The above construction gives geometric realization of such degeneration corresponding to the mutation $(1, 1, 1) \to (1, 1, 2)$. We expect there is similar geometric realization of every mutation.

5.2. Calabi–Yau cone metrics on three dimensional A_2 singularity. Through a stimulating discussion with Dr. Hans-Joachim Hein, we learned that Theorem 1.5 has the following application. Recall it was discovered by Gauntlett–Martelli–Sparks–Yau [29] that there may not exist Calabi–Yau cone metrics on certain isolated quasi-homogeneous hypersurface singularities, with the obvious Reeb vector field. In particular, there are two constraints: Bishop obstruction and Lichnerowicz obstruction. As an example, the cases of three dimensional A_{k-1} singularities were studied. Recall a three dimensional A_{k-1} singularity is the hypersurface in $\mathbb{C}^4$ defined by the following equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^k = 0.$$

There is a standard Reeb vector field ξ_k which generates the $\mathbb{C}^*$ action with weights $(k, k, k, 2)$. Let L_k be the Sasaki link of the A_{k-1} singularity. Then the existence of a Calabi–Yau cone metric with Reeb vector field ξ_k is equivalent to the existence of a Sasaki–Einstein metric on L_k. In [29], using the Bishop obstruction, it was proved that L_k admits no Sasaki–Einstein metric for $k > 20$, and using Lichnerowicz obstruction this bound was improved to $k > 3$. For $k = 2$ this is the well-known conifold singularity and there is a homogeneous Sasaki–Einstein metric on the link L_2. For $k = 3$ by Matsushima's theorem the possible Sasaki–Einstein metric on L_3 must be invariant under $SO(3;\mathbb{R})$ action, and is of cohomogeneity one. The ordinary differential equation has been written down explicitly in [29], and it is an open question in [29] whether L_3 admits a Sasaki–Einstein metric.

In the language of Sasaki geometry, the above examples L_k are all quasi-regular, meaning that the Reeb vector field ξ_k generates an S^1 action on L_k, and the quotient M_k is a polarized orbifold M_k(in the sense of [56]). The existence of a Sasaki–Einstein metric on L_k is equivalent to the existence of an orbifold Käler–Einstein metric on M_k. In the above concrete cases, the orbifold M_k is the hypersurface in $\mathbb{P}(k, k, k, 2)$ defined by the same equation $x_1^2 + x_2^2 + x_3^2 + x_4^k = 0$. Note that $\mathbb{P}(k, k, k, 2)$ is not well-formed. When $k = 2m + 1$ is odd, then

$$\begin{aligned}\mathbb{P}(2m+1,2m+1,2m+1,2) &\xrightarrow{\cong} \mathbb{P}(2m+1,2m+1,2m+1,2(2m+1)) = \mathbb{P}(1,1,1,2)\\ [x_1, x_2, x_3, x_4] &\mapsto [x_1, x_2, x_3, x_4^{2m+1}].\end{aligned}$$

When $k = 2m$ is even, then

$$\begin{aligned}\mathbb{P}(2m, 2m, 2m, 2) = \mathbb{P}(m, m, m, 1) &\xrightarrow{\cong} \mathbb{P}(m, m, m, m) = \mathbb{P}(1, 1, 1, 1)\\ [x_1, x_2, x_3, x_4] &\mapsto [x_1, x_2, x_3, x_4^m].\end{aligned}$$

So M_k is isomorphic to $\{z_1^2+z_2^2+z_3^2+z_4 = 0\} \cong \mathbb{P}^2$ when k is odd, and to $\{z_1^2+z_2^2+z_3^2+z_4^2 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$ when k is even.

Regarding the non-well-formed orbifold structure it is not hard to see that when k is odd we get $(\mathbb{P}^2, (1 - 1/k)D)$ and when k is even we get $(\mathbb{P}^1 \times \mathbb{P}^1, (1 - 2/k)\Delta)$. Thus we see the close relationship between the existence of Sasaki–Einstein metric on L_k and the existence of conical Kähler–Einstein metric on $(\mathbb{P}^2, (1 - 1/k)D)$. In particular we know there is no Sasaki–Einstein metric on L_k for $k > 3$ by Example 3.12. This is not surprising, since by [56] the Lichnerowicz obstruction could be interpreted as slope stability for orbifolds. The new observation here is the case $k = 3$ follows from Theorem 1.5. So we know the three dimensional A_2 singularity admits a Calabi–Yau cone metric with the standard Reeb vector field. This is Corollary 1.6.

The corresponding Sasaki–Einstein metric on L_k is invariant under the $SO(3;\mathbb{R})$ action. It would be interesting to find an explicit solution by solving the ODE written in [29]. In [21] cohomogeneity one Sasaki–Einstein five manifolds were classified, but the above result suggests that the classification is incomplete.

Remark 5.4. In [42], the first author used numerical method to solve the ODE in [29]. The numerical results confirm our theoretical result. Moreover, numerical results show that Conjecture 5.2 is true. Also, by calculating the simplest examples of A_0 and A_1 singularities, one finds there are indeed cases which were ignored in [21]. For details, see [42].

6. Kähler–Einstein Metrics from Branched Cover

One of our motivations for this paper is to construct smooth Kähler–Einstein metrics using branch covers(see [1,30] for such kind of constructions). If $D \sim mD_1$ with D_1 being an integral divisor, we can construct branch cover of X with branch locus D.

$$\begin{array}{ccc} B & \subset & Y \\ \downarrow & & \downarrow \pi \\ D & \subset & X \end{array}$$

The canonical divisors of X and Y are related by Hurwitz formula:

$$K_Y = \pi^* \left(K_X + \frac{m-1}{m} D \right).$$

In our setting, since $D \sim -\lambda K_X$, we get

$$K_Y^{-1} = \left(1 - \frac{m-1}{m}\lambda\right) \pi^* K_X^{-1} = r(1/m)\pi^* K_X^{-1}. \tag{32}$$

We have the following 3 cases to consider.

(1) (Positive Ricci) $-(K_X + (1 - 1/m)D)$ is ample. This is equivalent to $r(1/m) > 0$.
Example: $X = \mathbb{P}^2$. Define $\deg Y = (K_Y^{-1})^2$.
- $\deg D = 2$, $m = 2$, $\lambda = 2/3$. $\deg Y = 4$. $Y = \mathbb{P}^1 \times \mathbb{P}^1$.
- $\deg D = 3$, $m = 3$, $\lambda = 1$. $\deg Y = 3$. Y is a cubic surface.
- $\deg D = 4$, $m = 2$, $\lambda = 4/3$. $\deg Y = 2$. Y is $Bl_7\mathbb{P}^2$.

(2) (Ricci flat) $K_X + (1 - 1/m)D \sim 0$. This is equivalent to $r(1/m) = 0$.
Example: $X = \mathbb{P}^2$.
- $\deg D = 4$, $m = 4$, $\lambda = 4/3$. Y is a K3 surface in $\mathbb{P}^3$.
- $\deg D = 6$, $m = 2$, $\lambda = 2$. Y is a K3 surface.

(3) (Negative Ricci) $K_X + (1 - 1/m)D$ is ample.
Example: X is $\mathbb{P}^2$ and D is a general smooth, degree d curve such that $\lambda = d/3$. Choose $m|d$. Except for the cases already listed above, $K_X + (1 - 1/m)D$ is ample.

