Stability of Wave Patterns to the Inflow Problem of Full Compressible Navier-Stokes Equations

Xiaohong Qin* **Yi Wang†**

*Department of Mathematics, Nanjing University of Science and Technology, Nanjing, China
†Institute of Applied Mathematics, Academy of Mathematics and Systems Science, CAS, Beijing 100190, China

Abstract

The inflow problem of full compressible Navier-Stokes equations is considered on the half line $(0,+\infty)$. Firstly, we give the existence (or non-existence) of the boundary layer solution to the inflow problem when the right end state (ρ_+,u_+,θ_+) belongs to the subsonic, transonic and supersonic regions respectively. Then the asymptotic stability of not only the single contact wave but also the superposition of the boundary layer solution, the contact wave and the rarefaction wave to the inflow problem are investigated under some smallness conditions. Note that the amplitude of the rarefaction wave can be arbitrarily large. The proofs are given by the elementary energy method.

1 Introduction

In this paper, we consider the half space problem of the full (or non-isentropic) compressible Navie-Stokes equations in Eulerian coordinate:

$$\begin{cases} \rho_t+(\rho u)_x=0, & x>0,\quad t>0,\\ (\rho u)_t+(\rho u^2+p)_x=\mu u_{xx}, & x>0,\quad t>0,\\ [\rho(e+\dfrac{u^2}{2})]_t+[\rho u(e+\dfrac{u^2}{2})+pu]_x=\kappa\theta_{xx}+\mu(uu_x)_x, & x>0,\quad t>0, \end{cases} \tag{1.1}$$

where $\rho(t,x)>0$ is the density, $u(t,x)$ is the velocity, $\theta(t,x)$ is the absolute temperature of the gas, and $p=p(\rho,\theta)$ is the pressure, $e=e(\rho,\theta)$ is the internal energy, $\mu>0$ is the

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†Y. Wang is supported by NSFC grant (No. 10801128) and the Knowledge Innovation Program of the Chinese Academy of Sciences. E-mail: wangyi@amss.ac.cn.

viscosity constant, and $\kappa > 0$ is the coefficient of heat conduction. Here we consider the perfect gas, that is

$$p = R\rho\theta = A\rho^{\gamma}\exp\left(\frac{\gamma-1}{R}s\right), \quad e = \frac{R\theta}{\gamma-1} + const., \tag{1.2}$$

where s is the entropy, $\gamma > 1$ is the adiabatic exponent, and $A, R > 0$ are gas constants.

The initial values are given by

$$(\rho, u, \theta)(t=0, x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_+, u_+, \theta_+), x \to +\infty, \tag{1.3}$$

where (ρ_+, u_+, θ_+) is a constant state with ρ_+, θ_+ positive. The boundary values are the following:

$$(\rho, u, \theta)(t, x=0) = (\rho_-, u_-, \theta_-), \tag{1.4}$$

where $\rho_- > 0,\ \theta_- > 0,\ u_-$ are given constants. And of course the initial values (1.3) and the boundary condition (1.4) satisfy the compatible condition at the origin $(0, 0)$.

According to the sign of the velocity u_- on the boundary $\{x = 0\}$, the following three types of problems are proposed [17]:

(1) the inflow problem, i.e., the velocity $u_- > 0$;
(2) the outflow problem with $u_- < 0$;
(3) the impermeable wall problem, i.e., $u_- = 0$.

It should be remarked that in the cases (2) and (3), the density ρ_- can not be given on the boundary by the theory of well-posedness on the hyperbolic equation $(1.1)_1$.

In this paper, we are interested in the case of the inflow problem (1.1), (1.3)-(1.4). When $\kappa = \mu = 0$, the compressible system (1.1) becomes the inviscid Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \left[\rho(e + \frac{u^2}{2})\right]_t + [\rho u(e + \frac{u^2}{2}) + pu]_x = 0. \end{cases} \tag{1.5}$$

The Euler system (1.5) is a typical example of the hyperbolic conservation laws. It is well-known that the main feature of the solutions to the hyperbolic conservation laws is the formation of the shock wave no matter how smooth the initial values are. The Euler system (1.5) contains three basic wave patterns in the solutions to the Riemann problem. They are two nonlinear waves, called shock wave and rarefaction wave, and one linear wave called contact discontinuity. The above three dilation invariant wave solutions and their linear superpositions in the increasing order of characteristic speed, i.e., Riemann solutions, govern both local and large-time behavior of solutions to the Euler system. The invscid Euler system (1.5) is an ideal model in gas dynamics when the dissipation effects are neglected, thus it is of great importance to study the corresponding viscous system (1.1).

There has been a large literature on the large-time behavior of the solutions to Cauchy problem of the compressible Navier-Stokes equations (1.1) toward the viscous versions of the three basic wave patterns. In 1985, Matsumura-Nishihara [18] firstly proved the stability of the viscous shock wave to the isentropic compressible Navier-Stokes equations (i.e., the entropy s is assumed to be constant and the energy conservation law is not considered). Since then, many authors had been attracted to study the stability of the viscous wave patterns and much progress has been made. We refer to [3], [6], [7], [9], [11], [12], [13], [14], [15], [19], [21], [25], [26] and some references therein. All these results show that the large-time behavior of the solutions to Cauchy problem are basically governed by the Riemann problem of the corresponding Euler equations.

Recently, the initial-boundary value problem (IBVP) of (1.1) attracts increasing interest because it has more physical meanings and of course produces some new mathematical difficulties due to the boundary effect. Not only basic wave patterns but also a new wave, which is called boundary layer solution (BL-solution for brevity) [17], may appear in the IVBP case. Matsumura [17] proposes a criterion on the question when the BL-solution forms to the isentropic Navier-Stokes equations. The argument is also true to the full Navier-Stokes equations (1.1). Consider the Riemann problem to the Euler equations (1.5), where the initial right end state is given by the far field state (ρ_+, u_+, θ_+) in (1.3), and the left end state (ρ_-, u_-, θ_-) is given by the all possible states which are consistent with the boundary condition (1.4) at $\{x = 0\}$. Note that for the outflow problem, ρ_- can not be prescribed and is free on the boundary. On the one hand, when the left end state is uniquely determined so that the value at the boundary $\{x = 0\}$ of the solution to the Riemann problem is consistent with the boundary condition, we expect no BL-solution occurs. On the other hand, if the value of the Riemann problem's solution on the boundary is not consistent with the boundary condition for any admissible left end state, we expect a BL-solution which compensates the gap comes up. Such BL-solution could be constructed by the stationary solution to Navier-Stokes equations. The existence and stability of the BL-solution (to the inflow or outflow problems, to the isentropic or full Navier-Stokes equations) are studied extensively by many authors, see [2], [4], [8], [16], [17] [20], [22], [27], etc.. For the inflow problem of the full Naier-Stokes equation (1.1)–(1.4), Huang-Li-Shi [2] proved the stability of the BL-solution in some cases. More precisely, they show that when $(\rho_\pm, u_\pm, \theta_\pm)$ both belong to the subsonic region, the BL-solution is expected and the stability of this BL-solution and its superposition with the 3-rarefaction wave is proved under some smallness assumptions. Notice that both the BL-solution and the rarefaction wave must be weak enough. When the boundary value (ρ_-, u_-, θ_-) belongs to the supersonic region, there is no BL-solution. Thus the large-time behavior of the solution is expected to be same as that of the Cauchy problem and the stability of the rarefaction waves is given.

In this paper, firstly we give the existence (or non-existence) of the BL-solution when the right end state (ρ_+, u_+, θ_+) belongs to the subsonic, transonic and supersonic regions, respectively. The rigorous proof is given in Appendix. Notice that it is more natural to present the classifications according to the locations of the right end state (ρ_+, u_+, θ_+) from the qualitative theory of the autonomous ODE system. Then we prove the stability of not only the single contact wave but also the superposition of the BL-solution (subsonic case), the viscous contact wave and the 3-rarefaction wave to the inflow problem (1.1)–(1.4). Here the amplitude of the rarefaction wave can be arbitrarily large.

Now we briefly review some key analytic techniques in studying the stability of the basic wave patterns. The strict monotonicity of the corresponding characteristic speed along the wave profiles plays a crucial role in stability analysis of the viscous shock wave and rarefaction wave. Precisely speaking, the shock wave is a compression wave, so the characteristic speed is monotone decreasing in the shock profile. Thus anti-derivative variable to the perturbation should be introduced in the stability analysis. While the rarefaction wave is an expansion wave and the characteristic speed is monotone increasing along the rarefaction wave, thus the direct energy estimates to the perturbation itself are available. However, the characteristic speed along the contact wave is constant, and the spatial derivative of the velocity changes signs along the contact wave profile. Due to the degenerate characteristics, the stability of the contact wave profile to the compressible Navier-Stokes system (1.1) is just proved by [6] and [9] in 2005, twenty years later than the nonlinear wave in 1985. In [6] and [9], the anti-derivative variable to the perturbation is introduced and the proof framework is motivated by the viscous shock profile. Notice that a convergence rate of the order of $(1+t)^{-\frac{1}{4}}$ in sup-norm is a by-product of the estimation. However, there is no convergence rate obtained so far for the viscous shock wave and the rarefaction wave.

Recently, Huang-Matsumura-Xin [7] obtained a new estimate on the heat kernel which can be applied to the study of the stability of the viscous contact wave in the framework of the rarefaction wave, see [3] or Lemma 3.4 in the present paper. Namely, the anti-derivative variable of the perturbation is not needed and the estimations to the perturbation itself are also suitable to get the stability of the viscous contact wave. But the time-decay rate can not be gotten as a compensation. More importantly, the advantage of this framework is that it can be used to study the stability of the contact wave to the IBVP of (1.1) since the boundary terms could be treated conveniently. We will make full use of this new estimate on heat kernel to study the inflow problem (1.1) and get the expected results.

The novelty of the paper lies in the following three aspects: (1) The rigorous proof and the classifications of the existence (or non-existence) of the BL-solution to the inflow problem. (2) The stability of the superposition of three different wave patterns (the BL-solution, the viscous contact wave and the rarefaction wave). (3) The large amplitude of the rarefaction wave in the superposition wave. The main difficulties in our proofs are how to deal with the boundary terms and the interactions of three different wave patterns.

Because the system (1.1) we consider is in one dimension of the space variable x, it is convenient to use the following Lagrangian coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y,t)dy, \qquad t \Rightarrow t.$$

Thus the system (1.1) can be transformed into the following moving boundary problem of Navier-Stokes equations in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, & x > \sigma_- t, \quad t > 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, & x > \sigma_- t, \quad t > 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = \kappa(\frac{\theta_x}{v})_x + \mu(\frac{uu_x}{v})_x, & x > \sigma_- t, \quad t > 0, \\ (v,u,\theta)(t, x = \sigma_- t) = (v_-, u_-, \theta_-), & u_- > 0, \\ (v,u,\theta)(t=0,x) = (v_0,u_0,\theta_0)(x) \to (v_+,u_+,\theta_+), & \text{as} \quad x \to +\infty, \end{cases} \tag{1.6}$$

where $v(t,x) = \frac{1}{\rho(t,x)}$ represents the specific volume of the gas, and the boundary moves with the constant speed $\sigma_- = -\frac{u_-}{v_-} < 0$. Now we have that for the perfect gas,

$$p = \frac{R\theta}{v} = Av^{-\gamma} \exp{(\frac{\gamma - 1}{R}s)}, \qquad e = \frac{R}{\gamma - 1}\theta + \text{const}. \tag{1.7}$$

In order to fix the moving boundary $x = \sigma_- t$, we introduce a new variable $\xi = x - \sigma_- t$. Then we have the half space problem

$$\begin{cases} v_t - \sigma_- v_\xi - u_\xi = 0, & \xi > 0, \quad t > 0, \\ u_t - \sigma_- u_\xi + p_\xi = \mu(\frac{u_\xi}{v})_\xi, & \xi > 0, \quad t > 0, \\ (e + \frac{u^2}{2})_t - \sigma_-(e + \frac{u^2}{2})_\xi + (pu)_\xi = \kappa(\frac{\theta_\xi}{v})_\xi + \mu(\frac{uu_\xi}{v})_\xi, & \xi > 0, \quad t > 0, \\ (v,u,\theta)(t, \xi = 0) = (v_-, u_-, \theta_-), & u_- > 0, \\ (v,u,\theta)(t=0,\xi) = (v_0,u_0,\theta_0)(\xi) \to (v_+,u_+,\theta_+), & \text{as} \quad \xi \to +\infty. \end{cases} \tag{1.8}$$

Given the right end state (v_+, u_+, θ_+), we can define the following wave curves in the phase space (v, u, θ) with $v > 0$ and $\theta > 0$.

