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NOTE ON K-STABILITY OF PAIRS

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ABSTRACT. We prove that a pair (X, D) with X Fano and D an anti-canonical divisor is K-unstable for negative angles, and is K-semistable for zero angle.

1. INTRODUCTION

Let X be a Fano manifold. It was first proposed by Yau [20] that finding Kähler-Einstein metrics on X should be related to a certain algebro-geometric stability. In [17], the notion of K-stability was introduced by Tian. This has been conjectured to be equivalent to the existence of a Kähler-Einstein metric. One direction is essentially known, in a wider context of constant scalar curvature Kähler metrics [3]. Namely, it is proved by Donaldson [4] that the existence of a constant scalar curvature metric implies K-semistability. This was later strengthened by Stoppa [15] to K-stability in the absence of continuous automorphism group, and by Mabuchi [9] to K-polystability in general.

Recently in [6](see also, [16], [7]) K-stability has been defined for a pair (X, D), where X is a Fano manifold and D is a smooth anti-canonical divisor. The definition involves a parameter $\beta \in \mathbb{R}$. At least when $\beta \in (0, 1]$, the K-stability of a pair (X, D) with parameter β is conjectured to be equivalent to the existence of a Kähler-Einstein metric on X with cone singularities of angle $2\pi\beta$ transverse to D. This generalization grew out of a new continuity method for dealing with the other direction of the above conjecture, as outlined in [5]. Note heuristically the case $\beta = 0$ corresponds to a complete Ricci flat metric on the complement $X \setminus D$. By the work of Tian-Yau [18] such a metric always exists if D is smooth. In this short article we prove the following theorem, which may be viewed as an algebraic counterpart of the differential geometric result of Tian-Yau.

Theorem 1.1. *Any pair (X, D) is strictly K-semistable with respect to angle $\beta = 0$, and K-unstable with respect to angle $\beta < 0$.*

By the definition of K-stability for pairs which will be recalled in the next section, the Futaki invariant depends linearly on the angle β. Thus Theorem 1.1 leads immediately to the following

Corollary 1.2. *If X is K-stable(semi-stable), then for any smooth anti-canonical divisor D, the pair (X, D) is K-stable(semi-stable) with respect to angle $\beta \in (0, 1]$.*

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This corollary provides evidence to the picture described in [5] that a smooth Kähler-Einstein metric on X should come from a complete Calabi-Yau metric on $X \setminus D$ by increasing the angle from 0 to 2π. The relevant definitions will be given in the next section. The strategy to prove K-unstability for negative angles is by studying a particular test configuration, namely the deformation to the normal cone of D. To deal with the zero angle case we shall construct "approximately balanced" embeddings using the Calabi-Yau metric on D. In [11], Odaka proved that a Calabi-Yau manifold is K-stable, by a purely algebro-geometric approach. It is very likely that his method can give an alternative proof of the above theorem, but the one we take seems to be more quantitative.

2. K-stability for pairs

We first recall the definition of K-stability.

Definition 2.1. Let (X, L) be a polarized manifold. A *test configuration for* (X, L) is a $\mathbb{C}^*$ equivariant flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1)$ is isomorphic to (X, L). $(\mathcal{X}, \mathcal{L})$ is called *trivial* if it is isomorphic to the product $(X, L) \times \mathbb{C}$ with the trivial action on (X, L) and the standard action on $\mathbb{C}$.

Suppose D is a smooth divisor in X, then any test configuration $(\mathcal{X}, \mathcal{L})$ induces a test configuration $(\mathcal{D}, \mathcal{L})$ by simply taking the flat limit of the $\mathbb{C}^*$ orbit of D in $\mathcal{X}_1$. We call $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ a *test configuration for* (X, D, L). Given any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ for (X, D, L), we denote by A_k and $\tilde{A}_k$ the infinitesimal generators for the $\mathbb{C}^*$ action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ and $H^0(\mathcal{D}_0, \mathcal{L}_0^k)$ respectively. By general theory for k large enough we have the following expansions

$$d_k := h^0(\mathcal{X}_0, \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$
$$w_k := tr(A_k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),$$
$$\tilde{d}_k := h^0(\mathcal{D}_0, \mathcal{L}_0^k) = \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + O(k^{n-3}),$$
$$\tilde{w}_k := tr(\tilde{A}_k) = \tilde{b}_0 k^n + \tilde{b}_1 k^{n-1} + O(k^{n-2}).$$

