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FLUID DYNAMIC LIMIT TO THE RIEMANN SOLUTIONS OF EULER EQUATIONS: I. SUPERPOSITION OF RAREFACTION WAVES AND CONTACT DISCONTINUITY

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ABSTRACT. Fluid dynamic limit to compressible Euler equations from compressible Navier-Stokes equations and Boltzmann equation has been an active topic with limited success so far. In this paper, we consider the case when the solution of the Euler equations is a Riemann solution consisting two rarefaction waves and a contact discontinuity and prove this limit for both Navier-Stokes equations and the Boltzmann equation when the viscosity, heat conductivity coefficients and the Knudsen number tend to zero respectively. In addition, the uniform convergence rates in terms of the above physical parameters are also obtained. It is noted that this is the first rigorous proof of this limit for a Riemann solution with superposition of three waves even though the fluid dynamic limit for a single wave has been proved.

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1. Introduction. This paper concerns the fluid dynamic limit to the compressible Euler equations for two physical models, that is, the compressible Navier-Stokes equations and the Boltzmann equation. In the first part, we consider zero dissipation

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limit of the compressible Navier-Stokes system for viscous and heat conductive fluid in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon(\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x = (\kappa \frac{\theta_x}{v} + \varepsilon \frac{uu_x}{v})_x, \end{cases}$$
(1.1)

where the functions v(t,x) > 0, u(t,x), $\theta(t,x) > 0$ represent the specific volume, velocity and the absolute temperature of the gas respectively. And $p = p(v, \theta)$ is the pressure, $e = e(v, \theta)$ is the internal energy, $\varepsilon > 0$ is the viscosity coefficient, $\kappa > 0$ is the coefficient of the heat conductivity. Here, both ε and κ are taken as positive constants. And we consider the perfect gas where

$$p = \frac{R\theta}{v} = Av^{-\gamma} \exp\left(\frac{\gamma - 1}{R}s\right), \qquad e = \frac{R\theta}{\gamma - 1}, \tag{1.2}$$

with s denoting the entropy of the gas and A,R>0 , $\gamma>1$ being the gas parameters.

Formally, as the coefficients κ and ε tend to zero, the limiting system of (1.1) is the compressible Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = 0. \end{cases}$$
(1.3)

The study of this limiting process of viscous flows when the viscosity and heat conductivity coefficients tend to zero, is one of the important problems in the theory of the compressible fluid. When the solution of the inviscid flow is smooth, the zero dissipation limit can be solved by classical scaling method. However, the inviscid compressible flow usually contains discontinuities, such as shock waves and contact discontinuities. Therefore, how to justify the zero dissipation limit to the Euler equations with basic wave patterns is a natural and difficult problem.

Keeping in mind that the Navier-Stokes equations can be derived from the Boltzmann equation through the Chapman-Enskog expansion when the Knudsen number is close to zero, we assume the following condition on the viscosity constant ε and the heat conductivity coefficient κ in the system (1.1), cf. also [17]:

$$\begin{cases} \kappa = O(\varepsilon) & \text{as} \quad \varepsilon \to 0; \\ \nu \doteq \frac{\kappa(\varepsilon)}{\varepsilon} \ge c > 0 & \text{for some positive constant } c, \quad \text{as} \quad \varepsilon \to 0. \end{cases}$$
(1.4)

Now we briefly review some recent results on the zero dissipation limit of the compressible fluid with basic wave patterns. For the hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx},$$

Goodman-Xin [9] verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. For the compressible isentropic Navier-Stokes equations, Hoff-Liu [12] first proved the vanishing viscosity limit for piecewise constant solutions separated by noninteracting shocks even with initial layer. Later Xin [30] obtained the zero dissipation limit for rarefaction waves and Wang [28] generalized the result of Goodmann-Xin [9] to the isentropic Navier-Stokes equations. For the inviscid limit of the full compressible Navier-Stokes equations (1.1), Jiang-Ni-Sun [17] justified the zero dissipation limit of the system (1.1) for centered rarefaction waves. Wang [29] proved the zero dissipation limit of the system (1.1) for piecewise smooth solutions separated by shocks using the matched asymptotic expansion method introduced in [9]. Recently, Xin-Zeng [31] considered the zero dissipation limit of the system (1.1) for single rarefaction wave with well prepared initial data and obtained a uniform decay rate in terms of the dissipation coefficients. And Ma [22] obtained the zero dissipation limit of a single strong contact discontinuity in any fixed time interval with a decay rate.

However, to our knowledge, so far there is no result on the zero dissipation limit of the system (1.1) for superposition of different types of basic wave patterns. In the first part of this paper, we investigate the fluid dynamic limit of the compressible Navier-Stokes equations when the corresponding Euler equations have the Riemann solution as a superposition of two rarefaction waves and a contact discontinuity. For this, we need to study the interaction between the rarefaction waves and contact discontinuity.

In the second part of the paper, we study the hydrodynamic limit of the Boltzmann equation [2] with slab symmetry

$$f_t + \xi_1 f_x = \frac{1}{\varepsilon} Q(f, f), \ (f, t, x, \xi) \in \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^3,$$
(1.5)

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, $f(t, x, \xi)$ is the density distribution function of particles at time t with location x and velocity ξ , and $\varepsilon > 0$ is called the Knudsen number which is proportional to the mean free path. Remark that the notation ε here is same as the viscosity of the compressible Navier-Stokes equations (1.1), but it has different physical meanings from (1.1) in different equations and related contexts.

For monatomic gas, the rotational invariance of the particles leads to the following bilinear form for the collision operator

$$Q(f,g)(\xi) = \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2_+} \left(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi) \right) \\ B(|\xi - \xi_*|, \hat{\theta}) \, d\xi_* d\Gamma,$$

where ξ', ξ'_* are the velocities after an elastic collision of two particles with velocities ξ, ξ_* before the collision. Here, $\hat{\theta}$ is the angle between the relative velocity $\xi - \xi_*$ and the unit vector Γ in $\mathbf{S}^2_+ = \{\Gamma \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Gamma \ge 0\}$. The conservation of momentum and energy gives the following relation between the velocities before and after collision:

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Gamma] \Gamma, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Gamma] \Gamma. \end{cases}$$

In this paper, we consider the Boltzmann equation for two basic models, that is, the hard sphere model and the hard potential including Maxwellian molecules under the assumption of angular cut-off. For this, we assume that the collision kernel $B(|\xi - \xi_*|, \hat{\theta})$ takes one of the following two forms,

$$B(|\xi - \xi_*|, \hat{\theta}) = |(\xi - \xi_*, \Gamma)| = |\xi - \xi_*|\cos\theta,$$

and

$$B(|\xi - \xi_*|, \hat{\theta}) = |\xi - \xi_*|^{\frac{n-5}{n-1}} b(\hat{\theta}), \quad b(\hat{\theta}) \in L^1([0, \pi]), \ n \ge 5.$$

Here, n is the index in the potential of inverse power law which is proportional to r^{1-n} with r being the distance between two concerned particles.

Formally, when the Knudsen number ε tends to zero, the limit of the Boltzmann equation (1.5) is the classical system of Euler equations

$$\begin{cases}
\rho_t + (\rho u_1)_x = 0, \\
(\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\
(\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\
[\rho(E + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = 0,
\end{cases}$$
(1.6)

where

$$\rho(t,x) = \int_{\mathbf{R}^{3}} \varphi_{0}(\xi) f(t,x,\xi) d\xi,
\rho u_{i}(t,x) = \int_{\mathbf{R}^{3}} \varphi_{i}(\xi) f(t,x,\xi) d\xi, \quad i = 1,2,3,
\rho(E + \frac{|u|^{2}}{2})(t,x) = \int_{\mathbf{R}^{3}} \varphi_{4}(\xi) f(t,x,\xi) d\xi.$$
(1.7)

Here, ρ is the density, $u = (u_1, u_2, u_3)$ is the macroscopic velocity, E is the internal energy of the gas, and $p = R\rho\theta$ with R being the gas constant is the pressure. Note that the temperature θ is related to the internal energy by $E = \frac{3}{2}R\theta$, and $\varphi_i(\xi)(i=0,1,2,3,4)$ are the collision invariants given by

$$\begin{cases} \varphi_0(\xi) = 1, \\ \varphi_i(\xi) = \xi_i \text{ for } i = 1, 2, 3, \\ \varphi_4(\xi) = \frac{1}{2} |\xi|^2, \end{cases}$$

that satisfy

$$\int_{\mathbf{R}^3} \varphi_i(\xi) Q(h,g) d\xi = 0, \quad \text{for } i = 0, 1, 2, 3, 4.$$

How to justify the above limit, that is, the Euler equation (1.6) from Boltzmann equation (1.5) when Knudsen number ε tends to zero is an open problem going way back to the time of Maxwell. For this, Hilbert introduced the famous Hilbert expansion to show formally that the first order approximation of the Boltzmann equation gives the Euler equations. On the other hand, it is important to verify this limit process rigorously in mathematics. For the case when the Euler equation has smooth solutions, the zero Knudsen number limit of the Boltzmann equation has been studied even in the case with an initial layer, cf. Ukai-Asano [26], Caflish [3], Lachowicz [18] and Nishida [24] etc. However, as is well-known, solutions of the Euler equations (1.6) in general develop singularities, such as shock waves and contact discontinuities. Therefore, how to verify the fluid limit from Boltzmann equation to the Euler equations with basic wave patterns becomes an natural problem. In this direction, Yu [32] showed that when the solution of the Euler equations (1.6) contains only non-interacting shocks, there exists a sequence of solutions to the Boltzmann equation that converge to a local Maxwellian defined by the solution of the Euler equations (1.6) uniformly away from the shock in any fixed time interval. In this work, the inner and outer expansions developed by Goodman-Xin [9] for conservation laws and the Hilbert expansion were skillfully and cleverly used. Recently, Huang-Wang-Yang [15] proved the fluid dynamic limit of the Boltzmann equation to the Euler equations for a single contact discontinuity where the uniform decay rate was also obtained. And Xin-Zeng [31] proved the fluid dynamic limit of the compressible Navier-Stokes equations and Boltzmann equation to the Euler equations with non-interacting rarefaction waves. About the detailed introductions of the Boltzmann equation and its hydrodynamic limit, see the books [4], [7] etc.

In this paper, we will study the hydrodynamic limit of the Boltzmann equation when the corresponding Euler equations have a Riemann solution as a superposition of two rarefaction waves and a contact discontinuity. More precisely, given a Riemann solution of the Euler equations (1.6) with superposition of two rarefaction waves and a contact discontinuity, we will show that there exists a family of solutions to the Boltzmann equation that converge to a local Maxwellian defined by the Euler solution uniformly away from the contact discontinuity for strictly positive time as $\varepsilon \to 0$. Moreover, a uniform convergence rate in ε is also given.

As mentioned above for the compressible Navier-Stokes equations, we also need to study the detailed wave interactions through this limiting process.

For later use, we now briefly present the micro-macro decomposition around the local Maxwellian defined by the solution to the Boltzmann equation, cf. [19] and [21]. For a solution $f(t, x, \xi)$ of the Boltzmann equation (1.5), set

$$f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi),$$

where the local Maxwellian $\mathbf{M}(t, x, \xi) = \mathbf{M}_{[\rho, u, \theta]}(\xi)$ represents the macroscopic (fluid) component of the solution, which is naturally defined by the five conserved quantities, i.e., the mass density $\rho(t, x)$, the momentum $\rho u(t, x)$, and the total energy $\rho(E + \frac{1}{2}|u|^2)(t, x)$ in (1.7), through

$$\mathbf{M} = \mathbf{M}_{[\rho, u, \theta]}(t, x, \xi) = \frac{\rho(t, x)}{\sqrt{(2\pi R\theta(t, x))^3}} e^{-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)}}.$$
(1.8)

And $\mathbf{G}(t, x, \xi)$ being the difference between the solution and the above local Maxwellian represents the microscopic (non-fluid) component.

For convenience, we denote the inner product of h and g in $L^2_{\xi}(\mathbf{R}^3)$ with respect to a given Maxwellian $\tilde{\mathbf{M}}$ by:

$$\langle h,g \rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} h(\xi) g(\xi) d\xi$$

If \mathbf{M} is the local Maxwellian \mathbf{M} defined in (1.8), with respect to the corresponding inner product, the macroscopic space is spanned by the following five pairwise orthogonal base

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} \mathbf{M} \text{ for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}} (\frac{|\xi - u|^2}{R\theta} - 3) \mathbf{M}, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \ i, j = 0, 1, 2, 3, 4. \end{cases}$$

In the following, if $\tilde{\mathbf{M}}$ is the local Maxwellian \mathbf{M} , we just use the simplified notation $\langle \cdot, \cdot \rangle$ to denote the inner product $\langle \cdot, \cdot \rangle_{\mathbf{M}}$. The macroscopic projection \mathbf{P}_0 and microscopic projection \mathbf{P}_1 can be defined as follows

$$\begin{cases} \mathbf{P}_0 h = \sum_{j=0}^4 \langle h, \chi_j \rangle \chi_j, \\ \mathbf{P}_1 h = h - \mathbf{P}_0 h. \end{cases}$$

The projections \mathbf{P}_0 and \mathbf{P}_1 are orthogonal and satisfy

$$\mathbf{P}_0\mathbf{P}_0 = \mathbf{P}_0, \mathbf{P}_1\mathbf{P}_1 = \mathbf{P}_1, \mathbf{P}_0\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_0 = 0.$$

Note that a function $h(\xi)$ is called microscopic or non-fluid if

$$\int h(\xi)\varphi_i(\xi)d\xi = 0, \ i = 0, 1, 2, 3, 4$$

where $\varphi_i(\xi)$ is the collision invariants.

Under the above micro-macro decomposition, the solution $f(t, x, \xi)$ of the Boltzmann equation (1.5) satisfies

$$\mathbf{P}_0 f = \mathbf{M}, \ \mathbf{P}_1 f = \mathbf{G},$$

and the Boltzmann equation (1.5) becomes

$$(\mathbf{M} + \mathbf{G})_t + \xi_1 (\mathbf{M} + \mathbf{G})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}, \mathbf{G}) + Q(\mathbf{G}, \mathbf{G})].$$
(1.9)

By multiplying the equation (1.9) by the collision invariants $\varphi_i(\xi)(i = 0, 1, 2, 3, 4)$ and integrating the resulting equations with respect to ξ over \mathbf{R}^3 , one has the following fluid-type system for the fluid components:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\int \xi_1^2 \mathbf{G}_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = -\int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ [\rho(E + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = -\int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi. \end{cases}$$
(1.10)

Note that the above fluid-type system is not closed and one more equation for the non-fluid component **G** is needed and it can be obtained by applying the projection operator \mathbf{P}_1 to the equation (1.9):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} \left[\mathbf{L}_{\mathbf{M}} \mathbf{G} + Q(\mathbf{G}, \mathbf{G}) \right].$$
(1.11)

Here $\mathbf{L}_{\mathbf{M}}$ is the linearized collision operator of Q(f, f) with respect to the local Maxwellian \mathbf{M} :

$$\mathbf{L}_{\mathbf{M}}h = 2Q(\mathbf{M},h) = Q(\mathbf{M},h) + Q(h,\mathbf{M})$$

Note that the null space \mathfrak{N} of $\mathbf{L}_{\mathbf{M}}$ is spanned by the macroscopic variables:

$$\chi_i(\xi), \ j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant $\sigma_0 > 0$ such that for any function $h(\xi) \in \mathfrak{N}^{\perp}$, cf. [10],

$$< h, \mathbf{L}_{\mathbf{M}}h > \leq -\sigma_0 < \nu(|\xi|)h, h >,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere model and the hard potential including Maxwellian molecules with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0 < \nu_0 < \nu(|\xi|) \le c(1+|\xi|)^{\kappa_0},$$

for some positive constants ν_0, c and $0 \le \kappa_0 \le 1$.

Consequently, the linearized collision operator $\mathbf{L}_{\mathbf{M}}$ is a dissipative operator on $L^2(\mathbf{R}^3)$, and its inverse $\mathbf{L}_{\mathbf{M}}^{-1}$ exists in \mathfrak{N}^{\perp} .

It follows from (1.11) that

$$\mathbf{G} = \varepsilon \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_x)] + \Pi, \qquad (1.12)$$

with

$$\Pi = \mathbf{L}_{\mathbf{M}}^{-1}[\varepsilon(\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x)) - Q(\mathbf{G}, \mathbf{G})].$$
(1.13)

Plugging the equation (1.12) into (1.10) gives

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4\varepsilon}{3} (\mu(\theta) u_{1x})_x - \int \xi_1^2 \Pi_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = \varepsilon (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Pi_x d\xi, \quad i = 2, 3, \\ [\rho(E + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = \varepsilon (\lambda(\theta) \theta_x)_x + \frac{4\varepsilon}{3} (\mu(\theta) u_1 u_{1x})_x \\ + \varepsilon \sum_{i=2}^3 (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_x d\xi, \end{cases}$$
(1.14)

where the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\lambda(\theta) > 0$ are smooth functions of the temperature θ . Here, we normalize the gas constant R to be $\frac{2}{3}$ so that $E = \theta$ and $p = \frac{2}{3}\rho\theta$. The explicit formula of $\mu(\theta)$ and $\lambda(\theta)$ can be found for example in [5], we omit it here for brevity.

Since the problem considered in this paper is one dimensional in the space variable $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite the equation (1.5) and the system (1.6) in the *Lagrangian* coordinates as in the study of conservation laws. That is, set the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(t, y) dy, \qquad t \Rightarrow t.$$

We will still denote the Lagrangian coordinates by (t, x) for simplicity of notation. Then (1.5) and (1.6) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = \frac{1}{\varepsilon} Q(f, f),$$
(1.15)

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \ i = 2, 3, \\ (\theta + \frac{|u|^2}{2})_t + (pu_1)_x = 0. \end{cases}$$
(1.16)

Also, (1.10)-(1.14) take the form

$$v_{t} - u_{1x} = 0,$$

$$u_{1t} + p_{x} = -\int \xi_{1}^{2} \mathbf{G}_{x} d\xi,$$

$$u_{it} = -\int \xi_{1} \xi_{i} \mathbf{G}_{x} d\xi, \quad i = 2, 3,$$

$$\left(\theta + \frac{|u|^{2}}{2}\right)_{t} + (pu_{1})_{x} = -\int \frac{1}{2} \xi_{1} |\xi|^{2} \mathbf{G}_{x} d\xi,$$
(1.17)

$$\mathbf{G}_t - \frac{u_1}{v}\mathbf{G}_x + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_x) + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_x) = \frac{1}{\varepsilon}(\mathbf{L}_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G})), \quad (1.18)$$

with

$$\mathbf{G} = \varepsilon \mathbf{L}_{\mathbf{M}}^{-1} \left(\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x)\right) + \Pi_1, \qquad (1.19)$$

$$\Pi_1 = \mathbf{L}_{\mathbf{M}}^{-1} [\varepsilon(\mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x)) - Q(\mathbf{G}, \mathbf{G})].$$
(1.20)

$$v_{t} - u_{1x} = 0,$$

$$u_{1t} + p_{x} = \frac{4\varepsilon}{3} (\frac{\mu(\theta)}{v} u_{1x})_{x} - \int \xi_{1}^{2} \Pi_{1x} d\xi,$$

$$u_{it} = \varepsilon (\frac{\mu(\theta)}{v} u_{ix})_{x} - \int \xi_{1} \xi_{i} \Pi_{1x} d\xi, \quad i = 2, 3,$$

$$(\theta + \frac{|u|^{2}}{2})_{t} + (pu_{1})_{x} = \varepsilon (\frac{\lambda(\theta)}{v} \theta_{x})_{x} + \frac{4\varepsilon}{3} (\frac{\mu(\theta)}{v} u_{1} u_{1x})_{x}$$

$$+ \varepsilon \sum_{i=2}^{3} (\frac{\mu(\theta)}{v} u_{i} u_{ix})_{x} - \int \frac{1}{2} \xi_{1} |\xi|^{2} \Pi_{1x} d\xi.$$
(1.21)

With the above preparation, the main results in this paper for both the compressible Navier-Stokes equations and the Boltzmann equation will be given in the next section. And the proof of the zero dissipation limit for the compressible Navier-Stokes equations will be given in Section 3 while the proof of hydrodynamic limit for the Boltzmann equation will be given in the last section.

2. Main results.

2.1. Compressible Navier-Stokes equations. It is well known that for the Euler equations, there are three basic wave patterns, shock, rarefaction wave and contact discontinuity. And the Riemann solution to the Euler equations has a basic wave pattern consisting the superposition of these three waves with the contact discontinuity in the middle. For later use, let us firstly recall the wave curves for the two types of basic waves studied in this paper.

Given the right end state (v_+, u_+, θ_+) , the following wave curves in the phase space (v, u, θ) are defined with v > 0 and $\theta > 0$ for the Euler equations.