- Contact wave curve:

$$CD(v_+, u_+, \theta_+) = \{(v, u, \theta) | u = u_+, p = p_+, v \not\equiv v_+\}. \tag{1.9}$$

- BL-solution curve (subsonic case, i.e., $(v_+, u_+, \theta_+) \in \Omega^+_{sub}$):

$$BL(v_+, u_+, \theta_+) = \Big\{(v, u, \theta) \Big| \frac{u}{v} = -\sigma_- = \frac{u_+}{v_+}, (u, \theta) \in \mathcal{M}(u_+, \theta_+)\Big\}, \tag{1.10}$$

where $\mathcal{M}(u_+,\theta_+)$ is the center-stable manifold defined in Lemma 2.1 below.

• 3-Rarefaction wave curve:

$$R_3(v_+,u_+,\theta_+) := \left\{ (v,u,\theta) \middle| v > v_+,\ u = u_+ - \int_{v_+}^{v} \lambda_3(\eta,s_+)\, d\eta,\ s(v,\theta) = s_+ \right\}, \quad (1.11)$$

where $s_+ = s(v_+,\theta_+)$ and $\lambda_3 = \lambda_3(v,s)$ is the third characteristic speed given in (2.1).

Our main stability results are, roughly speaking, as follows:

(I). If the state $(v_-,u_-,\theta_-) \in \mathrm{CD}(v_+,u_+,\theta_+)$, then the viscous contact wave is asymptotic stable under some smallness conditions which are given in Theorem 2.1.

(II). If the state $(v_-,u_-,\theta_-) \in \mathrm{BL}\text{–}\mathrm{CD}\text{–}\mathrm{R}_3(v_+,u_+,\theta_+)$, then there exist a unique state $(v_*,u_*,\theta_*) \in \Omega^+_{sub}$ and a unique state (v^*,u^*,θ^*), such that $(v_-,u_-,\theta_-) \in \mathrm{BL}(v_*,u_*,\theta_*)$, $(v_*,u_*,\theta_*) \in \mathrm{CD}(v^*,u^*,\theta^*)$ and $(v^*,u^*,\theta^*) \in \mathrm{R}_3(v_+,u_+,\theta_+)$ and the superposition of the BL-solution, the viscous contact wave and the rarefaction wave is asymptotically stable provided that $|(u_- - u_*, \theta_- - \theta_*)|$ and $|v_* - v^*|$ are suitably small and the conditions in Theorem 2.2 hold. It is remarked that the BL-solution and the viscous contact wave must be weak but the rarefaction wave is not necessarily weak.

Notations: Throughout the paper several positive generic constants are denoted by $c, c_0, c_1, \cdots$ or $C, C_1, C_2, \cdots$ without confusions. The small constant $\nu > 0$ is used in Young inequality

$$ab \le \nu a^{p_1} + C_\nu b^{p_2}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1,$$

where C_ν is the constant depending on ν. For functional space, $H^l(\mathbf{R}^+)$ denotes the l-order Sobolev space with the norm

$$\|f\|_l = \left(\sum_{i=0}^{l} \|\partial_x^i f\|^2 \right)^{\frac{1}{2}}, \quad \text{where} \|f\| = \|f\|_{L^2}.$$

2 Preliminaries and main results

In this section, we will show the wave patterns considered in the paper. We start with the BL-solution.

2.1 BL-solution

It is known that the hyperbolic system (1.5) has three characteristic speeds

$$\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{\gamma p}{v}}. \qquad (2.1)$$

The sound speed $c(v,\theta)$ and the Mach number M are defined by

$$c(v,\theta) = v\sqrt{\frac{\gamma p}{v}} = \sqrt{R\gamma\theta}, \tag{2.2}$$

and

$$M(v,u,\theta) = \frac{|u|}{c(v,\theta)} = \frac{|u|}{\sqrt{R\gamma\theta}}, \tag{2.3}$$

respectively.

We divide the phase space $\{(v,u,\theta), v>0, \theta>0\}$ into three regions:

$$\begin{array}{ll} \Omega_{sub} & := \{(v,u,\theta)|\ M(v,u,\theta) < 1\ \}, \\ \Gamma_{trans} & := \{(v,u,\theta)|\ M(v,u,\theta) = 1\ \}, \\ \Omega_{super} & := \{(v,u,\theta)|\ M(v,u,\theta) > 1\ \}. \end{array} \tag{2.4}$$

Call them the subsonic, transonic and supersonic regions, respectively. If adding the alternative condition $u>0$ or $u<0$, then we have six connected subsets $\Omega^{\pm}_{sub}$, $\Gamma^{\pm}_{trans}$, and $\Omega^{\pm}_{super}$.

When $(v_-,u_-,\theta_-) \in \Omega^{+}_{sub}$, we have

$$\lambda_1(v_-,u_-,\theta_-) < \sigma_- < 0,$$

hence the existence of the traveling wave solution

$$\begin{array}{l} (V^B,U^B,\Theta^B)(\xi), \quad \xi = x - \sigma_- t, \\ (V^B,U^B,\Theta^B)(0) = (v_-,u_-,\theta_-), \quad (V^B,U^B,\Theta^B)(+\infty) = (v_+,u_+,\theta_+), \end{array} \tag{2.5}$$

to (1.6), or the stationary solution to (1.8) is expected. We call this traveling wave solution $(V^B,U^B,\Theta^B)(\xi)$ the boundary layer solution to the inflow problem (1.6). Note that the speed of the traveling wave is just the speed of the moving boundary of (1.6).

In the following, we will give the existence (or non-existence) of BL-solution to the inflow problem (1.6). From (2.5), BL-solution $(V^B,U^B,\Theta^B)(\xi)$ satisfies the following ODE system

$$\begin{cases} -\sigma_-(V^B)' - (U^B)' = 0, \qquad {}' := \dfrac{d}{d\xi} & \xi > 0, \\ -\sigma_-(U^B)' + (P^B)' = \mu(\dfrac{(U^B)'}{V^B})', & \xi > 0, \\ -\sigma_-(\dfrac{R}{\gamma-1}\Theta^B + \dfrac{(U^B)^2}{2})' + (P^B U^B)' = \kappa(\dfrac{(\Theta^B)'}{V^B})' + \mu(\dfrac{U^B(U^B)'}{V^B})', & \xi > 0, \\ (V^B,U^B,\Theta^B)(0) = (v_-,u_-,\theta_-), \quad (V^B,U^B,\Theta^B)(+\infty) = (v_+,u_+,\theta_+). \end{cases} \tag{2.6}$$

where $P^B := p(V^B,\Theta^B) = \frac{R\Theta^B}{V^B}$.

Integrating the system (2.6) over $(\xi, +\infty)$ implies that

$$\begin{cases} -\sigma_-(V^B - v_+) - (U^B - u_+) = 0, \\ \mu \dfrac{(U^B)'}{V^B} = -\sigma_-(U^B - u_+) + R\Big(\dfrac{\Theta^B}{V^B} - \dfrac{\theta_+}{v_+}\Big), \\ \kappa \dfrac{(\Theta^B)'}{V^B} = -\sigma_- \dfrac{R}{\gamma - 1}(\Theta^B - \theta_+) + p_+(U^B - u_+) + \dfrac{\sigma_-}{2}(U^B - u_+)^2, \\ (U^B, \Theta^B)(0) = (u_-, \theta_-), \quad (U^B, \Theta^B)(+\infty) = (u_+, \theta_+). \end{cases} \tag{2.7}$$

Let $\xi = 0$ in $(2.7)_1$, we have

$$\sigma_- = -\frac{u_-}{v_-} = -\frac{U^B}{V^B} = -\frac{u_+}{v_+}, \tag{2.8}$$

which is the first condition of BL-solution curve in (1.10).

From the fact $u_- > 0$ and $v_\pm > 0$, we find that u_+ must satisfy

$$u_+ = \frac{u_-}{v_-} v_+ > 0. \tag{2.9}$$

Rewrite $(2.7)_2$–$(2.7)_4$ as

$$\begin{cases} (U^B)' = -\dfrac{\sigma_-}{\mu} V^B (U^B - u_+) + \dfrac{R}{\mu}\Big(\Theta^B - \dfrac{\theta_+}{v_+} V^B\Big), \\ (\Theta^B)' = -\dfrac{R\sigma_- V^B}{\kappa(\gamma - 1)}(\Theta^B - \theta_+) + \dfrac{p_+}{\kappa} V^B (U^B - u_+) + \dfrac{\sigma_- V^B}{2\kappa}(U^B - u_+)^2, \\ (U^B, \Theta^B)(0) = (u_-, \theta_-), \quad (U^B, \Theta^B)(+\infty) = (u_+, \theta_+). \end{cases} \tag{2.10}$$

Denote

$$\bar{U}^B = U^B - u_+, \quad \bar{\Theta}^B = \Theta^B - \theta_+, \tag{2.11}$$

and

$$J = \begin{pmatrix} \frac{u_+^2 - R\theta_+}{\mu u_+} & \frac{R}{\mu} \\ \frac{R\theta_+}{\kappa} & \frac{R u_+}{\kappa(\gamma - 1)} \end{pmatrix} = \begin{pmatrix} \frac{(M_+^2\gamma - 1)u_+}{M_+^2 \gamma \mu} & \frac{R}{\mu} \\ \frac{u_+^2}{M_+^2 \gamma \kappa} & \frac{R u_+}{\kappa(\gamma - 1)} \end{pmatrix}, \tag{2.12}$$

where $M_+ = M(v_+, u_+, \theta_+)$.

Then we obtain the automatous ODE system

$$\begin{cases} \begin{pmatrix} \bar{U}^B \\ \bar{\Theta}^B \end{pmatrix}' = J \begin{pmatrix} \bar{U}^B \\ \bar{\Theta}^B \end{pmatrix} + \begin{pmatrix} F_1(\bar{U}^B, \bar{\Theta}^B) \\ F_2(\bar{U}^B, \bar{\Theta}^B) \end{pmatrix} \\ (\bar{U}^B, \bar{\Theta}^B)(0) = (u_- - u_+, \theta_- - \theta_+), \quad (\bar{U}^B, \bar{\Theta}^B)(+\infty) = (0, 0). \end{cases} \tag{2.13}$$

where

$$
\begin{aligned}
F_1(\bar{U}^B, \bar{\Theta}^B) &= \frac{1}{\mu}(\bar{U}^B)^2, \\
F_2(\bar{U}^B, \bar{\Theta}^B) &= \left(\frac{R\theta_+}{\kappa u_+} - \frac{u_+}{2\kappa}\right)(\bar{U}^B)^2 + \frac{R}{\kappa(\gamma-1)}\bar{U}^B\bar{\Theta}^B - \frac{1}{2\kappa}(\bar{U}^B)^3.
\end{aligned}
\tag{2.14}
$$

Now we state the existence results of the solution to (2.13) while its proof will be shown in Appendix.

Lemma 2.1 (Existence of BL-solution) *Suppose that* $v_\pm > 0$, $u_- > 0$, $\theta_\pm > 0$ *and let* $\delta^B = |(u_+ - u_-, \theta_+ - \theta_-)|$. *If* $u_+ \le 0$, *then there is no solution to (2.10) or (2.13). If* $u_+ > 0$, *then there exists a suitably small constant* $\delta > 0$ *such that if* $0 < \delta^B \le \delta$, *then*

Case I. Supersonic case: $M_+ > 1$. *Then there is no solution to (2.10) or (2.13).*

Case II. Transonic case: $M_+ = 1$. *Then* (u_+, θ_+) *is a saddle-knot point to (2.13). Precisely, there exists a unique trajectory* Γ *tangent to the line*

$$
\mu u_+(U^B - u_+) - \kappa(\gamma-1)(\Theta^B - \theta_+) = 0
$$

at the point (u_+, θ_+). *For each* $(u_-, \theta_-) \in \Gamma$, *there exists a unique solution* (U^B, Θ^B) *satisfying*

$$
\left|\frac{d^n}{d\xi^n}(U^B - u_+, \Theta^B - \theta_+)\right| \le C\frac{(\delta^B)^n}{(1+\delta^B\xi)^n}, \quad n = 0, 1, 2, \ldots, \qquad \xi \in \mathbb{R}_+. \tag{2.15}
$$

Case III. Subsonic case: $M_+ < 1$. *Then the equilibrium point* (u_+, θ_+) *is a saddle point of (2.13). Precisely, there exists a center-stable manifold* $\mathcal{M}$ *tangent to the line*

$$
(1 + a_2 c_2 u_+)(U^B - u_+) - a_2(\Theta^B - \theta_+) = 0
$$

on the opposite directions at the point (u_+, θ_+). *Here* c_2 *is one of the solutions to the equation*

$$
y^2 + \left(\frac{M_+^2\gamma - 1}{M_+^2 R\gamma} - \frac{\mu}{\kappa(\gamma-1)}\right)y - \frac{\mu}{M_+^2 R\gamma\kappa} = 0
$$

and $a_2 = -\frac{R}{\mu(\lambda_A^1 - \lambda_A^2)}$ *with* $\lambda_A^1 > 0$, $\lambda_A^2 < 0$ *are the two eigenvalues of the matrix* A. *Only when* $(u_-, \theta_-) \in \mathcal{M}$, *does there exist a unique solution* $(U^B, \Theta^B) \subset \mathcal{M}$ *satisfying*

$$
\left|\frac{d^n}{d\xi^n}(U^B - u_+, \Theta^B - \theta_+)\right| \le C\delta^B e^{-c\xi}, \quad n = 0, 1, 2, \ldots, \qquad \xi \in \mathbb{R}_+. \tag{2.16}
$$

Remark: This Lemma is the first one for the classifications of the BL-solution to the inflow problem (1.1). The stability of the single BL-solution in subsonic case (Case III) is proved in [2]. In this paper, we are concerned with the stability of the superposition of this subsonic BL-solution with viscous contact wave and rarefaction wave. As for the stability of the BL-solution in transonic case (Case II) and its superposition with other wave patterns, it is in consideration [23].

