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Vanishing Viscosity Limit of the Compressible Navier–Stokes Equations for Solutions to a Riemann Problem

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Abstract

We study the vanishing viscosity limit of the compressible Navier–Stokes equations to the Riemann solution of the Euler equations that consists of the superposition of a shock wave and a rarefaction wave. In particular, it is shown that there exists a family of smooth solutions to the compressible Navier–Stokes equations that converges to the Riemann solution away from the initial and shock layers at a rate in terms of the viscosity and the heat conductivity coefficients. This gives the first mathematical justification of this limit for the Navier–Stokes equations to the Riemann solution that contains these two typical nonlinear hyperbolic waves.

1. Introduction

Consider the compressible Navier–Stokes equations in the Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon \left(\dfrac{u_x}{v}\right)_x, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = \kappa \left(\dfrac{\theta_x}{v}\right)_x + \varepsilon \left(\dfrac{uu_x}{v}\right)_x, \end{cases} \tag{1.1}$$

where the functions $v(x, t) > 0$, $u(x, t)$, $\theta(t, x) > 0$ represent the specific volume, velocity and temperature of the gas, respectively. Here, we consider a perfect gas with pressure $p = p(v, \theta)$ and internal energy $e = e(v, \theta)$ given by

$$p = p(v, \theta) = \frac{R\theta}{v}, \qquad e = \frac{R}{\gamma - 1}\theta,$$

respectively, with $\gamma > 1$, $R > 0$ being the gas constants. By using the entropy s, one can write

$$p = p(v, s) = Av^{-\gamma} \exp\left(\frac{\gamma - 1}{R} s\right)$$

for some constant $A > 0$.

The main concern of this paper is the vanishing viscosity limit, which is an unsolved and still challenging problem with a long history. The main difficulty comes from the singularity, that is, shock waves, in the solutions to the inviscid compressible fluid, which has so far prevented solving the problem in the general setting by means of known analytic techniques and tools. Essential new ideas are needed to tackle this open problem; however, if the solution to the inviscid fluid is assumed to be smooth, then the problem is substantially easier and can be solved by a standard scaling method. Therefore, any attempt on this problem that involves the singularity in the inviscid solution can be viewed as progress to the solution for the general case.

Let us now review some related previous works along these lines. In fact, there are many results on the vanishing viscosity limit for a compressible fluid. First, for a system of hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{1.2}$$

GOODMAN and XIN [6] verified the limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. Later, YU [20] proved it for the hyperbolic conservation laws (1.2) with both shock and initial layers. In 2005, important progress made by BIANCHINI and BRESSAN [1] justified the vanishing viscosity limit in BV space, even though the problem is still unsolved for physical systems such as the Navier–Stokes equations.

For compressible isentropic Navier–Stokes equations in which the conservation of energy in (1.1) is neglected in the isentropic regime, HOFF and LIU [7] first proved the vanishing viscosity limit for piecewise constant shock with an initial layer. Later XIN [18] justified the limit for rarefaction waves. Then WANG [16] generalized the results of GOODMANN and XIN [6] to the isentropic Navier–Stokes equations.

Recently, CHEN and PEREPELITSA [2] proved the vanishing viscosity to the compressible Euler equations for the isentropic compressible Navier–Stokes equations by using a compensated compactness method for the general initial data if the far field does not contain a vacuum. Note that this result is very general because it allows initial data containing a vacuum in the interior domain. However, the framework of compensated compactness is basically limited to 2×2 systems so far, so this result does not apply to the full compressible Navier–Stokes equations (1.1).

It is well known that the solution to the Riemann problem for the Euler equations consists of three basic wave patterns: shock, rarefaction wave and contact discontinuity. Moreover, the Riemann solution is essential in the theory for Euler equations as it captures both the local and global behaviors of general solutions.

For the full Navier–Stokes equations, there are results on the limits to the Euler system for the basic wave patterns. We refer to JIANG ET AL. [10] and XIN and ZENG [19] for the rarefaction wave, WANG [17] for the shock wave, MA [12] for the contact discontinuity and HUANG ET AL. [9] for the superposition of two rarefaction waves and a contact discontinuity. One can also refer to Huang et al. [8] for the hydrodynamic limit of Boltzmann equation to the compressible Euler system with contact discontinuity.

The limit of the full compressible Navier–Stokes equations with basic wave patterns is, in fact, closely related to the stability of viscous wave patterns. The strict monotonicity of the corresponding characteristic speed along the wave profile plays an important role in the studies on the shock wave and rarefaction wave. Precisely, the shock wave is compressive so that the characteristic speed is monotone decreasing along the shock profile. To cope with this in the energy estimate, the anti-derivative variable of the perturbation should be used; see Goodman [5] and Matsumura and Nishihara [13]. On the other hand, the rarefaction wave is expansive and the characteristic speed is monotone increasing along the profile, so the approach used for shock profile does not apply; see Matsumura and Nishihara [14]. Hence, the different frameworks for stability analyses on the viscous shock wave and rarefaction wave render the stability of the Riemann solution with both rarefaction and shock waves still unsolved.

In order to verify the vanishing viscosity limit to the compressible Euler equations for the Riemann solution as a superposition of a rarefaction and a shock wave for any fixed time T, one main idea in this paper is to introduce "hyperbolic waves" that capture the propagation of the extra mass created by the approximate hyperbolic rarefaction wave profile in the viscous setting. With this tool, the vanishing viscosity limit can be formulated as a stability problem so that the energy method can be applied after some suitable scalings.

We now briefly explain why the hyperbolic waves is essential for the proof. First of all, in the regime of the vanishing viscosity limit of the compressible Navier–Stokes equations (1.1) to the compressible Euler equations for the Riemann solution as a superposition of a rarefaction and a shock wave, we will carry out the analysis in the framework of the shock profile, that is, using anti-derivative variables of the perturbation and using the monotonicity of the characteristic speed along the shock profile. This is needed because of the compressibility of the shock profile. By doing so, the error coming from the inviscid approximate rarefaction wave in the setting of viscous system (1.1) is not good enough to get a decay rate with respect to the viscosity because we use the anti-derivative variables. Note that for the stability of the rarefaction wave for the Navier–Stokes equations, one cannot take the anti-derivative because the rarefaction wave is expansive, that is, opposite to the shock.

Therefore, to overcome this difficulty, we introduce "hyperbolic waves" to recover the dissipation terms for the inviscid rarefaction profile. We will also show that the "hyperbolic waves" decay like the first-order derivative of the rarefaction wave profile so that the decay properties given in statement (1) of Lemma 2.3 are good enough to carry out the analysis. In this way, we circumvent the difficulty mentioned above posed by the inviscid rarefaction wave. On the other hand, we also need to treat the interactions of the hyperbolic wave with both the shock and the rarefaction profiles. To this end, we observe that the hyperbolic wave has the same pointwise estimates as the 1-rarefaction wave away from the wave fan on the right side due to the underlying rarefaction wave structure. This also plays a very important role when dealing with the wave interactions. With these new estimates, we can verify the vanishing viscosity limit of compressible Navier–Stokes equations to the Riemann solutions of an Euler system containing a rarefaction wave and a shock wave.