Definition 2.2. For any real number β, the *Futaki invariant* of a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ with respect to angle β is

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1-\beta)(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0).$$

When $\beta = 1$ we get the usual Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$

$$Fut(\mathcal{X}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0}.$$

Definition 2.3. A polarized manifold (X, L) is called *K-stable(semistable)* if $Fut(\mathcal{X}, \mathcal{L}) > 0 (\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{L})$. Similarly, (X, D, L) is called *K-stable(semistable) with respect to angle* β if $Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) > 0 (\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$.

When the central fiber $(\mathcal{X}_0, \mathcal{D}_0)$ is smooth, by Riemann-Roch the Futaki invariant then has a differential geometric expression as

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) = \int_{\mathcal{X}_0} (S - \underline{S}) H \frac{\omega^n}{n!} - (1-\beta)\left(\int_{\mathcal{D}_0} H \frac{\omega^{n-1}}{(n-1)!} - \frac{Vol(\mathcal{D}_0)}{Vol(\mathcal{X}_0)} \int_{\mathcal{X}_0} H \frac{\omega^n}{n!}\right),$$

where ω is an S^1 invariant Kähler metric in $2\pi c_1(\mathcal{L}_0)$ and H is the Hamiltonian function generating the S^1 action on $\mathcal{L}_0$. This differs from the usual Futaki invariant by an extra term which reflects the cone angle.

The above abstract notion of K-stability is closely related to Chow stability for projective varieties, which we now recall. Given a $\mathbb{C}^*$ action on $\mathbb{CP}^N$, and suppose the induced S^1 action preserves the Fubini-Study metric. Then the infinitesimal generator is given by a Hermitian matrix, say A. The Hamiltonian function for the S^1 action on $\mathbb{CP}^N$ is

$$H_A(z) = \frac{z^* A z}{|z|^2}.$$

Given a projective manifold V in $\mathbb{CP}^N$, we define the *center of mass* of V

$$\mu(V) = \int_V \frac{zz^*}{|z|^2} d\mu_{FS} - \frac{Vol(V)}{N+1} Id \in \sqrt{-1}\mathfrak{su}(N+1),$$

viewing $\mathbb{CP}^N$ as a co-adjoint orbit in $\mathfrak{su}(N+1)$. Define the *Chow weight of V with respect to A* to be

$$CH(V, A) = -Tr(\mu(V) \cdot A) = -\int_V H_A d\mu_{FS} + \frac{Vol(V)}{N+1} Tr A.$$

Notice this vanishes if A is a scalar matrix. The definition is not sensitive to singularities of V so one may define the Chow weight of any algebraic cycles in a natural way. It is well-known that the $CH(e^{tA}.V, A)$ is a decreasing function of t, see for example [4]. So

$$CH(V, A) \leq CH(V_\infty, A), \tag{2.1}$$

where V_∞ is the limiting Chow cycle of $e^{tA}.V$ as $t \to -\infty$. V_∞ is fixed by the $\mathbb{C}^*$ action and then $CH(V_\infty, A)$ is an algebraic geometric notion, i.e. independent of the Hermitian metric we choose on $\mathbb{C}^{N+1}$.

