THE BOGOMOLOV-MIYAOKA-YAU INEQUALITY FOR STACKY SURFACES

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Abstract. We present a generalization of the Bogomolov-Miyaoka-Yau inequality to Deligne-Mumford surfaces of general type.

1. Introduction

We work over $\mathbb{C}$.

For a smooth complex projective surface S of general type, the Bogomolov-Miyaoka-Yau inequality for S reads (see [8])

$$3c_2(T_S) \geq c_1(T_S)^2. \tag{1.1}$$

Together with Noether's inequality, this puts constraints on the topology of surfaces of general types. Generalizations of (1.1) to singular surfaces and surface pairs have been found, see for example [9], [5, 6].

In this paper we present a generalization of (1.1) to Deligne-Mumford stacks. Let $\mathcal{X}$ be a smooth proper Deligne-Mumford $\mathbb{C}$-stack of dimension 2. Let $\pi : \mathcal{X} \to X$ be the natural map to the coarse moduli space. We assume that X is a projective variety. Since $\mathcal{X}$ is assumed to be smooth, it has a tangent bundle $T_{\mathcal{X}}$. A good theory of Chern classes is available for Deligne-Mumford stacks, see for example [14], [4].

Main Theorem 1.1. *Let $\mathcal{X}$ be as above. Assume that the canonical bundle $K_{\mathcal{X}} := \wedge^2 T_{\mathcal{X}}^\vee$ is numerically effective, then*

$$3c_2(T_{\mathcal{X}}) \geq c_1(T_{\mathcal{X}})^2. \tag{1.2}$$

Certainly (1.2) takes the same shape as (1.1). A proof of (1.2), along the lines of Miyaoka's original proof of (1.1) in [8], is given in Section 2. Section 3 contains examples of (1.2). In Section 3.1 we consider (1.2) for a class of stacks $\mathcal{X}$ with stack structures in codimension 1, recovering [6, Theorem 0.1]. In Section 3.2 we consider (1.2) for Gorenstein stacks $\mathcal{X}$ with isolated stack points, recovering [9, Corollary 1.3].

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2. Proof of (1.2)

In this Section we give a proof of (1.2). Our proof is adapted from Miyaoka's original proof in [8].

Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack of dimension 2. If $\mathcal{X}$ has non-trivial stack structures at generic points, then $\mathcal{X}$ is an étale gerbe over a stack with trivial generic stack structure, see for example [2, Proposition 4.6]. More precisely, there is a finite group G, a stack $\mathcal{X}'$ with trivial generic stabilizers, and a morphism $f : \mathcal{X} \to \mathcal{X}'$ realizing $\mathcal{X}$ as a G-gerbe over $\mathcal{X}'$. Since $T_{\mathcal{X}} = f^*T_{\mathcal{X}'}$, we see that (1.2) for $\mathcal{X}$ is equivalent to (1.2) for $\mathcal{X}'$. Therefore it suffices to consider only those $\mathcal{X}$ with stack structures in codimension ≥ 1. For the rest of this section we assume this.

Let $\mathcal{F}$ be a locally free sheaf of rank 2 on $\mathcal{X}$. Let $\mathcal{V} := \mathbb{P}(\mathcal{F})$ be the projectivization, with natural projection $p : \mathcal{V} \to \mathcal{X}$. Let $\mathcal{H}$ be the divisor associated to the tautological sheaf on $\mathcal{V}$.

Lemma 2.1. *Assume that $\mathcal{W} \subset \mathcal{V}$ is linearly equivalent to $\mathcal{H} - p^*\mathcal{D}$, where $\mathcal{D} \subset \mathcal{X}$ is a divisor on $\mathcal{X}$. Then we have*

$$\mathcal{D} \cdot det\mathcal{F} \leq c_2(\mathcal{F}) + \mathcal{D}^2. \tag{2.1}$$

