K-THEORETIC QUASIMAP INVARIANTS AND THEIR WALL-CROSSING

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ABSTRACT. For each positive rational number ϵ, we define K-theoretic ϵ-stable quasimaps to certain GIT quotients $W /\!\!/ G$. For $\epsilon > 1$, this recovers the K-theoretic Gromov-Witten theory of $W /\!\!/ G$ introduced in more general context by Givental and Y.-P. Lee.

For arbitrary ϵ_1 and ϵ_2 in different stability chambers, these K-theoretic quasimap invariants are expected to be related by wall-crossing formulas. We prove wall-crossing formulas for genus zero K-theoretic quasimap theory when the target $W /\!\!/ G$ admits a torus action with isolated fixed points and isolated one-dimensional orbits.

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1. Introduction

For an affine algebraic variety $W = \mathrm{Spec}(A)$ that admits an action by a reductive algebraic group G, a choice of the polarization $\mathcal{O}(\theta)$ determines a GIT quotient $W \mathbin{/\!\!/}_\theta G$. In [2] and [1], the authors define the moduli space

$$Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$$

parametrizing maps of class d from genus g nodal curves with k-marked points to the quotient stack $[W/G]$ with ϵ-stability. Assuming W has only lci singularity, the canonical obstruction theory of $Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$ is perfect and hence yields a virtual fundamental class, see [2] and [1]. When $W \mathbin{/\!\!/} G$ is projective the cohomological quasimap invairants are defined in [2] and [1], using evaluation maps and descendant classes ψ at the markings.

By [7, Section 2.3], the perfect obstruction theory yields a *virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}}_{Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)}$$

in the K-theory of $Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$. K-theoretic (descendant) ϵ-quasimap invariants of $W \mathbin{/\!\!/} G$ are defined to be holomorphic Euler characteristics of vector bundles on the moduli space $Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$:

$$\begin{aligned}&\langle \gamma_1 L_1^{a_1}, \ldots, \gamma_k L_k^{a_k} \rangle^{W \mathbin{/\!\!/} G, \epsilon}_{g,k,d}\\ :=&\chi\left(Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d), \mathcal{O}^{\mathrm{vir}}_{Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)} \otimes (\otimes_{i=1}^k L_i^{\otimes a_i} \mathrm{ev}_i^*(\gamma_i))\right)\end{aligned}$$

where a_i are nonnegative integers, $\gamma_i \in K^0(W \mathbin{/\!\!/} G)$ and L_i are tautological line bundles over $Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$ corresponding to the i-th marked points.

As explained in [2] and [1], for each fixed class d, the set of positive rational numbers can be divided into chambers by finitely many walls $1, \frac{1}{2}, \cdots, \frac{1}{d(L_\theta)}$, such that the moduli space $Q^{\epsilon}_{g,k}(W \mathbin{/\!\!/} G, d)$ stays constant when ϵ is changing within a chamber, where $d(L_\theta)$ may be considered as the degree of the map with respect to the polarization $\mathcal{O}(\theta)$. We write $\epsilon = 0+$ for ϵ being sufficiently small and being in the first chamber $(0, \frac{1}{d(L_\theta)}]$, for all d. Changes of quasimap invariants as ϵ varies, termed *wall-crossing formulas*, is proved in [1] for genus 0 equivariant cohomological theory.

The goal of this paper is to study wall-crossing behavior for K-theoretic genus 0 quasimap theory.

We will consider permutation equivariant version of quantum K-theory, which takes into account the S_n-action on the moduli space by permuting the marked points, developed by Givental [4]. This permutation-equivariant theory works better in our context.

As in [1], genus zero wall-crossing formulas are naturally stated via generating functions of K-theoretic quasimap invariants. Let $\{\phi_\alpha\}$ be a basis of

$K^0(W /\!\!/ G) \otimes \mathbb{Q}$, $\{\phi^\alpha\}$ be the dual basis and $t = \sum_\alpha t^\alpha \phi_\alpha \in K^0(W /\!\!/ G) \otimes \mathbb{Q}$. Let Λ be the λ-algebra described in Section 2.3 and $\gamma \in K^0(W /\!\!/ G, \Lambda)$. For $\epsilon \geq 0+$, we define the S-operator:

$$(S^\epsilon)(q)(\gamma) = \gamma + \sum_\alpha \left(\sum_{(n,d)\neq(0,0)} Q^d \langle \frac{\phi^\alpha}{1-qL}, \gamma, t, \ldots, t \rangle^{\epsilon, S_n}_{0,n+2,d} \right) \phi_\alpha,$$

where q is a formal variable.

Suppose that $W /\!\!/ G$ admits a torus T action with isolated fixed points and isolated one-dimensional orbits. The main result of this paper, Theorem 4.4, is a wall-crossing formula which relates S-operators for ϵ_1 and ϵ_2 in different stability chambers and the invertible classes γ of the form $\gamma = \mathbb{1} + O(Q)$.

For each $\epsilon \geq 0+$, we also defined the permutation-equivariant $\mathcal{J}^\epsilon$-function and prove the following identity

$$\mathcal{J}^\epsilon(q) = S^\epsilon(q)(P^\epsilon)$$

where P^ϵ is a generating series on the quasimap graph space, see Proposition 3.2. Theorem 4.5 below shows, for each $\epsilon \geq 0+$, the permutation-equivariant $\mathcal{J}^\epsilon$-function lies in the Lagrangian cone $\mathcal{L}_{S_\infty, W/\!\!/G}$ of the permutation-equivariant K-theoretic Gromov-Witten theory of $W /\!\!/ G$.

Theorem 4.4 and Theorem 4.5 generalize the main theorems of [1] to K-theory when the torus action on $W /\!\!/ G$ has only isolated fixed points and isolated one-dimensional orbits. They can also be considered as generalizations of the K-theoretic mirror theorem for toric manifolds due to Givental [4].

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2. Constructions

2.1. *K*-theoretic Quasimap Invariants.

In this section, we review some basic definitions in quasimap theory following [2, 1]. We then define the K-theoretic version of quasimap invariants.

Consider an affine algebraic variety $W = \mathrm{Spec}(A)$ with G-action, where G is a reductive algebraic group. An element ξ of the character group $\chi(G)$, determines a one-dimensional G-representation $\mathbb{C}_\xi$, and hence an element

$$L_\xi = W \times \mathbb{C}_\xi$$

of the group $\mathrm{Pic}_G(W)$ of isomorphism classes of G-linearized line bundles on W.

Fixing a character $\theta \in \chi(G)$, we obtain the GIT quotient $W /\!\!/ G := W /\!\!/_\theta G$, which is a quasiprojective variety and the morphism

$$W /\!\!/ G \to W/_{\mathrm{aff}}G := \mathrm{Spec}(A^G)$$

is a projective morphism.

We write $W^s = W^s(\theta)$ (respectively $W^{ss} = W^{ss}(\theta)$) for the stable (respectively semistable) determined by θ. Following [2] and [1], we require the following assumptions for the rest of the paper:

- $W^s = W^{ss} \neq \varnothing$;
- W^s is nonsingular;
- G acts freely on W^s.

Let $(C, x_1, \ldots, x_k)$ be a prestable k-pointed curve, a map

$$[u] : C \to [W/G]$$

is represented by a pair (P, u), where $P \to C$ is a principal G-bundle over C and

$$u : C \to P \times_G W$$

is a section of the bundle $\rho : P \times_G W \to C$.

The numerical class d of a map (P, u) is the group homomorphism

$$d : \mathrm{Pic}_G(W) \to \mathbb{Z}, \quad L \mapsto \deg_C(u^*(P \times_G L))$$

Definition 2.1 ([1], Definition 2.4.1)**.** A quasimap to $W /\!\!/ G$ is a map from $((C, x_1, \ldots, x_k), P, u)$ to the quotient stack $[W/G]$ such that generic points of C land on the stable locus of W, i.e. for a generic point p of C, we have $u(p) \in P \times_G W^s$.