This well-known theory readily extends to pairs, see [5], [1]. We consider a pair of varieties (V, W) in $\mathbb{CP}^N$ where W is a subvariety of V. Given a parameter $\lambda \in [0, 1]$, we define the *center of mass of* (V, W) *with parameter* λ

$$\mu(V, W, \lambda) = \lambda \int_V \frac{zz^*}{|z|^2} d\mu_{FS} + (1-\lambda) \int_W \frac{zz^*}{|z|^2} d\mu_{FS} - \frac{\lambda Vol(V) + (1-\lambda) Vol(W)}{N+1} Id,$$

and the *Chow weight* with parameter λ:

$$CH(V, W, A, \lambda) = -Tr(\mu(V, W, \lambda) \cdot A).$$

A pair (V, W) with vanishing center of mass with parameter λ is called a *λ-balanced embedding.*

Now given a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$, it is explained in [13] and [4](see also [12]) that for k large enough one can realize it by a family of projective schemes in $\mathbb{P}(H^0(X, L^k)^*)$ with a one parameter group action. Moreover one could arrange that the fiber $(\mathcal{X}_1, \mathcal{D}_1, \mathcal{L}_1)$ is embedded into $\mathbb{P}(H^0(X, L^k)^*)$ with a prescribed Hermitian metric, and the $\mathbb{C}^*$ action is generated by a Hermitian matrix $-A_k$(negative sign because we are taking the dual). Then as in [4] the Futaki invariant is the limit of Chow weight:

$$\lim_{k\to\infty} k^{-n} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, \lambda)) = Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta), \tag{2.2}$$

with $\beta = \frac{3\lambda-2}{\lambda}$.

3. Proof of the main theorem

From now on we assume X is a Fano manifold of dimension n, D is a smooth anti-canonical divisor and the polarization is given by $L = -K_X$. We first prove the part of unstability in theorem 1.1, by considering the deformation to the normal cone of D, as studied by Ross-Thomas [14]. We blow up $D \times \{0\}$ in the total space $X \times \mathbb{C}$ and get a family $\pi : \mathcal{X} \to \mathbb{C}$. The exceptional divisor P is equal to the projective completion $\mathbb{P}(\nu_D \oplus \mathbb{C})$ of the normal bundle ν_D in X. The central fiber $\mathcal{X}_0$ is the gluing of P to X along $D = \mathbb{P}(\nu_D)$. There is a $\mathbb{C}^*$ action on $\mathcal{X}$ coming from the trivial action on X and the standard $\mathbb{C}^*$ action on $\mathbb{C}$. Let $\mathcal{D}$ be the proper transform of $D \times \mathbb{C}$. This is $\mathbb{C}^*$ invariant, and its intersection with the central fiber is the zero section $\mathbb{P}(\mathbb{C}) \subset \mathbb{P}(\nu_D \oplus \mathbb{C})$(The readers are referred to [14] for a very nice picture of a deformation to the normal cone). The line bundle we use is $\mathcal{L}_c = L(-cP)$(c is rational). It is shown in [14] that $\mathcal{L}_c$ is ample when $c \in (0, 1)$. There is also a natural lift of the $\mathbb{C}^*$ action to $\mathcal{L}_c$, so that we get test configurations $(\mathcal{X}, \mathcal{D}, \mathcal{L}_c)$ parametrized by c. We follow [14] to compute the Futaki invariant. Pick a sufficiently large integer k so that ck is an integer. We have the decomposition

$$H^0(\mathcal{X}, \mathcal{L}_c^k) = \bigoplus_{i=1}^{ck} t^{ck-i} H^0(X, L^{k-i}) \oplus t^{ck}\mathbb{C}[t]H^0(X, L^k),$$

where t is the standard holomorphic coordinate on $\mathbb{C}$. Using the short exact sequence

$$0 \to H^0(X, L^{i-1}) \to H^0(X, L^i) \to H^0(D, L^i) \to 0,$$

we obtain

$$\begin{aligned} H^0(\mathcal{X}_0, \mathcal{L}_c^k) &= H^0(\mathcal{X}, \mathcal{L}_c^k)/tH^0(\mathcal{X}, \mathcal{L}_c^k) \\ &= H^0(X, L^{(1-c)k}) \oplus \bigoplus_{i=0}^{ck-1} t^{ck-i} H^0(D, L^{k-i}). \end{aligned}$$

