Calabi flow, Geodesic rays, and uniqueness of constant scalar curvature Kähler metrics

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Abstract

We prove that constant scalar curvature Kähler metric "adjacent" to a fixed Kähler class is unique up to isomorphism. This extends the uniqueness theorem of Donaldson and Chen-Tian, and formally fits into the infinite dimensional G.I.T picture described by Donaldson. We prove that the Calabi flow near a cscK metric exists globally and converges uniformly to a cscK metric in a polynomial rate. Viewed in fixed a Kähler class, the Calabi flow is also shown to be asymptotic to a smooth geodesic ray at infinity. This latter fact is also interesting in the finite dimensional case, where we show that the downward gradient flow of the Kempf-Ness function in a semi-stable orbit is asymptotic to the direction of optimal degeneration.

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1 Introduction

The Kempf-Ness theorem relates complex quotient to symplectic reduction. Suppose a compact connected group G acts on a compact Kähler manifold X. We assume the action preserves the Kähler structure, with a moment map $\mu : X \to \mathfrak{g}^*$. Then the action extends to a holomorphic action of the complexified group $G^{\mathbb{C}}$. Under proper hypothesis the notion of stability could be defined. Then the Kempf-Ness theorem says that as sets:

$$X^{ss}/G^{\mathbb{C}} \simeq \mu^{-1}(0)//G.$$

To be more precise,

(1). A $G^{\mathbb{C}}$ -orbit is poly-stable if and only if it contains a zero of the moment map. The zeroes within it form a unique G orbit.

(2). A $G^{\mathbb{C}}$ -orbit is semi-stable if and only if its closure contains a zero of the moment map. We call such a zero a *de-stabilizer* of the original $G^{\mathbb{C}}$ orbit. The de-stabilizers all lie in the unique poly-stable orbit in the closure of the original orbit.

In Kähler geometry according to S. Donaldson([D1])(see also [Fu]) the problem of finding cscK (constant scalar curvature Kähler) metrics formally fits into a similar picture. However the spaces involved are infinite dimensional. Given a compact Kähler manifold (M, ω, J) , denote by \mathcal{G} the group of Hamiltonian diffeomorphisms of (M, ω) and by \mathcal{J} the space consists of almost complex structures on M which are compatible with ω . \mathcal{J} admits a natural Kähler structure which is invariant under the action of \mathcal{G} . The moment map is given by the Hermitian scalar curvature. The complexification of \mathcal{G} may not exist, since \mathcal{G} is infinite dimensional. Nevertheless, it still makes sense talking about the orbits of $\mathcal{G}^{\mathbb{C}}$ -it is simply the leaf of the foliation obtained by complexifying the infinitesimal actions of \mathcal{G} . Then the $\mathcal{G}^{\mathbb{C}}$ leaf of an integrable complex structure can be viewed as a principal \mathcal{G} -bundle over the Kähler class $[\omega]$. Thus an analogue to the Kempf-Ness theorem should relate the stability of the leaves to the existence of cscK metrics in the corresponding Kähler class. This was made more precise as the Yau-Tian-Donaldson conjecture (see [Th]). The notion of "stability" in this case is the so-called "K-stability", see [Ti1], [D5]. There are also other related notion of stability, see for example [RT], [Pa], etc.

Note that the Kempf-Ness theorem consists of both the existence and uniqueness part. It is known that the existence of cscK metrics implies various kinds of stability, however the converse is fairly difficult, due to the appearance of fourth order non-linear P.D.E's. Recently Donaldson([D6]) proved a general result that the conjecture is true for toric surfaces. The uniqueness part corresponding to the poly-stable case is known by

Theorem 1.1. (Donaldson[D3], Chen-Tian[CT]) Constant scalar Kähler metric in a fixed Kähler class, if exists, is unique up to holomorphic isometry.

Remark 1.2. When the manifold is Fano, the uniqueness of Kähler-Einstein metrics was previously proved by Bando-Mabuchi([BM]), and it was later generalized to the case of Kähler-Ricci solitons by Tian-Zhu([TZ1]). The uniqueness of cscK metrics was first proved by the first author in the case when $c_1(X) \leq 0$ ([Ch1]).

The purpose of this paper is to prove the uniqueness in the semi-stable case.

Theorem 1.3. If there are two cscK structures J_1 and J_2 both lying in the (C^{∞}) closure of the $\mathcal{G}^{\mathbb{C}}$ leaf of a complex structure $J \in \mathcal{J}^{int}$, then there is a symplectic diffeomorphism f such that $f^*J_1 = J_2$.

Definition 1.4. Let (M, ω, J) be a Kähler manifold and \mathcal{H} be the space of Kähler metrics in the Kähler class of ω . We say another Kähler structure (ω', J') on M is adjacent to \mathcal{H} if there is a sequence of Kähler metrics $\omega_i \in \mathcal{H}$ and diffeomorphisms f_i of M such that

$$f_i^* \omega_i \to \omega', f_i^* J \to J'$$

in C^{∞} sense. So in particular, the corresponding sequence of Riemannian metrics g_i converges to g' in the Cheeger-Gromov sense. Similarly, let (M, J)be a Fano manifold. We say another complex structure J' on M is adjacent to J if there is a sequence of diffeomorphisms f_i such that

$$f_i^* J \to J'.$$

Remark 1.5. The above definition is related to the "jumping" phenomenon of complex structures, i.e. the space of isomorphism classes of complex

structures on a fixed manifold is in general not Hausdorff. As a simple example, we can consider the blown-up of \mathbb{P}^2 at three points p_1 , p_2 , and p_3 . The underlying differential manifold is fixed, and a choice of the three points defines a complex structure. A choice of three points in a general position gives rise to the same complex structure, while a choice of three points on a line provides an example of an adjacent complex structure.

It follows theorem 1.3 that

Theorem 1.6. Let (M, ω, J) be a Kähler manifold. Assume $[\omega]$ is integral. Suppose there are two csc Kähler structures (ω_1, J_1) and (ω_2, J_2) both adjacent to the Kähler class of (ω, J) , then they are isomorphic.

Corollary 1.7. Let (M, J) be a Fano manifold. Suppose there are two complex structures J_1 and J_2 both adjacent to J and both admitting Kähler-Einstein metrics, then (M, J_1) and (M, J_2) are bi-holomorphic.

Remark 1.8. After finishing this paper, we learned that our theorem 1.6 and corollary 1.7 partially confirmed a conjecture of G. Tian([Ti2]) in the case of constant scalar curvature Kähler metric.

The main technical ingredient in the proof of the above theorems is to obtain some C^0 bound. We shall study the asymptotic behavior of the Calabi flow near a cscK metric. The global existence and convergence are established by using the Lojasiewicz inequality which controls the gradient of a real analytic function near a critical point. Suppose now we have two cscK metrics adjacent to a fixed Kähler class, then there are two Calabi flows in the neighborhoods of the corresponding cscK metrics. Since the Calabi flow decreases geodesic distance, we get a bound on the two Calabi flows in terms of geodesic distance. It is not known whether this bound implies C^0 bound automatically. Here we get around this difficulty by showing that the previous Calabi flow is *asymptotic* to a smooth geodesic ray. This involves a local study of the infinite dimensional Hamiltonian action of \mathcal{G} , which is the main technical part of this paper. We shall first look at the analogous finite dimensional problem. Finally we are able to derive C^0 bound for the two parallel geodesic rays.

The organization of this paper is as follows. In section 2, we review Donaldson's infinite dimensional moment map picture in Kähler geometry, and recall some known results for our later use. In section 3, we state the Lojasiewicz inequality and "Lojasiewicz arguments" for the gradient flow of a real analytic function. In section 4, we prove that in the finite dimensional case, the Kempf-Ness flow for a semi-stable point is asymptotic to a rational geodesic ray. In section 5, we study the stability of the Calabi flow near a cscK metric when the complex structure is deformed. In section 6, we generalize the arguments in section 4 to the infinite dimensional setting by considering the "reduced" Calabi flow. In section 7, the relative C^0 bound for two smooth parallel geodesic rays tamed by bounded geometry is derived. In section 8, we prove the main theorems. In Section 9, we shall discuss some further problems related to this study. The appendix contains the proof of the technical lemmas used in sections 4 and 6.

Acknowledgements: This paper was essentially finished in the October of 2009 during a conference in honor of Simon Donaldson at Northwestern University. With admiration, we want to dedicate this modest paper to him for his teaching of Kähler geometry to the first author in the last 12 years. Part of this work was done while both authors were visiting Stony Brook. We wish to thank both the department of Mathematics and the Simons Center for Geometry and Physics for their generous hospitality. We also thank Professors Blaine Lawson, Claude Lebrun, and Gang Tian for their interest in this work. The second author would also like to thank Joel Fine, Sean Paul and Zhan Wang for interesting discussions. Both authors are partially supported by an NSF grant.

2 The space of Kähler structures

Here we review the infinite dimensional moment map picture discovered by Fujiki([Fu]) and Donaldson([D1]). Let (M, ω, J_0) be a compact Kähler manifold. Denote by \mathcal{J} the space of almost complex structures on M which are compatible with ω , and by \mathcal{J}^{int} the subspace of \mathcal{J} consisting of integrable almost complex structures compatible with ω . Then \mathcal{J} is the space of smooth sections of an Sp(2n)/U(n) bundle over M, so it carries a natural Kähler structure. Indeed, there is a global holomorphic coordinate chart if we use the ball model of the Siegel upper half space in the usual way. J_0 determines a splitting $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ such that ω induces a positive definite Hermitian inner product on $T^{1,0}$, then \mathcal{J} could be identified with the space

$$\Omega_S^{0,1}(T^{1,0}) = \{ \mu \in \Omega^{0,1}(T^{1,0}) | \mathcal{A}(\mu) = 0, Id - \bar{\mu} \circ \mu > 0 \},\$$

where \mathcal{A} is the composition $\Omega^{0,p}(T^{1,0}) \to \Omega^{0,p}(T^{*0,1}) \to \Omega^{0,p+1}$. An element μ corresponds to an almost complex structure J whose corresponding (1,0) tangent space consists of vectors of the form $X - \bar{\mu}(X)(X \in T^{1,0})$. \mathcal{J}^{int} is a subvariety of \mathcal{J} cut out by quadratic equations:

$$N(\mu) = \bar{\partial}\mu + [\mu, \mu] = 0.$$

Denote by \mathcal{G} the group of Hamiltonian diffeomorphisms of (M, ω) . Its Lie algebra is $C_0^{\infty}(M; \mathbb{R})$. \mathcal{G} will be the infinite dimensional analogue of a compact

group, though the exponential map is not locally surjective for \mathcal{G} . \mathcal{G} acts naturally on \mathcal{J} , keeping \mathcal{J}^{int} invariant. A. Fujiki[Fu] and S. Donaldson([D1]) independently discovered that the \mathcal{G} action has a moment map given by the Hermitian scalar curvature functional $S - \underline{S}^1$, which can be viewed as an element in $(C_0^{\infty}(M; \mathbb{R}))^*$ through the L^2 inner product with respect to the measure $d\mu = \omega^n$. When J is integrable S(J) is simply the Riemannian scalar curvature of the Riemannian metric induced by ω and J. We say $J_0 \in \mathcal{J}$ is cscK if J_0 is integrable and (ω, J_0) has constant scalar curvature. So in the symplectic theory we are naturally lead to consider cscK metrics.

In the complex story, we need to look at $\mathcal{G}^{\mathbb{C}}$. Since \mathcal{G} is infinite dimensional, there may not exist a genuine complexification $\mathcal{G}^{\mathbb{C}}$. Nevertheless, we can still define the $\mathcal{G}^{\mathbb{C}}$ leaf of an integral complex structure J_0 , as follows. The infinitesimal action of \mathcal{G} at a point $J \in \mathcal{J}$ is given by

$$\mathcal{D}_J: C_0^\infty(M; \mathbb{R}) \to \Omega_S^{0,1}(T^{1,0}); \phi \to \bar{\partial}_J X_\phi.$$

This operator can be naturally complexified to an operator from $C_0^{\infty}(M; \mathbb{C}) = C_0^{\infty}(M; \mathbb{R}) \oplus \sqrt{-1}C_0^{\infty}(M; \mathbb{R})$ to $\Omega_S^{0,1}(T^{1,0})$. Then a complex structure J is on the $\mathcal{G}^{\mathbb{C}}$ leaf of J_0 if there is a smooth path $J_t \in \mathcal{J}^{int}$ such that \dot{J}_t lies in the image of \mathcal{D}_{J_t} . \mathcal{G} acts on the leaf naturally and the quotient is the space of Kähler metrics cohomologous to $[\omega]_{J_0}$. So the latter could be viewed as " $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ ". We define the space of Kähler potentials

$$\mathcal{H} = \{ \phi \in C^{\infty}(M; \mathbb{R}) | \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}.$$

Then \mathcal{H}/\mathbb{R} is formally the "dual" symmetric space of \mathcal{G} . This was made more precise by Mabuchi([M1]), Semmes([Se]) and Donaldson([D2]). Define a Weil-Petersson type Riemannian metric on \mathcal{H} by

$$(\psi_1,\psi_2)_\phi = \int_M \psi_1 \psi_2 d\mu_\phi$$

for $\psi_1, \psi_2 \in T_{\phi}\mathcal{H}$. It can be shown that the Riemannian curvature tensor is co-variantly constant and the sectional curvature is non-positive. A path $\phi(t)$ in \mathcal{H} is a *geodesic* if it satisfies the equation

$$\ddot{\phi}(t) - |\nabla_{\phi(t)}\dot{\phi}(t)|^2_{\phi(t)} = 0.$$

The first author([Ch1]) proved the existence of a unique $C^{1,1}$ geodesic connecting any two points in \mathcal{H} , and consequently that \mathcal{H} is a metric space with the distance given by the length of the $C^{1,1}$ geodesics. It is proved in [CC] that under this metric \mathcal{H} is non-positively curved in the sense of Alexanderov.

¹Here <u>S</u> is the average of scalar curvature, which indeed depends only on $[\omega]$ and $c_1(\omega)$, not on the choice of any compatible J.

So far the best regularity for the Dirichlet problems of the geodesic equation was obtained by Chen-Tian([CT]). The initial value problem for the geodesic equation is in general not well-posed. But by the non-positiveness of the curvature of \mathcal{H} , there should be lots of geodesic rays in \mathcal{H} . In [Ch3], the first author proved the following general theorem which we shall use later:

Theorem 2.1. Given a smooth geodesic ray $\phi(t)$ in \mathcal{H} which is tamed by a bounded geometry, there is a unique relative $C^{1,1}$ geodesic ray $\psi(t)$ emanating from any point ψ in \mathcal{H} such that

$$|\phi(t) - \psi(t)|_{C^{1,1}} \le C.$$

Remark 2.2. For the precise definition of "tameness" we refer to [Ch3]. But we point out that this is merely a technical condition imposed on the behavior of $\phi(t)$ at infinity so that the analysis on non-compact manifolds work. In our later applications where the geodesic ray $\phi(t)$ arises naturally from a test configuration with smooth total space, this assumption is always satisfied.

Definiton 2.3. Two geodesic rays $\phi(t)$ and $\psi(t)$ in \mathcal{H} are said to be parallel if

$$d_{\mathcal{H}}(\phi(t),\psi(t)) \le C.$$

Hence it is clear by definition that if $|\phi(t) - \psi(t)|_{C^0} \leq C$, then ϕ and ψ are parallel.

Analogous to the finite dimensional Kempf-Ness setting, there is a relevant functional E defined on \mathcal{H} , called the Mabuchi *K*-energy. It is the anti-derivative of the following closed one-form:

$$dE_{\phi}(\psi) = -\int_{M} (S(\phi) - \underline{S})\psi d\mu_{\phi}.$$
 (1)

So the norm square of the gradient of E is the Calabi energy:

$$Ca(\phi) = \int_M (S(\phi) - \underline{S})^2 d\mu_{\phi}.$$

By a direct calculation, along a smooth geodesic $\phi(t)$, we have

$$\frac{d^2}{dt^2}E(\phi(t)) = \int_M |\mathcal{D}_t \dot{\phi}(t)|^2 d\mu_{\phi(t)} \ge 0.$$

According to [Ch2], E can be extended to a continuous function on all $C^{1,1}$ potentials in \mathcal{H} . However, it is not clear why E is still convex. The first author proved some weak versions of convexity. In the case when $[\omega]$ is integral, we gave simplified proofs in [CS] using quantization(See also [Be]). We recall them for our later purpose.

Lemma 2.4. ([Ch3], [CS]). Given any $\phi_0, \phi_1 \in \mathcal{H}$, we have

$$E(\phi_1) - E(\phi_0) \le \sqrt{Ca(\phi_1)} \cdot d(\phi_0, \phi_1).$$

Lemma 2.5. ([Ch3], [CS]) Given any ϕ_0 , $\phi_1 \in \mathcal{H}$, let $\phi(t)$ be the $C^{1,1}$ geodesic connecting them. Then the derivatives of $E(\phi(t))$ at the end-points are well-defined and they satisfy the following inequality:

$$\frac{d}{dt}|_{t=0}E(\phi(t)) \leq \frac{d}{dt}|_{t=1}E(\phi(t))$$

This lemma implies that

Lemma 2.6. ([CC]) The Calabi flow on \mathcal{H} decreases geodesic distance.

3 Lojasiewicz inequality

In this section we recall Lojasiewicz's theory for the structure of a real analytic function. The following fundamental structure theorem for real analytic functions is well-known:

Theorem 3.1. (Lojasiewicz inequality) Suppose f is a real analytic function defined in a neighborhood U of the origin in \mathbb{R}^n . If f(0) = 0 and $\nabla f(0) = 0$, then there exist constants C > 0, and $\alpha \in [\frac{1}{2}, 1)$, and shrinking U if necessary, depending on n and f, such that for any $x \in V$, it holds that

$$|\nabla f(x)| \ge C \cdot |f(x)|^{\alpha}.$$
(2)

This type of inequality is crucial in controlling the behavior of the gradient flow. If $\alpha = \frac{1}{2}$, then we get exponential convergence. If $\alpha > \frac{1}{2}$, then we can obtain polynomial convergence:

Corollary 3.2. Suppose f is a non-negative real-analytic function defined in a neighborhood U of the origin in \mathbb{R}^n with f(0) = 0. Then there exists a neighborhood $V \subset U$ of the origin such that for any $x_0 \in V$, the downward gradient flow of f:

$$\begin{cases} \frac{d}{dt}x(t) = -\nabla f(x(t)) \\ x(0) = x_0. \end{cases}$$

converges uniformly to a limit $x_{\infty} \in U$ with $f(x_{\infty}) = 0$. Moreover, we have the following estimate:

1.

$$f(x(t)) \le C \cdot t^{-\frac{1}{2\alpha - 1}};$$

2.

$$d(x(t), x(\infty)) \le C \cdot t^{-\frac{1-\alpha}{2\alpha-1}},$$

where we assume the Lojasiewicz exponent $\alpha > \frac{1}{2}$.

