Note on geodesic rays tamed by simple test configurations

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Abstract

In this short note, we give a new proof of a theorem of Arezzo-Tian on the existence of smooth geodesic rays tamed by a special degeneration.

In [7], S. K. Donaldson proposed an ambitious program to tackle the problem of the existence and uniqueness of extremal metrics on a Kähler manifold from the perspective of the infinite dimensional space of Kähler potentials. He observed that the existence of smooth geodesics connecting two arbitrary Kähler potentials implies the uniqueness of Kähler metrics in the given class with constant scalar curvature. In [3], the first named author proved the existence of $C^{1,1}$ geodesics joining two arbitrary points in $\mathcal{H}$. Consequently, this established the uniqueness of extremal Kähler metrics when the first Chern class of the manifold is non-positive. At present, there is extensive research in this direction. In particular, the uniqueness problem has been completely settled (c.f. [12],[10] and [6]).

We shall first give a very brief outline about a small part of this program which is directly relevant to the problem at hand. For more detailed accounts, readers are referred to [7], [3], [6] and [4].

Let (M, ω, J) be an n dimensional Kähler manifold. Define the infinite dimensional space of Kähler potentials as

$$\mathcal{H} = \{\phi \in C^\infty(M) | \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0\}.$$

In [11](c.f. [7], [18]), T. Mabuchi first introduced a Weil-Peterson type metric on $\mathcal{H}$:

$$(\phi_1, \phi_2)_\phi = \int_M \phi_1\phi_2 \frac{\omega_\phi^n}{n!},$$

where $\phi_1, \phi_2 \in T_\phi\mathcal{H} \simeq C^\infty(M)$. It is easy to see that the geodesic equation in $\mathcal{H}$ is

$$\ddot{\phi} = \frac{1}{2}|\nabla_\phi \dot{\phi}|^2_\phi \tag{1}$$

A straightforward calculation shows (c.f. [7], [11], [18]) that the space $\mathcal{H}$ is formally of non-positive curvature. This fact was made rigorously in [2], where E. Calabi and the first named author proved that $\mathcal{H}$ is a non-positively curved space in the sense of Alexanderov.

According to S.Semmes [18], by adding a trivial S^1 factor, the geodesic equation could be written as a degenerate complex Monge-Ampére equation in $M \times ([0,1] \times S^1)$. Suppose X is a Riemann surface with boundary. Denote $\pi_1 : M \times X \to M$ and $\pi_2 : M \times X \to X$ as the two natural projection maps, and let $\Omega = \pi_1^*\omega$. Then, given $\phi_0 \in C^\infty(M \times \partial X)$ such that $\Omega + \sqrt{-1}\partial\overline{\partial}\phi_0 > 0$ on each slice $M \times \{x\}$ for all $x \in \partial X$, we consider the Dirichlet boundary value problem:

$$\begin{cases} (\Omega + \sqrt{-1}\partial\overline{\partial}\phi)^{n+1} = 0, & \text{on } M \times X\ ; \\ \phi = \phi_0, & \text{on } M \times \partial X. \end{cases} \tag{2}$$

A solution is of geometric interest if $\Omega + \sqrt{-1}\partial\overline{\partial}\phi > 0$ when restricted on each slice $M \times \{x\}$ for all $x \in X$. Since the target manifold $\mathcal{H}$ is an infinitesimal symmetric space, any smooth solution of (2) can be re-interpreted (c.f. [7]) as a harmonic map from X to $\mathcal{H}$ with prescribed boundary map $\phi_0 : \partial X \to \mathcal{H}$. Any geodesic segment connecting ϕ_1 with ϕ_2 corresponds to an S^1 invariant solution of (2) with $X = [0,1] \times S^1$ and $\phi_0(0,\tau) = \phi_1(\tau)$, $\phi_0(1,\tau) = \phi_2(\tau)$. The notion of a geodesic ray is similar to the finite dimensional case: a geodesic ray in $\mathcal{H}$ is a geodesic segment which can be infinitely extended in one direction. In other words, a geodesic ray corresponds to an S^1 invariant solution of the following:

$$(\Omega + \sqrt{-1}\partial\overline{\partial}\phi)^{n+1} = 0, \ \text{ on } \ M \times ([0,\infty) \times S^1) \simeq M \times (D \setminus \{0\}), \tag{3}$$

where D is the closed unit disk.