• Contact discontinuity wave curve:

$$CD(v_+, u_+, \theta_+) = \{(v, u, \theta) | u = u_+, p = p_+, v \not\equiv v_+\}.$$
(2.1)

• *i*-Rarefaction wave curve (i = 1, 3):

$$R_i(v_+, u_+, \theta_+) := \left\{ (v, u, \theta) \middle| v < v_+, \ u = u_+ - \int_{v_+}^v \lambda_i(\eta, s_+) \, d\eta, \ s(v, \theta) = s_+ \right\}$$
(2.2)

where $s_{+} = s(v_{+}, \theta_{+})$ and $\lambda_{i} = \lambda_{i}(v, s)$ is *i*-th characteristic speed of the Euler system (1.3) or (1.16).

Accordingly, when we study the Navier-Stokes equations, the corresponding wave profiles can be defined approximately as follows, cf. [16], [30].

2.1.1. Contact discontinuity. If $(v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+)$, i.e.,

$$u_- = u_+, \ p_- = p_+,$$

then the following Riemann problem of the Euler system $\left(1.3\right)$ with Riemann initial data

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_{-}, u_{-}, \theta_{-}), & x < 0, \\ (v_{+}, u_{+}, \theta_{+}), & x > 0 \end{cases}$$

admits a single contact discontinuity solution

$$(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \ t > 0, \\ (v_+, u_+, \theta_+), & x > 0, \ t > 0. \end{cases}$$
(2.3)

As in [14], the viscous version of the above contact discontinuity, called viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$, can be defined as follows. Since we expect that

 $P^{CD}\approx p_+=p_-, \quad {\rm and} \quad |\mathbf{U}^{\rm CD}|\ll 1,$

the leading order of the energy equation $(1.1)_3$ is

$$\frac{R}{\gamma - 1}\Theta_t + p_+ U_x = \kappa (\frac{\Theta_x}{V})_x.$$

Thus, we can get the following nonlinear diffusion equation

$$\Theta_t = a\varepsilon(\frac{\Theta_x}{\Theta})_x, \quad \Theta(t,\pm) = \theta_{\pm}, \quad a = \frac{\nu p_{\pm}(\gamma-1)}{R^2\gamma},$$

which has a unique self-similar solution $\hat{\Theta}(t, x) = \hat{\Theta}(\eta), \ \eta = \frac{x}{\sqrt{\varepsilon(1+t)}}.$

Now the viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ can be defined by

$$V^{CD}(t,x) = \frac{R\Theta(t,x)}{p_+},$$

$$U^{CD}(t,x) = u_+ + \frac{\kappa(\gamma-1)}{R\gamma} \frac{\hat{\Theta}_x(t,x)}{\hat{\Theta}(t,x)},$$

$$\Theta^{CD}(t,x) = \hat{\Theta}(t,x) + \frac{\varepsilon[R\gamma - \nu(\gamma-1)]}{\gamma p_+} \hat{\Theta}_t.$$
(2.4)

Here, it is straightforward to check that the viscous contact wave defined in (2.4) satisfies

$$|\Theta^{CD} - \theta_{\pm}| + [\varepsilon(1+t)]^{\frac{1}{2}} |\Theta_x^{CD}| + \varepsilon(1+t) |\Theta_{xx}^{CD}| = O(1)\delta^{CD} e^{-\frac{C_0 x^2}{\varepsilon(1+t)}}, \qquad (2.5)$$

as $|x| \to +\infty$, where $\delta^{CD} = |\theta_+ - \theta_-|$ represents the strength of the viscous contact wave and C_0 is a positive generic constant. Note that in the above definition, the higher order term $\frac{\varepsilon[R\gamma-\nu(\gamma-1)]}{\gamma p_+}\hat{\Theta}_t$ is used in $\Theta^{CD}(t,x)$ so that the viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t,x)$ satisfies the momentum equation exactly. Precisely, $(V^{CD}, U^{CD}, \Theta^{CD})(t,x)$ satisfies the system

$$\begin{cases} V_{t}^{CD} - U_{x}^{CD} = 0, \\ U_{t}^{CD} + P_{x}^{CD} = \varepsilon (\frac{U_{x}^{CD}}{V^{CD}})_{x}, \\ \frac{R}{\gamma - 1} \Theta_{t}^{CD} + P^{CD} U_{x}^{CD} = \kappa (\frac{\Theta_{x}^{CD}}{V^{CD}})_{x} + \varepsilon \frac{(U_{x}^{CD})^{2}}{V^{CD}} + Q^{CD}, \end{cases}$$
(2.6)

where $P^{CD} = \frac{R\Theta^{CD}}{V^{CD}}$ and the error term Q^{CD} has the property that

$$Q^{CD} = O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{C_0x^2}{\varepsilon(1+t)}}, \quad \text{as } |x| \to +\infty.$$
(2.7)

Remark 1. The viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ defined in (2.4) is different from the one used in [14] and [16]. Here, this ansatz is chosen such that the mass equation and the momentum equation are satisfied exactly while the error term occurs only in the energy equation. However, note that the approximate energy equation that the viscous contact wave satisfies is not in the conservative form. 2.1.2. Rarefaction waves. We now turn to the rarefaction waves. Since there is no exact rarefaction wave profile for either the Navier-Stokes equations or the Boltzmann equation, the following approximate rarefaction wave profile satisfying the Euler equations was motivated by [23] and [30]. For the completeness of the presentation, we include its definition and the properties in this subsection.

If $(v_-, u_-, \theta_-) \in R_i(v_+, u_+, \theta_+)$ (i = 1, 3), then there exists a *i*-rarefaction wave $(v^{r_i}, u^{r_i}, \theta^{r_i})(x/t)$ which is a global solution of the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x(v, \theta) = 0, \\ \frac{R}{\gamma - 1} \theta_t + p(v, \theta) u_x = 0, \\ (v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), x < 0, \\ (v_+, u_+, \theta_+), x > 0. \end{cases}$$
(2.8)

Consider the following inviscid Burgers equation with Riemann data

$$\begin{cases} w_t + ww_x = 0, \\ w(t = 0, x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases}$$
(2.9)

If $w_{-} < w_{+}$, then the above Riemann problem admits a rarefaction wave solution

$$w^{r}(t,x) = w^{r}(\frac{x}{t}) = \begin{cases} w_{-}, & \frac{x}{t} \le w_{-}, \\ \frac{x}{t}, & w_{-} \le \frac{x}{t} \le w_{+}, \\ w_{+}, & \frac{x}{t} \ge w_{+}. \end{cases}$$
(2.10)

Obviously, we have the following Lemma,

Lemma 2.1. For any shift $t_0 > 0$ in the time variable, we have

$$|w^{r}(t+t_{0},x) - w^{r}(t,x)| \le \frac{C}{t}t_{0},$$

where C is a positive constant depending only on w_{\pm} .

Remark that Lemma 2.1 plays an important role in the wave interaction estimates for the rarefaction waves.

As in [30], the approximate rarefaction wave $(V^R, U^R, \Theta^R)(t, x)$ to the problem (1.1) can be constructed by the solution of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_{\sigma}(x) = w(\frac{x}{\sigma}) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\sigma}, \end{cases}$$
(2.11)

where $\sigma > 0$ is a small parameter to be determined. Note that the solution $w_{\sigma}^{r}(t, x)$ of the problem (2.11) is given by

$$w_{\sigma}^r(t,x) = w_{\sigma}(x_0(t,x)), \qquad x = x_0(t,x) + w_{\sigma}(x_0(t,x))t.$$

And $w_{\sigma}^{r}(t, x)$ has the following properties:

Lemma 2.2. ([30]) Let $w_{-} < w_{+}$, (2.11) has a unique smooth solution $w_{\sigma}^{r}(t, x)$ satisfying

- (1) $w_{-} < w_{\sigma}^{r}(t, x) < w_{+}, \ (w_{\sigma}^{r})_{x}(t, x) \ge 0;$
- (2) For any p $(1 \le p \le +\infty)$, there exists a constant C such that $\|\frac{\partial}{\partial x}w_{\sigma}^{r}(t,\cdot)\|_{L^{p}(\mathbf{R})} \le C \min\left\{(w_{+}-w_{-})\sigma^{-1+1/p}, (w_{+}-w_{-})^{1/p}t^{-1+1/p}\right\},$ $\|\frac{\partial^{2}}{\partial x^{2}}w_{\sigma}^{r}(t,\cdot)\|_{L^{p}(\mathbf{R})} \le C \min\left\{(w_{+}-w_{-})\sigma^{-2+1/p}, \sigma^{-1+1/p}t^{-1}\right\};$

(3) If $x - w_{-}t < 0$ and $w_{-} > 0$, then

$$|w_{\sigma}^{r}(t,x) - w_{-}| \leq (w_{+} - w_{-})e^{-\frac{2|x-w_{-}t|}{\sigma}},$$
$$|\frac{\partial}{\partial x}w_{\sigma}^{r}(t,\cdot)| \leq \frac{2(w_{+}-w_{-})}{\sigma}e^{-\frac{2|x-w_{-}t|}{\sigma}};$$

If $x - w_{+}t > 0$ and $w_{+} < 0$, then

$$|w_{\sigma}^{r}(t,x) - w_{+}| \leq (w_{+} - w_{-})e^{-\frac{2|x-w_{+}t|}{\sigma}},$$
$$|\frac{\partial}{\partial x}w_{\sigma}^{r}(t,\cdot)| \leq \frac{2(w_{+}-w_{-})}{\sigma}e^{-\frac{2|x-w_{+}t|}{\sigma}};$$

(4) $\sup_{x \in \mathbf{R}} |w_{\sigma}^{r}(t,x) - w^{r}(\frac{x}{t})| \le \min\left\{(w_{+} - w_{-}), \frac{\sigma}{t}[\ln(1+t) + |\ln\sigma|]\right\}.$

Then the smooth approximate rarefaction wave profile denoted by $(V^{R_i}, U^{R_i}, \Theta^{R_i})(t, x)$ (i = 1, 3) can be defined by

$$\begin{cases}
S^{R_{i}}(t,x) = s(V^{R_{i}}(t,x), \Theta^{R_{i}}(t,x)) = s_{+}, \\
w_{\pm} = \lambda_{i\pm} := \lambda_{i}(v_{\pm}, \theta_{\pm}), \\
w_{\sigma}^{r}(t+t_{0},x) = \lambda_{i}(V^{R_{i}}(t,x), s_{+}), \\
U^{R_{i}}(t,x) = u_{+} - \int_{v_{+}}^{V^{R_{i}}(t,x)} \lambda_{i}(v, s_{+})dv,
\end{cases}$$
(2.12)

where t_0 is the shift used to control the interaction between waves in different families with the property that $t_0 \to 0$ as $\varepsilon \to 0$. In the following, we choose

$$t_0 = \varepsilon^{\frac{1}{5}}, \quad \text{and} \quad \sigma = \varepsilon^{\frac{2}{5}}.$$
 (2.13)

Note that $(V^{R_i}, U^{R_i}, \Theta^{R_i})(t, x)$ defined above satisfies

$$\begin{cases} V_t^{R_i} - U_x^{R_i} = 0, \\ U_t^{R_i} + P_x^{R_i} = 0, \\ \frac{R}{\gamma - 1} \Theta_t^{R_i} + P^{R_i} U_x^{R_i} = 0, \end{cases}$$
(2.14)

where $P^{R_i} = p(V^{R_i}, \Theta^{R_i})$.

By Lemmas 2.1 and 2.2, the properties on the rarefaction waves can be summarized as follows.

Lemma 2.3. The approximate rarefaction waves $(V^{R_i}, U^{R_i}, \Theta^{R_i})(t, x)$ (i = 1, 3) constructed in (2.12) have the following properties:

- (1) $U_x^{R_i}(t,x) > 0$ for $x \in \mathbf{R}, t > 0$;
- (2) For any $1 \le p \le +\infty$, the following estimates holds,

$$\begin{aligned} \| (V^{R_i}, U^{R_i}, \Theta^{R_i})_x \|_{L^p(dx)} &\leq C(t+t_0)^{-1+\frac{1}{p}}, \\ \| (V^{R_i}, U^{R_i}, \Theta^{R_i})_{xx} \|_{L^p(dx)} &\leq C\sigma^{-1+\frac{1}{p}}(t+t_0)^{-1}, \\ \| (V^{R_i}, U^{R_i}, \Theta^{R_i})_{xxx} \|_{L^p(dx)} &\leq C\sigma^{-2+\frac{1}{p}}(t+t_0)^{-1}, \end{aligned}$$

where the positive constant C only depends on p and the wave strength; (3) If $x \ge \lambda_{1+}(t+t_0)$, then

$$\begin{aligned} |(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x) - (v_-, u_-, \theta_-)| &\leq C e^{-\frac{2|x - \lambda_{1+}(t+t_0)|}{\sigma}}, \\ |(V^{R_1}, U^{R_1}, \Theta^{R_1})_x(t, x)| &\leq \frac{C}{\sigma} e^{-\frac{2|x - \lambda_{1+}(t+t_0)|}{\sigma}}; \end{aligned}$$

If
$$x \leq \lambda_{3-}(t+t_0)$$
, then
 $|(V^{R_3}, U^{R_3}, \Theta^{R_3})(t, x) - (v_+, u_+, \theta_+)| \leq Ce^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}},$
 $|(V^{R_3}, U^{R_3}, \Theta^{R_3})_x(t, x)| \leq \frac{C}{\sigma}e^{-\frac{2|x-\lambda_{3-}(t+t_0)|}{\sigma}};$

(4) There exist positive constants C and σ_0 such that for $\sigma \in (0, \sigma_0)$ and $t, t_0 > 0$,

$$\sup_{x \in \mathbf{R}} |(V^{R_i}, U^{R_i}, \Theta^{R_i})(t, x) - (v^{r_i}, u^{r_i}, \theta^{r_i})(\frac{x}{t})| \le \frac{C}{t} [\sigma \ln(1 + t + t_0) + \sigma |\ln \sigma| + t_0].$$

2.1.3. Superposition of rarefaction waves and contact discontinuity. In this subsection, we will define the solution profile that consists of the superposition of two rarefaction waves and a contact discontinuity. Let $(v_-, u_-, \theta_-) \in R_1 - CD - R_3(v_+, u_+, \theta_+)$. Then there exist uniquely two intermediate states (v_*, u_*, θ_*) and (v^*, u^*, θ^*) such that $(v_-, u_-, \theta_-) \in R_1(v_*, u_*, \theta_*), (v_*, u_*, \theta_*) \in CD(v^*, u^*, \theta^*)$ and $(v^*, u^*, \theta^*) \in R_3(v_+, u_+, \theta_+).$

So the wave pattern $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ consisting of 1-rarefaction wave, 2-contact discontinuity and 3-rarefaction wave that solves the corresponding Riemann problem of the Euler system (1.3) can be defined by

$$\begin{pmatrix} \bar{V} \\ \bar{U} \\ \bar{\Theta} \end{pmatrix} (t,x) = \begin{pmatrix} v^{r_1} + v^{cd} + v^{r_3} \\ u^{r_1} + u^{cd} + u^{r_3} \\ \theta^{r_1} + \theta^{cd} + \theta^{r_3} \end{pmatrix} (t,x) - \begin{pmatrix} v_* + v^* \\ u_* + u^* \\ \theta_* + \theta^* \end{pmatrix},$$
(2.15)

where $(v^{r_1}, u^{r_1}, \theta^{r_1})(t, x)$ is the 1-rarefaction wave defined in (2.8) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) , $(v^{cd}, u^{cd}, \theta^{cd})(t, x)$ is the contact discontinuity defined in (2.3) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v^*, u^*, θ^*) respectively, and $(v^{r_3}, u^{r_3}, \theta^{r_3})(t, x)$ is the 3-rarefaction wave defined in (2.8) with the left state (v_-, u_-, θ_-) replaced by (v^*, u^*, θ^*) .

Correspondingly, the approximate wave pattern $(V, U, \Theta)(t, x)$ of the compressible Navier-Stokes equations can be defined by

$$\begin{pmatrix} V\\ U\\ \Theta \end{pmatrix}(t,x) = \begin{pmatrix} V^{R_1} + V^{CD} + V^{R_3}\\ U^{R_1} + U^{CD} + U^{R_3}\\ \Theta^{R_1} + \Theta^{CD} + \Theta^{R_3} \end{pmatrix}(t,x) - \begin{pmatrix} v_* + v^*\\ u_* + u^*\\ \theta_* + \theta^* \end{pmatrix}, \quad (2.16)$$

where $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ is the approximate 1-rarefaction wave defined in (2.12) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) , $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ is the viscous contact wave defined in (2.4) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v^*, u^*, θ^*) respectively, and $(V^{R_3}, U^{R_3}, \Theta^{R_3})(t, x)$ is the approximate 3-rarefaction wave defined in (2.12) with the left state (v_-, u_-, θ_-) replaced by (v^*, u^*, θ^*) .

Thus, from the construction of the contact wave and Lemma 2.3, we have the following relation between the approximate wave pattern $(V, U, \Theta)(t, x)$ of the compressible Navier-Stokes equations and the exact inviscid wave pattern $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ to the Euler equations

$$|(V, U, \Theta)(t, x) - (V, \bar{U}, \Theta)(t, x)| \le \frac{C}{t} [\sigma \ln(1 + t + t_0) + \sigma |\ln \sigma| + t_0] + C\delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}},$$
(2.17)

with $t_0 = \varepsilon^{\frac{1}{5}}$ and $\sigma = \varepsilon^{\frac{2}{5}}$.

Moreover, $(V, U, \Theta)(t, x)$ satisfies the following system

$$\begin{cases} V_t - U_x = 0, \\ U_t + P_x = \varepsilon(\frac{U_x}{V})_x + Q_1, \\ \frac{R}{\gamma - 1}\Theta_t + PU_x = \kappa(\frac{\Theta_x}{V})_x + \varepsilon\frac{U_x^2}{V} + Q_2, \end{cases}$$
(2.18)

where $P = p(V, \Theta)$, and

$$\begin{aligned} Q_1 &= (P - P^{R_1} - P^{CD} - P^{R_3})_x - \varepsilon (\frac{U_x}{V} - \frac{U_x^{CD}}{V^{CD}})_x, \\ Q_2 &= (PU_x - P^{R_1}U_x^{R_1} - P^{CD}U_x^{CD} - P^{R_3}U_x^{R_3}) - \kappa (\frac{\Theta_x}{V} - \frac{\Theta_x^{CD}}{V^{CD}})_x \\ &- \varepsilon (\frac{U_x^2}{V} - \frac{(U_x^{CD})^2}{V^{CD}}) - Q^{CD}. \end{aligned}$$

Direct calculation shows that

$$Q_{1} = O(1) \left[|(V_{x}^{R_{1}}, \Theta_{x}^{R_{1}})||(V^{CD} - v_{*}, \Theta^{CD} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + |(V_{x}^{R_{3}}, \Theta_{x}^{R_{3}})||(V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{CD} - v^{*}, \Theta^{CD} - \theta^{*})| + |(V_{x}^{CD}, \Theta_{x}^{CD}, U_{xx}^{CD})||(V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + \varepsilon |(U_{x}^{CD}, V_{x}^{CD})||(U_{x}^{R_{1}}, V_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}})||(U_{x}^{R_{3}}, V_{x}^{R_{3}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}})||(U_{x}^{R_{3}}, V_{x}^{R_{3}})| \right] + O(1)\varepsilon \left[|U_{xx}^{R_{1}}| + |U_{xx}^{R_{3}}| + |U_{x}^{R_{1}}||V_{x}^{R_{1}}| + |U_{x}^{R_{3}}||V_{x}^{R_{3}}| \right] := Q_{11} + Q_{12}.$$

$$(2.19)$$

Similarly, we have

$$Q_{2} = O(1) \left[|U_{x}^{R_{1}}|| (V^{CD} - v_{*}, \Theta^{CD} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + |U_{x}^{R_{3}}|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{CD} - v^{*}, \Theta^{CD} - \theta^{*})| + |(U_{x}^{CD}, V_{x}^{CD}, \Theta_{x}^{CD})|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + \varepsilon |(U_{x}^{CD}, V_{x}^{CD}, \Theta_{x}^{CD})|| (U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{1}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}})|| (U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{3}})| \right] + O(1)\varepsilon \left[|\Theta_{xx}^{R_{1}}| + |\Theta_{xx}^{R_{3}}| + |(U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{3}})|^{2} \right] + |Q^{CD}| := Q_{21} + Q_{22} + |Q^{CD}|.$$

$$(2.20)$$

Here Q_{11} and Q_{21} represent the interactions coming from different wave patterns, Q_{12} and Q_{22} represent the error terms coming from the approximate rarefaction wave profiles, and Q^{CD} is the error term defined in (2.7) due to the viscous contact wave.