Remark 2.2. Points on C which map to the unstable locus of W are called base points, hence a quasimap has at most finitely many base points.

Definition 2.3 ([1], Definition 2.4.2)**.** A group homomorphism $d : \mathrm{Pic}_G(W) \to \mathbb{Z}$ is called L_θ-effective if it is a finite sum of classes of quasimaps, we write $\mathrm{Eff}(W, G, \theta)$ for the semigroup of L_θ-effective classes.

Let $\mathrm{Qmap}_{g,k}(W /\!\!/ G, d)$ be the moduli space of genus g, k-pointed quasimaps of class d to $W /\!\!/ G$.

Definition 2.4 ([1], Definition 2.4.4)**.** We say that a quasimap is prestable if its base points are away from the nodes and marked points of the underlying curve.

Definition 2.5 ([1], Definition 2.4.5)**.** Given a prestable quasimap

$$((C, x_1, \ldots, x_k), P, u)$$

to $W /\!\!/ G$, the length $l(x)$ at a point $x \in C$ is the contact order of $u(C)$ with the unstable subscheme $P \times_G W^{us}$ at $u(x)$. More precisely,

$$l(x) := \mathrm{length}_x(\mathrm{coker}(u^*\mathcal{J} \to \mathcal{O}_C)),$$

where $\mathcal{J}$ is the ideal sheaf of the closed subscheme $P \times_G W^{us}$ of $P \times_G W$.

Definition 2.6 ([1], Definition 2.4.6)**.** Given a positive rational number ϵ, a quasimap

$$((C, x_1, \dots, x_k), P, u)$$

to $W /\!\!/ G$ is called ϵ-stable if it is prestable and satisfies the following conditions

- $\omega_C(\sum_{i=1}^k x_i) \otimes \mathcal{L}_\theta^\epsilon$ is ample, where $\mathcal{L}_\theta = P \times_G \mathbb{C}_\theta = u^*(P \times_G L_\theta)$;
- $\epsilon l(x) \leq 1$ for every point $x \in C$.

The moduli space of ϵ-stable quasimaps $Q^\epsilon_{g,k}(W /\!\!/ G, d)$ is an open substack of $\mathrm{Qmap}_{g,k}(W /\!\!/ G, d)$. The universal family

$$((\mathcal{C}^\epsilon, x_1, \dots, x_k), \mathcal{P}, u)$$

over $Q^\epsilon_{g,k}(W /\!\!/ G, d)$ is obtained as follows. Let $\mathfrak{M}_{g,k}$ be the moduli space of prestable curves and $\mathfrak{Bun}_G$ be the moduli stack of principal G-bundles on the fibers of the universal curve $\pi : \mathfrak{C}_{g,k} \to \mathfrak{M}_{g,k}$. The universal curve

$$\pi : \mathcal{C}^\epsilon \to Q^\epsilon_{g,k}(W /\!\!/ G, d)$$

is the pull-back of $\mathfrak{C}_{g,k}$ via the natural forgetful morphism

$$\mu : Q^\epsilon_{g,k}(W /\!\!/ G, d) \to \mathfrak{M}_{g,k}.$$

$\mathcal{P}$ is the pull-back of the universal curve over $\mathfrak{Bun}_G$ via the natural forgetful morphism

$$\nu : Q^\epsilon_{g,k}(W /\!\!/ G, d) \to \mathfrak{Bun}_G,$$

and u is a section of the bundle

$$\rho : \mathcal{P} \times_G W \to \mathcal{C}^\epsilon$$

We write $\mathbb{R}T_\rho$ for the relative tangent complex of ρ. The canonical obstruction theory of $Q^\epsilon_{g,k}(W /\!\!/ G, d)$ relative to the smooth Artin stack $\mathfrak{Bun}_G$ is given by the complex

$$(R^\bullet \pi_*(u^* \mathbb{R}T_\rho))^\vee \tag{1}$$

Theorem 2.7 ([2], Theorem 7.1.6 & [1], Theorem 2.5.1)**.** *If W has only lci singularities, then the obstruction theory* (1) *is perfect.*

Hence, when W has lci singularities, by [7, Section 2.3], there is a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}_{Q^\epsilon_{g,k}(W /\!\!/ G, d)}$, which is an element of $K(Q^\epsilon_{g,k}(W /\!\!/ G, d))$, the Grothendieck group of coherent sheaves on $Q^\epsilon_{g,k}(W /\!\!/ G, d)$.

Since base points are away from the marking, we have the evaluation maps

$$\mathrm{ev}_i : Q^\epsilon_{g,k}(W /\!\!/ G, d) \to W /\!\!/ G, \quad i = 1, \dots, k.$$

We assume that $W /\!\!/ G$ is projective. Let L_i be the i-th tautological cotangent line bundle over $Q^\epsilon_{g,k}(W /\!\!/ G, d)$.

Definition 2.8. Given $\gamma_i \in K^0(W /\!\!/ G) \otimes \mathbb{Q}$ and nonnegative integers a_i, $1 \le i \le k$, we define the K-theoretic descendant ϵ-stable quasimap invariant of $W /\!\!/ G$:

(2)

$$\langle \gamma_1 L_1^{a_1}, \dots, \gamma_k L_k^{a_k} \rangle^{W/\!\!/G,\epsilon}_{g,k,d} := \chi\left(Q^\epsilon_{g,k}(W /\!\!/ G, d), \mathcal{O}^{\mathrm{vir}}_{Q^\epsilon_{g,k}(W/\!\!/G,d)} \otimes \left(\otimes_{i=1}^k L_i^{\otimes a_i}\, \mathrm{ev}_i^*(\gamma_i)\right)\right)$$

Remark 2.9 (see [1], Remark 2.4.8)**.**

(i) For $\epsilon > 1$, assume $(g,k) \neq (0,0)$, we have

$$Q^\epsilon_{g,k}(W /\!\!/ G, d) = \overline{M}_{g,k}(W /\!\!/ G, d)$$

and the corresponding K-theoretic ϵ-stable quasimap invariant is a K-theoretic Gromov-Witten invariant as defined by A. Givental and Y.-P. Lee [5] and [7]

(ii) Assume that $(g,k) \neq (0,0), (0,1)$ and fixed the numerial data (g,k,d), for each integer $1 \le e \le d(L_\theta) - 1$, the moduli space $Q^\epsilon_{g,k}(W /\!\!/ G, d)$ of ϵ-stable quasimaps to $W /\!\!/ G$ stays constant when $1/(e+1) < \epsilon \le 1/e$. Therefore, for each fixed d, the set of positive rational number is divided into chambers by finitely many walls $1, \frac{1}{2}, \cdots, \frac{1}{d(L_\theta)}$.

As explained in [1], the quasimap theory applies to a large class of targets, including toric and flag varieties, zero loci of sections of homogeneous bundles on toric and flag varieties, local targets with base a GIT quotient, Nakajima quiver varieties etc.

2.2. **Quasimap Graph Spaces.** We write $QG^\epsilon_{g,k}(W /\!\!/ G, d)$ for ϵ-stable quasimap graph spaces. The data

$$((C, x_1, \dots, x_k), P, u, \varphi)$$

represents the maps

$$C \longrightarrow [W/G] \times \mathbb{P}^1$$

of class $(d,1)$ and the morphism φ maps from C to $\mathbb{P}^1$ with degree 1. Namely, there is an irreducible component C_0 of C such that the restriction $\varphi|_{C_0} \to \mathbb{P}^1$ is an isomorphism and the remaining components $C \backslash C_0$ are contracted by φ. Elements of $QG^\epsilon_{g,k}(W /\!\!/ G, d)$ is given by the prestable quasimaps $((C, x_1, \dots, x_k), P, u, \varphi)$ with the stability conditions:

- $\omega_{\overline{C\backslash C_0}}(\sum x_i + \sum y_j) \otimes \mathcal{L}^\epsilon_\theta$ is ample, where x_i are marked points on $\overline{C\backslash C_0}$ and y_j are the nodes $\overline{C\backslash C_0} \cap C_0$;

- the inequality $\epsilon l(x) \le 1$ holds for every point x on C.