Assume we have already constructed an orbifold Kähler–Einstein metric $\widehat{\omega}_{KE}$ on $(X, (1 - 1/m)D)$. Then $\pi^*\widehat{\omega}_{KE}$ is a smooth Kähler–Einstein metric on Y. Note that orbifold Kähler metric can be seen as a special case of conical Kähler metric, i.e. when the cone angle is equal to $2\pi/m$ for some $m \in \mathbb{Z}$. So existence of conical Kähler-Einstein metrics with angle $2\pi/m$ will give rise to smooth Kähler–Einstein metrics. Using the existence theory for conical Kähler–Einstein metrics, we can construct a lot of smooth Kähler–Einstein metrics on Fano manifolds using branch covers. More precisely, using the notation of branch-covering above, we have

Theorem 6.1. *If there is conical Kähler–Einstein metric on $(X, (1-1/m)D)$, then there is a smooth Kähler–Einstein metric on Y. In particular, if X admits a Kähler–Einstein metric and $\lambda \geq 1$, then there exists smooth Kähler–Einstein metric on Y.*

To begin the proof, we first observe the following

Proposition 16. *Fix an orbifold Kähler metric ω on $(X, (1-1/m)D)$. The branch cover π induces a map from $\mathcal{PSH}(\omega)$ to $\mathcal{PSH}(\pi^*\omega)$ by pulling back. The energy functionals are compatible with this pull back.*

$$F^Y_{r(1/m)\pi^*\omega}(r(1/m)\pi^*\phi) = mF^X_{\omega,(1-1/m)D}(\phi),$$
$$\mathcal{M}^Y_{r(1/m)\pi^*\omega}(r(1/m)\pi^*\phi) = m\mathcal{M}^X_{\omega,(1-1/m)D}(\phi).$$

Similar relation holds for the functionals $F^0_\omega(\phi)$, I and J.

Proof. For any orbifold Kähler metric $\omega \in 2\pi c_1(X)$, there exists $H_{\omega,(1-1/m)D}$ such that

$$Ric(\omega) - r(1/m)\omega - (1-1/m)\{D\} = \sqrt{-1}\partial\bar{\partial} H_{\omega,(1-1/m)D}. \tag{33}$$

$\tilde{\omega} = r(1/m)\pi^*\omega$ is a smooth Kähler metric in $c_1(Y)$ (see (32)). Note that ω^n has poles along D, but $\pi^*\omega^n$ is a smooth volume form. From (33), we get

$$Ric(\tilde{\omega}) - \tilde{\omega} = \sqrt{-1}\partial\bar{\partial}\pi^* H_{\omega,(1-1/m)D}.$$

So $h_{\tilde{\omega}} := H_{\tilde{\omega},0} = \pi^* H_{\omega,(1-1/m)D}$ and $e^{h_{\tilde{\omega}}}\tilde{\omega}^n = \pi^*(e^{H_{\omega,(1-1/m)D}}\omega^n)$.

$$\int_X e^{H_{\omega,(1-\beta)D} - r(1/m)\phi}\omega^n/n! = \frac{1}{m}\int_Y e^{h_{\tilde{\omega}} - \pi^*(r(1/m)\phi)}\tilde{\omega}^n/n!.$$

So we get the identity for log-Ding-energy on X and Y. Similarly, by the defining formula for the $F^0_\omega(\phi)$, I, J functional in Definition 2.6, the relation stated in the proposition holds. □

Proof of Theorem 6.1. We can choose the reference metric ω on X to be orbifold metric. Then the pull back $\tilde{\omega} = r(1/m)\pi^*\omega$ is a smooth Kähler metric on Y. If $\omega_{KE} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_{KE}$ is the conical Kähler–Einstein metric on $(X, (1-1/m)D, c_1(X))$, then $\tilde{\phi}_{KE} = r(1/m)\pi^*\phi_{KE}$ is the bounded continuous solution of the following Monge–Ampère equation on Y.

$$(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\phi})^n = e^{h_{\tilde{\omega}} - \tilde{\phi}}\tilde{\omega}^n.$$

By the regularity result in [62] (see also [8]), $\tilde{\phi}_{KE}$ is indeed a smooth solution of Kähler–Einstein equation on Y. □

7. Convergence of Conical KE to Smooth KE

In this section, we prove the convergence statement in Theorem 1.2 and related discussions following it. So we assume there exists smooth Kähler–Einstein metric on X. When $Aut(X)$ is discrete, then ω_{KE} is invariant under $Aut(X)$. In this case, the Mabuchi energy is proper on $\hat{\mathcal{H}}(\omega)$.

Theorem 7.1. *Assume $D \sim_{\mathbb{Q}} -\lambda K_X$ with $\lambda \geq 1$, $\omega_\beta = \omega + \sqrt{-1}\partial\bar{\partial}\phi_\beta$ is the conical Kähler–Einstein metric on $(X, (1-\beta)D)$, and $\omega_{KE} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_{KE}$, then ϕ_β converges to ϕ_{KE} in C^0-norm. Moreover, ϕ_β converges smoothly on any compact set away from D.*

Proof. Choose any smooth reference Kähler metric ω. By Proposition 1, the log-Mabuchi-energy $\mathcal{M}_{\omega,(1-\beta)D}$ is proper on $\hat{\mathcal{H}}$ for $\beta \in (0,1]$. Furthermore, from the interpolation, we see that there exists constants C_1 and C_2 independent of β such that

$$\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) \geq C_1 I_\omega(\omega_\phi) - C_2. \tag{34}$$

Since ω_β obtains the minimum of log-Mabuchi-energy, we have

$$\mathcal{M}_{\omega,(1-\beta)D}(\omega_\beta) \leq \mathcal{M}_{\omega,(1-\beta)D}(\omega) = 0.$$

So from (34), we see that there exists a constant C independent of β such that

$$I_\omega(\omega_\beta) \le C.$$

Assume $\omega_\beta = \omega + \sqrt{-1}\partial\bar\partial\phi_\beta$. By [4] and [34], there exists a constant C independent of β such that

$$Vol(X) \cdot Osc(\phi_\beta) \le I_\omega(\omega_\beta) + C = \int_X \phi(\omega^n - \omega_\beta^n)/n! + C.$$

So $\|\phi_\beta\|_{C^0}$ is uniformly bounded. Now the first statement follows from standard pluripotential theory. For the last statement, we can use Chern–Lu's inequality:

$$\Delta_\beta(\log tr_{\omega_\beta}\omega - C\phi_\beta) \ge (C_1 - \lambda n) + (\lambda - C_2 tr_{\omega_\beta}\omega).$$

to get C^2-estimate on ϕ_β. (Note that it's easy to verify from the calculation in the Appendix of [34] that we indeed have the upper bound on bisectional curvature to be independent of β at least when $\beta \ge \delta > 0$.) Then we can use Krylov–Evan's estimate to get uniform higher order estimates on any compact set away from D. The smooth convergence away from D follows from these uniform estimates. □

When $Aut(X)$ is continuous, then $Aut(X)$ is the complexification of $G := Isom(X, \omega_{KE})$. By Bando–Mabuchi's theorem [4], the moduli space of Kähler–Einstein metrics (denoted by $\mathscr{M}_{KE}$) is isomorphic to the symmetric space $G^{\mathbb{C}}/G$. In particular

$$T_{\omega_{KE}}\mathscr{M}_{KE} = \mathfrak{g} = LieG.$$

Recall that Matsushima's theorem ([46]) says that

$$\mathfrak{g} = (\Lambda_1^{\mathbb{R}})_0 = \{\theta \in C^\infty(X, \mathbb{R}); (\Delta_{KE} + 1)\theta = 0, \int_X \theta\omega_{KE}^n/n! = 0\}.$$

Now we want to identify the limit ω_{KE}^D as $\beta \to 1$. ω_{KE}^D turns out to be the critical point of the following functional, which is a part of log-Mabuchi-functional.