We are now ready to formulate the problem. For the Navier–Stokes equations (1.1), formally, as ε and κ tend to zero, the limit system consists of the following compressible Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0. \end{cases} \tag{1.3}$$

Keeping in mind that the Navier–Stokes equations (1.1) can be derived from the Boltzmann equation through the Chapman–Enskog expansion when the Knudsen number is close to zero, the following assumptions on the viscosity coefficient ε and the heat conductivity coefficient κ in system (1.1) are natural (see also [10]):

$$\begin{cases} \kappa = O(\varepsilon) & \text{as } \varepsilon \to 0; \\ \nu \doteq \dfrac{\kappa(\varepsilon)}{\varepsilon} \geqq c > 0 & \text{for some positive constant } c, \text{ as } \varepsilon \to 0. \end{cases} \tag{1.4}$$

Note that the eigenvalues of the Jacobi matrix of the Euler system (1.3) are

$$\lambda_1(v,\theta) = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 \equiv 0, \quad \lambda_3(v,\theta) = \sqrt{\frac{\gamma p}{v}}. \tag{1.5}$$

It is known that the first and third characteristic fields of (1.3) are genuinely nonlinear and the second characteristic field is linearly degenerate (see [3,15]). Consider the Riemann problem of (1.3) with Riemann initial data

$$(v,u,\theta)(0,x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases} \tag{1.6}$$

where $v_\pm, \theta_\pm > 0$ and $u_\pm$ are given constants. Then, in general, either a shock wave or a rarefaction wave is generated in the genuinely nonlinear fields, which contacts the discontinuity in the linearly degenerate field. In this paper, we consider the case when the Riemann solution of (1.3) contains a rarefaction wave and a shock wave, and without loss of generality, we assume the rarefaction wave is in the first family and the shock wave is in the third family.

Now recall the corresponding 1-rarefaction and 3-shock wave curves in the phase space, as follows.

- 1-Rarefaction wave curve:

$$R_1(v_+,u_+,\theta_+) := \left\{ (v,u,\theta) \middle| v < v_+, s(v,\theta) = s_+, \right. \\ \left. u = u_+ + \int_v^{v_+} \lambda_1(v,s_+)\,\mathrm{d}v \right\}, \tag{1.7}$$

  where $s_+ = s(v_+,\theta_+)$, and $\lambda_1 = \lambda_1(v,s) = \lambda_1(v,s(v,\theta))$ is the first characteristic speed given in (1.5).

- 3-Shock wave curve:
  It is well known that the Riemann problem (1.3), (1.6) admits a 3-shock wave if and only if the two states $(v_\pm, u_\pm, \theta_\pm)$ satisfy the Rankine–Hugoniot condition

$$\begin{cases} -s_3(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s_3(u_+ - u_-) + (p_+ - p_-) = 0, \\ -s_3(E_+ - E_-) + (p_+u_+ - p_-u_-) = 0, \end{cases} \tag{1.8}$$

and Lax's entropy condition

$$0 < \lambda_3^+ < s_3 < \lambda_3^-, \tag{1.9}$$

where $p_\pm = p(v_\pm, \theta_\pm)$, $E_\pm = \frac{R}{\gamma-1}\theta_\pm + \frac{u_\pm^2}{2}$ and $\lambda_3^\pm = \lambda_3(v_\pm, \theta_\pm)$. The shock speed s_3 is uniquely determined by $(v_\pm, u_\pm, \theta_\pm)$ in (1.8). Denote the above 3-shock curve starting from (v_+, u_+, θ_+) by $S_3(v_+, u_+, \theta_+)$.

Use $(v_-, u_-, \theta_-) \in R_1(S_3(v_+, u_+, \theta_+))$ to denote the case in which there exists a unique state (v_*, u_*, θ_*) such that $(v_-, u_-, \theta_-) \in R_1(v_*, u_*, \theta_*)$ and $(v_*, u_*, \theta_*) \in S_3(v_+, u_+, \theta_+)$. Then in this case, the wave pattern $(\bar V, \bar U, \bar\Theta)(t, x)$ consisting of a 1-rarefaction wave and a 3-shock wave that solves the corresponding Riemann problem of the Euler system (1.3) can be defined by

$$\bar V = v^{r_1} + v^{s_3} - v_*, \quad \bar U = u^{r_1} + u^{s_3} - u_*, \quad \bar{\mathcal{E}} = E^{r_1} + E^{s_3} - E_*, \tag{1.10}$$

where $E^{r_1} = \frac{R}{\gamma-1}\theta^{r_1} + \frac{(u^{r_1})^2}{2}$, $E^{s_3} = \frac{R}{\gamma-1}\theta^{s_3} + \frac{(u^{s_3})^2}{2}$, $E_* = \frac{R}{\gamma-1}\theta_* + \frac{u_*^2}{2}$, and $(v^{r_1}, u^{r_1}, \theta^{r_1})(t, x)$ is the 1-rarefaction wave defined in (2.1) with the right state $(v_+, u_+, \theta_+) = (v_*, u_*, \theta_*)$, and $(v^{s_3}, u^{s_3}, \theta^{s_3})(t, x)$ is the 3-shock wave defined in (2.11) with the left state $(v_-, u_-, \theta_-) = (v_*, u_*, \theta_*)$.

Consequently, we can define

$$\bar\Theta = \frac{\gamma - 1}{R}\left(\bar{\mathcal{E}} - \frac{\bar U^2}{2}\right). \tag{1.11}$$

Due to the singularity of the rarefaction wave at $t = 0$, in this paper, we consider the problem on the time interval $[h, T]$ for any small fixed $h > 0$ up to any arbitrarily large but fixed time $T > 0$. Investigation of the interaction between the waves and the initial layer is another interesting topic, but will not be discussed in this paper. The main theorem of this paper can be stated as follows.

Theorem 1. *Suppose that the Riemann problem* (1.3), (1.6) *admits a solution* $(\bar V, \bar U, \bar\Theta)(t, x)$ *defined in* (1.10)–(1.11) *which is a superposition of a 1-rarefaction wave and a 3-shock wave up to time* T. *If* $\gamma \in (1, 3)$ *and the viscosity and heat conductivity satisfy the relation* (1.4)*, then there exist positive constants* ε_0 *and* δ_0 *with* δ_0 *independent of* h *and* T*, such that if* $\varepsilon \in (0, \varepsilon_0]$*, and*

$$(\gamma - 1)|v_+ - v_*| \leqq \delta_0,$$

then for any small $h > 0$, the compressible Navier–Stokes system (1.1) *admits a family of smooth solutions* $\{(v^{\varepsilon,h}, u^{\varepsilon,h}, \theta^{\varepsilon,h})(t,x)\}$ *satisfying*

$$\sup_{h \leqq t \leqq T} \sup_{|x-s_3 t| \geqq h} |(v^{\varepsilon,h}, u^{\varepsilon,h}, \theta^{\varepsilon,h})(t,x) - (\bar V, \bar U, \bar \Theta)(t,x)| \leqq C_{h,T}\, \varepsilon^{\frac{1}{5}} |\ln \varepsilon|. \tag{1.12}$$

Here, the constant $C_{h,T}$ is independent of ε, but depends on h and T. Moreover,

$$(v^{\varepsilon,h}, u^{\varepsilon,h}, \theta^{\varepsilon,h})(t,x) \to (\bar V, \bar U, \bar \Theta)(t,x), \quad a.e. \text{ in } (0,T) \times \mathbf{R}, \quad \text{as } \varepsilon \to 0 \text{ and then } h \to 0.$$

Remark 1. Note that the analysis in this paper can be applied to the case when the Riemann solution is a superposition of one shock and a contact discontinuity. Hence, together with our previous work on the superposition of two rarefaction waves and a contact discontinuity [9], the only Riemann solutions remaining unsolved are two cases: two shocks and one contact discontinuity, and the general case, that is, one shock, one rarefaction wave and a contact discontinuity. These two cases will be pursued by the authors in the future.

The rest of the paper is organized as follows. In Section 2, we construct the approximate solution to the compressible Navier–Stokes equations (1.1) corresponding to the basic wave patterns to Euler system (1.3). Note that here we introduce the hyperbolic wave to recover the viscous terms to the inviscid approximate rarefaction wave. Thus we have detailed information of the difference between the Riemann solution to Euler system and the approximate solution to the Navier–Stokes system by the construction. In Section 3, we look for some exact solution to the compressible Navier–Stokes equations (1.1) around the approximate solution by energy methods and give the proof of the main result. Finally, we give the detailed proof of the estimates of the hyperbolic waves in the Appendix.