The K-theoretic ϵ-stable quasimap invariants for graph spaces can be defined the same way as (2).

2.3. **Permutation Equivariant Quantum K-theory.** In this section we review the permutation equivariant K-theoretic Gromov-Witten theory developed by Givental in [4].

By a λ-algebra Λ, we mean an algebra over $\mathbb{Q}$ equipped with abstract Adams operations Ψ^k, $k = 1, 2, \ldots$, that is, ring homomorphisms $\Lambda \to \Lambda$ that satisfy $\Psi^r \Psi^s = \Psi^{rs}$ and $\Psi^1 = \mathrm{id}$. We assume that the λ-algebra Λ includes the Novikov variables, the algebra of symmetric polynomials in a given number of variables and the torus equivariant K-ring of a point. We also assume Λ has a maximal ideal Λ_+ with the corresponding Λ_+-adic topology. We write $\mathcal{K}$ for the space of rational functions of q with coefficients from $K^0(X) \otimes \Lambda$, the space $\mathcal{K}$ is equipped with a symplectic form

$$\Omega(f, g) := -[\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}](f(q^{-1}), g(q))\frac{dq}{q}.$$

where $(\cdot, \cdot)$ is the K-theoretic intersection pairing on $K^0(X) \otimes \Lambda$:

$$(a, b) := \chi(X; a \otimes b).$$

Then $\mathcal{K}$ is a symplectic linear space. It can be decomposed into the direct sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$$

where $\mathcal{K}_+$ is the subspace of Laurent polynomials in q and $\mathcal{K}_-$ is the complementary subspace of rational functions of q regular at $q = 0$ and vanishing at $q = \infty$.

Given a compact Kähler manifold X, consider the S_n modules

$$[t(L), \ldots, t(L)] := \sum_m (-1)^m H^m(\overline{M}_{0,n}(X, d); \mathcal{O}^{vir}_{\overline{M}_{0,n}(X,d)} \otimes_{i=1}^n t(L_i)),$$

Where the input $t(q)$ is a Laurent polynomial in q with coefficients in $K^0(X) \otimes \Lambda$ and Λ is a λ-algebra.

The correlators of permutation equivariant quantum K-theory is defined as

$$\langle t(L), \ldots, t(L)\rangle^{S_n}_{0,n,d} := \pi_*(\mathcal{O}^{vir}_{\overline{M}_{0,n}(X,d)} \otimes_{i=1}^n t(L_i)),$$

where π_* is the K-theoretic push forward along the projection

$$\pi : \overline{M}_{0,n}(X, d)/S_n \to [pt].$$

The permutation equivariant K-theoretic quasimap invariants can be defined similarly, replacing moduli spaces of stable maps by moduli spaces of stable quasimaps. When X admits a torus action, the extension of K-theoretic quasimap theory to torus equivariant setting is also straightforward.

2.4. **The J^ϵ Functions.** Consider the K-theoretic Poincaré metric

$$g_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle := \chi(W /\!\!/ G, \phi_\alpha \phi_\beta),$$

where $\{\phi_\alpha\}$ is a basis of $K^0(W /\!\!/ G) \otimes \mathbb{Q}$, we define permutation-equivariant K-theoretic J^ϵ-function of $X = W /\!\!/ G$ as

(3)

$$\mathcal{J}^\epsilon_X(t,q) := \mathbb{1} + \frac{t}{1-q} + \sum_{\alpha,\beta} \sum_{(n,d)\neq(0,0),(1,0)} Q^d \langle \frac{\phi_\alpha}{(1-q)(1-qL)}, t, \ldots, t \rangle^{\epsilon, S_n}_{0,n+1,d} g^{\alpha\beta} \phi_\beta.$$

where $t = \sum_\alpha t^\alpha \phi_\alpha \in K^0(W /\!\!/ G) \otimes \mathbb{Q}$ and the unstable terms in the summation, that is, when $n = 0, d \neq 0, d(L_\theta) \leq 1/\epsilon$, are defined the same way as in [1, Definition 5.1.1], via $\mathbb{C}^*$-localization on graph spaces: Choose coordinates $[x_0, x_1]$ on $\mathbb{P}^1$, then the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ is

$$t[x_0, x_1] = [tx_0, x_1], \quad \forall t \in \mathbb{C}^*.$$

This action naturally induces an action on the ϵ-stable quasimap graph spaces $QG^\epsilon_{g,k}(W /\!\!/ G, d)$. For $k = 0$ and $\epsilon \leq \frac{1}{d(L_\theta)}$, we write

$$F_0 \cong Q_{0,1}(W /\!\!/ G, d)_0$$

for the fixed locus parametrizing quasimaps of class d of the form $(C = \mathbb{P}^1, P, u)$ with a principal G-bundle P on $\mathbb{P}^1$, a section $u : \mathbb{P}^1 \to P \times_G W$ such that $u(x) \in P \times_G W^s$ for $x \neq 0 \in \mathbb{P}^1$ and $0 \in \mathbb{P}^1$ is a base-point of length $d(L_\theta)$. Then the unstable terms in (3) is defined as

$$\sum_{\alpha,\beta} \sum_{d\neq 0, d(L_\theta)\leq 1/\epsilon} Q^d \chi\left(F_0, \mathcal{O}^{\mathrm{vir}}_{F_0} \otimes \left(\frac{ev_1^*(\phi_\alpha)}{\mathrm{tr}_{\mathbb{C}^*} \wedge^* N^*_{F_0}}\right)\right) g^{\alpha\beta} \phi_\beta,$$

where $N^*_{F_0}$ is the conormal bundle of the fixed locus F_0.

The permutation-equivariant big J-function of X is

$$\mathcal{J}_X(t(q), q) := \mathbb{1} + \frac{t(q)}{1-q} + \sum_{\alpha,\beta} \sum_{(n,d)\neq(0,0),(1,0)} Q^d \langle \frac{\phi_\alpha}{(1-q)(1-qL)}, t(L), \ldots, t(L) \rangle^{S_n}_{0,n+1,d} g^{\alpha\beta} \phi_\beta,$$

where $t(q)$ is a Laurent polynomial in q with coefficients in $K^0(X) \otimes \Lambda$. We write $\mathcal{L}_{S_\infty, X}$ for the range of the big J-function in permutation-equivariant quantum K-theory of X.

3. S & P

3.1. **The S-operator.** We use double brackets to denote the generating function

$$\langle\!\langle \gamma_1 L_1^{a_1}, \ldots, \gamma_k L_k^{a_k} \rangle\!\rangle^\epsilon_{0,k} := \sum_{n\geq 0, d\geq 0} Q^d \langle \gamma_1 L_1^{a_1}, \ldots, \gamma_k L_k^{a_k}, t, \ldots, t \rangle^{\epsilon, S_n}_{0,k+n,d}$$

where $t \in K^0(W /\!\!/ G) \otimes \Lambda$ and summing over terms when $Q^\epsilon_{0,k+n}(W /\!\!/ G, d)/S_n$ exists (does not include unstable terms).