Firstly, we estimate the interaction terms Q_{11} and Q_{21} by dividing the whole domain $\Omega = \{(t, x) | (t, x) \in \mathbf{R}^+ \times \mathbf{R}\}$ into three regions:

$$\Omega_{-} = \{(t,x) | 2x \le \lambda_{1*}(t+t_0)\},
\Omega_{CD} = \{(t,x) | \lambda_{1*}(t+t_0) < 2x < \lambda_3^*(t+t_0)\},
\Omega_{+} = \{(t,x) | 2x \ge \lambda_3^*(t+t_0)\},$$

where $\lambda_{1*} = \lambda_1(v_*, \theta_*)$ and $\lambda_3^* = \lambda_3(v^*, \theta^*)$.

Now from Lemma 2.3, we have the following estimates in each section:

• In Ω_{-} ,

$$|V^{R_3} - v^*| = O(1)e^{-\frac{2|x| + 2\lambda_3^*(t+t_0)}{\sigma}}$$

= $O(1)e^{-\lambda_3^* \varepsilon^{-1/5}} e^{-\frac{2|x| + \lambda_3^*(t+t_0)}{\varepsilon^{2/5}}},$

$$\begin{aligned} |(V^{CD} - v_*, V^{CD} - v^*)| &= O(1)\delta^{CD}e^{-\frac{C[\lambda_{1*}(t+t_0)]^2}{4\varepsilon(1+t)}} \\ &= O(1)e^{-\frac{Ct_0(t+t_0)}{\varepsilon}} \\ &= O(1)e^{-\frac{Ct_0(x+t+t_0)}{\varepsilon}} \\ &= O(1)e^{-C\varepsilon^{-3/5}}e^{-\frac{C(|x|+t+t_0)}{\varepsilon^{4/5}}}; \end{aligned}$$

• In Ω_{CD} ,

$$\begin{aligned} |V^{R_1} - v_*| &= O(1)e^{-\frac{2|x|+2|\lambda_{1*}|(t+t_0)}{\sigma}} \\ &= O(1)e^{-|\lambda_{1*}|\varepsilon^{-1/5}}e^{-\frac{2|x|+|\lambda_{1*}|(t+t_0)}{\varepsilon^{2/5}}}, \\ |V^{R_3} - v^*| &= O(1)e^{-\frac{2|x|+2\lambda_3^*(t+t_0)}{\sigma}} \\ &= O(1)e^{-\lambda_3^*\varepsilon^{-1/5}}e^{-\frac{2|x|+\lambda_3^*(t+t_0)}{\varepsilon^{2/5}}}; \end{aligned}$$

• In
$$\Omega_+$$
,
 $|V^{R_1} - v_*| = O(1)e^{-\frac{2|x|+2|\lambda_{1*}|(t+t_0)}{\sigma}}$
 $= O(1)e^{-|\lambda_{1*}|\varepsilon^{-1/5}}e^{-\frac{2|x|+2|\lambda_{1*}|(t+t_0)}{\varepsilon^{2/5}}},$
 $|(V^{CD} - v_*, V^{CD} - v^*)| = O(1)\delta^{CD}e^{-\frac{C[\lambda_3^*(t+t_0)]^2}{4\varepsilon(1+t)}}$
 $= O(1)e^{-\frac{Ct_0(t+t_0)}{\varepsilon}}$
 $= O(1)e^{-\frac{Ct_0(1x|+t+t_0)}{\varepsilon}}$
 $= O(1)e^{-C\varepsilon^{-3/5}}e^{-\frac{C(|x|+t+t_0)}{\varepsilon^{4/5}}}$

Hence, in summary, we have

$$|(Q_{11}, Q_{21})| = O(1)e^{-C\varepsilon^{-1/5}}e^{-\frac{C(|x|+t+t_0)}{\varepsilon^{2/5}}},$$
(2.21)

for some positive constants C.

Now we consider the system (1.1) with the initial values

$$(v, u, \theta)(t = 0, x) = (V, U, \Theta)(t = 0, x).$$
(2.22)

Introduce the following scaled variables

$$y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}.$$
 (2.23)

In the following, we will use the notations $(v, u, \theta)(\tau, y)$ and $(V, U, \Theta)(\tau, y)$ for the unknown functions and the approximate wave profiles in the scaled variables. Set the perturbation around the composite wave pattern $(V, U, \Theta)(\tau, y)$ by

$$(\phi, \psi, \zeta)(\tau, y) = (v - V, u - U, \theta - \Theta)(\tau, y).$$

Then the perturbation $(\phi, \psi, \zeta)(\tau, y)$ satisfies the system

$$\begin{pmatrix}
\phi_{\tau} - \psi_{y} = 0, \\
\psi_{\tau} + (p - P)_{y} = \left(\frac{u_{y}}{v} - \frac{U_{y}}{V}\right)_{y} - \varepsilon Q_{1}, \\
\frac{R}{\gamma - 1}\zeta_{\tau} + \left(pu_{y} - PU_{y}\right) = \nu\left(\frac{\theta_{y}}{v} - \frac{\Theta_{y}}{V}\right)_{y} + \left(\frac{u_{y}^{2}}{v} - \frac{U_{y}^{2}}{V}\right) - \varepsilon Q_{2}, \\
(\phi, \psi, \zeta)(\tau = 0, y) = 0.
\end{cases}$$
(2.24)

And this system will be studied in Section 3.

2.1.4. *Main result to the compressible Navier-Stokes equations.* We are now ready to state the main result on the compressible Navier-Stokes equations as follows.

Theorem 2.4. Given a Riemann solution $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ defined in (2.15), which is a superposition of two rarefaction waves and a contact discontinuity for the Euler system (1.3), there exist small positive constants δ_0 and ε_0 such that if the contact wave strength $\delta^{CD} \leq \delta_0$ and the viscosity coefficient $\varepsilon \leq \varepsilon_0$, then the compressible Navier-Stokes equations (1.1) with (1.2) and (1.4) admits a unique global solution $(v^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon})(t, x)$ satisfying

$$\sup_{t,x)\in\Sigma_h} |(v^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon})(t, x) - (\bar{V}, \bar{U}, \bar{\Theta})(t, x)| \le C_h \ \varepsilon^{\frac{1}{5}}, \quad \forall h > 0,$$
(2.25)

where $\Sigma_h = \{(t, x) | t \ge h, \frac{x}{\sqrt{1+t}} \ge h\varepsilon^{\alpha}, 0 < \alpha < \frac{1}{2}\}$, and the positive constant C_h depends only on h but is independent of ε .

Remark 2. Theorem 2.4 shows that, away from the initial time t = 0 and the contact discontinuity located at x = 0 with the expansion rate $\frac{x^2}{\varepsilon(1+t)}$, for the viscosity coefficient $\varepsilon < \varepsilon_0$, there exists a unique global solution $(v^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon})(t, x)$ of the compressible Navier-Stokes equations (1.1) which tends to the Riemann solution $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ consisting of two rarefaction waves and a contact discontinuity when $\varepsilon \to 0$ and $\kappa = O(\varepsilon) \to 0$. Moreover, a uniform convergence rate $\varepsilon^{\frac{1}{5}}$ holds on the set Σ_h for any h > 0.

Remark 3. Theorem 2.4 holds uniformly when $(t, x) \in \Sigma_h$ for any fixed h > 0if the contact wave strength δ^{CD} and the viscosity coefficient ε are suitably small. However, if we restrict the problem to a set $\Sigma_h \cap \{t \leq T\}$ for any fixed T > 0, then we do not need to impose the smallness condition on the contact wave strength δ^{CD} because one can apply the Gronwall inequality to get an estimate depending on time T rather than the uniform estimate in time.

2.2. Boltzmann equation. We now turn to the Boltzmann equation. Similarly, we also define individual wave pattern, and then the superposition and finally state the main result in this subsection.

2.2.1. Contact discontinuity. We first recall the construction of the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ for the Boltzmann equation in [16]. Consider the Euler system (1.16) with a Riemann initial data

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_{-}, u_{-}, \theta_{-}), & x < 0, \\ (v_{+}, u_{+}, \theta_{+}), & x > 0, \end{cases}$$
(2.26)

where $u_{\pm} = (u_{1\pm}, 0, 0)$ and $v_{\pm} > 0, \theta_{\pm} > 0, u_{1\pm}$ are given constants. It is known (cf. [25]) that the Riemann problem (1.16), (2.26) admits a contact discontinuity solution

$$(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v_-, u_-, \theta_-), \ x < 0, \\ (v_+, u_+, \theta_+), \ x > 0, \end{cases}$$
(2.27)

provided that

$$u_{1+} = u_{1-}, \qquad p_- := \frac{2\theta_-}{3v_-} = p_+ := \frac{2\theta_+}{3v_+}.$$
 (2.28)

Motivated by (2.27) and (2.28), we expect that for the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$,

$$P^{CD} = \frac{2\Theta^{CD}}{3V^{CD}} \approx p_+, \ |U^{CD}|^2 \ll 1$$

Then the leading order of the energy equation $(1.21)_4$ is

$$\theta_t + p_+ u_{1x} = \varepsilon (\frac{\lambda(\theta)\theta_x}{v})_x. \tag{2.29}$$

By using the mass equation $(1.21)_1$ and $v \approx \frac{R\theta}{p_+}$, we obtain the following nonlinear diffusion equation

$$\theta_t = \varepsilon(a(\theta)\theta_x)_x, \ a(\theta) = \frac{9p_+\lambda(\theta)}{10\theta}.$$
(2.30)

From [1] and [6], we know that the nonlinear diffusion equation (2.30) admits a unique self-similar solution $\hat{\Theta}(\eta)$, $\eta = \frac{x}{\sqrt{\varepsilon(1+t)}}$ with the following boundary conditions

$$\hat{\Theta}(-\infty,t) = \theta_{-}, \ \hat{\Theta}(+\infty,t) = \theta_{+}$$

Let $\delta = |\theta_+ - \theta_-|$. $\hat{\Theta}(t, x)$ has the property

$$\hat{\Theta}_x(t,x) = \frac{O(1)\delta^{CD}}{\sqrt{\varepsilon(1+t)}} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \text{ as } x \to \pm \infty,$$
(2.31)

with some positive constant c depending only on θ_{\pm} . Now the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ can be defined by

$$\begin{split} V^{CD} &= \frac{2}{3p_{+}} \hat{\Theta}, \\ U_{1}^{CD} &= u_{1+} + \frac{2\varepsilon a(\hat{\Theta})}{3p_{+}} \hat{\Theta}_{x}, \ U_{i}^{CD} &= 0, (i = 2, 3), \\ \Theta^{CD} &= \hat{\Theta} + \frac{2\varepsilon}{3p_{+}} \hat{\Theta}_{t} [\frac{4}{3} \mu(\hat{\Theta}) - \frac{3}{5} \lambda(\hat{\Theta})]. \end{split}$$
(2.32)

Note that the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ satisfies the following system

$$\begin{cases} V_{t}^{CD} - U_{1x}^{CD} = 0, \\ U_{1t}^{CD} + P_{x}^{CD} = \frac{4\varepsilon}{3} (\frac{\mu(\Theta^{CD})}{V^{CD}} U_{1x}^{CD})_{x} + Q_{1}^{CD}, \\ U_{it}^{CD} = \varepsilon (\frac{\mu(\Theta^{CD})}{V^{CD}} U_{ix}^{CD})_{x}, i = 2, 3, \\ \Theta_{t}^{CD} + P^{CD} U_{1x}^{CD} = \varepsilon (\frac{\lambda(\Theta^{CD})}{V^{CD}} \Theta_{x}^{CD})_{x} + \frac{4\varepsilon}{3} \frac{\mu(\Theta^{CD})}{V^{CD}} (U_{1x}^{CD})^{2} \\ + \varepsilon \sum_{i=2}^{3} \frac{\mu(\Theta^{CD})}{V^{CD}} (U_{ix}^{CD})^{2} + Q_{2}^{CD}, \end{cases}$$
(2.33)

where

$$Q_1^{CD} = \frac{4\varepsilon}{3} \left(\frac{\mu(\Theta^{CD}) - \mu(\hat{\Theta})}{V^{CD}} U_{1x}^{CD} \right)_x = O(1)\delta^{CD}\varepsilon^{\frac{3}{2}}(1+t)^{-\frac{5}{2}}e^{-\frac{\varepsilon x^2}{\varepsilon(1+t)}}, \qquad (2.34)$$

$$Q_{2}^{CD} = \left[\frac{2\varepsilon}{3p_{+}}\hat{\Theta}_{t}\left(\frac{4}{3}\mu(\hat{\Theta}) - \frac{3}{5}\lambda(\hat{\Theta})\right)\right]_{t} + \frac{2\varepsilon}{3p_{+}V^{CD}}\hat{\Theta}_{t}\left[\frac{4}{3}\mu(\hat{\Theta}) - \frac{3}{5}\lambda(\hat{\Theta})\right]U_{1x}^{CD} + \frac{\varepsilon}{V^{CD}}(\lambda(\hat{\Theta})\hat{\Theta}_{x} - \lambda(\Theta^{CD})\Theta_{x}^{CD})_{x} - \frac{4\varepsilon\mu(\Theta^{CD})}{3V^{CD}}(U_{1x}^{CD})^{2} = O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{cx^{2}}{\varepsilon(1+t)}},$$

$$(2.35)$$

with some positive constant c > 0 depending only on θ_{\pm} .

Remark 4. The viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ for the Boltzmann equation (1.5) defined in (2.32) is different from the one used in [16]. Here, this ansatz is chosen such that the momentum equation is satisfied with a higher order error term. This is also different from the compressible Navier-Stokes equations where the ansatz satisfies the momentum equation exactly. But similar to the compressible Navier-Stokes cases, the approximate energy equation that the viscous contact wave satisfies is not in the conservative form.

From (2.31), we have

$$\begin{cases} |\hat{\Theta} - \theta_{-}| = O(1)\delta^{CD}e^{-\frac{cx^{2}}{2\varepsilon(1+t)}}, \text{ if } x < 0, \\ |\hat{\Theta} - \theta_{+}| = O(1)\delta^{CD}e^{-\frac{cx^{2}}{2\varepsilon(1+t)}}, \text{ if } x > 0. \end{cases}$$
(2.36)

Therefore,

$$|(V^{CD}, U^{CD}, \Theta^{CD})(t, x) - (v^{cd}, u^{cd}, \theta^{cd})(t, x)| = O(1)\delta^{CD}e^{-\frac{cx^2}{2\varepsilon(1+t)}}.$$
 (2.37)

2.2.2. Rarefaction waves. The construction of the *i*-rarefaction wave $(V^{R_i}, U^{R_i}, \Theta^{R_i})(t, x)$ (i = 1, 3) to the Boltzmann equation is almost same as the one defined in (2.14) for the compressible Navier-Stokes equations in the previous section. By setting $U_j^{R_i} = 0$ for i = 1, 3 and j = 2, 3, all the properties of the approximate rarefaction waves $(V^{R_i}, U_1^{R_i}, \Theta^{R_i})(t, x)$ (i = 1, 3) given in Lemma 2.3 will also be used later.

2.2.3. Superposition of rarefaction waves and contact discontinuity. We now consider the superposition of two rarefaction waves and a contact discontinuity. Set $(v_-, u_-, \theta_-) \in R_1$ -CD- $R_3(v_+, u_+, \theta_+)$. Then there exist uniquely two intermediate states (v_*, u_*, θ_*) and (v^*, u^*, θ^*) such that $(v_*, u_*, \theta_*) \in R_1(v_-, u_-, \theta_-)$, $(v_*, u_*, \theta_*) \in CD(v^*, u^*, \theta_-^*)$ and $(v^*, u^*, \theta^*) \in R_3(v_+, u_+, \theta_+)$.

So the wave pattern $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ consisting of 1-rarefaction wave, 2-contact discontinuity and 3-rarefaction wave as a Riemann solution to the Euler system (1.16) can be defined by

$$\begin{pmatrix} \bar{V} \\ \bar{U}_1 \\ \bar{\Theta} \end{pmatrix} (t,x) = \begin{pmatrix} v^{r_1} + v^{cd} + v^{r_3} \\ u_1^{r_1} + u_1^{cd} + u_1^{r_3} \\ \theta^{r_1} + \theta^{cd} + \theta^{r_3} \end{pmatrix} (t,x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_1^* \\ \theta_* + \theta^* \end{pmatrix},$$

$$\bar{U}_i = 0, (i = 2, 3).$$
(2.38)

where $(v^{r_1}, u_1^{r_1}, \theta^{r_1})(t, x)$ is the approximate 1-rarefaction wave defined in (2.8) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_{1*}, θ_*) , $(v^{cd}, u_1^{cd}, \theta^{cd})(t, x)$ is the contact discontinuity defined in (2.27) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v^*, u^*, θ^*) respectively, and $(v^{r_3}, u_1^{r_3}, \theta^{r_3})(t, x)$ is the 3-rarefaction wave defined in (2.8) with the left state (v_-, u_-, θ_-) replaced by (v^*, u_1^*, θ^*) .

Correspondingly, the approximate superposition wave $(V, U, \Theta)(t, x)$ can be defined by

$$\begin{pmatrix} V\\ U_1\\ \Theta \end{pmatrix}(t,x) = \begin{pmatrix} V^{R_1} + V^{CD} + V^{R_3}\\ U_1^{R_1} + U_1^{CD} + U_1^{R_3}\\ \Theta^{R_1} + \Theta^{CD} + \Theta^{R_3} \end{pmatrix}(t,x) - \begin{pmatrix} v_* + v^*\\ u_{1*} + u_1^*\\ \theta_* + \theta^* \end{pmatrix}, \quad (2.39)$$
$$U_i = 0, (i = 2, 3).$$

where $(V^{R_1}, U_1^{R_1}, \Theta^{R_1})(t, x)$ is the 1-rarefaction wave defined in (2.12) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_{1*}, θ_*) , $(V^{CD}, U_1^{CD}, \Theta^{CD})(t, x)$ is the viscous contact wave defined in (2.32) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v^*, u^*, θ^*) respectively, and $(V^{R_3}, U_1^{R_3}, \Theta^{R_3})(t, x)$ is the approximate 3-rarefaction wave defined in (2.12) with the left state (v_-, u_-, θ_-) replaced by (v^*, u_1^*, θ^*) .

Thus, from the construction of the contact wave and Lemma 2.3, we have the following relation between the approximate wave pattern $(V, U, \Theta)(t, x)$ of the Boltzmann equation and the exact inviscid wave pattern $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ to the Euler equations

$$\begin{aligned} |(V, U, \Theta)(t, x) - (\bar{V}, \bar{U}, \bar{\Theta})(t, x)| \\ &\leq \frac{C}{t} [\sigma \ln(1 + t + t_0) + \sigma |\ln \sigma| + t_0] + C \delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \end{aligned}$$
(2.40)

with $t_0 = \varepsilon^{\frac{1}{5}}$ and $\sigma = \varepsilon^{\frac{2}{5}}$.