In [7], Donaldson also conjectured that the existence of smooth geodesic rays where the K energy is strictly decreasing at the infinity is equivalent to the non-existence of constant scalar curvature metrics in $[\omega]$. Donaldson's conjecture certainly motivated the study of the existence of geodesic rays and related problems. However, the existence of geodesic rays is quite different from that of geodesic segments since the domain involved is naturally non-compact. More importantly, Donaldson [7] pointed out that the initial value problem for the geodesic ray equation is not always solvable in the smooth category. So we need to impose an alternative condition in order to solve equation (3) properly. Following Donaldson's program [7], this issue was discussed in [4]. According to [4], the initial Kähler potential together with the asymptotic direction (given by either an existing geodesic ray or an algebraic ray associated to a test configuration) forms a well-posed Dirichlet boundary value for equation (3). A set of new problems were discussed there which represents a mild attempt by the first named author to develop the existence theory for geodesic rays. In particular, he proved the existence of relative $C^{1,1}$ geodesic rays parallel to a given smooth geodesic ray under natural geometrical constraints. Unfortunately, there are few examples of the existence of geodesic rays in the literature, which creates serious problem for pushing the general existence theory further. In 2002, using Cauchy-Kowalewski theorem, C. Arezzo and G. Tian [1] proved the existence of a smooth geodesic ray asymptotically parallel to a special degeneration, or equivalently, to a simple test configuration(c.f. [9], [5]). One would like to see a more direct PDE proof of this important theorem. The main purpose of this note is to reprove the same theorem using the implicit function theorem.

Now we introduce the definition of *Kähler fibration* and *simple test configuration*.

Definition 1. A *Kähler fibration* (over the closed unit disk) is a map $\pi : (\mathcal{M}, J, \Omega) \to D$, where J is an integrable complex structure on $\mathcal{M}$, π is a holomorphic submersion, Ω is a closed two form on $\mathcal{M}$ which is compatible with J and it is a Kähler form on each fiber $M_z (z \in D)$(which is assumed to be compact without boundary).

Definition 2(c.f. [5], [9]). A (truncated) *simple test configuration* for a polarized Kähler manifold $L \to M$ is a Kähler fibration $\pi : (\mathcal{M}, J, \Omega) \to D$ together with a very ample line bundle Ł and a $\mathbb{C}^*$ equivariant embedding $\{Ł \to \mathcal{M} \to D\} \hookrightarrow \{\O(1) \to \mathbb{P}^N \times \mathbb{C} \to \mathbb{C}\}$, such that $\{L \to M\}$ is isomorphic $\{Ł|_{M_1} \to M_1\}$, where we denote $M_t = \pi^{-1}(t)$. Also, the $\mathbb{C}^*$ action on $\mathbb{C}$ is given by the standard multiplication, and the map $\mathbb{P}^N \times \mathbb{C} \to \mathbb{C}$ is simply the projection to the second factor, In addition, Ω should coincide with the restriction of the Fubini-Study metric on $\mathbb{P}^N$, while the induced S^1 actions on all these spaces are assumed to be unitary. Clearly all the fibers $\pi^{-1}(t)$ for $t \neq 0$ are biholomorphic to each other. A simple test configuration is called *product* if $\mathcal{M}$ is biholomorphic to $M \times \mathbb{C}$, and the $\mathbb{C}^*$ action on $\mathcal{M}$ is also a product action coming from $\mathbb{C}^*$ action on M and the standard multiplication on $\mathbb{C}$. It is called *trivial* if the $\mathbb{C}^*$ action on $\mathcal{M}$ is also trivial.

Remark 3. The above definition of a simple test configuration is essentially the same as the special degeneration studied by G. Tian first in [20].

Theorem 4 (Arezzo-Tian [1]). Given a non-trivial simple test configuration for $L \to M$, there exists a non trivial geodesic ray tamed by this test configuration.

According to [4], a geodesic ray is said to be *tamed by a test configuration* if it is asymptotically parallel to the algebraic ray defined by pulling back the Kähler potentials through the $\mathbb{C}^*$ action on $\mathcal{M}$.

We want to take a different route to prove this theorem. Following [8] and [5], smooth regular solutions to (3) are related to foliations of punctured holomorphic discs with some control on the total area. There is a Fredholm theory associated to the moduli space of holomorphic discs with totally real boundary condition. Deformation of this moduli space is the central topic of this note.

Arezzo-Tian's theorem is a consequence of the following proposition.

Proposition 5. let $\pi : (\mathcal{M}, J, \Omega) \to D$ be a Kähler fibration, there exists a smooth function ϕ defined in a neighborhood of the central fiber M_0 that solves the complex Monge-Ampére equation $(\Omega + \sqrt{-1}\partial\bar{\partial}\phi)^{n+1} = 0$ with $\Omega + \sqrt{-1}\partial\bar{\partial}\phi$ being positive on each fiber.