Proof. We closely follow Miyaoka's original proof [8]. Let $i : \mathcal{W} \subset \mathcal{V}$ be the inclusion morphism. Note that the composition $p \circ i : \mathcal{W} \to \mathcal{X}$ is birational by our assumption on the linear equivalence class of $\mathcal{W}$. Since resolutions can be chosen such that they are compatible with étale base change, there is a sequence of blow-ups

$$\mu : \mathcal{V}_s \xrightarrow{\mu_s} \mathcal{V}_{s-1} \to \cdots \to \mathcal{V}_1 \xrightarrow{\mu_1} \mathcal{V}_0 = \mathcal{V} \tag{2.2}$$

such that the proper transform $\mathcal{W}'$ of $\mathcal{W}$ is a smooth Deligne-Mumford stack in $\mathcal{V}_s$. Let $i' : \mathcal{W}' \subset \mathcal{V}_s$ and $\rho : \mathcal{W}' \to \mathcal{X}$ be the natural maps.

Let $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_s$ be the exceptional divisors on $\mathcal{V}_s$. The divisor $\mathcal{W}'$ is linearly equivalent to $\mu^*(\mathcal{H} - p^*\mathcal{D}) - \sum a_i\mathcal{E}_i$. It can be seen[1] that the canonical bundle $K_{\mathcal{W}'}$ satisfies $K_{\mathcal{W}'} = \rho^*K_{\mathcal{X}} + \sum \mathcal{C}_i$ where C_i is a curve and $\rho(\mathcal{C}_i) = point$. By the Hodge index theorem (for a stacky version see [7, Theorem 3.1.3]), it follows that $(K_{\mathcal{W}'} - \rho^*K_{\mathcal{X}} + \sum c_i i'^*\mathcal{E}_i)^2 \leq 0$ for any $c_i \in \mathbb{R}$.

Write $K_{\mathcal{V}_s} = \mu^*(-2\mathcal{H} + p^*K_{\mathcal{X}} + p^*(det\mathcal{F})) + \sum b_i\mathcal{E}_i$. The adjunction formula implies that

$$K_{\mathcal{W}'} = i'^*[\mu^*(-\mathcal{H}) + (p \circ \mu)^*(K_{\mathcal{X}} + det\mathcal{F} - \mathcal{D}) + \sum (b_i - a_i)\mathcal{E}_i].$$

Thus $i'^*\mu^*(-\mathcal{H} + p^*(det\mathcal{F} - \mathcal{D}))^2 \leq 0$. Set $k := i'^*\mu^*(-\mathcal{H} + p^*(det\mathcal{F} - \mathcal{D}))^2$. We can also compute this self-intersection number k in another way:

$$\begin{aligned} k &= \mu^*(-\mathcal{H} + p^*(det\mathcal{F}) - p^*\mathcal{D})^2(\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D} - \sum a_i\mathcal{E}_i) \\ &= \mu^*(-\mathcal{H} + p^*(det\mathcal{F}) - p^*\mathcal{D}))^2(\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D}) \qquad \text{(since } \mathcal{E}_i \text{ is exceptional)} \\ &= \mathcal{H}^3 - \mathcal{H}^2 \cdot p^*(\mathcal{D} - 2det\mathcal{F}) + \mathcal{H} \cdot (p^*(det\mathcal{F})^2 - (p^*\mathcal{D})^2). \end{aligned}$$

[1] The argument is similar to that of [8, Lemma 7] and is omitted.

Using the standard relations for the intersection numbers on the projectivization of a rank 2 vector bundle, we calculate that

$$k = c_1^2(det(\mathcal{F})) - c_2(\mathcal{F}) - (det\mathcal{F})^2 + det\mathcal{F}\cdot\mathcal{D} - \mathcal{D}^2 = -c_2(\mathcal{F}) + det\mathcal{F}\cdot\mathcal{D} - \mathcal{D}^2.$$

The result follows. □

Let $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ be a subsheaf of $\Omega^1_{\mathcal{X}}$. One key observation used in Miyaoka's original proof is that the Iitaka dimension of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is at most 1.

Theorem 2.1. *If $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is a subsheaf of $\Omega^1_{\mathcal{X}}$ of a projective Deligne-Mumford stack $\mathcal{X}$, then $h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n\mathcal{D})) \leq cn$ for some positive constant c and $n >> 0$.*

The proof of Theorem 2.1 is very similar to that of [8, Theorem 2"]. We only remark that we need to use Riemann-Roch for stacks proved in [11].

One can prove the following result using Theorem 2.1.