Proof. The proof is quite standard, and we call it "Lojasiewicz arguments" for later reference. Denote

$$V_{\delta} = \{ x \in \mathbb{R}^n | |x| \le \delta \},\$$

and fix $\delta > 0$ small so that inequality (2) holds for $x \in V_{\delta}$. In our calculation the constant C may vary from line to line. If $x(t) \in V_{\delta}$ for $t \in [0,T]$, then we compute

$$\frac{d}{dt}f^{1-\alpha}(x(t)) = -(1-\alpha) \cdot f^{-\alpha}(x(t)) \cdot |\nabla f(x(t))|^2 \le -C \cdot |\dot{x}(t)|,$$

thus for any T > 0,

$$\int_0^T |\dot{x}(t)| dt \le \frac{1}{C} \cdot f^{1-\alpha}(x_0).$$

For any $\epsilon \leq \frac{\delta}{2}$ small, we choose $\delta_2 \leq \delta$ small such that $f(x) \leq (C \cdot \epsilon)^{\frac{1}{1-\alpha}}$ for $x \in V_{\delta_2}$, and $\delta_1 = \min\{\epsilon, \delta_2\}$, then the flow initiating from any point $x_0 \in V_{\delta_1}$ will stay in $V_{2\epsilon}$. So the Lojasiewicz inequality holds for all x(t). Now

$$\frac{d}{dt}f^{1-2\alpha}(x(t)) = -(1-2\alpha) \cdot f^{-2\alpha}(x(t)) \cdot |\nabla f(x(t))|^2 \ge (2\alpha-1) \cdot C^2$$

 \mathbf{SO}

$$f(x(t)) \le C \cdot t^{-\frac{1}{2\alpha - 1}}.$$

For any $T_1 \leq T_2$, we get

$$d(x(T_1), x(T_2)) \le \int_{T_1}^{T_2} |\dot{x}(t)| dt \le C \cdot T_1^{-\frac{1-\alpha}{2\alpha-1}}$$

Therefore we obtain polynomial convergence and the required estimates. \Box

4 Finite dimensional case

4.1 Kempf-Ness theorem

Let (M, ω, J) be a Kähler manifold and assume there is an action of a compact connected group G on M which preserves the Kähler structure. Let μ be the corresponding moment map. This induces a holomorphic action of the complexified group $G^{\mathbb{C}}$. Then the Kempf-Ness theorem relates the complex quotient by $G^{\mathbb{C}}$ to the symplectic reduction by G([DK]). **Theorem 4.1.** (Kempf-Ness) $A \ G^{\mathbb{C}}$ -orbit contains a zero of the moment map if and only if it is poly-stable. It is unique up to the action of G. $A \ G^{\mathbb{C}}$ -orbit is semi-stable if and only if its closure contains a zero of the moment map; this zero is in the unique poly-stable orbit in the closure of the original orbit.

In this paper we are only interested in the uniqueness problem. We will first give a proof in the finite dimensional case, using an analytic approach. An essential ingredient in the proof of the Kempf-Ness theorem is the existence of a function E, called the *Kempf-Ness function*. Given a point $x \in M$, one can define a one-form α on $G^{\mathbb{C}}$ as:

$$\alpha_q(R_q\xi) = -\langle \mu(g.x), J\xi \rangle,$$

where R_g is the right translation by g and $\xi \in \mathfrak{g}_{\mathbb{C}}$. It is easy to check that α is closed and invariant under the left G-action. Then α is the pull back of a closed one-form $\bar{\alpha}$ from $G^{\mathbb{C}}/G$. It is well known that $G^{\mathbb{C}}/G$ is always contractible, so α gives rise to a function E, up to an additive constant. Notice if the G action is linearizable, this coincides with the usual definition given by the logarithm of the length of a vector on the induced line bundle. It is a standard fact that E is geodesically convex, i.e. $\bar{\alpha}$ is monotone along geodesics in $G^{\mathbb{C}}/G$. The critical points of E consist exactly of the zeroes of μ in the given $G^{\mathbb{C}}$ orbit. So any $G^{\mathbb{C}}$ orbit contains at most one zero of the moment map, up to the action of G. In the semi-stable case, we consider the function $f(x) = |\mu(x)|^2$ on M, and its downward gradient flow x(t). The flow line is tangent to the $G^{\mathbb{C}}$ orbit and the induced flow in $G^{\mathbb{C}}/G$ is exactly the downward gradient flow of E. We call either flow the Kempf-Ness flow. As we will see more explicitly later, a theorem of Duistermaat([Le]) says that for x(0) close to a zero of μ , the flow x(t) converges polynomially fast to a limit in $\mu^{-1}(0)$. Now suppose x is semi-stable, and x_1, x_2 are two poly-stable points in $\overline{G^{\mathbb{C}}.x}$. W.L.O.G, we can assume $\mu(x_1) = \mu(x_2) = 0$. Take $y_1, y_2 \in G^{\mathbb{C}}$ x such that y_i is close to x_i . Then the gradient flows $x_i(t)$ converges to a point $z_i \in \mu^{-1}(0)$ near x_i . Denote by $\gamma_i(t)$ the corresponding flow in $G^{\mathbb{C}}/G$. Since the gradient flow of a geodesically convex function decreases the geodesic distance, $d(\gamma_1(t), \gamma_2(t))$ is uniformly bounded. By compactness, we conclude that z_1 and z_2 must be in the same $G^{\mathbb{C}}$ orbit and by the uniqueness in the poly-stable case, we see that z_1 and z_2 must lie in the same G orbit. By choosing y_i arbitrarily close to x_i , we conclude that x_1 and x_2 are in the same G orbit.

The above argument proves the uniqueness of the poly-stable orbits in the closure of a semi-stable orbit. There are technical difficulties to extend this argument to the infinite dimensional setting, due to the loss of compactness. As a result, we need to investigate more about the gradient flow in the



Figure 1: a curve asymptotic to a geodesic ray

finite dimensional case. What we shall show next is that for a semi-stable point, the gradient flow is asymptotic to an "optimal" geodesic ray at infinity.

Definition 4.2. We say a curve $\gamma(t)(t \in [0, \infty))$ in a simply-connected nonpositively curved space is asymptotic to a geodesic ray $\chi(t)$ if for any fixed s > 0, $d(\gamma_t(s), \chi(s))$ tends to zero as t tends to ∞ , where γ_t is the geodesic connecting $\chi(0)$ and $\gamma(t)$ which is parametrized by arc-length. In other words, $\chi(t)$ is the point in the sphere at infinity induced by $\gamma(t)$ as $t \to \infty$ (see figure 1).

It follows from the definition that any two geodesic rays $\chi_1(t)$ and $\chi_2(t)$ that are both asymptotic to a given curve $\gamma(t)$ must be *parallel*, i.e. $d(\chi_1(t), \chi_2(t))$ is uniformly bounded.

4.2 Standard case

Let (V, J_0, g_0) be an *n* dimensional unitary representation of a compact connected Lie group *G*, so we have a group homomorphism: $G \to U(n)$. *V* is then a representation of the complexified group $G^{\mathbb{C}}$. Denote by Ω_0 the induced Kähler form on *V*. It is easy to see that the *G* action always has a moment map $\mu: V \to \mathfrak{g}^* \simeq \mathfrak{g}$, where we have identified \mathfrak{g} with \mathfrak{g}^* by fixing an invariant metric. It is defined as

$$(\mu(v),\xi) = \frac{1}{2}\Omega_0(\xi.v,v).$$
 (3)

For any $v \in V$, denote the infinitesimal action of G at v by

$$L_v: \mathfrak{g} \to V; \xi \mapsto \xi. v_s$$

then it is easy to see that

$$\mu(v) = \frac{1}{2}L_v^*(J_0v).$$

 L_v can also be viewed as a map from $\mathfrak{g}^{\mathbb{C}}$ to V, and then $\mu(v) = -\frac{1}{2}JL_v^*v$.

Now consider the function $f: V \to \mathbb{R}; v \mapsto |\mu(v)|^2$, whose downward gradient flow equation is:

$$\frac{d}{dt}v = -\nabla f(v) = -J_0 L_v(\mu(v)).$$
(4)

Since f is a homogeneous polynomial, and thus real analytic, the Lojasiewicz inequality holds for f, i.e. there exists constant C > 0 and $\alpha \in [\frac{1}{2}, 1)$, such that for v close to zero,

$$|\nabla f(v)| \ge C \cdot |f(v)|^{\alpha}.$$

The previous Lojasiewicz arguments show that for v close to 0, the flow (4) starting from v will converge polynomially fast to a critical point of f.

From now on we assume 0 de-stabilizes v, i.e. $0 \in \overline{G^{\mathbb{C}}.v}$. Thus the gradient flow (4) converges to the origin by the uniqueness in the previous section. Since everything is homogeneous, we can study the induced flow on $\mathbb{P}(V)$. The action of G is then holomorphic and Hamiltonian with respect to the Fubini-Study metric on $\mathbb{P}(V)$, with moment map $\hat{\mu} : \mathbb{P}(V) \to \mathfrak{g}$. It is then easy to see that

$$\hat{\mu}([v]) = \frac{\mu(v)}{|v|^2}$$

Let $\hat{f} = |\hat{\mu}|^2$, then we can study the downward gradient flow of \hat{f} on $\mathbb{P}(V)$:

$$\frac{d}{ds}[v] = -\nabla \hat{f}([v]) = -J_0 L_{[v]}(\hat{\mu}([v])).$$
(5)

Let $\pi: V \to \mathbb{P}(V)$ be the quotient map, then clearly

$$\pi_*(\nabla f(v)) = |v|^2 \nabla \tilde{f}([v]).$$

So the flow (5) is just a re-parametrization of the image under π of the flow (4): if v(t) satisfies (4), then [v(s)] satisfies (5), with $\frac{ds}{dt} = |v(t)|^2$. Since \hat{f} is also real analytic, the flow [v(s)] converges polynomially fast to a unique limit $[v]_{\infty}$.

Lemma 4.3.

$$\hat{\mu}([v]_{\infty}) \neq 0$$

Proof. Otherwise [v] is semi-stable with respect to the action of $G^{\mathbb{C}}$ on $\mathbb{P}(V)$, thus the corresponding Kempf-Ness function $\log |g.v|^2$ is bounded below on $G^{\mathbb{C}}$. This contradicts the assumption that $0 \in \overline{G^{\mathbb{C}}.v}$.

Thus we know that

$$\frac{\mu(v(s))}{|v(s)|^2} = \hat{\mu}([v]_{\infty}) + O(s^{-\gamma})(\gamma > 0)$$

is bounded away from zero when s is large enough. So for t sufficiently large, we have

$$|\nabla f(v(t))|^4 \ge C \cdot |f(v(t))|^3.$$

The Lojasiewicz arguments then ensure that v(t) actually converges to 0 in the order $O(t^{-\frac{1}{2}})$. So we obtain $s \leq C \cdot \log t$.

Now since the gradient flow of f is tangent to the $G^{\mathbb{C}}$ orbit, it can also be viewed as a flow on $G^{\mathbb{C}}/G$. This is given by a path $\gamma(t) = [g(t)]$, where $g(t) \in G^{\mathbb{C}}$ satisfies

$$\dot{g}(t)g(t)^{-1} = -J\mu(g(t).v),$$

and the re-parameterized path corresponding to (5) is

$$\dot{g}(s)g(s)^{-1} = -J\hat{\mu}(g(s).[v]),$$

and

$$\frac{d}{ds}\gamma(s) = -J\hat{\mu}([v]_{\infty}) + O(s^{-\gamma}).$$

In the following we shall use the re-parameterized version as $\left|\frac{d}{ds}\gamma(s)\right|$ has a lower bound as $s \to \infty$ which makes it more convenient to analyze the asymptotic behavior.

Theorem 4.4. γ is asymptotic to a geodesic ray χ in $G^{\mathbb{C}}/G$. Moreover, the direction of γ is conjugate to $\frac{\hat{\mu}([v]_{\infty})}{|\hat{\mu}([v]_{\infty})|}$ under the adjoint action of G.

Proof. We already know $\dot{\gamma}(s)$ is getting close to $\hat{\mu}([v]_{\infty})$, but this is not sufficient to conclude that γ is asymptotic to a geodesic ray with direction $\hat{\mu}([v]_{\infty})$. We shall analyze this more carefully, by elementary geometry. First it is easy to see that

$$|\ddot{\gamma}(s)| = |L_{[v](s)}^* L_{[v](s)} \hat{\mu}([v](s))|,$$

where $L_{[v](s)}$ is the infinitesimal action of \mathfrak{g} at [v](s). Since $[v](s) \to [v]_{\infty}$ as $s \to \infty$, by corollary 3.2 we get

$$\int_t^\infty |\ddot{\gamma}(s)| ds \le C \int_t^\infty |L_{[v](s)}\hat{\mu}([v](s))| ds = C \int_t^\infty |\nabla \hat{f}(s)| ds \le C \cdot t^{-\beta},$$

where $\beta = \frac{1-\alpha}{2\alpha-1} > 0$. Notice that here α is the exponent appearing in the Lojasiewicz inequality for \hat{f} , not the original f. From the above we know $\lim_{s\to\infty} |\dot{\gamma}(s)| = |\hat{\mu}([v]_{\infty})| > 0$, so if we parameterize γ by arc-length and denote the resulting path by $\tilde{\gamma}(u)$, then we have

$$|\ddot{\gamma}(u)| = |\dot{\gamma}(s)|^{-2} |\ddot{\gamma}(s) - \frac{\langle \ddot{\gamma}(s), \dot{\gamma}(s) \rangle}{|\dot{\gamma}(s)|^2} \dot{\gamma}(s)| \le C \cdot |\ddot{\gamma}(s)|$$

Therefore

$$\int_t^\infty |\ddot{\widetilde{\gamma}}(u)| du \le C \cdot t^{-\beta},$$

Now for any u > 0, let $\tilde{\gamma}_u(v)(v \in [0,1)$ be the geodesic in $G^{\mathbb{C}}/G$ connecting $\tilde{\gamma}(0)$ and $\tilde{\gamma}(u)$. Denote by $L_u(v)(v \in [0,u])$ the distance between $\tilde{\gamma}(v)$ and $\tilde{\gamma}_u(v)$. Then $L_u(0) = L_u(u) = 0$ and a standard calculation of the second variation of length(using the non-positivity of the sectional curvature of $G^{\mathbb{C}}/G$) gives

$$\frac{d^2}{dv^2}L_u(v) \ge -|\ddot{\widetilde{\gamma}}(v)|$$

Now define the function

$$f_u(v) = \int_0^v \int_w^\infty |\ddot{\widetilde{\gamma}}(r)| dr dw - \frac{v}{u} \int_0^u \int_w^\infty |\ddot{\widetilde{\gamma}}(r)| dr dw.$$

Then it is well-defined by the decay of $|\ddot{\gamma}|$, and $f_u(0) = f_u(u) = 0$ and

$$\frac{d^2}{dv^2}f_u(v) = -|\ddot{\tilde{\gamma}}(v)|.$$

Thus by maximum principle $L_u(v) \leq f_u(v)$ for all u > 0 and $v \in [0, u]$. Fix v we see

$$\sup_{u} L_{u}(v) \leq \int_{0}^{v} \int_{w}^{\infty} |\ddot{\widetilde{\gamma}}(r)| dr dw \leq C \cdot v^{1-\beta}.$$

Moreover, for any $u_2 > u_1 >> 1$, by comparison argument the angle between $\tilde{\gamma}_{u_1}$ and $\tilde{\gamma}_{u_2}$ is bounded by $d(\tilde{\gamma}_{u_1}(u_1), \tilde{\gamma}_{u_2}(u_1))/u_1 = L_{u_2}(u_1)/u_1$, which is controlled by $C \cdot u_1^{\beta-1}$. Thus we conclude that the direction of $\tilde{\gamma}_u$ is converging uniformly to some limit direction and so $\tilde{\gamma}(\text{and thus } \gamma)$ is asymptotic to a geodesic ray χ starting from $\gamma(0)$. Now for any s > 0 by the same way we get a geodesic ray χ_s starting from $\gamma(s)$ which is asymptotic to γ . So the rays χ_s are all asymptotic to each other and one could easily see that they are all parallel, and then $\dot{\chi}_s(0)$ are all conjugate to each other under the action of G. On the other hand, if we denote by $\gamma_{s,t}(u)(u \in [0,1])$ the geodesic connecting $\gamma(s)$ and $\gamma(t)$ for s < t, then again by second variation,

$$\frac{d}{dt} \langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \frac{\dot{\gamma}_{s,t}(1)}{|\dot{\gamma}_{s,t}(1)|} \rangle \ge -C \frac{|\ddot{\gamma}(t)|}{|\dot{\gamma}(t)|} \ge -C |\ddot{\gamma}(t)|.$$

So we get

$$\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \frac{\dot{\gamma}_{s,t}(1)}{|\dot{\gamma}_{s,t}(1)|} \rangle \ge 1 - \int_s^t |\ddot{\gamma}(u)| du \ge 1 - C \cdot s^{-\beta}.$$

We know $\dot{\gamma}(t) = J\hat{\mu}([v]_{\infty}) + O(t^{-\alpha})$, and as $t \to \infty$ up to the adjoint action of G we have

$$\frac{\dot{\gamma}_{s,t}(1)}{|\dot{\gamma}_{s,t}(1)|} \to \dot{\chi}(s).$$

So let $s \to \infty$ we see $\dot{\chi}(0)$ is conjugate to $\frac{\hat{\mu}([v]_{\infty})}{|\hat{\mu}([v]_{\infty})|}$ under the adjoint action of G.

From the proof of the above theorem we see that $\chi(s)$ also degenerates v to the origin since the path v(t) is of order $O(t^{-\frac{1}{2}}) = O(e^{-C \cdot s})$. By Kempf([Ke]) and Ness([Ne]), the direction $\hat{\mu}([v]_{\infty})$ is indeed rational, i.e. it generates an algebraic one-parameter subgroup $\lambda : \mathbb{C}^* \to G^{\mathbb{C}}$. Moreover, the direction $\hat{\mu}([v]_{\infty})$ is the unique(up to the adjoint action of G) optimal direction for v in the sense of Kirwan([Ki])) and Ness([Ne]).

4.3 Linear Case

Now we suppose G acts linearly on $(V = \mathbb{C}^n, \Omega, J_0)$ where J_0 is the standard complex structure on \mathbb{C}^n and Ω is a real-analytic symplectic form compatible with J_0 . Then the action has a real-analytic moment map μ with $\mu(0) = 0$. μ is not necessarily standard but the Lojasiewicz inequality still holds for $f = |\mu|^2$. Suppose $0 \in \overline{G^{\mathbb{C}} \cdot v}$, then the downward gradient flow v(t) of $f(v) = |\mu(v)|^2$ converges to the origin polynomially fast. Let $\hat{v}(t)$ be the downward gradient flow of $\hat{f}(v) = |\hat{\mu}(v)|^2$, where $\hat{\mu}$ is the moment map for the linearized G action on $(V = T_0 V, \Omega_0, J_0)$. By the arguments in the previous section, $\hat{v}(t)$ converges to zero in the order $O(t^{-\frac{1}{2}})$ and the corresponding flow $\hat{\gamma}(t)$ is asymptotic to a rational geodesic ray $\chi(t)$. Let $\gamma(t)$ in $G^{\mathbb{C}}/G$ be the flow corresponding to v(t), we want to show $\gamma(t)$ is also asymptotic to $\chi(t)$. It suffices to bound the distance L(t) between $\gamma(t)$ and $\hat{\gamma}(t)$. Let $\psi_t(s)(s \in [0, 1])$ be the geodesic connecting $\gamma(t)$ and $\hat{\gamma}(t)$, then

$$\begin{aligned} \frac{d}{dt}L(t) &= \frac{1}{L(t)}\langle \dot{\psi}(1), \hat{\mu}(\hat{v}(t)) \rangle - \frac{1}{L(t)} \langle \dot{\psi}(0), \mu(v(t)) \rangle \\ &= \frac{1}{L(t)} \langle \dot{\psi}(1), \mu(\hat{v}(t)) \rangle - \langle \dot{\psi}(0), \mu(v(t)) \rangle) + \frac{1}{L(t)} \langle \dot{\psi}(1), \hat{\mu}(\hat{v}(t)) - \mu(\hat{v}(t)) \rangle \\ &\leq |\hat{\mu}(\hat{v}(t)) - \mu(\hat{v}(t))|, \end{aligned}$$

where we used the fact that the Kempf-Ness function is geodesically convex. To estimate the last term, notice since the G action is linear, we have for any $\xi \in \mathfrak{g}$

$$\begin{aligned} \langle \mu(v), \xi \rangle &= \langle \mu(0) + \int_0^1 \frac{d}{dt} \mu(tv) dt, \xi \rangle \\ &= \int_0^1 \Omega_{tv}(\xi . tv, v) dt \\ &= \frac{1}{2} \Omega_0(\xi . v, v) dt + O(|v|^3) \\ &= \langle \hat{\mu}(v), \xi \rangle + O(|v|^3). \end{aligned}$$

From the previous secton we know $\hat{v}(t) = O(t^{-\frac{1}{2}})$, so we obtain

$$\frac{d}{dt}L(t) \leq C \cdot t^{-\frac{3}{2}}$$

and so L(t) is uniformly bounded. Therefore, we conclude the following theorem:

Theorem 4.5. Suppose G acts Hamiltonian linearly on (V, Ω, J_0) , with the moment map given by μ . Suppose also a vector v_0 is de-stabilized by the origin. Let v(t) be the downward gradient flow of $|\mu|^2$ emanating from v, then v(t) converges to 0 in the order $O(t^{-\frac{1}{2}})$. Let $\gamma(t)$ be the corresponding flow in $G^{\mathbb{C}}/G$, then there exists a geodesic ray χ in $G^{\mathbb{C}}/G$, which is asymptotic to γ . Moreover, χ is rational.