Notations. In this paper, we will use the notations $c, C, C_i (i = 1, 2, 3, \ldots)$ to denote generic constants. We use $\|\cdot\|$ to denote the standard $L_2(\mathbf{R}; dy)$ norm, and $\|\cdot\|_{H^i}$ $(i = 1, 2, 3, \ldots)$ to denote the Sobolev $H^i(\mathbf{R})$ norm. Sometimes, we also use $O(1)$ to denote a uniform bounded constant.

2. Construction of the Approximate Solution

2.1. Approximate Rarefaction Wave

Since there is no exact rarefaction wave profile for the compressible Navier–Stokes equations, the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Xin [18]. For completeness of presentation, we include its definition and properties in this subsection.

If $(v_-, u_-, \theta_-) \in R_1(v_+, u_+, \theta_+)$ as defined in (1.7), then there exists a 1-rarefaction wave $(v^{r_1}, u^{r_1}, \theta^{r_1})(x/t)$ which is a global solution of the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0, \\ (v, u, \theta)(0, x) = \begin{cases} (v_-, u_-, \theta_-), x < 0, \\ (v_+, u_+, \theta_+), x > 0. \end{cases} \end{cases} \tag{2.1}$$

Consider the following inviscid Burgers equation with Riemann data

$$\begin{cases} w_t + ww_x = 0, \\ w(t = 0, x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases}$$

If $w_- < w_+$, then the above Riemann problem admits a rarefaction wave solution

$$w^r(t, x) = w^r(\frac{x}{t}) = \begin{cases} w_-, & \frac{x}{t} \leqq w_-, \\ \frac{x}{t}, & w_- \leqq \frac{x}{t} \leqq w_+, \\ w_+, & \frac{x}{t} \geqq w_+. \end{cases}$$

As in [18], the approximate rarefaction wave $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ to the problem (1.1) can be constructed through the solution of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0, x) = w_\sigma(x) = w(\dfrac{x}{\sigma}) = \dfrac{w_+ + w_-}{2} + \dfrac{w_+ - w_-}{2} \tanh \dfrac{x}{\sigma}, \end{cases} \tag{2.2}$$

where $\sigma > 0$ is a small parameter to be determined. In fact, we take $\sigma = \varepsilon^{\frac{1}{5}}$ in the following parts of this paper. Note that the solution $w^r_\sigma(t, x)$ of the problem (2.2) is given by

$$w^r_\sigma(t, x) = w_\sigma(x_0(t, x)), \qquad x = x_0(t, x) + w_\sigma(x_0(t, x))t.$$

Note that $w^r_\sigma(t, x)$ has the following properties:

Lemma 2.1. *Let $w_- < w_+$, then (2.2) has a unique smooth solution $w^r_\sigma(t, x)$ satisfying*

(1) $w_- < w^r_\sigma(t, x) < w_+,\ (w^r_\sigma)_x(t, x) > 0$*;*

(2) *For any $p(1 \leqq p \leqq +\infty)$, there exists a constant C such that*

$$\begin{aligned} &\|\tfrac{\partial}{\partial x} w^r_\sigma(t, \cdot)\|_{L^p(\mathbf{R})} \leqq C \min\left\{(w_+ - w_-)\sigma^{-1+1/p},\ (w_+ - w_-)^{1/p} t^{-1+1/p}\right\}, \\ &\|\tfrac{\partial^2}{\partial x^2} w^r_\sigma(t, \cdot)\|_{L^p(\mathbf{R})} \leqq C \min\left\{(w_+ - w_-)\sigma^{-2+1/p},\ \sigma^{-1+1/p} t^{-1}\right\}; \end{aligned}$$

(3) *If $x - w_- t < 0$ and $w_- > 0$, then*

$$\begin{aligned} &|w^r_\sigma(t, x) - w_-| \leqq (w_+ - w_-) e^{-\frac{2|x - w_- t|}{\sigma}}, \\ &|\tfrac{\partial^k}{\partial x^k} w^r_\sigma(t, x)| \leqq \tfrac{2(w_+ - w_-)}{\sigma^k} e^{-\frac{2|x - w_- t|}{\sigma}},\ k = 1, 2; \end{aligned}$$

If $x - w_+ t > 0$ and $w_+ < 0$, then

$$\begin{aligned} &|w^r_\sigma(t, x) - w_+| \leqq (w_+ - w_-) e^{-\frac{2|x - w_+ t|}{\sigma}}, \\ &|\tfrac{\partial^k}{\partial x^k} w^r_\sigma(t, x)| \leqq \tfrac{2(w_+ - w_-)}{\sigma^k} e^{-\frac{2|x - w_+ t|}{\sigma}},\ k = 1, 2; \end{aligned}$$

(4) $\sup_{x\in\mathbf{R}} |w^r_\sigma(t,x) - w^r(\frac{x}{t})| \leqq C \min\left\{(w_+ - w_-), \frac{\sigma}{t}[\ln(1+t) + |\ln\sigma|]\right\}$.

The proof of statements (1), (2) and (4) can be found in [18], while the proof of statement (3) can be obtained similarly, as in [14].

Then the smooth approximate rarefaction wave profile denoted by $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ can be defined by

$$\begin{cases} w_\pm = \lambda_{1\pm} := \lambda_1(v_\pm, \theta_\pm), \\ w^r_\sigma(t,x) = \lambda_1(V^{R_1}(t,x), \Theta^{R_1}(t,x)), \\ s(V^{R_1}(t,x), \Theta^{R_1}(t,x)) = s_\pm = s(v_\pm, \theta_\pm), \\ U^{R_1}(t,x) = u_+ - \displaystyle\int_{v_+}^{V^{R_1}(t,x)} \lambda_1(v, s_+)\mathrm{d}v. \end{cases} \tag{2.3}$$

Note that $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$, defined above, satisfies

$$\begin{cases} V^{R_1}_t - U^{R_1}_x = 0, \\ U^{R_1}_t + P^{R_1}_x = 0, \\ \mathcal{E}^{R_1}_t + (P^{R_1}U^{R_1})_x = 0, \end{cases} \tag{2.4}$$

where $\mathcal{E}^{R_1} = \frac{R}{\gamma-1}\Theta^{R_1} + \frac{(U^{R_1})^2}{2}$ and $P^{R_1} = p(V^{R_1}, \Theta^{R_1})$.

By Lemma 2.1, the properties on the approximate rarefaction waves $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ can be summarized as follows.

