

# THE BOGOMOLOV-MIYAOKA-YAU INEQUALITY FOR STACKY SURFACES

JIUN-CHENG CHEN AND HSIAN-HUA TSENG

ABSTRACT. We present a generalization of the Bogomolov-Miyaoka-Yau inequality to Deligne-Mumford surfaces of general type.

## 1. INTRODUCTION

We work over  $\mathbb{C}$ .

For a smooth complex projective surface  $S$  of general type, the Bogomolov-Miyaoka-Yau inequality for  $S$  reads (see [8])

$$(1.1) \quad 3c_2(T_S) \geq c_1(T_S)^2.$$

Together with Noether's inequality, this puts constraints on the topology of surfaces of general types. Generalizations of (1.1) to singular surfaces and surface pairs have been found, see for example [9], [5, 6].

In this paper we present a generalization of (1.1) to Deligne-Mumford stacks. Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford  $\mathbb{C}$ -stack of dimension 2. Let  $\pi : \mathcal{X} \rightarrow X$  be the natural map to the coarse moduli space. We assume that  $X$  is a projective variety. Since  $\mathcal{X}$  is assumed to be smooth, it has a tangent bundle  $T_{\mathcal{X}}$ . A good theory of Chern classes is available for Deligne-Mumford stacks, see for example [14], [4].

**Main Theorem 1.1.** *Let  $\mathcal{X}$  be as above. Assume that the canonical bundle  $K_{\mathcal{X}} := \wedge^2 T_{\mathcal{X}}^{\vee}$  is numerically effective, then*

$$(1.2) \quad 3c_2(T_{\mathcal{X}}) \geq c_1(T_{\mathcal{X}})^2.$$

Certainly (1.2) takes the same shape as (1.1). A proof of (1.2), along the lines of Miyaoka's original proof of (1.1) in [8], is given in Section 2. Section 3 contains examples of (1.2). In Section 3.1 we consider (1.2) for a class of stacks  $\mathcal{X}$  with stack structures in codimension 1, recovering [6, Theorem 0.1]. In Section 3.2 we consider (1.2) for Gorenstein stacks  $\mathcal{X}$  with isolated stack points, recovering [9, Corollary 1.3].

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## 2. PROOF OF (1.2)

In this Section we give a proof of (1.2). Our proof is adapted from Miyaoka's original proof in [8].

Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford stack of dimension 2. If  $\mathcal{X}$  has non-trivial stack structures at generic points, then  $\mathcal{X}$  is an étale gerbe over a stack with trivial generic stack structure, see for example [2, Proposition 4.6]. More precisely, there is a finite group  $G$ , a stack  $\mathcal{X}'$  with trivial generic stabilizers, and a morphism  $f : \mathcal{X} \rightarrow \mathcal{X}'$  realizing  $\mathcal{X}$  as a  $G$ -gerbe over  $\mathcal{X}'$ . Since  $T_{\mathcal{X}} = f^*T_{\mathcal{X}'}$ , we see that (1.2) for  $\mathcal{X}$  is equivalent to (1.2) for  $\mathcal{X}'$ . Therefore it suffices to consider only those  $\mathcal{X}$  with stack structures in codimension  $\geq 1$ . For the rest of this section we assume this.

Let  $\mathcal{F}$  be a locally free sheaf of rank 2 on  $\mathcal{X}$ . Let  $\mathcal{V} := \mathbb{P}(\mathcal{F})$  be the projectivization, with natural projection  $p : \mathcal{V} \rightarrow \mathcal{X}$ . Let  $\mathcal{H}$  be the divisor associated to the tautological sheaf on  $\mathcal{V}$ .

**Lemma 2.1.** *Assume that  $\mathcal{W} \subset \mathcal{V}$  is linearly equivalent to  $\mathcal{H} - p^*\mathcal{D}$ , where  $\mathcal{D} \subset \mathcal{X}$  is a divisor on  $\mathcal{X}$ . Then we have*

$$(2.1) \quad \mathcal{D} \cdot \det \mathcal{F} \leq c_2(\mathcal{F}) + \mathcal{D}^2.$$

*Proof.* We closely follow Miyaoka's original proof [8]. Let  $i : \mathcal{W} \subset \mathcal{V}$  be the inclusion morphism. Note that the composition  $p \circ i : \mathcal{W} \rightarrow \mathcal{X}$  is birational by our assumption on the linear equivalence class of  $\mathcal{W}$ . Since resolutions can be chosen such that they are compatible with étale base change, there is a sequence of blow-ups

$$(2.2) \quad \mu : \mathcal{V}_s \xrightarrow{\mu_s} \mathcal{V}_{s-1} \rightarrow \cdots \rightarrow \mathcal{V}_1 \xrightarrow{\mu_1} \mathcal{V}_0 = \mathcal{V}$$

such that the proper transform  $\mathcal{W}'$  of  $\mathcal{W}$  is a smooth Deligne-Mumford stack in  $\mathcal{V}_s$ . Let  $i' : \mathcal{W}' \subset \mathcal{V}_s$  and  $\rho : \mathcal{W}' \rightarrow \mathcal{X}$  be the natural maps.