We consider the permutation-equivariant $\mathcal{J}^\epsilon$-function of $X = W /\!\!/ G$,

$$\mathcal{J}^\epsilon_X(t,q) := \mathbb{1} + \frac{t}{1-q} + \sum_{\alpha,\beta} \sum_{(n,d)\neq(0,0),(1,0)} Q^d \langle \frac{\phi_\alpha}{(1-q)(1-qL)}, t, \dots, t \rangle^{\epsilon, S_n}_{0,n+1,d} g^{\alpha\beta}\phi_\beta.$$

with unstable terms defined as before and $t \in K^0(W /\!\!/ G)\otimes\mathbb{Q}$. Givental [3] introduced a non-constant metric for permutation-equivariant quantum K-theory. Similarly, we can define a non-constant metric for each $\epsilon \geq 0+$:

$$G^\epsilon_{\alpha\beta} := g_{\alpha\beta} + \langle\!\langle \phi_\alpha, \phi_\beta \rangle\!\rangle^\epsilon_{0,2},$$

and the inverse tensor

$$\begin{aligned} G^{\alpha\beta}_\epsilon = & g^{\alpha\beta} - \langle\!\langle \phi^\alpha, \phi^\beta \rangle\!\rangle^\epsilon_{0,2} + \sum_\mu \langle\!\langle \phi^\alpha, \phi^\mu \rangle\!\rangle^\epsilon_{0,2} \langle\!\langle \phi_\mu, \phi^\beta \rangle\!\rangle^\epsilon_{0,2} \\ & - \sum_{\mu,\nu} \langle\!\langle \phi^\alpha, \phi^\mu \rangle\!\rangle^\epsilon_{0,2} \langle\!\langle \phi_\mu, \phi^\nu \rangle\!\rangle^\epsilon_{0,2} \langle\!\langle \phi_\nu, \phi^\beta \rangle\!\rangle^\epsilon_{0,2} + \cdots \end{aligned}$$

where $\{\phi^\alpha\}$ is the Poincaré dual basis of $\{\phi_\alpha\}$.

The operator $S^\epsilon_t : \mathcal{K} \to \mathcal{K}$ is defined as

$$(S^\epsilon_t)(q)(\gamma) = \sum_{\alpha,\beta} \left(\langle \phi_\alpha, \gamma \rangle + \sum_{(n,d)\neq(0,0)} Q^d \langle \frac{\phi_\alpha}{1-qL}, \gamma, t, \dots, t \rangle^{\epsilon, S_n}_{0,n+2,d} \right) g^{\alpha\beta}\phi_\beta$$

The operator $(S^\epsilon_t)^* : \mathcal{K} \to \mathcal{K}$ is defined as

$$(S^\epsilon_t)^*(q)(\gamma) = \sum_{\alpha,\beta} \left(\langle \gamma, \phi_\alpha \rangle + \sum_{(n,d)\neq(0,0)} Q^d \langle \frac{\gamma}{1-qL}, \phi_\alpha, t, \dots, t \rangle^{\epsilon, S_n}_{0,n+2,d} \right) G^{\alpha\beta}_\epsilon \phi_\beta,$$

Proposition 3.1. *The operator S^ϵ_t is a unitary operator:*

$$(S^\epsilon_t)^*(q) \circ (S^\epsilon_t)(1/q) = Id,$$

for all $\epsilon \geq 0+$, where $0+$ represents a sufficiently small positive rational number such that $\epsilon = 0+$ is in the first chamber $(0, \frac{1}{d(L_\theta)}]$ for all class d.

Proof. Consider elements $p_0, p_\infty \in K^0_{\mathbb{C}^*}(\mathbb{P}^1)$ defined by the restriction to the fixed points:

$$p_0|_0 = q, p_0|_\infty = 1, \quad \text{and} \quad p_\infty|_0 = 1, p_\infty|_\infty = 1/q.$$

For arbitrary elements $\gamma, \delta \in K(W /\!\!/ G)$, we consider the generating series

$$\sum_{n\geq0, d\geq0} Q^d \langle \gamma(1-p_0), t, \dots, t, \delta(1-p_\infty) \rangle^{QG^\epsilon, S_n}_{0,n+2,d} \tag{4}$$

Applying $\mathbb{C}^*$-localization, the generating series (4) can be written as

$$\begin{aligned}
&\sum_{\alpha,\beta}\left(\langle\phi_\alpha,\gamma\rangle+\langle\!\langle\frac{\phi_\alpha}{1-qL},\gamma\rangle\!\rangle^\epsilon_{0,2}\right)g^{\alpha\beta}\left(\langle\delta,\phi_\beta\rangle+\langle\!\langle\delta,\frac{\phi_\beta}{1-L/q}\rangle\!\rangle^\epsilon_{0,2}\right)\\
=&\sum_{\alpha,\beta}\langle\phi_\alpha,\gamma\rangle g^{\alpha\beta}\left(\langle\delta,\phi_\beta\rangle+\langle\!\langle\delta,\phi_\beta\rangle\!\rangle^\epsilon_{0,2}\right)+O\left(\frac{1}{1-q}\right)\\
=&\sum_{\alpha}\langle\phi_\alpha,\gamma\rangle(\langle\delta,\phi^\alpha\rangle+\langle\!\langle\delta,\phi^\alpha\rangle\!\rangle^\epsilon_{0,2})+O\left(\frac{1}{1-q}\right)\\
=&\langle\delta,\gamma\rangle+\langle\!\langle\delta,\gamma\rangle\!\rangle^\epsilon_{0,2}+O\left(\frac{1}{1-q}\right)
\end{aligned}$$

where the first equation comes from the expansions

$$\frac{1}{1-qL}=\sum_{i\geq 0}\frac{q^i(L-1)^i}{(1-q)^{i+1}}$$

and

$$\begin{aligned}
\frac{1}{1-L/q}&=\sum_{i\geq 0}\frac{(1/q)^i(L-1)^i}{(1-1/q)^{i+1}}=\sum_{i\geq 0}\frac{q(L-1)^i}{(q-1)^{i+1}}\\
&=1+\frac{1}{q-1}+\sum_{i\geq 1}\frac{q(L-1)^i}{(q-1)^{i+1}}=1+O\left(\frac{1}{1-q}\right).
\end{aligned}$$

On the other hand, the generating series (4) has no pole at $q=1$, hence

$$\sum_{\alpha,\beta}\left(\langle\phi_\alpha,\gamma\rangle+\langle\!\langle\frac{\phi_\alpha}{1-qL},\gamma\rangle\!\rangle^\epsilon_{0,2}\right)g^{\alpha\beta}\left(\langle\delta,\phi_\beta\rangle+\langle\!\langle\delta,\frac{\phi_\beta}{1-L/q}\rangle\!\rangle^\epsilon_{0,2}\right)=\langle\delta,\gamma\rangle+\langle\!\langle\delta,\gamma\rangle\!\rangle^\epsilon_{0,2}.$$

Therefore

$$\begin{aligned}
&(S^\epsilon_t)^*(q)\circ(S^\epsilon_t)(1/q)(\gamma)\\
=&\sum_{\alpha,\beta}G^{\alpha\beta}\phi_\beta\sum_{a,b}\left(\langle\phi_b,\phi_\alpha\rangle+\langle\!\langle\frac{\phi_b}{1-qL},\phi_\alpha\rangle\!\rangle^\epsilon_{0,2}\right)\left(\langle\phi_a,\gamma\rangle+\langle\!\langle\frac{\phi_a}{1-L/q},\gamma\rangle\!\rangle^\epsilon_{0,2}\right)g^{ab}\\
=&\sum_{\alpha,\beta}G^{\alpha\beta}\phi_\beta(\langle\phi_\alpha,\gamma\rangle+\langle\!\langle\phi_\alpha,\gamma\rangle\!\rangle^\epsilon_{0,2})\\
=&\sum_\beta\phi_\beta\langle\phi^\beta,\gamma\rangle\\
=&\gamma
\end{aligned}$$

□

3.2. **The P-series.** We also consider a generating series on the graph space known as P-series.

$$P^\epsilon(t,q):=\sum_{\alpha,\beta}\phi_\beta G^{\alpha\beta}_\epsilon\langle\!\langle\phi_\alpha(1-p_\infty)\rangle\!\rangle^{QG^\epsilon}_{0,1}$$

where $p_\infty\in K^0_{\mathbb{C}^*}(\mathbb{P}^1)$ is defined by the restriction to the fixed points:

$$p_\infty|_0=1,\quad p_\infty|_\infty=1/q.$$

Then we have

Proposition 3.2. *For every* $\epsilon \geq 0+$, *the equation* $\mathcal{J}^\epsilon_t(q) = S^\epsilon_t(q)(P^\epsilon(t,q))$ *holds.*