2.2 Viscous contact wave

If $(v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+)$, i.e.,

$$u_- = u_+,\ p_- = p_+, \tag{2.17}$$

then the following Riemann problem of the Euler system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = 0, \end{cases} \tag{2.18}$$

with Riemann initial value

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$

admits a single contact discontinuity solution

$$(v, u, \theta)(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0,\ t > 0, \\ (v_+, u_+, \theta_+), & x > 0,\ t > 0. \end{cases}$$

From [6], the viscous version of the above contact discontinuity, called viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$, could be defined by

$$\begin{aligned} &\Theta^{CD}(t, x) = \Theta^{Sim}(\frac{x}{\sqrt{1+t}}), \\ &V^{CD}(t, x) = \frac{R\Theta^{CD}(t, x)}{p_+}, \\ &U^{CD}(t, x) = u_+ + \frac{\kappa(\gamma - 1)}{R\gamma} \frac{\Theta_x^{CD}(t, x)}{\Theta^{CD}(t, x)}, \end{aligned} \tag{2.19}$$

where $\Theta^{Sim}(\eta)$, $\eta = \dfrac{x}{\sqrt{1+t}}$, is the unique self-similar solution of the nonlinear diffusion equation

$$\begin{cases} \Theta_t = \dfrac{\kappa p_+(\gamma - 1)}{R^2\gamma} \left(\dfrac{\Theta_x}{\Theta}\right)_x, \\ \Theta(\pm\infty) = \theta_\pm. \end{cases} \tag{2.20}$$

Thus the viscous contact wave defined in (2.19) satisfies the following property

$$|\Theta^{CD} - \theta_\pm| + (1+t)^{\frac{1}{2}}|\Theta_x^{CD}| + (1+t)|\Theta_{xx}^{CD}| + (1+t)^{\frac{3}{2}}|\Theta_{xxx}^{CD}| = O(1)\delta^{CD} e^{-\frac{c_0 x^2}{1+t}}, \tag{2.21}$$

as $|x| \to +\infty$, where $\delta^{CD} = |\theta_+ - \theta_-|$ is the amplitude of the viscous contact wave and c_0 is a positive constant. Note that $\xi = x - \sigma_- t$, then $(V^{CD}, U^{CD}, \Theta^{CD})(t, x) =$

$(V^{CD}, U^{CD}, \Theta^{CD})(t, \xi + s_- t)$ satisfies the system

$$\begin{cases} V_t^{CD} - \sigma_- V_\xi^{CD} - U_\xi^{CD} = 0, \\ U_t^{CD} - \sigma_- U_\xi^{CD} + P_\xi^{CD} = \mu(\dfrac{U_\xi^{CD}}{V_C})_\xi + \bar{Q}_1, \\ \dfrac{R}{\gamma - 1}(\Theta_t^{CD} - \sigma_- \Theta_\xi^{CD}) + P^{CD} U_\xi^{CD} = \kappa(\dfrac{\Theta_\xi^{CD}}{V^{CD}})_\xi + \mu \dfrac{(U_\xi^{CD})^2}{V^{CD}} + \bar{Q}_2, \end{cases} \tag{2.22}$$

where $P^{CD} = \dfrac{R\Theta^{CD}}{V^{CD}} = p_+ = p_-$ and the error terms $\bar{Q}_1,\ \bar{Q}_2$ are given by

$$\begin{aligned} \bar{Q}_1 &= (U_t^{CD} - \sigma_- U_\xi^{CD}) - \mu(\frac{U_\xi^{CD}}{V^{CD}})_\xi = O(1)(|\Theta_\xi^{CD}|^3 + |\Theta_{\xi\xi\xi}^{CD}| + |\Theta_{\xi\xi}^{CD}||\Theta_\xi^{CD}|) \\ &= O(1)\delta^{CD}(1+t)^{-\frac{3}{2}} e^{-\frac{c_0(\xi+\sigma_- t)^2}{1+t}}, \qquad \text{as } |\xi + \sigma_- t| \to +\infty, \\ \bar{Q}_2 &= -\mu \frac{(U_\xi^{CD})^2}{V^{CD}} = O(1)(|\Theta_\xi^{CD}|^4 + |\Theta_{\xi\xi}^{CD}|^2) \\ &= O(1)(\delta^{CD})^2(1+t)^{-2} e^{-\frac{c_0(\xi+\sigma_- t)^2}{1+t}}, \qquad \text{as } |\xi + \sigma_- t| \to +\infty. \end{aligned} \tag{2.23}$$

2.3 Rarefaction wave

If $(v_-, u_-, \theta_-) \in R_3(v_+, u_+, \theta_+)$, then there exists a 3-rarefaction wave $(v^r, u^r, s^r)(x/t)$ which is the global (in time) weak solution of the following Riemann problem

$$\begin{cases} v_t^r - u_x^r = 0, \\ u_t^r + p_x(v^r, \theta^r) = 0, \qquad\qquad t > 0, x \in \mathbf{R}, \\ \dfrac{R}{\gamma - 1}\theta_t^r + p(v^r, \theta^r) u_x^r = 0, \\ (v^r, u^r, \theta^r)(0, x) = \begin{cases} (v_-, u_-, \theta_-), x < 0, \\ (v_+, u_+, \theta_+), x > 0. \end{cases} \end{cases} \tag{2.24}$$

From [5], it is convenient to construct the approximated rarefaction wave $(V^R, U^R, \Theta^R)(t, x)$ to the inflow problem (1.6) by the solution of the Burgers equation

$$\begin{cases} w_t + w w_x = 0, \\ w(0, x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q \delta^r \displaystyle\int_0^{\varepsilon x} y^q e^{-y}\, dy, & x \geq 0, \end{cases} \end{cases} \tag{2.25}$$

where $\delta^r = w_+ - w_-$, $q \geq 16$ is some fixed constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y} dy = 1$, and $\varepsilon \leq 1$ is a small positive constant to be determined later. The solution of the Burgers equation (2.25) have the following properties:

Lemma 2.2 ([5]) *Let $0 < w_- < w_+$, Burgers equation (2.25) has a unique smooth solution $w(t,x)$ satisfying*

i) $w_- \le w(t,x) < w_+,\ w_x(t,x) \ge 0$,

ii) For any p $(1 \le p \le \infty)$, there exists a constant C_{pq} such that

$$\begin{aligned}
&\| w_x(t) \|_{L^p} \le C_{pq} \min\{\delta_r \varepsilon^{1-1/p},\ \delta_r^{1/p} t^{-1+1/p}\},\\
&\| w_{xx}(t) \|_{L^p} \le C_{pq} \min\{\delta_r \varepsilon^{2-1/p},\ (\delta_r^{1/p} + \delta_r^{1/q}) t^{-1+1/q}\},
\end{aligned}$$

iii) If $x < w_- t$, then $w(t,x) \equiv w_-$,

iv) $\sup\limits_{x\in\mathbb{R}} |w(t,x) - w^r(x/t)| \to 0$, as $t \to \infty$.

Thus we construct the approximated rarefaction wave $(V^R, U^R, \Theta^R)(t,x)$ by

$$\begin{cases}
S^R = s(V^R, \Theta^R) = s_+,\\
w(1+t,x) = \lambda_3(V^R(t,x), s_+),\\
U^R(t,x) = u_+ - \displaystyle\int_{v_+}^{V^R(t,x)} \lambda_3(v, s_+) dv.
\end{cases} \tag{2.26}$$

Note that $\xi = x - \sigma_- t$, then the smoothed 3-rarefaction wave $(V^R, U^R, \Theta^R)(t,\xi)$ defined above satisfies

$$\begin{cases}
V_t^R - \sigma_- V_\xi^R - U_\xi^R = 0,\\
U_t^R - \sigma_- U_\xi^R + P_\xi^R = 0, \qquad\qquad \xi > 0,\ t > 0,\\
\dfrac{R}{\gamma - 1}(\Theta_t^R - \sigma_- \Theta_\xi^R) + P^R U_\xi^R = 0,\\
(V^R, U^R, \Theta^R)(t,0) = (v_-, u_-, \theta_-), \quad (V^R, U^R, \Theta^R)(t,+\infty) = (v_+, u_+, \theta_+),
\end{cases} \tag{2.27}$$

where $P^R := p(V^R, \Theta^R)$.

Lemma 2.3 ([5]) Let $\delta^R = |(v_+, u_+, \theta_+) - (v_-, u_-, \theta_-)|$. The approximated rarefaction wave $(V^R, U^R, \Theta^R)(t,\xi)$ satisfies

i) For $\xi > 0$, $,t > 0$, $U_\xi^R(t,\xi) \ge 0$,

ii) For any p $(1 \le p \le \infty)$, there exists a constant C_{pq} such that for $t \ge 0$,

$$\begin{aligned}
&\| (V_\xi^R, U_\xi^R, \Theta_\xi^R)(t) \|_{L^p} \le C_{pq} \min\{\delta^R \varepsilon^{1-1/p},\ (\delta^R)^{1/p}(1+t)^{-1+1/p}\},\\
&\| (V_{\xi\xi}^R, U_{\xi\xi}^R, \Theta_{\xi\xi}^R)(t) \|_{L^p} \le C_{pq} \min\{\delta^R \varepsilon^{2-1/p},\ ((\delta^R)^{1/p} + (\delta^R)^{1/q})(1+t)^{-1+1/q}\},
\end{aligned}$$

iii) If $\xi + \sigma_- t \le \lambda_3(v_-, u_-, \theta_-)(1+t)$, then $(V^R, U^R, \Theta^R)(t,\xi) \equiv (v_-, u_-, \theta_-)$,

iv) $\sup\limits_{\xi\in\mathbb{R}_+} |(V^R, U^R, \Theta^R)(t,\xi) - (v^r, u^r, \theta^r)(\frac{\xi}{1+t})| \to 0$, as $t \to \infty$.

2.4 Main results

Now we can state our main results. The first one is in the following.

Theorem 2.1 If $(v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+)$. Let $(V^{CD}, U^{CD}, \Theta^{CD})(t,x)$ be the viscous contact wave defined in (2.19). There exists a small constant δ_0 such that if the wave amplitude δ^{CD} and the initial values satisfy

$$\delta^{CD} + \|(v_0 - V_0^{CD}, u_0 - U_0^{CD}, \theta_0 - \Theta_0^{CD})\|_1 \leq \delta_0,$$

then the moving boundary problem (1.6) or the half space problem (1.8) admits a unique global solution $(v, u, \theta)(t, \xi)$ satisfying

$$\begin{aligned}
&(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(t,\xi) \in C([0,+\infty), H^1(\mathbf{R}^+)),\\
&(v - V^{CD})_\xi(t,\xi) \in L^2(0,+\infty; L^2(\mathbf{R}^+)),\\
&((u - U^{CD})_\xi, (\theta - \Theta^{CD})_\xi)(t,\xi) \in L^2(0,+\infty; H^1(\mathbf{R}^+)),
\end{aligned}$$

and

$$\lim_{t\to+\infty} \sup_{\xi\in\mathbf{R}^+} |(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(t,\xi)| = 0.$$

Now we state our second result. If $(v_-, u_-, \theta_-) \in$ BL-CD-R$_3(v_+, u_+, \theta_+)$, then there exist states $(v_*, u_*, \theta_*) \in \Omega^+_{sub}$ and (v^*, u^*, θ^*), such that $(v_-, u_-, \theta_-) \in$ BL(v_*, u_*, θ_*), $(v_*, u_*, \theta_*) \in$ CD(v^*, u^*, θ^*) and $(v^*, u^*, \theta^*) \in$ R$_3(v_+, u_+, \theta_+)$. In fact, by$(v_-, u_-, \theta_-) \in$ BL(v_*, u_*, θ_*) and (1.10), we have

$$u_* = -\sigma_- v_*, \tag{2.28}$$

and by $(v^*, u^*, \theta^*) \in$ R$_3(v_+, u_+, \theta_+)$, (1.11) gives

$$u^* = u_+ - \int_{v_+}^{v^*} \lambda_3(\eta, s_+) d\eta, \quad v^* > v_+. \tag{2.29}$$

Thus the two curves (2.28) and (2.29) have a unique intersection point $u = \tilde{u}$ in (v, u) space. If $u_* = u^* = \tilde{u}$, then $v_* = v^*$, thus there is no contact wave. By $(v_*, u_*, \theta_*) \in$ CD(v^*, u^*, θ^*), we have

$$u_* = u^*, \qquad v_* \neq v^*.$$

Thus if $u_* = u^* \neq \tilde{u}$, then $v_* \neq v^*$ and there exists a contact wave. Among the three values $u_*(= u^*)$, v_* and v^*, only one is independent, the other two can be determined accordingly.