Lemma 2.2. *The approximate rarefaction waves* $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ *constructed in* (2.3) *have the following properties:*

(1) $U^{R_1}_x(t,x) > 0$ *for* $x \in \mathbf{R}, t > 0$*;*

(2) *For any* $1 \leqq p \leqq +\infty$, *the following estimates hold,*

$$\begin{aligned} \|(V^{R_1}, U^{R_1}, \Theta^{R_1})_x\|_{L^p(dx)} &\leqq C \min\left\{\delta^{R_1}\sigma^{-1+1/p},\ (\delta^{R_1})^{1/p}t^{-1+1/p}\right\}, \\ \|(V^{R_1}, U^{R_1}, \Theta^{R_1})_{xx}\|_{L^p(dx)} &\leqq C \min\left\{\delta^{R_1}\sigma^{-2+1/p},\ \sigma^{-1+1/p}t^{-1}\right\}, \end{aligned}$$

where $\delta^{R_1} = |v_+ - v_-|$ *is the rarefaction wave strength and the positive constant* C *depends only on* p*;*

(3) *If* $x \geqq \lambda_{1+}t$, *then*

$$\begin{aligned} &|(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x) - (v_+, u_+, \theta_+)| \leqq C\mathrm{e}^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \\ &|\partial_x^k(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)| \leqq \frac{C}{\sigma^k}\mathrm{e}^{-\frac{2|x-\lambda_{1+}t|}{\sigma}},\ k = 1, 2; \end{aligned}$$

(4) *There exist positive constants* C *and* σ_0 *such that for* $\sigma \in (0, \sigma_0)$ *and* $t > 0$,

$$\sup_{x\in\mathbf{R}} |(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x) - (v^{r_1}, u^{r_1}, \theta^{r_1})(\frac{x}{t})| \leqq \frac{C}{t}\left[\sigma\ln(1+t) + \sigma|\ln\sigma|\right].$$

2.2. Hyperbolic Waves

Since we consider the case of the combination of a rarefaction wave and a shock wave, it is suitable to introduce anti-derivative variables to the perturbations of the solutions near the wave profiles. From (2.4), we know that the approximate rarefaction wave $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t,x)$ satisfies the compressible Euler equations exactly without viscous terms. Thus if we carry out the energy estimates to the anti-derivative variables, the error terms due to the viscous terms from the approximate rarefaction wave are not good enough to get the desired estimates. In order to overcome these difficulties, we introduce a new wave, called a hyperbolic wave. These hyperbolic waves play a crucial role in our analysis. Now we give a detailed description of these hyperbolic waves. Let the hyperbolic waves $(d_1, d_2, d_3)(t,x)$ satisfy the linear system

$$\begin{cases} d_{1t} - d_{2x} = 0, \\ d_{2t} + (p_v^{R_1} d_1 + p_u^{R_1} d_2 + p_E^{R_1} d_3)_x = \varepsilon \left(\dfrac{U_x^{R_1}}{V^{R_1}} \right)_x, \\ d_{3t} + [(pu)_v^{R_1} d_1 + (pu)_u^{R_1} d_2 + (pu)_E^{R_1} d_3]_x = \kappa \left(\dfrac{\Theta_x^{R_1}}{V^{R_1}} \right)_x + \varepsilon \left(\dfrac{U^{R_1} U_x^{R_1}}{V^{R_1}} \right)_x, \end{cases} \tag{2.5}$$

where $p = \frac{R\theta}{v} = p(v,u,E) = \frac{(\gamma-1)(2E-u^2)}{2v}$ and $p_v^{R_1} = p_v(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})$, etc. Now we want to solve this linear hyperbolic system (2.5) on the time interval $[h, T]$. First we diagonalize the above system. Rewrite the system (2.5) as

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_t + \left[A^{R_1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \right]_x = \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix},$$

where $B_1 = \varepsilon(\frac{U_x^{R_1}}{V^{R_1}})_x$, $B_2 = \kappa(\frac{\Theta_x^{R_1}}{V^{R_1}})_x + \varepsilon(\frac{U^{R_1}U_x^{R_1}}{V^{R_1}})_x$ and the matrix

$$A^{R_1} = \begin{pmatrix} 0 & -1 & 0 \\ p_v^{R_1} & p_u^{R_1} & p_E^{R_1} \\ (pu)_v^{R_1} & (pu)_u^{R_1} & (pu)_E^{R_1} \end{pmatrix}$$

with three distinct eigenvalues $\lambda_1^{R_1} := \lambda_1(V^{R_1}, s_\pm) < 0 = \lambda_2^{R_1} < \lambda_3(V^{R_1}, s_\pm) := \lambda_3^{R_1}$ and the corresponding left and right eigenvectors $l_j^{R_1}, r_j^{R_1}$ $(j = 1,2,3)$ satisfying

$$\begin{aligned} L^{R_1} A^{R_1} R^{R_1} &= \mathrm{diag}(\lambda_1^{R_1}, 0, \lambda_3^{R_1}) \equiv \Lambda^{R_1}, \\ L^{R_1} R^{R_1} &= \mathrm{Id.}, \end{aligned}$$

Here $L^{R_1} = (l_1^{R_1}, l_2^{R_1}, l_3^{R_1})^t$, $R^{R_1} = (r_1^{R_1}, r_2^{R_1}, r_3^{R_1})$ with $l_i^{R_1} = l_i(V^{R_1}, U^{R_1}, s_\pm)$ and $r_i^{R_1} = r_i(V^{R_1}, U^{R_1}, s_\pm)$ $(i = 1,2,3)$ and Id. is the 3×3 identity matrix. Now we set

$$(D_1, D_2, D_3)^t = L^{R_1}(d_1, d_2, d_3)^t, \tag{2.6}$$

then

$$(d_1, d_2, d_3)^t = R^{R_1}(D_1, D_2, D_3)^t, \tag{2.7}$$

and (D_1, D_2, D_3) satisfies the system

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}_t + \left[\Lambda^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} \right]_x = L^{R_1} \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix} + L_t^{R_1} R^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + L_x^{R_1} R^{R_1} \Lambda^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}. \tag{2.8}$$

Because the 1-Riemann invariant is constant along the approximate rarefaction wave curve, we have that

$$L_t^{R_1} = -\lambda_1^{R_1} L_x^{R_1}.$$

Substituting the above equation into (2.8), we obtain the diagonalized system

$$\begin{cases} D_{1t} + (\lambda_1^{R_1} D_1)_x = b_{12} B_1 + b_{13} B_2 + a_{12} V_x^{R_1} D_2 + a_{13} V_x^{R_1} D_3, \\ D_{2t} = b_{22} B_1 + b_{23} B_2 + a_{22} V_x^{R_1} D_2 + a_{23} V_x^{R_1} D_3, \\ D_{3t} + (\lambda_3^{R_1} D_3)_x = b_{32} B_1 + b_{33} B_2 + a_{32} V_x^{R_1} D_2 + a_{33} V_x^{R_1} D_3, \end{cases} \tag{2.9}$$

where a_{ij}, b_{ij} are given functions of V^{R_1}, U^{R_1} and $S^{R_1} = s_- = s_+$. Note that in the diagonalized system (2.9), the equations of D_2, D_3 are decoupled with D_1 due to the rarefaction wave structure of the system.

Now we impose the following conditions to the above linear hyperbolic system (2.9) on the domain $(t, x) \in [h, T] \times \mathbf{R}$:

$$D_1(t = h, x) = 0, \qquad D_2(t = T, x) = D_3(t = T, x) = 0. \tag{2.10}$$

Now we can solve the linear diagonalized hyperbolic system (2.9) under the conditions (2.10). Moreover, we have the following estimates:

Lemma 2.3. *There exist positive constant $C_{h,T}$ independent of ε, such that*

(1)

$$\left\| \frac{\partial^k}{\partial x^k} d_i(t, \cdot) \right\|_{L^2(\mathbf{R})}^2 \leqq C_{h,T} \frac{\varepsilon^2}{\sigma^{2k+1}}, \quad i = 1, 2, 3, \ k = 0, 1, 2, 3.$$

(2) *If $x > \lambda_{1+}t$, then we have*

$$|d_i(x, t)| \leqq C_{h,T} \frac{1}{\sigma} e^{-\frac{|x - \lambda_{1+}t|}{\sigma}},$$
$$|d_{ix}(x, t)| \leqq C_{h,T} \frac{1}{\sigma^2} e^{-\frac{|x - \lambda_{1+}t|}{\sigma}}, \ i = 1, 2, 3.$$

The proof of Lemma 2.3 will be shown in the Appendix.