Remark 6. *Sphere at infinity*. The space of Kähler potentials is a (non-compact) infinite dimensional manifold with non-positive curvature. Like in the finite dimension case, we can formulate the notion of the geodesic sphere at infinity. Two geodesic rays determine the same point at infinity if and only if they are parallel to each other (or their distance stays bounded). In this sense, the asymptotic direction determined by either a geodesic ray or an algebraic test configuration, should define a point in the "sphere at infinity". The geodesic ray problems discussed in [4] can be viewed as a Dirichlet boundary value problem: Given a "point" in the "sphere at infinity" and an initial Kähler potential in $\mathcal{H}$, can we always find a geodesic ray connecting them? A geodesic ray is just a "geodesic segment" connecting a "finite point" with a point in the "sphere at infinity". Using this language, what Arezzo-Tian proved is that there is some geodesic connecting some "finite point" to the point in the "sphere at infinity" defined by the test configuration. Our proof using perturbation method actually provides slightly more: there are open sets in $\mathcal{H}$ asymptotically along algebraic degeneration of the test configuration, such that every point in these open sets emanates a geodesic ray to the given point at infinity. This is also a consequence of theorem 1.4 in [5].

The proof of Proposition 5 is based on a perturbation theory first introduced in [8] by Donaldson in the case of a trivial test configuration. In this note, we follow its generalization in [5]. By the definition of a Kähler fibration, $\mathcal{M}$ is always diffeomorphic to the product $M_0 \times D$. So we can for simplicity assume $\mathcal{M} = M \times D$ for a $2n$ dimensional smooth manifold M and the map π involved in the definition is the projection map to the second factor . Fix once and for all a cover of $M \times D$ by small balls, say $\{U_i\}_{i \in I}$. Following Donaldson's construction, we can associate a manifold $\mathcal{W}$ to any Kähler fibration, as follows: On each U_i, we choose local holomorphic coordinates to be $(z_1, \ldots, z_n, z)$, where z is simply given by π. Then Ω could be written as $\sqrt{-1}\partial\bar{\partial}\rho_i$ for some locally defined function ρ_i. $\mathcal{W}$ is obtained by twisting the vertical holomorphic cotangent bundle $E = T^*(M \times D)/\pi^*T^*D$. More precisely, we glue ξ in $E|_{U_i}$ with $\xi + \partial(\rho_i - \rho_j)$ in $E|_{U_j}$ over the corresponding fiber. It is easy to see that $\mathcal{W}$ is also a fibration over D and the canonical complex-symplectic structure on the holomorphic cotangent bundle induces a fiber-wise complex-symplectic form on $\mathcal{W}$. Furthermore, Ω defines an exact LS-graph[1] on each vertical fiber.

Of course, our construction of $\mathcal{W}$ is not canonical. However, if we fix an open cover and an initial Kähler fibration, then ρ_i could be chosen to depend smoothly on the data Ω and J for a small variation(Indeed, by the well known theorem of Newlander-Nirenberg, holomorphic coordinates could be made to vary smoothly. Then, one can follow the proof of Dolbeault's lemma to show this). Moreover, by definition, $\mathcal{W}$ is always diffeomorphic to E, or further, to the real vertical cotangent bundle, still denoted by E, which is independent of Ω and J. Therefore, if we pull back everything to the latter, a perturbation of Ω and J really gives us a perturbation of the complex-symplectic structure on E.

Now let $\phi_0 : \partial D \to \mathbb{R}$ be a smooth function such that $\Omega + \sqrt{-1}\partial\bar{\partial}\phi_0$ is positive on fibers over ∂D. Then it defines exact LS-graphs Λ_{z,ϕ_0} over any $z \in \partial D$. Following [8], [5], we have a one-to-one correspondence:

(A) A C^∞ solution ϕ to the homogeneous Monge-Ampére equation: $(\Omega + \sqrt{-1}\partial\bar{\partial}\phi_0)^{n+1} = 0$ satisfying the boundary condition $\phi|_{\partial D} = \phi_0$ and such that $\Omega + \sqrt{-1}\partial\bar{\partial}\phi$ still defines a Kähler fibration(together with J).