Now assume that $u_*(= u^*)$ is given, then from (2.28) and (2.29), we can determine v_* and v^* by

$$v_* = \frac{u_*}{-\sigma_-}, \qquad v^* = v_+ \left[\frac{\gamma - 1}{2A\sqrt{R\gamma\theta_+}}(u_* - u_+) + 1\right]^{\frac{2}{1-\gamma}}.$$

By the definition of the rarefaction wave curve R_3 in (1.11), we have $s(v_*, \theta_*) = s_+$, i.e.,

$$\theta^* = \theta_+ \left(\frac{v_+}{v^*}\right)^{\gamma-1} = \theta_+ \left[\frac{\gamma - 1}{2A\sqrt{R\gamma\theta_+}}(u_* - u_+) + 1\right]^2.$$

Again by the contact wave curve (1.9), one must have

$$p_* = p^*,$$

thus

$$\theta_* = \frac{\theta^*}{v^*} v_* = \frac{\theta_+}{-\sigma_- v_+} u_* \left[\frac{\gamma - 1}{2A\sqrt{R\gamma\theta_+}} (u_* - u_+) + 1 \right]^{\frac{2\gamma}{\gamma-1}}. \tag{2.30}$$

So if u_* large enough, then (u_*, θ_*) in (2.30) must belong to the region $\Omega_{sub}^+ := \{(u_*, \theta_*) | 0 < u_* < \sqrt{R\gamma\theta_*}\}$. Moreover, from the definition of the BL-solution, $(u_-, \theta_-) \in \mathcal{M}(u_*, \theta_*)$. Thus (u_*, θ_*) can be determined uniquely if (u_-, θ_-) is given suitably.

Define the superposition wave $(V, U, \Theta)(t, \xi)$ by

$$\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix} (t, \xi) = \begin{pmatrix} V^B + V^{CD} + V^R \\ U^B + U^{CD} + U^R \\ \Theta^B + \Theta^{CD} + \Theta^R \end{pmatrix} (t, \xi) - \begin{pmatrix} v_* + v^* \\ u_* + u^* \\ \theta_* + \theta^* \end{pmatrix}, \tag{2.31}$$

where $(V^B, U^B, \Theta^B)(t, \xi)$ is the subsonic BL-solution defined in Lemma 2.1 (Case II) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*), $(V^{CD}, U^{CD}, \Theta^{CD})(t, \xi)$ is the viscous contact wave defined in (2.19) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v^*, u^*, θ^*), respectively, and $(V^R, U^R, \Theta^R)(t, \xi)$ is the smoothed 3-rarefaction wave defined in (2.26) with the left state (v_-, u_-, θ_-) replaced by (v^*, u^*, θ^*).

Theorem 2.2 If $(v_-, u_-, \theta_-) \in \text{BL-CD-R}_3(v_+, u_+, \theta_+)$. Let $(V, U, \Theta)(t, x)$ be the superposition of the BL-solution, the viscous contact wave and the rarefaction wave defined in (2.31). There exists a small constant δ_0 such that if the BL-solution amplitude δ^B, the contact discontinuity amplitude δ^{CD} and the initial values satisfy

$$\delta^B + \delta^{CD} + \|(v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0)\|_1 \leq \delta_0,$$

then the inflow problem (1.6) or the half space problem (1.8) admits a unique global solution $(v, u, \theta)(t, \xi)$ satisfying

$$\begin{aligned} &(v - V, u - U, \theta - \Theta)(t, \xi) \in C([0, +\infty), H^1(\mathbf{R}^+)), \\ &(v - V)_\xi(t, \xi) \in L^2(0, +\infty; L^2(\mathbf{R}^+)), \\ &((u - U)_\xi, (\theta - \Theta)_\xi)(t, \xi) \in L^2(0, +\infty; H^1(\mathbf{R}^+)), \end{aligned}$$

and

$$\lim_{t \to +\infty} \sup_{\xi \in \mathbf{R}^+} |(v - V, u - U, \theta - \Theta)(t, \xi)| = 0.$$

3 Stability analysis

In this section we will prove our main stability results Theorem 2.1 and Theorem 2.2. We will focus on the proof of Theorem 2.2, i.e., the stability of the superposition wave. The proof of Theorem 2.1 is almost same as Theorem 2.2 and we will omit it for brevity.

Besides the intrinsic properties of the BL-solution, the viscous contact wave and the rarefaction wave in the stability analysis, the interaction between the wave patterns should be dealt with carefully in the stability analysis. Here we will use the elementary energy methods to prove Theorem 2.2 by the classical continuum procedure.

Firstly we will reformulate the system of the superposition wave $(V, U, \Theta)(t, \xi)$ defined in (2.31).

3.1 Reformulation of the problem

Recall the definition of the superposition wave $(V, U, \Theta)(t, \xi)$ defined in (2.31). Then we have

$$\begin{cases} V_t - \sigma_- V_\xi - U_\xi = 0, \\ U_t - \sigma_- U_\xi + P_\xi = \mu(\frac{U_\xi}{V})_\xi + Q_1, \qquad \xi > 0,\ t > 0, \\ \frac{R}{\gamma - 1}(\Theta_t - \sigma_- \Theta_\xi) + PU_\xi = \kappa(\frac{\Theta_\xi}{V})_\xi + \mu\frac{U_\xi^2}{V} + Q_2, \\ (V, U, \Theta)(t, 0) = (v_- + V^{CD} - v_*, u_- + U^{CD} - u_*, \theta_- + \Theta^{CD} - \theta_*)(t, 0), \end{cases} \tag{3.1}$$

where $P = p(V, \Theta) = \frac{R\Theta}{V}$, and the error terms Q_i $(i = 1, 2)$ are given by

$$\begin{aligned} Q_1 = \ & (P - P^B - P^{CD} - P^R)_\xi - \mu\left[(\frac{U_\xi}{V})_\xi - (\frac{U_\xi^B}{V^B})_\xi - (\frac{U_\xi^{CD}}{V^{CD}})_\xi\right] + \bar{Q}_1, \\ Q_2 = \ & (PU_\xi - P^B U_\xi^B - P^{CD} U_\xi^{CD} - P^R U_\xi^R) - \kappa\left[(\frac{\Theta_\xi}{V})_\xi - (\frac{\Theta_\xi^B}{V^B})_\xi - (\frac{\Theta_\xi^{CD}}{V^{CD}})_\xi\right] \\ & -\mu\left[\frac{U_\xi^2}{V} - \frac{(U_\xi^B)^2}{V^B} - \frac{(U_\xi^{CD})^2}{V^{CD}}\right] + \bar{Q}_2. \end{aligned} \tag{3.2}$$

and $\bar{Q}_i$ $(i = 1, 2)$ are the error terms defined in (2.23) to the viscous contact wave.

Due to the different propagation speeds of the BL-solution, the viscous contact wave and the rarefaction wave, we can get the following estimates of the error terms $Q_i (i = 1, 2)$:

$$\begin{aligned} Q_1 \ = \ & O(1)\Big[|(U_\xi^B, V_\xi^B, \Theta_\xi^B, U_{\xi\xi}^B)||(V - V^B, \Theta - \Theta^B, V_\xi^{CD}, U_\xi^{CD}, V_\xi^R, U_\xi^R)| \\ & + |(U_\xi^{CD}, V_\xi^{CD}, \Theta_\xi^{CD}, U_{\xi\xi}^{CD})||(V - V^{CD}, \Theta - \Theta^{CD}, V_\xi^R, U_\xi^R)| \\ & + |(V_\xi^R, \Theta_\xi^R)||(V - V^R, \Theta - \Theta^R)| + |(U_{\xi\xi}^R, U_\xi^R V_\xi^R)|\Big] + |\bar{Q}_1| \\ = \ & O(1)(\delta^B + \delta^{CD})e^{-c(|\xi| + t)} + O(1)(|U_{\xi\xi}^R|, |(U_\xi^R, V_\xi^R)|^2) + |\bar{Q}_1|, \end{aligned} \tag{3.3}$$

for some positive constant c independent of ξ and t. Similarly,

$$Q_2 = O(1)(\delta^B + \delta^{CD})e^{-c(\xi+t)} + O(1)(|\Theta^R_{\xi\xi}|, |(\Theta^R_\xi, V^R_\xi, U^R_\xi)|^2) + |\bar{Q}_2|. \tag{3.4}$$

Denote the perturbation by

$$(\phi, \psi, \zeta)(t, \xi) = (v, u, \theta)(t, \xi) - (V, U, \Theta)(t, \xi),$$

then we have the initial boundary value problem of the perturbation $(\phi, \psi, \zeta)(t, \xi)$:

$$\begin{cases} \phi_t - \sigma_-\phi_\xi - \psi_\xi = 0, & \xi > 0,\ t > 0, \\ \psi_t - \sigma_-\psi_\xi + (p - P)_\xi = \mu(\dfrac{u_\xi}{v} - \dfrac{U_\xi}{V})_\xi - Q_1, & \xi > 0,\ t > 0, \\ \dfrac{R}{\gamma-1}(\zeta_t - \sigma_-\zeta_\xi) + (pu_\xi - PU_\xi) = \kappa(\dfrac{\theta_\xi}{v} - \dfrac{\Theta_\xi}{V})_\xi + \mu(\dfrac{u^2_\xi}{v} - \dfrac{U^2_\xi}{V}) - Q_2, & \xi > 0,\ t > 0, \\ (\phi, \psi, \zeta)(t, \xi = 0) = (v_- - V, u_- - U, \theta_- - \Theta)(t, \xi = 0), \\ (\phi, \psi, \zeta)(t = 0, \xi) = (\phi_0, \psi_0, \zeta_0)(\xi) \to (0, 0, 0), \quad \text{as} \ \ \xi \to +\infty. \end{cases} \tag{3.5}$$

Since the local existence of the solution of (3.5) is well-known, we just state it and omit its proof for brevity.

Denote that

$$N(t) = \sup_{\tau\in[0,t]} \|(\phi, \psi, \zeta)(\tau, \cdot)\|^2_1, \tag{3.6}$$

and define the solution space by

$$X_{\underline{m},\overline{m}}(0, T) = \left\{ (\phi, \psi, \zeta)(t, \xi) \left| \begin{array}{l} (\phi, \psi, \zeta)(t, \xi) \in C([0, T]; H^1(\mathbf{R}^+)), \\ (\psi_\xi, \zeta_\xi) \in L^2(0, T; H^1(\mathbf{R}^+)), \\ \phi_\xi \in L^2(0, T; L^2(\mathbf{R}^+)), \quad N(T) \le \overline{m} \\ \quad \inf\limits_{[0,T]\times\mathbf{R}^+} \{(V + \phi), (\Theta + \zeta)\}(t, \xi) \ge \underline{m}. \end{array} \right. \right\} \tag{3.7}$$

for some positive constants $\underline{m}, \overline{m}$.

Proposition 3.1 (Local existence) Let $(\phi_0, \psi_0, \zeta_0) \in H^1(\mathbf{R}^+)$. If $\|(\phi_0, \psi_0, \zeta_0)\|_1 \le \overline{m}$ and $\inf_{[0,T]\times\mathbf{R}^+}\{(V + \phi), (\Theta + \zeta)\}(t, \xi) \ge \underline{m}$, then there exist δ_1 and $t_0 = t_0(\underline{m}, \overline{m}) > 0$ such that if the wave amplitude satisfies $\delta^B + \delta^{CD} < \delta_1$, then the half space problem (3.5) admits a unique solution $(\phi, \psi, \zeta)(t, \xi) \in X_{\frac{\underline{m}}{2},2\overline{m}}(0, t_0)$.

To prove Theorem 2.1, it is sufficient to prove the following a priori estimate.

Proposition 3.2 (A priori estimate) Suppose that the half space problem (3.5) has a solution $(\phi, \psi, \zeta)(t, \xi) \in X_{\frac{\underline{m}}{2},\varepsilon_0}[0, T]$ for a suitably small constant $\varepsilon_0 > 0$. There exists a positive constant δ_2, such that if the wave amplitude satisfies $\delta^B + \delta^{CD} < \delta_2$, then the solution $(\phi, \psi, \zeta)(t, \xi)$ satisfy that for $\forall\ t \in [0, T]$,

$$N(t) + \int_0^t \|\phi_\xi(\tau, \cdot)\|^2 + \|(\psi_\xi, \zeta_\xi)(\tau, \cdot)\|^2_1 d\tau \le C(N(0) + \delta_2 + \varepsilon^{\frac{1}{8}}), \tag{3.8}$$

where the positive constant C is independent of t.