Then we have

$$\begin{cases} V_t - U_{1x} = 0, \\ U_{1t} + P_x = \varepsilon \left(\frac{\mu(\Theta)U_{1x}}{V}\right)_x + Q_1, \\ U_{it} = \varepsilon \left(\frac{\mu(\Theta)U_{ix}}{V}\right)_x, \ i = 2, 3, \\ \Theta_t + PU_{1x} = \varepsilon \left(\frac{\lambda(\Theta)\Theta_x}{V}\right)_x + \varepsilon \frac{\mu(\Theta)U_{1x}^2}{V} + Q_2, \end{cases}$$
(2.41)

where $P = p(V, \Theta)$ and

$$\begin{array}{rcl} Q_{1} & = (P - P^{R_{1}} - P^{CD} - P^{R_{3}})_{x} - \varepsilon (\frac{\mu(\Theta)U_{1x}}{V} - \frac{\mu(\Theta^{CD})U_{1x}^{CD}}{V^{CD}})_{x} - Q_{1}^{CD}, \\ Q_{2} & = (PU_{1x} - P^{R_{1}}U_{1x}^{R_{1}} - P^{CD}U_{1x}^{CD} - P^{R_{3}}U_{1x}^{R_{3}}) - \varepsilon (\frac{\lambda(\Theta)\Theta_{x}}{V} - \frac{\lambda(\Theta^{CD})\Theta_{x}^{CD}}{V^{CD}})_{x} \\ & -\varepsilon (\frac{\mu(\Theta)U_{1x}^{2}}{V} - \frac{\mu(\Theta^{CD})(U_{1x}^{CD})^{2}}{V^{CD}}) - Q_{2}^{CD}. \end{array}$$

Direct computation yields

$$\begin{aligned} Q_{1} &= O(1) \Big[|(V_{x}^{R_{1}}, \Theta_{x}^{R_{1}})|| (V^{CD} - v_{*}, \Theta^{CD} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| \\ &+ |(V_{x}^{R_{3}}, \Theta_{x}^{R_{3}})|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{CD} - v^{*}, \Theta^{CD} - \theta^{*})| \\ &+ |(V_{x}^{CD}, \Theta_{x}^{CD}, U_{xx}^{CD})|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| \\ &+ \varepsilon |(U_{x}^{CD}, V_{x}^{CD})|| (U_{x}^{R_{1}}, V_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}})|| (U_{x}^{R_{3}}, V_{x}^{R_{3}})| \\ &+ O(1)\varepsilon \Big[|U_{xx}^{R_{1}}| + |U_{xx}^{R_{3}}| + |U_{x}^{R_{1}}||V_{x}^{R_{1}}| + |U_{x}^{R_{3}}||V_{x}^{R_{3}}| \Big] + |Q_{1}^{CD}| \\ &:= Q_{11} + Q_{12} + |Q_{1}^{CD}|, \end{aligned}$$

$$(2.42)$$

and

$$Q_{2} = O(1) \left| |U_{x}^{R_{1}}|| (V^{CD} - v_{*}, \Theta^{CD} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + |U_{x}^{R_{3}}|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{CD} - v^{*}, \Theta^{CD} - \theta^{*})| + |(U_{x}^{CD}, V_{x}^{CD}, \Theta_{x}^{CD})|| (V^{R_{1}} - v_{*}, \Theta^{R_{1}} - \theta_{*}, V^{R_{3}} - v^{*}, \Theta^{R_{3}} - \theta^{*})| + \varepsilon |(U_{x}^{CD}, V_{x}^{CD}, \Theta_{x}^{CD})|| (U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{1}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{1}})| + \varepsilon |(U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{3}}, W_{x}^{R_{3}}, \Theta_{x}^{R_{3}})| \right] + O(1)\varepsilon \left[|\Theta_{xx}^{R_{1}}| + |\Theta_{xx}^{R_{3}}| + |(U_{x}^{R_{1}}, V_{x}^{R_{1}}, \Theta_{x}^{R_{1}}, U_{x}^{R_{3}}, V_{x}^{R_{3}}, \Theta_{x}^{R_{3}})|^{2} \right] + |Q_{2}^{CD}| \\ := Q_{21} + Q_{22} + |Q_{2}^{CD}|.$$

$$(2.43)$$

Here, Q_{11} and Q_{21} represent the interaction of waves in different families, Q_{12} and Q_{22} represent the error terms coming from the approximate rarefaction wave profiles, and $Q_i^{CD}(i=1,2)$ are the error terms defined in (2.34) and (2.35) due to the viscous contact wave.

Similar to the compressible Navier-Stokes equations case, for the interaction terms, we have

$$|(Q_{11}, Q_{21})| = O(1)e^{-C\varepsilon^{-1/5}}e^{-\frac{C(|x|+t+t_0)}{\varepsilon^{2/5}}},$$
(2.44)

for some positive constants C.

We now reformulate the system by introducing a scaling for the independent variables. Set

$$y = \frac{x}{\varepsilon}, \ \tau = \frac{t}{\varepsilon}$$

as in the previous section for the compressible Navier-Stokes equations. We also use the notations $(v, u, \theta)(\tau, y)$, $\mathbf{G}(\tau, y, \xi)$, $\Pi_1(\tau, y, \xi)$ and $(V, U, \Theta)(\tau, y)$ in the scaled independent variables. Set the perturbation around the composite wave $(V, U, \Theta)(\tau, y)$ by

$$(\phi, \psi, \zeta)(\tau, y) = (v - V, u - U, \theta - \Theta)(\tau, y)$$

Under this scaling, the hydrodynamic limit problem is reduced to a time asymptotic stability problem of the composite wave to the Boltzmann equation. Notice that the hydrodynamic limit proved here is global in time compared to the case on shock profile studied in [32] which is locally in time.

From (1.21) and (2.42), we have the following system for the perturbation (ϕ, ψ, ζ)

$$\begin{cases} \phi_{\tau} - \psi_{1y} = 0, \\ \psi_{1\tau} + (p - P)_y = \frac{4}{3} (\frac{\mu(\theta)u_{1y}}{v} - \frac{\mu(\Theta)U_{1y}}{V})_y - \int \xi_1^2 \Pi_{1y} d\xi - \varepsilon Q_1, \\ \psi_{i\tau} = (\frac{\mu(\theta)u_{i1y}}{v} - \frac{\mu(\Theta)U_{iy}}{V})_y - \int \xi_1 \xi_i \Pi_{1y} d\xi, \ i = 2, 3, \\ \zeta_{\tau} + (pu_{1y} - PU_{1y}) = (\frac{\lambda(\theta)\theta_y}{v} - \frac{\lambda(\Theta)\Theta_y}{V})_y + \frac{4}{3} (\frac{\mu(\theta)u_{1y}^2}{v} - \frac{\mu(\Theta)U_{1y}^2}{V}) \\ + \sum_{i=2}^3 \frac{\mu(\theta)u_{iy}^2}{v} + \sum_{i=1}^3 u_i \int \xi_1 \xi_i \Pi_{1y} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1y} d\xi - \varepsilon U_1 Q_1 - \varepsilon Q_2, \end{cases}$$

$$(2.45)$$

where the error terms Q_i (i = 1, 2) are given in (2.42) and (2.43) respectively.

We now derive the equation for the non-fluid component $\mathbf{G}(\tau, y, \xi)$ in the scaled independent variables. From (1.18), we have

$$\mathbf{G}_{\tau} - \frac{u_1}{v}\mathbf{G}_y + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) + \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_y) = \mathbf{L}_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G}).$$
(2.46)

Thus, we obtain

$$\mathbf{G} = \frac{1}{v} \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_y)] + \Pi_1, \qquad (2.47)$$

and

$$\Pi_{1}(\tau, y, \xi) = \mathbf{L}_{\mathbf{M}}^{-1}[\mathbf{G}_{\tau} - \frac{u_{1}}{v}\mathbf{G}_{y} + \frac{1}{v}\mathbf{P}_{1}(\xi_{1}\mathbf{G}_{y}) - Q(\mathbf{G}, \mathbf{G})].$$
(2.48)

Let

$$\mathbf{G}_{0}(\tau, y, \xi) = \frac{3}{2v\theta} \mathbf{L}_{\mathbf{M}}^{-1} \{ \mathbf{P}_{1}[\xi_{1}(\frac{|\xi - u|^{2}}{2\theta}\Theta_{y} + \xi \cdot U_{y})\mathbf{M}] \},$$
(2.49)

and

$$\mathbf{G}_1(\tau, y, \xi) = \mathbf{G}(\tau, y, \xi) - \mathbf{G}_0(\tau, y, \xi).$$
(2.50)

Then $\mathbf{G}_1(\tau, y, \xi)$ satisfies

$$\mathbf{G}_{1\tau} - \mathbf{L}_{\mathbf{M}} \mathbf{G}_{1} = -\frac{3}{2v\theta} \mathbf{P}_{1} [\xi_{1} (\frac{|\xi - u|^{2}}{2\theta} \zeta_{y} + \xi \cdot \psi_{y}) \mathbf{M}] + \frac{u_{1}}{v} \mathbf{G}_{y} - \frac{1}{v} \mathbf{P}_{1} (\xi_{1} \mathbf{G}_{y}) + Q(\mathbf{G}, \mathbf{G}) - \mathbf{G}_{0\tau}.$$

$$(2.51)$$

Notice that in (2.50) and (2.51), \mathbf{G}_0 is subtracted from \mathbf{G} because $\|(\Theta_y, U_y)\|^2 \sim (1 + \varepsilon^{\frac{1}{2}} \tau)^{-1/2}$ is not integrable globally in τ .

Finally, from (1.15) and the scaling transformation (2.23), we have

$$f_{\tau} - \frac{u_1}{v} f_y + \frac{\xi_1}{v} f_y = Q(f, f).$$
(2.52)

The estimation on the fluid and non-fluid components governed by the above systems will be given in the last section.

2.2.4. *Main result to Boltzmann equation*. With the above preparation, we are now ready to state the main result on the Boltzmann equation as follows.

Theorem 2.5. Given a Riemann solution $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ defined in (2.38), which is a superposition of two rarefaction waves and a contact discontinuity to the Euler system (1.16), there exist small positive constants δ_0 , ε_0 and a global Maxwellian $\mathbf{M}_{\star} = \mathbf{M}_{[v_{\star}, u_{\star}, \theta_{\star}]}$, such that if the contact wave strength $\delta^{CD} \leq \delta_0$, and the Knudsen number $\varepsilon \leq \varepsilon_0$, then the Boltzmann equation (1.5) admits a unique global solution $f^{\varepsilon}(t, x, \xi)$ satisfying

$$\sup_{(t,x)\in\Sigma_h} \|f^{\varepsilon}(t,x,\xi) - \mathbf{M}_{[\bar{V},\bar{U},\bar{\Theta}]}(t,x,\xi)\|_{L^2_{\xi}(\frac{1}{\sqrt{\mathbf{M}_{\star}}})} \le C_h \varepsilon^{\frac{1}{5}}, \qquad \forall h > 0, \qquad (2.53)$$

where $\Sigma_h = \{(t,x)|t \ge h, \frac{x}{\sqrt{1+t}} \ge h\varepsilon^{\alpha}, 0 < \alpha < \frac{1}{2}\}$, the norm $\|\cdot\|_{L^2_{\xi}(\frac{1}{\sqrt{M_{\star}}})}$ is $\|\frac{\cdot}{\sqrt{M_{\star}}}\|_{L^2_{\xi}(\mathbf{R}^3)}$ and the positive constant C_h depends only on h but is independent of ε .

Remark 5. Theorem 2.5 shows that, away from the initial time t = 0 and the contact discontinuity located at x = 0 with the expansion rate $\frac{x^2}{\varepsilon(1+t)}$, for Knudsen number $\varepsilon < \varepsilon_0$, there exists a unique global solution $f^{\varepsilon}(t, x, \xi)$ of the Boltzmann equation (1.5) which tends to the Maxwellian $\mathbf{M}_{[\bar{V},\bar{U},\bar{\Theta}]}(t, x, \xi)$ with $(\bar{V}, \bar{U}, \bar{\Theta})(t, x)$ being the Riemann solution to the Euler equation with the combination of two rarefaction waves and a contact discontinuity when $\varepsilon \to 0$. Moreover, a uniform convergence rate $\varepsilon^{\frac{1}{5}}$ in the norm $L^2_{\xi}(\frac{1}{\sqrt{M_*}})$ holds on the set Σ_h for any fixed h > 0.

Remark 6. Theorem 2.5 holds uniformly on the $(t, x) \in \Sigma_h$ for any h > 0 if the contact wave strength δ^{CD} and Knudsen number ε are suitably small. But if we restrict the problem to the set $\Sigma_h \cap \{t \leq T\}$ for any fixed T > 0, then we don't need the smallness condition on the contact wave strength δ^{CD} by using Gronwall inequality to get a time dependent estimate rather than the uniform estimation in time.

Notations: Throughout this paper, the positive generic constants which are independent of T, ε are denoted by c, C or C_0 . For function spaces, $H^l(\mathbf{R})$ denotes the l-th order Sobolev space with its norm

$$||f||_l = (\sum_{j=0}^l ||\partial_y^j f||^2)^{\frac{1}{2}}, \text{ and } ||\cdot|| := ||\cdot||_{L^2(dy)}$$

where $L^2(dz)$ means the L^2 integral over **R** with respect to the Lebesgue measure dz, and z = x or y.

3. Proof of Theorem 2.4: Zero dissipation limit of Navier-Stokes equations. We will prove Theorem 2.4 about the fluid dynamic limit for the compressible Navier-Stokes equations to the Riemann solution of the Euler equations in this section. The proof is based on the energy estimates on the perturbation in the scaled independent variables. In fact, to prove Theorem 2.4, it is sufficient to prove the following theorem.

Theorem 3.1. There exist small positive constants δ_1 and ε_1 such that if the initial values and the contact wave strength δ^{CD} satisfy

$$\mathcal{N}(\tau)|_{\tau=0} + \delta^{CD} \le \delta_1, \tag{3.1}$$

and the Knudsen number ε satisfies $\varepsilon \leq \varepsilon_1$, then the problem (2.24) admits a unique global solution $(v^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon})(\tau, y)$ satisfying

$$\sup_{\tau,y} |(v^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon})(\tau, y) - (V, U, \Theta)(\tau, y)| \le C\varepsilon^{\frac{1}{5}}.$$
(3.2)

Here $\mathcal{N}(\tau)$ is defined by (3.3) below.

We will focus on the reformulated system (2.24). Since the local existence of the solution to (2.24) is standard, to prove the global existence, we only need to close the following a priori estimate by the continuity argument

$$\mathcal{N}(\tau) = \sup_{0 \le \tau' \le \tau} \|(\phi, \psi, \zeta)(\tau', \cdot)\|_1^2 \le \chi^2, \tag{3.3}$$

where χ is a small positive constant depending only on the initial values and the strength of the contact wave. And the proof of the above a priori estimate is given by the following energy estimations.

Firstly, multiplying $(2.24)_2$ by ψ yields

$$\left(\frac{1}{2}\psi^{2}\right)_{\tau} - (p-P)\psi_{y} + \left(\frac{u_{y}}{v} - \frac{U_{y}}{V}\right)\psi_{y} = -\varepsilon Q_{1}\psi + \left[\left(\frac{u_{y}}{v} - \frac{U_{y}}{V}\right)\psi - (p-P)\psi\right]_{y}.$$
(3.4)

Since $p - P = R\Theta(\frac{1}{v} - \frac{1}{V}) + \frac{R\zeta}{v}$ and $\phi_{\tau} = \psi_y$, we get

$$\frac{(\frac{1}{2}\psi^2)_{\tau} - R\Theta(\frac{1}{v} - \frac{1}{V})\phi_{\tau} - \frac{R}{v}\zeta\psi_y + \frac{\psi_y^2}{v} \\ = -(\frac{1}{v} - \frac{1}{V})U_y\psi_y - \varepsilon Q_1\psi + \left[(\frac{u_y}{v} - \frac{U_y}{V})\psi - (p - P)\psi\right]_y.$$
(3.5)

 Set

$$\Phi(z) = z - 1 - \ln z.$$
(3.6)

It is easy to check that $\Phi(1) = \Phi'(1) = 0$ and $\Phi(z)$ is strictly convex around z = 1. Moreover,

$$[R\Theta\Phi(\frac{v}{V})]_{\tau} = R\Theta_{\tau}\Phi(\frac{v}{V}) - R\Theta(\frac{1}{v} - \frac{1}{V})\phi_{\tau} - \frac{PV_{\tau}}{vV}\phi^{2}.$$
(3.7)

On the other hand, note that

$$\left[\frac{R}{\gamma-1}\Theta\Phi(\frac{\theta}{\Theta})\right]_{\tau} = \frac{R}{\gamma-1}\left(1-\frac{\Theta}{\theta}\right)\zeta_{\tau} + \frac{R}{\gamma-1}\Phi(\frac{\theta}{\Theta})\Theta_{\tau} - \frac{R}{\gamma-1}\frac{\Theta_{\tau}\zeta^{2}}{\theta\Theta}, \quad (3.8)$$

$$\begin{aligned} \frac{R}{\gamma-1}(1-\frac{\Theta}{\theta})\zeta_{\tau} \\ &= (1-\frac{\Theta}{\theta})[-(pu_y-PU_y)+\nu(\frac{\theta_y}{v}-\frac{\Theta_y}{V})_y+(\frac{u_y^2}{v}-\frac{U_y^2}{V})-\varepsilon Q_2] \\ &= -\frac{R}{v}\zeta\psi_y-\frac{\zeta}{\theta}(p-P)U_y-\nu(\frac{\zeta}{\theta})_y(\frac{\theta_y}{v}-\frac{\Theta_y}{V})+\frac{\zeta}{\theta}(\frac{u_y^2}{v}-\frac{U_y^2}{V}) \\ &-\varepsilon\frac{\zeta}{\theta}Q_2+\left[\nu\frac{\zeta}{\theta}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})\right]_y \\ &= -\frac{R}{v}\zeta\psi_y-\frac{\zeta}{\theta}(p-P)U_y-\frac{\nu\zeta_y^2}{v\theta}-\nu\frac{\zeta_y}{\theta}(\frac{1}{v}-\frac{1}{V})\Theta_y \\ &+\frac{\nu\zeta\theta_y}{\theta^2}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})+\frac{\zeta}{\theta}(\frac{u_y^2}{v}-\frac{U_y^2}{V})-\varepsilon Q_2\frac{\zeta}{\theta}+\left[\frac{\nu\zeta}{\theta}(\frac{\theta_y}{v}-\frac{\Theta_y}{V})\right]_y. \end{aligned}$$
(3.9)

Substituting (3.7)-(3.9) into (3.5) gives

$$\begin{aligned} & [\frac{1}{2}\psi^2 + R\Theta\Phi(\frac{v}{V}) + \frac{R}{\gamma - 1}\Theta\Phi(\frac{\theta}{\Theta})]_{\tau} + \frac{\psi_y^2}{v} + \frac{\nu\zeta_y^2}{v\theta} + J_1 \\ &= -U_y(\frac{1}{v} - \frac{1}{V})\psi_y - \nu\frac{\zeta_y}{\theta}(\frac{1}{v} - \frac{1}{V})\Theta_y + \frac{\nu\zeta\theta_y}{\theta^2}(\frac{\theta_y}{v} - \frac{\Theta_y}{V}) \\ &+ \frac{\zeta}{\theta}(\frac{u_y^2}{v} - \frac{U_y^2}{V}) - \varepsilon Q_1\psi - \varepsilon Q_2\frac{\zeta}{\theta} + (\cdots)_y, \end{aligned}$$
(3.10)

where

$$J_1 = \frac{\zeta}{\theta} (p - P) U_y - R\Theta_\tau \Phi(\frac{v}{V}) - \frac{R}{\gamma - 1} \Theta_\tau \Phi(\frac{\theta}{\Theta}) + \frac{PV_\tau}{vV} \phi^2 + \frac{R}{\gamma - 1} \frac{\Theta_\tau \zeta^2}{\theta \Theta}.$$
 (3.11)

Direct calculation shows that

$$J_{1} = PU_{y}\left[\Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V})\right] - \left[\frac{U_{y}^{2}}{V} + \nu(\frac{\Theta_{y}}{V})_{y} + \varepsilon Q_{2}\right]\left[(\gamma - 1)\Phi(\frac{v}{V}) - \Phi(\frac{\Theta}{\theta})\right]$$
$$= PU_{y}\left[\Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V})\right] - \left[\frac{U_{y}^{2}}{V} + \nu(\frac{\Theta_{y}}{V})_{y} + \varepsilon Q_{2}\right]\left[(\gamma - 1)\Phi(\frac{v}{V}) - \Phi(\frac{\Theta}{\theta})\right].$$
(3.12)

Thus, substituting (3.12) into (3.10) gives

$$[\frac{1}{2}\psi^{2} + R\Theta\Phi(\frac{v}{V}) + \frac{R}{\gamma - 1}\Theta\Phi(\frac{\theta}{\Theta})]_{\tau} + \frac{\psi_{y}^{2}}{v} + \frac{\nu\zeta_{y}^{2}}{v\theta} + P(U_{y}^{R_{1}} + U_{y}^{R_{3}})[\Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V})] = J_{2} - \varepsilon Q_{1}\psi - \varepsilon Q_{2}\frac{\zeta}{\theta} + (\cdots)_{y},$$
(3.13)

where

$$J_{2} = -PU_{y}^{CD}\left[\Phi\left(\frac{\theta V}{v\Theta}\right) + \gamma\Phi\left(\frac{v}{V}\right)\right] + \left[\frac{U_{y}^{2}}{V} + \nu\left(\frac{\Theta_{y}}{V}\right)_{y} + \varepsilon Q_{2}\right]\left[(\gamma - 1)\Phi\left(\frac{v}{V}\right) - \Phi\left(\frac{\Theta}{\theta}\right)\right] \\ -U_{y}\left(\frac{1}{v} - \frac{1}{V}\right)\psi_{y} - \nu\frac{\zeta_{y}}{\theta}\left(\frac{1}{v} - \frac{1}{V}\right)\Theta_{y} + \frac{\nu\zeta\theta_{y}}{\theta^{2}}\left(\frac{\theta_{y}}{v} - \frac{\Theta_{y}}{V}\right) + \frac{\zeta}{\theta}\left(\frac{u_{y}^{2}}{v} - \frac{U_{y}^{2}}{V}\right).$$

$$(3.14)$$

Here, $(\cdots)_y$ represents the conservative terms which vanishes after integrating in y over **R**.