Let  $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s$  be the exceptional divisors on  $\mathcal{V}_s$ . The divisor  $\mathcal{W}'$  is linearly equivalent to  $\mu^*(\mathcal{H} - p^*\mathcal{D}) - \sum a_i \mathcal{E}_i$ . It can be seen<sup>1</sup> that the canonical bundle  $K_{\mathcal{W}'}$  satisfies  $K_{\mathcal{W}'} = \rho^*K_{\mathcal{X}} + \sum \mathcal{C}_i$  where  $\mathcal{C}_i$  is a curve and  $\rho(\mathcal{C}_i) = \text{point}$ . By the Hodge index theorem (for a stacky version see [7, Theorem 3.1.3]), it follows that  $(K_{\mathcal{W}'} - \rho^*K_{\mathcal{X}} + \sum c_i i'^* \mathcal{E}_i)^2 \leq 0$  for any  $c_i \in \mathbb{R}$ .

Write  $K_{\mathcal{V}_s} = \mu^*(-2\mathcal{H} + p^*K_{\mathcal{X}} + p^*(\det \mathcal{F})) + \sum b_i \mathcal{E}_i$ . The adjunction formula implies that

$$K_{\mathcal{W}'} = i'^*[\mu^*(-\mathcal{H}) + (p \circ \mu)^*(K_{\mathcal{X}} + \det \mathcal{F} - \mathcal{D}) + \sum (b_i - a_i) \mathcal{E}_i].$$

Thus  $i'^*\mu^*(-\mathcal{H} + p^*(\det \mathcal{F} - \mathcal{D}))^2 \leq 0$ . Set  $k := i'^*\mu^*(-\mathcal{H} + p^*(\det \mathcal{F} - \mathcal{D}))^2$ . We can also compute this self-intersection number  $k$  in another way:

$$\begin{aligned} k &= \mu^*(-\mathcal{H} + p^*(\det \mathcal{F}) - p^*\mathcal{D})^2 (\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D} - \sum a_i \mathcal{E}_i) \\ &= \mu^*(-\mathcal{H} + p^*(\det \mathcal{F}) - p^*\mathcal{D})^2 (\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D}) \quad (\text{since } \mathcal{E}_i \text{ is exceptional}) \\ &= \mathcal{H}^3 - \mathcal{H}^2 \cdot p^*(\mathcal{D} - 2\det \mathcal{F}) + \mathcal{H} \cdot (p^*(\det \mathcal{F})^2 - (p^*\mathcal{D})^2). \end{aligned}$$

<sup>1</sup>The argument is similar to that of [8, Lemma 7] and is omitted.

Using the standard relations for the intersection numbers on the projectivization of a rank 2 vector bundle, we calculate that

$$k = c_1^2(\det(\mathcal{F})) - c_2(\mathcal{F}) - (\det \mathcal{F})^2 + \det \mathcal{F} \cdot \mathcal{D} - \mathcal{D}^2 = -c_2(\mathcal{F}) + \det \mathcal{F} \cdot \mathcal{D} - \mathcal{D}^2.$$

The result follows.  $\square$

Let  $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$  be a subsheaf of  $\Omega_{\mathcal{X}}^1$ . One key observation used in Miyaoka's original proof is that the Iitaka dimension of  $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$  is at most 1.

**Theorem 2.1.** *If  $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$  is a subsheaf of  $\Omega_{\mathcal{X}}^1$  of a projective Deligne-Mumford stack  $\mathcal{X}$ , then  $h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n\mathcal{D})) \leq cn$  for some positive constant  $c$  and  $n \gg 0$ .*

The proof of Theorem 2.1 is very similar to that of [8, Theorem 2'']. We only remark that we need to use Riemann-Roch for stacks proved in [11].

One can prove the following result using Theorem 2.1.