This is indeed the weight decomposition of $H^0(\mathcal{X}_0, \mathcal{L}_c^k)$ under the $\mathbb{C}^*$ action. Note the weight is -1 on t. So

$$\dim H^0(\mathcal{X}_0, \mathcal{L}_c^k) = \dim H^0(X, L^{(1-c)k}) + \sum_{i=0}^{ck-1} \dim H^0(D, L^{k-i}) = \dim H^0(X, L^{ck}).$$

This actually shows the flatness of the family $(\mathcal{X}, \mathcal{D}, \mathcal{L})$. Thus by Riemann-Roch,

$$a_0 = \frac{1}{n!}\int_X c_1(L)^n,$$

and

$$a_1 = \frac{1}{2(n-1)!}\int_X c_1(-K_X)\cdot c_1(L)^{n-1} = \frac{na_0}{2}.$$

The weight is given by

$$\begin{aligned} w_k &= -\sum_{i=0}^{ck-1}(ck-i)\dim H^0(D, L^{k-i}) \\ &= -\sum_{i=0}^{ck-1}(ck-i)(\frac{(k-i)^{n-1}}{(n-1)!}\int_D c_1(L)^{n-1} + O(k^{n-3})) \\ &= -na_0\int_0^c (c-x)(1-x)^{n-1}dx\cdot k^{n+1} - \frac{nca_0}{2}k^n + O(k^{n-1}). \end{aligned}$$

So

$$b_0 = (\frac{1-(1-c)^{n+1}}{n+1} - c)a_0,$$

and

$$b_1 = -\frac{nca_0}{2}.$$

Thus the ordinary Futaki invariant for the test configuration $(\mathcal{X}, \mathcal{L})$ is given by

$$Fut_c(\mathcal{X}, \mathcal{L}) = \frac{2(a_1b_0 - a_0b_1)}{a_0} = n(\frac{1-(1-c)^{n+1}}{n+1})a_0.$$

Note

$$H^0(\mathcal{D}, \mathcal{L}_c^k) = H^0(D\times\mathbb{C}, L^k\otimes(t)^{ck}) = t^{ck}\mathbb{C}[t]H^0(D, L^k).$$

So

$$H^0(\mathcal{D}_0, \mathcal{L}_c^k) = H^0(\mathcal{D}, \mathcal{L}_c^k)/tH^0(\mathcal{D}, \mathcal{L}_c^k) = t^{ck}H^0(D, L^k).$$

Thus we see

$$\tilde{a}_0 = \int_D \frac{c_1(L)^{n-1}}{(n-1)!} = na_0,$$

and

$$\tilde{b}_0 = -c\int_D \frac{c_1(L)^{n-1}}{(n-1)!} = -nca_0.$$

Therefore,

$$\begin{aligned} Fut_c(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) &= Fut_c(\mathcal{X}, \mathcal{L}) + (1-\beta)(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0}b_0) \\ &= [n(\frac{1-(1-c)^{n+1}}{n+1}) + (1-\beta)(-nc + n(c - \frac{1-(1-c)^{n+1}}{n+1}))]a_0 \\ &= n\beta\frac{1-(1-c)^{n+1}}{n+1}a_0. \end{aligned}$$

Therefore for $\beta < 0$ this particular test configuration gives rise to unstability, and for $\beta = 0$ the pair (X, D) can not be stable.

Now we move on to prove K-semistability for $\beta = 0$. Using again the short exact sequence

$$0 \to H^0(X, L^{j-1}) \to H^0(X, L^j) \to H^0(D, L^j) \to 0$$

successively we can choose a splitting

$$H^0(X, L^k) = H^0(X, L^{s-1}) \oplus \bigoplus_{j=s}^{k} H^0(D, L^j) \tag{3.1}$$