By the strict convexity of $\Phi(z)$ around z = 1, under the a priori assumption (3.3) with sufficiently small $\chi > 0$, there exist positive constants c_1 and c_2 such that,

$$c_1\phi^2 \le \Phi(\frac{v}{V}) \le c_2\phi^2, \quad c_1\zeta^2 \le \Phi(\frac{\Theta}{\theta}), \Phi(\frac{\theta}{\Theta}) \le c_2\zeta^2, c_1(\phi^2 + \zeta^2) \le \Phi(\frac{\theta V}{v\Theta}) \le c_2(\phi^2 + \zeta^2).$$
(3.15)

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and

Thus, we have

$$\int_{\mathbf{R}} |J_2| dy \leq \int_{\mathbf{R}} (\frac{\psi_y^2}{4v} + \frac{\nu \zeta_y^2}{4v\theta}) dy + C(\tau + \tau_0)^{-2} ||(\phi, \zeta)||^2
+ C \int_{\mathbf{R}} \delta^{CD} \varepsilon (1 + \varepsilon \tau)^{-1} e^{-\frac{\varepsilon \varepsilon y^2}{1 + \varepsilon \tau}} |(\phi, \zeta)|^2 dy + \int_{\mathbf{R}} \varepsilon |Q_2| |(\phi, \zeta)|^2 dy.$$
(3.16)

Notice that the last term $\varepsilon |Q_2| |(\phi, \zeta)|^2$ on the right hand side of (3.16) can be estimated similarly as for the terms $\varepsilon Q_1 \psi$ and $\varepsilon Q_2 \frac{\zeta}{\theta}$ under the a priori assumption (3.3). Now we estimate the terms $\varepsilon Q_1 \psi$ and $\varepsilon Q_2 \frac{\zeta}{\theta}$ on the right hand side of (3.13). First,

$$\int_{\mathbf{R}} \varepsilon |Q_1| |\psi| dy = \int_{\mathbf{R}} \varepsilon (|Q_{11}| + |Q_{12}|) |\psi| dy.$$

From the estimation on the interaction given in (2.21), we get

$$\int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon |Q_{11}| |\psi| d\tau dy
\leq \int_{0}^{\tau} ||\psi||_{L_{y}^{\infty}} \int_{\mathbf{R}} |Q_{11}| dx d\tau
\leq C \int_{0}^{\tau} e^{-C\varepsilon^{-1/5}} e^{-\frac{C(t+t_{0})}{\varepsilon^{2/5}}} ||\psi||^{\frac{1}{2}} ||\psi_{y}||^{\frac{1}{2}} d\tau
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + C_{\beta} \int_{0}^{\tau} e^{-C\varepsilon^{-1/5}} e^{-C\varepsilon^{3/5}(\tau+\tau_{0})} ||\psi||^{\frac{2}{3}} d\tau
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + C_{\beta} e^{-C\varepsilon^{-1/5}} \sup_{[0,\tau]} ||\psi(\tau)||^{\frac{2}{3}}
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + \beta \sup_{[0,\tau]} ||\psi(\tau)||^{2} + C_{\beta} e^{-C\varepsilon^{-1/5}},$$
(3.17)

and

$$\int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon |Q_{12}| |\psi| d\tau dy
\leq \varepsilon^{2} \int_{0}^{\tau} \int_{\mathbf{R}} (|(w_{\delta}^{r})_{xx}|, |(w_{\delta}^{r})_{x}|^{2}) |\psi| d\tau dy
\leq \varepsilon \int_{0}^{\tau} (||(w_{\delta}^{r})_{xx}||_{L^{1}(dx)}, ||(w_{\delta}^{r})_{x}||_{L^{2}(dx)}^{2}) ||\psi||_{L_{y}^{\infty}} d\tau
\leq \int_{0}^{\tau} (\tau + \tau_{0})^{-1} ||\psi||^{\frac{1}{2}} ||\psi_{y}||^{\frac{1}{2}} d\tau
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + C_{\beta} \int_{0}^{\tau} (\tau + \tau_{0})^{-\frac{4}{3}} ||\psi||^{\frac{2}{3}} d\tau
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + 3C_{\beta} \tau_{0}^{-\frac{1}{3}} \sup_{[0,\tau]} ||\psi(\tau)||^{\frac{2}{3}}
\leq \beta \int_{0}^{\tau} ||\psi_{y}||^{2} d\tau + \beta \sup_{[0,\tau]} ||\psi(\tau)||^{2} + C_{\beta} \varepsilon^{\frac{2}{5}},$$
(3.18)

where $\tau_0 = \frac{t_0}{\varepsilon} = \varepsilon^{-\frac{4}{5}}$, and $\beta > 0$ is a small constant to be determined later and C_{β} is a positive constant depending on β .

The term $\varepsilon Q_2 \frac{\zeta}{\theta}$ can be estimated similarly because the only difference is about the error term Q^{CD} coming from the viscous contact wave in Q_2 . For this, we have

$$\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} |Q^{CD}||\zeta| dy d\tau
\leq \varepsilon^{2} \int_{0}^{\tau} \left[\|\zeta\|_{L_{y}^{\infty}} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-2} e^{-\frac{\varepsilon\varepsilon y^{2}}{1+\varepsilon\tau}} dy \right] d\tau
\leq \varepsilon^{\frac{3}{2}} \int_{0}^{\tau} \left[\|\zeta\|_{L_{y}^{2}}^{\frac{1}{2}} \|\zeta_{y}\|_{L_{y}^{2}}^{\frac{1}{2}} (1+\varepsilon\tau)^{-\frac{3}{2}} \right] d\tau$$

$$\leq \beta \int_{0}^{\tau} \|\zeta_{y}\|^{2} d\tau + C_{\beta} \varepsilon^{2} \sup_{[0,\tau]} \|\zeta\|_{L_{y}^{2}}^{\frac{2}{3}} \int_{0}^{\tau} (1+\varepsilon\tau)^{-2} d\tau$$

$$\leq \beta \|\zeta_{y}\|^{2} + \beta \sup_{[0,\tau]} \|\zeta\|_{L_{y}^{2}}^{2} + C_{\beta} \varepsilon^{\frac{3}{2}}.$$

$$(3.19)$$

By substituting (3.15)-(3.19) into (3.13) and choosing β suitably small, we can get

$$\begin{aligned} \|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \left[\|(\psi_{y},\zeta_{y})\|^{2} + \|\sqrt{(U_{y}^{R_{1}},U_{y}^{R_{3}})}(\phi,\zeta)\|^{2} \right] d\tau \\ &\leq C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau. \end{aligned}$$
(3.20)

Now we need to estimate $\|\phi_y\|^2$. Let $\tilde{v} = \frac{v}{V}$, then

$$\frac{\tilde{v}_{\tau}}{\tilde{v}} = \frac{u_y}{v} - \frac{U_y}{V}.$$

Rewrite the equation $(2.24)_2$ as

$$\frac{(v_y)}{\tilde{v}})_{\tau} - \psi_{\tau} - (p - P)_y - \varepsilon Q_1 = 0.$$
(3.21)

By multiplying (3.21) by $\frac{\tilde{v}_y}{\tilde{v}}$ and noticing that

$$-(p-P)_y = \frac{R\theta}{v}\frac{\tilde{v}_y}{\tilde{v}} - \frac{R\zeta_y}{v} + (p-P)\frac{V_y}{V} + R\Theta_y(\frac{1}{v} - \frac{1}{V}), \qquad (3.22)$$

we get

$$\begin{bmatrix} \frac{1}{2} (\frac{\tilde{v}_y}{\tilde{v}})^2 - \psi \frac{\tilde{v}_y}{\tilde{v}} \end{bmatrix}_{\tau} + \begin{bmatrix} \psi \frac{\tilde{v}_\tau}{\tilde{v}} \end{bmatrix}_y + \frac{R\theta}{v} (\frac{\tilde{v}_y}{\tilde{v}})^2 \\ = \psi_y (\frac{u_y}{v} - \frac{U_y}{V}) + \begin{bmatrix} \frac{R\zeta_y}{v} - (p-P)\frac{V_y}{V} - R\Theta_y (\frac{1}{v} - \frac{1}{V}) + \varepsilon Q_1 \end{bmatrix} \frac{\tilde{v}_y}{\tilde{v}}.$$

Integrating the above equality over $[0, \tau] \times \mathbf{R}$ in τ and y, we obtain

$$\int_{\mathbf{R}} \left[\frac{1}{2} (\frac{\tilde{v}_y}{\tilde{v}})^2 - \psi \frac{\tilde{v}_y}{\tilde{v}} \right] (\tau, y) dy + \int_0^\tau \int_{\mathbf{R}} \frac{R\theta}{2v} (\frac{\tilde{v}_y}{\tilde{v}})^2 dy d\tau$$

$$\leq C \int_0^\tau \left[\| (\psi_y, \zeta_y) \|^2 + \varepsilon^2 \| Q_1 \|^2 \right] d\tau + C \int_0^\tau \int_{\mathbf{R}} | (V_y, U_y, \Theta_y) |^2 | (\phi, \zeta) |^2 dy d\tau.$$
(3.23)

The by using the equality

$$\frac{\tilde{v}_y}{\tilde{v}} = \frac{v_y}{v} - \frac{V_y}{V} = \frac{\phi_y}{v} - \frac{V_y\phi}{vV},$$

$$C^{-1}(|\phi_y|^2 - |V_y\phi|^2) \le (\frac{\tilde{v}_y}{\tilde{v}})^2 \le C(|\phi_y|^2 + |V_y\phi|^2).$$
(3.24)

we have

By the estimation on Q_{11} in (2.21) and Lemma 2.3, we have

$$\int_{0}^{\tau} \varepsilon^{2} \|Q_{1}\|^{2} d\tau \leq C \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon^{2} (|Q_{11}|^{2} + |Q_{12}|^{2}) dy d\tau \\
\leq C \int_{0}^{t} \int_{\mathbf{R}} (|Q_{11}|^{2} + \varepsilon^{2}|(w_{\delta}^{r})_{xx}|^{2} + \varepsilon^{2}|(w_{\delta}^{r})_{x}|^{4}) dx dt \qquad (3.25) \\
\leq C e^{-C\varepsilon^{-1/5}} + C\varepsilon^{2} (t_{0}^{-2} + \delta^{-1} t_{0}^{-1}) \\
\leq C\varepsilon^{\frac{7}{5}}.$$

Moreover, we have

$$\begin{aligned} |(V_y, U_y, \Theta_y)|^2 &= \varepsilon^2 |(V_x, U_x, \Theta_x)|^2 \\ &\leq \varepsilon^2 \sum_{i=1,3} |(V_x^{R_i}, U_x^{R_i}, \Theta_x^{R_i})|^2 + \varepsilon^2 |(V_x^{CD}, U_x^{CD}, \Theta_x^{CD})|^2 \\ &\leq C \varepsilon^2 (t+t_0)^{-2} + C \delta^{CD} \varepsilon (1+t)^{-1} e^{-\frac{C_0 \varepsilon^2}{\varepsilon(1+t)}} \\ &= C (\tau + \tau_0)^{-2} + C \delta^{CD} \varepsilon (1+\varepsilon \tau)^{-1} e^{-\frac{C_0 \varepsilon y^2}{1+\varepsilon \tau}}. \end{aligned}$$
(3.26)

Substituting (3.24)-(3.26) into (3.23) gives

$$\begin{aligned} \|\phi_{y}(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \|\phi_{y}\|^{2} d\tau &\leq C \|(\phi,\psi)(\tau,\cdot)\|^{2} \\ + C \int_{0}^{\tau} \|(\psi_{y},\zeta_{y})\|^{2} d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{\tau}{5}} \\ + C\delta^{CD} \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau. \end{aligned}$$
(3.27)

Now we estimate the higher order derivatives of (ψ, ζ) . Multiplying $(2.24)_2$ by $-\psi_{yy}$ and $(2.24)_3$ by $-\zeta_{yy}$, and then adding the resulting equations together yield

$$\begin{aligned} & [\frac{1}{2}\psi_y^2 + \frac{R}{2(\gamma - 1)}\zeta_y^2]_\tau + \frac{\psi_{yy}^2}{v} + \nu \frac{\zeta_{yy}^2}{v} \\ &= \{(p - P)_y + \frac{v_y}{v^2}\psi_y + [U_y(\frac{1}{v} - \frac{1}{V})]_y + \varepsilon Q_1\}\psi_{yy} \\ &+ \{(pu_y - PU_y) + \frac{\nu v_y}{v^2}\zeta_y + [\nu\Theta_y(\frac{1}{v} - \frac{1}{V})]_y + (\frac{u_y^2}{v} - \frac{U_y^2}{V}) + \varepsilon Q_2\}\zeta_{yy}. \end{aligned}$$
(3.28)

The right hand side of (3.28) will be estimated terms by terms as follows. From (3.22) and (3.26), we get

$$\int_{0}^{\tau} \int_{\mathbf{R}} (p-P)_{y} \psi_{yy} dy d\tau
\leq C \int_{0}^{\tau} \int_{\mathbf{R}} \left[|(\phi_{y}, \zeta_{y})| + |(V_{y}, \Theta_{y})||(\phi, \zeta)| \right] |\psi_{yy}| dy d\tau
\leq \beta \int_{0}^{\tau} \|\psi_{yy}\|^{2} d\tau + C_{\beta} \int_{0}^{\tau} \|(\phi_{y}, \zeta_{y})\|^{2} d\tau + C_{\beta} \int_{0}^{\tau} (\tau + \tau_{0})^{-2} \|(\phi, \zeta)\|^{2} d\tau
+ C_{\beta} \delta^{CD} \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1 + \varepsilon \tau)^{-1} e^{-\frac{C \varepsilon y^{2}}{1 + \varepsilon \tau}} |(\phi, \zeta)|^{2} dy d\tau.$$
(3.29)

Similar estimate holds for the term $\int_0^\tau \int_{\mathbf{R}} (pu_y - PU_y) \zeta_{yy} dy d\tau$.

Notice that

$$\int_{0}^{\tau} \int_{\mathbf{R}} \frac{v_{y}}{v^{2}} \psi_{y} \psi_{yy} dy d\tau
\leq C \int_{0}^{\tau} \int_{\mathbf{R}} (|\phi_{y}| + |V_{y}|) |\psi_{y}| |\psi_{yy}| dy d\tau
\leq C \int_{0}^{\tau} (\|\phi_{y}\| \|\psi_{yy}\| \|\psi_{y}\| \|L_{y}^{\infty} + \|V_{y}\| \|L_{y}^{\infty}\| \|\|\psi_{y}\| \psi_{yy}\|) d\tau
\leq C \int_{0}^{\tau} \|\psi_{yy}\|^{\frac{3}{2}} \|\psi_{y}\|^{\frac{1}{2}} \|\phi_{y}\| d\tau + C\varepsilon^{\frac{1}{2}} \int_{0}^{\tau} \|\psi_{y}\| \psi_{yy}\| d\tau
\leq \beta \int_{0}^{\tau} \|\psi_{yy}\|^{2} d\tau + C_{\beta} (\sup_{[0,\tau]} \|\phi_{y}\|^{4} + \varepsilon) \int_{0}^{\tau} \|\psi_{y}\|^{2} d\tau
\leq \beta \int_{0}^{\tau} \|\psi_{yy}\|^{2} d\tau + C_{\beta} (\chi^{4} + \varepsilon) \int_{0}^{\tau} \|\psi_{y}\|^{2} d\tau,$$
(3.30)

where in the third inequality we have used the fact that $||V_y||_{L^{\infty}} \leq C\varepsilon^{\frac{1}{2}}$ because of (3.26).

Similarly, we have

$$\int_{0}^{\tau} \int_{\mathbf{R}} \nu \frac{v_y}{v^2} \zeta_y \zeta_{yy} dy d\tau
\leq \beta \int_{0}^{\tau} \|\zeta_{yy}\|^2 d\tau + C_{\beta} (\chi^4 + \varepsilon) \int_{0}^{\tau} \|\zeta_y\|^2 d\tau.$$
(3.31)

The remaining terms can be estimated directly by using (3.25) and the fact that

$$\begin{split} & [U_y(\frac{1}{v} - \frac{1}{V})]_y = O(1)[|(U_{yy}, U_y V_y)||\phi| + |U_y||\phi_y|], \\ & [\nu\Theta_y(\frac{1}{v} - \frac{1}{V})]_y = O(1)[|(\Theta_{yy}, \Theta_y V_y)||\phi| + |\Theta_y||\phi_y|]. \end{split}$$

Hence, if we take β suitably small, then we obtain

$$\begin{aligned} \|(\psi_{y},\zeta_{y})(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \|(\psi_{yy},\zeta_{yy})\|^{2} d\tau \\ &\leq C \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})\|^{2} d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{7}{5}} \\ &+ C\delta^{CD} \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau. \end{aligned}$$
(3.32)

The combination of (3.20), (3.27) and (3.32) yields that

$$\begin{aligned} \|(\phi,\psi,\zeta)(\tau,\cdot)\|_{1}^{2} &+ \int_{0}^{\tau} \left[\|\phi_{y}\|^{2} + \|(\psi_{y},\zeta_{y})\|_{1}^{2} \right] d\tau \\ &\leq C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C\delta^{CD} \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau. \end{aligned}$$
(3.33)

In order to close the estimate, we only need to control the last term in (3.33), which comes from the viscous contact wave. For this, we will apply the following technique by using the heat kernel motivated by [13].

Lemma 3.2. Suppose that $h(\tau, y)$ satisfies

$$h \in L^{\infty}(0, +\infty; L^{2}(\mathbf{R})), \ h_{y} \in L^{2}(0, +\infty; L^{2}(\mathbf{R})), \ h_{\tau} \in L^{2}(0, +\infty; H^{-1}(\mathbf{R})),$$

Then

$$\int_{0}^{\tau} \int_{\mathbf{R}^{+}} \varepsilon (1 + \varepsilon \tau)^{-1} e^{-\frac{2a\varepsilon y^{2}}{1 + \varepsilon \tau}} h^{2}(\tau, y) dy d\tau
\leq C_{a} \left[\|h(0, y)\|^{2} + \int_{0}^{\tau} \|h_{y}\|^{2} d\tau + \int_{0}^{\tau} \langle h_{\tau}, hg_{a}^{2} \rangle_{H^{-1} \times H^{1}} d\tau \right]$$
(3.34)

where

$$g_a(\tau, y) = \varepsilon^{\frac{1}{2}} (1 + \varepsilon \tau)^{-\frac{1}{2}} \int_{-\infty}^{y} e^{-\frac{a\varepsilon \eta^2}{1 + \varepsilon \tau}} d\eta,$$

and a > 0 is the constant to be determined later.