**Proposition 2.2.** *Let  $\mathcal{F} \subset \Omega_{\mathcal{X}}^1$  be a locally free sheaf of rank 2 and assume that  $\det(\mathcal{F})^{\otimes n}$  is generated by global sections for some  $n > 0$ . If  $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  has a non-trivial section, then*

$$\mathcal{D} \cdot \det(\mathcal{F}) \leq \max\{c_2(\mathcal{F}), 0\}.$$

*Proof.* Consider  $p : \mathcal{V} = \mathbb{P}(\mathcal{F}) \rightarrow \mathcal{X}$ . The canonical isomorphism gives us

$$H^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})) = H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(\mathcal{H} - p^*\mathcal{D})).$$

If  $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  has a non-trivial section, then  $|\mathcal{H} - p^*\mathcal{D}|$  is non-empty. Pick  $\mathcal{W} \in |\mathcal{H} - p^*\mathcal{D}|$ . Decompose  $\mathcal{W}$  as  $\mathcal{W} = \mathcal{W}_0 + p^*\mathcal{D}'$  where  $\mathcal{W}_0$  is effective and irreducible which is linearly equivalent to  $\mathcal{H} - p^*(\mathcal{D} + \mathcal{D}')$  and  $\mathcal{D}'$  is effective. Note that  $(\det \mathcal{F})^{\otimes n}$  is generated by global sections, so the intersection number  $\mathcal{D}' \cdot \det(\mathcal{F}) \geq 0$ . It follows that  $\mathcal{D} \cdot \det(\mathcal{F}) \leq (\mathcal{D} + \mathcal{D}') \cdot \det(\mathcal{F})$  and it suffices to prove  $(\mathcal{D} + \mathcal{D}') \cdot \det(\mathcal{F}) \leq \max\{c_2(\mathcal{F}), 0\}$ . Set  $\mathcal{D}'' = \mathcal{D} + \mathcal{D}'$  to simplify notation. By Lemma 2.1,  $\mathcal{D}'' \cdot \det(\mathcal{F}) \leq c_2(\mathcal{F}) + \mathcal{D}'' \cdot \mathcal{D}''$ . Observe that  $\mathcal{O}_{\mathcal{X}}(\mathcal{D}'')$  is a subsheaf of  $\Omega_{\mathcal{X}}^1$ . Indeed, the effectiveness of  $\mathcal{W}_0$  ensures the existence of a non-trivial section of  $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$ , i.e. an injection  $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$ . Twisting by  $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$ , embeds  $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$  into  $\mathcal{F} \subset \Omega_{\mathcal{X}}^1$ . By Theorem 2.1,  $\mathcal{D}''$  has Iitaka dimension at most 1. It follows that  $\mathcal{D}'' \cdot \det(\mathcal{F}) \leq 0$  or  $\mathcal{D}'' \cdot \mathcal{D}'' \leq 0$ .<sup>2</sup> This completes the proof.  $\square$

Assuming  $c_2(\mathcal{F})$  is positive for the time being, we can obtain an upper bound on  $c_2$  provided the sheaf  $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  has no sections. This can then be used to derive a contradiction. To be more precise, one needs a modified version of Proposition 2.2, in which the condition on the sheaf  $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  having a section is replaced by the condition that some symmetric power  $S^m \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  having a section.

**Theorem 2.2.** *Let  $\mathcal{F} \subset \Omega_{\mathcal{X}}^1$  be a locally free sheaf of rank 2 and assume that  $\det(\mathcal{F})^{\otimes n}$  is generated by global sections for some  $n > 0$ . If  $S^m \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  has a non-trivial section, then*

$$\mathcal{D} \cdot \det(\mathcal{F}) \leq \max\{mc_2(\mathcal{F}), 0\}.$$

The proof of Theorem 2.2 follows from Proposition 2.2 and the following easy lemma (which is analogous to [8, Lemma 11]).

<sup>2</sup>Arguing as in [8, Lemma 10].

**Lemma 2.3.** *Let  $p : \mathcal{V} = \mathbb{P}(\mathcal{F}) \rightarrow \mathcal{X}$  be the projective bundle of a locally free sheaf of rank 2. Let  $\mathcal{W} \in |m\mathcal{H} - p^*\mathcal{D}|$ . Then there is a surjective morphism  $\beta : \mathcal{X}' \rightarrow \mathcal{X}$  such that  $\beta^*\mathcal{W}$  is decomposed to  $\mathcal{W}_1 + \cdots + \mathcal{W}_m$  where  $\mathcal{W}_i$  is an effective divisor linear equivalent to  $\mathcal{H}' - p^*\mathcal{D}_i$ .*