Lemma 7.2. *Define the functional*

$$\mathscr{F}_{\omega,D}(\omega_\phi) = \lambda(I - J)_\omega(\omega_\phi) + \int_X \log|s|_h^2(\omega_\phi^n - \omega^n)/n!$$

where $\lambda\omega = -\sqrt{-1}\partial\bar\partial\log|\cdot|_h^2$. *Then* $\mathscr{F}$ *satisfies the following properties:*

(1)

$$\mathcal{M}_{\omega,(1-\beta)D}(\omega_\phi) = \mathcal{M}_\omega(\omega_\phi) + (1-\beta)\mathscr{F}_{\omega,D}(\omega_\phi). \tag{35}$$

(2) $\mathscr{F}_\omega$ *satisfies the cocycle condition. More precisely, for* $\phi, \psi \in \mathcal{PSH}_\infty(\omega)$, *we have*

$$\mathscr{F}_{\omega,D}(\omega_\phi) - \mathscr{F}_{\omega_\psi,D}(\omega_\phi) = \mathscr{F}_{\omega,D}(\omega_\psi),$$
$$\mathscr{F}_\omega(\omega_\psi) = -\mathscr{F}_{\omega_\psi}(\omega).$$

(3) $\mathscr{F}$ *is convex along geodesics of Kähler metrics..*

Proof. The first item follows from the expression for log-Mabuchi energy in, for example, formula (14). The second statement follows from the cocycle properties of $\mathcal{M}_{\omega,(1-\beta)D}$ and $\mathcal{M}_{\omega}$. It can also be verified by direct calculations.

For the last statement, first it is well known that $\mathscr{M}$ is a totally geodesic submanifold of the space of smooth Kähler metrics in $2\pi c_1(X)$ and $(I-J)_{\omega}(\omega_{\phi})$ is convex on the space of smooth Kähler metrics. Assume $\phi(t)$ is a geodesic, i.e. $\ddot{\phi}-|\nabla\dot{\phi}|^2_{\omega_\phi}=0$. Then we can calculate

$$\begin{aligned}\frac{d}{dt}(I-J)_{\omega}(\omega_{\phi}) &= -\int_X \phi\Delta_{\omega_\phi}\dot{\phi}\omega_\phi^n/n! = -\int_X \dot{\phi}(\omega_\phi-\omega)\wedge\omega_\phi^{n-1}/(n-1)!\\ &= n\frac{d}{dt}F^0_{\omega}(\phi)+\int_X\dot{\phi}\omega\wedge\omega_\phi^{n-1}/(n-1)!.\\ \frac{d}{dt}\int_X \log|s|_h^2(\omega_\phi^n-\omega^n)/n! &= \int_X \log|s|_h^2\Delta_{\omega_\phi}\dot{\phi}\omega_\phi^n/n!\\ &= \int_X(-\lambda\omega+2\pi\{D\})\dot{\phi}\omega_\phi^{n-1}/(n-1)!.\end{aligned}$$

So combining the above two identities, we get

$$\frac{d}{dt}\mathscr{F}_{\omega,D}(\omega_\phi)=\frac{d}{dt}(n\lambda F^0_{\omega}(\phi)-F^0_{\omega,2\pi D}(\phi)).$$

This is certainly true by the way how we integrate the log-Futaki invariant to get the log Mabuchi energy. Now since $F^0_{\omega}(\phi)$ is affine along geodesics of Kähler metrics, we get

$$\begin{aligned}\frac{d^2}{dt^2}\mathscr{F}_{\omega,2\pi D}(\omega_\phi) &= -\frac{d^2}{dt^2}F^0_{\omega,D}(\phi)=\frac{\int_{2\pi D}(\omega+\partial\bar{\partial}\Phi)^n/n!}{dt\wedge d\bar{t}}\\ &= \int_{2\pi D}\ddot{\phi}\omega_{KE}^{n-1}/(n-1)!-\int_{2\pi D}\partial\dot{\phi}\wedge\bar{\partial}\dot{\phi}\wedge\omega_{KE}^{n-2}/(n-2)!\\ &= \int_{2\pi D}(|\nabla\dot{\phi}|^2_{\omega_{KE}}-|\nabla^D\dot{\phi}|^2_{\omega_{KE}|_D})\omega_{KE}^{n-1}/(n-1)!\\ &= \int_{2\pi D}|(\nabla\dot{\phi})^{\perp}|^2_{\omega_{KE}}\omega_{KE}^{n-1}/(n-1)!\geq 0. \qquad (36)\end{aligned}$$

□

Lemma 7.3. *We have the following different formulas for the Hessian of $\mathscr{F}_{\omega,D}$ on $\mathscr{M}_{KE}$: for any $\theta\in(\Lambda_1^{\mathbb{R}})_0$,*

$$\begin{aligned}Hess\mathscr{F}(\theta,\theta) &= \int_{2\pi D}|\nabla\theta^{\perp}|^2\omega_{KE}^{n-1}/(n-1)! \qquad (37)\\ &= \lambda\int_X\theta^2\omega_{KE}^n/n!+\int_X(-\theta^2+\theta^i\theta_i)(\lambda\phi-\log|s|_h^2)\omega_{KE}^n/n!\\ &= \lambda\int_X\theta^2\omega_{KE}^n/n!+\int_X(\theta^2-\theta^i\theta_i)(\log|s|^2_{he^{-\lambda\phi}})\omega_{KE}^n/n!. \qquad (38)\end{aligned}$$

Proof. The first identity follows from (36) because $\theta=\frac{\partial\phi}{\partial t}|_{t=0}$. Let's prove the 2nd identity. For $\theta\in\Lambda_1=Ker(\Delta_{KE}+1)$, $\nabla\theta$ is a holomorphic vector field generating a one parameter subgroup σ_t in $Aut(X)$. Let $\sigma_t^*\omega_{KE} = \omega_{KE} + \sqrt{-1}\partial\bar\partial\phi_t$. Then ϕ_t satisfies the geodesic equation: $\ddot\phi - |\nabla\dot\phi|^2_{\omega_\phi} = 0$ with initial velocity $\frac{d}{dt}\phi|_{t=0} = \theta$.

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0}\omega_t^n &= \frac{d}{dt}(\Delta\dot\phi\omega_t^n) = (\dot\Delta\dot\phi + \Delta\ddot\phi + (\Delta\dot\phi)^2)\omega_{KE}^n\\ &= (-\theta^{i\bar j}\theta_{i\bar j} + (\theta^i\theta_i)_j{}^j + \theta^2)\omega_{KE}^n\\ &= (-\theta^{i\bar j}\theta_{i\bar j} + \theta^i{}_j\theta_i{}^j + \theta^i{}_j{}^j\theta_i + \theta^2)\omega_{KE}^n/n!\\ &= (\theta^2 - \theta^i\theta_i)\omega_{KE}^n\end{aligned}$$

Note that in the last identity, the relation $\Delta\theta = \theta_i{}^i = -\theta$ was used. So we get

$$\begin{aligned}Hess\mathscr{F}(\theta,\theta) &= \frac{d^2}{dt^2}\left(\lambda(I-J)_\omega(\omega_t) + \int_X \log|s|_h^2(\omega_t^n - \omega^n)/n!\right)\\ &= -\lambda\int_X \dot\phi\Delta\dot\phi\omega_{KE}^n/n! + \int_X(-\lambda\phi + \log|s|_h^2)\frac{d^2}{dt^2}\omega_t^n/n!\\ &= \lambda\int_X\theta^2\omega_{KE}^n/n! + \int_X(\log|s|^2_{he^{-\lambda\phi}})(\theta^2 - \theta^i\theta_i)\omega_{KE}^n/n!.\end{aligned}$$

□

Lemma 7.4. *If there is no holomorphic vector field on X which is tangent to D, i.e. $Aut(X, D)$ is discrete, then $Hess\mathscr{F}$ is non-degenerate at any point $\omega_{KE} \in \mathscr{M}_{KE}$. In particular, this holds when $\lambda \geq 1$.*

Proof. We have seen $Hess\mathscr{F}$ is non-negative at any point $\omega_{KE} \in \mathscr{M}_{KE}$ using formula (37). $Hess\mathscr{F}$ is degenerate if and only if $\int_{2\pi D}|(\nabla\theta)^\perp|^2\omega_{KE}^{n-1}/(n-1)! = 0$. This happens if and only if $(\nabla\theta)^\perp \equiv 0$ on D, i.e. when the holomorphic vector field $\nabla\theta$ is tangent to D. The last statement follows from Corollary 2.21 (see also [58]). □