Proposition 2.2. *Let $\mathcal{F} \subset \Omega^1_{\mathcal{X}}$ be a locally free sheaf of rank 2 and assume that $det(\mathcal{F})^{\otimes n}$ is generated by global sections for some $n > 0$. If $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then*

$$\mathcal{D}\cdot det(\mathcal{F}) \leq max\{c_2(\mathcal{F}), 0\}.$$

Proof. Consider $p : \mathcal{V} = \mathbb{P}(\mathcal{F}) \to \mathcal{X}$. The canonical isomorphism gives us

$$H^0(\mathcal{X}, \mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})) = H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(\mathcal{H} - p^*\mathcal{D})).$$

If $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then $|\mathcal{H} - p^*\mathcal{D}|$ is non-empty. Pick $\mathcal{W} \in |\mathcal{H} - p^*\mathcal{D}|$. Decompose $\mathcal{W}$ as $\mathcal{W} = \mathcal{W}_0 + p^*\mathcal{D}'$ where $\mathcal{W}_0$ is effective and irreducible which is linearly equivalent to $\mathcal{H} - p^*(\mathcal{D}+\mathcal{D}')$ and $\mathcal{D}'$ is effective. Note that $(det\mathcal{F})^{\otimes n}$ is generated by global sections, so the intersection number $\mathcal{D}'\cdot det(\mathcal{F}) \geq 0$. It follows that $\mathcal{D}\cdot det(\mathcal{F}) \leq (\mathcal{D}+\mathcal{D}')\cdot det(\mathcal{F})$ and it suffices to prove $(\mathcal{D}+\mathcal{D}')\cdot det(\mathcal{F}) \leq max\{c_2(\mathcal{F}), 0\}$. Set $\mathcal{D}'' = \mathcal{D}+\mathcal{D}'$ to simplify notation. By Lemma 2.1, $\mathcal{D}''\cdot det(\mathcal{F}) \leq c_2(\mathcal{F}) + \mathcal{D}''\cdot\mathcal{D}''$. Observe that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}'')$ is a subsheaf of $\Omega^1_{\mathcal{X}}$. Indeed, the effectiveness of $\mathcal{W}_0$ ensures the existence of a non-trivial section of $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$, i.e. an injection $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$. Twisting by $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$, embeds $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$ into $\mathcal{F}\subset\Omega^1_{\mathcal{X}}$. By Theorem 2.1, $\mathcal{D}''$ has Iitaka dimension at most 1. It follows that $\mathcal{D}''\cdot det(\mathcal{F}) \leq 0$ or $\mathcal{D}''\cdot\mathcal{D}'' \leq 0$. [2] This completes the proof. □

Assuming $c_2(\mathcal{F})$ is positive for the time being, we can obtain an upper bound on c_2 provided the sheaf $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has no sections. This can then be used to derive a contradiction. To be more precise, one needs a modified version of Proposition 2.2, in which the condition on the sheaf $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ having a section is replaced by the condition that some symmetric power $S^m\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ having a section.

Theorem 2.2. *Let $\mathcal{F} \subset \Omega^1_{\mathcal{X}}$ be a locally free sheaf of rank 2 and assume that $det(\mathcal{F})^{\otimes n}$ is generated by global sections for some $n > 0$. If $S^m\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then*

$$\mathcal{D}\cdot det(\mathcal{F}) \leq max\{mc_2(\mathcal{F}), 0\}.$$

The proof of Theorem 2.2 follows from Proposition 2.2 and the following easy lemma (which is analogous to [8, Lemma 11]).

[2] Arguing as in [8, Lemma 10].