*Proof of Theorem 2.2.* The following argument is taken from [8, Theorem 3]. Let  $f$  be a global section of  $S^m\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ . Lemma 2.3 implies that after a suitable cover  $\beta : \mathcal{Y} \rightarrow \mathcal{X}$ , we can decompose  $\beta^*f$  can be written as  $f_1 f_2 \cdots f_m \in H^0(\mathcal{Y}, S^m\beta^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}))$ , where  $f_i \in H^0(\mathcal{Y}, \beta^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}_i))$  and  $(\det\beta^*\mathcal{F})^{\otimes m} \cong (\beta^*\det(\mathcal{F}))^{\otimes m}$  is generated by global sections. From Proposition 2.2, it follows that  $\beta^*\mathcal{D}_i \cdot (\det(\beta^*\mathcal{F})) \leq \max\{c_2(\beta^*\mathcal{F}), 0\}$ . Summing over all  $i$ 's, we have  $\beta^*\mathcal{D} \cdot \det(\beta^*\mathcal{F}) \leq \max\{mc_2(\beta^*\mathcal{F}), 0\}$ . Let  $d$  be the mapping degree of  $\beta$ . Clearly,  $\beta^*\mathcal{D} \cdot \det(\beta^*\mathcal{F}) = dD \cdot \det(\mathcal{F})$  and  $c_2(\beta^*\mathcal{F}) = d\beta^*c_2(\mathcal{F})$ .  $\square$

We now come to (1.2).

**Theorem 2.3.** *Let  $\mathcal{X}$  be a non-singular Deligne-Mumfords stack with the projective coarse space  $X$  of general type and  $c_1(\mathcal{X})$  nef. Then  $c_1^2(\mathcal{X}) \leq 3c_2(\mathcal{X})$  holds.*

*Proof.* As in [8], we consider two cases: (1)  $c_1^2(\mathcal{X}) \leq 2c_2(\mathcal{X})$  and (2)  $c_1^2(\mathcal{X}) > 2c_2(\mathcal{X})$ . The first case is obvious. For the second case, set  $\alpha := \frac{c_2(\mathcal{X})}{c_1^2(\mathcal{X})}$ . Note that  $\alpha < 1/2$ . Pick  $\delta > 0$  sufficiently small and rational. By Theorem 2.2 applied to  $\mathcal{D} = m(\alpha + \delta)K_{\mathcal{X}}$ ,  $\mathcal{F} = \Omega_{\mathcal{X}}^1$ , we can find a positive integer  $m$  such that  $m(\alpha + \delta) \in \mathbb{Z}$ , and

$$h^0(\mathcal{X}, S^m\Omega_{\mathcal{X}}^1 \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha + \delta)K_{\mathcal{X}})) = 0.$$

By Serre duality for smooth projective Deligne-Mumford stacks [10, Theorem 2.22], we have

$$h^2(\mathcal{X}, S^m\Omega_{\mathcal{X}}^1 \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha + \delta)K_{\mathcal{X}})) = h^0(\mathcal{X}, S^m\Omega_{\mathcal{X}}^1 \otimes \mathcal{O}_{\mathcal{X}}(-m(1 - \alpha - \delta)K_{\mathcal{X}}) \otimes K_{\mathcal{X}}).$$

As  $\alpha < 1/2$  and  $\delta$  is small, we have  $1 - \alpha - \delta > \alpha$ . We apply Theorem 2.2 to  $\mathcal{D} = m(2 - \alpha - \delta)K_{\mathcal{X}}$ ,  $\mathcal{F} = \Omega_{\mathcal{X}}^1$ , to get

$$h^2(\mathcal{X}, S^m\Omega_{\mathcal{X}}^1 \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha + \delta)K_{\mathcal{X}})) = 0.$$

Hence

$$\chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha + \delta)K_{\mathcal{X}})) = -h^1(\mathcal{X}, S^m\Omega_{\mathcal{X}}^1 \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha + \delta)K_{\mathcal{X}})) \leq 0.$$

Note that to compute the cohomology groups of a (subsheaf of) symmetric power of a vector bundle, one can work on the the projectivized vector bundle and computing the cohomology groups of relevant line bundles. Thus

$$0 \geq \chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha + \delta)K_{\mathcal{X}})) = \chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H} - (\alpha + \delta)\pi^*K_{\mathcal{X}}))).$$

By Riemann-Roch for stacks [11], we have  $\chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H} - (\alpha + \delta)\pi^*K_{\mathcal{X}})))$  grows like  $\frac{1}{6}(\mathcal{H} - (\alpha + \delta)\pi^*K_{\mathcal{X}})^3 m^3$  as  $m \rightarrow \infty$ . It implies that  $(\mathcal{H} - (\alpha + \delta)\pi^*K_{\mathcal{X}})^3 \leq 0$ . Taking  $\delta$  to 0, we obtain

$$\begin{aligned} 0 \geq (\mathcal{H} - \alpha\pi^*K_{\mathcal{X}})^3 &= c_1^2(\mathcal{X}) - c_2(\mathcal{X}) - 3\alpha c_1^2(\mathcal{X}) + 3\alpha^2 c_1^2(\mathcal{X}) \\ &= (1 - \alpha - 3\alpha + 3\alpha^2)c_1^2(\mathcal{X}) \\ &= (1 - \alpha)(1 - 3\alpha)c_1^2(\mathcal{X}). \end{aligned}$$