4.4 General Case

In general we need to linearize the problem, using the Marle-Guillemin-Sternberg normal form. Let (M, ω, J, G, μ) be a real analytic Hamiltonian G-action on a real analytic Kähler manifold. Choosing a bi-invariant metric on \mathfrak{g} we can identify \mathfrak{g} with \mathfrak{g}^* . Suppose $x \in M$ is a zero of μ . Let G_0 be the isotropy group of x and \mathfrak{g}_0 be its Lie algebra. The bi-invariant product on \mathfrak{g} allows a G_0 invariant splitting:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}.$$

Notice $\mathfrak{g}.x \subset (\mathfrak{g}.x)^{\omega}$. Denote by N the orthogonal complement of $\mathfrak{g}.x \oplus J\mathfrak{g}.x$ in T_xM , then N is G_0 -invariant and the linear G_0 action on N has a canonical moment map $\mu_N : N \to \mathfrak{g}_0$. Let

$$Y = G \times_{G_0} (\mathfrak{m} \times N),$$

then G acts naturally on Y on the left.

Lemma 4.6. (Marle-Guillemin-Sternberg [GS], [OR]) There exists a symplectic form ω defined in a neighborhood U of [e, 0, 0] in Y, under which the G action is Hamiltonian with a moment map given by

$$\mu: U \to \mathfrak{g}; [g, \rho, v] \to Ad_g^*(\mu_N(v) + \rho).$$

There exists a local G equivariant symplectic diffeomorphism $\Phi: Y \to M$ which respects the moment maps, and satisfies $\Phi([e, 0, 0]) = x$, $\Phi^*J - J_0 = O(r^2)$) on N and $\Phi^*J = J_0$ at [e, 0, 0]. Here J_0 is the canonical G-invariant almost complex structure on Y induced by J, which will be more explicit in the proof. Moreover, we can take Φ to be real analytic if everything we start with is so.

The only new feature here is the control on the complex structure. The proof of this theorem is a bit technical and will be deferred to the appendix.

From now on we will work on (U, Ω_0, J) where we also denote by J the pullback Φ^*J .

Theorem 4.7. Suppose $y \in U$ is de-stabilized by x, then the Kempf-Ness flow y(t) of $|\mu|^2$ converges to $y_{\infty} \in G.x$ polynomially fast. Moreover the corresponding flow $\gamma(t)$ in $G^{\mathbb{C}}/G$ is asymptotic to a geodesic ray $\chi(t)$ which is rational and also degenerates y to y_{∞} .

Remark 4.8. Here we could define $\chi(t)$ as the "optimal" degeneration of y, generalizing the usual definition in the linear case.

To prove the theorem, we study the function $f = |\mu|^2$ on U. By definition,

$$f([g, \rho, v]) = |\rho|^2 + |\mu_N(v)|^2,$$
$$\nabla f([g, \rho, v]) = J[L_g \rho, ad_{\mu_N(v)} \rho, \mu_N(v).v],$$

Since f is real analytic, we have for some $\alpha \in [\frac{1}{2}, 1)$ that

$$|\nabla f| \ge C \cdot |f|^{\alpha}.$$

Therefore y(t) converges to a zero y_{∞} of μ polynomially fast. By uniqueness, $y_{\infty} \in G.x$. Without loss of generality, we will assume $y_{\infty} = x$ from now on, and we shall distinguish between two cases.

In the first case we assume $G_0 = G$, then $\mathfrak{m} = 0$, and we are essentially reduced to the linear case. What we obtain is a Kähler manifold $(U \subset N, \Omega_0, J)$. We just need to holomorphically linearize the G action:

Lemma 4.9. There exits a G-equivariant holomorphic embedding

$$\Phi: (V \subset T_0U, J_0) \hookrightarrow (U, J); 0 \mapsto x.$$

Proof. Shrinking U if necessaray, we can first choose a holomorphic embedding

$$\Psi: (U,J) \hookrightarrow (T_0U,J_0); x \mapsto 0.$$

Again Shrinking U if necessary, define

$$\hat{\Psi}: (U,J) \to (T_0U,J_0); y \mapsto \frac{1}{|G|} \int_G g^{-1} \cdot \Psi(g.y) d\mu,$$

where μ is a Harr measure on G. Then $\hat{\Psi}$ is holomorphic, and $d\hat{\Psi}_x = d\Psi_x$, so $\hat{\Psi}$ is an embedding near x. Then we can just take $\Phi = \hat{\Psi}^{-1}$.

Now using Φ we can work on $(V_1, \Omega = \Phi^*\Omega_0, J_0)$ with a linear Hamiltonian of G, and the linear theory in the previous section applies to conclude the theorem in this case.

In the second case we assume G_0 is a proper subgroup of G. We will try to reduce to the first case. It is easy to see that the G_0 action on Y is also Hamiltonian, with a moment map $\hat{\mu}$ equal to the orthogonal projection of μ to \mathfrak{g}_x . Therefore,

$$\hat{\mu}([g,\rho,v]) = Ad_q^*\mu_N(v).$$

Denote by $G_0^{\mathbb{C}}$ the isotropy group of x.

Lemma 4.10. $G_0^{\mathbb{C}}$ is the complexification of G_0 (hence is reductive).

Proof. This lemma is well-known. In the Lie algebra level, we just need to show if $\xi . x + J\eta . x = 0$ for some $\xi, \eta \in \mathfrak{g}$, then $\xi . x = \eta . x = 0$. This follows easily from the definition of the moment map:

$$\omega(\eta.x, J\eta.x) = (d\mu(J\eta.x), \eta) = (d\mu(J\eta.x + \xi.x), \eta) - (Ad_{\xi}^*\mu(x), \eta) = 0.$$

Hence $\eta.x = 0$ and $\xi.x = 0.$

Lemma 4.11. We can choose a point in the $G^{\mathbb{C}}$ orbit of y, denoted by \hat{y} , so that x de-stabilizes \hat{y} for the group G_0 .

Proof. It suffices to find \hat{y} in the $G^{\mathbb{C}}$ orbit of y such that x lies in the closure of $G_0^{\mathbb{C}}.\hat{y}$. To do this, we first choose an arbitrary holomorphic map $\Psi: T_x M \to M$ with $\Psi(0) = x$ and $d\Psi(0) = Id$. As before we can linearize the action so that Ψ is G_0 -equivariant. $T_x M$ has a \mathbb{C} -linear decomposition

$$T_x M = \mathfrak{g}^{\mathbb{C}}.x \oplus N,$$

where N is as before the orthogonal complement of $\mathfrak{g}^{\mathbb{C}}.x = \mathfrak{g}.x \otimes \mathbb{C} = \mathfrak{g}.x \oplus J_0(\mathfrak{g}.x)$. Then we define

$$\Phi: G^{\mathbb{C}} \times_{G^{\mathbb{C}}_{0}} N \to M; [(g, v)] \to g. \Psi(v)$$

This is a local diffeomorphism around [(Id, 0)]. So for any y close to x, there is a unique $(g, v) \in G^{\mathbb{C}} \times N$ which is close to [(Id, 0)] such that $y = g.\Psi(Id, v)$. Let $\hat{y} = \Psi(Id, v)$. We claim $x \in \overline{G_0^{\mathbb{C}}}.\hat{y}$. Notice that the Kempf-Ness flow y(t) converges to x, so this gives rise to a smooth family (g(t), v(t)) with $y(t) = g(t).\Psi(Id, v(t))$. Let $\hat{y}(t) = \Psi(Id, v(t))$. Since y(t) all lie in the same $G^{\mathbb{C}}$ orbit, so are $\hat{y}(t)$. Thus all v(t) lie in the $G_0^{\mathbb{C}}$ orbit of v, and

$$\lim_{t \to \infty} v(t) = 0.$$

Therefore, $x \in \overline{G_0.\hat{y}}$.

Let $\tilde{y}(t)$ be the downward gradient flow of f with $\tilde{y}(0) = \hat{y}$, and $\hat{y}(t)$ be the downward gradient flow of $\hat{f} = |\hat{\mu}|^2$ with $\hat{y}(0) = \hat{y}$. Let $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$ be the corresponding path in $G^{\mathbb{C}}/G$ and $G_0^{\mathbb{C}}/G_0$ respectively. Then the previous linear theory tells that $\hat{y}(t)$ converges to x in the order $O(t^{-\frac{1}{2}})$ and $\hat{\gamma}(t)$ is asymptotic to a rational geodesic ray $\chi(t)$ with the same degeneration limit. On the other hand $G_0^{\mathbb{C}}/G_0$ is naturally a totally geodesic submanifold of $G^{\mathbb{C}}/G$, and next we will prove that the distance between $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$ in $G^{\mathbb{C}}/G$ is uniformly bounded.

We denote by $\psi_t(s)(s \in [0, 1])$ the geodesic in $G^{\mathbb{C}}/G$ connecting $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$, and L(t) the length of ψ_t , then it is easy to see that

$$\begin{aligned} \frac{d}{dt}L(t) &= \frac{1}{L(t)}(\mu(y(t)), \dot{\psi}_t(0)) - \frac{1}{L(t)}(\hat{\mu}(\hat{y}(t)), \dot{\psi}_t(1)) \\ &= \frac{1}{L(t)}(\mu(y(t)), \dot{\psi}_t(0)) - \frac{1}{L(t)}(\mu(\hat{y}(t)), \dot{\psi}_t(1)) + \frac{1}{L(t)}(\mu(\hat{y}(t)) - \hat{\mu}(\hat{y}(t)), \dot{\psi}_t(1)) \\ &\leq |\mu(\hat{y}(t)) - \hat{\mu}(\hat{y}(t))|, \end{aligned}$$

where again we have used the convexity of the Kempf-Ness function. In our situation, $\mu - \hat{\mu} = Ad_g^*\rho$. Here g(t) are $\rho(t)$ are uniquely determined by the choice at t = 0 if we require $\dot{\rho}(t) \in \mathfrak{m}$ and $g(t)^{-1}\dot{g}(t) \in \mathfrak{m}$. Now at $\hat{y}(t) = [g(t), \rho(t), v(t)]$, we have

$$\nabla \hat{f} = J.([0, ad^*_{\mu_N(v)}\rho, \mu_N(v).v])$$

= $[ad^*_{\mu_N(v)}\rho, 0, J_0 \cdot (\mu_N(v).v)] + (J - J_0)ad^*_{\mu_N(v)}\rho + (J - J_0)\mu_N(v).v]$

Therefore,

$$\begin{aligned} |\frac{d}{dt}\rho(\hat{y}(t))| &= |\Pi_{\mathfrak{m}}(\nabla\hat{f})| \\ &\leq C \cdot |J - J_0||\mu_N(v)||\rho| + C \cdot d(\hat{y}(t), x)^2 |\mu_N(v).v|) \\ &\leq C \cdot (t^{-\frac{3}{2}}|\rho| + t^{-\frac{5}{2}}). \end{aligned}$$

Since $\rho(\infty) = 0$, we first get

$$|\rho(t)| \le C \cdot t^{-\frac{1}{2}}$$

Then plug back into the previous inequality and repeat to obtain

$$\frac{d}{dt}\rho(\hat{x}(t)) \le C \cdot t^{-\frac{5}{2}},$$

and then

$$|\rho(\hat{x}(t))| \le C \cdot t^{-\frac{3}{2}}$$

So

$$L(t) \le \int_1^t s^{-\frac{3}{2}} ds + C \le C.$$

Therefore L(t) is uniformly bounded.

By definition, we see that $\tilde{\gamma}(t)$ is also asymptotic to the geodesic ray $\chi(t)$. Now the original $\gamma(t)$ is also asymptotic to $\chi(t)$ again because that the Kempf-Ness flow in $G^{\mathbb{C}}/G$ decreases the geodesic distance.

Then it is easy to see that $\chi(t)$ has the same degeneration limit as $\gamma(t)$. So this completes the proof of theorem 4.7.

5 Stability of the Calabi flow

We first recall the definition of the Calabi flow. It is an infinite dimensional analogue of the previously mentioned Kempf-Ness flow. Let (M, ω, J_0) be a Kähler manifold. As before, we have the group \mathcal{G} acting on \mathcal{J} and preserves

 $\mathcal{J}^{int}.$ The action of $\mathcal G$ on $\mathcal J$ has a moment map given by the Hermitian scalar curvature functional

$$S - \underline{S} : \mathcal{J} \to C_0^\infty(M; \mathbb{R}).$$

Its norm is called the *Calabi functional*:

$$Ca(J) = \int_M (S(J) - \underline{S})^2 d\mu_{\omega}.$$

The gradient of Ca under the natural metric on \mathcal{J} is given by

$$\nabla Ca(J) = \frac{1}{2} J \mathcal{D}_J S(J).^2$$

The *Calabi flow* is the downward gradient flow of *Ca* on \mathcal{J}^{int} . Its equation is given by

$$\frac{d}{dt}J(t) = -\frac{1}{2}J(t)\mathcal{D}_{J(t)}S(J(t)).$$
(6)

As in the finite dimensional space, the Calabi flow can be lifted to $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$, which in this case is just the space of Kähler metrics

$$\mathcal{H}_J = \{ \phi \in C_0^\infty(M; \mathbb{R}) | \omega + \sqrt{-1} \partial_J \bar{\partial}_J \phi > 0 \}.$$

The equation reads:

$$\frac{d}{dt}\phi(t) = S(\phi(t)) - \underline{S}.$$
(7)

By (1), this is also the downward gradient flow of the Mabuchi functional E. The two equations (6) and (7) are essentially equivalent:

Lemma 5.1. Any solution of (7) naturally gives rise to a solution of (6); any solution J(t) of (6) induces a solution of (7), if J(t) all lie in \mathcal{J}^{int} .

Proof. Given a path $\phi(t) \in \mathcal{H}$, we consider the time-dependent vector fields $X(t) = -\frac{1}{2} \nabla_{\phi(t)} \dot{\phi}(t)$. Let f_t be the family of diffeomorphisms generated by X(t). Then $f_t^*(\omega + \sqrt{-1}\partial \bar{\partial} \phi(t)) = \omega$. Let $J(t) = f_t^* J$. Then

$$\frac{d}{dt}J(t) = -\frac{1}{2}J(t)\mathcal{D}_{J(t)}\dot{\phi}(t).$$

This proves the first half of the lemma. For the second half, if J(t) is a solution to (6). We again consider the vector fields $X(t) = \frac{1}{2} \nabla_{J(t)} S(J(t))$ and the induced diffeomorphisms f_t . Then $f_t^* J(t) = J(0)$ since $J(t) \in \mathcal{J}^{int}$, and $f_t^* \omega = \omega + \sqrt{-1} dJ(0) d\phi(t)$, with $\frac{d}{dt} \phi(t) = S(\phi(t)) - \underline{S}$.

²The factor comes from the fact that the metric we choose on \mathcal{J} is $(\mu_1, \mu_2)_J := 2Re \int_M \langle \mu_1, \mu_2 \rangle_J \omega^n$.

Equation (6) is not parabolic, due to the \mathcal{G} invariance. But (7) is parabolic and we have the following estimates:

Lemma 5.2. (see [CH2]) Suppose there are constants $C_1, C_2 > 0$ such that along the Calabi flow:

$$\begin{cases} \frac{\partial \phi}{\partial t} = S - \underline{S}\\ \phi(0) = \phi_0, \end{cases}$$
(8)

we have

$$||Rm(g(t))||_{L^{\infty}(g(t))} \le C_1,$$

and the Sobolev constant of g(t) is bounded by C_2 for all $t \in [0,T)$, then for any l > 0, and $t \in [1,T)$, we have

$$||\nabla_t^l Rm(g(t))||_{L^{\infty}(q(t))} \le C,$$

where C > 0 depends only C_1, C_2, l, n .

The Calabi flow equation in the form (7) was first proposed by E. Calabi([Ca1], [Ca2]) to find extremal metrics in a fixed Kähler class. The short time existence was established by Chen-He([CH1]). They also proved the global existence assuming Ricci curvature bound.

The equation (6) also has its own advantage. Namely, when the space \mathcal{H} does not admit any cscK metric, the solution of equation (7) must diverge when $t \to \infty$. However, it is still possible that the corresponding J(t) still converges in the bigger ambient space \mathcal{J} . In this section we are interested in the Calabi flow (6) starting from an integrable complex structure in a neighborhood of a cscK metric. We shall prove the following theorem:

Theorem 5.3. Suppose $J_0 \in \mathcal{J}$ is cscK. Then there exists a small $C^{k,\lambda}(k \gg 1)$ neighborhood \mathcal{U} of J_0 in \mathcal{J}^{int} , such that the Calabi flow J(t) starting from any $J \in \mathcal{U}$ exists globally and converges polynomially fast to a cscK metric $J_{\infty} \in \mathcal{J}$ in $C^{k,\lambda}$ topology. Up to a Hamiltonian diffeomorphism we can assume J_{∞} is smooth, then the convergence is also in C^{∞} .

Remark 5.4. When J lies on the leaf of J_0 , i.e. the corresponding Kähler metrics are in the same Kähler classes, this was proved in [CH1] and the convergence is indeed exponential. In general, the convergence is exponential if and only if J_0 and J_{∞} are on the same $\mathcal{G}^{\mathbb{C}}$ leaf.

Remark 5.5. There are also studies of stability of other geometrical flows (such as Kähler-Ricci flow) in Kähler geometry when the complex structure is deformed, see for example [CLW], [TZ2]... We believe the idea in this section could also apply to other settings. In a sequel to this paper([SW]),

the second author and Y-Q. Wang proved a similar stability theorem for the Kähler-Ricci flow on Fano manifolds. We should mention that two alternative approaches in the study of the stability of Kähler-Ricci flow have been announced by C.Arezzo-G. La Nave and G. Tian-X. Zhu.

In general this type of stability result is based on a very rough a priori estimate of the length of the flow and the parabolicity. Here the key ingredient is the following Lojasiewicz type inequality which yields the required a priori estimate.

Theorem 5.6. Suppose $J_0 \in \mathcal{J}^{int}$ is cscK, then there exists a $L_k^2(k \gg 1)$ neighborhood \mathcal{U} of J_0 in \mathcal{J}^{int} and constants C > 0, $\alpha \in [\frac{1}{2}, 1)$ such that for any $J \in \mathcal{U}$, the following inequality holds:

$$||\mathcal{D}_J S(J)||_{L^2} \ge C \cdot ||S(J) - \underline{S}||_{L^2}^{2\alpha},$$
(9)

where $\mathcal{D}_J \phi = \bar{\partial}_J X_{\phi} + \bar{X}_{\phi} N_J$. When J is integrable, $\mathcal{D}_J \phi = \bar{\partial}_J X_{\phi}$ is the Lichnerowicz operator.