for s large enough and all $k > s$. By Yau's theorem [19] there is a unique Ricci flat metric ω_0 in $c_1(L)|_D$. This defines a Hermitian metric on $H^0(D, L^j)$ by the L^2 inner product. We can put an arbitrary metric on $H^0(X, L^{s-1})$, and make the splitting (3.1) orthogonal. We also identify the vector spaces with their duals using these metrics. Take s large enough so that D embeds into $\mathbb{P}(H^0(D, L^j))$ and X embeds into $\mathbb{P}(H^0(X, L^j))$ for all $j \geq s-1$. Choosing an orthonormal basis of $H^0(D, L^j)$ we get an embedding $f_j : D \to \mathbb{P}(H^0(D, L^j)) \cong \mathbb{P}^{n_j-1}$(here $n_j = \dim H^0(D, L^j)$). We also pick an arbitrary embedding $f_{s-1} : X \to \mathbb{P}(H^0(X, L^{s-1}))$. Denote by D_j the image of f_j, and let $N(D_{j-1}, D_j)$ be the variety consisting of all points in $\mathbb{P}(H^0(D, L^{j-1}) \oplus H^0(D, L^j)) \subset \mathbb{P}(H^0(X, L^k))$ of the form $[uf_{j-1}(p) : vf_j(p)]$ for $p \in D$ and $u, v \in \mathbb{C}$. The projection map $\pi_j : N(D_{j-1}, D_j) \to D$ makes it a $\mathbb{P}^1$ bundle over D. This is isomorphic to the projective completion of the normal bundle of D in X. Let X_k be the union of all these $N(D_{j-1}, D_j)(s \leq j \leq k)$ together with $f_{s-1}(X)$. Then it is not hard to see that as a pair of Chow cycles (X_k, D_k) lies in the closure of the $PGL(d_k; \mathbb{C})$ orbit of a smooth embedding of (X, D) in $\mathbb{P}(H^0(X, L^k))$. We want to estimate its center of mass. The following two lemmas involve some calculation and the proof will be deferred to the end of this section.

Lemma 3.1. *For $s \leq j \leq k$ we have*

$$\pi_{j*}\omega_{FS}^n = \sum_{i=0}^{n-1} \omega_j^i \wedge \omega_{j-1}^{n-1-i},$$

where $\omega_j = f_j^* \omega_{FS}$.

This lemma implies that

$$Vol(N(D_{j-1}, D_j)) = \frac{1}{n!}\sum_{i=0}^{n-1} j^i (j-1)^{n-1-i} \cdot (n-1)! Vol(D) = (j^n - (j-1)^n) Vol(X).$$

Summing over j we see that $Vol(X_k) = k^n Vol(X)$.

Notice $N(D_{j-1}, D_j)$ can only contribute to the $H^0(D, L^{j-1})$ and $H^0(D, L^j)$ components of the center of mass of X_k. Denote by $Z_j = (Z_j^1, \cdots, Z_j^{n_j})$ the homogeneous coordinates on $H^0(D, L^j)$ for $s \leq j \leq k$, and by Z_{s-1} the homogeneous coordinate on $H^0(X, L^{s-1})$. Then we have

Lemma 3.2. *For $s \leq j \leq k$ we have*

$$\pi_{j*}\frac{Z_j Z_{j-1}^*}{|Z_j|^2 + |Z_{j-1}|^2}\omega_{FS}^n = 0,$$

$$\pi_{j*}\frac{Z_{j-1}Z_j^*}{|Z_j|^2+|Z_{j-1}|^2}\omega_{FS}^n=0,$$

$$\pi_{j*}\frac{Z_jZ_j^*}{|Z_j|^2+|Z_{j-1}|^2}\omega_{FS}^n=\frac{Z_jZ_j^*}{|Z_j|^2}\cdot\sum_{i=0}^{n-1}\frac{i+1}{n+1}\omega_j^i\wedge\omega_{j-1}^{n-1-i},$$

$$\pi_{j*}\frac{Z_{j-1}Z_{j-1}^*}{|Z_j|^2+|Z_{j-1}|^2}\omega_{FS}^n=\frac{Z_{j-1}Z_{j-1}^*}{|Z_{j-1}|^2}\cdot\sum_{i=0}^{n-1}\frac{n-i}{n+1}\omega_j^i\wedge\omega_{j-1}^{n-1-i},$$