2.3. Viscous Shock Wave

If $(v_-, u_-, \theta_-) \in S_3(v_+, u_+, \theta_+)$, then the Riemann problem (1.3), (1.6) admits a 3-shock wave

$$(v^{s_3}, u^{s_3}, \theta^{s_3})(t,x) = \begin{cases} (v_-, u_-, \theta_-), & x < s_3 t, \\ (v_+, u_+, \theta_+), & x > s_3 t. \end{cases} \tag{2.11}$$

It is well known that the compressible Navier–Stokes system (1.1) admits a smooth traveling wave solution with shock profile $(V^{S_3}, U^{S_3}, \Theta^{S_3})(x - s_3 t)$ under the conditions (1.8) and (1.9) (see [15]).

We first recall some properties of the viscous 3-shock wave. The shock profile $(V^{S_3}, U^{S_3}, \Theta^{S_3})(\xi)$, $\xi = x - s_3 t$, is determined by

$$\begin{cases} -s_3 (V^{S_3})' - (U^{S_3})' = 0, \\ -s_3 (U^{S_3})' + (P^{S_3})' = \varepsilon (\frac{(U^{S_3})'}{V^{S_3}})', \\ -s_3 (\mathcal{E}^{S_3})' + (P^{S_3} U^{S_3})' = \kappa (\frac{(\Theta^{S_3})'}{V^{S_3}})' + \varepsilon (\frac{U^{S_3} (U^{S_3})'}{V^{S_3}})', \\ \left(V^{S_3}, U^{S_3}, \Theta^{S_3}\right)(\pm\infty) = (v_\pm, u_\pm, \theta_\pm), \end{cases} \tag{2.12}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$, $P^{S_3} = p(V^{S_3}, \Theta^{S_3})$, $\mathcal{E}^{S_3} = \frac{R}{\gamma - 1} \Theta^{S_3} + \frac{(U^{S_3})^2}{2}$ and $(v_\pm, u_\pm, \theta_\pm)$ satisfy R-H condition (1.8) and Lax entropy condition (1.9), and s_3 is uniquely determined by (1.8). Integrating (2.12) on $(\pm\infty, \xi)$ gives

$$\begin{cases} \dfrac{s_3 \varepsilon V^{S_3}_\xi}{V^{S_3}} = -\left[P^{S_3} + s_3^2 \left(V^{S_3} - \dfrac{b_1}{s_3^2} \right) \right], \\ \dfrac{\kappa \Theta^{S_3}_\xi}{s_3 V^{S_3}} = -\left[\mathcal{E}^{S_3} - \dfrac{s_3^2}{2} \left(V^{S_3} - \dfrac{b_1}{s_3^2} \right)^2 + \dfrac{b_1^2}{2 s_3^2} - b_2 \right], \\ U^{S_3} = -(s_3 V^{S_3} + a), \end{cases} \tag{2.13}$$

where $a = -(s_3 v_\pm + u_\pm)$, $b_1 = p_\pm + s_3^2 v_\pm$, $b_2 = e_\pm + p_\pm v_\pm + s_3^2 \frac{v_\pm^2}{2}$ and $p_\pm = p(v_\pm, \theta_\pm)$. From [4,11], we have the following Lemma:

Lemma 2.4. *Assume that the two states* $(v_\pm, u_\pm, \theta_\pm)$ *satisfy R-H condition* (1.8) *and Lax entropy condition* (1.9)*; then there exists a unique shock profile* $(V^{S_3}, U^{S_3}, \Theta^{S_3})(\xi)$*, up to a shift, of the ODE system* (2.12)*. Moreover, there are positive constants C and c independent of* $\gamma > 1$ *such that*

$$\begin{cases} s_3 V_\xi^{S_3} = -U_\xi^{S_3} > 0, \quad \Theta_\xi^{S_3} < 0, \\ \left(|V^{S_3} - v_\pm|, |U^{S_3} - u_\pm|, \dfrac{|\Theta^{S_3} - \theta_\pm|}{\gamma - 1} \right) \leqq C\delta^{S_3} \mathrm{e}^{-c\frac{\delta^{S_3}|\xi|}{\varepsilon}}, \quad \text{as } \xi \to \pm\infty, \\ \left| \left(V_\xi^{S_3}, U_\xi^{S_3}, \dfrac{\Theta_\xi^{S_3}}{\gamma - 1} \right) \right| \leqq C \dfrac{(\delta^{S_3})^2}{\varepsilon} \mathrm{e}^{-c\frac{\delta^{S_3}|\xi|}{\varepsilon}}, \quad \text{as } \xi \to \pm\infty, \\ \left| \dfrac{\Theta_\xi^{S_3}}{V_\xi^{S_3}} \right| \leqq C(\gamma - 1), \\ s_3^2 = \dfrac{\gamma R \theta_-}{v_+ v_-}(1 - q_+), \quad \theta_+ = \theta_- \left(1 - \dfrac{v_+ + v_-}{v_+} q_+ \right), \end{cases} \tag{2.14}$$

where $\delta^{S_3} = O(1)|v_+ - v_-|$ is the strength of the 3-shock wave and $q_+ = \frac{m_+}{1+m_+}$ with $m_+ = \frac{(\gamma-1)\delta^{S_3}}{2v_+}$.

2.4. Superposition of Rarefaction Wave and Shock Wave

The approximate wave profile $(V, U, \Theta)(t, x)$ with the superposition of the 1-rarefaction and 3-shock waves to the compressible Navier–Stokes equations can be defined by

$$\begin{pmatrix} V \\ U \\ \mathcal{E} \end{pmatrix} (t, x) = \begin{pmatrix} V^{R_1} + d_1 + V^{S_3} - v_* \\ U^{R_1} + d_2 + U^{S_3} - u_* \\ \mathcal{E}^{R_1} + d_3 + \mathcal{E}^{S_3} - E_*, \end{pmatrix} (t, x) \tag{2.15}$$

where $(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})(t, x)$ is the approximate 1-rarefaction wave defined in (2.3) with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*), $(d_1, d_2, d_3)(t, x)$ are the hyperbolic waves defined in (2.5), and $(V^{S_3}, U^{S_3}, \mathcal{E}^{S_3})(t, x)$ is the viscous 3-shock wave defined in (2.14) with the left state (v_-, u_-, θ_-) replaced by (v_*, u_*, θ_*).

From (2.15), we have

$$\Theta = \Theta^{R_1} + \Theta^{S_3} - \theta_* + \frac{\gamma - 1}{R} d_3 - \frac{\gamma - 1}{2R} [U^2 - (U^{R_1})^2 - (U^{S_3})^2 + u_*^2]. \tag{2.16}$$

Thus, from the construction of the approximate rarefaction wave and Lemmas 2.2–2.4, we have the following relation between the approximate wave pattern $(V, U, \mathcal{E}, \Theta)(t, x)$ of the compressible Navier–Stokes equations and the exact inviscid wave pattern $(\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)$ to the Euler equations

$$\begin{aligned} &|(V, U, \mathcal{E}, \Theta)(t, x) - (\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)| \\ &\quad \leqq C\Big[|(V^{R_1} - v^{r_1}, U^{R_1} - u^{r_1}, \mathcal{E}^{R_1} - E^{r_1}, \Theta^{R_1} - \theta^{r_1})| + |(d_1, d_2, d_3)| \\ &\qquad + |(V^{S_3} - v^{s_3}, U^{S_3} - u^{s_3}, \mathcal{E}^{S_3} - E^{s_3}, \Theta^{S_3} - \theta^{s_3})| \Big] \\ &\quad \leqq C\Big[\frac{1}{t}\big(\sigma \ln(1 + t) + \sigma |\ln \sigma|\big) + \frac{\varepsilon}{\sigma} + \delta^{S_3} \mathrm{e}^{-c\delta^{S_3}\frac{|x - s_3 t|}{\varepsilon}} \Big], \end{aligned} \tag{2.17}$$

with $\sigma = \varepsilon^{\frac{1}{5}}$.