Since  $\alpha < 1/2$  and  $c_1^2(\mathcal{X})$  is non-negative, we get  $1 - 3\alpha \leq 0$  as desired.  $\square$

## 3. EXAMPLES OF (1.2)

**3.1. Codimension 1 stack structure.** We consider (1.2) for an example of stack  $\mathcal{X}$  with stack structures in codimension 1.

Let  $X$  be a smooth complex projective surface and  $D$  a simple normal crossing  $\mathbb{Q}$ -divisor of the form  $D = \sum_i (1 - 1/r_i)D_i$  with  $r_i \geq 2$  integers. Let  $\mathcal{X}$  be the natural stack cover of the pair  $(X, D)$ , see [3, Definition 2.1] for its definition. By construction the coarse moduli space of  $\mathcal{X}$  is  $X$ . The natural map  $\pi : \mathcal{X} \rightarrow X$  is an isomorphism outside  $\pi^{-1}(\text{Supp } D)$ , which is where  $\mathcal{X}$  has non-trivial stack structures. Furthermore we have the following formula for the canonical bundle:

$$(3.1) \quad K_{\mathcal{X}} = \pi^*(K_X + D).$$

We now examine (1.2) for this  $\mathcal{X}$ . By (3.1),

$$c_1(T_{\mathcal{X}})^2 = c_1(K_{\mathcal{X}})^2 = (K_X + D)^2.$$

By Gauss-Bonnet theorem for Deligne-Mumford stacks [12, Corollaire 3.44] we have

$$c_2(T_{\mathcal{X}}) = \chi(\mathcal{X}),$$

the Euler characteristic of  $\mathcal{X}$  as defined in [12, Definition 3.43] (note that the notation  $\chi^{orb}$  is used in [12]). Put

$$\mathcal{D}_i := \pi^{-1}(D_i), \quad \mathcal{D}_i^\circ := \mathcal{D}_i \setminus (\cup_{j \neq i} (\mathcal{D}_i \cap \mathcal{D}_j)).$$

Then we have

$$\chi(\mathcal{X} \setminus \pi^{-1}(\text{Supp } D)) = \chi(\mathcal{X}) - \sum_i \chi(\mathcal{D}_i^\circ) - \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Similarly, put  $D_i^\circ = D_i \setminus (\cup_{j \neq i} (D_i \cap D_j))$ , we have

$$\chi(X \setminus \text{Supp } D) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\bar{p} \in D_i \cap D_j} \chi(\bar{p}).$$

Since  $\mathcal{X} \setminus \pi^{-1}(\text{Supp } D) \simeq X \setminus \text{Supp } D$ , we have  $\chi(\mathcal{X} \setminus \pi^{-1}(\text{Supp } D)) = \chi(X \setminus \text{Supp } D)$ . Equivalently,

$$\chi(\mathcal{X}) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\bar{p} \in D_i \cap D_j} \chi(\bar{p}) + \sum_i \chi(\mathcal{D}_i^\circ) + \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Since the map  $\mathcal{D}_i^\circ \rightarrow D_i^\circ$  is of degree  $1/r_i$  and the map  $\mathcal{D}_i \cap \mathcal{D}_j \rightarrow D_i \cap D_j$  is of degree  $1/r_i r_j$ , we have

$$\chi(\mathcal{D}_i) = \frac{1}{r_i} \chi(D_i), \quad \chi(\mathcal{D}_i \cap \mathcal{D}_j) = \frac{1}{r_i r_j} \chi(D_i \cap D_j).$$

This implies that

$$(3.2) \quad \chi(\mathcal{X}) = \chi(X) - \sum_i (1 - 1/r_i) \chi(D_i^\circ) + \sum_{\bar{p} \in D_i \cap D_j} (1/r_i r_j - 1).$$

By [6, Theorem 8.7], for  $\bar{p} \in D_i \cap D_j$  the local orbifold Euler number of the pair  $(X, D)$  at  $\bar{p}$  is given by  $e_{orb}(\bar{p}; X, D) = 1/r_i r_j$ . Together with (3.2) this implies that  $\chi(\mathcal{X})$  coincides with the orbifold Euler number  $e_{orb}(X, D)$  of the pair  $(X, D)$ , as defined in [6]. Thus if  $K_{\mathcal{X}}$  is numerically effective, then (1.2) is equivalent to [6, Theorem 0.1] applied to the pair  $(X, D)$ .