The proof of Lemma 3.2 is similar to the one given in [13]. The only difference here is that we need to be careful about the parameter ε in the estimation. Therefore, we omit its proof for brevity. Based on Lemma 3.2, we can obtain

Lemma 3.3. There exists a constant C > 0 such that if δ^{CD} and ε_0 are small enough, then we have

$$\int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_0 \varepsilon y^2}{1+\varepsilon\tau}} |\langle \phi, \psi, \zeta \rangle|^2 dy d\tau$$

$$\leq C \|\langle \phi, \psi, \zeta \rangle(\tau, \cdot)\|^2 + C \int_{0}^{\tau} \|\langle \phi_y, \psi_y, \zeta_y \rangle\|^2 d\tau$$

$$+ C \int_{0}^{\tau} (\tau+\tau_0)^{-\frac{3}{2}} \|\langle \phi, \psi, \zeta \rangle\|^2 d\tau + C \varepsilon^{\frac{2}{5}}.$$
(3.35)

Proof. From the equation $(2.24)_2$ and the fact that $p - P = \frac{R\zeta - P\phi}{v}$, we have

$$\psi_{\tau} + (\frac{R\zeta - P\phi}{v})_y = (\frac{u_y}{v} - \frac{U_y}{V})_y - \varepsilon Q_1.$$

Then

$$(R\zeta - P\phi)_y = \frac{R\zeta - P\phi}{v}(V_y + \phi_y) - v\psi_\tau + v(\frac{u_y}{v} - \frac{U_y}{V})_y - v\varepsilon Q_1.$$
(3.36)

Let

$$G_b(\tau, y) = \varepsilon (1 + \varepsilon \tau)^{-1} \int_{-\infty}^{y} e^{-\frac{b\varepsilon \eta^2}{1 + \varepsilon \tau}} d\eta,$$

where b is a positive constant to be determined later. Multiplying the equation (3.36) by $G_b(R\zeta - P\phi)$ gives

$$\left[\frac{G_b(R\zeta - P\phi)^2}{2}\right]_y - (G_b)_y \frac{(R\zeta - P\phi)^2}{2} \\
= \frac{G_b(R\zeta - P\phi)^2}{v} (V_y + \phi_y) - G_b v(R\zeta - P\phi)\psi_\tau \\
+ G_b v(R\zeta - P\phi)(\frac{u_y}{v} - \frac{U_y}{V})_y - \varepsilon G_b v(R\zeta - P\phi)Q_1.$$
(3.37)

Note that

$$-G_b v (R\zeta - P\phi)\psi_{\tau} = -[G_b v (R\zeta - P\phi)\psi]_{\tau} + [G_b v (R\zeta - P\phi)\psi]_y + (G_b v)_{\tau} (R\zeta - P\phi)\psi + G_b v \psi (R\zeta - P\phi)_{\tau},$$
(3.38)

$$(R\zeta - P\phi)_{\tau}$$

$$= R\zeta_{\tau} - P_{\tau}\phi - P\phi_{\tau}$$

$$= (\gamma - 1) \left[-(p - P)(U_y + \psi_y) + \left(\frac{u_y^2}{v} - \frac{U_y^2}{V}\right) + \nu\left(\frac{\theta_y}{v} - \frac{\Theta_y}{V}\right)_y - \varepsilon Q_2 \right]$$

$$(3.39)$$

$$-\gamma P\psi_y - P_{\tau}\phi.$$

By using the equality

$$-G_{b}v\gamma P\psi_{y}\psi = -[\gamma G_{b}vP\frac{\psi^{2}}{2}]_{y} + \gamma vP(G_{b})_{y}\frac{\psi^{2}}{2} + \gamma (vP)_{y}G_{b}\frac{\psi^{2}}{2}, \qquad (3.40)$$

we have

$$\varepsilon(1+\varepsilon\tau)^{-1}e^{-\frac{b\varepsilon y^2}{1+\varepsilon\tau}}[(R\zeta - P\phi)^2 + \gamma Pv\psi^2] = [G_bv(R\zeta - P\phi)\psi]_{\tau} + (\cdots)_y + Q_4, \quad (3.41)$$

where

$$Q_{4} = -(vG_{b})_{\tau}v(R\zeta - P\phi)\psi - \frac{\gamma\psi^{2}}{2}(Pv)_{y}G_{b} + G_{b}v\psi P_{\tau}\phi + (\gamma - 1)G_{b}v\psi \left[(p - P)(U_{y} + \psi_{y}) - (\frac{u_{y}^{2}}{v} - \frac{U_{y}^{2}}{V}) + \varepsilon Q_{2} \right] + [G_{b}v(R\zeta - P\phi)]_{y}(\frac{u_{y}}{v} - \frac{U_{y}}{V}) + (\gamma - 1)\nu(G_{b}v\psi)_{y}(\frac{\theta_{y}}{v} - \frac{\Theta_{y}}{V}) - \frac{G_{b}(R\zeta - P\phi)^{2}}{v}(V_{y} + \phi_{y}) + \varepsilon G_{b}v(R\zeta - P\phi)Q_{1}.$$
(3.42)

Note that

$$\|G_b(\tau,\cdot)\|_{L^{\infty}} \le C_{\alpha} \varepsilon^{\frac{1}{2}} (1+\varepsilon\tau)^{-\frac{1}{2}}.$$

Thus, integrating (3.41) over $(0, \tau) \times \mathbf{R}$ gives

$$\int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{b\varepsilon y^{2}}{1+\varepsilon\tau}} [(R\zeta - P\phi)^{2} + \psi^{2}] dy d\tau$$

$$\leq C \|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} + C \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})(\tau,\cdot)\|^{2} d\tau$$

$$+ C \int_{0}^{\tau} (\tau+\tau_{0})^{-\frac{3}{2}} \|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} d\tau + C\varepsilon^{\frac{7}{5}}$$

$$+ C\delta^{CD} \int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau.$$
(3.43)

In order to get the desired estimate stated in Lemma 3.3, set

$$h = \frac{R}{\gamma - 1}\zeta + P\phi$$

in Lemma 3.2. We only need to compute the last term on the right hand side of (3.34) for this given function h. From the energy equation $(2.24)_3$, we have

$$h_{\tau} = -(p-P)\psi_{y} + [P_{\tau}\phi - (p-P)U_{y}] + \nu(\frac{\theta_{y}}{v} - \frac{\Theta_{y}}{V})_{y} + (\frac{u_{y}^{2}}{v} - \frac{U_{y}^{2}}{V}) - \varepsilon Q_{2}$$

$$:= \sum_{i=1}^{5} H_{i}.$$
(3.44)

Thus

$$\int_0^\tau \langle h_\tau, hg_a^2 \rangle_{H^1 \times H^{-1}} d\tau = \sum_{i=1}^5 \int_0^\tau \int_{\mathbf{R}} hg_a^2 H_i dy d\tau.$$
(3.45)

By noticing that

$$\|g_a(\tau, \cdot)\|_{L^{\infty}} \le C_a,$$

we can estimate $\int_0^{\tau} \int_{\mathbf{R}} hg_a^2 H_i dy d\tau (i = 2, \dots, 6)$ directly. The estimation on $\int_0^{\tau} \int_{\mathbf{R}} hg_a^2 H_1 dy d\tau$ is more subtle. Firstly, by using the mass equation $(2.24)_1$, we have

$$\begin{split} hg_{a}^{2}H_{1} &= -(p-P)\psi_{y}hg_{a}^{2} \\ &= -\frac{(\gamma-1)h+\gamma P\phi}{v}hg_{a}^{2}\phi_{\tau} \\ &= -\frac{(\gamma-1)h^{2}g_{a}^{2}}{v}\phi_{\tau} - \frac{\gamma Phg_{a}^{2}}{2v}(\phi^{2})_{\tau} \\ &= -[\frac{(\gamma-1)h^{2}\phi g_{a}^{2}}{v} + \frac{\gamma Ph\phi^{2}g_{a}^{2}}{2v}]_{\tau} + \frac{2(\gamma-1)h^{2}\phi + \gamma Ph\phi^{2}}{v}g_{a}(g_{a})_{\tau} \\ &- \frac{2(\gamma-1)h^{2}\phi + \gamma Ph\phi^{2}}{2v^{2}}g_{a}^{2}v_{\tau} + \frac{\gamma h\phi^{2}g_{a}^{2}}{2v}P_{\tau} + [\frac{2(\gamma-1)\phi h}{v} + \frac{\gamma P\phi^{2}}{2v}]g_{a}^{2}h_{\tau} \\ &:= \sum_{i=1}^{5} J_{i}. \end{split}$$

Now the terms $J_i(i = 1, \dots, 4)$ can be estimated directly, cf. [13]. Here we only calculate the term J_5 . From (3.44), we have

$$J_5 = \sum_{i=1}^{6} \left[\frac{2(\gamma - 1)\phi h}{v} + \frac{\gamma P \phi^2}{2v} \right] g_a^2 H_i := \sum_{i=1}^{5} J_5^i.$$

Now J_5^1 can be estimated as follows:

$$\begin{split} \int_0^\tau \int |J_5^1| dy d\tau &\leq C \int_0^\tau \int |\psi_y| |(\phi,\zeta)|^3 dy d\tau \\ &\leq C \int_0^\tau \|(\phi,\zeta)\|_{L_\infty}^2 \|\psi_y\| \|(\phi,\zeta)\| d\tau \\ &\leq C \int_0^\tau \|(\phi,\zeta)_y\| \|\psi_y\| \|(\phi,\zeta)\|^2 d\tau \\ &\leq C \sup_{[0,\tau]} \|(\phi,\zeta)(\tau,\cdot)\|^2 \int_0^\tau \|(\phi,\psi,\zeta)_y\|^2 d\tau \\ &\leq C \chi^2 \int_0^\tau \|(\phi,\psi,\zeta)_y\|^2 d\tau. \end{split}$$

Note that the other terms $J_5^i (i = 2, \dots, 5)$ can be estimated directly, we omit the details for brevity.

Therefore, by taking the constant $a = \frac{C_0}{2}$, we obtain

$$\int_{0}^{\tau} \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} h^{2} dy d\tau$$

$$\leq C \|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} + C \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})\|^{2} d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-\frac{3}{2}} \|(\phi,\psi)\|^{2} d\tau$$

$$+ C\varepsilon^{\frac{2}{5}} + C(\delta^{CD}+\chi) \int_{0}^{\tau} \int_{\mathbf{R}^{+}}^{\tau} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\zeta)|^{2} dy d\tau.$$
(3.46)

By taking $b = C_0$ in (3.43) and by combining the estimates (3.43) with (3.46), we yield the desired estimation in Lemma 3.3 if we choose suitably small positive constants δ^{CD} , ε_0 and χ .

Now from (3.33) and Lemma 3.3, if the strength of the contact wave δ^{CD} and the parameter χ on the a priori estimate are suitably small, we can get

$$\begin{aligned} \|(\phi,\psi,\zeta)(\tau,\cdot)\|_{1}^{2} &+ \int_{0}^{\tau} \left[\|\phi_{y}\|^{2} + \|(\psi_{y},\zeta_{y})\|_{1}^{2} \right] d\tau \\ &\leq C \Big[\int_{0}^{\tau} (\tau+\tau_{0})^{-\frac{3}{2}} \|(\phi,\psi,\zeta)\|^{2} d\tau + \varepsilon^{\frac{2}{5}} \Big] \end{aligned}$$

With this, the Gronwall inequality gives

$$\|(\phi,\psi,\zeta)(\tau,\cdot)\|_{1}^{2} + \int_{0}^{\tau} \left[\|\phi_{y}\|^{2} + \|(\psi_{y},\zeta_{y})\|_{1}^{2}\right] d\tau \leq C\varepsilon^{\frac{2}{5}}.$$

And then we complete the proof of Theorem 3.1 by Sobolev imbedding.

4. Proof of Theorem 2.5: Hydrodynamic limit of Boltzmann equation. In the last section, we will prove the fluid dynamic limit for the Boltzmann equation to the Riemann solution for the Euler equations as stated in Theorem 2.5. Again, the proof is based on energy estimates for the Boltzmann equation (2.52) in the scaled independent variables. For this, it is sufficient to prove the following theorem.

Theorem 4.1. There exist two small positive constants δ_1 , ε_1 , and a global Maxwellian $\mathbf{M}_{\star} = \mathbf{M}_{[v_{\star}, u_{\star}, \theta_{\star}]}$ such that if the initial data and the strength of the contact wave δ^{CD} satisfy

$$\mathcal{N}(\tau)|_{\tau=0} + \delta^{CD} \le \delta_1, \tag{4.1}$$

and the Knudsen number $\varepsilon \leq \varepsilon_1$, then the problem (2.52) admits a unique global solution $f^{\varepsilon}(\tau, y, \xi)$ satisfying

$$\sup_{\tau,y} \|f^{\varepsilon}(\tau,y,\xi) - \mathbf{M}_{[V,U,\Theta]}(\tau,y,\xi)\|_{L^{2}_{\xi}(\frac{1}{\sqrt{\mathbf{M}_{\star}}})} \le C\varepsilon^{\frac{1}{5}}.$$
(4.2)

Here, $\mathcal{N}(\tau)$ is defined by (4.5) below.

Remark 7. If we choose the initial data for the Boltzmann equation (2.52) as

$$f^{\varepsilon}(0, y, \xi) = \mathbf{M}_{[V, U, \Theta]}(0, y, \xi) = \mathbf{M}_{[V(0, y), U(0, y), \Theta(0, y)]}(\xi),$$
(4.3)

then

$$\mathcal{N}(\tau)|_{\tau=0} = O(1) \left[\|(\Theta_y, U_y)\|^2 + \|(V_{yy}, \Theta_{yy}, U_{yy})\|^2 \right] \Big|_{\tau=0} = O(1)\varepsilon^{\frac{1}{2}}.$$
 (4.4)

In this case, the functional measuring the perturbation $\mathcal{N}(\tau)$ at $\tau = 0$ is smaller than the estimate given in Theorem 4.1 that is of the order of $O(\varepsilon^{\frac{2}{5}})$ because ε is small.

Consider the reformulated system (2.45) and (2.51). Since the local existence of solution to (2.45) and (2.51) is now standard, cf. [11] and [27], to prove the global

existence, we only need to close the following a priori estimate by the continuity argument:

$$\mathcal{N}(\tau) = \sup_{0 \le \tau' \le \tau} \left\{ \|(\phi, \psi, \zeta)(\tau', \cdot)\|_{1}^{2} + \int \int \frac{|\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi dy + \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy + \sum_{|\alpha|=2} \int \int \frac{|\partial^{\alpha} f|^{2}}{\mathbf{M}_{\star}} d\xi dy \right\} \le \chi^{2},$$

$$(4.5)$$

where $\partial^{\alpha}, \partial^{\alpha'}$ denote the derivatives with respect to y and τ respectively, and χ is a small positive constant depending on the initial data and the strength of the contact wave, and \mathbf{M}_{\star} is a global Maxwellian to be chosen later.

Note that the a priori assumption (4.5) implies that

$$\|(\phi,\psi,\zeta)\|_{L_{\infty}}^2 \le C\chi^2,\tag{4.6}$$

and

$$\|\int \frac{\mathbf{G}_1^2}{\mathbf{M}_{\star}} d\xi\|_{L^y_{\infty}} \le C \left(\int \int \frac{\mathbf{G}_1^2}{\mathbf{M}_{\star}} d\xi dy\right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\mathbf{G}_{1y}|^2}{\mathbf{M}_{\star}} d\xi dy\right)^{\frac{1}{2}} \le C(\varepsilon + \chi^2), \quad (4.7)$$

and for $|\alpha| = 1$,

$$\|\int \frac{|\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi\|_{L^{y}_{\infty}} \leq C \left(\int \int \frac{|\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy\right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\partial^{\alpha} \mathbf{G}_{y}|^{2}}{\mathbf{M}_{\star}} d\xi dy\right)^{\frac{1}{2}} \leq C(\varepsilon + \chi^{2}).$$

$$(4.8)$$

From (1.17) and (2.41), we have

$$\begin{cases} \phi_{\tau} - \psi_{1y} = 0, \\ \psi_{1\tau} + (p - P)_y = -\frac{4}{3} (\frac{\mu(\Theta)}{V} U_{1y})_y - \varepsilon Q_1 - \int \xi_1^2 \mathbf{G}_y d\xi, \\ \psi_{i\tau} = -(\frac{\mu(\Theta)}{V} U_{iy})_y - \int \xi_1 \xi_i \mathbf{G}_y d\xi, \ i = 2, 3, \\ \zeta_{\tau} + (pu_{1y} - PU_{1y}) = -(\frac{\lambda(\Theta)}{V} \Theta_y)_y - \frac{4}{3} \frac{\mu(\Theta)}{V} U_{1y}^2 - \varepsilon Q_2 \\ -\varepsilon Q_1 U_1 - \frac{1}{2} \int \xi_1 |\xi|^2 \mathbf{G}_y d\xi + \sum_{i=1}^3 u_i \int \xi_1 \xi_i \mathbf{G}_y d\xi. \end{cases}$$
(4.9)

Thus

$$\|(\phi_{\tau},\psi_{\tau},\zeta_{\tau})\|^2 \le C(\varepsilon+\chi^2). \tag{4.10}$$

Hence, we have

$$\|(v_{\tau}, u_{\tau}, \theta_{\tau})\|^{2} \leq C \|(\phi_{\tau}, \psi_{\tau}, \zeta_{\tau})\|^{2} + C \|(V_{\tau}, U_{\tau}, \Theta_{\tau})\|^{2} \leq C(\varepsilon + \chi^{2}).$$
(4.11)

In addition, (4.5) also implies that

$$\|(v_y, u_y, \theta_y)\|^2 \le C \|(\phi_y, \psi_y, \zeta_y)\|^2 + C \|(V_y, U_y, \Theta_y)\|^2 \le C(\varepsilon + \chi^2).$$
(4.12)

Since

$$\|\partial^{\alpha}\left(\rho,\rho u,\rho(E+\frac{|u|^2}{2})\right)\|^2 \le C \int \int \frac{|\partial^{\alpha}f|^2}{\mathbf{M}_{\star}} d\xi dy \le C\chi^2, \tag{4.13}$$

the inequalities (4.11)-(4.13) give

$$\begin{aligned} \|\partial^{\alpha}(v, u, \theta)\|^{2} &\leq C \|\partial^{\alpha} \left(\rho, \rho u, \rho(E + \frac{|u|^{2}}{2})\right)\|^{2} \\ &+ C \sum_{|\alpha|=1} \int |\partial^{\alpha} \left(\rho, \rho u, \rho(E + \frac{|u|^{2}}{2})\right)|^{4} dy \\ &\leq C(\varepsilon + \chi^{2}). \end{aligned}$$
(4.14)

Thus, for $|\alpha| = 2$, we have

$$\|\partial^{\alpha}(\phi,\psi,\zeta)\|^{2} \leq C(\|\partial^{\alpha}(v,u,\theta)\|^{2} + \|\partial^{\alpha}(V,U,\Theta)\|^{2}) \leq C(\varepsilon+\chi^{2}).$$

$$(4.15)$$

Finally, from the fact that $f = \mathbf{M} + \mathbf{G}$, we can obtain for $|\alpha| = 2$,

$$\int \int \frac{|\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy \leq C \int \int \frac{|\partial^{\alpha} f|^{2}}{\mathbf{M}_{\star}} d\xi dy + C \int \int \frac{|\partial^{\alpha} \mathbf{M}|^{2}}{\mathbf{M}_{\star}} d\xi dy \\
\leq C \int \int \frac{|\partial^{\alpha} f|^{2}}{\mathbf{M}_{\star}} d\xi dy + C \|\partial^{\alpha} (v, u, \theta)\|^{2} + C \sum_{|\alpha'|=1} \int |\partial^{\alpha'} (v, u, \theta)|^{4} dy \quad (4.16) \\
\leq C (\varepsilon + \chi^{2}).$$

Before proving the a priori estimate (4.5), we list some basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [8].

Lemma 4.2. There exists a positive constant C such that

$$\int \frac{\nu(|\xi|)^{-1}Q(f,g)^2}{\tilde{\mathbf{M}}} d\xi \le C \left\{ \int \frac{\nu(|\xi|)f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{\nu(|\xi|)g^2}{\tilde{\mathbf{M}}} d\xi \right\},$$

where \mathbf{M} can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 4.2, the following three lemmas are taken from [20]. And the proofs are straightforward by using Cauchy inequality.

Lemma 4.3. If $\theta/2 < \theta_{\star} < \theta$, then there exist two positive constants $\sigma = \sigma(v, u, \theta; v_{\star}, u_{\star}, \theta_{\star})$ and $\eta_0 = \eta_0(v, u, \theta; v_{\star}, u_{\star}, \theta_{\star})$ such that if $|v - v_{\star}| + |u - u_{\star}| + |\theta - \theta_{\star}| < \eta_0$, we have for $h(\xi) \in \mathfrak{N}^{\perp}$,

$$-\int \frac{h\mathbf{L}_{\mathbf{M}}h}{\mathbf{M}_{\star}}d\xi \ge \sigma \int \frac{\nu(|\xi|)h^2}{\mathbf{M}_{\star}}d\xi.$$

Lemma 4.4. Under the assumptions in Lemma 4.3, we have for each $h(\xi) \in \mathfrak{N}^{\perp}$,

$$\begin{cases} \int \frac{\nu(|\xi|)}{\mathbf{M}} |\mathbf{L}_{\mathbf{M}}^{-1}h|^2 d\xi \leq \sigma^{-2} \int \frac{\nu(|\xi|)^{-1}h^2}{\mathbf{M}} d\xi, \\ \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{L}_{\mathbf{M}}^{-1}h|^2 d\xi \leq \sigma^{-2} \int \frac{\nu(|\xi|)^{-1}h^2}{\mathbf{M}_{\star}} d\xi. \end{cases}$$

Lemma 4.5. Under the conditions in Lemma 4.3, for any positive constants k and λ , it holds that

$$|\int \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_{\star}} d\xi - \int \frac{g_1 |\xi|^k g_2}{\mathbf{M}_{\star}} d\xi| \le C_{k,\lambda} \int \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_{\star}} d\xi,$$

where the constant $C_{k,\lambda}$ depends on k and λ .