Remark 5.7. The Lojasiewicz inequality was first used by L. Simon([Si]) in the study of convergence of parabolic P.D.E's. Råde([Ra]) used Simon's idea to study the convergence of the Yang-Mills flow on two or three dimensional manifold. It also appeared in the study of asymptotic behavior in Floer theory in [D4]. Here we follow [Ra] closely.

We begin the proof by reducing the problem to a finite dimensional one and then use Lojasiewicz's inequality(theorem 3.1).

To simplify the notation, we assume the function spaces appearing below consist of normalized functions, i.e. functions with average zero. We have the elliptic complex at J_0 (see [FS]):

$$L^2_{k+2}(M;\mathbb{C}) \xrightarrow{\mathcal{D}_0} T_{J_0}\mathcal{J} = L^2_k(\Omega^{0,1}_S(T^{1,0})) \xrightarrow{\bar{\partial}_0} L^2_{k-1}(\Omega^{0,2}_S(T^{1,0})),$$

where $\Omega_S^{0,p}(T^{1,0})$ is the kernel of the operator \mathcal{A} in section 3. So we have an L^2 orthogonal decomposition:

$$\Omega_S^{0,1}(T^{1,0}) = Im\mathcal{D}_0 \oplus Ker\mathcal{D}_0^*$$

On the other hand, the infinitesimal action of the gauge group \mathcal{G} is just the restriction of \mathcal{D}_0 to $L^2_{k+2}(M;\mathbb{R})$, which we denote by Q_0 . Since J_0 is cscK, $\mathcal{D}_0^*\mathcal{D}_0$ is a real operator. Thus

$$Im(\mathcal{D}_0) = \mathcal{D}_0(L^2_{k+2}(M;\mathbb{R})) \oplus \mathcal{D}_0(L^2_{k+2}(M;\sqrt{-1}\mathbb{R}))$$

is an L^2 orthogonal decomposition, so

$$L_k^2(\Omega_S^{0,1}(T^{1,0})) = ImQ_0 \oplus KerQ_0^*,$$

where explicitly, $Q_0^* \mu = Re \mathcal{D}_0^* \mu$.

Now as in section 2 we identify a L_k^2 neighborhood of J_0 with an open set in the Hilbert space $L_k^2(\Omega_S^{0,1}(T^{1,0}))$. By the implicit function theorem, any integrable complex structure $J = J_0 + \mu \in \mathcal{J}^{int}$ with $||\mu||_{L_k^2}$ small is in the \mathcal{G} orbit of an integrable complex structure $J_0 + \nu$ with $\nu \in KerQ_0^*$ and $||\nu||_{L_k^2}$ small. Since both sides of (9) are invariant under the action of \mathcal{G} , it suffices to prove it for $\mu \in KerQ_0^*$.

We still need to fix another gauge so that the problem becomes elliptic. Recall that \mathcal{J}^{int} is the subvariety of \mathcal{J} cut out by the equation:

$$N(\mu) = \bar{\partial}_0 \mu + [\mu, \mu] = 0.$$

We would like to linearize this space to $Ker\bar{\partial}_0$. Let $W = KerQ_0^* \cap Ker\bar{\partial}_0$. Consider the operator

$$\Phi: (W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))) \times (Im\bar{\partial}_0 \cap L^2_{k+1}(\Omega^{0,2}_S(T^{1,0}))) \to Im\bar{\partial}_0 \cap L^2_{k-1}(\Omega^{0,2}_S(T^{1,0}))$$

by sending (μ, α) to the orthogonal projection to $Im\bar{\partial}_0$ of $N(\mu + \bar{\partial}_0^* \alpha)$. Since the linearization

$$D\Phi_0(\nu,\beta) = \partial_0 \partial_0^* \beta$$

whose second component is an isomorphism, by the implicit function theorem, for any $\nu \in W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))$ with $||\nu||_{L^2_k}$ small, there exists a unique $\alpha = \alpha(\nu) \in Im\bar{\partial}_0 \subset L^2_{k+1}(\Omega^{0,2}_S(T^{1,0}))$ with $||\alpha||_{L^2_{k+1}}$ small such that $\mu = \nu + \bar{\partial}_0^* \alpha$ satisfies $\Phi(\mu) = 0$. Furthermore, we have

$$||\alpha(\nu)||_{L^2_{k+1}} \le C \cdot ||\nu||^2_{L^2_k}$$

Define a map L from $B_{\epsilon_1}(W \cap L_k^2(\Omega_S^{0,1}(T^{1,0})))$ to $KerQ_0^* \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ by sending ν to μ , then L is real analytic and a neighborhood of J_0 in $\mathcal{J}^{int} \cap KerQ_0^* \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ is contained in the image of L. Moreover we have that for all $\nu \in B_{\epsilon_1}W \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ and $\lambda \in W \cap L_l^2(\Omega_S^{0,1}(T^{1,0}))$ (for any $l \leq k$),

$$c_l \cdot ||\lambda||_{L^2_l} \le ||(DL)_{\nu}(\lambda)||_{L^2_l} \le C_l \cdot ||\lambda||_{L^2_l}, \tag{10}$$

and

$$c_l \cdot ||\lambda||_{L^2_l} \le ||(DL)^*_{\nu}(DL)_{\nu}(\lambda)||_{L^2_l} \le C_l \cdot ||\lambda||_{L^2_l}.$$
(11)

To be explicit, the differential of α at ν is given by

$$(D\alpha)_{\nu}(\lambda) = (D\Phi)_{L(\nu)}(0, -)^{-1} \circ (D\Phi)_{L(\nu)}(\lambda, 0)$$

So if we denote $\mu = L(\nu)$ and $\beta = (D\alpha)_{\nu}(\lambda)$, then β satisfies:

$$\bar{\partial}_0 \bar{\partial}_0^* \beta + \Pi_{Im\bar{\partial}_0} [\mu, \bar{\partial}_0^* \beta] = \bar{\partial}_0 \lambda + \Pi_{Im\bar{\partial}_0} [\mu, \lambda] = \Pi_{Im\bar{\partial}_0} [\mu, \lambda].$$

Thus by ellipticity we obtain for ν small that

$$||(D\alpha)_{\nu}(\lambda)||_{L^{2}_{l+1}} \leq C \cdot ||\nu||_{L^{2}_{k}} \cdot ||\lambda||_{L^{2}_{l}}.$$
(12)

(10) follows from (12) and similarly we can prove (11).

Now consider the Hilbert space $W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))$ with the constant L^2 metric defined by J_0 . Define the functional \widetilde{Ca} on on a small neighborhood of the origin in $W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))$ by pulling back Ca through L, i.e.

$$\widetilde{Ca}(\nu) = \frac{1}{2}Ca(L(\nu)) = \frac{1}{2}\int (S(L(\nu)) - \underline{S})^2 \omega^n.$$

It is easy to see that

$$\delta_{\lambda}S(L(\nu)) = 2Im\mathcal{D}^*_{L(\nu)}((DL)_{\nu}(\lambda))$$

So the gradient is

$$\nabla \widetilde{Ca} = (DL)^*_{\nu} (J\mathcal{D}_{L(\nu)} S(L(\nu))).$$

We first prove that in a neighborhood of 0 in W,

$$||\nabla \widetilde{Ca}(\nu)||_{L^2} \ge C \cdot (\widetilde{Ca}(\nu))^{\alpha}.$$
(13)

The linearization of the gradient is the Hessian:

$$H_0 := \delta_{\cdot} \nabla \widetilde{Ca} : L^2_k(W) \to L^2_{k-4}(W); \lambda \mapsto 2J_0 \mathcal{D}_0 \mathcal{D}_0^* \lambda.$$

 H_0 is an elliptic operator, so it has a finite dimensional kernel W_0 consisting of smooth elements, and W has the following decomposition:

$$W = W_0 \oplus W',$$

where H_0 restricts to invertible operators from $L_k^2(W')$ to $L_{k-4}^2(W')$. So there exists a c > 0, such that for any $\mu' \in W'$, we have

$$||H_0(\mu')||_{L^2_{k-4}} \ge C \cdot ||\mu'||_{L^2_k}.$$

By the implicit function theorem, for any $\mu_0 \in W_0$ with $||\mu_0||_{L^2}^3$ small, there exists a unique element $\mu' = G(\mu_0) \in W'$ with $||\mu'||_{L^2_k}$ small, such that $\nabla \widetilde{Ca}(\mu_0 + \mu') \in W_0$. Moreover the map $G : B_{\epsilon_1}W_0 \to B_{\epsilon_2}W'$ is real analytic. Now consider the function

$$f: W_0 \to \mathbb{R}; \mu_0 \mapsto \widetilde{Ca}(\mu_0 + G(\mu_0)).$$

By construction, this is a real analytic function. For any $\mu_0 \in W_0$, it is easy to see that $\nabla f(\mu_0) = \nabla \widetilde{Ca}(\mu_0 + G(\mu_0)) \in W_0$.

Now we shall estimate the two sides of inequality (13) separately. For any $\mu \in W$ with $||\mu||_{L^2_k} \leq \epsilon$, we can write $\mu = \mu_0 + G(\mu_0) + \mu'$, where $\mu_0 \in W_0, \, \mu' \in W'$, and

$$\begin{aligned} ||\mu_0||_{L^2_k} &\leq c \cdot ||\mu||_{L^2_k}, \\ ||G(\mu_0)||_{L^2_k} &\leq c \cdot ||\mu||_{L^2_k} \\ ||\mu'||_{L^2_k} &\leq c \cdot ||\mu||_{L^2_k}. \end{aligned}$$

For the left hand side of (13), we have:

$$\begin{aligned} \nabla \widetilde{Ca}(\mu) &= \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + \mu') \\ &= \nabla \widetilde{Ca}(\mu_0 + G(\mu_0)) + \int_0^1 \delta_{\mu'} \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + s\mu') ds \\ &= \nabla f(\mu_0) + \delta_{\mu'} \nabla \widetilde{Ca}(0) + \int_0^1 (\delta_{\mu'} \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + s\mu') - \delta_{\mu'} \nabla \widetilde{Ca}(0)) ds \end{aligned}$$

The first two terms are L^2 orthogonal to each other. For the second term we have

$$||\delta_{\mu'}\nabla \widetilde{Ca}(0)||_{L^2}^2 = ||H_0(\mu')||_{L^2}^2 \ge C \cdot ||\mu'||_{L^2_4}^2.$$

For the last term, we have

$$||\delta_{\mu'}\nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + s\mu') - \delta_{\mu'}\nabla \widetilde{Ca}(0)|| \le C \cdot ||\mu||_{L^2_k} ||\mu'||_{L^2_4} \le C \cdot \epsilon \cdot ||\mu'||_{L^2_4}.$$

Therefore, we have

$$\|\nabla \widetilde{Ca}(\mu)\|_{L^2}^2 \ge |\nabla f(\mu_0)|_{L^2}^2 + C \cdot \|\mu'\|_{L^2_4}^2.$$
(14)

³Since W_0 is finite dimensional, any two norms on it are equivalent. We use the L^2 norm for our later purpose.

For the right hand side of (13), we have

$$\begin{aligned} \widetilde{Ca}(\mu) &= \widetilde{Ca}(\mu_0 + G(\mu_0) + \mu') \\ &= \widetilde{Ca}(\mu_0 + G(\mu_0)) + \int_0^1 \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + s\mu')\mu' ds \\ &= f(\mu_0) + \nabla f(\mu_0)\mu' + \int_0^1 \int_0^1 \delta_{\mu'} \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + st\mu')\mu' dt ds \\ &= f(\mu_0) + H_0(\mu')\mu' + \int_0^1 \int_0^1 (\delta_{\mu'} \nabla \widetilde{Ca}(\mu_0 + G(\mu_0) + st\mu') - \delta_{\mu'} \nabla \widetilde{Ca}(0))\mu' dt ds \end{aligned}$$

So

$$\widetilde{Ca}(\mu) \le |f(\mu_0)|_{L^2} + C \cdot ||\mu'||_{L^2_4}^2.$$
(15)

Now we apply the Lojasiewicz inequality to f, and obtain that

$$|\nabla f(\mu_0)|_{L^2} \ge C \cdot |f(\mu_0)|^{\alpha},$$

for some $\alpha \in [\frac{1}{2}, 1)$. Together with (14) and (15) we have proved (13).

To prove (9), we need to compare $||\nabla Ca(L(\nu))||_{L^2}$ and $||\nabla \widetilde{Ca}(\nu)||_{L^2}$, i.e. we want

$$||(DL)_{\nu}^{*}(\mathcal{D}_{L(\nu)}S(L(\nu)))||_{L^{2}} \leq C \cdot ||\mathcal{D}_{L(\nu)}S(L(\nu))||_{L^{2}}.$$
(16)

We can take L^2 decomposition

$$\mathcal{D}_{L(\nu)}S(L(\nu)) = (DL)_{\nu}\lambda + \beta,$$

where $\lambda \in W$ and $\beta \in Ker(DL)^*_{\nu}$. So we just need to prove

$$||(DL)_{\nu}^{*}(DL)_{\nu}\lambda||_{L^{2}} \leq C \cdot ||(DL)_{\nu}\lambda||_{L^{2}}$$

for any λ . This follows from (10) and (11). \Box

Now we follow the Lojasiewicz arguments. Suppose we have a Calabi flow J(t) along an integral leaf staying in a L_k^2 neighborhood of J_0 , then by (6)

$$\frac{d}{dt}Ca(J)^{1-\alpha} = -(1-\alpha)Ca(J)^{-\alpha} ||\nabla Ca(J)||_{L^2(t)}^2 \le -C \cdot ||\nabla Ca(J)||_{L^2(t)}.$$

Thus

$$\int_0^t ||\dot{J}||_{L^2(s)} ds = \int_0^t ||\nabla Ca(J(s))||_{L^2(s)} ds \le C \cdot Ca(J(0))^{1-\alpha}.$$
 (17)

So we get L^2 length estimate for the Calabi flow in terms of the initial Calabi energy. For γ slightly bigger than α , we have for $\beta = 2 - \frac{\gamma}{\alpha} < 1$,

$$\frac{d}{dt}Ca(J)^{1-\gamma} = -(1-\gamma)Ca(J)^{-\gamma} ||\nabla Ca(J)||_{L^2(t)}^2 \le -C \cdot ||\nabla Ca(J)||_{L^2(t)}^{\beta}$$

So for $\beta \in (2 - \frac{1}{\alpha}, 1)$ we have

$$\int_{0}^{t} ||\dot{J}(s)||_{L^{2}(s)}^{\beta} ds = \int_{0}^{t} ||\nabla Ca(J(s))||_{L^{2}(s)}^{\beta} ds \le C(\beta) \cdot Ca(J(0))^{1-(2-\beta)\alpha}.$$
(18)

Also we have polynomial decay:

$$\frac{d}{dt}Ca(t)^{1-2\alpha} \ge C > 0,$$

 \mathbf{SO}

$$Ca(J(t)) \le C \cdot (t+1)^{-\frac{1}{2\alpha-1}}.$$
 (19)

Now we define

$$\mathcal{U}_{k}^{\delta} = \{ J \in C^{k,\lambda}(\mathcal{J}^{int}) \mid ||\mu_{J}||_{C^{k,\lambda}} \leq \delta \}$$

where again we identify J close to J_0 with $\mu_J \in \Omega_S^{0,1}(T^{1,0})$. Notice that if $\delta \ll 1$, then for any tensor ξ , the $C_J^{k,\lambda}$ norms defined by (J,ω) are equivalent for any $J \in \mathcal{U}_k^{\delta}$. We omit the subscript J if $J = J_0$. Also for k sufficiently large, the Sobolev constant is uniformly bounded in \mathcal{U}_k^{δ} .

Theorem 5.8. Suppose J_0 is a cscK metric in \mathcal{J}^{int} . Then there exist $\delta_2 > \delta_1 > 0$, such that for any $J(0) \in \mathcal{U}_k^{\delta_1}$, the Calabi flow J(t)(t > 0) starting from J(0) will stay in $\mathcal{U}_k^{\delta_2}$ all the time.

Proof. Choose $\delta > 0$ such that the previous a priori estimates hold in \mathcal{U}_k^{δ} . If suffices to prove that there exists $\delta_1 < \delta_2 < \delta$ such that for any Calabi flow J(t) with $J(0) \in \mathcal{U}_k^{\delta_1}$, if $J(t) \in \mathcal{U}_k^{\delta}$ for $t \in [0, T)$, then $J(T) \in \mathcal{U}_k^{\delta_2}$. By lemma 5.2, for $t \geq 1$ and l, we have

$$||Rm(J(t))||_{C^{l,\lambda}_{\iota}} \le C(l).$$

Now fix $\beta \in (2 - \frac{1}{\alpha}, 1)$, for any p, there is an N(p) (independent of $t \ge 1$), such that the following interpolation inequality holds

$$||\dot{J}(t)||_{L^{2}_{p}(t)} \leq C(p) \cdot ||\dot{J}(t)||^{\beta}_{L^{2}(t)} \cdot ||\mathcal{D}_{J}S(J)||^{1-\beta}_{L^{2}_{N(p)}(t)} \leq C(p) \cdot ||\dot{J}(t)||^{\beta}_{L^{2}(t)},$$

So by (18) we have

$$\int_{1}^{T} ||\dot{J}(t)||_{L_{p}^{2}(t)} dt \leq C(p) \cdot Ca(J(1))^{1-(2-\beta)\alpha} \leq C(p) \cdot Ca(J(0))^{1-(2-\beta)\alpha} \leq C(p) \cdot \epsilon(\delta_{1}).$$

Since the Sobolev constant is uniformly bounded in \mathcal{U}_k^{δ} , we obtain for any l,

$$\int_{1}^{T} ||\dot{J}(t)||_{C_{t}^{l,\lambda}} dt \leq C(l) \cdot \epsilon(\delta_{1}).$$

Therefore,

$$||J(T) - J(1)||_{C^{k,\lambda}} \le \int_{1}^{T} ||\dot{J}(t)||_{C^{k,\lambda}} dt \le \epsilon(\delta_1)$$

By the finite time stability of the Calabi flow, we have

$$||J(1) - J_0||_{C^{k,\lambda}} = \epsilon(\delta_1).$$

Thus

$$||J(T) - J_0||_{C^{k,\lambda}} \le \epsilon(\delta_1).$$

Now choose $\delta_2 = \frac{\delta}{2}$, and $\epsilon(\delta_1) \leq \delta_2$, then the theorem is concluded. \Box

From theorem 5.8, we know the Calabi flow exists globally in $C^{k,\lambda}$ and thus by sequence converges to J_{∞} in $C^{k,\beta}$ for $\beta < \alpha$. Now again by the Lojasiewicz arguments we see the limit must be unique and the convergence is in a polynomial rate in $C^{k,\lambda}$.

Now we assume that $J_{\infty} = J_0$ is smooth. Then we can prove smooth convergence. We first use the ellipticity to obtain a priori estimates in \mathcal{U}_k^{δ} for $k \gg 1$. Any $\mu \in \mathcal{U}_k^{\delta}$ satisfies the following elliptic system:

$$\begin{cases}
Im \mathcal{D}_0^* \mu = S(\mu) + O(||\mu||_{L_2^2}^2), \\
Re \mathcal{D}_0^* \mu = Q_0^*(\mu), \\
\bar{\partial}\mu + [\mu, \mu] = 0.
\end{cases}$$
(20)

So we have the following a priori estimate:

$$||\mu||_{C^{l+2,\alpha}} \le C \cdot (||\mu||_{C^{l,\lambda}} + ||S(\mu)||_{C^{l,\lambda}} + ||Q_0^*(\mu)||_{C^{l,\lambda}}).$$
(21)

From the proof of theorem 5.8, we know that $||\mu(t)||_{C^{k,\lambda}}$ and $||S(\mu(t))||_{C^{k,\lambda}}$ are uniformly bounded. Since

$$||Q_0^*(\mu(t))||_{C^{k,\lambda}} \le \int_t^\infty ||Q_0^*(\dot{\mu}(s))||_{C_s^{k,\lambda}} ds \le \epsilon(Ca(J(s)))$$

is bounded, we obtain $||\mu(t)||_{C^{k+2,\alpha}}$ bound, so we can derive smooth convergence by bootstrapping argument. This finishes the proof of theorem 5.3.