Then the superposition wave $(V, U, \mathcal{E})(t, x)$ satisfies the system

$$\begin{cases} V_t - U_x = 0, \\ U_t + P_x = \varepsilon \left(\dfrac{U_x}{V}\right)_x + Q_{1x}, \\ \mathcal{E}_t + (PU)_x = \kappa \left(\dfrac{\Theta_x}{V}\right)_x + \varepsilon \left(\dfrac{UU_x}{V}\right)_x + Q_{2x}, \end{cases} \tag{2.18}$$

where $P = p(V, \Theta)$ and

$$\begin{aligned} Q_1 &= [P - P^{R_1} - P^{S_3} + p_* - (p_v^{R_1} d_1 + p_u^{R_1} d_2 + p_E^{R_1} d_3)] - \varepsilon \left(\frac{U_x}{V} - \frac{U_x^{R_1}}{V^{R_1}} - \frac{U_x^{S_3}}{V^{S_3}}\right) \\ &= O(1)\Big[(d_1, d_2, d_3)^2 + \varepsilon|d_{2x}| + \varepsilon|U_x^{R_1} d_1|\Big] \\ &\quad + O(1)\Big[|(V^{S_3} - v_*, \Theta^{S_3} - \theta_*)||(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3)| \\ &\quad + \varepsilon|U_x^{S_3}||(V^{R_1} - v_*, d_1)| + \varepsilon|(U_x^{R_1}, d_{2x})||V^{S_3} - v_*|\Big] \\ &:= Q_{11} + Q_{12}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} Q_2 &= \Big[PU - P^{R_1} U^{R_1} - P^{S_3} U^{S_3} + p_* u_* - ((pu)_v^{R_1} d_1 + (pu)_u^{R_1} d_2 + (pu)_E^{R_1} d_3)\Big] \\ &\quad - \kappa \left(\frac{\Theta_x}{V} - \frac{\Theta_x^{R_1}}{V^{R_1}} - \frac{\Theta_x^{S_3}}{V^{S_3}}\right) - \varepsilon \left(\frac{UU_x}{V} - \frac{U^{R_1} U_x^{R_1}}{V^{R_1}} - \frac{U^{S_3} U_x^{S_3}}{V^{S_3}}\right) \\ &= O(1)\Big[(d_1, d_2, d_3)^2 + \varepsilon|(d_{2x}, d_{3x})| + \varepsilon|(U_x^{R_1}, \Theta_x^{R_1})(d_1, d_2)|\Big] \\ &\quad + O(1)\Big[|(V^{S_3} - v_*, \Theta^{S_3} - \theta_*, U^{S_3} - u_*)|| \\ &\quad (V^{R_1} - v_*, \Theta^{R_1} - \theta_*, U^{R_1} - u_*, d_1, d_2, d_3)| \\ &\quad + \varepsilon|(U_x^{S_3}, \Theta_x^{S_3})||(V^{R_1} - v_*, U^{R_1} - u_*, d_1, d_2)| \\ &\quad + \varepsilon|(U_x^{R_1}, \Theta_x^{R_1}, d_{2x}, d_{3x})||(V^{S_3} - v_*, U^{S_3} - u_*)|\Big] \\ &:= Q_{21} + Q_{22}. \end{aligned} \tag{2.20}$$

Note that the error terms $Q_{i1} (i = 1, 2)$ come from the hyperbolic waves $(d_1, d_2, d_3)(t, x)$ and the error terms $Q_{i2} (i = 1, 2)$ are the wave interactions of the viscous shock wave and the approximate rarefaction wave and hyperbolic waves. Fortunately, the estimates to these error terms are good enough to close the energy analysis by suitably choosing σ in the approximate rarefaction wave. In fact, we take $\sigma = \varepsilon^{\frac{1}{5}}$ in this paper.

Now we calculate the estimations of the superposition terms in (2.19) and (2.20). For this, we divide the whole space $[h, T] \times \mathbf{R}$ by the following two parts:

$$\Omega^{R_1} = \{(t, x) | x \leqq \frac{(\lambda_1^* + s_3)t}{2}, \quad t \in [h, T]\},$$

and

$$\Omega^{S_3} = \{(t, x) | x > \frac{(\lambda_1^* + s_3)t}{2}, \quad t \in [h, T]\}.$$

Then by the Lax entropy condition, we have

$$s_3 < \lambda_3^* = -\lambda_1^*.$$

So we have that in Ω^{R_1}, $x \leqq \frac{(\lambda_1^*+s_3)t}{2} < 0$. Thus we can get

- in Ω^{R_1},

$$|(V^{S_3} - v_*, \Theta^{S_3} - \theta_*, U^{S_3} - u_*)| = O(1)\delta^{S_3} \mathrm{e}^{-c\frac{\delta^{S_3}|x-s_3t|}{\varepsilon}},$$
$$|(U_x^{S_3}, \Theta_x^{S_3})| = O(1)\frac{(\delta^{S_3})^2}{\varepsilon} \mathrm{e}^{-c\frac{\delta^{S_3}|x-s_3t|}{\varepsilon}}.$$

Since $s_3 > 0 > \lambda_1^*$, we have in Ω^{S_3},

$$x > \frac{(\lambda_1^* + s_3)t}{2} > \lambda_1^* t.$$

Then by Lemma 2.3, we can get

- in Ω^{S_3},

$$|(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3)| = O(1)\frac{1}{\sigma} \mathrm{e}^{-\frac{c|x-\lambda_{1*}t|}{\sigma}},$$
$$|(U_x^{R_1}, \Theta_x^{R_1}, d_{1x}, d_{2x}, d_{3x})| = O(1)\frac{1}{\sigma^2} \mathrm{e}^{-\frac{c|x-\lambda_{1*}t|}{\sigma}}.$$

So we have the following wave interaction estimates:

$$|(Q_{12}, Q_{22})| = O(1)\mathrm{e}^{-\frac{c|x|}{\sigma}} \mathrm{e}^{-\frac{C_h}{\sigma}}. \tag{2.21}$$

3. Proof of Theorem 1

In this section, we will prove our main result, Theorem 1. First we reformulate the system by introducing a scaling for the independent variables. Set

$$y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}.$$

We also use the notations $(v, u, , \theta, E)(\tau, y)$, $(V, U, \Theta, \mathcal{E})(\tau, y)$ in the scaled independent variables. Set the perturbation around the composite wave profile $(V, U, \Theta)(\tau, y)$ by

$$(\phi, \psi, \zeta, \omega)(\tau, y) = (v - V, u - U, \theta - \Theta, E - \mathcal{E})(\tau, y).$$

From (1.1) and (2.18), we have the following system for the perturbation (ϕ, ψ, ω):

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + (p - P)_y = \left(\dfrac{u_y}{v} - \dfrac{U_y}{V}\right)_y - Q_{1y}, \\ \omega_\tau + (pu - PU)_y = \nu\left(\dfrac{\theta_y}{v} - \dfrac{\Theta_y}{V}\right)_y + \left(\dfrac{uu_y}{v} - \dfrac{UU_y}{V}\right)_y - Q_{2y}, \end{cases} \tag{3.1}$$

where the error terms Q_i $(i = 1, 2)$ are given in (2.19) and (2.20) respectively. In particular, we can choose the initial data of the system (3.1) as

$$(\phi, \psi, \omega)(\tau = h/\varepsilon, y) = (0, 0, 0).$$

Introduce the anti-derivative variables

$$(\Phi, \Psi, \bar{W})(\tau, y) = \int_{-\infty}^{y} (\phi, \psi, \omega)(\tau, y')\mathrm{d}y'.$$