3.2 Boundary estimates

In this section we will obtain the boundary estimates needed in the analysis below. From the definition of the viscous contact wave (2.19) and its property (2.21), we have the following estimates, which are very important in the boundary estimates,

$$(v_* - V^{CD}, u_* - U^{CD}, \theta_* - \Theta^{CD})(t,\xi) = O(1)\delta^{CD} e^{-\frac{c_0(\xi+\sigma_- t)^2}{1+t}}, \quad \text{as} \quad |\xi + \sigma_- t| \to \infty. \tag{3.9}$$

So on the boundary $\xi = 0$,

$$\begin{aligned}(\phi,\psi,\zeta)(t,\xi=0) \quad &= (v_- - V, u_- - U, \theta_- - \Theta)(t,\xi=0)\\ &= (v_* - V^{CD}, u_* - U^{CD}, \theta_* - \Theta^{CD})(t,\xi=0)\\ &= O(1)\delta^{CD} e^{-\frac{c_0(\sigma_- t)^2}{1+t}}\\ &= O(1)\delta^{CD} e^{-c_1 t}, \quad \text{as} \quad t \to +\infty.\end{aligned} \tag{3.10}$$

Thus we have the following lemma.

Lemma 3.1(Boundary estimates) There exists the positive constant C such that for any $t > 0$,

$$\begin{aligned}&\int_0^t |(\phi,\psi,\zeta)|^2(\tau,\xi=0)d\tau \le C(\delta^{CD})^2,\\ &\int_0^t [\mu(\frac{u_\xi}{v} - \frac{U_\xi}{V})\psi](\tau,\xi=0)d\tau \le \nu \int_0^t (\|\psi_{\xi\xi}\|^2 + \|\psi_\xi\|^2)d\tau + C_\nu(\delta^{CD})^2,\\ &\int_0^t [\kappa(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V})\frac{\zeta}{\theta}](\tau,\xi=0)d\tau \le \nu \int_0^t (\|\zeta_{\xi\xi}\|^2 + \|\zeta_\xi\|^2)d\tau + C_\nu(\delta^{CD})^2,\\ &\int_0^t [-\frac{\mu\sigma_-}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 + \psi\frac{\tilde{v}_\tau}{\tilde{v}}](\tau,\xi=0)d\tau \le \nu \int_0^t \|\psi_{\xi\xi}\|^2 d\tau + C_\nu \int_0^t \|\psi_\xi\|^2 d\tau + C(\delta^{CD})^2,\end{aligned} \tag{3.11}$$

where $\tilde{v} = \dfrac{v}{V}$, ν is a positive small constant to be determined later and C_ν is a positive constant depending on ν.

Proof: The proof of $(3.11)_1$ is a direct consequence of (3.10).

Now we prove $(3.11)_2$.

$$
\begin{aligned}
&\int_0^t [\mu(\frac{u_\xi}{v} - \frac{U_\xi}{V})\psi](\tau, \xi = 0) d\tau \\
&\leq C \int_0^t |\psi|(|\psi_\xi| + |U_\xi||\phi|)(\tau, \xi = 0) d\tau \\
&\leq C \int_0^t |\psi_\xi||\psi|(\tau, \xi = 0) d\tau + C \int_0^t (|\psi|^2 + |\phi|^2)(\tau, \xi = 0) d\tau \\
&\leq C \int_0^t \sup_{\xi \in [0, +\infty)} |\psi_\xi(\tau, \xi)| \cdot |\psi(t, \xi = 0)| d\tau + C(\delta^{CD})^2 \\
&\leq C \int_0^t \|\psi_\xi(\tau, \cdot)\|^{\frac{1}{2}} \cdot \|\psi_{\xi\xi}(\tau, \cdot)\|^{\frac{1}{2}} \cdot |\psi(\tau, \xi = 0)| d\tau + C(\delta^{CD})^2 \\
&\leq \nu \int_0^t (\|\psi_{\xi\xi}(\tau, \cdot)\|^2 + \|\psi_\xi(\tau, \cdot)\|^2) d\tau + C_\nu \int_0^t |\psi(\tau, \xi = 0)|^2 d\tau + C(\delta^{CD})^2 \\
&\leq \nu \int_0^t (\|\psi_{\xi\xi}(\tau, \cdot)\|^2 + \|\psi_\xi(\tau, \cdot)\|^2) d\tau + C_\nu(\delta^{CD})^2.
\end{aligned}
$$

So the proof of $(3.11)_2$ is completed. Similarly, we can obtain $(3.11)_3$.

Then we will verify the inequality $(3.11)_4$. Notice that

$$
\frac{\tilde{v}_\xi}{\tilde{v}} = \frac{v_\xi}{v} - \frac{V_\xi}{V} = \frac{\phi_\xi}{v} - \frac{V_\xi \phi}{vV}, \tag{3.12}
$$

and

$$
\begin{aligned}
\frac{\tilde{v}_t}{\tilde{v}} &= \frac{v_t}{v} - \frac{V_t}{V} = \frac{\sigma_- v_\xi + u_\xi}{v} - \frac{\sigma_- V_\xi + U_\xi}{V} \\
&= \sigma_- \frac{\tilde{v}_\xi}{\tilde{v}} + (\frac{u_\xi}{v} - \frac{U_\xi}{V}).
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int_0^t [-\frac{\mu\sigma_-}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 + \psi\frac{\tilde{v}_\tau}{\tilde{v}}](\tau, \xi = 0) d\tau \\
&\leq C \int_0^t (|\phi_\xi|^2 + |\psi_\xi|^2 + |\phi|^2 + |\psi|^2)(\tau, \xi = 0) d\tau \\
&\leq C \int_0^t (|\phi_\tau|^2 + |\psi_\xi|^2 + |\phi|^2 + |\psi|^2)(\tau, \xi = 0) d\tau \\
&\leq C \int_0^t \sup_{\xi \in [0, +\infty)} |\psi_\xi(\tau, \xi)|^2 d\tau + C(\delta^{CD})^2 \\
&\leq C \int_0^t \|\psi_\xi(\tau, \cdot)\| \|\psi_{\xi\xi}(\tau, \cdot)\| d\tau + C(\delta^{CD})^2 \\
&\leq \nu \int_0^t \|\psi_{\xi\xi}(\tau, \cdot)\|^2 d\tau + C_\nu \int_0^t \|\psi_\xi(\tau, \cdot)\|^2 d\tau + C(\delta^{CD})^2,
\end{aligned}
$$

where in the second inequality we have used the fact

$$\phi_\xi = \frac{\phi_\tau - \psi_\xi}{\sigma_-}.$$

Now Lemma 3.1 is proved.

3.3 Energy estimates

In this section we will prove the a priori estimate in Proposition 3.2. Firstly we have the following Lemma:

Lemma 3.2 There exist a constant $C > 0$ such that if the wave amplitudes δ^B, δ^{CD} and the constants ε, ε_0 are small enough, then we have $\forall\ t \in [0, T]$,

$$\begin{aligned} & \|(\phi,\psi,\zeta,\phi_\xi)(t,\cdot)\|^2 + \int_0^t \|(\phi_\xi,\psi_\xi,\zeta_\xi)(\tau,\cdot)\|^2 d\tau + \int_0^t \int_{\mathbf{R}^+} U_\xi^R(\phi^2+\zeta^2) d\xi d\tau \\ \leq\ & C\|(\phi_0,\psi_0,\zeta_0,\phi_{0\xi})\|^2 + C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi,\psi,\zeta)(\cdot,\tau)\|^2 d\tau + 1\Big] \\ & + C\nu \int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})(\tau,\cdot)\|^2 d\tau + C\delta^{CD} \int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} e^{\frac{-c_0(\xi+\sigma_-\tau)^2}{1+\tau}} |(\phi,\zeta)|^2 d\xi d\tau. \end{aligned} \tag{3.13}$$

Proof: Let

$$\Phi(z) = z - 1 - \ln z.$$

Similar in [9], we can get the following estimate

$$\begin{aligned} & I_{1t}(t,\xi) + H_{1\xi}(t,\xi) + \mu\frac{\Theta\psi_\xi^2}{v\theta} + \kappa\frac{\Theta\zeta_\xi^2}{v\theta^2} + PU_\xi^R\left[\Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V})\right] \\ & = Q_3 - Q_1\psi - Q_2\frac{\zeta}{\theta}, \end{aligned} \tag{3.14}$$

where

$$I_1(t,\xi) = R\Theta\Phi(\frac{v}{V}) + \frac{\psi^2}{2} + \frac{R\Theta}{\gamma-1}\Phi(\frac{\theta}{\Theta}), \tag{3.15}$$

$$H_1(t,\xi) = -\sigma_- I_1(t,\xi) + (p-P)\psi - \mu(\frac{u_\xi}{v} - \frac{U_\xi}{V})\psi - \kappa(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V})\frac{\zeta}{\theta}, \tag{3.16}$$

$$\begin{aligned} Q_3 = \ & -P(U_\xi^B + U_\xi^{CD})\left[\Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V})\right] + \left[\kappa(\frac{\Theta_\xi}{V})_\xi + \mu\frac{U_\xi^2}{V} + Q_2\right]\Big[(\gamma-1)\Phi(\frac{v}{V}) \\ & + \Phi(\frac{\theta}{\Theta}) - \frac{\zeta^2}{\theta\Theta}\Big] - \mu(\frac{1}{v} - \frac{1}{V})U_\xi\psi_\xi + \mu(\frac{1}{v} - \frac{1}{V})U_\xi^2\frac{\zeta}{\theta} + 2\mu\frac{\zeta\psi_\xi U_\xi}{v\theta} + \kappa\frac{\Theta_\xi\zeta_\xi\zeta}{v\theta^2} \\ & -\kappa(\frac{1}{v} - \frac{1}{V})\frac{\Theta\Theta_\xi\zeta_\xi}{\theta^2} + \kappa(\frac{1}{v} - \frac{1}{V})\frac{\zeta\Theta_\xi^2}{\theta^2} \end{aligned} \tag{3.17}$$

Note that

$$\Phi(1) = \Phi'(1) = 0, \qquad \Phi''(z) = z^{-2} > 0.$$

So there exists a positive constant C such that

$$C^{-1}(z-1)^2 \leq \Phi(z) \leq C(z-1)^2,$$

if z is near 1.

Using the a priori assumptions $N(T) \leq \varepsilon_0$ for suitably small constant ε_0, we can get

$$C^{-1}|\phi|^2 \leq \Phi(\frac{v}{V}) \leq C|\phi|^2, \qquad C^{-1}|\zeta|^2 \leq \Phi(\frac{\theta}{\Theta}) \leq C|\zeta|^2, \tag{3.18}$$

and

$$C^{-1}|(\phi,\zeta)|^2 \leq \Phi(\frac{\theta V}{v\Theta}) + \gamma\Phi(\frac{v}{V}) \leq C|(\phi,\zeta)|^2. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17) and using Cauchy inequality imply

$$\begin{aligned} Q_3 \leq \quad & \frac{\mu\Theta\psi_\xi^2}{4v\theta} + \frac{\kappa\Theta\zeta_\xi^2}{4v\theta^2} + O(1)\Big[|(V_\xi^B, U_\xi^B, \Theta_\xi^B, \Theta_{\xi\xi}^B)| + (|\Theta_\xi^{CD}|^2, |\Theta_{\xi\xi}^{CD}|) \\ & +(|(V_\xi^R, U_\xi^R, \Theta_\xi^R)|^2, |\Theta_{\xi\xi}^R|) + |Q_2|\Big](\phi^2+\zeta^2) \end{aligned} \tag{3.20}$$

By the fact

$$|f(\xi)| = |f(0) + \int_0^\xi f_y dy| \leq |f(0)| + \sqrt{\xi}\ \|f_\xi\|, \tag{3.21}$$

we have

$$\begin{aligned} & \int_0^t \int_{R_+} |(V_\xi^B, U_\xi^B, \Theta_\xi^B, \Theta_{\xi\xi}^B)|(\phi^2+\zeta^2) d\xi d\tau \\ & \leq C\delta^B \int_0^t \int_{R_+} e^{-c\xi}\left(|(\phi,\zeta)|^2(\tau,0) + \xi\|(\phi_\xi,\zeta_\xi)\|^2\right) d\xi d\tau \\ & \leq C\delta^B \int_0^t |(\phi,\zeta)|^2(\tau,0) d\tau + C\delta^B \int_0^t \|(\phi_\xi,\zeta_\xi)\|^2 d\tau \\ & \leq C\delta^B(\delta^{CD})^2 + C\delta^B \int_0^t \|(\phi_\xi,\zeta_\xi)\|^2 d\tau. \end{aligned} \tag{3.22}$$

By the properties of the viscous contact wave, we can obtain

$$\int_0^t \int_{R_+} (|\Theta_\xi^{CD}|^2, |\Theta_{\xi\xi}^{CD}|)(\phi^2+\zeta^2) d\xi \ \leq C\delta^{CD} \int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}} |(\phi,\zeta)|^2 d\xi. \tag{3.23}$$

Using the definition of the approximate rarefaction wave, we have

$$
\begin{aligned}
&\int_0^t \int_{R_+} (|(V_\xi^R, U_\xi^R, \Theta_\xi^R)|^2, |\Theta_{\xi\xi}^R|)(\phi^2+\zeta^2)d\xi \\
&\le \int_0^t (\|(V_\xi^R, U_\xi^R, \Theta_\xi^R)\|^2 + \|\Theta_{\xi\xi}^R\|_{L^1})\|(\phi,\zeta)\|_{L^\infty}^2 d\tau \\
&\le C\varepsilon^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{16}} \|(\phi,\zeta)\|\|(\phi_\xi,\zeta_\xi)\| d\tau \\
&\le C\varepsilon^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{8}} \|(\phi,\zeta)\|^2 d\tau + C\varepsilon^{\frac{1}{8}} \int_0^t \|(\phi_\xi,\zeta_\xi)\|^2 d\tau
\end{aligned}
\tag{3.24}
$$

where in the second inequality we have used

$$
\|(\Theta_\xi^R, V_\xi^R, U_\xi^R)\|^2 \le C\varepsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}},
$$

and

$$
\|(\Theta_{\xi\xi}^R, V_{\xi\xi}^R, U_{\xi\xi}^R)\|_{L_\xi^1} \le C\varepsilon^{\frac{1}{8}}(1+t)^{-\frac{13}{16}},
$$

if we let $q \ge 16$ in Lemma 2.3.