Proof. Apply $\mathbb{C}^*$-localization to $P^\epsilon(t,q)$, we have

$$P^\epsilon(t,q) = \sum_{\alpha,\beta}\phi_\beta G_\epsilon^{\alpha\beta}\sum_{a,b}\left(\phi_a + \frac{1}{1-q}\langle\phi_a, t\rangle + \langle\langle\frac{\phi_a}{(1-q)(1-qL)}\rangle\rangle^\epsilon_{0,1}\right)g^{ab}$$
$$\cdot\left(\langle\phi_\alpha,\phi_b\rangle + \langle\langle(1-1/q)\phi_\alpha, \frac{\phi_b}{((1-1/q)(1-L/q)}\rangle\rangle^\epsilon_{0,2}\right)$$

Hence

$$\begin{aligned}P^\epsilon(t,q) =& \sum_{\alpha,\beta}\phi_\beta G_\epsilon^{\alpha\beta}\left(\langle\phi_\alpha,\mathcal{J}^\epsilon\rangle + \langle\langle\phi_\alpha, \frac{\mathcal{J}^\epsilon}{1-L/q}\rangle\rangle^\epsilon_{0,2}\right)\\ =&(S^\epsilon_t)^*(1/q)(J^\epsilon)\end{aligned}$$

Then the proposition follows from the unitary property of the S-operators (Proposition 3.1). ☐

Since we have the expansion

$$J^\epsilon(t,q) = \mathbb{1} + O(Q) + O\left(\frac{1}{1-q}\right)$$

and

$$\begin{aligned}(S^\epsilon_t)^*(1/q)(\gamma) =& \sum_{\alpha,\beta}\left(\langle\gamma,\phi_\alpha\rangle + \langle\langle\gamma,\phi_\alpha\rangle\rangle^\epsilon_{0,2}\right)G_\epsilon^{\alpha\beta}\phi_\beta + O\left(\frac{1}{1-q}\right)\\ =&\gamma + O\left(\frac{1}{1-q}\right)\end{aligned}$$

Hence

$$(S^\epsilon_t)^*(1/q)(J^\epsilon(t,q)) = \mathbb{1} + O(Q) + O\left(\frac{1}{1-q}\right)$$

The fact that $P^\epsilon(t,q)$ has no pole at $q=1$ implies

$$P^\epsilon(t,q) = (S^\epsilon_t)^*(1/q)(J^\epsilon(t,q)) = \mathbb{1} + O(Q)$$

In particular, for $\epsilon > 1$, we have $J(t,q) = \mathbb{1} + O\left(\frac{1}{1-q}\right)$ and $P(t,q) = \mathbb{1}$. Then Proposition 3.2 becomes

$$J(t,q) = S_t(q)(\mathbb{1}),$$

which is a consequence of the K-theoretic string equation in [4].

Let $\gamma \in K^0(W /\!\!/ G)\otimes\Lambda$ be an invertible element of the form $\gamma = \mathbb{1} + O(Q)$. For each $\epsilon \geq 0+$, we have

$$S^\epsilon_t(q)(\gamma) = \gamma + \tau^\epsilon_\gamma(t)\frac{1}{1-q} + \left(\frac{1}{1-q}\right)^2 p_\epsilon\left(\frac{q}{1-q}\right),$$

where $p_\epsilon\left(\frac{q}{1-q}\right)$ stands for a power series in the variable $q/(1-q)$ with coefficients in $K^0(W /\!\!/ G)\otimes\Lambda\{\{t_i\}\}$. For $\gamma = \mathbb{1} + O(Q)$, we have

$$\begin{aligned} S^\epsilon_t(\gamma) =& S^\infty_t(\gamma) && \mod Q\\ =& S^\infty_t(\mathbb{1}) && \mod Q\\ =& J^\infty(t) && \mod Q\\ =& \mathbb{1} + \frac{t}{1-q} + \left(\frac{1}{1-q}\right)^2 p_\infty\left(\frac{q}{1-q}\right) && \mod Q, \end{aligned}$$

where $p_\infty\left(\frac{q}{1-q}\right)$ stands for a power series in $q/(1-q)$ with coefficients in $K^0(W /\!\!/ G)\otimes\Lambda\{\{t_i\}\}$. Therefore

$$\tau^\epsilon_\gamma(t) = t + O(Q).$$

is an invertible transformation on $K^0(W /\!\!/ G)\otimes\Lambda$. For $0+\leq \epsilon_1\leq\epsilon_2\leq\infty$, let

$$\tau^{\epsilon_1,\epsilon_2}_\gamma(t) := (\tau^{\epsilon_1}_\gamma)^{-1}\circ\tau^{\epsilon_2}_\gamma(t),$$

then

$$S^{\epsilon_1}_{\tau^{\epsilon_1,\epsilon_2}_\gamma(t)}(q)(\gamma) = S^{\epsilon_2}_t(q)(\gamma) + \frac{1}{(1-q)^2}p_{\epsilon_1,\epsilon_2}\left(\frac{q}{1-q}\right),$$

where $p_{\epsilon_1.\epsilon_2}\left(\frac{q}{1-q}\right)$ stands for a power series in $q/(1-q)$ with coefficient in $K^0(W /\!\!/ G)\otimes\Lambda\{\{t_i\}\}$. In general, we have the following lemma.

Lemma 3.3. *For every $\epsilon\geq 0+$, there exist a uniquely determined element $P^{\infty,\epsilon}(t,q)\in\mathcal{K}_+$, convergent in the Q-adic topology for each t, and a uniquely determined map $t\mapsto\tau^{\infty,\epsilon}(t)$ on $K^0(W /\!\!/ G)\otimes\Lambda$ satisfying the following properties:*

- $\tau^{\infty,\epsilon}(t) = t \mod Q$;
- $P^{\infty,\epsilon}(t,q) = \mathbb{1} \mod Q$;
- $S^\epsilon_t(q)(P^\epsilon(t,q)) = S^\infty_{\tau^{\infty,\epsilon}(t)}(q)(P^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t),q)) + \frac{1}{(1-q)^2}p_{\epsilon,\infty}\left(\frac{q}{1-q}\right)$, *where $p_{\epsilon,\infty}\left(\frac{q}{1-q}\right)$ stands for a power series in the variable $q/(1-q)$ with coefficient in $K^0(W /\!\!/ G)\otimes\Lambda\{\{t_i\}\}$.*

Proof. Elements $\tau^{\infty,\epsilon}$ and $P^{\infty,\epsilon}(t,q)$ can be constructed by induction on the degree d and the construction is unique. The construction is not difficult but messy, and is therefore omitted. □

Remark 3.4. We write

$$P^{\infty,\epsilon}(t,z) = \sum_i C_i(Q,\{t_i\},q)\phi_i,$$

where the coefficients $C_i(Q,\{t_i\},q)\in\Lambda[[\{t_i\},q,1/q]]$. Hence by general properties of K-theoretic Gromov-Witten invariants (see [4]), the element

$$S^\infty_t(q)(P^{\infty,\epsilon}(t,q)) = \sum_i C_i(1-q)\partial_{t_i}J^\infty(t,q)$$

is on the Lagrangian cone $\mathcal{L}_{S_\infty, W/\!\!/G}$ associated to the genus 0 K-theoretic Gromov-Witten theory of $W /\!\!/ G$.

4. Genus 0 Wall-Crossing

The purpose of this section is to establish a wall-crossing result that relates genus 0 K-theoretic quasimap invariants for different stability parameters. The proof, which is parallel to the treatment for cohomological quasimap theory in [1] and the toric case in [4], is based on localization. To this end, we assume in this section that there is a torus T action on W and the action commutes with the G-action. This induces T-actions on $[W/G]$ and $W /\!\!/ G$. We assume that the T-action on $W /\!\!/ G$ has isolated fixed points and isolated 1-dimensional orbits.

4.1. **Fixed Point Localization.** Following the analysis of fixed loci in [1], we can describe the fixed loci of the T-equivariant quasimaps.