Then $(\Phi, \Psi, \bar{W})(\tau, y)$ satisfies that

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau + (p - P) = \left(\dfrac{u_y}{v} - \dfrac{U_y}{V}\right) - Q_1, \\ \bar{W}_\tau + (pu - PU) = \nu\left(\dfrac{\theta_y}{v} - \dfrac{\Theta_y}{V}\right) + \left(\dfrac{uu_y}{v} - \dfrac{UU_y}{V}\right) - Q_2. \end{cases} \tag{3.2}$$

To capture the dissipation of the perturbed energy equation $(3.2)_3$, we introduce another variable related to the absolute temperature

$$W = \frac{\gamma - 1}{R}(\bar{W} - U\Psi),$$

then

$$\zeta = W_y - \frac{\gamma - 1}{R}\left(\frac{\Psi_y^2}{2} - U_y\Psi\right). \tag{3.3}$$

Now we linearize system (3.2) around the superposition wave profile $(V, U, \Theta)(\tau, y)$ defined in (2.15). For the perturbed momentum equation $(3.2)_2$, noting that

$$\begin{aligned} p - P &= \frac{R\zeta}{V} - \frac{p}{V}\Phi_y \\ &= \frac{R}{V}\zeta - \frac{P}{V}\Phi_y - \frac{p - P}{V}\Phi_y \\ &= \frac{R}{V}W_y - \frac{P}{V}\Phi_y + \frac{\gamma - 1}{V}U_y\Psi - \frac{\gamma - 1}{2V}\Psi_y^2 - \frac{p - P}{V}\Phi_y, \end{aligned}$$

and

$$\begin{aligned} \frac{u_y}{v} - \frac{U_y}{V} &= \frac{\Psi_{yy}}{V} + (U_y + \Psi_{yy})\left(\frac{1}{v} - \frac{1}{V}\right) \\ &= \frac{\Psi_{yy}}{V} - \frac{U_y}{V^2}\Phi_y + \frac{U_y\Phi_y^2}{vV^2} - \frac{\Phi_y\Psi_{yy}}{vV}, \end{aligned}$$

we have

$$\Psi_\tau - \frac{\eta}{V}\Phi_y + \frac{R}{V}W_y + \frac{\gamma - 1}{V}U_y\Psi = \frac{\Psi_{yy}}{V} + F_1 - Q_1,$$

with

$$\eta = P - \frac{U_y}{V}, \tag{3.4}$$

and the nonlinear term

$$\begin{aligned} F_1 &= \frac{p-P}{V}\Phi_y + \frac{\gamma-1}{2V}\Psi_y^2 - \frac{\Psi_{yy}\Phi_y}{vV} + \frac{U_y\Phi_y^2}{vV^2} \\ &= O(1)|(\phi, \psi, \zeta, \psi_y)|^2. \end{aligned} \tag{3.5}$$

For the perturbed energy equation $(3.2)_3$, we can first derive the equation for the variable W with

$$\frac{R}{\gamma-1}W_\tau + U_\tau\Psi + p(u-U) = \nu\left(\frac{\theta_y}{v} - \frac{\Theta_y}{V}\right) + \frac{u_y}{v}(u-U) - Q_2 + UQ_1. \tag{3.6}$$

We can compute that

$$\begin{aligned} p(u-U) &= P\Psi_y + (p-P)\Psi_y, \\ \nu\left(\frac{\theta_y}{v} - \frac{\Theta_y}{V}\right) &= \frac{\nu\zeta_y}{V} + \nu(\Theta_y + \zeta_y)\left(\frac{1}{v} - \frac{1}{V}\right) \\ &= \frac{\nu}{V}W_{yy} + \frac{(\gamma-1)\nu}{RV}(U_y\Psi)_y - \frac{\nu\Theta_y}{V^2}\Phi_y \\ &\quad -\frac{(\gamma-1)\nu}{RV}\Psi_y\Psi_{yy} - \frac{\nu\Phi_y\zeta_y}{vV} + \frac{\nu\Theta_y}{vV^2}\Phi_y^2, \end{aligned}$$

and

$$\frac{u_y}{v}(u-U) = \frac{U_y}{V}\Psi + \left(\frac{u_y}{v} - \frac{U_y}{V}\right)\Psi_y.$$

Thus, substituting the above three equalities into (3.6), we have

$$\frac{R}{\gamma-1}W_\tau + \eta\Psi_y + U_\tau\Psi - \frac{(\gamma-1)\nu}{RV}(U_y\Psi)_y + \frac{\nu\Theta_y}{V^2}\Phi_y = \frac{\nu}{V}W_{yy} + F_2 - Q_2 + UQ_1,$$

with the nonlinear term

$$\begin{aligned} F_2 &= -(p-P)\Psi_y - \frac{(\gamma-1)\nu}{RV}\Psi_y\Psi_{yy} - \frac{\nu\Phi_y\zeta_y}{vV} + \frac{\nu\Theta_y}{vV^2}\Phi_y^2 + \left(\frac{u_y}{v} - \frac{U_y}{V}\right)\Psi_y \\ &= O(1)|(\phi, \psi, \zeta, \psi_y, \zeta_y)|^2. \end{aligned} \tag{3.7}$$

In summary, we obtain the following linearized system for (Φ, Ψ, W):

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau - \frac{\eta}{V}\Phi_y + \frac{R}{V}W_y + \frac{\gamma-1}{V}U_y\Psi = \frac{\Psi_{yy}}{V} + F_1 - Q_1, \\ \frac{R}{\gamma-1}W_\tau + \eta\Psi_y + U_\tau\Psi - \frac{(\gamma-1)\nu}{RV}(U_y\Psi)_y + \frac{\nu\Theta_y}{V^2}\Phi_y \\ \qquad = \frac{\nu}{V}W_{yy} + F_2 - Q_2 + UQ_1, \end{cases} \tag{3.8}$$

with η defined in (3.4) and F_i $(i = 1, 2)$ being the nonlinear terms given by (3.5) and (3.7), respectively.

To prove Theorem 1, it is sufficient to prove the following proposition due to the fact of (2.17).

Proposition 3.1. *Suppose that $\gamma \in (1, 3)$. Then there exist positive constants ε_0 and δ_0 with δ_0 independent of h and T, such that if $\varepsilon \in (0, \varepsilon_0]$ and*

$$(\gamma - 1)|v_+ - v_*| \leqq \delta_0,$$

then for any $h > 0$ small enough, system (3.8) admits a smooth solution $(\Phi, \Psi, W)(\tau, y)$ on the time interval $\left[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}\right]$ satisfying

$$\sup_{\tau\in[\frac{h}{\varepsilon},\frac{T}{\varepsilon}]} \Big[\|(\Phi, \Psi, W)(\tau, \cdot)\|_{L^2} + \|(\phi, \psi, \zeta)(\tau, \cdot)\|_{H^1}\Big] \leqq C_{h,T}\, \varepsilon^{\frac{1}{5}}. \tag{3.9}$$

where the constant $C_{h,T}$ is independent of ε, but may depend on h and T.

Thus, to prove Proposition 3.1 we will focus on the reformulated system (3.8). Since the local existence of the solution to (3.8) is standard, to prove global existence on the interval $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$, we need only to close the following a priori estimate by the continuity argument

$$\mathcal{N}(\tau) = \sup_{\frac{h}{\varepsilon}\leqq\tau'\leqq\tau} \Big[\|(\Phi, \Psi, W)(\tau, \cdot)\|^2_{L^2} + \|(\phi, \psi, \zeta)(\tau, \cdot)\|^2_{H^1}\Big] \leqq \chi^2, \quad \tau \in \left[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}\right] \tag{3.10}$$

where χ is a small positive constant depending only on the initial data and the viscosity ε. Proof of the above a priori estimate (3.10) and Proposition 3.1 are given by the following energy estimates in Lemmas 3.1–3.5 as [11].