Theorem 5.3 has its own interest. This yields a purely analytical proof of an extension of a theorem due to Chen [Ch4] and Székelyhidi [Sz]. This is inspired by an observation of Tosatti [To]. In particular, we do not require the Kähler class to be integral. **Theorem 5.9.** ([Ch4]) For any $J \in U$, the Mabuchi functional E on the space of Kähler metrics compatible with J is bounded below, and the lower bound is achieved by the infimum along the Calabi flow initiating from J.

Proof. From the proof of theorem 5.3 we know the Calabi flow $J(t) \in \mathcal{J}^{int}$ starting from J converges to a limit J_{∞} with estimate

$$Ca(J(t)) \le C \cdot (t+1)^{-\frac{1}{2\alpha-1}}.$$

By lemma 5.1, this is equivalent to the Calabi flow $\phi(t)$ in the space of Kähler metrics compatible with J. Then

$$E(\phi(t)) = E(\phi(0)) - \int_0^t Ca(\phi(s)) ds \ge E(\phi(0)) - C \cdot \frac{2\alpha - 1}{2\alpha - 2} \cdot [1 - (t+1)^{\frac{2\alpha - 2}{2\alpha - 1}}] \ge -C'.$$

For any other Kähler potential ϕ , we have by lemma 2.4 that

$$E(\phi) \ge E(\phi(t)) - \sqrt{Ca(\phi(t))} \cdot d(\phi, \phi(t)).$$

Since

$$d(\phi,\phi(t)) \le d(\phi,\phi(0)) + d(\phi(0),\phi(t)) \le C + \int_0^t \sqrt{Ca(\phi(s))} ds \le C \cdot [1 + (t+1)^{\frac{4\alpha-3}{4\alpha-2}}],$$

we have

$$E(\phi) \ge \liminf_{t \to \infty} E(\phi(t)) - C \cdot (t+1)^{-\frac{1}{4\alpha-2}} \cdot [1 + (t+1)^{\frac{4\alpha-3}{4\alpha-2}}] = \lim_{t \to \infty} E(\phi(t))$$

is bounded below.

6 Reduced Calabi flow

In this section we shall discuss a reduced finite dimensional problem. The usual Kuranishi method provides a local slice as follows. Assume J_0 is cscK. We have as before the following elliptic complex:

$$C_0^{\infty}(M; \mathbb{C}) \xrightarrow{\mathcal{D}_0} T_{J_0}\mathcal{J} = \Omega_S^{0,1}(T^{1,0}) \xrightarrow{\partial_0} \Omega_S^{0,2}(T^{1,0}).$$

Let $\Box_0 = \mathcal{D}_0 \mathcal{D}_0^* + (\bar{\partial}_0^* \bar{\partial}_0)^2$, and $H^1 = Ker \Box_0$. Let G be the isotropy group of J_0 , which is the group of Hamiltonian isometries of (M, ω, J_0) , with Lie algebra $\mathfrak{g} = Ker \mathcal{D}_0 \cap C_0^{\infty}(M; \mathbb{R})$. By the classical Matsushima-Lichnerowicz theorem, $Ker \mathcal{D}_0$ is the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} , and so the complexification $G^{\mathbb{C}}$ of G is a subgroup of the group of holomorphic transformations of (M, J_0) , with Lie algebra $\mathfrak{g}^{\mathbb{C}} = Ker \mathcal{D}_0$. Then the linear G action on H^1 extends to an action of $G^{\mathbb{C}}$. For convenience, we include a proof of the following standard fact. **Lemma 6.1.** (Kuranishi) There exists a neighborhood B of 0 in H^1 , and a G-equivariant holomorphic embedding

$$\Phi: B \to \mathcal{J}$$

such that:

(1). $\Phi(0) = J_0;$

(2). If v_1 and v_2 in B are in the same $G^{\mathbb{C}}$ orbit and $\Phi(v_1)$ is integrable, then $\Phi(v_2)$ is integrable, and $\Phi(v_1)$ and $\Phi(v_2)$ are in the same $\mathcal{G}^{\mathbb{C}}$ leaf. Conversely, if $\Phi(v)$ is integrable and $(d\Phi)_v(u)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf at $\Phi(v)$, then u is tangent to the $\mathcal{G}^{\mathbb{C}}$ orbit at v.

(3). Any integrable J sufficiently close to J_0 lies in the $\mathcal{G}^{\mathbb{C}}$ leaf of some element in the image of Φ .

Proof. We can identify any J close to J_0 with an element μ in $\Omega_S^{0,1}(T^{1,0})$, and J is integrable if and only if

$$N(\mu) = \bar{\partial}_0 \mu + [\mu, \mu] = 0.$$

We can first choose a *G*-equivariant holomorphic embedding Ψ from a ball *B* in $\Omega_S^{0,1}(T^{1,0})$ into \mathcal{J} with $d\Psi_0 = Id$, by using the same "average trick" as in the proof of lemma 4.9. Let

$$V = \{ \mu \in \Omega_S^{0,1}(T^{1,0}) | \mathcal{D}_0^* \mu = 0 \},\$$

and

$$U = \{\mu \in \Omega_S^{0,1}(T^{1,0}) | N(\mu) = 0, \mathcal{D}_0^* \mu = 0\}.$$

Denote by G the Green operator for \Box_0 and $H : \Omega_S^{0,1}(T^{1,0}) \to H^1$ the orthogonal projection. Then for any $\mu \in U$, we have

$$\mu = G\Box_0\mu + H\mu = -G\bar{\partial}_0^*\bar{\partial}_0\bar{\partial}_0^*[\mu,\mu] + H\mu.$$

Define a G-equivariant map

$$F: \Omega_S^{0,1}(T^{1,0}) \to \Omega_S^{0,1}(T^{1,0}); \mu \mapsto \mu + G\bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [\mu, \mu],$$

where both spaces are endowed with the Sobolev L_k^2 norm. Its derivative at 0 is the identity map, so by the implicit function theorem, there is an inverse holomorphic map $F^{-1}: V_1(\subset \Omega_S^{0,1}(T^{1,0})) \to V_2(\subset \Omega_S^{0,1}(T^{1,0}))$. Let Qbe restriction of F^{-1} on $B = V_1 \cap H^1$ and Φ be the composition

$$\Phi: B \to \mathcal{J}; v \mapsto \Psi \circ Q(v).$$

Since H^1 consists of smooth elements, the image of Φ also consists of smooth elements.

Now we check Φ is the desired map. For any $v \in B$, we have

$$\mathcal{D}_0^*Q(v) = -\mathcal{D}_0^*G\bar{\partial}_0^*\bar{\partial}_0\bar{\partial}_0^*[Q(v),Q(v)] = 0,$$

and

$$N(Q(v)) = -\bar{\partial}_0 G \bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [Q(v), Q(v)] + [Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] + [Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v)] = G (\bar{\partial}_0^* \bar{\partial}_0)^2 [Q($$

So N(Q(v)) = 0 if and only if H[Q(v), Q(v)] = 0, as in [Ku]. Therefore a neighborhood of 0 in U is an analytic set contained in the image of Q. Since both Ψ and F are G-equivariant and holomorphic, the first part of (2) is true. Following [Sz], we define a map P from a neighborhood of $(J_0, 0)$ in $\mathcal{J} \times C_0^{\infty}(M; \mathbb{C})$ to \mathcal{J} as follows. Given $\mu \in \Omega_S^{0,1}(T^{1,0})$ representing an element in \mathcal{J} close to J_0 , and $\phi = \phi_1 + \sqrt{-1}\phi_2 \in C_0^{\infty}(M; \mathbb{C})$ small. There is a family of Hamiltonian diffeomorphisms f_t with

$$\dot{f}_t = X_{\phi_1}$$

Denote $J_1 = f_1^* J$. Since $\omega_{\phi} = \omega + \sqrt{-1} dJ_1 d\phi_2$ is isotopic to ω through the path $\omega_t = (1-t)\omega + t\omega_{\phi_2}$. Then there is a canonical path of diffeomorphisms g_t such that $g_t^* \omega_t = \omega$. Now $g_1^* J_1$ is the image under Ψ of an element $\mu_1 \in \Omega_S^{0,1}(T^{1,0})$. Then define

$$P(\mu, \phi) = G\mathcal{D}_0^* \mu_1.$$

Then P is a smooth function from $L^2_k(V) \times L^2_k(M; \mathbb{C})$ to the orthogonal complement $L^2_k(A^0)$ of $\mathfrak{g}^{\mathbb{C}}$ in $L^2_k(M; \mathbb{C})$. It is easy to calculate the derivative of P at $(J_0, 0)$ is

$$(DP)_0(\nu,\psi) = G\mathcal{D}_0^*\nu + G\mathcal{D}_0^*\mathcal{D}_0\psi.$$

The derivative with respect to the second variable is surjective with a finite dimensional kernel $0 \times \mathfrak{g}^{\mathbb{C}}$. Thus by implicit function theorem, any integrable complex structure close to J_0 lies in the $\mathcal{G}^{\mathbb{C}}$ leaf of an element in U, and thus is contained in the $\mathcal{G}^{\mathbb{C}}$ leaf of the image of Φ . So (3) is proved.

It suffices to prove the last statement in (2). Suppose $\mu = \Phi(v)$, and $\nu = (d\Phi)_v(u)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf, i.e $\nu = \mathcal{D}_{\mu}\phi$ for some complex valued function ϕ . Then $DP_{(\mu,0)}(0,\phi) = 0$. On the other hand, the kernel of $DP_{(\mu,0)}(0,-)$ has the same dimension as dim $\mathfrak{g}^{\mathbb{C}}$ if μ is sufficiently close to zero. Thus, $\phi \in \mathfrak{g}^{\mathbb{C}}$ and u is tangent to the $G^{\mathbb{C}}$ orbit of v.

By [D1] the action of \mathcal{G} on \mathcal{J} has a moment map given by the scalar curvature functional $\mu = S - \underline{S} : \mathcal{J} \to C_0^{\infty}(M; \mathbb{R})$. The downward gradient flow of $|\mu|^2$ is just the Calabi flow. Now we reduce this flow to a finite dimensional flow. Note G as a subgroup of \mathcal{G} acts on \mathcal{J} with induced moment map $\bar{\mu} = \Pi_{\mathfrak{g}}(S - \underline{S})$. It is the L^2 projection of μ to \mathfrak{g} with respect to the natural volume form. We can consider the gradient flow of $|\bar{\mu}|^2$, whose equation reads

$$\frac{d}{dt}J = -\frac{1}{2}J\mathcal{D}_J\bar{\mu}(J).$$
(22)

If we have a solution to equation (22) such that J_t is integrable for all $t \in [0, T]$, then we can translate it to a flow in \mathcal{H} given by

$$\frac{d}{dt}\phi = \Pi_{f_t^*\mathfrak{g}}(S(\phi) - \underline{S}), \qquad (23)$$

where f_t is the family of diffeomorphism satisfying

$$\frac{d}{dt}f_t = -\frac{1}{2}J_t X_{S(J_t)},$$

and the projection is taken with respect to the volume form of $f_t^*\omega$. We will study the relation between this flow and the Calabi flow later on. Let us call the flow (22) or (23) the *reduced Calabi flow*. It is the gradient flow of the norm squared of the moment map of a finite dimensional compact group action.

Now we can pull back the Kähler structure on \mathcal{J} to B, denoted by $(\hat{\Omega}, \hat{J})$. By the previous lemma, we know G acts on (B, Ω, J) holomorphically and isometrically, with moment map $\tilde{\mu}$ equal to $\Phi^* \bar{\mu}$. We can then study the reduced Calabi flow on a finite dimensional ambient space B. Let J be an integrable complex structure J close to J_0 such that the Calabi flow J(t)converges to J_0 . Suppose J_0 is not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J. By property (3) in lemma 6.1, we can smoothly perturb J(t) to $\bar{J}(t)$ in the $\mathcal{G}^{\mathbb{C}}$ orbit such that $\overline{J}(t) = \Phi(v(t))$ for $v(t) \to 0 \in B$. Since $\overline{J}(t)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf, by property (2) in lemma 6.1, we see that $\dot{v}(t)$ is tangent to the $G^{\mathbb{C}}$ orbit. So v is de-stabilized by 0 in B under the $G^{\mathbb{C}}$ action. By our previous study of the finite dimensional case, the reduced Calabi flow starting from v exists for all time and converges to 0 in the order $O(t^{-\frac{1}{2}})$, and the corresponding flow $\hat{J}(t)$ in $G^{\mathbb{C}}/G$ is asymptotic to a rational geodesic ray χ which also degenerate v to zero. We can view χ as a geodesic ray in \mathcal{H} as well, so the reduced Calabi flow in \mathcal{H} is asymptotic to a smooth geodesic ray with the same degeneration limit. This needs a bit more clarification. First of all, for any element g in $G^{\mathbb{C}}$, one can choose a path g(t) in $G^{\mathbb{C}}$ with g(0) equal to identity and g(1) = g. Then we have

$$\frac{d}{dt}g(t) \cdot g(t)^{-1} = \xi(t) + \sqrt{-1}\eta(t).$$

We can choose a path h(t) in G with h(0) being identity, such that

$$\frac{d}{dt}(h(t)g(t)) \in \sqrt{-1}\mathfrak{g}$$

This is equivalent to

$$\frac{d}{dt}h(t) \cdot h(t)^{-1} + h(t)\xi(t)h(t)^{-1} = 0.$$

Now we define a map F from an open set in $G^{\mathbb{C}}/G$ to \mathcal{H} as follows. This open set is a geodesic convex open set \mathcal{U} in $G^{\mathbb{C}}/G$ such that [g].v still lies in the previously constructed Kuranishi slice. Let v(t) = g(t).v, and $J(t) = \Phi(v(t))$. Then J(t) are all integrable and

$$\frac{d}{dt}J(t) = -(\mathcal{D}_{J(t)}\xi(t) + J(t)\mathcal{D}_{J(t)}\eta(t)),$$

where $\xi(t)$ and $\eta(t)$ are viewed as functions on M through the inclusion $\mathfrak{g} \subset C_0^{\infty}(M;\mathbb{R})$. Choose an isotopy of Hamiltonian diffeomorphisms f_t such that

$$\frac{d}{dt}f_t = X_{\xi(t)}.$$

Then $\widetilde{J}(t) = f_t^* J(t)$ satisfies

$$\frac{d}{dt}\widetilde{J}(t) = \widetilde{J}(t)\mathcal{D}_{\widetilde{J}(t)}\widetilde{\eta}(t),$$

where $\tilde{\eta}(t) = f_t^* \eta(t)$. In fact, $\tilde{J}(t) = \Phi(h(t)g(t).v)$. Then by Lemma 5.1 if we choose an isotopy of diffeomorphisms k_t with

$$\frac{d}{dt}k_t = -\nabla_{\widetilde{J}(t)}\widetilde{\eta}(t),$$

then

$$k_t^*\widetilde{J}(t) = J$$

and $k_t^*\omega = \omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi(t)$. We define F([g]) to be $\phi(1)$. Of course we need to show this is well-defined, it suffices to show the definition is independent of the path chosen in $G^{\mathbb{C}}/G$. Since $G^{\mathbb{C}}/G$ is always simply connected, we only to show it is invariant under based homotopy. Fo this, we choose a two parameter family $g_{s,t}$ in $G^{\mathbb{C}}$ such that $g_{s,0}$ is equal to identity, and $g_{s,1} = g$. Correspondingly we have h(s,t) in G with h(s,0) equal to identity. Let $\tilde{g}_{s,t} = h_{s,t} \cdot g_{s,t}$, then we have

$$\frac{\partial}{\partial t}\widetilde{g}_{s,t}\cdot\widetilde{g}_{s,t}^{-1}=\sqrt{-1}\eta(s,t)\in\sqrt{-1}\mathfrak{g}.$$

Also we have

$$\frac{\partial}{\partial s}\widetilde{g}_{s,t}\cdot\widetilde{g}_{s,t}^{-1} = \xi(s,t) + \sqrt{-1}\zeta(s,t) \in \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}.$$

So we have the relation

$$\sqrt{-1}\frac{\partial}{\partial s}\eta(s,t) = \frac{\partial}{\partial t}\xi(s,t) + \sqrt{-1}\frac{\partial}{\partial t}\zeta(s,t) + [\sqrt{-1}\eta(s,t),\xi(s,t) + \sqrt{-1}\zeta(s,t)].$$

In particular

$$\frac{\partial}{\partial s}\eta(s,t) = \frac{\partial}{\partial t}\zeta(s,t) + [\eta(s,t),\xi(s,t)].$$

Also $\xi(s,0) = \zeta(s,0) = \xi(s,1) = \zeta(s,1) = 0$. Let $J_{s,t} = \Phi(g_{s,t}.v)$, and $f_{s,t}$ be the two parameter family of diffeomorphisms obtained by fixing s and integrate along the t direction as before. In particular, f(s,0) is equal to identity for all s. We compute

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}f_{s,t}^*\omega = -\frac{\partial}{\partial s}f_{s,t}^*dJ_{s,t}d\eta(s,t) = -\frac{\partial}{\partial s}dJdf_{s,t}^*\eta(s,t).$$

We have

$$\begin{aligned} &\frac{\partial}{\partial s} f_{s,t}^* \eta(s,t) \\ &= f_{s,t}^* (\frac{\partial}{\partial s} \eta(s,t) + \mathcal{L}_{J_{s,t} \nabla_{s,t} \xi(s,t) - \nabla_{s,t} \zeta(s,t)} \eta(s,t)) \\ &= f_{s,t}^* (\frac{\partial}{\partial s} \eta(s,t) + \{\xi(s,t), \eta(s,t)\} - \langle \nabla_{s,t} \zeta(s,t), \nabla_{s,t} \eta(s,t) \rangle) \\ &= f_{s,t}^* (\frac{\partial}{\partial t} \zeta(s,t) - \langle \nabla_{s,t} \zeta(s,t), \nabla_{s,t} \eta(s,t) \rangle)) \\ &= \frac{\partial}{\partial t} F_{s,t}^* \zeta(s,t). \end{aligned}$$

Thus

$$\frac{\partial}{\partial s}|_{t=1}f_{s,t}^*\omega = -dJd(\int_0^1 \frac{\partial}{\partial t}f_{s,t}^*\zeta(s,t)dt) = -dJd(f_{s,1}^*\zeta(s,1)) = 0.$$

Thus the map F depends only on the point [g], not on the path chosen. So F is a well-defined smooth map. From this it is clear that F is a local isometric embedding, in particular, the image is totally geodesic. Thus we have proved that the reduced Calabi flow in \mathcal{H} is asymptotic to a smooth geodesic ray with the same degeneration limit. By Section 4.2 this geodesic ray is indeed rational, i.e. extends to a \mathbb{C}^* action. Then it follows from arguments in [Sz] that χ is tamed by a smooth test configuration, so it is tamed by a bounded geometry in the sense of [Ch3].