With the above preparation, we are ready to perform the energy estimation as follows. Firstly, similar to (3.13), we can get

$$\begin{split} &\left(\sum_{i=1}^{3} \frac{1}{2} \psi_{i}^{2} + R\Theta\Phi(\frac{v}{V}) + \Theta\Phi(\frac{\theta}{\Theta})\right)_{\tau} + \frac{4}{3} \frac{\mu(\theta)}{v} \psi_{1y}^{2} + \sum_{i=2}^{3} \frac{\mu(\theta)}{v} \psi_{iy}^{2} + \frac{\lambda(\theta)}{v\theta} \zeta_{y}^{2} \\ &+ P(U_{1y}^{R_{1}} + U_{1y}^{R_{3}}) \left[\Phi(\frac{\theta V}{v\Theta}) + \frac{5}{3} \Phi(\frac{v}{V}) \right] = -PU_{1y}^{CD} \left[\Phi(\frac{\theta V}{v\Theta}) + \frac{5}{3} \Phi(\frac{v}{V}) \right] \\ &+ \left[\left(\frac{\lambda(\Theta)\Theta_{y}}{V}\right)_{y} + \frac{4}{3} \frac{\mu(\Theta)U_{1y}^{2}}{V} + \varepsilon Q_{2} \right] \left[\frac{2}{3} \Phi(\frac{v}{V}) - \Phi(\frac{\Theta}{\theta}) \right] - \frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V}\right) U_{1y} \psi_{1y} \\ &- \frac{\zeta_{y}}{\theta} \left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\Theta)}{V}\right) \Theta_{y} + \frac{\zeta\theta_{y}}{\theta^{2}} \left(\frac{\lambda(\theta)\theta_{y}}{v} - \frac{\lambda(\Theta)\Theta_{y}}{V}\right) + \frac{4\zeta}{3\theta} \left(\frac{\mu(\theta)}{v} u_{1y}^{2} - \frac{\mu(\Theta)}{V} U_{1y}^{2}\right) \\ &+ \frac{\zeta}{\theta} \sum_{i=2}^{3} \frac{\mu(\theta)}{v} u_{iy}^{2} - \frac{\zeta}{\theta} (\varepsilon Q_{2} - \varepsilon Q_{1}U_{1}) - \varepsilon Q_{1}\psi_{1} + N_{1} + (\cdots)_{y}, \end{split}$$

$$\tag{4.17}$$

where

$$N_1 = -\sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi + \frac{\zeta}{\theta} (\sum_{i=1}^3 u_i \int \xi_1 \xi_i \Pi_{1y} d\xi - \frac{1}{2} \int \xi_1 |\xi|^2 \Pi_{1y} d\xi).$$
(4.18)

The estimation on the macroscopic terms in (4.17) is almost same as (3.20) for the compressible Navier-Stokes equations so that we have

$$\begin{aligned} \|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \left[\|(\psi_{y},\zeta_{y})\|^{2} + \|\sqrt{(U_{1y}^{R_{1}},U_{1y}^{R_{3}})}(\phi,\zeta)\|^{2} \right] d\tau \\ &\leq C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} (\phi^{2}+\zeta^{2}) dy d\tau + \int_{0}^{\tau} \int N_{1} dy d\tau. \end{aligned}$$
(4.19)

Now we estimate the microscopic term $\int_0^{\tau} \int N_1 dy d\tau$ in (4.19). For this, we only estimate the term $T_1 =: -\int_0^{\tau} \int \psi_1 \int \xi_1^2 \Pi_{1y} d\xi dy d\tau$ because other terms in $\int_0^{\tau} \int N_1 dy d\tau$ can be estimated similarly. For T_1 , integration by parts with respect to y and Cauchy inequality yield

$$T_{1} = \int_{0}^{\tau} \int \psi_{1y} \int \xi_{1}^{2} \Pi_{1} d\xi dy d\tau \leq \beta \int_{0}^{\tau} \|\psi_{1y}\|^{2} d\tau + C_{\beta} \int_{0}^{\tau} \int |\int \xi_{1}^{2} \Pi_{1} d\xi|^{2} dy d\tau.$$
(4.20)

By (2.49), we have

$$\int_{0}^{\tau} \int |\int \xi_{1}^{2} \Pi_{1} d\xi|^{2} dy d\tau$$

$$\leq C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1}(\mathbf{G}_{\tau}) d\xi|^{2} dy d\tau + C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1}(\frac{u_{1}}{v}\mathbf{G}_{y}) d\xi|^{2} dy d\tau$$

$$+ C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1}[\frac{1}{v}\mathbf{P}_{1}(\xi_{1}\mathbf{G}_{y})] d\xi|^{2} dy d\tau + C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1}[Q(\mathbf{G},\mathbf{G})] d\xi|^{2} dy d\tau$$

$$:= \sum_{i=1}^{4} T_{1}^{i}.$$
(4.21)

(4.21) Let \mathbf{M}_{\star} be a global Maxwellian with its state $(v_{\star}, u_{\star}, \theta_{\star})$ satisfying $\frac{1}{2}\theta < \theta_{\star} < \theta$ and $|v - v_{\star}| + |u - u_{\star}| + |\theta - \theta_{\star}| \le \eta_0$ so that Lemma 4.3 holds. Then we can obtain

$$T_{1}^{1} \leq C \int_{0}^{\tau} \int |\int \frac{\nu(|\xi|) |\mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi \cdot \int \nu^{-1}(|\xi|) \xi_{1}^{4} \mathbf{M}_{\star} d\xi | dy d\tau$$

$$\leq C \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau.$$
(4.22)

Similarly,

$$T_1^2 \le C \int_0^\tau \int \int \frac{\nu^{-1}(|\xi|) |\mathbf{G}_y|^2}{\mathbf{M}_\star} d\xi dy d\tau.$$
(4.23)

Moreover,

$$T_{1}^{3} \leq C \int_{0}^{\tau} \int |\int \frac{\nu(|\xi|) |\mathbf{L}_{\mathbf{M}}^{-1}[\frac{1}{v}\mathbf{P}_{1}(\xi_{1}\mathbf{G}_{y})]|^{2}}{\mathbf{M}_{[2v_{\star},2u_{\star},2\theta_{\star}]}} d\xi \cdot \int \nu^{-1}(|\xi|) \xi_{1}^{4} \mathbf{M}_{[2v_{\star},2u_{\star},2\theta_{\star}]} d\xi |dyd\tau|$$

$$\leq C \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\frac{1}{v}\mathbf{P}_{1}(\xi_{1}\mathbf{G}_{y})|^{2}}{\mathbf{M}_{[2v_{\star},2u_{\star},2\theta_{\star}]}} d\xi dyd\tau$$

$$\leq C \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\mathbf{G}_{y}|^{2}}{\mathbf{M}_{\star}} d\xi dyd\tau.$$
(4.24)

From Lemma 4.2, we have

$$T_{1}^{4} \leq C \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)|Q(\mathbf{G},\mathbf{G})|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau$$

$$\leq C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)|\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \cdot \int \frac{|\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau$$

$$\leq C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)(|\mathbf{G}_{0}|^{2} + |\mathbf{G}_{1}|^{2})}{\mathbf{M}_{\star}} d\xi \cdot \int \frac{|\mathbf{G}_{0}|^{2} + |\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau$$

$$\leq C(\varepsilon + \chi^{2}) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)|\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau + C\varepsilon^{\frac{1}{2}}.$$

$$(4.25)$$

Substituting (4.20)-(4.25) into (4.19) yields that

$$\begin{split} &\|(\phi,\psi,\zeta)(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \left[\|(\psi_{y},\zeta_{y})\|^{2} + \|\sqrt{(U_{1y}^{R_{1}},U_{1y}^{R_{3}})}(\phi,\zeta)\|^{2}\right]d\tau \\ &\leq C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}}(\phi^{2}+\zeta^{2}) dy d\tau \\ &+ C \sum_{|\alpha'|=1} \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)|\partial^{\alpha'}\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &+ C(\chi^{2}+\varepsilon) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)|\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{split}$$
(4.26)

To recover the term $\|\phi_y\|^2$ in the integral $\int_0^{\tau} \cdots d\tau$ in (4.26), as in the previous subsection for the compressible Navier-Stokes equations, we firstly rewrite the equation $(2.47)_2$ as

$$\frac{4}{3} \frac{\mu(\Theta)}{V} \phi_{y\tau} - \psi_{1\tau} - (p - P)_y \\
= -\frac{4}{3} (\frac{\mu(\Theta)}{V})_y \psi_{1y} - \frac{4}{3} [(\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V}) u_{1y}]_y + \varepsilon Q_1 + \int \xi_1^2 \Pi_{1y} d\xi,$$
(4.27)

by using the equation of conservation of the mass $(2.47)_1$.

Since

$$-(p-P)_y = \frac{P}{V}\phi_y - \frac{2}{3V}\zeta_y + (\frac{p}{v} - \frac{P}{V})v_y - \frac{2}{3}(\frac{1}{v} - \frac{1}{V})\theta_y,$$

and

$$\phi_y \psi_{1\tau} = (\phi_y \psi_1)_\tau - (\phi_\tau \psi_1)_y + \psi_{1y}^2,$$

by multiplying (4.27) by ϕ_y , we get

$$(\frac{2\mu(\Theta)}{3V}\phi_{y}^{2} - \phi_{y}\psi_{1})_{\tau} + \frac{P}{V}\phi_{y}^{2} = (\frac{2\mu(\Theta)}{3V})_{\tau}\phi_{y}^{2} + \psi_{1y}^{2} + \frac{2}{3V}\zeta_{y}\phi_{y} - (\frac{P}{v} - \frac{P}{V})v_{y}\phi_{y} + \frac{2}{3}(\frac{1}{v} - \frac{1}{V})\theta_{y}\phi_{y} - \frac{4}{3}(\frac{\mu(\Theta)}{V})_{y}\psi_{1y}\phi_{y} - \frac{4}{3}[(\frac{\mu(\Theta)}{v} - \frac{\mu(\Theta)}{V})u_{1y}]_{y}\phi_{y} + \varepsilon Q_{1}\phi_{y} + \int \xi_{1}^{2}\Pi_{1y}d\xi\phi_{y}.$$

$$(4.28)$$

Integrating (4.28) with respect to τ, y and using the Cauchy inequality yield

$$\begin{aligned} \|\phi_{y}(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \|\phi_{y}\|^{2} d\tau &\leq C \|\psi_{1}(\tau,\cdot)\|^{2} + C \int_{0}^{\tau} \|(\psi_{y},\zeta_{y})\|^{2} d\tau \\ + C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\psi,\zeta)|^{2} dy d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau \\ + C\varepsilon^{\frac{\tau}{5}} + C\chi \int_{0}^{\tau} \|\psi_{1yy}\|^{2} d\tau + \int_{0}^{\tau} \int |\int \xi_{1}^{2} \Pi_{1y} d\xi|^{2} dy d\tau. \end{aligned}$$

$$(4.29)$$

For the microscopic term $\int_0^\tau \int |\int \xi_1^2 \Pi_{1y} d\xi|^2 dy d\tau$, by (2.50), we have

$$\begin{split} &\int_{0}^{\tau} \int |\int \xi_{1}^{2} \Pi_{1y} d\xi|^{2} dy d\tau \\ &\leq C \Big[\int_{0}^{\tau} \int |\int \xi_{1}^{2} (\mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_{\tau})_{y} d\xi|^{2} dy d\tau + \int_{0}^{\tau} \int |\int \xi_{1}^{2} (\mathbf{L}_{\mathbf{M}}^{-1} \frac{u_{1}}{v} \mathbf{G}_{y})_{y} d\xi|^{2} dy d\tau \\ &+ \int_{0}^{\tau} \int |\int \xi_{1}^{2} [\mathbf{L}_{\mathbf{M}}^{-1} \frac{1}{v} \mathbf{P}_{1}(\xi_{1} \mathbf{G}_{y})]_{y} d\xi|^{2} dy d\tau + \int_{0}^{\tau} \int |\int \xi_{1}^{2} [\mathbf{L}_{\mathbf{M}}^{-1} Q(\mathbf{G}, \mathbf{G})]_{y} d\xi|^{2} dy d\tau \Big] \\ &:= \sum_{i=1}^{4} T_{2}^{i}. \end{split}$$

(4.30) Note that the inverse of the linearized operator ${\bf L}_{\bf M}^{-1}$ satisfies that , for any $h\in \mathcal{N}^{\perp},$

$$(\mathbf{L}_{\mathbf{M}}^{-1}h)_{\tau} = \mathbf{L}_{\mathbf{M}}^{-1}(h_{\tau}) - 2\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}h, \mathbf{M}_{\tau})\}, (\mathbf{L}_{\mathbf{M}}^{-1}h)_{y} = \mathbf{L}_{\mathbf{M}}^{-1}(h_{y}) - 2\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}h, \mathbf{M}_{y})\}.$$

$$(4.31)$$

Then we have

$$T_{2}^{1} \leq C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_{y\tau} d\xi|^{2} dy d\tau + C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1} \{Q(\mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_{\tau}, \mathbf{M}_{y})\} d\xi|^{2} dy d\tau \leq C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau + C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) |\mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi \int \frac{\nu(|\xi|) |\mathbf{M}_{y}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \leq C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau + C \int_{0}^{\tau} \int |(v_{y}, u_{y}, \theta_{y})|^{2} \int \frac{\nu(|\xi|) |\mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \leq C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau + C(\varepsilon + \chi^{2}) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) |\mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau.$$

$$(4.32)$$

Similar estimates hold for T_2^i (i = 2, 3). Moreover,

$$T_{2}^{4} \leq C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1} Q(\mathbf{G}, \mathbf{G}_{y}) d\xi|^{2} dy d\tau + C \int_{0}^{\tau} \int |\int \xi_{1}^{2} \mathbf{L}_{\mathbf{M}}^{-1} \{Q(\mathbf{L}_{\mathbf{M}}^{-1} Q(\mathbf{G}, \mathbf{G}), \mathbf{M}_{y})\} d\xi|^{2} dy d\tau \leq C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) |\mathbf{G}_{y}|^{2}}{\mathbf{M}_{\star}} d\xi \int \frac{|\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) |\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \int \frac{|\mathbf{G}_{y}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{0}^{\tau} \int |(v_{y}, u_{y}, \theta_{y})|^{2} \int \frac{\nu(|\xi|) |\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \int \frac{|\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \leq C(\chi^{2} + \varepsilon) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) (|\mathbf{G}_{1}|^{2} + |\mathbf{G}_{y}|^{2})}{\mathbf{M}_{\star}} d\xi dy d\tau.$$

$$(4.33)$$

Substituting (4.30)-(4.33) into (4.29) gives

$$\begin{split} \|\phi_{y}(\tau,\cdot)\|^{2} &+ \int_{0}^{\tau} \|\phi_{y}\|^{2} d\tau \leq C \|\psi_{1}(\tau,\cdot)\|^{2} + C \int_{0}^{\tau} \|(\psi_{y},\zeta_{y})\|^{2} d\tau \\ &+ C\delta^{CD} \varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\psi,\zeta)|^{2} dy d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\zeta)\|^{2} d\tau \\ &+ C\varepsilon^{\frac{2}{5}} + C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2} d\xi dy d\tau + C\chi \int_{0}^{\tau} \|\psi_{1yy}\|^{2} d\tau \\ &+ C(\varepsilon+\chi^{2}) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)(\sum_{|\alpha'|=1} |\partial^{\alpha'}\mathbf{G}|^{2} + |\mathbf{G}_{1}|^{2})}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{split}$$

$$(4.34)$$

We now turn to the time derivatives. To estimate $\|(\phi_{\tau}, \psi_{\tau}, \zeta_{\tau})\|^2$, we need to use the system (4.9). By multiplying (4.9)₁ by ϕ_{τ} , (4.9)₂ by $\psi_{1\tau}$, (4.9)₃ by $\psi_{i\tau}$ (i = 2, 3) and (4.9)₄ by ζ_{τ} respectively, and adding them together, after integrating with respect to τ and y, we have

$$\int_{0}^{\tau} \|(\phi_{\tau},\psi_{\tau},\zeta_{\tau})(\tau,\cdot)\|^{2} d\tau \leq C \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} + \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\psi,\zeta)\|^{2} d\tau + C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau \qquad (4.35) + C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C_{0}\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\psi,\zeta)|^{2} dy d\tau.$$

The microscopic component \mathbf{G}_1 can be estimated by using the equation (2.52). Multiplying (2.52) by $\frac{v\mathbf{G}_1}{\mathbf{M}_{\star}}$ gives

$$(v\frac{\mathbf{G}_{1}^{2}}{2\mathbf{M}_{\star}})_{\tau} - \frac{v\mathbf{G}_{1}}{\mathbf{M}_{\star}}\mathbf{L}_{\mathbf{M}}\mathbf{G}_{1} = v_{\tau}\frac{|\mathbf{G}_{1}|^{2}}{2\mathbf{M}_{\star}} + \left\{-\frac{3}{2v\theta}\mathbf{P}_{1}[\xi_{1}(\frac{|\xi-u|^{2}}{2\theta}\zeta_{y}+\xi\cdot\psi_{y})\mathbf{M}] + \frac{u_{1}}{v}\mathbf{G}_{y} - \frac{1}{v}\mathbf{P}_{1}(\xi_{1}\mathbf{G}_{y}) + Q(\mathbf{G},\mathbf{G}) - \mathbf{G}_{0\tau}\right\}\frac{v\mathbf{G}_{1}}{\mathbf{M}_{\star}}.$$

$$(4.36)$$

Integrating (4.36) with respect to τ, ξ and y and using the Cauchy inequality and Lemma 4.2-4.5 yield that

$$\int \int \frac{\mathbf{G}_{1}^{2}}{\mathbf{M}_{\star}}(\tau, y, \xi) d\xi dy + \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) |\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \\
\leq C\varepsilon^{\frac{2}{5}} + C \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^{2} d\tau + C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau,$$
(4.37)

where we have used the fact that

$$\int \int \frac{v \mathbf{G}_1^2}{\mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy = \int \int \frac{v \mathbf{G}_0^2}{\mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy$$
$$\leq C \| (\Theta_y, U_y) (\tau = 0, \cdot) \|^2 \leq C \varepsilon^{\frac{1}{2}}.$$

Next we derive the estimate on the higher order derivatives. By multiplying $(2.46)_2$ by $-\psi_{1yy}$, $(2.46)_3$ by $-\psi_{iyy}$ (i = 2, 3), $(2.46)_4$ by $-\zeta_{yy}$, and adding them

together, we obtain

$$\begin{split} &(\sum_{i=1}^{3} \frac{\psi_{iy}^{2}}{2} + \frac{\zeta_{y}^{2}}{2})_{\tau} + \frac{4}{3} \frac{\mu(\theta)}{v} \psi_{1yy}^{2} + \sum_{i=2}^{3} \frac{\mu(\theta)}{v} \psi_{iyy}^{2} + \frac{\lambda(\theta)}{v} \zeta_{yy}^{2} = \\ &-\frac{4}{3} (\frac{\mu(\theta)}{v})_{y} \psi_{1y} \psi_{1yy} - \sum_{i=2}^{3} (\frac{\mu(\theta)}{v})_{y} \psi_{iy} \psi_{iyy} - (\frac{\lambda(\theta)}{v})_{y} \zeta_{y} \zeta_{yy} \\ &-\frac{4}{3} [(\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V}) U_{1y}]_{y} \psi_{1yy} - [(\frac{\lambda(\theta)}{v} - \frac{\lambda(\Theta)}{V}) \Theta_{y}]_{y} \zeta_{yy} + (p - P)_{y} \psi_{1yy} \\ &+ \varepsilon Q_{1} \psi_{1yy} + (p u_{1y} - P U_{1y}) \zeta_{yy} - [\frac{4}{3} (\frac{\mu(\theta)}{v} u_{1y}^{2} - \frac{\mu(\Theta)}{V} U_{1y}^{2}) \\ &+ \sum_{i=2}^{3} \frac{\mu(\theta)}{v} u_{iy}^{2} - (\varepsilon Q_{2} - \varepsilon Q_{1} U_{1})] \zeta_{yy} + \sum_{i=1}^{3} \psi_{iyy} \int \xi_{1} \xi_{i} \Pi_{1y} d\xi \\ &- \zeta_{yy} (\sum_{i=1}^{3} u_{i} \int \xi_{1} \xi_{i} \Pi_{1y} d\xi - \frac{1}{2} \int \xi_{1} |\xi|^{2} \Pi_{1y} d\xi). \end{split}$$

$$(4.38)$$

Integrating (4.38) with respect to τ, y and ξ yields

$$\begin{aligned} \|(\psi_{y},\zeta_{y})(\tau,\cdot)\|^{2} + \int_{0}^{\tau} \|(\psi_{yy},\zeta_{yy})\|^{2}d\tau \\ &\leq C \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})\|^{2}d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\psi,\zeta)\|^{2}d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C\delta^{CD}\varepsilon \int_{0}^{\tau} \int_{\mathbf{R}} (1+\varepsilon\tau)^{-1} e^{-\frac{C\varepsilon y^{2}}{1+\varepsilon\tau}} |(\phi,\psi,\zeta)|^{2}dyd\tau \\ &+ C(\varepsilon^{\frac{1}{2}}+\chi) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{1}|^{2}d\xi dyd\tau + C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2}d\xi dyd\tau \\ &+ C(\varepsilon^{\frac{1}{2}}+\chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha'}\mathbf{G}|^{2}d\xi dyd\tau. \end{aligned}$$

$$(4.39)$$

(4.39) Again, to recover $\|\phi_{yy}\|^2$ in the time integral in (4.39), by applying ∂_y to (2.46)₂, we get

$$\psi_{1y\tau} + (p-P)_{yy} = -\frac{4}{3} (\frac{\mu(\Theta)}{V} U_{1y})_{yy} - \varepsilon Q_{1y} - \int \xi_1^2 \mathbf{G}_{yy} d\xi.$$
(4.40)

Note that

$$(p-P)_{yy} = -\frac{p}{v}\phi_{yy} + \frac{R}{v}\zeta_{yy} - \frac{1}{v}(p-P)V_{yy} - \frac{\phi}{v}P_{yy} - \frac{2v_y}{v}(p-P)_y - \frac{2P_y}{v}\phi_y.$$
 (4.41)