This lemma implies that the center of mass $\mu(X_k)$ also splits as the direct sum of μ_j's. For j between s and $k-1$ we have

$$\mu_j(X_k)=\int_{X_k}\frac{Z_jZ_j^*}{|Z|^2}\frac{\omega_{FS}^n}{n!}=\frac{1}{n!}\int_D\frac{Z_jZ_j^*}{|Z_j|^2}\sum_{i=0}^{n-1}(\frac{i+1}{n+1}\omega_j^i\wedge\omega_{j-1}^{n-1-i}+\frac{n-i}{n+1}\omega_{j+1}^i\wedge\omega_j^{n-1-i}),$$

while

$$\mu_k(X_k)=\int_{X_k}\frac{Z_kZ_k^*}{|Z|}\frac{1}{n!}\omega_{FS}^n=\frac{1}{n!}\int_D\frac{Z_kZ_k^*}{|Z_k|^2}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\omega_k^i\wedge\omega_{k-1}^{n-1-i},$$

and

$$\mu_{s-1}(X_k)=\frac{1}{n!}\int_D\frac{Z_{s-1}Z_{s-1}^*}{|Z_{s-1}|^2}\sum_{i=0}^{n-1}\frac{n-i}{n+1}\omega_s^i\wedge\omega_{s-1}^{n-1-i}+\int_{X_{s-1}}\frac{Z_{s-1}Z_{s-1}^*}{|Z_{s-1}|^2}\frac{\omega_{FS}^n}{n!}.$$

The induced metric ω_j is related to the original metric ω_0 by the "density of state" function:

$$\omega_j=j\omega_0+\sqrt{-1}\partial\bar\partial\log\rho_j(\omega_0).$$

It is well-known that we have the following expansion(see [2], [21], [8], [10])

$$\rho_j(\omega_0)=j^{n-1}+\frac{S(\omega_0)}{2}j^{n-2}+O(j^{n-3})=j^{n-1}+O(j^{n-3}),$$

since ω_0 is Ricci flat. Thus

$$\omega_j^i\omega_{j-1}^{n-1-i}=j^i(j-1)^{n-1-i}\omega_0^{n-1}(1+O(j^{-3})).$$

To estimate μ_j recall we have chosen an orthonormal basis $\{s_j^l\}$ of $H^0(D,L^j)$ and we can assume μ_j is a diagonal matrix. Then for $s\le j\le k-1$ we obtain

$$\mu_j^l(X_k)=\int_D\frac{|s_j^l|^2(1+O(j^{-3}))}{j^{n-1}+O(j^{n-3})}\sum_{i=0}^{n-1}(\frac{i+1}{n+1}j^i(j-1)^{n-1-i}+\frac{n-i}{n+1}(j+1)^ij^{n-1-i})\frac{\omega_0^{n-1}}{n!}.$$

It is easy to see that

$$\sum_{i=0}^{n-1}(\frac{i+1}{n+1}j^i(j-1)^{n-1-i}+\frac{n-i}{n+1}(j+1)^ij^{n-1-i})=nj^{n-1}+O(j^{n-3}).$$

Thus

$$\mu_j^l(X_k)=1+O(j^{-2}).$$

For $j=k$, we have

$$\mu_k^l(X_k)=1/2+O(k^{-1}).$$

For $j=s-1$, we have

$$\mu_{s-1}^l(X_k)=O(1).$$

The center of mass of the pair (X_k, D_k) with respect to $\lambda = 2/3$ is given by

$$\mu(X_k, D_k, 2/3) = \frac{2}{3}\mu(X_k) + \frac{1}{3}\mu(D_k) - \underline{\mu} \cdot Id,$$

where we denote

$$\underline{\mu} = \frac{2Vol(X_k) + Vol(D_k)}{3d_k} = \frac{2}{3} + O(k^{-2}).$$