Lemma 2.3. *Let $p : \mathcal{V} = \mathbb{P}(\mathcal{F}) \to \mathcal{X}$ be the projective bundle of a locally free sheaf of rank 2. Let $\mathcal{W} \in |m\mathcal{H} - p^*\mathcal{D}|$. Then there is a surjective morphism $\beta : \mathcal{X}' \to \mathcal{X}$ such that $\beta^*\mathcal{W}$ is decomposed to $\mathcal{W}_1 + \cdots \mathcal{W}_m$ where $\mathcal{W}_i$ is an effective divisor linear equivalent to $\mathcal{H}' - p^*\mathcal{D}_i$.*

Proof of Theorem 2.2. The following argument is taken from [8, Theorem 3]. Let f be a global section of $S^m\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$. Lemma 2.3 imples that after a suitable cover $\beta : \mathcal{Y} \to \mathcal{X}$, we can decompose β^*f can be written as $f_1 f_2 \cdots f_m \in H^0(\mathcal{Y}, S^m\beta^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}))$, where $f_i \in H^0(\mathcal{Y}, \beta^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}_i))$ and $(det\beta^*\mathcal{F})^{\otimes m} \cong (\beta^*det(\mathcal{F})^{\otimes m})$ is generated by global sections. From Proposition 2.2, it follows that $\beta^*\mathcal{D}_i \cdot (det(\beta^*\mathcal{F})) \le max\{c_2(\beta^*\mathcal{F}), 0\}$. Summing over all i's, we have $\beta^*\mathcal{D} \cdot det(\beta^*\mathcal{F}) \le max\{mc_2(\beta^*\mathcal{F}), 0\}$. Let d be the mapping degree of β. Clearly, $\beta^*\mathcal{D} \cdot det(\beta^*\mathcal{F}) = dD \cdot det(\mathcal{F})$ and $c_2(\beta^*\mathcal{F}) = d\beta^*c_2(\mathcal{F})$. □

We now come to (1.2).

Theorem 2.3. *Let $\mathcal{X}$ be a non-singular Deligne-Mumfors stack with the projective coarse space X of general type and $c_1(\mathcal{X})$ nef. Then $c_1^2(\mathcal{X}) \le 3c_2(\mathcal{X})$ holds.*

Proof. As in [8], we consider two cases: (1) $c_1^2(\mathcal{X}) \le 2c_2(\mathcal{X})$ and (2) $c_1^2(\mathcal{X}) > 2c_2(\mathcal{X})$. The first case is obvious. For the second case, set $\alpha := \frac{c_2(\mathcal{X})}{c_1^2(\mathcal{X})}$. Note that $\alpha < 1/2$. Pick $\delta > 0$ sufficiently small and rational. By Theorem 2.2 applied to $\mathcal{D} = m(\alpha+\delta)K_{\mathcal{X}}$, $\mathcal{F} = \Omega^1_{\mathcal{X}}$, we can find a positive integer m such that $m(\alpha+\delta) \in \mathbb{Z}$, and

$$h^0(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = 0.$$

By Serre duality for smooth projective Deligne-Mumford stacks [10, Theorem 2.22], we have

$$h^2(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = h^0(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(1-\alpha-\delta)K_{\mathcal{X}}) \otimes K_{\mathcal{X}}).$$

As $\alpha < 1/2$ and δ is small, we have $1-\alpha-\delta > \alpha$. We apply Theorem 2.2 to $\mathcal{D} = m(2-\alpha-\delta)K_{\mathcal{X}}$, $\mathcal{F} = \Omega^1_{\mathcal{X}}$, to get

$$h^2(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = 0.$$

Hence

$$\chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha+\delta)K_{\mathcal{X}})) = -h^1(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) \le 0.$$

Note that to compute the cohomology groups of a (subsheaf of) symmetric power of a vector bundle, one can work on the the projectivized vector bundle and computing the cohomology groups of relevant line bundles. Thus

$$0 \ge \chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha+\delta)K_{\mathcal{X}})) = \chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H} - (\alpha+\delta)\pi^*K_{\mathcal{X}})).$$

By Riemann-Roch for stacks [11], we have $\chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H} - (\alpha+\delta)\pi^*K_{\mathcal{X}}))$ grows like $\frac{1}{6}(\mathcal{H} - (\alpha+\delta)\pi^*K_{\mathcal{X}})^3m^3$ as $m \to \infty$. It implies that $(\mathcal{H} - (\alpha+\delta)\pi^*K_{\mathcal{X}})^3 \le 0$. Taking δ to 0, we obtain

$$\begin{aligned} 0 \ge (\mathcal{H} - \alpha\pi^*K_{\mathcal{X}})^3 =& c_1^2(\mathcal{X}) - c_2(\mathcal{X}) - 3\alpha c_1^2(\mathcal{X}) + 3\alpha^2 c_1^2(\mathcal{X}) \\ =& (1 - \alpha - 3\alpha + 3\alpha^2)c_1^2(\mathcal{X}) \\ =& (1-\alpha)(1-3\alpha)c_1^2(\mathcal{X}). \end{aligned}$$