Lemma 7.5. *When restricted to $\mathscr{M}_{KE}$, there exists a unique minimum ω_{KE}^D of $\mathscr{F}_{\omega,D}(\omega_\phi)$.*

Proof. By the previous Lemma, $\mathscr{F}_{\omega,D}$ is a convex functional on the space $\mathscr{M}_{KE} \cong G^{\mathbb{C}}/G$. To prove the existence of critical point, we only need to show it's proper on $G^{\mathbb{C}}/G$. Because we assumed $\lambda \geq 1$ and there exists Kähler–Einstein on X, by Theorem 2.10, $\mathcal{M}_{\omega,(1-\beta)D}$ is proper for $\beta \in (0, 1)$. Because the Mabuchi energy is constant on $\mathscr{M}_{KE}$, by equality (35), $\mathscr{M}_{\omega,(1-\beta)D} = (1-\beta)\mathscr{F}_{\omega,D}$+constant is proper on $\mathscr{M}_{KE}$. □

Write $\omega_{KE}^D = \omega + \sqrt{-1}\partial\bar\partial\phi_{KE}^D$, then it satisfies the critical point equation

$$\int_X(\log|s|_h^2 - \lambda\phi_{KE}^D)\psi(\omega_{KE}^D)^n/n! = 0.$$

for any $\psi \in T_{\omega_{KE}^D}\mathscr{M}_{KE} \cong \Lambda_1(\omega_{KE}^D)/\mathbb{R}$. In other words, $\lambda\phi_{KE}^D - \log|s|_h^2 \in \Lambda_1^\perp$.

Proposition 17. *As $\beta \to 1$, the conical Kähler–Einstein metrics ω_β converges to a unique smooth Kähler–Einstein metric $\omega_{KE}^D \in \mathscr{M}_{KE}$ (under one technical assumption that the implicit function theorem is valid on admissible function space when $\beta = 1$).*

Proof. Recall that the conical Kähler–Einstein equation can be written as

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{h_\omega - r(\beta)\phi} \frac{\omega^n}{|s|^{2(1-\beta)}}.$$

Any $\omega_{KE} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_{KE} \in \mathscr{M}_{KE}$ satisfies the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi_{KE})^n = e^{h_\omega - \phi_{KE}}\omega^n.$$

By Lemma 7.5, there exists a unique minimum ω_{KE}^D of the functional $\mathscr{F}_{\omega,D}$ on $\mathscr{M}_{KE}$. We will choose $\omega_{KE} = \omega_{KE}^D$ in the following argument. Divide the above two equations to get

$$\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_{KE})^n} = \phi_{KE} - r(\beta)\phi - (1-\beta)\log |s|_h^2.$$

Let $\phi = \phi_{KE} + \psi$ and $\psi = \theta + \psi'$ with $\theta \in \Lambda_1$ and $\psi' \in \Lambda_1^\perp$, then

$$\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'))^n}{\omega_{KE}^n} + r(\beta)(\theta + \psi') = (1-\beta)(\lambda\phi_{KE} - \log |s|_h^2). \quad (39)$$

We use Bando–Mabuchi's bifurcation method to solve the equation for β close to 1. First project to $\Lambda_1^\perp$ to get

$$(1-P_0)\left(\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'))^n}{\omega_{KE}^n}\right) + r(\beta)\psi' = (1-\beta)(\lambda\phi_{KE} - \log |s|_h^2). \quad (40)$$

The equation is satisfied for $(\beta, \psi, \theta) = (1, 0, 0)$. The linearization of the left side of this equation with respect to ψ' is

$$(1 - P_0)(\tilde{\Delta}_\theta + r(\beta))\psi'$$

where $\tilde{\Delta}_\theta$ is the Laplacian with respect to $\omega_{KE} + \sqrt{-1}\partial\bar{\partial}\theta$. Since $\Lambda_1 = Ker(\Delta_{KE} + 1)$, there exists a positive constant $\delta > 0$, such that

$$(1 - P_0)(-\Delta_{\omega_{KE}} - 1) \geq \delta > 0.$$

By continuity, it's easy to see that

$$(1 - P_0)(-\tilde{\Delta}_\theta - r(\beta)) \geq \delta/2 > 0.$$

for (β, θ) close to $(1, 0)$. In other words, the inverse of $(1 - P_0)(\tilde{\Delta}_\theta + r(\beta))$ has uniformly bounded operator norm for (β, θ) close to $(1, 0)$. So by implicit function theorem, there exists solution $\psi'_{\beta,\theta}$ for β near 1 and θ small. Now to solve the Eq. (39), we only need to solve the following equation, obtained by projecting to Λ_1,

$$P_0\left(\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'))^n}{\omega_{KE}^n}\right) = -r(\beta)\theta. \quad (41)$$

To solve this, we need to take the gauge group $G = Isom(X, \omega_{KE})$ into account and rewrite (41) in another form. For any $\sigma \in G$ near Id, we have a function $\theta = \theta_\sigma$

satisfying $\sigma^*\omega_{KE} = \omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta})$. Because $\sigma^*\omega_{KE}$ is a smooth Kähler–Einstein metric, we have the equation

$$\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta}))^n}{\omega_{KE}^n} = -(\theta + \psi'_{1,\theta})$$

Now let $\psi'_{\beta,\theta} = \psi'_{1,\theta} + (1-\beta)\xi_{\beta,\theta}$. We can rewrite the Eq. (39) in the following form

$$\begin{aligned}
&\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta} + (1-\beta)\xi_{\beta,\theta}))^n}{\omega_{KE}^n} \\
&= -(1-\lambda(1-\beta))(\theta + \psi'_{1,\theta} + (1-\beta)\xi_{\beta,\theta}) + (1-\beta)(\lambda\phi_{KE} - \log|s|_h^2) \\
&= \log\left(\frac{\omega_\theta^n}{\omega_{KE}^n}\right) + (1-\beta)\lambda(\theta + \psi'_{1,\theta}) - (1-\beta)r(\beta)\xi_{\beta,\theta} + (1-\beta)(\lambda\phi_{KE} - \log|s|_h^2).
\end{aligned}$$

where $\omega_\theta = \omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta})$. In particular, it's easy to see that (40) is equivalent to

$$\begin{aligned}
&\frac{1}{1-\beta}(1-P_0)\left(\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta} + (1-\beta)\xi_{\beta,\theta}))^n}{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta}))^n}\right) \\
&= \lambda\psi'_{1,\theta} - r(\beta)\xi_{\beta,\theta} + (\lambda\phi_{KE} - \log|s|_h^2).
\end{aligned}$$

Let $\beta \to 1$ to get

$$(1-P_0)\left((\Delta_\theta + 1)\xi_{1,\theta}\right) - \lambda\psi'_{1,\theta} = \lambda\phi_{KE} - \log|s|_h^2.$$

where Δ_θ is the Laplacian with respect to the metric ω_θ. In particular, $\Delta_0 = \Delta_{KE}$. Since $Im(\Delta_0 + 1) = (Ker(\Delta_0 + 1))^\perp = \Lambda_1^\perp$, so in particular,

$$(\Delta_0 + 1)\xi_{1,0} = \lambda\phi_{KE} - \log|s|_h^2. \tag{42}$$

Now the Eq. (41) is equivalent to

$$P_0\left(\frac{1}{1-\beta}\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta} + (1-\beta)\xi_{\beta,\theta}))^n}{(\omega_{KE} + \sqrt{-1}\partial\bar{\partial}(\theta + \psi'_{1,\theta}))^n}\right) - \lambda\theta = 0 \tag{43}$$

Denote by $\Gamma(\beta,\theta)$ the term on the left side, Then

$$\Gamma(1,0) = 0, \quad \Gamma(1,\theta) = P_0(\Delta_\theta\xi_{1,\theta}) - \lambda\theta.$$