Consider the T-equivariant version of the permutation-equivariant K-theoretic S^ϵ-operator, denoted by $S^\epsilon_{t,T}$. Consider the T-fixed point basis

$$\{\phi_\beta\}_\beta \subset K^0(W /\!\!/ G)_T.$$

Given a T-fixed point $\beta \in (W /\!\!/ G)^T$, we have the restriction of $S^\epsilon_{t,T}$ to β:

$$\beta^* S^\epsilon_{t,T}(q)(\gamma) = \sum_\alpha \left(\langle \phi_\alpha, \gamma \rangle + \sum_{(n,d)\neq(0,0)} Q^d \langle \frac{\phi_\alpha}{1-qL}, \gamma, t, \dots, t \rangle^{\epsilon, S_n, T}_{0,n+2,d} \right) g^{\alpha\beta}$$

Fix $\epsilon > 0$, an effective class $d \neq 0$ and a non-negative integer n. For a T-fixed quasimap

$$((C, x_1, x_2, \dots, x_{n+2}), P, \mu),$$

and (P', μ'), the restriction of the pair (P, μ) to an irreducible component C' of C, the rational map

$$[\mu'] : C' \dashrightarrow W /\!\!/ G,$$

induces a regular map

$$[\mu']_{reg} : C' \to W /\!\!/ G.$$

Then the regular map $[\mu']$ satisfies one of the following three conditions:

- $[\mu']_{reg}$ is a constant map and maps to a T-fixed point of $W /\!\!/ G$, in this case, we call C' a contracted component;
- there are no base points (i.e. $[\mu']_{reg} = [\mu']$) and it is a cover of a 1-dimensional orbit of the T-action on $W /\!\!/ G$, totally ramified over the two fixed points of the orbit;
- $[\mu']_{reg}$ is a ramified cover of an 1-dimensional orbit as in the second case, but $[\mu']$ has a base-point at one of the fixed point and a node at the other fixed point.

Let M be a connected component of the fixed locus $Q^{\epsilon}_{0,n+2}(W /\!\!/ G, d)^T$. Following [1], M is of *initial type* if the first marking is on a contracted irreducible component of the domain curve, of *recursion type* if the first marking is on a non-contracted irreducible component.

Lemma 4.1 (Poles of $\beta^* S^{\epsilon}_{t,T}$). *The restriction $\beta^* S^{\epsilon}_{t,T}$ is a rational function of q with possibly poles only at 0, ∞, roots of unity and at most simple poles at $q = (\lambda(\beta,\mu)^{-1/m})$, where $\lambda(\beta,\mu)$ is the character of the torus action on the tangent line at the fixed point μ corresponding to the 1-dimensional orbit connecting the fixed points β and μ; for some $m = 1, 2, \ldots$.*

Proof. We apply virtual localization to the sum

$$\sum_{\alpha} Q^d \langle \frac{\phi_\alpha}{1-qL}, \gamma, t, \ldots, t\rangle^{\epsilon, S_n, T}_{0,n+1,d} g^{\alpha\beta}$$

Note that we have the formal expansion

$$\frac{1}{1-xL} = \sum_{i\geq 0} \frac{x^i}{(1-x)^{i+1}} (L-1)^i.$$

For each initial component M with the first marking lying over β, we claim that it contributes polynomials of $\frac{1}{1-\xi q}$, where ξ is a root of unity:

The vertex factor of the initial component M that corresponds to the fixed point β can be written as the fiber product

$$\Big(\big(\overline{Q}_{0,val(\beta)+k}(W /\!\!/ G, 0)^T \times_{(W/\!\!/G)^T} \overline{Q}_{0,1}(W /\!\!/ G, d_1)^T_\infty\big) \times \cdots\Big) \times_{(W/\!\!/G)^T} \overline{Q}_{0,1}(W /\!\!/ G, d_k)^T_\infty,$$

where the moduli space $\overline{Q}_{0,1}(W /\!\!/ G, d_i)_\infty$ parametrizes the quasimaps $(\mathbb{P}^1, P, u)$ of class d_i with a principal G-bundle P on $\mathbb{P}^1$, a section $u : \mathbb{P}^1 \to P \times_G W$ such that $u(x) \in W^s$ for $x \neq \infty \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$ is a base-point of length $d_i(L_\theta)$. We also have

$$\overline{Q}_{0,val(\beta)+k}(W /\!\!/ G, 0)^T = \overline{M}_{0,val(\beta)+k}$$

is a finite dimensional complex manifold, hence L restricts to a unipotent element and the trace $\mathrm{tr}_h(\frac{1}{1-qL}) = \frac{1}{1-q\xi\tilde{L}}$, where $h \in S_n$, ξ is the eigenvalue of h on L and $\tilde{L}$ is the restriction of L to the fixed point locus of h.

For each recursion component M with the first marking lies over β, then the restriction of L to M is $\lambda(\beta,\mu)^{1/m}$. Hence $\beta^* S^{\epsilon}_{t,T}$ has simple poles at $q = \lambda(\beta,\mu)^{-1/m}$. $\square$

Lemma 4.2 (Recursion Relation). *The restriction $\beta^* S^{\epsilon}_{t,T}$ of $S^{\epsilon}_{t,T}$ to the fixed point β satisfies the recursion relation*

(5)

$$\beta^* S^{\epsilon}_{t,T}(q) = I^{\epsilon}_{\beta}(q) + \sum_{\mu \in o(\beta)} \sum_{m=1}^{\infty} \frac{Q^{md(\beta,\mu)}}{m} \frac{\phi^\beta}{C_{\beta,\mu,m}} \frac{1}{1-\lambda(\beta,\mu)^{1/m} q} \mu^* S^{\epsilon}_{t,T}(\lambda(\beta,\mu)^{1/m})$$

where

(i) *$I^\epsilon_\beta(q)$ is the sum of the contribution of all the components of initial type. Each $Q,\{t_i\}$ coefficient of $I^\epsilon_\beta(q)$ is of the form*

$$\sum_{\xi:root\ of\ unity}\sum_{i\geq 0} c_{i,\xi}(\xi q)^i/(1-\xi q)^{i+1}.$$

(ii) *$o(\beta)$ is the set of all fixed points μ connected to β via a 1-dimensional orbit, $d(\beta,\mu)$ is the homology class of the orbit and $\lambda(\beta,\mu)$ is the character of the torus representation on the tangent line at β corresponding to the orbit.*

(iii) *The recursion coefficient $C_{\beta,\mu,m}$ is the T-equivariant K-theoretic Euler class of the virtual cotangent space to the moduli space $\overline{M}_{0,2}(W/\!\!/G,m)$ at the corresponding fixed point and this recursion coefficient does not depend on ϵ.*

Proof. We again apply virtual localization to the summand

$$\sum_\alpha Q^d\langle\frac{\phi_\alpha}{1-qL},\gamma,t,\dots,t\rangle^{\epsilon,S_n,T}_{0,n+1,d}g^{\alpha\beta}.$$

Consider fixed components of recursion type, there are two possibilities when the component has contributions to the pole at $q=\lambda(\beta,\mu)^{-1/m}$:

The first possibility is that M is an one-dimensional orbit with multiplicity m connecting the the fixed points β,μ and has a marked point at the other end. It is an isolated fixed point p in $\overline{M}_{0,2}(W/\!\!/G,m)$. Note that this component does not dependent on the choose of ϵ. The contribution is

$$(\sum_\alpha\phi_\alpha g^{\alpha\beta})\frac{1}{1-\lambda(\beta,\mu)^{1/m}q}\frac{Q^{md(\beta,\mu)}}{mC_{\beta,\mu,m}}\sum_\nu\langle\phi_\nu,\gamma\rangle g^{\nu\mu}$$

where the recursion coefficient

$$C_{\beta,\mu,m}=\mathrm{Euler}^K_T(T^*_p\overline{M}_{0,2}(W/\!\!/G,m)) \tag{6}$$

is the T-equivariant K-theoretic Euler class of the virtual cotangent space to the moduli space at the point p corresponding to the m-multiple cover of the 1-dimensional orbit connecting fixed points β and μ. Hence $C_{\beta,\mu,m}$ does not dependent on the choose of ϵ.