Since $\alpha < 1/2$ and $c_1^2(\mathcal{X})$ is non-negative, we get $1 - 3\alpha \le 0$ as desired. □

3. Examples of (1.2)

3.1. **Codimension 1 stack structure.** We consider (1.2) for an example of stack $\mathcal{X}$ with stack structures in codimension 1.

Let X be a smooth complex projective surface and D a simple normal crossing $\mathbb{Q}$-divisor of the form $D = \sum_i (1-1/r_i)D_i$ with $r_i \geq 2$ integers. Let $\mathcal{X}$ be the natural stack cover of the pair (X, D), see [3, Definition 2.1] for its definition. By construction the coarse moduli space of $\mathcal{X}$ is X. The natural map $\pi : \mathcal{X} \to X$ is an isomorphism outside $\pi^{-1}(\operatorname{Supp} D)$, which is where $\mathcal{X}$ has non-trivial stack structures. Furthermore we have the following formula for the canonical bundle:

$$K_{\mathcal{X}} = \pi^*(K_X + D). \tag{3.1}$$

We now examine (1.2) for this $\mathcal{X}$. By (3.1),

$$c_1(T_{\mathcal{X}})^2 = c_1(K_{\mathcal{X}})^2 = (K_X + D)^2.$$

By Gauss-Bonnet theorem for Deligne-Mumford stacks [12, Corollaire 3.44] we have

$$c_2(T_{\mathcal{X}}) = \chi(\mathcal{X}),$$

the Euler characteristic of $\mathcal{X}$ as defined in [12, Definition 3.43] (note that the notation χ^{orb} is used in [12]). Put

$$\mathcal{D}_i := \pi^{-1}(D_i), \quad \mathcal{D}_i^\circ := \mathcal{D}_i \setminus (\cup_{j \neq i}(\mathcal{D}_i \cap \mathcal{D}_j)).$$

Then we have

$$\chi(\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D)) = \chi(\mathcal{X}) - \sum_i \chi(\mathcal{D}_i^\circ) - \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Similarly, put $D_i^\circ = D_i \setminus (\cup_{j \neq i}(D_i \cap D_j))$, we have

$$\chi(X \setminus \operatorname{Supp} D) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\bar{p} \in D_i \cap D_j} \chi(\bar{p}).$$

Since $\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D) \simeq X \setminus \operatorname{Supp} D$, we have $\chi(\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D)) = \chi(X \setminus \operatorname{Supp} D)$. Equivalently,

$$\chi(\mathcal{X}) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\bar{p} \in D_i \cap D_j} \chi(\bar{p}) + \sum_i \chi(\mathcal{D}_i^\circ) + \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Since the map $\mathcal{D}_i^\circ \to D_i^\circ$ is of degree $1/r_i$ and the map $\mathcal{D}_i \cap \mathcal{D}_j \to D_i \cap D_j$ is of degree $1/r_i r_j$, we have

$$\chi(\mathcal{D}_i) = \frac{1}{r_i}\chi(D_i), \quad \chi(\mathcal{D}_i \cap \mathcal{D}_j) = \frac{1}{r_i r_j}\chi(D_i \cap D_j).$$

This implies that

$$\chi(\mathcal{X}) = \chi(X) - \sum_i (1 - 1/r_i)\chi(D_i^\circ) + \sum_{\bar{p} \in D_i \cap D_j} (1/r_i r_j - 1). \tag{3.2}$$

By [6, Theorem 8.7], for $\bar{p} \in D_i \cap D_j$ the local orbifold Euler number of the pair (X, D) at $\bar{p}$ is given by $e_{orb}(\bar{p}; X, D) = 1/r_i r_j$. Together with (3.2) this implies that $\chi(\mathcal{X})$ coincides with the orbifold Euler number $e_{orb}(X, D)$ of the pair (X, D), as defined in [6]. Thus if $K_{\mathcal{X}}$ is numerically effective, then (1.2) is equivalent to [6, Theorem 0.1] applied to the pair (X, D).