**3.2. Condimension 2 stack structure.** Let  $\mathcal{X}$  be a smooth proper Deligne-Mumford  $\mathbb{C}$ -stack of dimension 2 with isolated stack structures. Let  $\pi : \mathcal{X} \rightarrow X$  be the natural map to the coarse moduli space  $X$ . Let  $p_1, p_2, \dots, p_k \in \mathcal{X}$  be the stacky points. Suppose that  $\mathcal{X}$  is Gorenstein, i. e. each stacky point  $p_i$  has a neighborhood  $p_i \in U_i \subset \mathcal{X}$  of the form  $U_i \simeq [\mathbb{C}^2/G_i]$  with  $G_i \subset SU(2)$  a finite subgroup, identifying  $p_i$  with  $[0/G_i] \in [\mathbb{C}^2/G_i]$ . It is a standard fact that the coarse moduli space  $X$  is a projective surface with canonical singularities.

Suppose further that  $K_{\mathcal{X}}$  is numerically effective. We consider (1.2) for such  $\mathcal{X}$ .

By assumption we have  $K_{\mathcal{X}} = \pi^* K_X$ . Thus

$$c_1(T_{\mathcal{X}})^2 = c_1(K_{\mathcal{X}})^2 = c_1(K_X)^2.$$

We now consider the term  $c_2(T_{\mathcal{X}})$ . The first step is to consider  $\chi(\mathcal{O}_{\mathcal{X}})$  by using Riemann-Roch theorem for stacks [12, 11]. We follow [13, Appendix A] for the presentation of the Riemann-Roch theorem. We have

$$\chi(\mathcal{O}_{\mathcal{X}}) = \int_{I\mathcal{X}} \tilde{c}h(\mathcal{O}_{\mathcal{X}}) \tilde{T}d(T_{\mathcal{X}}).$$

Here  $I\mathcal{X}$  is the inertia stack of  $\mathcal{X}$ . By our assumption on  $\mathcal{X}$ , we have the following description of  $I\mathcal{X}$ :

$$I\mathcal{X} = \mathcal{X} \cup \bigcup_{i=1}^k (Ip_i \setminus p_i).$$

Here the term  $Ip_i \setminus p_i$  is the inertia stack of  $p_i \simeq BG_i$  with the main component removed, namely

$$Ip_i \setminus p_i \simeq \bigcup_{(g) \neq (1): \text{conjugacy class of } G_i} BC_{G_i}(g),$$

where  $C_{G_i}(g) \subset G_i$  is the centralizer subgroup of  $g \in G_i$  and  $BC_{G_i}(g)$  is its classifying stack. By the definition of the Chern character  $\tilde{c}h$ , we have  $\tilde{c}h(\mathcal{O}_{\mathcal{X}}) = 1$  on every component of  $I\mathcal{X}$ . Hence

$$(3.3) \quad \chi(\mathcal{O}_{\mathcal{X}}) = \int_{I\mathcal{X}} \tilde{T}d(T_{\mathcal{X}}) = \int_{\mathcal{X}} \tilde{T}d(T_{\mathcal{X}})|_{\mathcal{X}} + \sum_{i=1}^k \int_{Ip_i \setminus p_i} \tilde{T}d(T_{\mathcal{X}})|_{Ip_i \setminus p_i}.$$

Note that  $\tilde{T}d(T_{\mathcal{X}})|_{\mathcal{X}} = Td(T_{\mathcal{X}})$ , and we only need its degree 2 component. Hence

$$(3.4) \quad \int_{\mathcal{X}} \tilde{T}d(T_{\mathcal{X}})|_{\mathcal{X}} = \frac{1}{12} \int_{\mathcal{X}} (c_2(T_{\mathcal{X}}) + c_1(T_{\mathcal{X}})^2).$$

The contribution coming from  $Ip_i \setminus p_i$  can be also evaluated.

**Lemma 3.3.** *Let  $E_i$  be the exceptional divisor of the minimal resolution of  $\mathbb{C}^2/G_i$ . Then*

$$\int_{Ip_i \setminus p_i} \tilde{T}d(T_{\mathcal{X}})|_{Ip_i \setminus p_i} = \frac{1}{12} (\chi(E_i) - \frac{1}{|G_i|}).$$

This Lemma is proved in the Appendix.