Now we estimate the terms $Q_1\psi$, $Q_2\frac{\zeta}{\theta}$ on the right-hand side of (3.14) and the term $|Q_2|(\phi^2+\zeta^2)$ on the right-hand side of (3.20). Due to the estimation of Q_1 in (3.3), we have

$$
\begin{aligned}
\int_0^t \int_{\mathbf{R}^+} |Q_1\psi| d\xi d\tau &\le C\int_0^t \|\psi\|_{L_\xi^\infty} \|Q_1\|_{L_\xi^1} d\tau \\
&\le C\int_0^t \|\psi\|^{\frac{1}{2}} \|\psi_\xi\|^{\frac{1}{2}} \Big[\delta^B e^{-c\tau} + \delta^{CD}(1+\tau)^{-1} + C\varepsilon^{\frac{1}{8}}(1+\tau)^{-\frac{13}{16}}\Big] d\tau \\
&\le C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t \|\psi_\xi\|^2 d\tau + \int_0^t \|\psi\|^{\frac{2}{3}}(1+\tau)^{-\frac{13}{12}} d\tau\Big] \\
&\le C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t \|\psi_\xi\|^2 d\tau + \int_0^t \|\psi\|^2(1+\tau)^{-\frac{13}{12}} d\tau + 1\Big].
\end{aligned}
\tag{3.25}
$$

Similarly we can calculate the term $Q_2\frac{\zeta}{\theta}$ and $|Q_2|(\phi^2+\zeta^2)$.

Integrating (3.15) over $\mathbf{R}^+ \times [0,t]$ and using the boundary estimates in Lemma 3.1, we can obtain

$$
\begin{aligned}
&\|(\phi,\psi,\zeta)(t,\cdot)\|^2 + \int_0^t \|(\psi_\xi,\zeta_\xi)(\tau,\cdot)\|^2 d\tau + \int_0^t \int_{\mathbf{R}^+} U_\xi^R(\phi^2+\zeta^2) d\xi d\tau \\
\le\ & C\|(\phi_0,\psi_0,\zeta_0)\|^2 + C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi,\psi,\zeta)\|^2 d\tau + 1\Big] \\
&+ C\nu \int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})(\tau,\cdot)\|^2 d\tau + C\delta^{CD} \int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}} |(\phi,\zeta)|^2 d\xi d\tau.
\end{aligned}
\tag{3.26}
$$

Now we estimate $\|\phi_\xi\|^2$. Let

$$\tilde{v} = \frac{v}{V}.$$

From the system $(3.5)_2$, we have

$$\mu(\frac{\tilde{v}_\xi}{\tilde{v}})_t - \mu\sigma_-(\frac{\tilde{v}_\xi}{\tilde{v}})_\xi - \psi_t + \sigma_-\psi_\xi - (p-P)_\xi - Q_1 = 0.$$

Multiplying the above equation by $\frac{\tilde{v}_\xi}{\tilde{v}}$ and noticing that

$$-(p-P)_\xi = \frac{R\theta}{v}\frac{\tilde{v}_\xi}{\tilde{v}} - \frac{R\zeta_\xi}{v} + (p-P)\frac{V_\xi}{V} - R\Theta_\xi(\frac{1}{V} - \frac{1}{v}),$$

we can get

$$\begin{aligned} &\left[\frac{\mu}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 - \psi\frac{\tilde{v}_\xi}{\tilde{v}}\right]_t - \left[\frac{\mu\sigma_-}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 - \psi\frac{\tilde{v}_t}{\tilde{v}}\right]_\xi + \frac{R\theta}{v}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 \\ = \ & \psi_\xi(\frac{u_\xi}{v} - \frac{U_\xi}{V}) + \left[\frac{R\zeta_\xi}{v} - (p-P)\frac{V_\xi}{V} + R\Theta_\xi(\frac{1}{V} - \frac{1}{v}) - Q_1\right]\frac{\tilde{v}_\xi}{\tilde{v}}. \end{aligned}$$

Integrating the above equality and using the boundary estimate (3.11), we obtain

$$\begin{aligned} &\int_{\mathbf{R}^+}\left[\frac{\mu}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 - \psi\frac{\tilde{v}_\xi}{\tilde{v}}\right](t,\xi)d\xi + \int_0^t\int_{\mathbf{R}^+}\frac{R\theta}{2v}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 d\xi d\tau \\ \leq \ & \int_{\mathbf{R}^+}\left[\frac{\mu}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 - \psi\frac{\tilde{v}_\xi}{\tilde{v}}\right](0,\xi)d\xi + C(\delta^{CD})^2 + \nu\int_0^t\|\psi_{\xi\xi}(\tau,\cdot)\|^2 d\tau \\ &+C\int_0^t\left[\|(\psi_\xi,\zeta_\xi)\|^2 + \|Q_1\|^2\right]d\tau + C\int_0^t\int_{\mathbf{R}^+}|(V_\xi,U_\xi,\Theta_\xi)|^2|(\phi,\zeta)|^2 d\xi d\tau. \end{aligned} \tag{3.27}$$

Using the equality (3.12), we can get

$$C^{-1}(|\phi_\xi|^2 - |V_\xi\phi|^2) \leq (\frac{\tilde{v}_\xi}{\tilde{v}})^2 \leq C(|\phi_\xi|^2 + |V_\xi\phi|^2). \tag{3.28}$$

By the estimation of Q_1 in (3.3), we have

$$\int_0^t\|Q_1\|^2 d\tau \leq C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}}). \tag{3.29}$$

Similar to (3.22)-(3.24), we can compute the last term in the right hand side of (3.27). Thus we can obtain

$$\begin{aligned} &\|\phi_\xi(t,\cdot)\|^2 + \int_0^t\|\phi_\xi\|^2 d\tau \leq C\|(\phi_0,\psi_0,\phi_{0\xi})\|^2 + C\|(\phi,\psi)(t,\cdot)\|^2 \\ &+C\nu\int_0^t\|\psi_{\xi\xi}(\tau,\cdot)\|^2 d\tau + C\delta^{CD}\int_0^t\int_{\mathbf{R}^+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}|(\phi,\zeta)|^2 d\xi d\tau \\ &+C\int_0^t\|(\psi_\xi,\zeta_\xi)\|^2 d\tau + C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t(1+\tau)^{-\frac{13}{12}}\|(\phi,\psi,\zeta)\|^2 d\tau + 1\Big]. \end{aligned} \tag{3.30}$$

Multiplying the inequality (3.26) by a large constant $C_1 > 0$, and adding it to (3.30), we complete the proof of Lemma 3.2.

Now we derive the higher order estimates. Multiplying the equation $(3.5)_2$ by $-\psi_{\xi\xi}$, we can get

$$(\frac{\psi_\xi^2}{2})_t-[\psi_t\psi_\xi-\frac{\sigma_-\psi_\xi^2}{2}]_\xi+\mu\frac{\psi_{\xi\xi}^2}{v}=\mu\frac{\psi_\xi}{v^2}v_\xi\psi_{\xi\xi}+\Big\{(p-P)_\xi-\mu[U_\xi(\frac{1}{v}-\frac{1}{V})]_\xi+Q_1\Big\}\psi_{\xi\xi}. \quad (3.31)$$

From the boundary estimate

$$\begin{aligned}
&\int_0^t[\psi_\tau\psi_\xi-\frac{\sigma_-\psi_\xi^2}{2}](\tau,\xi=0)d\tau\\
\leq\ & C\int_0^t(|\psi_\xi(\tau,0)|^2+|\psi_\tau(\tau,0)|^2)d\tau\\
\leq\ & C\int_0^t\|\psi_\xi\|\|\psi_{\xi\xi}\|d\tau+C(\delta^{CD})^2\\
\leq\ & \nu\int_0^t\|\psi_{\xi\xi}\|^2d\tau+C_\nu\int_0^t\|\psi_\xi\|^2d\tau+C(\delta^{CD})^2,
\end{aligned}$$

we can get the following inequality by integrating (3.31) over $\mathbf{R}^+\times(0,t)$

$$\begin{aligned}
\|\psi_\xi\|^2(t)+\int_0^t\|\psi_{\xi\xi}\|^2d\tau\leq C\|\psi_{0\xi}\|^2+C(\delta^{CD})^2+C\varepsilon^{\frac15}\int_0^t(1+\tau)^{-\frac32}\|(\phi,\zeta)\|^2d\tau\\
+C\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2d\tau+C(\delta^{CD})^2\int_0^t\int_{\mathbf{R}^+}(1+\tau)^{-1}e^{-\frac{-c_0(\xi+\sigma_-\tau)^2}{1+\tau}}|(\phi,\zeta)|^2d\xi d\tau,
\end{aligned} \quad (3.32)$$

where we have used the following estimation

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}^+}|\phi_\xi||\psi_\xi||\psi_{\xi\xi}|d\xi d\tau &\leq C\int_0^t\|\phi_\xi\|\|\psi_{\xi\xi}\|\|\psi_\xi\|_{L_\xi^\infty}d\tau\\
&\leq C\int_0^t\|\phi_\xi\|\|\psi_{\xi\xi}\|^{\frac32}\|\psi_\xi\|^{\frac12}d\tau\\
&\leq \nu\int_0^t\|\psi_{\xi\xi}\|^2d\tau+C_\nu\sup_t\|\phi_\xi\|^4\int_0^t\|\psi_\xi\|^2d\tau\\
&\leq \nu\int_0^t\|\psi_{\xi\xi}\|^2d\tau+C_\nu\ \varepsilon_0^4\int_0^t\|\psi_\xi\|^2d\tau.
\end{aligned}$$

Multiplying $(3.5)_3$ by $-\zeta_{\xi\xi}$, almost similar to the estimates for $\|\psi_\xi\|^2(t)$, we can obtain

$$\begin{aligned}
\|\zeta_\xi\|^2(t)+\int_0^t\|\zeta_{\xi\xi}\|^2d\tau\leq C\|\zeta_{0\xi}\|^2+C(\delta^{CD})^2+C\varepsilon^{\frac15}\int_0^t(1+\tau)^{-\frac32}\|(\phi,\zeta)\|^2d\tau\\
+C\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2d\tau+C(\delta^{CD})^2\int_0^t\int_{\mathbf{R}^+}(1+\tau)^{-1}e^{-\frac{-c_0(\xi+\sigma_-\tau)^2}{1+\tau}}|(\phi,\zeta)|^2d\xi d\tau.
\end{aligned} \quad (3.33)$$

Combining Lemma 3.2 and the higher order estimations (3.32) and (3.33), we have the following Lemma:

Lemma 3.3 If the wave amplitudes δ^B, δ^{CD} and the constants ε, ε_0 are small enough, then we have $\forall\ t \in [0,T]$,

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(t,\cdot)\|_1^2 + \int_0^t \|\phi_\xi\|^2 + \|(\psi_\xi,\zeta_\xi)\|_1^2 d\tau + \int_0^t \int_{\mathbf{R}^+} U_\xi^R(\phi^2+\zeta^2) d\xi d\tau \\
\leq\ & C\|(\phi_0,\psi_0,\zeta_0)\|_1^2 + C(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}})\Big[\int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi,\psi,\zeta)\|^2 d\tau + 1\Big] \\
& + C\delta^{CD} \int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} e^{-\frac{-c_0(\xi+\sigma_-\tau)^2}{1+\tau}} |(\phi,\zeta)|^2 d\xi d\tau.
\end{aligned} \tag{3.34}$$

In order to close the estimate, we only need to control the last term in (3.34), which comes from the viscous contact wave. So we will use the estimation on the heat kernel in [3] and [7].