3.2. **Condimension 2 stack structure.** Let $\mathcal{X}$ be a smooth proper Deligne-Mumford $\mathbb{C}$-stack of dimension 2 with isolated stack structures. Let $\pi : \mathcal{X} \to X$ be the natural map to the coarse moduli space X. Let $p_1, p_2, ..., p_k \in \mathcal{X}$ be the stacky points. Suppose that $\mathcal{X}$ is Gorenstein, i. e. each stacky point p_i has a neighborhood $p_i \in U_i \subset \mathcal{X}$ of the form $U_i \simeq [\mathbb{C}^2/G_i]$ with $G_i \subset SU(2)$ a finite subgroup, identifying p_i with $[0/G_i] \in [\mathbb{C}^2/G_i]$. It is a standard fact that the coarse moduli space X is a projective surface with canonical singularities.

Suppose further that $K_\mathcal{X}$ is numerically effective. We consider (1.2) for such $\mathcal{X}$.

By assumption we have $K_\mathcal{X} = \pi^* K_X$. Thus

$$c_1(T_\mathcal{X})^2 = c_1(K_\mathcal{X})^2 = c_1(K_X)^2.$$

We now consider the term $c_2(T_\mathcal{X})$. The first step is to consider $\chi(\mathcal{O}_\mathcal{X})$ by using Riemann-Roch theorem for stacks [12, 11]. We follow [13, Appendix A] for the presentation of the Riemann-Roch theorem. We have

$$\chi(\mathcal{O}_\mathcal{X}) = \int_{I\mathcal{X}} \widetilde{ch}(\mathcal{O}_\mathcal{X}) \widetilde{Td}(T_\mathcal{X}).$$

Here $I\mathcal{X}$ is the inertia stack of $\mathcal{X}$. By our assumption on $\mathcal{X}$, we have the following description of $I\mathcal{X}$:

$$I\mathcal{X} = \mathcal{X} \cup \bigcup_{i=1}^{k} (Ip_i \setminus p_i).$$

Here the term $Ip_i \setminus p_i$ is the inertia stack of $p_i \simeq BG_i$ with the main component removed, namely

$$Ip_i \setminus p_i \simeq \bigcup_{(g)\neq(1):\text{conjugacy class of } G_i} BC_{G_i}(g),$$

where $C_{G_i}(g) \subset G_i$ is the centralizer subgroup of $g \in G_i$ and $BC_{G_i}(g)$ is its classifying stack. By the definition of the Chern character $\widetilde{ch}$, we have $\widetilde{ch}(\mathcal{O}_\mathcal{X}) = 1$ on every component of $I\mathcal{X}$. Hence

$$\chi(\mathcal{O}_\mathcal{X}) = \int_{I\mathcal{X}} \widetilde{Td}(T_\mathcal{X}) = \int_\mathcal{X} \widetilde{Td}(T_\mathcal{X})|_\mathcal{X} + \sum_{i=1}^{k} \int_{Ip_i \setminus p_i} \widetilde{Td}(T_\mathcal{X})|_{Ip_i \setminus p_i}. \tag{3.3}$$

Note that $\widetilde{Td}(T_\mathcal{X})|_\mathcal{X} = Td(T_\mathcal{X})$, and we only need its degree 2 component. Hence

$$\int_\mathcal{X} \widetilde{Td}(T_\mathcal{X})|_\mathcal{X} = \frac{1}{12} \int_\mathcal{X} (c_2(T_\mathcal{X}) + c_1(T_\mathcal{X})^2). \tag{3.4}$$

The contribution coming from $Ip_i \setminus p_i$ can be also evaluated.

Lemma 3.3. *Let E_i be the exceptional divisor of the minimal resolution of $\mathbb{C}^2/G_i$. Then*

$$\int_{Ip_i \setminus p_i} \widetilde{Td}(T_\mathcal{X})|_{Ip_i \setminus p_i} = \frac{1}{12}(\chi(E_i) - \frac{1}{|G_i|}).$$

This Lemma is proved in the Appendix.