Next, we reinterpret the term  $\chi(\mathcal{O}_{\mathcal{X}})$ . By definition,  $\chi(\mathcal{O}_{\mathcal{X}}) := \sum_{l \geq 0} (-1)^l \dim H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ . Since  $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$  (see e.g. [1, Theorem 2.2.1]), we have  $H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^l(X, \mathcal{O}_X)$  and

$$(3.5) \quad \chi(\mathcal{O}_{\mathcal{X}}) = \chi(\mathcal{O}_X).$$

Combining (3.3), (3.4), (3.5), and Lemma 3.3, we obtain the following expression of  $c_2(T_{\mathcal{X}})$ :

$$(3.6) \quad \int_{\mathcal{X}} c_2(T_{\mathcal{X}}) = 12\chi(\mathcal{O}_X) - \int_{\mathcal{X}} c_1(T_{\mathcal{X}})^2 - \sum_{i=1}^k (\chi(E_i) - 1/|G_i|).$$

Using this, we see that in the present situation, (1.2) is equivalent to

$$(3.7) \quad 12\chi(\mathcal{O}_X) \geq \frac{4}{3}c_1(K_X)^2 + \sum_{i=1}^k (\chi(E_i) - \frac{1}{|G_i|}).$$

On the other hand, it is clear that (3.7) is a special case of [9, Corollary 1.3].

### APPENDIX A. PROOF OF LEMMA 3.3

In this Appendix we prove Lemma 3.3. By our assumption on  $\mathcal{X}$ , for  $g \in G_i$ , the  $g$ -action on the tangent space  $T_{p_i} \mathcal{X}$  has two eigenvalues  $\xi_g$  and  $\xi_g^{-1}$ , where  $\xi_g$  is a certain root of unity. By the definition of  $\widetilde{Td}(T_{\mathcal{X}})$  we have

$$(A.1) \quad \int_{I_{p_i} \setminus p_i} \widetilde{Td}(T_{\mathcal{X}})|_{I_{p_i} \setminus p_i} = \sum_{(g) \neq (1): \text{conjugacy class of } G_i} \frac{1}{|C_{G_i}(g)|} \frac{1}{2 - \xi_g - \xi_g^{-1}}.$$

We now evaluate (A.1) using the *ADE* classification of  $\mathbb{C}^2/G_i$ .

**A.1. Type A.** If  $\mathbb{C}^2/G_i$  is of type  $A_{n-1}$ , then  $G_i \simeq \mathbb{Z}_n$  and the action on  $\mathbb{C}^2$  is given as follows. If we identify  $\mathbb{Z}_n$  with the group of  $n$ -th roots of 1, then an element  $\xi \in \mathbb{Z}_n$  acts on  $\mathbb{C}^2$  via the matrix

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

It follows that (A.1) is given by

$$(A.2) \quad \frac{1}{n} \sum_{l=1}^{n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}l/n) - \exp(2\pi\sqrt{-1}l/n)^{-1}}.$$

By [7, Lemma 3.3.2.1], (A.2) is equal to

$$\frac{n^2 - 1}{12n} = \frac{1}{12}(n - 1/n).$$

Since the exceptional divisor of the minimal resolution of  $\mathbb{C}^2/\mathbb{Z}_n$  is a chain of  $(n - 1)$  copies of  $\mathbb{CP}^1$ , its Euler characteristic is  $n$ . This proves the Lemma in type A case.

A.2. **Type D.** If  $\mathbb{C}^2/G_i$  is of type  $D_{n+2}$  (here  $n \geq 2$ ), then  $G_i$  is isomorphic to the binary dihedral group  $Dic_n$ . The group  $Dic_n$  is of order  $4n$  and may be presented as follows:

$$Dic_n = \langle a, x | a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle.$$

The action of  $Dic_n$  on  $\mathbb{C}^2$  is given as follows:

$$(A.3) \quad a \mapsto \begin{pmatrix} \exp(\pi\sqrt{-1}/n) & 0 \\ 0 & \exp(-\pi\sqrt{-1}/n) \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An element calculation shows that the conjugacy classes of  $Dic_n$  and the orders of their centralizer subgroups are given as follows:

$$(A.4) \quad \begin{aligned} &\{1\}, \quad \{a^n\} \quad (\text{order of centralizer group} = 4n) \\ &\{a^l, a^{-l}\}, 1 \leq l \leq n-1, \quad (\text{order of centralizer group} = 2n) \\ &\{xa, xa^3, xa^5, \dots, xa^{2n-1}\}, \quad \{x, xa^2, xa^4, \dots, xa^{2n-2}\} \quad (\text{order of centralizer group} = 4). \end{aligned}$$

Using (A.3) and (A.4) it is easy to identify the contribution from each conjugacy class. It follows that (A.1) is given by

$$(A.5) \quad \frac{1}{2n} \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{16n} + \frac{1}{8} + \frac{1}{8}.$$