Let $\theta(t) = t\theta \in \Lambda_1 = Ker(\Delta_0 + 1)$. For any $\theta' \in \Lambda_1$,

$$\begin{aligned}
\int_X \frac{d}{dt}\Gamma(1,\theta)\Big|_{t=0} \theta'\omega_{KE}^n/n! &= -\lambda\int_X \theta\theta'\omega_{KE}^n/n! + \int_X (\dot{\Delta}_\theta\xi_{1,0} + \Delta_0\dot{\xi}_{1,0})\theta'\omega_{KE}^n/n! \\
&= -\lambda\int_X \theta\theta'\omega_{KE}^n/n! + \int_X -\theta^{i\bar{j}}(\xi_{1,0})_{i\bar{j}}\theta'\omega_{KE}^n/n!.
\end{aligned}$$

Let $\xi = \xi_{1,0} \in \Lambda_1^{\perp}$. As the calculation in [4], we have

$$\begin{aligned}\int_X \theta^{i\bar{j}}\xi_{i\bar{j}}\theta'\omega_{KE}^n/n! &= -\int_X (\theta^i\xi_{i\bar{j}}^{\bar{j}}\theta' + \theta^i\xi_{i\bar{j}}\theta'^{\bar{j}})\omega_{KE}^n/n! = -\int_X (\theta^i\xi_{\bar{j}\,i}^{\bar{j}}\theta' - \theta^i\xi_i\theta'^{\bar{j}}_{\bar{j}})\omega_{KE}^n/n!\\ &= -\int_X \theta^i((\Delta+1)\xi)_i\theta'\omega_{KE}^n/n! = \int_X (-\theta\theta' + \theta^i\theta'_i)(\Delta+1)\xi\omega_{KE}^n/n!\\ &= -\int_X (\theta\theta' - \theta^i\theta'_i)(\lambda\phi - \log|s|_h^2)\omega_{KE}^n/n!\end{aligned}$$

In the last identiy, we used the relation in (42). So

$$\begin{aligned}D_2\Gamma(1,0)(\theta)\theta' &= -\lambda\int_X \theta\theta'\omega_{KE}^n/n! + \int_X (\theta\theta' - \theta^i\theta'_i)(\lambda\phi - \log|s|_h^2)\omega_{KE}^n/n!\\ &= -Hess\mathscr{F}(\theta,\theta'). \quad (\text{ by equation (38)})\end{aligned}$$

By Lemma 7.4, $D_2\Gamma(1,0)$ is invertible, so by implicit function theorem, (43) is solvable for β close to 1. So we get conical Kähler-Einstein metrics ω_β for β close to 1 and by continuity ϕ_β converges to ϕ_{KE}^D as $\beta \to 1$. $\square$

Remark 7.6. The above calculations are variations of Bando–Mabuchi's calculation. In their work [4], Bando–Mabuchi solved equations in Aubin's continuity method backwardly from the correctly identified Kähler–Einstein metric. To do this, in general, they needed to perturb the reference metric to make sure some linear map similar to $D_2\Gamma(1,0)$ is invertible. In the conical case, when there is no holomorphic vector field tangent to D, by Lemma 7.4 and above calculations, we see that $D_2\Gamma(1,0)$ is always invertible. This is not very surprising because the conical continuity method is in some sense more canonically related to the background geometry. If we continue to calculate as in [4]:

$$\begin{aligned}\int_X \theta^{i\bar{j}}\xi_{i\bar{j}}\theta'\omega_{KE}^n/n! &= -\int_X (\theta\theta' - \theta^i\theta'_i)(\lambda\phi - \log|s|_h^2)\omega_{KE}^n/n!\\ &= \frac{1}{2}n\lambda\int_X \theta\theta'\omega_{KE}^n/n! - \frac{1}{2}\int_{2\pi D}\theta\theta'\omega_{KE}^{n-1}/(n-1)!.\end{aligned}$$

so that the linearized operator becomes

$$D_2\Gamma(1,0)(\theta)\theta' = -\lambda(1+n/2)\int_X \theta\theta'\omega_{KE}^n/n! + \frac{1}{2}\int_{2\pi D}\theta\theta'\omega_{KE}^{n-1}/(n-1)!,$$

it seems not straightforward to see that $D_2\Gamma(1,0)$ is nondegenerate using this formula.

Remark 7.7. One reason why we packed all the conical spaces together in the space of admissible functions is because that we need to work in different function space corresponding to different cone angles. Strictly speaking, there are subtleties in applying implicit functional theorem in this setting. However, we expect one can generalize Donaldson's argument to validate the application of implicit function theorem.

8. Relations to Song–Wang's Work

In this section, we will briefly explain Song–Wang's results and derive one of its implications.

For one thing, they also observe the interpolation property for the log-Ding-energy. Secondly, they considered pluri-anticanonical sections in their paper. This corresponds to the $\lambda \geq 1$ case in our paper. Recall that $R(X)$ in the introduction (see (2)) is defined to be the greatest lower bound of Ricci curvature of smooth Kähler metrics in $c_1(X)$. $R(X)$ was studied in [38,41,58,61,64]. Donaldson [25] made the following conjecture

Conjecture 8.1 (Donaldson). *Let $D \in |-K_X|$ be a smooth divisor, then there exists a conical Kähler–Einstein metric on $(X, (1-\beta)D)$ if and only if $\beta \in (0, R(X))$.*

Song–Wang proved a weak version of Donaldson's conjecture by allowing pluri-anticanonical divisor and its dependence on β. Translating their result in our notations, they proved

Theorem 8.2 (Song–Wang [58]). *For any $\gamma \in (0, R(X))$ there exists a large $\lambda \in \mathbb{Z}$ and a smooth divisor $D \in |\lambda K_X^{-1}|$ such that there exists a conical Kähler–Einstein metric on $(X, \lambda^{-1}(1-\gamma)D)$.*

Remark 8.3. Note that in general, λ and D may depend on γ. γ is related to the cone angle parameter β by the relation $\lambda^{-1}(1-\gamma) = 1-\beta$ or equivalently, $\gamma = r(\beta) = 1-\lambda(1-\beta)$.

The proof of this theorem can be explained through the Hölder's inequality

$$\int_X e^{h_\omega - \gamma\phi} \frac{\omega^n}{n!|s|^{2(1-\gamma)/\lambda}} \leq \left(\int_X e^{p(h_\omega-\gamma\phi)}\omega^n/n!\right)^{1/p} \left(\int_X |s|^{-2q(1-\gamma)/\lambda}\omega^n/n!\right)^{1/q}$$

where $p^{-1}+q^{-1} = 1$. To make contact with the invariant $R(X)$, one choose $p = \frac{t}{\gamma}$ for any $t \in (\gamma, R(X))$. (This is related to the characterization of $R(X)$ through the properness of twisted Ding-energy as in [41]) Then $q = (1-p^{-1})^{-1} = \frac{t}{t-\beta}$. Now the integrability of the second integral on the right gives the restriction on λ: $\frac{2q(1-\gamma)}{\lambda} - 1 < 1$. This gives the lower bound of λ in Song–Wang's theorem.

$$\lambda > (1-\gamma)\frac{R(X)}{R(X)-\gamma}.$$

One other result Song–Wang proved is the construction of toric conical Kähler–Einstein metrics. This can be combined with the strategy in our paper to prove a (weak) version of Donaldson's conjecture on toric Fano manifolds. We will explain this briefly.

Any toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle \subset \mathbb{R}^n$ containing only O as the interior lattice point. For any $P \in \mathbb{R}^n$, P determines a toric $\mathbb{R}$-divisor $D_P \sim_{\mathbb{R}} -K_X$. More concretely, assume that the polytope is defined by the inequalities $l_j(x) = \langle x, \nu_j\rangle + a_j \geq 0$. Then $D_P = \sum_j l_j(P)D_j$. If $P \in \overline{\triangle}$ is a rational point, then for any integer λ such that λP is an integral lattice point, there exist a genuine holomorphic section s_P^λ of $-\lambda K_X$ and an integral divisor λD_P.