To prove that the Calabi flow is asymptotic to the reduced Calabi flow, we need to generalize lemma 4.6 to the infinite dimensional case. Then by the same argument as before, together with lemma 2.5 that the Mabuchi functional is weakly convex, one can show **Lemma 6.2.** Let $\hat{J}(t)$ be the reduced Calabi flow as before and $\hat{\phi}(t)$ be the corresponding flow in \mathcal{H} . Then for any Calabi flow path $\phi(t) \in \mathcal{H}$, we have for all t that

$$d(\phi(t), \dot{\phi}(t)) \le C.$$

The proof will be given in the appendix. Combining all these we arrive at the following theorem:

Theorem 6.3. Let (M, ω_0, J_0) be a csc Kähler manifold. Let J be a complex structure in \mathcal{J} close to J_0 and the Calabi flow starting from J converges to J_0 at the infinity. Suppose J_0 is not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J. Then there is a smooth geodesic ray $\phi(t)$ in the space of Kähler metrics $\mathcal{H}_{\omega,J}$ which is tamed by bounded geometry and degenerates J to J_0 in the space \mathcal{J} . Furthermore, $\phi(t)$ is asymptotic to the Calabi flow with respect to the Mabuchi-Semmes-Donaldson metric in the sense of definition 4.2.

7 Relative Bound for parallel Geodesic rays

It is well-known that in a Riemannian manifold with non-positive curvature, the distance between two geodesics is a convex function. In this section we first justify this property for the infinite dimensional space \mathcal{H} .

Lemma 7.1. Let $\phi_1(t)$ and $\phi_2(t)$ be two $C^{1,1}$ geodesics in \mathcal{H} , then $d(\phi_1(t), \phi_2(t))$ is a convex function of t.

Proof. We first assume both geodesics are C^{∞} . Let $\gamma_{\epsilon}(t,s)$ be the ϵ -geodesic connecting $\gamma_1(t)$ and $\gamma_2(t)$ (see [Ch1]), then

$$\frac{d^2}{dt^2}L(\gamma_{\epsilon}(t)) = \int_0^1 \frac{1}{|\gamma_{\epsilon,s}|} \{|\gamma_{\epsilon,ts}^{\perp}|^2 - R(\gamma_{\epsilon,s},\gamma_{\epsilon,t})\} ds + \frac{1}{|\gamma_{\epsilon,s}|} \langle \gamma_{\epsilon,s},\gamma_{\epsilon,tt} \rangle|_0^1 \\ - \int_0^1 \frac{\langle \gamma_{\epsilon,ss},\gamma_{\epsilon,tt} \rangle}{|\gamma_{\epsilon,s}|} + \frac{\langle \gamma_{\epsilon,s},\gamma_{\epsilon,ss} \rangle \langle \gamma_{\epsilon,s},\gamma_{\epsilon,tt} \rangle}{|\gamma_{\epsilon,s}|^3} ds$$

Along the ϵ -geodesics, we have

$$|\gamma_{\epsilon,ss}| = \sqrt{\int_0^1 (\phi_{\epsilon,ss} - \nabla_{\phi_{\epsilon,s}} \phi_{\epsilon,s})^2 \omega_{\phi_{\epsilon}}^n} \le C(t)\sqrt{\epsilon},$$

where C(t) is uniformly bounded if t varies in a bounded interval. Also

$$|\gamma_{\epsilon,tt}| \le C(t)$$

and

$$\gamma_{\epsilon,s}| \to L_t$$

uniformly for $s \in [0, 1]$ and t bounded. Therefore, we have

$$\frac{d^2}{dt^2}L(\gamma_{\epsilon}(t)) \ge -C(t)\sqrt{\epsilon},$$

so for any $a \leq b$,

$$L_{\epsilon}(ta + (1-t)b) \le tL_{\epsilon}(a) + (1-t)L_{\epsilon}(b) + C\sqrt{\epsilon}(t-a)(b-t).$$

Let $\epsilon \to 0$,

$$L(ta + (1 - t)b) \le tL(a) + (1 - t)L(b).$$

So L(t) is still a convex function, and the argument of the lemma yields the same conclusion.

In the general case we need to define the distance between two $C^{1,1}$ potentials, which is just the infimum of the length of all $C^{1,1}$ paths connecting the two points. Clearly the distance between any two points is always non-negative.

Now we assume ϕ_1 and ϕ_2 are $C^{1,1}$ but $\phi_i(0)$ and $\phi_i(1)$ are smooth, we want to prove for $t \in [0, 1]$,

$$L(t) \le (1-t)L(0) + tL(1).$$
(24)

To prove this, choose a δ -geodesic ϕ_{δ}^i approximating ϕ_i with endpoints fixed. Let $\phi_{\epsilon,\delta}(t,s)$ be the geodesic connecting $\phi_{\delta}^1(t)$ and $\phi_{\delta}^2(t)$, and $L_{\epsilon,\delta}(t)$ be its length. Then similar calculation shows that

$$\frac{d^2}{dt^2}L_{\epsilon,\delta}(t) \ge -C\sqrt{\delta} - C(\delta,t)\sqrt{\epsilon}$$

 So

$$L_{\epsilon,\delta}(t) \le (1-t)L_{\epsilon,\delta}(0) + tL_{\epsilon,\delta}(1) + \frac{1}{2}(C\sqrt{\delta} + C(\delta,t)\sqrt{\epsilon})t(1-t).$$

Let $\epsilon \to 0$, we have

$$L_{\delta}(t) \le (1-t)L_{\delta}(0) + tL_{\delta}(1) + C\sqrt{\delta}.$$

Let $\delta \to 0$, we get the desired inequality. So the theorem is true in this case.

If $\phi_i(0)$ and $\phi_i(1)$ are not assumed to be smooth, we can approximate them weakly in $C^{1,1}$ by smooth potentials $\phi_i^{\epsilon}(0)$, $\phi_i^{\epsilon}(1)$ respectively. Let $\phi_i^{\epsilon}(t)$ be the geodesic connecting $\phi_i^{\epsilon}(0)$ and $\phi_i^{\epsilon}(1)$. Then we know $d(\phi_1^{\epsilon}(t), \phi_2^{\epsilon}(t))$ is a convex function. By maximum principle for the Monge-Ampère equations, we know

$$\phi_i^{\epsilon}(t) - \phi_i(t)|_{C^0} \le \max(|\phi_i^{\epsilon}(0) - \phi_i(0)|_{C^0}, |\phi_i^{\epsilon}(1) - \phi_i(1)|_{C^0}).$$

Hence $|\phi_i^{\epsilon}(t) - \phi_i(t)|_{C^0} \to 0$, in particular, $d(\phi_i^{\epsilon}(t), \phi_i(t)) \to 0$. Therefore, $d(\phi_1^{\epsilon}(t), \phi_2^{\epsilon}(t))$ converges uniformly to $d(\phi_1(t), \phi_2(t))$. So the latter is also convex.

Lemma 7.2. If ϕ_1 is in $\mathcal{H}(i.e. \phi_1 \text{ is smooth and } \omega_1 \text{ is positive})$ and ϕ_2 is $C^{1,1}$, then $d(\phi_1, \phi_2) = 0$ if and only if $\phi_1 = \phi_2$.

Proof. We can choose C^{∞} potential ϕ_2^{ϵ} converging to ϕ_2 weakly in $C^{1,1}$ as $\epsilon \to 0$. Then by [Ch1],

$$d(\phi_1, \phi_2^{\epsilon}) \ge \max(\int_{\phi_1 \ge \phi_2^{\epsilon}} (\phi_1 - \phi_2^{\epsilon}) \omega_{\phi_1}^n, \int_{\phi_2^{\epsilon} \ge \phi_1} (\phi_2^{\epsilon} - \phi_1) \omega_{\phi_2^{\epsilon}}^n)$$

Let $\epsilon \to 0$, we get

$$d(\phi_1, \phi_2) \ge \max(\int_{\phi_1 \ge \phi_2} (\phi_1 - \phi_2)\omega_1^n, \int_{\phi_2 \ge \phi_1} (\phi_2 - \phi_1)\omega_2^n).$$

So if $d(\phi_1, \phi_2) = 0$, then

$$\int_{\phi_1 \ge \phi_2} (\phi_1 - \phi_2) \omega_1^n = 0,$$

and

$$\int_{\phi_2 \ge \phi_1} (\phi_2 - \phi_1) \omega_2^n = 0.$$

The first equation implies $\phi_1 \leq \phi_2$. The second equation implies that

$$\int_{\phi_2 > \phi_1} \omega_2^n = 0.$$

Let $\Omega = \{x \in M | \phi_2(x) > \phi_1(x)\}$. Then by Stokes' formula,

$$\begin{split} \int_{\Omega} \omega_1^n &= \int_{\Omega} \omega_1^n - \omega_2^n \\ &= \int_{\Omega} \sqrt{-1} \partial \bar{\partial} (\phi_1 - \phi_2) \cdot \sum_{j=0}^{n-1} \omega_1^j \wedge \omega_2^{n-1-j} \\ &= \int_{\partial \Omega} \sqrt{-1} \bar{\partial} (\phi_1 - \phi_2) \cdot \sum_{j=0}^{n-1} \omega_1^j \wedge \omega_2^{n-1-j} \\ &= 0. \end{split}$$

So Ω is empty. Thus $\phi_1 = \phi_2$.

Corollary 7.3. Let ϕ_1 be a geodesic ray tamed by bounded geometry(see [Ch3]), and ϕ_2 another geodesic ray parallel to ϕ_1 with $\phi_2(0)$ smooth. Then $\phi_1 - \phi_2$ has a uniform relative $C^{1,1}$ bound(with respect to ω_{ϕ_1}).

Proof. By [Ch3], there is a $C^{1,1}$ geodesic ray ϕ_3 emanating from $\phi_2(0)$ such that $|\phi_3(t) - \phi_1(t)|_{C_{\phi_1}^{1,1}} \leq C$. Thus $d(\phi_2(t), \phi_3(t))$ is uniformly bounded. Since $\phi_2(0) = \phi_3(0)$, by lemma 7.1, $d(\phi_2(t), \phi_3(t)) = 0$. Lemma 7.2 then implies $\phi_2(t) = \phi_3(t)$. So $|\phi_2(t) - \phi_1(t)|_{C_{\phi_1}^{1,1}} \leq C$.

Corollary 7.4. Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth paths in \mathcal{H} with $d(\gamma_1(t), \gamma_2(t))$ uniformly bounded. Suppose $\phi(t)$ is a smooth geodesic ray in \mathcal{H} asymptotic to γ_1 , then it is also asymptotic to γ_2 .

Proof. Let $\gamma_i(t,s)$ be the geodesic connecting $\phi(0)$ and $\gamma_i(t)$ parametrized by arc-length. Fix s, by assumption, $d(\gamma_1(t,s),\phi(s)) \to 0$ as $t \to \infty$. So in particular, $d(\phi(0),\gamma_1(t)) \to \infty$. Suppose $d(\gamma_1(t),\gamma_2(t)) \leq C$. Choose T large enough so that $d(\phi(0),\gamma_1(T)) \gg s+C$. Then $d(\gamma_1(T,T-C),\gamma_2(T,T-C)) \leq$ 4C. By lemma 7.1, as $T \to \infty$,

$$d(\gamma_1(T,s),\gamma_2(T,s)) \le \frac{s}{T} \cdot 4C \to 0.$$

By definition, $\phi(t)$ is asymptotic to γ_2 .

Similarly we can prove

Corollary 7.5. Let $\gamma(t)$ be a smooth path in \mathcal{H} which is asymptotic to two smooth geodesic rays $\phi_1(t)$ and $\phi_2(t)$. Then ϕ_1 and ϕ_2 are parallel, i.e. $d(\phi_1(t), \phi_2(t))$ is uniformly bounded. If we assume one of them is tamed by bounded geometry, say ϕ_1 then by corollary 7.3, $|\phi_1(t) - \phi_2(t)|_{C^{1,1}_{\phi_1}} \leq C$.

8 Proof of the main theorems

Now we proceed to prove the main theorems.

Lemma 8.1. Suppose g_i is a sequence of Riemmanian metrics on a manifold M. If there are two sequences f_i and h_i of diffeomorphism of M such that $f_i^*g_i \to g_1$, and $h_i^*g_i \to g_2$ in C^{∞} , then $f_i \circ h_i^{-1}$ converges by subsequence to a diffeomorphism f in C^{∞} with $f^*g_2 = g_1$.

The proof is standard using compactness. We omit it here.

Corollary 8.2. The quotient \mathcal{J}/\mathcal{G} is Hausdorff in the C^{∞} topology.



PSfrag replacements

Figure 2: Calabi flows and asymptotic geodesic rays

Lemma 8.3. (C^0 bound implies no Kähler collapsing) Suppose there are two sequences $\phi_i, \psi_i \in \mathcal{H}$ converging in the Cheeger-Gromov sense, i.e. there are two sequences of diffeomorphisms f_i, h_i such that

$$f_i^*(J,\omega_{\phi_i}) \to (J_1,\omega_1)$$

and

$$h_i^*(J,\omega_{\psi_i}) \to (J_2,\omega_2)$$

in the C^{∞} topology. If $|\phi_i - \psi_i|_{C^0} \leq C$, then $|\phi_i - \psi_i|_{C_{\omega_{\phi_i}}^k}$ is bounded for all k, and there is a subsequence k_i such that $f_{k_i}^{-1} \circ h_{k_i}$ converges in C^{∞} to a diffeomorphism f with $f^*J_1 = J_2$ and $f^*\omega_1 = \omega_2 + \sqrt{-1}\partial_{J_2}\bar{\partial}_{J_2}\phi$.

The proof is quite standard now, given the volume estimates in [CH1]. We will omit it here.

Proof. (of theorem 1.3). We may assume J_1 and J_2 are not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J, the proof in the other case is similar. We proceed by contradiction. Suppose J_1 and J_2 were not in the same \mathcal{G} orbit. Then by corollary 8.2 we can assume there are disjoint \mathcal{G} invariant neighborhoods \mathcal{U}_1 , \mathcal{U}_2 of J_1 , J_2 respectively. Pick J'_i in the intersection of \mathcal{U}_i with $\mathcal{G}^{\mathbb{C}}$ leaf of J. Now by theorem 5.3, we know that the Calabi flow $J_i(t)$ starting from J'_i exits globally and converges to $J_i(\infty) \in \mathcal{U}_i$. So $J_1(\infty)$ and $J_2(\infty)$ are not in the same \mathcal{G} -orbit either. By theorem 6.3, the corresponding Calabi flow $\phi_i(t)$ in the space of Kähler metrics is asymptotic to a smooth geodesic ray which also degenerates some other \hat{J}_i to $J_i(\infty)$. Since J'_1 and J'_2 are both in the $\mathcal{G}^{\mathbb{C}}$ leaf of J, we can pull everything back to J and then we have two Calabi flows $\phi_i(t)$ each asymptotic to a smooth geodesic ray $\chi_i(t)$ tamed by bounded geometry. By [CC], $d(\phi_1(t), \phi_2(t))$ is decreasing, so by corollary 7.4, $\phi_1(t)$ is also asymptotic to $\chi_2(t)$. By corollary 7.3 and corollary 7.5

$$|\chi_1(t) - \chi_2(t)|_{C^{1,1}} \le C.$$

So lemma 8.3 implies, there is no Kähler collapsing, and there is a diffeomorphism f with $f^*J_1(\infty) = J_2(\infty)$, and $f^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. Since $(f^*\omega, J_2(\infty))$ and $(\omega, J_2(\infty))$ are both csc Kähler structures in the same Kähler class, by theorem 1.1, there is a diffeomorphism h with $h^*J_2(\infty) = J_2(\infty)$ and $h^*f^*\omega = \omega$, so $(f \circ h)^*(\omega, J_1(\infty)) = (\omega, J_2(\infty))$. Contradiction.

Proof. (of theorem 1.6). Suppose $f_i^*(\omega_{\phi_i}, J) \to (\omega_1, J_1)$, and $h_i^*(\omega_{\psi_i}, J) \to (\omega_2, J_2)$. Since $[\omega]$ is integral, we see that $[f_i^*\omega_{\phi_i}] = [\omega_1]$ for *i* large enough, so we can further assume that $f_i^*\omega_{\phi_i} = \omega_1$, and $h_i^*\omega_{\psi_i} = \omega_2$. Then we can follow the proof of theorem 1.3.

Proof. (of corollary 1.7). Suppose $f_i^*J \to J_1$. Since $c_1(J_1) > 0$, we have $c_1(f_i^*J) > 0$, and we can choose a sequence of Kähler metrics ω_i in $c_1(J)$ such that $f_i^*\omega_i \to \omega_1$. Then we can apply theorem 1.6.

9 Further Discussions

There are also some further interesting questions.

Problem 9.1. A general notion of optimal degenerations and its relation to the Calabi flow. Generalize the theorem to the uniqueness of some "canonical" objects in the closure, allowing the occurrence of singularities. On the other hand, by the Yau-Tian-Donaldson conjecture, one would like to know if there is a direct algebraic-geometric counterpart of Theorem 1.3, i.e. whether a K-polystable adjacent Kähler structure is unique.

Problem 9.2. Quantization approach([D4], [Fi]). In the case of discrete automorphism group, Donaldson [D4] proved the existence of cscK metric implies asymptotic Chow stability. Theorem 1.1 in this case follows immediately. It looks like one can use the finite dimensional Kempf-Ness theorem to deal with Theorem 1.3 also. However, this can not be straightforward. The reason is that for an adjacent cscK Kähler structure whose underlying complex structure is different from the original one, the automorphism group can not be finite; and it is known that the existence of cscK metric(or even KE metric) does not necessarily imply asymptotic Chow poly-stability, see the recent counter-example in [OSY], [DZ]. It seems to the authors that more delicate work is required to proceed by the quantization method.

Problem 9.3. It follows from our result that Tian's conjecture in [Ti2] is likely to hold for cscK metrics (The original conjecture allows mild singularities). In the case of general extremal metrics, we might need to modify the statement in Tian's conjecture a bit. This can be easily seen in the corresponding finite dimensional analogue. In that case any gradient flow can be reversed and we can get critical points in the limit along both directions of the flow. Clearly they are not in the same G orbit and therefore "adjacent" critical point is not necessarily unique. In our infinite dimensional case, the naive uniqueness also fails for adjacent extremal metrics. Such examples were already implicit in Calabi's seminal paper [Ca2]. Namely, we consider the blown up of \mathbb{P}^2 at three distinct points p_1 , p_2 and p_3 (denoted by $Bl_{p_1,p_2,p_3}(\mathbb{P}^2)$), then by [APS], the class $\pi^*[\omega_{FS}] - \epsilon^2([E_1] + [E_2] + [E_3])$ contains extremal metrics for ϵ small enough. If p_1 , p_2 and p_3 are in general position (i.e. they do not lie on a line), then $Bl_{p_1,p_2,p_3}(\mathbb{P}^2)$ are all biholomorphic and by [Ca2] the classes $\pi^*[\omega_{FS}] - \epsilon^2([E_1] + [E_2] + [E_3])$ have vanishing Futaki invariant thus the extremal metrics are cscK. If p_1 , p_2 and p_3 lie on a line, Calabi pointed out in [Ca2] that there is no cscK metric due to the Lichnérowicz-Matsushima theorem. It is easy to see that for a fixed Kähler class $\pi^*[\omega_{FS}] - \epsilon^2([E_1] + [E_2] + [E_3])$, the extremal metrics in the case p_1 , p_2 and p_3 lie on a line are adjacent to the cscK metrics in the case p_1 , p_2 , p_3 are in general position. So we can find proper extremal metrics even adjacent to cscK metrics. C. Lebrun also pointed out to us another example, where we can look at the Hirzebruch surfaces F_{2n} of even degree. If n > m, then with appropriate polarization, F_{2n} is adjacent to F_{2m} , while in [Ca1], Calabi explicitly constructed extremal metrics in any Kähler classes.

The problem where the uniqueness fails can be seen from the fact that our proof depends on the Calabi flow in an essential way. Since the Calabi flow can only detect de-stabilizing extremal metrics, we might want to consider only the uniqueness of de-stabilizing(i.e. energy minimizing) extremal metrics, as a modification of Tian's conjecture. This idea of de-stabilizing extremal metrics has already been implicitly discussed in [Ch3].