The second possibility is M has an one-dimensional orbit connecting β,μ and a subgraph M' attached to μ, i.e. the first marking of M' lies over μ. The contribution is

$$\frac{\sum_\alpha\phi_\alpha g^{\alpha\beta}}{1-\lambda(\beta,\mu)^{1/m}q}\frac{Q^{md(\beta,\mu)}}{mC_{\beta,\mu,m}}(\mu^*S^\epsilon_{t,T}(t,\lambda(\beta,\mu)^{-1/m})-\sum_\nu\langle\phi_\nu,\gamma\rangle g^{\nu\mu}).$$

Finally, the polynomiality of the coefficients of $I^\epsilon_\beta(q)$ follows from unipotency of L.

This completes the proof. ∎

Remark 4.3. Write

$$P^\epsilon(t,q) = P^\epsilon(t,(\lambda(\beta,\mu))^{-1/m}) + (1-\lambda(\beta,\mu)^{1/m}q)A^\epsilon_{\mu,m}(q),$$

where $A^\epsilon_{\mu,m}(q)$ is a power series in $(1-q)$. Applying localization as in the previous lemma, we have the following recursion relation for $S^\epsilon_t(q)(P^\epsilon(t,q))$:

$$\begin{aligned}
&\beta^* S^\epsilon_t(q)(P^\epsilon(t,q)) \\
&= \tilde{I}^\epsilon_\beta(q) + \sum_{\mu\in o(\beta)}\sum_{m=1}^\infty \frac{Q^{md(\beta,\mu)}}{m}\frac{\phi^\beta}{C_{\beta,\mu,m}}\frac{1}{1-\lambda(\beta,\mu)^{1/m}q}\mu^* S^\epsilon_{t,T}(\lambda(\beta,\mu)^{1/m})(P^\epsilon(t,q)) \\
&= I^\epsilon_\beta(q) + \sum_{\mu\in o(\beta)}\sum_{m=1}^\infty \frac{Q^{md(\beta,\mu)}}{m}\frac{\phi^\beta}{C_{\beta,\mu,m}}\frac{1}{1-\lambda(\beta,\mu)^{1/m}q}\mu^* S^\epsilon_{t,T}(\lambda(\beta,\mu)^{1/m})(P^\epsilon(t,\lambda(\beta,\mu)^{-1/m}))
\end{aligned}$$

where $\tilde{I}^\epsilon_\beta(q)$ is the summation of the contribution of all the components of initial type and

$$I^\epsilon_\beta(q) := \tilde{I}^\epsilon_\beta(q) + \sum_{\mu\in o(\beta)}\sum_{m=1}^\infty \frac{Q^{md(\beta,\mu)}}{m}\frac{\phi^\beta}{C_{\beta,\mu,m}}\mu^* S^\epsilon_{t,T}(\lambda(\beta,\mu)^{1/m})(A^\epsilon_{\mu,m}(q))$$

The recursion relation for $S^\infty_{\tau^{\infty,\epsilon}(t)}(q)(P^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t),q))$ works in the same way.

4.2. **Main Results.** This section, we state and prove the main theorems of this paper.

Theorem 4.4. *Assume the torus T action on $W /\!\!/ G$ has isolated fixed points and isolated 1-dimensional orbits. Let $0+ \le \epsilon_1 < \epsilon_2 \le \infty$, $\gamma \in K^0_T(W /\!\!/ G)\otimes\Lambda$ is of the form $\mathbb{1} + O(Q)$. Then*

$$S^{\epsilon_1}_{\tau^{\epsilon_1,\epsilon_2}_\gamma(t)}(q)(\gamma) = S^{\epsilon_2}_t(q)(\gamma).$$

Theorem 4.5. *Assume the torus T action on $W /\!\!/ G$ has isolated fixed points and isolated 1-dimensional orbits, then for all $\epsilon \ge 0+$,*

$$\mathcal{J}^\epsilon(t,q) = S^\epsilon_t(q)(P^\epsilon(t,q)) = S^\infty_{\tau^{\infty,\epsilon}(t)}(q)(P^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t),q)),$$

hence, lies on $\mathcal{L}_{S_\infty, W/\!\!/G}$, the Lagrangian cone of the permutation-equivariant K-theoretic Gromov-Witten theory of $W /\!\!/ G$.

Theorem 4.5 is a consequence of Theorem 4.4 and the arguments in Remark 3.4.

Remark 4.6. Certainly, it is conjectured that the statements in Theorems 4.4 and 4.5 hold for $W /\!\!/ G$ without torus actions.

Lemma 4.7. *For each torus fixed point $\beta \in (W /\!\!/ G)^T$, the series*

$$D(S^\epsilon_\beta) := S^\epsilon_\beta(Q,t,q)S^\epsilon_\beta(Q(1/q)^{aL_\theta},t,1/q)$$

has no pole at roots of unity, where a varies in integers and $(Q(1/q)^{aL_\theta})^d = Q^d(1/q)^{ad(L_\theta)}$.

Proof. Given a fixed point $\beta \in (W /\!\!/ G)^T$, we write $QG^{\epsilon}_{0,m+2,d}(W /\!\!/ G)_\beta$ for the T-fixed locus parametrizes quasimaps with the parametrized $\mathbb{P}^1$ contracted to the point β. Let κ be the inclusion map into the graph space and put

$$\operatorname{Res}_\mu \mathcal{O}^{\mathrm{vir}}_{QG^{\epsilon}_{0,m+2,d}(W/\!\!/G)} := \kappa_*\left(\frac{\mathcal{O}^{\mathrm{vir}}_{QG^{\epsilon}_{0,m+2,d}(W/\!\!/G)_\beta}}{\mathrm{Euler}^K_T(N^{\mathrm{vir}})}\right)$$

the T-equivariant residue at the fixed locus $QG^{\epsilon}_{0,m+2,d}(W /\!\!/ G)_\beta$. Furthermore, we write

$$\gamma = \sum_d Q^d \gamma_d$$

where $\gamma_d \in K^0_T(W /\!\!/ G) \otimes \mathbb{Q}$. Consider the generating series

$$\begin{aligned}
&\langle\!\langle \gamma(1-p_0), \gamma(1-p_\infty); U(L_\theta) \rangle\!\rangle^{QG,\epsilon}_{0,2;\beta} := \\
&\quad \sum_{m,d,d_1,d_2} Q^d Q^{d_1} Q^{d_2} \chi\Big(QG^{\epsilon}_{0,m+2,d}(W /\!\!/ G)_\beta / S_m,\ \operatorname{Res}_\mu \mathcal{O}^{\mathrm{vir}}_{QG^{\epsilon}_{0,m+2,d}(W/\!\!/G)} \\
&\qquad \otimes (U_{d_1,d_2}(L_\theta))^a \operatorname{ev}_1^*(\gamma_{d_1}(1-p_0)) \operatorname{ev}_2^*(\gamma_{d_2}(1-p_\infty)) \prod_{i=1}^m \operatorname{ev}_{i+2}^*(t)\Big)
\end{aligned}$$

where, $U_{d_1,d_2}(L_\theta)$ is the universal $\mathbb{C}^*$-equivaraint line bundle obtained from pulling back $\mathcal{O}(1)$ with the canonical linearization, as described in [1, Section 3.3]. This is defined without $\mathbb{C}^*$-localization, hence has no pole at $q = 1$. Applying $\mathbb{C}^*$-localization to this generating series, we have

$$\langle\!\langle \gamma(1-p_0), \gamma(1-p_\infty); U(L_\theta) \rangle\!\rangle^{QG,\epsilon}_{0,2;\beta} = \lambda(\mathcal{O}(\theta);\beta)^a D(S^{\epsilon}_\beta),$$

where $\lambda(\mathcal{O}(\theta);\beta)$ is the character of the torus representation on the fiber of $\mathcal{O}(\theta)$ at β.