Let P_c be the barycenter of $\triangle$, then the ray $\overrightarrow{P_c O}$ intersect the boundary $\partial\triangle$ at a unique point Q. Note that in general, Q is a rational point. In [38], the first author proved $R(X)$ is given by

$$R(X) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|}. \tag{44}$$

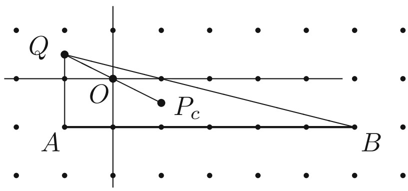
For any $\gamma \in (0, 1]$, define $P_\gamma = -\frac{\gamma}{1-\gamma} P_c$. Then $P_\gamma \in \overline{\triangle}$ if and only if $\gamma \in (0, R(X)]$, which is also equivalent to D_{P_γ} being effective. In particular, $P_{R(X)} = Q$. Using these notations, Song–Wang proved the following theorem by adapting the method in Wang–Zhu's work [73] on the existence of Kähler–Ricci solitons on toric Fano manifolds.

Theorem 8.4 (Song–Wang[58]). *For any $\gamma \in (0, R(X)]$, there exists toric solution to the following equation:*

$$Ric(\omega) = \gamma\omega + (1-\gamma)\{D_{P_\gamma}\}.$$

When $\gamma \in (0, R(X)]$ is rational, then the solution ω_γ is a conical Kähler–Einstein metric on $(X, (1-\gamma)D_{P_\gamma})$. In particular, when $\gamma = R(X)$, there exists a conical Kähler–Einstein metric on $(X, (1-R(X))D_Q)$.

Example 8.5. The above theorem can be generalized to toric orbifold case. (See [57] and [36] for related works) We will illustrate this by showing the conical Kähler–Einstein on $X = \mathbb{P}(1, 1, 4)$ considered in Sect. 5 in the toric language. The polytope determining $(X, -K_X)$ is the following rational polytope $\triangle$. Note that $-2K_X$ is Cartier because $2\triangle$ is a lattice polytope.



$Q = (-1, 1/2)$, $P_c = (1, -1/2)$. So $R(X) = |\overline{OQ}|/|\overline{P_c Q}| = 1/2$. $D_Q = 3/2D$, where the divisor D corresponds to the facet $\overline{AB}$. The conical Kähler–Einstein satisfies the equation:

$$Ric(\omega) = \frac{1}{2}\omega + (1 - \frac{1}{2}) \cdot \frac{3}{2}D.$$

So the cone angle along D is $2\pi\beta$ with $\beta = 1 - 3/4 = 1/4$.

Now we show that Song–Wang's existence result implies Theorem 2.

Proof of Theorem 2. Let $\mathcal{F}_Q$ be the minimal face of $\triangle$ containing Q. For any $\lambda \in \mathbb{Z}$ such that λQ is an integral point, define a set of rational points by

$$\mathcal{R}(Q, \lambda) = \{Q\} \bigcup \left((\overline{\triangle} \setminus \overline{\mathcal{F}_Q}) \bigcap \frac{1}{\lambda}\mathbb{Z}^n \right).$$

Then we define the linear system $\mathscr{L}_\lambda$ to be the linear subspace spanned by the holomorphic sections corresponding to rational points in $\mathcal{R}(Q, \lambda)$:

$$\mathscr{L}_\lambda = Span_{\mathbb{C}} \left\{ s_P^\lambda ; P \in \mathcal{R}(Q, \lambda) \right\}.$$

Choose any general element $D \in \mathscr{L}_\lambda$, the coefficient of the term s_Q^λ is nonzero. Because Q is a vertex of the convex hull of $\mathcal{R}(Q, \lambda)$, there exists a $\mathbb{C}^*$ action denoted by $\sigma(t)$ contained in the torus action, such that

$$\lim_{t \to 0} \sigma(t)^* D = \lambda D_Q.$$

In this way, we construct a degeneration $(\mathcal{X}, \frac{1-R(X)}{\lambda}\mathcal{D}, K_X^{-1})$ with $\mathcal{X} = X \times \mathbb{C}$ and $\mathcal{D}_t = \sigma_t^* D$. By Song–Wang's theorem in [58], the central fibre $(\mathcal{X}_0, \frac{1-R(X)}{\lambda}(\lambda D_Q)) = (X, (1 - R(X))D_Q)$ has conical Kähler–Einstein metric. So we use Theorem 1.8 to get the lower bound of $\mathcal{M}_{X, \frac{1-R(X)}{\lambda}D}$. (Here we are in a simpler situation. Since $\mathcal{X} = X \times \mathbb{C}$, we just need to use the trivial geodesic and apply Berndtsson's result in [11] to get the subharmonicity and use the argument as in the proof of Theorem 1.8) On the other hand, because $\lambda \geq 1$, we can use the interpolation result in Proposition 1 to see that $\mathcal{M}_{X, \frac{1-\gamma}{\lambda}D}$ is proper for any $\gamma \in (0, R(X))$ (actually for any $\gamma \in (1 - \lambda, R(X))$. So there exists a conical Käler–Einstein metric on $(X, \frac{1-\gamma}{\lambda}D)$ for any $\gamma \in (0, R(X))$. There can not be conical Kähler–Einstein metric for $\gamma \in (R(X), 1)$ is easy to get because the twisted energy is bounded from below by the log-Ding-energy. For details, see [58] and also [41]. The non-existence for $\gamma = R(X)$ is implied by Donaldson's openness theorem in [25] (see Theorem 2.4), since otherwise there exists conical Kähler–Einstein for some $\gamma \in (R(X), 1)$. □

Remark 8.6. The smoothness of the generic member seems to be more subtle than we first thought. We will discuss this a little bit using standard toric geometry. For this, we first denote $\{H_i\}_{i=1}^N$ to be the set of codimensional 1 face (i.e. facet) of $\triangle$. Define

$$\mathcal{B}(\mathcal{F}_Q) = \left(\bigcup_{\mathcal{F}_Q \not\subset H_i} H_i\right) \bigcap \mathcal{F}_Q.$$

Now it's easy to see that the base locus of $\mathscr{L}_\lambda$ is equal to

$$\mathbb{B}_Q = \bigcup_{\sigma \subset \mathcal{B}(\mathcal{F}_Q)} X_\sigma.$$

Here for any face of $\triangle$ we denote X_σ to be the toric subvariety determined by σ. Indeed, this follows from the following fact: if P is any lattice point and $\mathcal{F}_P$ is the minimal face containing P. Define

$$\mathbf{Star}(\mathcal{F}_P) = \bigcup_{\mathcal{F}_P \subset \sigma} \sigma.$$

where σ ranges over all the (closed) faces of $\triangle$. (including $\triangle$ itself). Then the zero set of the corresponding holomorphic section s_P is the toric divisor corresponding to the set

$$\overline{\triangle} \setminus (\mathbf{Star}(\mathcal{F}_P))^\circ = \bigcup_{\mathcal{F}_P \not\subset H_i} H_i \subset \partial\triangle.$$