Lemma 3.4 Suppose that $h(t,\xi)$ satisfies

$$h \in L^\infty(0,T;L^2(\mathbf{R}^+)), \quad h_\xi \in L^2(0,T;L^2(\mathbf{R}^+)), \quad h_t - \sigma_- h_\xi \in L^2(0,T;H^{-1}(\mathbf{R}^+)),$$

Then

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} h^2 e^{-\frac{\beta(\xi+\sigma_-\tau)^2}{1+\tau}} d\xi d\tau \\
&\leq C_\beta \Big[\|h(0,\xi)\|^2 + \int_0^t h^2(\tau,\xi=0) d\tau + \int_0^t \|h_\xi\|^2 d\tau + \int_0^t \langle h_t - \sigma_- h_\xi, h g_\beta^2 \rangle_{H^1\times H^{-1}} d\tau\Big]
\end{aligned} \tag{3.35}$$

where

$$g_\beta(t,\xi) = -(1+t)^{-\frac{1}{2}} \int_{\xi+\sigma_- t}^{+\infty} e^{-\frac{\beta\eta^2}{1+t}} d\eta,$$

and $\beta > 0$ is the constant to be determined.

The proof of Lemma 3.4 can be done similarly in [3]. The only difference is that the space we considered here is on the half line and the boundary terms should be treated.

Lemma 3.5 There exist a constant $C > 0$ such that if δ^{CD} and ε_0 are small enough, then we have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^+} \frac{e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}}{1+\tau} |(\phi,\psi,\zeta)|^2 d\xi d\tau \\
&\leq C\Big[(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}}) + \|(\phi_0,\psi_0,\zeta_0)\|^2 + \|(\phi,\psi,\zeta)(t,\cdot)\|^2\Big] \\
&\quad + C\nu \int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})\|^2 d\tau + C\int_0^t \|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2 d\tau + C\int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi,\psi)\|^2 d\tau.
\end{aligned} \tag{3.36}$$

Proof: From the equation $(3.5)_2$ and the fact $p-P=\frac{R\zeta-P\phi}{v}$, we have

$$\psi_t - s_-\psi_\xi + (\frac{R\zeta-P\phi}{v})_\xi = \mu(\frac{u_\xi}{v}-\frac{U_\xi}{V})_\xi - Q_1.$$

Then we get

$$(R\zeta-P\phi)_\xi = \frac{R\zeta-P\phi}{v}(V_\xi+\phi_\xi) - v(\psi_t-\sigma_-\psi_\xi) + \mu v(\frac{u_\xi}{v}-\frac{U_\xi}{V})_\xi - vQ_1. \tag{3.37}$$

Let

$$G_\alpha(t,\xi) = -(1+t)^{-1}\int_{\xi+\sigma_-t}^{+\infty} e^{-\frac{\alpha\eta^2}{1+t}}d\eta,$$

where α is a positive constant to be determined later. Multiplying the equation (3.37) by $G_\alpha(R\zeta-P\phi)$ gives

$$\begin{aligned}
&\left[\frac{G_\alpha(R\zeta-P\phi)^2}{2}\right]_\xi - (G_\alpha)_\xi\frac{(R\zeta-P\phi)^2}{2}\\
=\ & \frac{G_\alpha(R\zeta-P\phi)^2}{v}(V_\xi+\phi_\xi) - G_\alpha v(R\zeta-P\phi)(\psi_t-\sigma_-\psi_\xi)\\
&+\mu G_\alpha v(R\zeta-P\phi)(\frac{u_\xi}{v}-\frac{U_\xi}{V})_\xi - G_\alpha v(R\zeta-P\phi)Q_1.
\end{aligned} \tag{3.38}$$

Note that

$$\begin{aligned}
-G_\alpha v(R\zeta-P\phi)(\psi_t-\sigma_-\psi_\xi) &= -[G_\alpha v(R\zeta-P\phi)\psi]_t + [G_\alpha v(R\zeta-P\phi)\psi]_\xi\\
&+[(G_\alpha v)_t-\sigma_-(G_\alpha v)_\xi](R\zeta-P\phi)\psi + G_\alpha v\psi[(R\zeta-P\phi)_t-\sigma_-(R\zeta-P\phi)_\xi],
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
&(R\zeta-p_+\phi)_t - \sigma_-(R\zeta-P\phi)_\xi\\
&= (R\zeta_t - Rs_-\zeta_\xi) - (P_t-\sigma_-P_\xi)\phi - P(\phi_t-\sigma_-\phi_\xi)\\
&= -\gamma P\psi_\xi + (\gamma-1)\Big[-(p-P)(U_\xi+\psi_\xi) + \mu(\frac{u_\xi^2}{v}-\frac{U_\xi^2}{V}) + \kappa(\frac{\theta_\xi}{v}-\frac{\Theta_\xi}{V})_\xi - Q_2\Big]\\
&\quad -(P_t-\sigma_-P_\xi)\phi.
\end{aligned} \tag{3.40}$$

And using the equality

$$-G_\alpha v\gamma P\psi_\xi\psi = -[\gamma G_\alpha vP\frac{\psi^2}{2}]_\xi + \gamma vP(G_\alpha)_\xi\frac{\psi^2}{2} + \gamma(vP)_\xi\frac{\psi^2}{2}, \tag{3.41}$$

we can get

$$\frac{e^{-\frac{\alpha(\xi+\sigma_-t)^2}{1+t}}}{2(1+t)}[(R\zeta-P\phi)^2+\gamma Pv\psi^2] = [G_\alpha v(R\zeta-P\phi)\psi]_t + H_{2\xi}(t,\xi) + Q_4, \tag{3.42}$$

where

$$
\begin{aligned}
H_2(t,\xi) = \quad & \frac{G_\alpha(R\zeta - P\phi)^2}{2} + \gamma G_\alpha v P\frac{\psi^2}{2} - \sigma_- G_\alpha v(R\zeta - P\phi)\psi \\
& -\kappa(\gamma-1)G_\alpha v\psi(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V}) - \mu G_\alpha v(R\zeta - P\phi)(\frac{u_\xi}{v} - \frac{U_\xi}{V}),
\end{aligned}
\tag{3.43}
$$

and

$$
\begin{aligned}
Q_4 = \quad & -[(G_\alpha)_t - \sigma_-(G_\alpha)_\xi]v(R\zeta - P\phi)\psi - G_\alpha u_\xi(R\zeta - P\phi)\psi \\
& +(\gamma-1)G_\alpha v\psi\left[(p-P)(U_\xi + \psi_\xi) - \mu(\frac{u_\xi^2}{v} - \frac{U_\xi^2}{V}) + Q_2\right] \\
& +\mu[G_\alpha v(R\zeta - P\phi)]_\xi(\frac{u_\xi}{v} - \frac{U_\xi}{V}) + (\gamma-1)\kappa(G_\alpha v\psi)_\xi(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V}) \\
& +G_\alpha v(R\zeta - P\phi)Q_1 + G_\alpha v\psi(P_t - \sigma_- P_\xi)\phi,
\end{aligned}
\tag{3.44}
$$

From the boundary estimate (3.11), we have

$$
\int_0^t H_2(\tau,\xi=0)d\tau \le C(\delta^{CD})^2 + \nu\int_0^t \|(\psi_\xi,\zeta_\xi,\psi_{\xi\xi},\zeta_{\xi\xi})(\tau,\cdot)\|^2 d\tau. \tag{3.45}
$$

Note that

$$
\|G_\alpha(t,\cdot)\|_{L^\infty} \le C_\alpha(1+t)^{-\frac{1}{2}},
$$

thus integrating (3.42) over $\mathbf{R}^+ \times (0,t)$ gives

$$
\begin{aligned}
& \int_0^t\int_{\mathbf{R}^+} \frac{e^{-\frac{\alpha(\xi+\sigma_-\tau)^2}{1+\tau}}}{1+\tau}[(R\zeta - P\phi)^2 + \psi^2]d\xi d\tau \\
& \le C\left[(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}}) + \|\phi_0,\psi_0,\zeta_0\|^2\right] + C(1+t)^{-1}\|(\phi,\psi,\zeta)(t,\cdot)\|^2 \\
& +C\int_0^t (1+\tau)^{-\frac{13}{12}}\|(\phi,\psi,\zeta)(\tau,\cdot)\|^2 d\tau + C\int_0^t \|(\phi_\xi,\psi_\xi,\zeta_\xi)(\tau,\cdot)\|^2 d\tau \\
& +C\nu\int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})(\tau,\cdot)\|^2 d\tau + C\delta^{CD}\int_0^t\int_{\mathbf{R}^+} \frac{e^{-\frac{\alpha(\xi+\sigma_-\tau)^2}{1+\tau}}}{1+\tau}|(\phi,\zeta)|^2 d\xi d\tau.
\end{aligned}
\tag{3.46}
$$

In order to get the desired estimate in Lemma 3.5, we must derive the other similar estimates from the energy equation $(3.5)_3$. Set

$$
h = R\zeta + (\gamma-1)p_+\phi
$$

in Lemma 3.4. Thus we only need to compute the last term in (3.35). From the energy equation $(3.5)_3$, we have

$$
h_t - \sigma_- h_\xi = (P_t - \sigma_- P_\xi)\phi - (p-P)u_\xi + \kappa(\frac{\theta_\xi}{v} - \frac{U_\xi}{V})_\xi + \mu(\frac{u_\xi^2}{v} - \frac{U_\xi^2}{V}) - Q_2, \tag{3.47}
$$

Thus

$$
\begin{aligned}
& \int_0^t \langle h_t - \sigma_- h_\xi, hg_\beta^2 \rangle_{H^1\times H^{-1}} d\tau \\
= \ & \int_0^t \int_{\mathbf{R}^+} [(P_t - \sigma_- P_\xi)\phi - (p-P)U_\xi] hg_\beta^2 d\xi d\tau + \int_0^t \int_{\mathbf{R}^+} (p-P)\psi_\xi hg_\beta^2 d\xi d\tau \\
& + \int_0^t [\kappa(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V}) hg_\beta^2](\tau, \xi = 0) d\tau + \int_0^t \int_{\mathbf{R}^+} \kappa(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V})(hg_\beta^2)_\xi d\xi d\tau \\
& + \int_0^t \int_{\mathbf{R}^+} \mu(\frac{u_\xi^2}{v} - \frac{U_\xi^2}{V}) hg_\beta^2 d\xi d\tau + \int_0^t \int_{\mathbf{R}^+} Q_2 hg_\beta^2 d\xi d\tau \\
:= \ & \sum_{i=1}^6 J_i.
\end{aligned}
\tag{3.48}
$$

Note that

$$
\|g_\beta(t,\cdot)\|_{L^\infty} \le C_\beta,
$$

we can estimate $J_i (i = 1,3,4,5,6)$ directly. In order to estimate J_2, from the mass equation $(3.5)_1$, we have

$$
\begin{aligned}
& (p-P)\psi_\xi hg_\beta^2 \\
= \ & \frac{(\gamma-1)h - \gamma P\phi}{v} hg_\beta^2 (\phi_t - \sigma_-\phi_\xi) \\
= \ & \frac{(\gamma-1)h^2 g_\beta^2}{v}(\phi_t - \sigma_-\phi_\xi) - \frac{\gamma P h g_\beta^2}{2v}[(\phi^2)_t - \sigma_-(\phi^2)_\xi] \\
= \ & \left[\frac{2(\gamma-1)\phi h^2 g_\beta^2 - \gamma P h\phi^2 g_\beta^2}{2v}\right]_t - \sigma_- \left[\frac{2(\gamma-1)\phi h^2 g_\beta^2 - \gamma P h\phi^2 g_\beta^2}{2v}\right]_\xi \\
& + \frac{\gamma P h\phi^2 - 2(\gamma-1)h^2\phi}{v} g_\beta[(g_\beta)_t - \sigma_-(g_\beta)_\xi] - \frac{\gamma P h\phi^2 - 2(\gamma-1)h^2\phi}{v^2} g_\beta^2 (v_t - \sigma_- v_\xi) \\
& + \left[\frac{2(\gamma-1)g_\beta^2\phi h}{v} + \frac{\gamma P g_\beta^2 \phi^2}{2v}\right](h_t - \sigma_- h_\xi) + \frac{\gamma g_\beta^2 \phi^2 h}{2v}(P_t - \sigma_- P_\xi)
\end{aligned}
$$

Now each term can be estimated directly, the detailed proof can be seen in [3]. Remark that here we need to compute the boundary terms. Therefore, taking the constant $\beta = \frac{c_0}{2}$, we can get from Lemma 3.4 that

$$
\begin{aligned}
& \int_0^t \int_{\mathbf{R}^+} \frac{e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}}{1+\tau} h^2 d\xi d\tau \le C\Big[(\delta^B + \delta^{CD} + \varepsilon^{\frac{1}{8}}) + \|(\phi_0,\psi_0,\zeta_0)\|^2 + \|(\phi,\psi,\zeta)(t,\cdot)\|^2\Big] \\
& \quad + C\nu \int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})\|^2 d\tau + C\int_0^t \|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2 d\tau + C\int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi,\psi)\|^2 d\tau \\
& \quad + C(\delta^{CD} + \varepsilon_0) \int_0^t \int_{\mathbf{R}^+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}} |(\phi,\zeta)|^2 d\xi d\tau.
\end{aligned}
\tag{3.49}
$$

Taking $\alpha = c_0$ in (3.46) and combining the estimates (3.46) with (3.49) yield the desired estimation in Lemma 3.5 if we choose suitably small constants δ^{CD} and ε_0.