(B) A smooth map $G : M \times D \to E$ which covers the identity map on D, holomorphic in the second variable, and satisfies the boundary condition: for all $z \in \partial D$, $G(\cdot, z) \in \Lambda_{z,\phi_0}$ (Alternatively, we could view this as a family of holomorphic sections of the fibration $E \to D$ whose boundary lies in some totally real submanifold given by $\bigcup_{z \in \partial D} \Lambda_{z,\phi_0}$). In addition, we require that $p_1 \circ G(\cdot, 0)$ is the identity map,

[1] In a complex symplectic manifold (M, Θ), a submanifold L is called an *LS-submanifold* if L is Lagrangian with respect to $Re\Theta$, while the restriction of $Im\Theta$ on L is a symplectic form. For more details, see [8], [5].

and $p_1 \circ G(\cdot, z)$ is a diffeomorphism for any $z \in D$, where $p_1 : E \to M$ is the projection map.

Lemma 7. Perturbation of Ω, J and ϕ_0 preserves a smooth solution to the above equation, i.e. the compact family of normalized holomorphic discs in (B) is stable under perturbation.

To prove this Lemma, we need to set up a Fredholm theory for holomorphic discs with totally real boundary conditions. Denote by (X, Y) the space of maps from X to Y in an appropriate Sobolev space. Let $\mathcal{F}$ be the subspace of (D, E) which are sections of the fibration, i.e normalized maps. Fix J_0 on E, and a totally real submanifold R_0 of E with respect to J_0(For example, in our case, the exact Lagrangian graphs defined by the known smooth solution restricted on ∂D). Let $\mathcal{N}(R_0)$ be a neighborhood of R_0 in the space of all totally real submanifolds. For each $R \in \mathcal{N}(R_0)$, there is an associated diffeomorphism $\phi_R : R \to R_0$ which extends to a diffeomorphism of E. Moreover, We can choose ϕ_R to depend smoothly on R. Now let $\mathcal{B} = \cup_{u\in\mathcal{F}} u^*(TE)$ be an infinite dimensional vector bundle over $\mathcal{F}$, and $\mathcal{J}$ be the space of almost complex structures on E. Then $\mathcal{B} \times (\partial D, E)$ is a bundle over $\mathcal{F} \times \mathcal{J} \times \mathcal{N}(R_0)$, with a section $s(u, J, R) = (\bar{\partial}_J u, \phi_R^{-1} \circ u|_{\partial D})$. Fix J_0, and let s_0 be the restriction of s to the slice $\mathcal{F} \times \{J_0\} \times \mathcal{N}(R_0)$. A theorem of Oh [15] says that s_0 is transversal to the submanifold $\{0\} \times (\partial D, R_0)$ at a point (u_0, R_0) if u_0 is not multiply covered, i.e. there exists a $z \in \partial D$, such that $u_0^{-1}(u_0(z)) \cap \partial D = z$ and $Du_0(z) \neq 0$. So in our particular case s is transversal to $\{0\} \times (\partial D, R_0)$ at (u_0, J_0, R_0) for every disc coming from a solution of our previous equation (A). Therefore, $s^{-1}(\{0\} \times (\partial D, R_0))$ is smooth Banach manifold near (u_0, J_0, R_0).

Now consider the projection map $s^{-1}(\{0\} \times (\partial D, R_0)) \to \mathcal{J} \times \mathcal{N}(R_0)$, which is Fredholm of index $2n$(c.f [8], [5]). Given a smooth solution on $(M \times D, J, \Omega)$ as in (A), we have a 2n dimensional compact family of normalized holomorphic discs into (E, J_0), where J_0 is defined by J and Ω. Moreover, the holomorphic discs appearing in the family are all super-regular [2], and in particular regular. Now if we perturb J, Ω and ϕ_0, we are actually perturbing J_0 and R_0. Standard Fredholm theory ensures the existence of a nearby family of normalized regular holomorphic discs, which proves Lemma 7. $\square$

Proof of proposition 5. For $r \in (0, 1)$, let $\mathcal{M}(r)$ be the rescaled Kähler fibration defined by $(\mathcal{M}, J, \Omega)|_{|z|\leq r}$ with $\pi_r(w) = \pi(w)/r$. When r is small enough, $\mathcal{M}(r)$ is close to the trivial fibration given by the product $(M_0, J|_{M_0}, \Omega|_{M_0}) \times D$. The latter

[2]For a family of holomorphic discs $G : M \times D \to \mathcal{W}$ parameterized by M, we say that a disc $G_x(x \in M)$ is super-regular if the derivative $dp_1 \circ d_xG(\cdot, z) : T_xM \to T_{p_1\circ G(x,z)}M$ is surjective for all $z \in D$. It is proved in [8], [5] that a super-regular disc is automatically regular.

has an obvious solution to (A)(just take $\phi = 0$). Therefore by Lemma 7, for r small, we obtain a solution to the equation on $\mathcal{M}(r)$, which is the same as a solution near the central fiber on $\mathcal{M}$. $\square$