By Bertini's Theorem [31], the generic element $D \in \mathscr{L}_\lambda$ is smooth away from $\mathbb{B}_Q$. To analyze the situation near $\mathbb{B}_Q$, fix any vertex P of $\mathcal{F}$. We can choose integral affine coordinates $\{x_i\}_{i=1}^n$ such that

$$\mathcal{F}_Q = \bigcap_{i=m+1}^{n} \{x_i = 0\}.$$

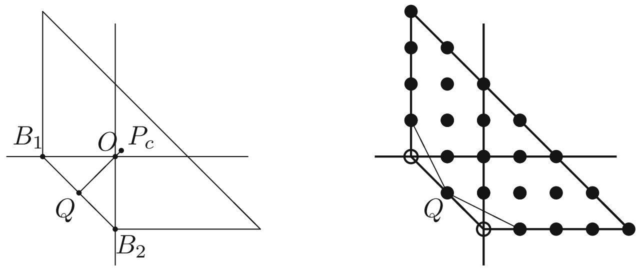
We can also write $Q = \lambda(d_1, \ldots, d_m, 0, \ldots, 0)$ with λd_i being positive integers. On the other hand, by standard toric geometry, the normal fan of $\triangle$ at P determines an affine chart $\mathcal{U}_P$ on X. There exists complex coordinate $\{z_i\}_{i=1}^n$ such that $X_{\mathcal{F}_Q} \cap \mathcal{U}_P = \{z_{m+1} = 0, \ldots, z_n = 0\}$. Locally, the generic member D in $\mathcal{L}_\lambda$ is given by the equation of the form:

$$\prod_{i=1}^{m} a_i z_i^{\lambda d_i} + \sum_{j=m+1}^{n} b_j z_j (1 + f_j(z_1, \ldots, z_m)) + \sum_{j,k=m+1}^{n} c_{jk} z_j z_k g(z_1, \ldots, z_n)).$$

where $a_i, b_j \neq 0$. If we delete the lattice points corresponding to terms $z_j f_j(z_1, \ldots, z_m)$, then C would be smooth near $X_{\mathcal{F}_Q} \cap \mathcal{U}_P$. Since $\mathbb{B}_Q \subset X_{\mathcal{F}_Q}$ and $\mathcal{U}_P$ covers $X_{\mathcal{F}_Q}$ as P ranges over all the vertices of $\mathcal{F}_Q$ we conclude that D is smooth at points in $\mathbb{B}_Q$ as well. This certainly puts a lot of restriction on the sub-linear system. However, even if we don't delete these lattice points, the generic member in $\mathcal{L}_\lambda$ could be smooth. For example, this is the case when $\mathcal{F}_Q$ has dimension ≤ 1 in which case the base locus consists of isolated points. In particular, this is true when the toric variety has dimension ≤ 2.

Remark 8.7. The degeneration behavior in the toric case is related to the study of degenerations in [39] where the current D_{P_γ} is replaced by $(1-\gamma)\omega$ with ω being a smooth reference metric.

Example 8.8. Let $X = Bl_p\mathbb{P}^2$. Let $[Z_0, Z_1, Z_2]$ be homogeneous coordinate on $\mathbb{P}^2$. We can assume $p = (1, 0, 0) \in \mathbb{C}^2 = \{Z_0 \neq 0\} \subset \mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^2$ be the blow down of exceptional divisor E. For simplicity we use H to denote both the hyperplane class on $\mathbb{P}^2$ and its pull-back on X. Then $-K_X = 3H - E$ and $-2K_X = 6H - 2E$. So divisors in $|-2K_X|$ correspond to the sextic curves on $\mathbb{P}^2$ whose vanishing order at 0 is at least 2. More precisely, if C is such a curve representing $6H$, then the corresponding divisor $D(C)$ in $|-2K_X| = |6H - 2E|$ is the strict transform of C. In toric language, X is determined by the following polytope:



The invariant $R(X) = 6/7$ was calculated in [61] and [38]. Since the point $Q = (-1/2, -1/2)$, it's easy to see that $D_Q = 1/2(F_1 + F_2) + 2D_\infty$. By Song-Wang [58], there is a conical Kähler–Einstein metric on $(X, (1 - R(X))D_Q) = (X, 1/7 D_Q)$.

Now λQ is integral when λ is even. The generic divisors in the linear system $\mathcal{L}_2$ correspond to the sextic curves given by degree 6 homogeneous polynomial of the form

$$C : Z_0^4 Z_1 Z_2 + \sum_{i=0}^{3} \sum_{j=0}^{6-i} a_{ij} Z_0^i Z_1^j Z_2^{6-i-j} = 0.$$

Let σ_t be the $\mathbb{C}^*$-action given by

$$(Z_0, Z_1, Z_2) \to (Z_0, t^{-1} Z_1, t^{-1} Z_2).$$

Then $\lim_{t\to 0}\sigma_t \cdot C = \{Z_0^4 Z_1 Z_2 = 0\}$. Equivalently, by taking strict transform, we get $\lim_{t\to 0}\sigma_t \cdot D(C) = 2D_Q$. The same argument applies to $\lambda = 2m$ being even, where the divisors in $\mathscr{L}_{2m}$ correspond to the degree $6m$ curves of the form:

$$Z_0^{4m} Z_1^m Z_2^m + \sum_{i=0}^{4m-1}\sum_{j=0}^{6m-i} a_{ij} Z_0^i Z_1^j Z_2^{6m-i-j} = 0.$$

Note that the strict transform of such generic curves are smooth at the base locus $\mathbb{B}_Q = B_1 \cup B_2$ and so smooth everywhere.

Remark 8.9. From the above discussion, we see that when λ is even, the divisor degenerates while the ambient space stays the same. The case when $\lambda = 1$, or more generally when λ is odd, is still open. From the point of view of our strategy, the problem is that the right degeneration to conical Kähler–Einstein pair is still missing. In this case, we expect the degeneration also happens to the ambient space, similar with the degree 2 plane curve case studied in Sect. 5.

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References

1. Arezzo, C., Ghigi, A., Pirola, G.P.: Symmetries, quotients and Kähler–Einstein metrics. J. Reine Angew. Math. **591**, 177–200 (2006)
2. Aubin, T.: Réduction du cas positif de l'équation de Monge–Ampère sur les variétés Kählériennes compactes à la démonstration d'une inégalité. J. Func. Anal. **57**, 143–153 (1984)
3. Aubin, T.: Equations du type Monge–Ampère sur les variétés compactes. C. R. Acad. Sci. Paris **283**, 119–121 (1976)
4. Bando, S., Mabuchi, T.: Uniqueness of Einstein Kähler metrics modulo connected group actions, Algebraic geometry, Sendai 1985. Adv. Stud. Pure Math. **10**, 11–40 (1987)
5. Bedford, E., Taylor, B.A.: The Dirichlet Problem for a complex Monge–Ampère equation. Invent. Math. **37**, 1–44 (1976)
6. Berman, R.J: A thermodynamic formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics. Adv. Math. **248**, 1254–1297 (2013)
7. Berman, R.J.: K-polystability of Q-Fano varieties admitting Kähler–Einstein metrics. arXiv:1205.6214
8. Berman, R.J., Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A.: Kähler–Ricci flow and Ricci iteration on log-Fano varieties. arXiv:1111.7158
9. Berndtsson, B.: Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains. Ann. Inst. Fourier Grenoble **56**(6), 1633–1662 (2006)
10. Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. Ann. Math. **169**, 531–560 (2009)
11. Berndtsson, B.: A Brunn–Minkowski type inequality for Fano manifolds and the Bando–Mabuchi uniqueness theorem. arXiv:1103.0923
12. Berndtsson, B., Paun, M.: Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J. **145**(2), 341–378 (2008)
13. Błocki, Z.: On geodesics in the space of Kähler metric, to appear in the Proceedings of the "Conference in Geometry" dedicated to Shing-Tung Yau (Warsaw, April 2009)
14. Błocki, Z., Kołodziej, S.: Regularization of plurisubharmonic functions on manifolds. Proc. Am. Math. Soc. **135**(7), 2089–2093 (2007)