Now from Lemma 3.3 and Lemma 3.5, if the wave amplitude δ^{CD} and the constant ν are suitably small, we can get

$$\begin{aligned}\|(\phi,\psi,\zeta)(t,\cdot)\|_1^2 \quad &+ \int_0^t \|\phi_\xi\|^2 + \|(\psi_\xi,\zeta_\xi)\|_1^2 d\tau \le C\|(\phi_0,\psi_0,\zeta_0)\|_1^2 \\ &+C(\delta^B+\delta^{CD}+\varepsilon^{\frac{1}{8}})\Big[\int_0^t (1+\tau)^{-\frac{13}{12}}\|(\phi,\psi,\zeta)\|^2 d\tau + 1\Big].\end{aligned}$$

Finally, Gronwall inequality gives the a priori estimate in Proposition 3.2:

$$\|(\phi,\psi,\zeta)(t,\cdot)\|_1^2 + \int_0^t \|\phi_\xi\|^2 + \|(\psi_\xi,\zeta_\xi)\|_1^2 d\tau \le C\|(\phi_0,\psi_0,\zeta_0)\|_1^2 + C(\delta^B+\delta^{CD}+\varepsilon^{\frac{1}{8}}).$$

Thus we complete the proof of Theorem 2.2.

The proof of Theorem 2.1 can be done along the same line as Theorem 2.2, we omit it for brevity.

Appendix: Proof of Lemma 2.1

Now we give the rigorous proof of Lemma 2.1. Firstly from (2.9), $u_+ > 0$. Thus if $u_+ \le 0$, then there is no solution to (2.10) or (2.13). Now assume that $u_+ > 0$. Then we can compute that the determinant of the matrix J defined in (2.12)

$$\det J = \frac{R(u_+^2 - R\gamma\theta_+)}{\kappa\mu(\gamma-1)} = \frac{R^2\gamma\theta_+(M_+^2-1)}{\kappa\mu(\gamma-1)}. \tag{A.1}$$

So we can divide it into three cases according to the sign of the quantity $M_+^2 - 1$.

Case I (Supersonic): $M_+ > 1$, then $\det J > 0$. We can easily know that J has two positive eigenvalues. Thus the ODE system (2.13) has no solution.

Case II (Transonic): $M_+ = 1$, then $\det J = 0$. One of the eigenvalues of the matrix J is zero, the other one is positive. This case is a little subtle. Firstly we can choose a nonsingular matrix P such that $P^{-1}JP$ changes into a standard form. For example, let

$$P = \begin{pmatrix} \frac{\kappa(\gamma-1)^2}{R\kappa\mu+\kappa(\gamma-1)^2} & \frac{R\kappa\gamma(\gamma-1)}{[R\kappa\mu+\kappa(\gamma-1)^2]u_+} \\ -\frac{\mu u_+}{\kappa(\gamma-1)} & 1 \end{pmatrix}.$$

then

$$P^{-1}JP = \begin{pmatrix} \lambda_J & 0 \\ 0 & 0 \end{pmatrix} := \Lambda_J,$$

where λ_J is the positive eigenvalue of J given by

$$\lambda_J = \left(\frac{\gamma-1}{\mu} + \frac{R}{\kappa(\gamma-1)}\right)u_+ > 0.$$

Let

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := P^{-1}\begin{pmatrix} \bar{U}^B \\ \bar{\Theta}^B \end{pmatrix}, \tag{A.2}$$

we have

$$W_\xi = \Lambda_J W + G(W), \tag{A.3}$$

where

$$G(W) = P^{-1}F(PW),$$

and

$$F(PW) = \begin{pmatrix} F_1(PW) \\ F_2(PW) \end{pmatrix}.$$

We can rewrite (2.13) as

$$\begin{cases} W_{1\xi} = \lambda_J W_1 + G_1(W_1, W_2), \\ W_{2\xi} = G_2(W_1, W_2), \end{cases} \tag{A.4}$$

Obviously, there exists a suitably small neighborhood $\Omega_{\bar{\delta}_0}(0,0)$ such that $(G_1, G_2)(W)$ is analytic. And in this neighborhood, if $|W_1| \ll |W_2|$, then

$$G_2(W) \;=\; -\frac{R^2\gamma^2\kappa(\gamma-1)(\gamma+1)}{2[R\gamma\mu+\kappa(\gamma-1)^2]^2u_+}W_2^2 + o(W_2^2). \tag{A.5}$$

From the geometric theory of the automatous ordinary differential systems, we know that the equilibrium state (0,0) is a saddle-node point to the system (A.4). And (0,0) is an attractor whose trajectory, denoted by Γ, is unique and tangent to W_2-axis at (0,0). From the uniqueness of the attractor trajectory Γ, we know that only when $(W_1, W_2)(0) \in \Gamma$, there exists a solution to (A.4), otherwise, there does not exist solution to (2.13). When $(W_1, W_2)(0) \in \Gamma$, the solution $(W_1, W_2)(\xi)$ satisfy that $|W_1(\xi)| \ll |W_2(\xi)|$ if ξ is large enough, thus we have

$$-\sigma_1 W_2^2 \le W_{2\xi} \le -\sigma_2 W_2^2, \tag{A.6}$$

where $0 < \sigma_1 < \sigma_2$ are two constants.

So we can get

$$|(W_1, W_2)(\xi)| \;\le\; C\frac{\delta^B}{1+\delta^B\xi} \quad \xi \in \mathbb{R}_+, \tag{A.7}$$

where $\delta^B = |(W_1, W_2)(0)| = O(1)|(u_+ - u_-, \theta_+ - \theta_-)|$ is small enough.

From (A.2), we can get the BL-solution $(U^B, \Theta^B)(\xi)$ in the transonic case ($M_+ = 1$) satisfy that

$$\begin{aligned}&\frac{\mu u_+}{\kappa(\gamma-1)}(U^B - u_+) - (\Theta^B - \theta_+)\\ =\ & \int_\xi^\infty \left[-\frac{u_+(\bar{U}^B)^2}{\kappa(\gamma-1)} + \frac{(2-\gamma)u_+}{2\gamma\kappa}(\bar{U}^B)^2 + \frac{R}{\kappa(\gamma-1)}\bar{U}^B\bar{\Theta}^B - \frac{1}{2\kappa}(\bar{U}^B)^3\right] d\xi, \quad \text{(A.8)}\end{aligned}$$

Case III (Subsonic): $M_+ < 1$, then $\det J < 0$. One can see that J has one positive and one negative eigenvalues. Similar to Case II, we can choose a nonsingular matrix P such that $P^{-1}JP$ is in a standard form. Now we give the detailed procedure for choosing the matrix P. Firstly, let

$$P_1 = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix},$$

where the constant a_1 is to be determined, then

$$P_1^{-1} = \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix},$$

Assume that

$$P_1^{-1}JP_1 = \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} := \mathfrak{M}, \quad \text{(A.9)}$$

where the constants m_{11}, m_{12}, m_{22} will be fixed when a_1 is determined. From $m_{21} = 0$ in (A.9), we get a equation of a_1:

$$\frac{R}{\mu}a_1^2 + \left(\frac{(M_+^2\gamma - 1)u_+}{M_+^2\gamma\mu} - \frac{Ru_+}{\kappa(\gamma-1)}\right)a_1 - \frac{u_+^2}{M_+^2\gamma\kappa} = 0,$$

i.e.,

$$\left(\frac{a_1}{u_+}\right)^2 + \left(\frac{M_+^2\gamma - 1}{M_+^2R\gamma} - \frac{\mu}{\kappa(\gamma-1)}\right)\frac{a_1}{u_+} - \frac{\mu}{M_+^2R\gamma\kappa} = 0.$$

Then we can solve the above equation to obtain

$$a_1 = c_1u_+ < 0 \quad \text{or} \quad a_1 = c_2u_+ > 0, \quad \text{(A.10)}$$

where $c_1 < \min\left\{0, -\frac{M_+^2\gamma-1}{M_+^2R\gamma}\right\}$, $c_2 > \max\left\{\frac{\mu}{\kappa(\gamma-1)}, \frac{\mu}{\kappa(\gamma-1)} - \frac{M_+^2\gamma-1}{M_+^2R\gamma}\right\} > 0$ are the solutions of the following equation

$$y^2 + \left(\frac{M_+^2\gamma - 1}{M_+^2R\gamma} - \frac{\mu}{\kappa(\gamma-1)}\right)y - \frac{\mu}{M_+^2R\gamma\kappa} = 0. \quad \text{(A.11)}$$

Without loss of generality, we choose $a_1 = c_2 u_+$, then we can compute that the matrix $\mathfrak{M}$ in (A.9)

$$\mathfrak{M} = \begin{pmatrix} \lambda_J^1 & \frac{R}{\mu} \\ 0 & \lambda_J^2 \end{pmatrix}$$

where $\lambda_J^1 = \left(\frac{M_+^2\gamma-1}{M_+^2R\gamma} + c_2\right)u_+ > 0$ and $\lambda_J^2 = \left(\frac{R}{\kappa(\gamma-1)} - c_2\right)u_+ < 0$ are the two eigenvalues of the matrix J.

Then we can choose a matrix

$$P_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix},$$

such that

$$P_2^{-1}\mathfrak{M}P_2 = \begin{pmatrix} \lambda_J^1 & 0 \\ 0 & \lambda_J^2 \end{pmatrix} := \Lambda_J, \tag{A.12}$$

Then we can get

$$a_2 = -\frac{R}{\mu(\lambda_J^1 - \lambda_J^2)}. \tag{A.13}$$

Now we set

$$P = P_1P_2 = \begin{pmatrix} 1 & \frac{a_2}{u_+} \\ c_2u_+ & 1 + a_2c_2 \end{pmatrix}$$

Then

$$P^{-1}JP = \Lambda_J = \mathrm{diag}\{\lambda_J^1, \lambda_J^2\}.$$

Let

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := P^{-1}\begin{pmatrix} \bar{U}^B \\ \bar{\Theta}^B \end{pmatrix}, \tag{A.14}$$

we have

$$W_\xi = \Lambda_J W + G(W), \tag{A.15}$$

where

$$G(W) = P^{-1}F(PW),$$

and

$$F(PW) = \begin{pmatrix} F_1(PW) \\ F_2(PW) \end{pmatrix}.$$

We can rewrite (A.15) as

$$\begin{cases} W_{1\xi} = \lambda_J^1 W_1 + G_1(W_1, W_2), \\ W_{2\xi} = \lambda_J^2 W_2 + G_2(W_1, W_2), \end{cases} \tag{A.16}$$

From above, one can easily know that G_1, G_2 are analytic with respect to (W_1, W_2) near $(0,0)$, then the equilibrium point $(0,0)$ is the saddle point of (A.16), i.e., in a suitably small neighborhood $\Omega_{\bar{\delta}_0}(0,0)$, there exist two opposite attractor trajectories Γ_1, Γ_2 tangent to W_2-axis at $(0,0)$. Let $\mathcal{M} = \Gamma_1 \cup \Gamma_2$, then $\mathcal{M}$ is a center-stable manifold. Only when $(W_1, W_2)(0) \in \mathcal{M}$, there exists a solution the ODE system (A.16). In such case, there exist two positive constants σ_3, σ_4 which is close to $-\lambda_A^2$ such that

$$-\sigma_3 W_2 \le W_2' \le -\sigma_4 W_2, \quad \xi \in \mathbb{R}_+. \tag{A.17}$$

So we have that there exist positive constants c and C such that

$$|(W_1, W_2)(\xi)| \le C\delta^B e^{-c\xi}, \quad \xi \in \mathbb{R}_+, \tag{A.18}$$

where $\delta^B = |(W_1, W_2)(0)| = O(1)|(u_+ - u_-, \theta_+ - \theta_-)|$ is the amplitude of the BL-solution.

The BL-solution (U^B, Θ^B) satisfies

$$\begin{aligned}
&(1 + a_2 c_2 u_+)(U^B - u_+) - a_2(\Theta^B - \theta_+) \\
= &\int_\xi^\infty e^{-\lambda_A^2 \xi}\Big\{\frac{1 + a_2 c_2}{\mu}(\bar{U}^B)^2 + \frac{R}{\kappa(\gamma - 1)}\bar{U}^B\bar{\Theta}^B \\
&\qquad - \frac{a_2}{u_+}\Big[\Big(\frac{Ru_+}{M_+^2\kappa\gamma} - \frac{u_+}{2\kappa}\Big)(\bar{U}^B)^2 - \frac{1}{2\kappa}(\bar{U}^B)^3\Big]\Big\}d\xi.
\end{aligned} \tag{A.19}$$

Now we complete the proof of Lemma 2.1.

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