Next, we reinterpret the term $\chi(\mathcal{O}_{\mathcal{X}})$. By definition, $\chi(\mathcal{O}_{\mathcal{X}}) := \sum_{l\geq 0}(-1)^l \dim H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Since $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ (see e.g. [1, Theorem 2.2.1]), we have $H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^l(X, \mathcal{O}_X)$ and

$$\chi(\mathcal{O}_{\mathcal{X}}) = \chi(\mathcal{O}_X). \tag{3.5}$$

Combining (3.3), (3.4), (3.5), and Lemma 3.3, we obtain the following expression of $c_2(T_{\mathcal{X}})$:

$$\int_{\mathcal{X}} c_2(T_{\mathcal{X}}) = 12\chi(\mathcal{O}_X) - \int_{\mathcal{X}} c_1(T_{\mathcal{X}})^2 - \sum_{i=1}^{k}(\chi(E_i) - 1/|G_i|). \tag{3.6}$$

Using this, we see that in the present situation, (1.2) is equivalent to

$$12\chi(\mathcal{O}_X) \geq \frac{4}{3}c_1(K_X)^2 + \sum_{i=1}^{k}(\chi(E_i) - \frac{1}{|G_i|}). \tag{3.7}$$

On the other hand, it is clear that (3.7) is a special case of [9, Corollary 1.3].

Appendix A. Proof of Lemma 3.3

In this Appendix we prove Lemma 3.3. By our assumption on $\mathcal{X}$, for $g \in G_i$, the g-action on the tangent space $T_{p_i}\mathcal{X}$ has two eigenvalues ξ_g and ξ_g^{-1}, where ξ_g is a certain root of unity. By the definition of $\widetilde{Td}(T_{\mathcal{X}})$ we have

$$\int_{Ip_i\setminus p_i} \widetilde{Td}(T_{\mathcal{X}})|_{Ip_i\setminus p_i} = \sum_{(g)\neq(1):\text{conjugacy class of } G_i} \frac{1}{|C_{G_i}(g)|}\frac{1}{2-\xi_g-\xi_g^{-1}}. \tag{A.1}$$

We now evaluate (A.1) using the ADE classification of $\mathbb{C}^2/G_i$.

A.1. **Type A.** If $\mathbb{C}^2/G_i$ is of type A_{n-1}, then $G_i \simeq \mathbb{Z}_n$ and the action on $\mathbb{C}^2$ is given as follows. If we identify $\mathbb{Z}_n$ with the group of n-th roots of 1, then an element $\xi \in \mathbb{Z}_n$ acts on $\mathbb{C}^2$ via the matrix

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

It follows that (A.1) is given by

$$\frac{1}{n}\sum_{l=1}^{n-1}\frac{1}{2-\exp(2\pi\sqrt{-1}l/n)-\exp(2\pi\sqrt{-1}l/n)^{-1}}. \tag{A.2}$$

By [7, Lemma 3.3.2.1], (A.2) is equal to

$$\frac{n^2-1}{12n} = \frac{1}{12}(n-1/n).$$

Since the exceptional divisor of the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$ is a chain of $(n-1)$ copies of $\mathbb{CP}^1$, its Euler characteristic is n. This proves the Lemma in type A case.

A.2. **Type D.** If $\mathbb{C}^2/G_i$ is of type D_{n+2} (here $n \geq 2$), then G_i is isomorphic to the binary dihedral group Dic_n. The group Dic_n is of order $4n$ and may be presented as follows:

$$Dic_n = \left\langle a, x | a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \right\rangle.$$

The action of Dic_n on $\mathbb{C}^2$ is given as follows:

$$a \mapsto \begin{pmatrix} \exp(\pi\sqrt{-1}/n) & 0 \\ 0 & \exp(-\pi\sqrt{-1}/n) \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.3}$$

An element calculation shows that the conjugacy classes of Dic_n and the orders of their centralizer subgroups are given as follows:

(A.4)

$\{1\}$, $\{a^n\}$ (order of centralizer group $= 4n$)

$\{a^l, a^{-l}\}, 1 \leq l \leq n-1$, (order of centralizer group $= 2n$)

$\{xa, xa^3, xa^5, ..., xa^{2n-1}\}$, $\{x, xa^2, xa^4, ..., xa^{2n-2}\}$ (order of centralizer group $= 4$).