For any positive integer m, $\Psi^m(D(S^{\epsilon}_\beta))$ is also defined without $\mathbb{C}^*$-localization, hence has no pole at $q = 1$, where the Adams operation extended from Λ by $\Psi^m(q) = q^m$. Therefore, $D(S^{\epsilon}_\beta)$ has no pole at the m-th root of unity. Hence the Lemma follows. □

Remark 4.8. Since P^∞ and $P^{\infty,\epsilon}$ in Theorem 4.5 have no pole at roots of unity, the proof also applies to the restrictions of $S^{\epsilon}_t(q)(P^{\epsilon}(t,q))$ and $S^{\infty}_{\tau^{\infty,\epsilon}(t)}(q)(P^{\infty,\epsilon}(\tau^{\infty,\epsilon}(t),q))$ to fixed points, after appropriate adjustments to the generating series.

Lemma 4.9 (Uniqueness Lemma)**.** *Let*

$$\{S_{1,\beta}\}_{\beta\in(W/\!\!/G)^T} \quad \text{and} \quad \{S_{2,\beta}\}_{\beta\in(W/\!\!/G)^T}$$

be two systems of power series in $\Lambda[[t_i]]\{\{q,1/q\}\}$ that satisfy the following properties:

(1): *For all $\beta \in (W /\!\!/ G)^T$, $S_{1,\beta}$ and $S_{2,\beta}$ are rational functions of q with possibly poles only at 0, ∞, roots of unity and at most simple poles at $q = (\lambda(\beta,\mu)^{-1/m})$, where $\lambda(\beta,\mu)$ is the character of the torus* *representation on the tangent line at the fixed point β corresponding to the 1-dimensional orbit connecting the fixed points β and μ, for $m = 1, 2, \ldots$.*

(2): *The systems*

$$\{S_{1,\beta}\}_{\beta\in(W/\!\!/G)^T} \quad \text{and} \quad \{S_{2,\beta}\}_{\beta\in(W/\!\!/G)^T}$$

both satisfy the recursion relation (5).

(3): *For all $\beta \in (W /\!\!/ G)^T$, the series*

$$D(S^\epsilon_{1,\beta}) := S^\epsilon_{1,\beta}(Q, t, q)S^\epsilon_{1,\beta}(Q(1/q)^{aL_\theta}, t, 1/q)$$

and

$$D(S^\epsilon_{2,\beta}) := S^\epsilon_{2,\beta}(Q, t, q)S^\epsilon_{2,\beta}(Q(1/q)^{aL_\theta}, t, 1/q)$$

have no pole at $q = 1$.

(4): *For all $\beta \in (W /\!\!/ G)^T$,*

$$S_{1,\beta} = S_{2,\beta} + \frac{1}{(1-q)^2} p_{1,2}\left(\frac{q}{1-q}\right),$$

where $p_{1,2}\left(\frac{q}{1-q}\right)$ stands for a power series in the variable $q/(1-q)$ with coefficient in $K^0(W /\!\!/ G) \otimes \Lambda\{\{t_i\}\}$.

(5): *For all $\beta \in (W /\!\!/ G)^T$,*

$$S_{1,\beta} = S_{2,\beta} \mod Q$$

Then $S_{1,\beta} = S_{2,\beta}$ for all $\beta \in (W /\!\!/ G)^T$.

Proof. We write

$$S_{1,\beta} = \sum_d Q^d \sum_k c_{1,\beta,d,k}(q) \prod_i t_i^{k_i} \quad \text{and} \quad S_{2,\beta} = \sum_d Q^d \sum_k c_{2,\beta,d,k}(q) \prod_i t_i^{k_i}$$

where $k := \{k_i\}_i$ and k_i are nonnegative integers. We define the bi-degree of the monomial $Q^d \prod_i t_i^{k_i}$ to be $(\sum_i k_i, d(L_\theta))$. We write $S^{(m,l)}_{1,\beta}$ and $S^{(m,l)}_{2,\beta}$ for the part of bi-degree (m, l) of $S_{1,\beta}$ and $S_{2,\beta}$, respectively. It suffices to show

$$S^{(m,l)}_{1,\beta} = S^{(m,l)}_{2,\beta}, \quad \text{for all } \beta \in (W /\!\!/ G)^T \text{ and } (m, l) \in \mathbb{N} \times \mathbb{N}$$

We prove it by induction on (m, l) using the lexicographic order

$$(m', l') < (m, l) \text{ if and only if } m' < m, \text{ or } m' = m \text{ and } l' < l.$$

The base case when $d = 0$ is true due to property (5).

For $l \geq 1$, we assume

$$S^{(m',l')}_{1,\beta} = S^{(m',l')}_{2,\beta} \text{ for all } \beta \in (W /\!\!/ G)^T \text{ and all } (m', l') < (m, l)$$

Denote by $D^{(m,l)}$ the part of bi-degree (m,l) of the difference $D(S_{1,\beta}) - D(S_{2,\beta})$. By induction, we have

$$D^{(m,l)} = S^{(m,l)}_{1,\beta}(q) - S^{(m,l)}_{2,\beta}(q) + (1/q)^{al}(S^{(m,l)}_{1,\beta}(1/q) - S^{(m,l)}_{2,\beta}(1/q)).$$

By properties (2), $\left(S_{1,\beta}^{(m,l)}(q) - S_{2,\beta}^{(m,l)}(q)\right)$ is the sum of monomials of the form $c_i(\xi q)^i/(1-\xi q)^{i+1}$, for $i \geq 0$ and roots of unity ξ, with coefficient in $K^0(W /\!\!/ G) \otimes \Lambda\{\{t_i\}\}$, we write

$$\left(S_{1,\beta}^{(m,l)}(q) - S_{2,\beta}^{(m,l)}(q)\right)_1$$

for the sum of terms of $S_{1,\beta}^{(m,l)}(q) - S_{2,\beta}^{(m,l)}(q)$ with $\xi = 1$, therefore, by property (4), we have

$$\left(S_{1,\beta}^{(m,l)}(q) - S_{2,\beta}^{(m,l)}(q)\right)_1 = \left(\frac{1}{1-q}\right)^n (Aq^{n-2} + O(1-q))$$

and

$$\left(S_{1,\beta}^{(m,l)}(1/q) - S_{2,\beta}^{(m,l)}(1/q)\right)_1 = \left(\frac{1}{1-q}\right)^n ((-1)^n Aq^2 + O(1-q)),$$

for an integer $n \geq 2$ and a nonzero element $A \in \Lambda\{\{t_i\}\}$. Moreover, we have

$$(1/q)^{al} = 1 + al(1-q) + O(1-q),$$

therefore

$$\left(D^{(m,l)}\right)_1 = \left(\frac{1}{1-q}\right)^n (Aq^{n-2} + (-1)^n Aq^2 + O(1-q))$$

For $n > 1$, then $D^{(m,l)}$ has a pole at $q = 1$. It contradicts property (3). Therefore, $S_{1,\beta}^{(m,l)} - S_{2,\beta}^{(m,l)}$ has no pole at $q = 1$. Similar argument shows $S_{1,\beta}^{(m,l)} - S_{2,\beta}^{(m,l)}$ has no pole at roots of unity. Therefore, $S_{1,\beta}^{(m,l)} = S_{2,\beta}^{(m,l)}$. $\square$

Theorem 4.4 now follows from the above uniqueness lemma applied to $\{S^{\epsilon_1}_{\tau_\gamma^{\epsilon_1,\epsilon_2}(t),\beta}\}$ and $\{S^{\epsilon_2}_{t,\beta}\}$. The required properties are checked before. Property (1) is in Lemma 4.1. Property (2) is in Lemma 4.2. Property (3) is in Lemma 4.7. Property (4) is in Lemma 3.3. Property (5) is clear.

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