15. Caffarelli, L., Kohn, J.J., Nirenberg, L., Spruck, J.: The Dirichlet problem for non-linear second order elliptic equations II: Complex Monge–Ampère, and uniformly elliptic equations. Commun. Pure Appl. Math. **38**, 209–252 (1985)
16. Cheltsov, I.A., Rubinstein, Y.A.: Asymptotically log Fano varieties. arXiv:1308.2503
17. Chen, X.: Space of Kähler metric. J. Differ. Geom. **56**, 189–234 (2000)
18. Chen, X.: Space of Kähler metrics (IV)—On the lower bound of the K-energy. arXiv:0809.4081
19. Chen, X., Donaldson, S.K., Sun, S.: Kahler–Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2π and completion of the main proof (to appear in J. Amer. Math. Soc.). arXiv:1302.0282
20. Chen, Q., Wang, W., Wu, Y., Xu, B.: Conformal metrics with constant curvature one and finite conical singularities on compact Riemann surfaces. arXiv:1302.6457
21. Conti, D.: Cohomogeneity one Einstein–Sasaki 5-manifolds. Commun. Math. Phys. **274**(3), 751-774 (2007)
22. Demailly, J.P.: Regularization of closed positive currents and intersection theory. J. Alg. Geom. **1**(3), 361–409 (1992)
23. Ding, W.Y.: Remarks on the existence problem of positive Kähler–Einstein metrics. Math. Ann. **282**, 463–471 (1988)
24. Donaldson, S.K.: Scalar curvature and stability of toric varieties. J. Differ. Geom. **62**, 289–349 (2002)
25. Donaldson, S.K.: Kähler metrics with cone singularities along a divisor. Essays in Mathematics and its applications, 49–79 (2012). arXiv:1102.1196
26. Donaldson, S.K., Sun, S.: Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry. arXiv:1206.2609
27. Eyssidieux, P., Guedj, V., Zeriahi, A.: Singular Kähler–Einstein metrics. J. Ame. Math. Soc. **22**(3), 607–639 (2009)
28. Futaki, A.: An obstruction to the existence of Einstein Kähler metrics. Invent. Math. **73**(3), 437–443 (1983)
29. Gauntlett, J., Martelli, D., Sparks, J., Yau, S-T.: Obstructions to the existence of Sasaki–Einstein metrics. Commun. Math. Phys. **273**(3), 803–827 (2007)
30. Ghigi, A., János Kollár, J.: Kähler–Einstein metrics on orbifolds and Einstein metrics on spheres. Comment. Math. Helv. **82**(4), 877–902 (2007)
31. Griffiths, B., Harris, J.: Principles of algebraic geometry. Wiley, New York (1978)
32. Guan, B.: The Dirichlet problem for complex Monge–Ampère equations and regularity of the pluricomplex Green function. Commun. Anal. Geom. **6**, 687–703 (1998)
33. Hacking, P., Prokhorov, Y.: Smoothable del Pezzo surfaces with quotient singularities. Compos. Math. **146**, 169–192 (2010)
34. Jeffres, T.D., Mazzeo, R., Rubinstein, Y.A.: Kähler–Einstein metrics with edge singularities, with an appendix by Li, C. and Rubinstein, Y.A. arXiv:1105.5216v3
35. Kołodziej, S.: The complex Monge–Ampère equation. Acta Math. **180**(1), 69–117 (1998)
36. Legendre, E.: Toric Kähler–Einstein metrics and convex compact polytopes. arXiv:1112.3239
37. Li, C.: Notes on Sean Paul's paper. http://www.math.sunysb.edu/chili
38. Li, C.: Greatest lower bounds on the Ricci curvature of toric Fano manifolds. Adv. Math. **226**, 4921–2932 (2011)
39. Li, C.: On the limit behavior of metrics in continuity method to Kahler-Einstein problem in toric Fano case. Compos. Math. **148**(06), 1985–2003. arXiv:1012.5229
40. Li, C.: Remarks on logarithmic K-stability. Commun. Contemp. Math. (to appear). arXiv:1104.0428
41. Li, C.: PhD thesis (2012). http://www.math.sunysb.edu/chili
42. Li, C.: Numerical solutions of Kähler–Einstein metrics on $\mathbb{P}^2$ with conical singularities along conic curve. J. Geom. Anal. (to appear). arXiv:1207.6592
43. Li, C.: Yau–Tian–Donaldson correspondence for K-semistable Fano manifolds. arXiv:1302.6681
44. Li, C., Xu, C.: Special test configurations and K-stability of Fano varieties. Ann. Math. **180**, 197–232 (2014). arXiv:1111.5398
45. Luo, F., Tian, G.: Liouville equation and spherical convex polytopes. Proc. Am. Math. Soc. **116**, 1119–1129 (1992)
46. Matsushima, Y.: Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kähléreine. Nagoya Math. J. **11**, 145–150 (1957)
47. Mabuchi, T.: K-energy maps integrating Futaki invariants. Tohoku Math. J. (2) **38**(4), 575–593 (1986)
48. McOwen, R.C.: Point singularties and conformal metrics on Riemann surfaces. Proc. Am. Math. Soc. **103**, 222–224 (1988)
49. Odaka, Y., Sun, S.: Testing log-K-stability by blowing up formalism. arXiv:1112.1353
50. Paul, S.T.: Geometric analysis of Chow Mumford stability. Adv. Math. **182**, 333–356 (2004)
51. Paul, S.T.: Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. Ann. Math. **175**, 255–296 (2012)
52. Paul, S.T., Tian, G.: CM stability and the generalized Futaki invariant II. Astérisque No. **328**, 339–354.

53. Phong, D.H., Sturm, J.: The Dirichlet problem for degenerate complex Monge–Ampere equations. Commun. Anal. Geom. **18**(1), 147–170 (2010). arXiv:0904.1898
54. Phong, D.H., Song, J., Sturm, J., Weinkove, B.: The Moser–Trudinger inequality on Kähler–Einstein manifolds. Ame. J. Math. **130**(4), 1067–1085 (2008)
55. Ross, J., Thomas, R.: A study of the Hilbert–Mumford criterion for the stability of projective varieties. J. Algebraic Geom. **16**, 201–255 (2007)
56. Ross, J., Thomas, R.: Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics. J. Differ. Geom. **88**, 109–160 (2011)
57. Shi, Y., Zhu, X.: Kähler–Ricci solitons on toric Fano orbifolds. Math. Z. **271**(3–4), 1241–1251 (2012).
58. Song, J., Wang, X.: The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality. (in preprint)
59. Sun, S.: Note on K-stability of pairs. Math. Annalen **355**(1), 259–272. arXiv:1108.4603
60. Sun, S., Wang, Y.: On the Kähler–Ricci flow near a Kähler–Einstein metric. J. für die reine und angewandte Mathematik (to appear). arxiv:1004.2018
61. Székelyhidi, G.: Greatest lower bounds on the Ricci curvature of Fano manifolds. Compos. Math. **147**, 319–331 (2011)
62. Székelyhidi, G., Tosatti, V.: Regularity of weak solutions of a complex Monge–Ampère equation. Anal. PDE **4**(3), 369–378 (2011)
63. Tian, G.: On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. Invent. Math. **89**, 225–246 (1987)
64. Tian, G.: On stability of the tangent bundles of Fano varieties. Int. J. Math. **3**(3), 401–413 (1992)
65. Tian, G.: Kähler–Einstein metrics on algebraic manifolds. In: Transcendental Methods in Algebraic Geometry (Cetraro 1994), Lecture Notes in Math. pp. 143–185 (1646)
66. Tian, G.: Bott–Chern forms and geometric stability. Discret Contin. Dyn. Syst. **6**(1), 211–220 (2000)
67. Tian, G.: The K-energy on hypersurfaces and stability. Commun. Anal. Geom. **2**(2), 239–265 (1994)
68. Tian, G.: Kähler–Einstein metrics with positive scalar curvature. Invent. math. **137**, 1–37 (1997)
69. Tian, G.: Canonical Metrics on Kähler Manifolds. In: Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel (2000)
70. Tian, G., Zhu, X.: Convergence of Khler–Ricci flow. J. Am. Math. Soc. **20**(3), 675–699 (2007)
71. Tian, G.: Existence of Einstein metrics on Fano manifolds. Prog. Math. **297**, Part 1, 119–159 (2012)
72. Troyanov, M.: Prescribing curvature on compact surfaces with conic singularities. Trans. Am. Math. Soc. **324**, 793–821 (1991)
73. Wang X., Zhu X.: Kähler–Ricci solitons on toric manifolds with positive first Chern class. Adv. Math. **188**, 87–103 (2004)
74. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. Commun. Pure Appl. Math. **31**, 339–441 (1978)
75. Zhang, S.: Heights and reductions of Semi-stable varieties. Compos. Math. **104**, 77–105 (1996)

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