Using (A.3) and (A.4) it is easy to identify the contribution from each conjugacy class. It follows that (A.1) is given by

$$\frac{1}{2n}\sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{16n} + \frac{1}{8} + \frac{1}{8}. \tag{A.5}$$

We need to evaluate the sum $\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}}$. Again by [7, Lemma 3.3.2.1], we have

$$\begin{aligned}\frac{(2n)^2-1}{12} =& \sum_{k=1}^{2n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}k/(2n)) - \exp(2\pi\sqrt{-1}k/(2n))^{-1}} \\ =& \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{4} \\ &+ \sum_{k=1}^{n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}}.\end{aligned}$$

Note that

$$\begin{aligned} & 2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1} \\ =& 2 + \exp(\pi\sqrt{-1}k/n) + \exp(\pi\sqrt{-1}k/n)^{-1} \\ =& 2 + 2\cos(\pi k/n) = 4\cos^2(\pi k/(2n)) = 4\sin^2((\pi(k+n)/(2n)); \\ & 2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1} \\ =& 2 - 2\cos(\pi k/n) = 4\sin^2(\pi k/(2n)).\end{aligned}$$

Since $\sin(\pi(k+n)/(2n)) = -\sin(\pi(k-n)/(2n))$, we see that

$$\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}}$$
$$= \sum_{k=1}^{n-1} \frac{1}{2-\exp(2\pi\sqrt{-1}(n+k)/(2n))-\exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}},$$

from which it follows that

$$2\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{4} = \frac{(2n)^2-1}{12}.$$

This shows that

$$\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}} = \frac{n^2-1}{6}$$

and (A.1) is given by

$$\frac{n^2-1}{12n} + \frac{1}{16n} + \frac{1}{8} + \frac{1}{8} = \frac{1}{12}(n+3-\frac{1}{4n}).$$

Since the exceptional divisor of the minimal resolution of $\mathbb{C}^2/Dic_n$ is a tree of $\mathbb{CP}^1$ whose dual graph is the Dynkin diagram D_{n+2}, its Euler characteristic is $n+3$ and the Lemma is proved in this case.

A.3. **Type E.** If $\mathbb{C}^2/G_i$ is of type E, then there are three possibilities: E_6, E_7, E_8. The group G_i is isomorphic to the binary tetrahedral group (for E_6), the binary octahedral group (for E_7), or the binary icosahedral group (for E_8). In each case the group and its action on $\mathbb{C}^2$ can be explicitly described, and the Lemma can be proved by computing (A.1) using this information. We work out the details for E_6 and leave the other two cases to the reader.

In the E_6 case, the group G_i is isomorphic to the binary tetrahedral group $2T$. This group is of order 24 and its elements can be identified with the following quaternion numbers:

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k), \quad \pm i, \quad \pm j, \quad \pm k, \quad , \pm 1.$$

The group $2T$ has 7 conjugacy classes:

Conjugacy Class	(1)	(-1)	(i)	$(\frac{1}{2}(1+i+j+k))$
Size	1	1	6	4
Conjugacy Class	$(\frac{1}{2}(1+i+j-k))$	$(\frac{1}{2}(-1+i+j+k))$	$(\frac{1}{2}(-1+i+j-k))$	
Size	4	4	4	

The action of $2T$ on $\mathbb{C}^2$ can be described using the following identification

$$x+yi+zj+wk \mapsto \begin{pmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{pmatrix}.$$

Now it is straightforward to see that (A.1) is given by

$$\frac{1}{24}\frac{1}{2-(-2)} + \frac{1}{4}\frac{1}{2-0} + \frac{1}{6}\frac{1}{2-1} + \frac{1}{6}\frac{1}{2-1} + \frac{1}{6}\frac{1}{2-(-1)} + \frac{1}{6}\frac{1}{2-(-1)} = \frac{167}{288} = \frac{1}{12}(7-\frac{1}{24}).$$

Since 7 is the Euler characteristic of the exceptional divisor of the minimal resolution of $\mathbb{C}^2/2T$, the result follows.

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