Thus for $s \leq j \leq k-1$ and $0 \leq l \leq n_j$ we have

$$\mu_j^l(X_k, D_k, 2/3) = O(j^{-2}) + O(k^{-2}).$$

Since n_j is a polynomial of degree $n-1$ in j, we obtain

$$|\mu_j(X_k, D_k, 2/3)|_2 = (\sum_{l=0}^{n_j} |\mu_j^l(X_k, D_k, 2/3)|^2)^{1/2} = O(j^{\frac{n-5}{2}}),$$

and

$$\sum_{j=s}^{k-1} |\mu_j(X_k, D_k, 2/3)|_2 = O(k^{\frac{n-3}{2}}).$$

For $j = k$, we have

$$\mu_k^l(D_k) = \int_D \frac{|s_k^l|^2}{k^{n-1} + O(k^{n-3})}(1 + O(k^{-2}))\frac{k^{n-1}\omega_0^{n-1}}{(n-1)!} = 1 + O(k^{-2}).$$

So

$$\mu_k^l(X_k, D_k) = O(k^{-1}),$$

and

$$|\mu_k(X_k, D_k)|_2 = O(k^{\frac{n-3}{2}}).$$

Therefore we obtain

$$|\mu(X_k, D_k)|_2 = O(k^{\frac{n-3}{2}}).$$

So for a smoothly embedded (X, D) in $\mathbb{P}(H^0(X, L^k))$ we have

$$\inf_{g \in PGL(d_k;\mathbb{C})} |\mu(g.(X, D))|_2 = O(k^{\frac{n-3}{2}}).$$

In particular there are embeddings $\iota_k : (X, D) \to \mathbb{P}(H^0(X, L^k))$ such that

$$|\mu(\iota_k(X, D))|_2 = O(k^{\frac{n-3}{2}}).$$

Now any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ can be represented by a family in $\mathbb{P}(H^0(X, L^k))$ such that the fiber $(\mathcal{X}_1, \mathcal{D}_1, \mathcal{L}_1)$ is embedded by ι_k and the $\mathbb{C}^*$ action is generated by a Hermitian matrix A_k. Again by general theory $|A_k|_2^2 = TrA_k^2 = O(k^{n+2})$. Therefore by monotonicity of the Chow weight we obtain

$$\begin{aligned} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, 2/3) &\geq CH_k(\mathcal{X}_1, \mathcal{D}_1, -A_k, 2/3) \\ &\geq -\inf_{g \in PGL(d_k;\mathbb{C})} |\mu(g.(X, D))|_2 \cdot |-A_k|_2 \\ &\geq -O(k^{n-\frac{1}{2}}). \end{aligned}$$

Thus by (2.2)

$$Fut(\mathcal{X}, \mathcal{D}, \mathcal{L}, 0) = \lim_{k\to\infty} k^{-n} CH_k(\mathcal{X}_0, \mathcal{D}_0, -A_k, \frac{2}{3}) \geq 0.$$

This finishes the proof of Theorem 1.1.

Now we prove Lemmas 3.1 and 3.2. In general suppose there are two embeddings $f_1 : D \to \mathbb{P}^l$ and $f_2 : D \to \mathbb{P}^m$. As before, let $N(D)$ be the variety in $\mathbb{P}^{l+m+1}$ containing all points of the form $(tf_1(x), sf_2(x))$ where $t, s \in \mathbb{C}$. Intuitively $N(D)$ is ruled by all lines connecting $f_1(x)$ and $f_2(x)$ for $x \in D$. Choose a local coordinate chart U in D such that the image $f_1(U)$ and $f_2(U)$ are contained in a standard coordinate chart for the projective spaces $\mathbb{P}^l$ and $\mathbb{P}^m$ respectively. Let $[1 : z]$ and $[1 : w]$ be local coordinates in $\mathbb{P}^l$ and $\mathbb{P}^m$. Under unitary transformations we may assume $f_1(x_0) = [1 : 0]$ and $f_2(x_0) = [1 : 0]$. The line connecting $f_1(x_0)$ and $f_2(x_0)$ is parametrized as $[1 : 0 : t : 0]$ for $t \in \mathbb{C}$. Along this line we have