Multiplying (4.40) by $-\phi_{yy}$ and using (4.41) imply

$$-\int \psi_{1y}\phi_{yy}(\tau,y)dy + \int_{0}^{\tau} \int \frac{p}{2v}\phi_{yy}^{2}dyd\tau \leq C \int_{0}^{\tau} \|(\psi_{1yy},\zeta_{yy})\|^{2}d\tau + C\varepsilon^{\frac{2}{5}} + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\psi,\zeta)\|^{2}d\tau + C(\varepsilon^{\frac{1}{2}}+\chi) \int_{0}^{\tau} \|(\phi_{y},\psi_{y},\zeta_{y})\|^{2}d\tau + C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2}d\xi dyd\tau.$$

$$(4.42)$$

To estimate $\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2$ and $\|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2$, we use the system (4.9) again. By applying ∂_y to (4.9), and multiplying the four equations of (4.9) by $\phi_{y\tau}$, $\psi_{1y\tau}, \psi_{iy\tau}$ (i = 2, 3), $\zeta_{y\tau}$ respectively, then adding them together and integrating

with respect to τ and y give

$$\int_{0}^{\tau} \|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^{2} d\tau \leq C \int_{0}^{\tau} \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\
+ C \int_{0}^{\tau} (\tau + \tau_{0})^{-2} \|(\phi, \psi, \zeta)\|^{2} d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \|(\phi_{y}, \psi_{y}, \zeta_{y})\|^{2} d\tau \\
+ C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau + C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2} d\xi dy d\tau.$$
(4.43)

Similarly, we have

$$\int_{0}^{\tau} \|(\phi_{\tau\tau},\psi_{\tau\tau},\zeta_{\tau\tau})\|^{2} d\tau \leq C \int_{0}^{\tau} \|(\phi_{y\tau},\psi_{y\tau},\zeta_{y\tau})\|^{2} d\tau + C\varepsilon^{\frac{2}{5}} + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\psi,\zeta)\|^{2} d\tau + C(\varepsilon^{\frac{1}{2}}+\chi) \sum_{|\alpha|=2} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi,\psi,\zeta)\|^{2} d\tau + C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau + C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2} d\xi dy d\tau.$$

$$(4.44)$$

A suitable linear combination of $\left(4.39\right)$ - $\left(4.44\right)$ gives

$$\begin{split} \|(\psi_{y},\zeta_{y},\phi_{yy})(\tau,\cdot)\|^{2} + \sum_{|\alpha|=2} \int_{0}^{\tau} \|\partial^{\alpha}(\phi,\psi,\zeta)\|^{2} d\tau \\ &\leq C \sum_{|\alpha|=2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha}\mathbf{G}|^{2} d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha'}\mathbf{G}|^{2} d\xi dy d\tau \\ &+ C(\varepsilon^{\frac{1}{2}}+\chi) \int_{0}^{\tau} \int \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{1}|^{2} d\xi dy d\tau + C \int_{0}^{\tau} (\tau+\tau_{0})^{-2} \|(\phi,\psi,\zeta)\|^{2} d\tau \\ &+ C(\varepsilon^{\frac{1}{2}}+\chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi,\psi,\zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}}. \end{split}$$

$$(4.45)$$

To close the a priori estimate, we also need to estimate the derivatives on the non-fluid component **G**, i.e., ∂^{α} **G**, ($|\alpha| = 1, 2$). Applying ∂_y on (2.47), we have

$$\mathbf{G}_{y\tau} - \left(\frac{u_1}{v}\mathbf{G}_y\right)_y + \left\{\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y)\right\}_y + \left\{\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{G}_y)\right\}_y$$

$$= \mathbf{L}_{\mathbf{M}}\mathbf{G}_y + 2Q(\mathbf{M}_y, \mathbf{G}) + 2Q(\mathbf{G}_y, \mathbf{G}).$$

$$(4.46)$$

Since

$$\mathbf{P}_1(\xi_1 \mathbf{M}_y) = \frac{3}{2v\theta} \mathbf{P}_1[\xi_1(\frac{|\xi - u|^2}{2\theta}\theta_y + \xi \cdot u_y)\mathbf{M}],$$

we have

$$|\{\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y)\}_y| \le C(v_y^2 + u_y^2 + \theta_y^2 + |\theta_{yy}| + |u_{yy}|)|\hat{B}(\xi)|\mathbf{M},$$

where $\hat{B}(\xi)$ is a polynomial of ξ . This yields that

$$\int_0^\tau \int \int |\{\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y)\}_y \frac{\mathbf{G}_y}{\mathbf{M}_\star} |d\xi dy d\tau \le \frac{\sigma}{8} \int_0^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{G}_y|^2 d\xi dy d\tau + C \int_0^\tau \|(\psi_{yy}, \zeta_{yy})\|^2 d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \int_0^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C\varepsilon^{\frac{2}{5}}.$$

Thus, multiplying (4.46) by $\frac{v G_y}{M_{\star}}$ and using the Cauchy inequality and Lemmas 4.2-4.5 yield

$$\int \int \frac{|\mathbf{G}_{y}|^{2}}{2\mathbf{M}_{\star}}(\tau, y, \xi) d\xi dy + \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau$$

$$\leq C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{1}|^{2} d\xi dy d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \|(\phi_{y}, \psi_{y}, \zeta_{y})\|^{2} d\tau$$

$$+ C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{yy}|^{2} d\xi dy d\tau + C \int_{0}^{\tau} \|(\phi_{yy}, \zeta_{yy})\|^{2} d\tau + C\varepsilon^{\frac{2}{5}}.$$
(4.47)

Similarly,

$$\int \int \frac{|\mathbf{G}_{\tau}|^{2}}{2\mathbf{M}_{\star}} (\tau, y, \xi) d\xi dy + \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{\tau}|^{2} d\xi dy d\tau \\
\leq C \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y\tau}|^{2} d\xi dy d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{1}|^{2} d\xi dy d\tau \\
+ C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{y}|^{2} d\xi dy d\tau \\
+ C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^{2} d\tau + C \int_{0}^{\tau} \|(\psi_{y\tau}, \zeta_{y\tau})\|^{2} d\tau + C\varepsilon^{\frac{2}{5}},$$
(4.48)

where we have used the fact that

$$\int \int \frac{v |\mathbf{G}_{\tau}|^2}{2\mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy = \int \int \frac{|\mathbf{P}_1(\xi_1 \mathbf{M}_y)|^2}{2v \mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy$$
$$\leq C \| (v, u, \theta)_y (\tau = 0, \cdot) \|^2$$
$$= C \| (V, U, \Theta)_y (\tau = 0, \cdot) \|^2 \leq C \varepsilon^{\frac{1}{2}}.$$

Finally, we estimate the highest order derivatives, that is, $\int \psi_{1y} \phi_{yy} dy$ and $\int_0^{\tau} \int \int \frac{\nu(|\xi|) |\partial^{\alpha} \mathbf{G}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau$ with $|\alpha| = 2$ in (4.45). To do so, it is sufficient to study $\int \int \frac{|\partial^{\alpha} f|^2}{\mathbf{M}_{\star}} d\xi dy$ ($|\alpha| = 2$) in view of (4.13)- (4.16). For this, from (2.53) we have

$$vf_{\tau} - u_1f_y + \xi_1f_y = vQ(f, f) = v[\mathbf{L}_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G})]$$

Applying ∂^{α} ($|\alpha| = 2$) to the above equation gives

$$v(\partial^{\alpha}f)_{\tau} - v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G} - u_{1}(\partial^{\alpha}f)_{y} + \xi_{1}(\partial^{\alpha}f)_{y}$$

$$= -\partial^{\alpha}vf_{\tau} + \partial^{\alpha}u_{1}f_{y} - \sum_{|\alpha'|=1} [\partial^{\alpha-\alpha'}v\partial^{\alpha'}f_{\tau} - \partial^{\alpha-\alpha'}u_{1}\partial^{\alpha'}f_{y}] \qquad (4.49)$$

$$+ [\partial^{\alpha}(v\mathbf{L}_{\mathbf{M}}\mathbf{G}) - v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G}] + \partial^{\alpha}[vQ(\mathbf{G},\mathbf{G})].$$

Multiplying (4.49) by $\frac{\partial^{\alpha} f}{\mathbf{M}_{\star}} = \frac{\partial^{\alpha} \mathbf{M}}{\mathbf{M}_{\star}} + \frac{\partial^{\alpha} \mathbf{G}}{\mathbf{M}_{\star}}$ yields

$$\left(\frac{v|\partial^{\alpha}f|^{2}}{2\mathbf{M}_{\star}}\right)_{\tau} - v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G} \cdot \frac{\partial^{\alpha}\mathbf{G}}{\mathbf{M}_{\star}} \\
= \frac{\partial^{\alpha}f}{\mathbf{M}_{\star}} \left\{ -\partial^{\alpha}vf_{\tau} + \partial^{\alpha}u_{1}f_{y} - \sum_{|\alpha'|=1} [\partial^{\alpha-\alpha'}v\partial^{\alpha'}f_{\tau} - \partial^{\alpha-\alpha'}u_{1}\partial^{\alpha'}f_{y}] \right. \\
\left. + [\partial^{\alpha}(v\mathbf{L}_{\mathbf{M}}\mathbf{G}) - v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G}] + \partial^{\alpha}[vQ(\mathbf{G},\mathbf{G})] \right\} + v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G} \cdot \frac{\partial^{\alpha}\mathbf{M}}{\mathbf{M}_{\star}} + (\cdots)_{y}.$$

$$(4.50)$$

Hence,

$$\begin{split} &\int_{0}^{\tau} \int \int |\partial^{\alpha} v f_{\tau} \frac{\partial^{\alpha} f}{\mathbf{M}_{\star}} |d\xi dy d\tau \\ &\leq \int_{0}^{\tau} \int \left[|\partial^{\alpha} v| \int (|\mathbf{M}_{\tau}| + |\mathbf{G}_{\tau}|) \frac{|\partial^{\alpha} \mathbf{M}| + |\partial^{\alpha} \mathbf{G}|}{\mathbf{M}_{\star}} d\xi \right] dy d\tau \\ &\leq C(\varepsilon + \chi^{2}) \int_{0}^{\tau} ||\partial^{\alpha} (\phi, \psi, \zeta)||^{2} d\tau + \frac{\sigma}{16} \int_{0}^{\tau} \int \int \frac{v |\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &+ C(\varepsilon + \chi^{2}) \int_{0}^{\tau} \int \int \int \frac{|\mathbf{G}_{\tau}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &+ C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} ||\partial^{\alpha'} (\phi, \psi, \zeta)||^{2} d\tau + C\varepsilon^{\frac{2}{5}}, \end{split}$$

and

$$\begin{split} &\sum_{|\alpha'|=1} \int_0^\tau \int \int |\partial^{\alpha-\alpha'} v \partial^{\alpha'} f_\tau \frac{\partial^{\alpha} f}{\mathbf{M}_{\star}} |d\xi dy d\tau \\ &\leq \sum_{|\alpha'|=1} \int_0^\tau \int |\partial^{\alpha-\alpha'} v| \int (|\partial^{\alpha'} \mathbf{M}_\tau| + |\partial^{\alpha'} \mathbf{G}_\tau|) \frac{|\partial^{\alpha} \mathbf{M}| + |\partial^{\alpha} \mathbf{G}|}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq \frac{\sigma}{16} \int_0^\tau \int \int \frac{v |\partial^{\alpha} \mathbf{G}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C(\delta + \gamma) \int_0^\tau \|\partial^{\alpha} (\phi, \psi, \zeta)\|^2 d\tau + C\varepsilon^{\frac{2}{5}}. \end{split}$$

Notice that similar estimates can be obtained for the terms $\partial^{\alpha} u_1 f_y \frac{\partial^{\alpha} f}{\mathbf{M}_{\star}}$ and $\sum_{\substack{|\alpha'|=1}} \partial^{\alpha-\alpha'} u_1 \partial^{\alpha'} f_y \frac{\partial^{\alpha} f}{\mathbf{M}_{\star}}$. Furthermore, we have

$$\partial^{\alpha}(v\mathbf{L}_{\mathbf{M}}\mathbf{G}) - v\mathbf{L}_{\mathbf{M}}\partial^{\alpha}\mathbf{G} = (\partial^{\alpha}v)\mathbf{L}_{\mathbf{M}}\mathbf{G} + 2vQ(\partial^{\alpha}\mathbf{M},\mathbf{G}) \\ + \sum_{|\alpha'|=1} \bigg\{ 2vQ(\partial^{\alpha-\alpha'}\mathbf{M},\partial^{\alpha'}\mathbf{G}) + \partial^{\alpha-\alpha'}v[\mathbf{L}_{\mathbf{M}}\partial^{\alpha'}\mathbf{G} + 2Q(\partial^{\alpha'}\mathbf{M},\mathbf{G})] \bigg\},$$

and

$$\partial^{\alpha} [vQ(\mathbf{G}, \mathbf{G})] = (\partial^{\alpha} v)Q(\mathbf{G}, \mathbf{G}) + 2vQ(\partial^{\alpha}\mathbf{G}, \mathbf{G}) \\ + \sum_{|\alpha'|=1} \bigg\{ vQ(\partial^{\alpha-\alpha'}\mathbf{G}, \partial^{\alpha'}\mathbf{G}) + 2(\partial^{\alpha-\alpha'}v)Q(\partial^{\alpha'}\mathbf{G}, \mathbf{G})] \bigg\}.$$

For illustration, we only estimate one of the above terms in the following because the other terms can be discussed similarly.

$$\begin{split} &\int_{0}^{\tau} \int \int \frac{\upsilon \partial^{\alpha} \mathbf{G} \cdot Q(\partial^{\alpha} \mathbf{G}, \mathbf{G})}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq \frac{\sigma}{16} \int_{0}^{\tau} \int \int \frac{\upsilon |\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &+ C \int_{0}^{\tau} \int \int \left(\int \frac{\nu(|\xi|) |\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \cdot \int \frac{|\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi + \int \frac{|\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \cdot \int \frac{\nu(|\xi|) |\mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi \right) dy d\tau \\ &\leq \frac{\sigma}{8} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} v |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau \\ &\quad + C \int_{0}^{\tau} |\sup_{y} \int \frac{\nu(|\xi|) |\mathbf{G}_{1}|^{2}}{\mathbf{M}_{\star}} d\xi \cdot \int \int \frac{|\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy | d\tau \\ &\leq \frac{\sigma}{8} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} v |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau \\ &\quad + C(\varepsilon + \chi^{2}) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) ||\mathbf{G}_{1y}|^{2} + |\mathbf{G}_{1}|^{2}]}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq \frac{\sigma}{8} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} v |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau + C(\varepsilon + \chi^{2}) \int_{0}^{\tau} ||(\phi_{y}, \psi_{y}, \zeta_{y})||^{2} d\tau + C\varepsilon^{\frac{2}{5}} \\ &\quad + C(\varepsilon + \chi^{2}) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|) ||\mathbf{G}_{y}|^{2} + |\mathbf{G}_{1}|^{2}]}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{split}$$

Now we estimate the term $\int_0^{\tau} \int \int v \mathbf{L}_{\mathbf{M}} \partial^{\alpha} \mathbf{G} \cdot \frac{\partial^{\alpha} \mathbf{M}}{\mathbf{M}_{\star}} d\xi dy d\tau$ in (4.50). First, note that $\mathbf{P}_1(\partial^{\alpha} \mathbf{M})$ does not contain the term $\partial^{\alpha}(v, u, \theta)$ for $|\alpha| = 2$. Thus, we have

$$\int_{0}^{\tau} \int \int \frac{v \mathbf{L}_{\mathbf{M}} \partial^{\alpha} \mathbf{G} \cdot \partial^{\alpha} \mathbf{M}}{\mathbf{M}} d\xi dy d\tau = \int_{0}^{\tau} \int \int \frac{v \mathbf{L}_{\mathbf{M}} \partial^{\alpha} \mathbf{G} \cdot \mathbf{P}_{1}(\partial^{\alpha} \mathbf{M})}{\mathbf{M}} d\xi dy d\tau$$
$$\leq \frac{\sigma}{16} \int_{0}^{\tau} \int \int \frac{v |\partial^{\alpha} \mathbf{G}|^{2}}{\mathbf{M}_{\star}} d\xi dy d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}}.$$
(4.51)

Also we can get

$$\int_{0}^{\tau} \int \int v \mathbf{L}_{\mathbf{M}} \partial^{\alpha} \mathbf{G} \cdot \partial^{\alpha} \mathbf{M} (\frac{1}{\mathbf{M}_{\star}} - \frac{1}{\mathbf{M}}) d\xi dy d\tau \leq \frac{\sigma}{16} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} v |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau + C \eta_{0}^{2} \int_{0}^{\tau} \|\partial^{\alpha}(\phi, \psi, \zeta)\|^{2} d\tau + C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^{2} d\tau + C\varepsilon^{\frac{2}{5}},$$

$$(4.52)$$

where the small constant η_0 is defined in Lemma 4.3. The combination of (4.52) and (4.52) gives the estimation on $\int_0^{\tau} \int \int v \mathbf{L}_{\mathbf{M}} \partial^{\alpha} \mathbf{G} \cdot \frac{\partial^{\alpha} \mathbf{M}}{\mathbf{M}_{\star}} d\xi dy d\tau$. Thus, integrating (4.50) and using the above estimates give

$$\begin{split} &\int \int \frac{v |\partial^{\alpha} f|^{2}}{2\mathbf{M}_{\star}}(\tau, y, \xi) d\xi dy + \frac{\sigma}{2} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} v |\partial^{\alpha} \mathbf{G}|^{2} d\xi dy d\tau \\ &\leq C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^{2} d\tau + C(\eta_{0} + \delta + \gamma) \sum_{|\alpha|=2} \int_{0}^{\tau} \|\partial^{\alpha}(\phi, \psi, \zeta)\|^{2} d\tau \\ &+ C(\varepsilon^{\frac{1}{2}} + \chi) \sum_{|\alpha'|=1} \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha'} \mathbf{G}|^{2} d\xi dy d\tau + C\varepsilon^{\frac{2}{5}} \\ &+ C(\varepsilon^{\frac{1}{2}} + \chi) \int_{0}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\mathbf{G}_{1}|^{2} d\xi dy d\tau, \end{split}$$

where we have used the fact that

$$\int \int \frac{v |\partial^{\alpha} f|^2}{2\mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy = \int \int \frac{v |\partial^{\alpha} \mathbf{M}_{[V,U,\Theta]}|^2}{2\mathbf{M}_{\star}} (\tau = 0, y, \xi) d\xi dy$$
$$\leq C \| (V,U,\Theta)_{yy} (\tau = 0, \cdot) \|^2 + C \| (V,U,\Theta)_y (\tau = 0, \cdot) \|_{L^4}^4$$
$$\leq C \varepsilon^{\frac{3}{2}}.$$

Finally, similar to Lemma 3.3 in the previous section, we can get

$$\begin{split} &\int_0^\tau \int_{\mathbf{R}} \varepsilon (1+\varepsilon\tau)^{-1} e^{-\frac{C_0 \varepsilon y^2}{1+\varepsilon\tau}} |\langle \phi, \psi, \zeta \rangle|^2 dy d\tau \\ &\leq C \| (\phi, \psi, \zeta)(\tau, \cdot) \|^2 + C \int_0^\tau \| (\phi_y, \psi_y, \zeta_y) \|^2 d\tau + C \varepsilon^{\frac{2}{5}} \\ &+ C \int_0^\tau (\tau+\tau_0)^{-\frac{4}{3}} \| (\phi, \psi, \zeta) \|^2 d\tau + C (\varepsilon^{\frac{1}{2}} + \chi) \int_0^\tau \int \int \frac{\nu(|\xi|) |\mathbf{G}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau \\ &+ C \sum_{|\alpha'|=1} \int_0^\tau \int \int \frac{\nu^{-1}(|\xi|) |\partial^{\alpha'} \mathbf{G}|^2}{\mathbf{M}_\star} d\xi dy d\tau. \end{split}$$

Note that here we need to estimate the microscopic terms.

In summary, by combining all the above estimates and by choosing the strength of the contact wave δ^{CD} , the bound on the a priori estimate χ and the Knudsen number ε to be suitably small, we obtain

$$\begin{split} \mathcal{N}(\tau) &+ \int_0^\tau \Big[\sum_{1 \le |\alpha| \le 2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + \|\sqrt{(U_{1y}^{R_1}, U_{1y}^{R_3})}(\phi, \zeta)\|^2 \Big] d\tau \\ &+ \int_0^\tau \int \int \frac{\nu(|\xi|) \mathbf{G}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + \sum_{|\alpha'|=1} \int \int \frac{\nu(|\xi|) |\partial^{\alpha'} \mathbf{G}|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy \\ &+ \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|) |\partial^\alpha f|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy \le C\varepsilon^{\frac{2}{5}}. \end{split}$$

With the energy estimate, we complete the proof of Theorem 4.1.

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