Problem 9.4. The integrality assumption in theorem 1.6 is just for fixing the symplectic form. It seems possible to remove this assumption.

A Marle-Guillemin-Sternberg normal form

In this appendix, we shall give a proof of the Marle-Guillemin-Sternberg normal form theorem for a Hamiltonian group action in the finite dimensional case(lemma 4.6). We shall also consider an infinite dimensional case for our purpose(lemma 6.2). We suppose that there is a compatible complex structure, which in general we can not standardize without some "errors".

A.1 Model case

We first look at a prototype. Suppose ω is a Kähler metric defined in a neighborhood of 0 in \mathbb{C}^n . Then we can not trivialize both the complex structure and the symplectic structure simultaneously, however, we can make either of them standard, with appropriate control on the other.

First, it is easy choose a holomorphic coordinate such that

$$\omega = \omega_0 + O(|z|^2),$$

where ω_0 is the standard symplectic form on \mathbb{C}^n . In this way the complex structure is made standard, while the error on the symplectic form is quadratic.

Now we denote $\alpha = \omega - \omega_0$. Let $f_t : z \to tz$ be the contraction map. Then

$$\alpha = f_1^* \alpha - f_0^* \alpha = d\theta,$$

where $\theta = \int_0^1 f_t^* (X \lrcorner \alpha) dt$, and $X = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}$. So
 $\theta = O(|z|^3).$

Let $\omega_t = (1 - t)\omega + t\omega_0$, then $\phi_t^* \omega_t = \omega_0$, where ϕ_t is the isotopy generated by the vector fields Y_t satisfying

$$Y_t \lrcorner \omega_t = -\theta.$$

Thus, $Y_t = O(|z|^3)$ and so

 $\phi_t(z) = z + O(|z|^3),$

and

$$\phi_t^* J_0 - J_0 = O(|z|^2).$$

In this way the symplectic structure is standard, with an quadratic error on the complex structure.

Proof of lemma 4.5 A.2

Suppose a compact group G acts on a Kähler manifold (M, Ω, J) with moment map μ , and z_0 is a zero of the moment map, but not fixed by the whole group G. We denote by G_0 the isotropy group of z_0 and \mathfrak{g}_0 its Lie algebra. We also fix an Ad_G -invariant metric on \mathfrak{g} . Now consider $Y = G \times_{G_0} (\mathfrak{m} \oplus N)$. Here N is the orthogonal complement of $\mathfrak{g}.z_0$ in $(\mathfrak{g}.z_0)^{\omega_0}$, and \mathfrak{m} is the orthogonal complement of \mathfrak{g}_0 in \mathfrak{g} . We identify $\rho \in \mathfrak{m}$ with $\tilde{\rho} \in J_0 \cdot (\mathfrak{g}.z_0)$ through

$$\langle \rho, \eta \rangle = \Omega_0(\tilde{\rho}, X_\eta).$$
 (25)

This also induces an identification between \mathfrak{m} and $\mathfrak{g}.z_0$ which is different from the one coming from the action. G_0 acts on $(N, \Omega_N = \Omega_0|_N)$ linearly with a natural moment map μ_N . Y is in fact the symplectic quotient of $G \times (\mathfrak{g}_0 \oplus \mathfrak{m} \oplus N) \simeq T^*G \times N$ by G_0 . The induced symplectic form on Y is given explicitly by (see [OR])

$$\begin{split} \tilde{\Omega}_{[g,\rho,v]}((L_g\xi_1,\rho_1,v_1),(L_g\xi_2,\rho_2,v_2)) \\ &:= \langle \rho_2 + d_v\mu_N(v_2),\xi_1 \rangle - \langle \rho_1 + d_v\mu_N(v_1),\xi_2 \rangle + \langle \rho + \mu_N(v),[\xi_1,\xi_2] \rangle \\ &\quad + \Omega_0(X_{\xi_1},X_{\xi_2}) + \Omega_0(v_1,v_2) \\ &= \langle \rho_2 + d_v\mu_N(v_2),\xi_1 \rangle - \langle \rho_1 + d_v\mu_N(v_1),\xi_2 \rangle + \langle \rho,[\xi_1,\xi_2] \rangle + \Omega_0(v_1,v_2) + \langle \mu_N(v),[\xi_1,\xi_2] \rangle \\ &= \Omega_0(X_1,X_2) + \langle \rho,[\xi_1,\xi_2] \rangle + (\langle d_v\mu_N(v_2),\xi_1 \rangle - \langle d_v\mu_N(v_1),\xi_2 \rangle) + \langle \mu_N(v),[\xi_1,\xi_2] \rangle, \end{split}$$

where we identify $T_q G$ with \mathfrak{g} through left translation, and $X_i = X_{\xi_i} + \alpha_i + v_i$ is viewed as a tangent vector at z_0 . The G action on Y is Hamiltonian with moment map:

$$\tilde{\mu}: Y \to \mathfrak{g}; [g, \rho, v] \to Ad_q^*(\mu_N(v) + \rho).$$

To prove lemma 4.6, we need to trace the proof of the relative Darboux theorem. Since Ω is Kähler, we can choose holomorphic coordinates on a neighborhood V of z_0 such that $\Omega - \Omega_0 = O(r^2)$. Let exp_{z_0} be the exponential map with respect to the metric induced from J and Ω . Then we have

$$exp_{z_0}(\rho + v) = z_0 + \rho + v + O(r^3).$$

Consider the map

~

$$exp: G \times_{G_0} (\mathfrak{m} \oplus N) \to M; (\xi, \rho, v) \mapsto e^{\xi} . exp_{z_0}(\rho + v).$$

This is a diffeomorphism from a G-invariant neighborhood U of $G \times 0$ to a neighborhood V of $G.z_0$. Indeed, its derivative at [e, 0, 0] is given by

$$dexp_{z_0}: \mathfrak{m} \oplus \mathfrak{m} \oplus N \to T_{z_0}M = \mathfrak{m} \oplus \mathfrak{m} \oplus N; (\xi, \rho, v) \mapsto (L(\xi), \rho, v),$$

where we have made use of the identification (25), and $L: \mathfrak{m} \to \mathfrak{m}$ is the the automorphism such that

$$(L(\xi),\eta) = g_0(X_{\xi}, X_{\eta})$$

for any $\xi, \eta \in \mathfrak{m}$. Denote $\Omega' = exp^*\Omega$ and $J' = exp^*J$, then we have

$$J'_{(0,0,0)}(\xi,\rho,v) = (L^{-1}(\rho), -L(\xi), J_0 \cdot v).$$

We can extend J' to an almost complex structure \tilde{J} defined on Y.

On V, denote by (z, \bar{z}) the coordinates for N, x for $\mathfrak{g}.z_0$ and y for $W = J_0 \cdot (\mathfrak{g}.z_0)$. The tangent space at z_0 is naturally identified with V. Let $(\frac{\partial}{\partial v}, \frac{\partial}{\partial \bar{v}})$, and $\frac{\partial}{\partial \rho}$ be the vector fields on U corresponding to $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial y}$ (on $\mathfrak{m} \oplus N$) respectively and $\frac{\partial}{\partial \xi}$ the vector fields induced by left translation of $\frac{\partial}{\partial x} \in T_{z_0}V$. These vector fields could also be viewed as vector fields on V through the map exp. Then at $[e, \rho, v]$ we have

$$\begin{split} \frac{\partial}{\partial v} &= \frac{\partial}{\partial z} + O(r^2);\\ \frac{\partial}{\partial \bar{v}} &= \frac{\partial}{\partial \bar{z}} + O(r^2);\\ \frac{\partial}{\partial \rho} &= \frac{\partial}{\partial y} + O(r^2);\\ L\frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x} + \xi.y + \xi.z + O(r^2). \end{split}$$

Now it is easy to see that

$$\begin{split} \tilde{\Omega}(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}) &= \Omega'(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}) + O(r^2), \\ \tilde{\Omega}(\frac{\partial}{\partial z},\frac{\partial}{\partial z}) &= \Omega'(\frac{\partial}{\partial z},\frac{\partial}{\partial z}) + O(r^2) = O(r^2); \\ \tilde{\Omega}(\frac{\partial}{\partial z},\frac{\partial}{\partial y}) &= \Omega'(\frac{\partial}{\partial z},\frac{\partial}{\partial y}) + O(r^2) = O(r^2); \\ \tilde{\Omega}(\frac{\partial}{\partial y},\frac{\partial}{\partial y}) &= \Omega'(\frac{\partial}{\partial y},\frac{\partial}{\partial y}) + O(r^2) = O(r^2). \end{split}$$

and

$$\begin{split} \tilde{\Omega}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) &= \tilde{\Omega}(\frac{\partial}{\partial v} + O(r^2), L\frac{\partial}{\partial \xi} - \xi . y - \xi . z + O(r^2)) \\ &= \Omega'(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) + O(r); \end{split}$$

similarly,

$$\tilde{\Omega}(\frac{\partial}{\partial y},\frac{\partial}{\partial x}) = \Omega'(\frac{\partial}{\partial y},\frac{\partial}{\partial x}) + O(r);$$

$$\begin{split} \tilde{\Omega}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) &= \tilde{\Omega}(L\frac{\partial}{\partial \xi} - \xi . y - \xi . z + O(r^2), L\frac{\partial}{\partial \xi} - \xi . y - \xi . z + O(r^2)) \\ &= \Omega'(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + O(r). \end{split}$$

Therefore, we obtain:

$$\begin{split} &\alpha = \Omega' - \tilde{\Omega} = O(r^2)(dzd\bar{z} + dzdy + d\bar{z}dy + dydy) + O(r)(dzdx + d\bar{z}dx + dydx + dxdx). \end{split}$$
 Now let $f_t: (g,\rho,v) \to (g,t\rho,tv)$, then

$$X_t = \dot{f}_t = t\rho \frac{\partial}{\partial \rho} + tv \frac{\partial}{\partial v} + t\bar{v} \frac{\partial}{\partial \bar{v}} = ty \frac{\partial}{\partial y} + tz \frac{\partial}{\partial z} + t\bar{z} \frac{\partial}{\partial \bar{z}} + O(r^2).$$

We have

$$\alpha = d\theta,$$

with

$$\theta = \int_0^1 f_t^*(X_t \lrcorner \alpha) dt$$

=
$$\int_0^1 (ty \frac{\partial}{\partial y} + tz \frac{\partial}{\partial z} + t\bar{z} \frac{\partial}{\partial \bar{z}} + O(r^2)) \lrcorner [O(r^2)(dz d\bar{z} + dz dy + d\bar{z} dy + dy dy) + O(r)(dz dx + d\bar{z} dx + dy dx + dx dx)] dt$$

$$= O(r^2)dx + O(r^3),$$

where the estimate is valid at $[e, \rho, v]$. Let $\Omega_t = (1 - t)\tilde{\Omega} + t\Omega'$, then

$$\phi_t^*\Omega_t = \tilde{\Omega},$$

where $\dot{\phi}_t = Y_t$ satisfies

$$Y_t \lrcorner \Omega_t = \theta.$$

Since

 $\Omega_t=\Omega_0+O(r^2)(dzd\bar{z}+dzdy+d\bar{z}dy+dydy)+O(r)(dzdx+d\bar{z}dx+dydx+dxdx).$ So at $[e,\rho,v],$ we have

$$Y_t = O(r^2)\frac{\partial}{\partial y} + O(r^3) = O(r^2)\frac{\partial}{\partial \rho} + O(r^3).$$

Since Y_t is *G*-invariant, this is also true at $[g, \rho, v]$ for g close to Id. Thus the integral curve of Y_t satisfies

$$v_t = v_0 + O(r_0^3);$$

 $\rho_t = \rho_0 + O(r_0^2).$

Therefore,

$$(\phi_t^*J')\frac{\partial}{\partial v} = \phi_t^{-1}{}_*J'((\phi_t){}_*\frac{\partial}{\partial v}) = \phi_t^{-1}{}_*J'(\frac{\partial}{\partial v} + O(r^2)) = \phi_t^{-1}{}_*J(\frac{\partial}{\partial z} + O(r^2)) = \tilde{J}\frac{\partial}{\partial v} + O(r^2),$$

and similarly

$$(\phi_t^* J') \frac{\partial}{\partial \bar{v}} = \tilde{J} \frac{\partial}{\partial \bar{z}} + O(r^2).$$

Let $\Phi = \phi_1$, then $\Phi^* \Omega' = \tilde{\Omega}$. We get the required estimate that

$$\Phi^* J' - \tilde{J} = O(r),$$

and

$$\Phi^* J' \cdot X - \tilde{J} \cdot X = O(r^2)|X|,$$

for $X \in N$. Hence lemma 4.6 is proved.

A.3 Proof of lemma 6.2

Now we proceed to our infinite dimensional problem, following the same route as in the finite dimensional setting. However, there are a few more technical issues, as we shall see below. Suppose (M, ω, J_0) is a csc Kähler manifold. Then the relevant group \mathcal{G} is the group of Hamiltonian diffeomorphisms of (M, ω) , which acts on the space \mathcal{J} of almost complex structures compatible with ω . Here in order to apply the implicit function theorem, we shall put C^{∞} topology on these infinite dimensional objects which makes them into tame Fréchet spaces([Ha]). \mathcal{J} inherits a natural (weak) Kähler structure (Ω, I) from the original Kähler manifold M. The action of \mathcal{G} preserves the Kähler structure and has a moment map given by the Hermitian scalar curvature functional $m(J) = S(J) - \underline{S}$. Denote by G the identity component of the holomorphic isometry group of (M, ω, J_0) . Let \mathfrak{g} and \mathfrak{g}_0 the Lie algebra of \mathcal{G} , G respectively. Then we have an L^2 orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m},$$

where \mathfrak{m} is the image of $Q^* = Re\mathcal{D}^*$. We want to show that a neighborhood V of J_0 in \mathcal{J} is \mathcal{G} -equivariantly Hamiltonian diffeomorphic to a neighborhood U in

$$Y = \mathcal{G} \times_G (\mathfrak{m} \oplus N),$$

where \mathcal{G} acts adjointly on \mathfrak{g} by

$$f.\phi = f^*\phi.$$

N is the orthogonal complement of the image of \mathcal{D} in $\Omega^{0,1}(T^{1,0})$, and G acts on N by pulling back: $g.\mu = g^*\mu$. This action is Hamiltonian with moment map given by

$$m_N: N \to \mathfrak{g}_0; (m_N(v), \xi) = \frac{1}{2}\Omega(\xi.v, v)$$

Similar to the finite dimensional case we can define a (weak) symplectic form on U. The left G action on Y is Hamiltonian with moment map given by

$$\tilde{m}: [g, \rho, v] = g^*(\rho + m_N(v)).$$

The exponential map Ψ on \mathcal{J} with respect to the natural Riemannian metric is well defined by fiber-wise exponential map of the symmetric space Sp(2n)/U(n), and it is easy to see that it is a local tame embedding of a neighborhood of the origin in $\Omega_S^{0,1}(T^{1,0})$ into \mathcal{J} . Using the local holomorphic coordinate chart of \mathcal{J} , the Kähler form satisfies

$$\Omega_{\mu} = \Omega_0 + O(|\mu|^2).$$

It is also clear that

$$\Psi(\mu) = \mu + O(|\mu|^3).$$

Here the norms on both sides could be taken to be the same. Now we can define a map

$$\Phi: U \to \mathcal{J}; [g, \rho, v] \mapsto g^* \Psi(\rho + v).$$

Lemma A.1. \mathcal{G} is a smooth tame Lie group.

Proof. We first prove it is a smooth tame space. We can identify a Hamiltonian diffeomorphism H with an exact Lagrangian graph G_H in $\mathcal{M} = M \times M$, i.e.

$$G_H = \{(x, H(x)) | x \in M\}.$$

Here \mathcal{M} is endowed with a canonical symplectic form $\omega' = \pi_1^* \omega - \pi_2^* \omega$, where π_i is the projection map to the *i*-th factor. A Lagrangian graph is called *ex*act if it can be deformed by exact Lagrangian isotopies to the identity. We can construct local charts for \mathcal{G} as follows. Given any $H \in \mathcal{G}$, by Weinstein's Lagrangian neighborhood theorem ([We]), we can choose a symplectic diffeomorphism between a tubular neighborhood \mathcal{U} of G_H in \mathcal{M} and a tubular neighborhood \mathcal{V} of 0 section in the cotangent bundle T^*M . Then locally any Hamiltonian diffeomorphism close to H is represented by the graph of an exact one-form, i.e. the differential of some real valued function on M. So locally \mathcal{U} can be identified with an open subset of $C_0^{\infty}(M;\mathbb{R})$. Thus \mathcal{G} is modelled on $C_0^{\infty}(M;\mathbb{R})$. Now we check the transition function is smooth tame. In our case locally between any two charts there is a symplectic diffeomorphism of the cotangent bundle $F: T^*M \to T^*M$ which is identity on the zero section. Then the induced transition map is smooth tame, by observing that the C^k distance between the graph of exact one-forms $d\phi_1$ and $d\phi_2$ is equivalent to the C^{k+1} distance between ϕ_1 and ϕ_2 . Similarly we can prove that the group multiplication and inverses are both smooth tame.

Since the finite dimensional group G acts smooth tame and freely on $\mathcal{G} \times (\mathfrak{m} \oplus N)$, we know that

$$Y = \mathcal{G} \times_G (\mathfrak{m} \oplus N)$$

is a tame space with a smooth tame \mathcal{G} - action.

Lemma A.2. The G-equivariant map

$$\Phi: \mathcal{G} \times_G (\mathfrak{m} \oplus N) \to \mathcal{J}; [g, \rho, v] \mapsto g^* \Psi(\rho + v)$$

is smooth tame with a local smooth tame inverse around [Id, 0, 0].

Proof. It is clear by definition that the map is smooth and tame. The k-th derivative of Φ is tame of degree k+1. To apply Hamilton's implicit function theorem, we need to study the derivative of Φ near [Id, 0, 0]. At $\delta = [g, \rho, v]$, we denote $\mu = \Phi(\delta)$. Then we have

$$D_{\delta}\Phi: \qquad \mathfrak{m} \oplus \mathfrak{m} \oplus N \to \Omega_{S}^{0,1}(T^{1,0}); [\phi, \psi, u] \mapsto \\ (Id - \bar{\mu}) \circ (I - \mu \circ \bar{\mu})^{-1} \circ [Q_{\mu}\phi + g^{*}D\Psi|_{\rho+v}(\sqrt{-1}\mathcal{D}_{0}\psi + u)] \circ (Id - \mu)^{-1}.$$

To find the inverse to D_{δ} , we need to first decompose $\Omega_S^{0,1}(T^{1,0})$ into the direct sum of $D\Psi^{-1} \circ ImQ_{\mu}|_{\mathfrak{m}}$ and $KerQ_0^*$ with estimate. This can be done using elliptic theory. We can obtain that

$$\nu = (D\Psi)^{-1} \circ Q_\mu \phi + \sqrt{-1}Q_0 \psi + \eta_\mu$$

where $\eta \in KerD_0^*$. Take the map $P_{\mu} : \nu \mapsto (\phi, \sqrt{-1}Q_0\psi + \eta)$. Then it is smooth tame again by elliptic estimates. Since the inverse of $D_{\delta}\Phi$ is the combination of P with some other smooth tame operator, it is also smooth tame. Then we can apply the Nash-Moser implicit function theorem([Ha]) to conclude the lemma.