$$\begin{aligned}\omega_{FS} &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(1 + |z|^2 + |t|^2 + |t|^2|w|^2)\\ &= \frac{\sqrt{-1}}{2\pi} \cdot \frac{(1 + |t|^2)\sum_i dz^i \wedge d\bar{z}^i + |t|^2(1 + |t|^2)\sum_j dw^j \wedge d\bar{w}^j + dt \wedge d\bar{t}}{(1 + |t|^2)^2}.\end{aligned}$$

Thus

$$\omega_{FS}^n = n(\frac{\sqrt{-1}}{2\pi})^n(1+|t|^2)^{-n-1}(\sum_i dz^i \wedge d\bar{z}^i + |t|^2 \sum_j dw^j \wedge d\bar{w}^j)^{n-1} \wedge dt \wedge d\bar{t}.$$

Hence integrating along the $\mathbb{P}^1$ we get

$$\begin{aligned}\int_{\mathbb{P}^1} \omega_{FS}^n &= \frac{1}{2\pi}\int_{\mathbb{C}} n(\omega_1 + |t|^2\omega_2)^{n-1} \wedge (1 + |t|^2)^{-n-1}\sqrt{-1}dt \wedge d\bar{t}\\ &= \frac{1}{2\pi}\int_0^\infty n\sum_{j=0}^{n-1}\binom{n-1}{j}\omega_1^j \wedge \omega_2^{n-1-j}x^j(1 + x)^{-n-1}dx\\ &= \sum_{j=0}^{n-1}\omega_1^j \wedge \omega_2^{n-1-j}.\end{aligned}$$

This proves lemma 3.1.

For the center of mass we compute

$$\int_{\mathbb{P}^1} \frac{1}{1 + |t|^2}\omega_{FS}^n = \sum_{j=0}^{n-1}\frac{j + 1}{n + 1}\omega_1^j \wedge \omega_2^{n-1-j},$$

and

$$\int_{\mathbb{P}^1} \frac{|t|^2}{1 + |t|^2}\omega_{FS}^n = \sum_{j=0}^{n-1}\frac{n - j}{n + 1}\omega_1^j \wedge \omega_2^{n-1-j}.$$

Thus globally we obtain

$$\int_{N(D)} \frac{zz^*}{|z|^2 + |w|^2}\omega_{FS}^n = \int_D \frac{zz^*}{|z|^2}\sum_{j=0}^{n-1}\frac{j + 1}{n + 1}\omega_1^j \wedge \omega_2^{n-1-j},$$

and

$$\int_{N(D)} \frac{ww^*}{|z|^2 + |w|^2}\omega_{FS}^n = \int_D \frac{ww^*}{|w|^2}\sum_{j=0}^{n-1}\frac{n - j}{n + 1}\omega_1^j \wedge \omega_2^{n-1-j}.$$

Also notice by symmetry of $N(D)$ under the map $w \mapsto -w$ we have

$$\int_{N(D)} \frac{zw^*}{|z|^2+|w|^2}\omega_{FS}^n = 0.$$

Similarly

$$\int_{N(D)} \frac{wz^*}{|z|^2+|w|^2}\omega_{FS}^n = 0.$$

This proves lemma 3.2.

Remark 3.3. *In the case when X is $\mathbb{P}^1$ and D consists of two points, one can indeed find the precise balanced embedding for $\lambda = 2/3$. In $\mathbb{P}^k$ let L be the chain of lines L_i connecting p_i and $p_{i+1}(0 \le i \le k-1)$, where p_i is the i-th coordinate point. Then it is easy to see that L is the degeneration limit of a smooth degree k rational curve, and it is exactly $\frac{2}{3}$-balanced. It is well-known that a rational normal curve in $\mathbb{P}^k$ is always Chow polystable, it follows by linearity that it is also Chow polystable for $\lambda \in (2/3, 1]$.*

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