We need to evaluate the sum  $\sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}}$ . Again by [7, Lemma 3.3.2.1], we have

$$\begin{aligned} \frac{(2n)^2 - 1}{12} &= \sum_{k=1}^{2n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}k/(2n)) - \exp(2\pi\sqrt{-1}k/(2n))^{-1}} \\ &= \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{4} \\ &\quad + \sum_{k=1}^{n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}}. \end{aligned}$$

Note that

$$\begin{aligned} &2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1} \\ &= 2 + \exp(\pi\sqrt{-1}k/n) + \exp(\pi\sqrt{-1}k/n)^{-1} \\ &= 2 + 2 \cos(\pi k/n) = 4 \cos^2(\pi k/(2n)) = 4 \sin^2((\pi(k+n))/(2n)); \\ &2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1} \\ &= 2 - 2 \cos(\pi k/n) = 4 \sin^2(\pi k/(2n)). \end{aligned}$$



Since  $\sin(\pi(k+n)/(2n)) = -\sin(\pi(k-n)/(2n))$ , we see that

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} \\ &= \sum_{k=1}^{n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}}, \end{aligned}$$

from which it follows that

$$2 \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{4} = \frac{(2n)^2 - 1}{12}.$$

This shows that

$$\sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} = \frac{n^2 - 1}{6}$$

and (A.1) is given by

$$\frac{n^2 - 1}{12n} + \frac{1}{16n} + \frac{1}{8} + \frac{1}{8} = \frac{1}{12} \left( n + 3 - \frac{1}{4n} \right).$$

Since the exceptional divisor of the minimal resolution of  $\mathbb{C}^2/Dic_n$  is a tree of  $\mathbb{C}\mathbb{P}^1$  whose dual graph is the Dynkin diagram  $D_{n+2}$ , its Euler characteristic is  $n + 3$  and the Lemma is proved in this case.

**A.3. Type E.** If  $\mathbb{C}^2/G_i$  is of type  $E$ , then there are three possibilities:  $E_6, E_7, E_8$ . The group  $G_i$  is isomorphic to the binary tetrahedral group (for  $E_6$ ), the binary octahedral group (for  $E_7$ ), or the binary icosahedral group (for  $E_8$ ). In each case the group and its action on  $\mathbb{C}^2$  can be explicitly described, and the Lemma can be proved by computing (A.1) using this information. We work out the details for  $E_6$  and leave the other two cases to the reader.

In the  $E_6$  case, the group  $G_i$  is isomorphic to the binary tetrahedral group  $2T$ . This group is of order 24 and its elements can be identified with the following quaternion numbers:

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k), \quad \pm i, \quad \pm j, \quad \pm k, \quad \pm 1.$$

The group  $2T$  has 7 conjugacy classes:

Conjugacy Class	(1)	(-1)	(i)	$(\frac{1}{2}(1+i+j+k))$
Size	1	1	6	4
Conjugacy Class	$(\frac{1}{2}(1+i+j-k))$	$(\frac{1}{2}(-1+i+j+k))$	$(\frac{1}{2}(-1+i+j-k))$	
Size	4	4	4	

The action of  $2T$  on  $\mathbb{C}^2$  can be described using the following identification

$$x + yi + zj + wk \mapsto \begin{pmatrix} x + yi & z + wi \\ -z + wi & x - yi \end{pmatrix}.$$

Now it is straightforward to see that (A.1) is given by

$$\frac{1}{24} \frac{1}{2 - (-2)} + \frac{1}{4} \frac{1}{2 - 0} + \frac{1}{6} \frac{1}{2 - 1} + \frac{1}{6} \frac{1}{2 - 1} + \frac{1}{6} \frac{1}{2 - (-1)} + \frac{1}{6} \frac{1}{2 - (-1)} = \frac{167}{288} = \frac{1}{12} \left( 7 - \frac{1}{24} \right).$$

Since 7 is the Euler characteristic of the exceptional divisor of the minimal resolution of  $\mathbb{C}^2/2T$ , the result follows.

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DEPARTMENT OF MATHEMATICS, THIRD GENERAL BUILDING, NATIONAL TSING HUA UNIVERSITY, NO. 101 SEC 2 KUANG FU ROAD, HSINCHU, TAIWAN 30043, TAIWAN

*E-mail address:* jcchen@math.nthu.edu, jcchenster@gmail.com

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 100 MATH TOWER, 231 WEST 18TH AVE., COLUMBUS, OH 43210, USA

*E-mail address:* hhtseng@math.ohio-state.edu