As in the finite dimensional case, there is a canonically defined (weak) symplectic form on \mathcal{U} given by

$$\begin{aligned} \Omega_{[g,\rho,v]}((L_g\xi_1,\rho_1,v_1),(L_g\xi_2,\rho_2,v_2)) \\ &:= \langle \rho_2 + d_v\mu_N(v_2),\xi_1 \rangle - \langle \rho_1 + d_v\mu_N(v_1),\xi_2 \rangle + \langle \rho + \mu_N(v),[\xi_1,\xi_2] \rangle + \Omega_0(v_1,v_2) \\ &= (\mathcal{D}_0^*\mathcal{D}_0\xi_1,\rho_2) - (\mathcal{D}_0^*\mathcal{D}_0\rho_1 + d_v\mu_N(v_1) - [\mathcal{D}_0^*\mathcal{D}_0\rho + \mu_N(v),\xi_1],\xi_2) + \Omega_0(v_1,v_2) + (\xi_1,d_v\mu_N(v_2)) \end{aligned}$$

By the above lemma we can pull back the symplectic form Ω and the complex structure I to \mathcal{U} , denoted by Ω' and I' respectively. There is also a canonical almost complex structure I_0 on \mathcal{U} defined by

$$I_0: \mathfrak{m} \oplus \mathfrak{m} \oplus N \to \mathfrak{m} \oplus \mathfrak{m} \oplus N; (\xi, \rho, v) \to ((\mathcal{D}_0^* \mathcal{D}_0)^{-1} \rho, -\mathcal{D}_0^* \mathcal{D}_0 \xi, I(0)(v)).$$

It is easy to see that $I' = I_0$ at [Id, 0, 0].

Proposition A.3. There are neighborhoods $\mathcal{U}_i, \mathcal{V}_i(i = 1, 2) (\mathcal{U}_2 \subset \mathcal{U}_1)$ of [Id, 0, 0] in Y and two \mathcal{G} -equivariant smooth tame maps

$$\Sigma_1: \mathcal{U}_1 \to \mathcal{V}_1,$$

 $\Sigma_2: \mathcal{V}_2 \to \mathcal{U}_2,$

which fixes the \mathcal{G} -orbit of [Id, 0, 0] such that $\Sigma_1 \circ \Sigma_2$ equal to the identity and such that

$$\Sigma_1^* \dot{\Omega} = \Omega',$$

$$\Sigma_2^* \Omega' = \tilde{\Omega},$$

and for any $X \in N$, and $[g, \rho, v] \in V_2$,

$$(D\Sigma_1) \circ I' \circ (D\Sigma_2)(X) - I_0(X) = O(r^2) \cdot |X|,$$

and at [Id, 0, 0],

$$(D\Sigma_1) \circ I' \circ (D\Sigma_2) = I_0.$$

Here the estimate is only in the tame sense, i.e. the norm on the left hand side might be weaker than that on the right, r is the norm of $[g, \rho, v]$.

Proof. The idea of the proof is the same as the finite dimensional case. The main difficulty is to show the existence of solutions to the involved O.D.E's in infinite dimension. Once this is established, then everything else will follow formally. First we have $(\mu = \Phi([Id, \rho, v]))$

$$\begin{aligned} \Omega'_{[g,\rho,v]}((L_g\xi_1,\rho_1,v_1),(L_g\xi_2,\rho_2,v_2)) \\ &= \Omega_{\mu}(\mathcal{D}_{\mu}\xi_1 + d\Phi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1),\mathcal{D}_{\mu}\xi_2 + d\Phi_*(\sqrt{-1}\mathcal{D}_0\rho_2 + v_2)) \\ &= -Im(\mathcal{D}_{\mu}\xi_1 + d\Phi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1),\mathcal{D}_{\mu}\xi_2 + d\Phi_*(\sqrt{-1}\mathcal{D}_0\rho_2 + v_2))_{L^2} \\ &= (-Im\mathcal{D}_{\mu}^*\mathcal{D}_{\mu}\xi_1 - Re\mathcal{D}_{\mu}^* \circ d\Phi_*(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1),\xi_2) \\ &+ (Re\mathcal{D}_0^* \circ (d\Phi_*)^t(\mathcal{D}_{\mu}\xi_1) + Im\mathcal{D}_0^* \circ (d\Phi_*)^t d\Phi_*(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1),\rho_2) \\ &+ \Omega_0((d\Phi_*)^t(\mathcal{D}_{\mu}\xi_1 + d\Phi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1)),v_2) \end{aligned}$$

As in the finite dimensional case, we need to solve an O.D.E. Let $\Omega_t = (1-t)\tilde{\Omega} + t\Omega'$. The isotopies $f_t : [g, \rho, v] \to [g, t\rho, tv]$ gives rise to timedependent vector field $X_t(f_t([g, \rho, v])) = [0, \rho, v]$. We first need to solve another time-dependent vector field Y_t through the following relation:

$$\Omega_{t\ [g,\rho,v]}(Y_t,Z) = \int_0^1 (\Omega' - \tilde{\Omega})_{[g,s\rho,sv]}((0,\rho,v), f_{s*}Z)ds$$
(26)

Notice that Y_t is \mathcal{G} -invariant. So we can assume g = Id. Let $Y_t = (\xi_1, \rho_1, v_1)$ and $Z = (\xi_2, \rho_2, v_2)$. By choosing Z arbitrarily, we get the following system of equations:

$$-tIm\mathcal{D}_{\mu}^{*}\mathcal{D}_{\mu}\xi_{1} - tRe\mathcal{D}_{\mu}^{*} \circ d\Phi_{*}(\mathcal{D}_{0}\rho_{1} - \sqrt{-1}v_{1})$$

$$-(1-t)\mathcal{D}_{0}^{*}\mathcal{D}_{0}\rho_{1} - (1-t)d_{v}\mu_{N}(v_{1}) + (1-t)[\mathcal{D}_{0}^{*}\mathcal{D}_{0}\rho + \mu_{N}(v),\xi_{1}]$$

$$+\int_{0}^{1}Re\mathcal{D}_{\mu_{s}}^{*} \circ d\Phi_{s*}(s\mathcal{D}_{0}\rho + \sqrt{-1}sv) + \mathcal{D}_{0}^{*}\mathcal{D}_{0}(s\rho) - d_{v}\mu_{N}(sv)ds \in \mathfrak{g}_{0}$$
(27)

$$tRe\mathcal{D}_{0}^{*} \circ (d\Phi_{*})^{t}\mathcal{D}_{\mu}\xi_{1} + tIm\mathcal{D}_{0}^{*} \circ (d\Phi_{*})^{t} \circ (d\Phi_{*})(\mathcal{D}_{0}\rho_{1} - \sqrt{-1}v_{1})$$
$$+ (1-t)\mathcal{D}_{0}^{*}\mathcal{D}_{0}\xi_{1} - \int_{0}^{1} Im\mathcal{D}_{0}^{*} \circ (d\Phi_{s})_{*}^{t} \circ (d\Phi_{s})_{*}(s\mathcal{D}_{0}\rho - \sqrt{-1}sv)ds \in \mathfrak{g}_{0}$$
(28)

$$t(d\Phi_{*})^{t} \circ (\mathcal{D}_{\mu}\xi_{1} + d\Phi_{*}(\sqrt{-1}\mathcal{D}_{0}\rho_{1} + v_{1})) + (1 - t)v_{1} + (1 - t)(d_{v}\mu_{N})^{*}(\xi_{1})$$
$$-\int_{0}^{1} (d\Phi_{s*})^{t} \circ d\Phi_{s*}(s\sqrt{-1}\mathcal{D}_{0}\rho + sv) - svds \in Im\mathcal{D}_{0}$$
(29)

Since Ω' and $\tilde{\Omega}$ are both non-degenerate, this system admits a (unique) weak solution. Then applying elliptic regularity, the solutions are smooth. Next we shall prove that there are two neighborhoods \mathcal{N}_1 , \mathcal{N}_2 of 0 in $m \times N$, and a smooth tame map F from \mathcal{N}_1 to $C^{\infty}([0,1], \mathfrak{m} \times N)$ such that the time 1 evaluation of the image of F is a smooth tame map from \mathcal{N}_1 to \mathcal{N}_2 and for any $(\rho, v) \in \mathcal{N}_1$,

$$\begin{cases} \frac{d}{dt}F_t(\rho, v) = (\rho_1(t), v_1(t)), & t \in [0, 1]; \\ F_0(\rho, v) = (\rho, v). \end{cases}$$
(30)

To prove this claim, we shall exploit Hamilton's implicit function theorem again. Define a map

$$H: C^{\infty}([0,1], \mathfrak{m} \times N) \to (\mathfrak{m} \times N) \times C^{\infty}([0,1], \mathfrak{m} \times N)$$

which sends $(\rho(t), v(t))$ to $(\rho(0), v(0)) \times (\dot{\rho}(t) - \rho_1(t), \dot{v}(t) - v_1(t))$. It is clear that H is a smooth tame map and H(0) = 0. We shall show that for $x = (\rho(t), v(t))$ close to zero, the derivative of H at x is invertible and its inverse is smooth tame. Let $\delta x = (\tilde{\rho}(t), \tilde{v}(t))$, then the derivative of H along δx is given by $(\tilde{\rho}(0), \tilde{v}(0)) \times (\dot{\rho} - \delta \rho_1(\tilde{\rho}), (\dot{v} - \delta v_1(\tilde{v}))$. So the invertibility of dH is equivalent to the solvability of the Cauchy problem of the following linear system along $(\rho(t), v(t))$:

$$\begin{cases} \frac{d}{dt}(\alpha, u) = (\delta \rho_1(\alpha), \delta v_1(u)) + (\beta, q), & t \in [0, 1] \\ (\alpha(0), u(0)) = (\tilde{\rho}(0), \tilde{v}(0)). \end{cases}$$
(31)

Thus we need to linearize equations (27) and (29). As a result, we get the following

$$\begin{cases} \dot{\alpha}(t) = A_1(\rho(t), v(t))\alpha(t) + A_2(\rho(t), v(t))u(t) + \beta(t), \\ \dot{u}(t) = B_2(\rho(t), v(t))\dot{\alpha}(t) + C_2(\rho(t), v(t))\alpha(t) + B_0(\rho(t), v(t))u(t) + q(t), \\ (\alpha(0), u(0)) = (\tilde{\rho}(0), \tilde{v}(0)), \end{cases}$$
(32)

where A_i , B_i , C_i are pseudo-differential operators of order *i* whose coefficients depend on $(\rho(t), v(t))$. Let $w(t) = u(t) - B_2(\rho(t), v(t))\alpha(t)$. Then the systems of equations for $(\alpha(t), w(t))$ become symmetric hyperbolic, for which the Cauchy problem is always solvable with estimates, see [AG]. From the proof we can check that the solution depends tamely on $(\rho(t), v(t))$, $(\beta(t), q(t))$ and the initial condition $(\tilde{\rho}(0), \tilde{v}(0))$. So by Hamilton's implicit function theorem *H* has a local smooth tame inverse. Let $F = H^{-1}(-, 0)$ and the claim is then proved. Now for any $[g, \rho, v]$ close to [Id, 0, 0], we obtain a path $(\rho(t), v(t)) = F_t(\rho, v)$. Then we can solve the O.D.E $\dot{g}(t) = L_{g(t)}\xi_1(t)$, where $\xi_1(t)$ is determined by $(\rho(t), v(t))$. $[g(t), \rho(t), v(t)]$ is then an integral curve of Y_t by the \mathcal{G} -invariancy. Now we define

$$\Sigma_2: \mathcal{V}_2 \to \mathcal{U}_2; [g, \rho, v] \mapsto [g_1, F_1(\rho, v)]$$

Then from the previous arguments we know that Σ_2 is smooth tame and fixes $\mathcal{G}.[Id, 0, 0]$. Moreover, $\Sigma^* \Omega' = \tilde{\Omega}$. It follows from equations (27), (28), (29) that we have a tame estimate

$$|v_1(t)| \le C \cdot (|\rho(t)| + |v(t)|)^3.$$

Since $|(v(t), \rho(t))| \leq C \cdot |(v(0), \rho(0))|$, we obtain

$$|v_1(t) - v_1(0)| \le C \cdot (|\rho(0)| + |v(0)|)^3.$$

By symmetry, we can obtain the map Σ_1 . Then one can check that the required estimates hold.

Now to prove lemma 6.2, we just need to apply the previous proposition to the path $\hat{J}(t)$, and use exactly the same argument as in the proof of theorem 4.7.

References

- [AG] S. Alinhac, P. Gérard. Pseudo-differential Operators and the Nash-Moser Theorem. Graduate Studies in Mathematics, Volume 82. American Mathematical Society, 2007.
- [APS] C. Arezzo, F. Pacard, M. Singer. Extremal metrics on blow ups, Duke Math. J. 157 (2011), no. 1, 1–51
- [BM] S. Bando, T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., 10 (1987), 11-40.

- [Be] B.Berndtsson. Probability measures related to geodesics in the space of Kahler metrics, arXiv:0907.1806.
- [Ca1] E. Calabi. Extremal Kähler metric, Seminar of Differential Geometry, ed. S. T. Yau, Annals of Mathematics Studies 102, Princeton University Press (1982), 259-290.
- [Ca2] E. Calabi. Extremal Kähler metric, II, Differential Geometry and Complex Analysis, eds. I. Chavel and H. M. Farkas, Spring Verlag (1985), 95-114.
- [CC] E. Calabi, X-X. Chen. Space of Kähler metrics and Calabi flow, J. Differential Geom. 61 (2002), no. 2, 173–193.
- [Ch1] X-X. Chen. The space of Kähler metrics, J. Differential Geom. 56 (2000), no. 2, 189–234.
- [Ch2] X-X. Chen. On the lower bound of the Mabuchi energy and its application, Internet. Math. Res. Notices 2000, no. 12, 607-623.
- [Ch3] X-X. Chen. Space of Kähler metrics III-The greatest lower bound of the Calabi energy, Invent. math. 175(2009), 453-680.
- [Ch4] X-X. Chen. The space of Kähler metrics (IV)—On the lower bound of the K energy. Preprint, arXiv:0809.4081.
- [CH1] X-X. Chen, W-Y. He. On the Calabi flow, Amer. J. Math. 130 (2008), no. 2, 539–570.
- [CH2] X-X. Chen, W-Y. He. The Calabi flow on Kähler surface with bounded Sobolev constant-(I), arxiv:math/0710.5159.
- [CLW] X-X. Chen, H. Li, B. Wang. Kähler-Ricci flow with small initial energy. Geom. Funct. Anal. 18 (2009), no. 5, 1525–1563.
- [CS] X-X. Chen, S. Sun. Space of Kähler metrics IV-Kähler quantization, To appear in Metric and Differential Geometry, a volume in honor of Jeff Cheeger for his 65th birthday.
- [CT] X-X. Chen, G. Tian. Geometry of Kähler metrics and foliations by holomorphic discs, Publications Mathématiques de L'IHÉS, Vol 107, 1-107.
- [DZ] A. Della Vedova, F. Zuddas, Scalar curvature and asymptotic Chow stability of projective bundles and blowups, arXiv:1009.5755.

- [D1] S. K. Donaldson. Remarks on gauge theory, complex geometry and 4manifold topology, Fields Medalists' Lectures, World Sci. Publ., Singapore, 1997, 384-403.
- [D2] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics, Northern California Symplectic Geometry Seminar, 13– 33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.
- [D3] S. K. Donaldson. Scalar curvature and projective embeddings, I. J. Differential. Geom. 59 (2001), no. 3, 479–522.
- [D4] S. K. Donaldson. Floer homology groups in Yang-Mills theory. (English summary) With the assistance of M. Furuta and D. Kotschick. Cambridge Tracts in Mathematics, 147. Cambridge University Press, Cambridge, 2002.
- [D5] S. K. Donaldson. Conjectures in Kähler geometry, Strings and geometry, Clay Math. Proc., 3, Amer. Math. Soc., Providence, RI, (2004), 71-78.
- [D6] S. K. Donaldson. Constant scalar curvature metrics on toric surfaces. Geom. Funct. Anal. 19 (2009), no. 1, 83–136.
- [DK] S. K. Donaldson, P. B. Kronheimer. The geometry of four-manifolds. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.
- [Fi] J. Fine. Calabi flow and projective embeddings, arxiv: math/0811.0155.
- [Fu] A. Fujiki. The moduli spaces and Kähler metrics of polarized algebraic varieties. Sugaku Expositions. Sugaku Expositions 5 (1992), no. 2, 173– 191.
- [FS] A. Fujiki, G. Schumacher, The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics. Publ. Res. Inst. Math. Sci. 26 (1990), no. 1, 101–183.
- [GS] V. Guillemin, S. Sternberg. A normal form for the moment map. Differential geometric methods in mathematical physics (Jerusalem, 1982), 161–175, Math. Phys. Stud., 6, Reidel, Dordrecht, 1984.
- [Ha] R. S. Hamilton. The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.
- [Ke] G. R. Kempf. Instability in invariant theory, Ann. of Math. (2) 108 (1978), no. 2, 299–316.

- [Ki] F. C. Kirwan. Cohomology of quotients in symplectic and algebraic geometry, Mathematical Notes, 31. Princeton University Press, Princeton, NJ, 1984.
- [Ku] M. Kuranishi. New proof for the existence of locally complete families of complex structures. 1965 Proc. Conf. Complex Analysis (Minneapolis, 1964) pp. 142–154 Springer, Berlin.
- [Le] E. Lerman. Gradient flow of the norm squared of a moment map, Enseign. Math. (2) 51(2005), no. 1-2, 117-127.
- [M1] T. Mabuchi. Some symplectic geometry on compact Kähler manifolds I. Osaka J. Math. 24 (1987), no. 2, 227–252.
- [Ne] L. Ness. A Stratification of the Null Cone Via the Moment Map, With an appendix by David Mumford. Amer. J. Math. 106 (1984), no. 6, 1281– 1329.
- [OSY] H. Ono, Y. Sano, N. Yotsutani, An example of asymptotically Chow unstable manifolds with constant scalar curvature, arXiv:0906.3836.
- [OR] J. Ortega, T. Ratiu. Momentum maps and Hamiltonian reduction. Progress in Mathematics, 222. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [Pa] S. Paul. Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. arxiv:math/0811.2548.
- [Ra] J. Råde. On the Yang-Mills heat equation in two and three dimensions.J. Reine Angew. Math. 431 (1992), 123–163.
- [RT] J. Ross, R. Thomas. A study of the Hilbert-Mumford criterion for the stability of projective varieties. J. Algebraic Geom. 16 (2007), no. 2, 201– 255.
- [Se] S. Semmes. Complex Monge-Ampère equations and sympletic manifolds, Amer. J. Math, no. 114, 495–550, 1992.
- [Si] L. Simon. Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. Ann. of Math. (2) 118 (1983), no. 3, 525–571.
- [SW] S. Sun, Y-Q. Wang. On the Kähler-Ricci flow near a Kähler-Einstein metric, arxiv:math/1004.2018.
- [Sz] G. Székelyhidi. The Kähler-Ricci flow and K-stability, Amer. J. Math. 132 (2010), 1077–1090.

- [Th] R. Thomas. Notes on GIT and symplectic reduction for bundles and varieties, arxiv: math/0512411.
- [Ti1] G. Tian. Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), 1-39.
- [Ti2] G. Tian. Extremal metrics and geometric stability. Special issue for S. S. Chern. Houston J. Math. 28 (2002), no. 2, 411–432.
- [To] V. Tosatti. The K-energy on small deformations of constant saclar curvature Kähler manifolds. arxiv: math/1010.1859.
- [TZ1] G. Tian, X-H. Zhu. A new holomorphic invariant and uniqueness of Kähler-Ricci solitons. Comment. Math. Helv., 77 (2002), no. 2, 297-325.
- [TZ2] G. Tian, X-H.Zhu. Perelman's W-functional and stability of Kähler-Ricci flow, arxiv: math/0801.3504.
- [We] A. Weinstein. Symplectic manifolds and their Lagrangian submanifolds. Advances in Math. 6 (1971), 329-346.

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