Proof of Theorem 4. If we use an S^1 invariant boundary condition as the initial perturbation data, the solution ϕ in proposition 5 will also be S^1 invariant(by the uniqueness as proved in [7]). Then we obtain a geodesic ray on the fiber M_1 by pulling back the restriction of $\Omega + \sqrt{-1}\partial\bar{\partial}\phi$ on each fiber to a fixed fiber by the $\mathbb{C}^*$ action, and we also get a foliation by punctured holomorphic discs on $M_1 \times (D \setminus \{0\})$. Furthermore, if the test configuration is non trivial, the corresponding foliation would not be trivial since the $\mathbb{C}^*$ action on $\mathcal{M}$ is not along the leaf direction given by the orthogonal complement of the tangent space of the fibers with respect to $\Omega + \sqrt{-1}\partial\bar{\partial}\phi$. Thus, in this case, we do get a nontrivial geodesic ray. Since ϕ is smooth on $\mathcal{M}$, the geodesic ray is parallel to the algebraic ray defined simply by pulling back Ω through the $\mathbb{C}^*$ action. $\square$

Remark 8. So far we have been talking about simple test configurations, which by definition have smooth central fiber. In general(c.f. [9]), we should allow singular central fibers. Our proof of the existence of a smooth geodesic ray does not directly extend to the general case since then we need to do perturbation theory on non smooth manifolds. However it might be possible to apply theorem 4 to test configurations with some mild singularities . We might hope to blow up the singularities to yield a simple test configuration which is biholomorphic to the original one everywhere except the central fiber, while the central fiber is a resolution of singularities of the original one. We do not know whether a general procedure exists to realize this. Here we only describe an approach by dealing with a specific example(called Atiyah's "flop"), as follows:

Let X be the singular hypersurface in $\mathbb{C}^4$ defined by the equation $x_1x_2 + x_3^2 - x_4^2 = 0$, and $\pi : X \to \mathbb{C}$ is the projection to the last factor. Clearly there is exactly one singularity of the total space X lying on the central fiber, and it is easy to see all the other fibers are biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with one line (of degree 2) removed. The central fiber is a singular quadric, whose minimal resolution is a line bundle over $\mathbb{P}^1$ of degree -2. We could obtain a smooth 3-fold $\widetilde{X}$ whose central fiber is also smooth, by blowing up the Weil divisor (which is not Cartier) defined by $x_2 = x_3 + x_4 = 0$. More precisely, Let $\widetilde{X}$ be the closure of $p^{-1}(X \setminus \{0\})$ in

$$Y = \{((x_1, x_2, x_3, x_4), [y_2 : y_3]) \in \mathbb{C}^4 \times \mathbb{P}^1 | x_2y_3 = (x_3 + x_4)y_2\}$$

where $p : Y \to \mathbb{C}^4$ is the projection map. The fibration structure naturally survives and the projection map $p : \widetilde{X} \to X$ is a biholomorphism away from the central fiber, while the central fiber is exactly the minimal resolution. Here the fibers are all noncompact, but we can in stead consider the compactified space $\mathcal{M}$ which is defined in $\mathbb{P}^4$ by the same equation. The map $\pi : \mathcal{M} \to \mathbb{P}^1$ sending $[x_1, \cdots, x_5]$ to $[x_4 : x_5]$ is well defined away from a $\mathbb{P}^1$ cut by equations $x_1 x_2 + x_3^2 = 0$ and $x_4 = x_5 = 0$. So by blowing up this $\mathbb{P}^1$ we can define the projection map. Now repeat the previous construction, we get a smooth resolution $\widetilde{\mathcal{M}}$. The central fiber is Σ_2(degree 2 Hirzebruch surface), while other fibers are all $\mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see there is a natural $\mathbb{C}^*$ action on these spaces. Therefore theorem 4 asserts there is a smooth geodesic ray on $\mathbb{P}^1 \times \mathbb{P}^1$ induced by $\widetilde{\mathcal{M}}$, which is the same as that induced by $\mathcal{M}$. Note here the geodesic ray lies in the Kähler class of the restriction of the Fubini-Study metric through the embedding of $\widetilde{\mathcal{M}}$, not $\mathcal{M}$ itself.

An interesting question is, to what extent, one can generalize this example to all dimensions. What are the natural conditions we should impose on the central fiber or the test configurations?

Another interesting question is: Given a sequence of Kähler potentials in $\mathcal{H}$ which is bounded in the sense of Cheeger-Gromov, but not bounded in the holomorphic category. Does there exist a point in the “sphere at infinity” which reflects this non-compactness or degeneracy?

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