Lemma 3.1. *Under the assumptions of Proposition 3.1, there exists a positive constant C independent of ε such that*

$$\begin{aligned}
&\sup_{\frac{h}{\varepsilon}\leqq\tau_1\leqq\tau} \left\|\left(\Phi, \Psi, \frac{W}{\sqrt{\gamma-1}}\right)(\tau_1, \cdot)\right\|^2 \\
&\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \left[\left\|\sqrt{|U^{S_3}_{1y}|}\left(\Psi, \frac{W}{\sqrt{\gamma-1}}\right)\right\|^2 + \left\|(\Psi_y, W_y)\right\|^2\right] \mathrm{d}\tau \\
&\leqq C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 \mathrm{d}\tau + C\mathcal{N}(\tau) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\zeta, \zeta_y, \psi_y)\|^2 \mathrm{d}\tau \\
&\quad + C\big[(\gamma-1)|v_+ - v_*| + \mathcal{N}(\tau)\big] \int_{\frac{h}{\varepsilon}}^{\tau} \|\Phi_y\|^2 \mathrm{d}\tau + C\varepsilon^{\frac{2}{5}}.
\end{aligned} \tag{3.11}$$

Proof. First, from the fact

$$v_- < V^{R_1} < v_* < V^{S_3} < v_+,$$

we have

$$v_- - \|d_1\|_{L^\infty} \leqq V \leqq v_+ + \|d_1\|_{L^\infty},$$

thus

$$\frac{v_-}{2} \leqq V \leqq 2v_+, \quad \text{if} \quad \varepsilon \ll 1.$$

From $(2.12)_1$, $(2.12)_2$ and recalling $b_1 = p_+ + s_3^2 v_+ = p_* + s_3^2 v_*$, we can obtain

$$P^{S_3} - \frac{U_y^{S_3}}{V^{S_3}} = b_1 - s_3^2 V^{S_3}. \tag{3.12}$$

Then by the fact that $v_* < V^{S_3} < v_+$, we have

$$0 < p_+ \leqq b_1 - s_3^2 V^{S_3} \leqq p_*. \tag{3.13}$$

We can compute that

$$\begin{aligned}
\eta = P - \frac{U_y}{V} &= \left(P^{S_3} - \frac{U_y^{S_3}}{V^{S_3}}\right) + \left(P^{R_1} - p_*\right) - \frac{U_y^{R_1}}{V^{R_1}} \\
&\quad + \left(P - P^{R_1} - P^{S_3} + p_*\right) - \left(\frac{U_y}{V} - \frac{U_y^{R_1}}{V^{R_1}} - \frac{U_y^{S_3}}{V^{S_3}}\right) \\
&= \left(b_1 - s_3^2 V^{S_3}\right) + \left(P^{R_1} - p_*\right) - \frac{U_y^{R_1}}{V^{R_1}} + Q_3,
\end{aligned} \tag{3.14}$$

where Q_3 is given by

$$\begin{aligned}
Q_3 &= (P - P^{R_1} - P^{S_3} + p_*) - \left(\frac{U_y}{V} - \frac{U_y^{R_1}}{V^{R_1}} - \frac{U_y^{S_3}}{V^{S_3}}\right) \\
&= Q_1 + (p_v^{R_1} d_1 + p_u^{R_1} d_2 + p_E^{R_1} d_3),
\end{aligned} \tag{3.15}$$

with $Q_1 := Q_{11} + Q_{12}$ defined in (2.19). So from (2.19) and wave interaction estimates (2.21) for Q_{12}, we have

$$\begin{aligned}
Q_3 &= |Q_{11}| + |Q_{12}| + O(1)|(d_1, d_2, d_3)| \\
&= O(1)\Big[|(d_1, d_2, d_3)|^2 + \varepsilon|d_{2x}| + \varepsilon|U_x^{R_1} d_1|\Big] + O(1)\mathrm{e}^{-\frac{C_h}{\sigma}} + O(1)|(d_1, d_2, d_3)| \\
&= O(1)\left(\frac{\varepsilon}{\sigma}\right)^2 + O(1)\mathrm{e}^{-\frac{C_h}{\sigma}} + O(1)\frac{\varepsilon}{\sigma} \\
&= O(1)\varepsilon^{\frac{4}{5}},
\end{aligned} \tag{3.16}$$

where we have used estimate (1) of the hyperbolic wave in Lemma 2.3 and the fact that $\sigma = \frac{1}{5}$.

On the other hand, from the properties of the approximate rarefaction waves in Lemma 2.2, we have

$$p_* \leqq P^{R_1} \leqq p_-, \quad \text{and} \;\; U_y^{R_1} = O(1)\varepsilon. \tag{3.17}$$

Substituting (3.13), (3.16) and (3.17) into (3.14), we have that there exist positive constants c and C such that

$$0 < c \leqq \eta \leqq C, \tag{3.18}$$

provided $\varepsilon \ll 1$.

Multiplying $(3.8)_1$ by Φ, $(3.8)_2$ by $\frac{V}{\eta}\Psi$, $(3.8)_3$ by $\frac{R}{\eta^2}W$, respectively, we can get the following three equalities:

$$\left(\frac{\Phi^2}{2}\right)_\tau - \Phi\Psi_y = 0, \tag{3.19}$$

$$\begin{aligned}&\left(\frac{V\Psi^2}{2\eta}\right)_\tau - \left(\frac{V}{2\eta}\right)_\tau \Psi^2 - \Phi_y\Psi + \frac{R}{\eta}W_y\Psi + \frac{\gamma-1}{\eta}U_y\Psi^2\\ &\quad = \left(\frac{\Psi\Psi_y}{\eta}\right)_y - \frac{\Psi_y^2}{\eta} - \left(\frac{1}{\eta}\right)_y \Psi_y^2 + \frac{V}{\eta}(F_1 - Q_1)\Psi,\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left[\frac{R^2}{2(\gamma-1)\eta^2}W^2\right]_\tau - \left[\frac{R^2}{2(\gamma-1)\eta^2}\right]_\tau W^2 + \frac{R}{\eta}W\Psi_y + \frac{R}{\eta^2}U_\tau\Psi W\\ &\quad -\left[\frac{(\gamma-1)\nu}{V\eta^2}U_y\Psi W\right]_y + \left[\frac{(\gamma-1)\nu}{V\eta^2}\right]_y U_y\Psi W + \frac{(\gamma-1)\nu}{V\eta^2}U_y\Psi W_y + \frac{R\nu}{V^2\eta^2}\Theta_y\Phi_y W\\ &= \left(\frac{R\nu}{V\eta^2}WW_y\right)_y - \frac{R\nu}{V\eta^2}W_y^2 - \left(\frac{R\nu}{V\eta^2}\right)_y WW_y + \frac{R}{\eta^2}(F_2 + Q_2 - UQ_1)W.\end{aligned} \tag{3.21}$$

Collecting the above three equations (3.19), (3.20) and (3.21) together, we can get

$$\begin{aligned}&I_1(\Phi, \Psi, W)_\tau + I_2(\Psi, \Psi_y) + I_3(W, W_y) = I_4(\Psi, W, \Phi_y, W_y)\\ &\quad + \frac{V}{\eta}(F_1 - Q_1)\Psi + \frac{R}{\eta^2}(F_2 + Q_2 - UQ_1)W + (\cdots)_y,\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}I_1(\Phi, \Psi, W) &= \frac{\Phi^2}{2} + \frac{V}{2\eta}\Psi^2 + \frac{R^2}{2(\gamma-1)\eta^2}W^2,\\ I_2(\Psi, \Psi_y) &= \left[\frac{\gamma-1}{\eta}U_y - \left(\frac{V}{2\eta}\right)_\tau\right]\Psi^2 + \left(\frac{1}{\eta}\right)_y \Psi\Psi_y + \frac{1}{\eta}\Psi_y^2,\\ I_3(W, W_y) &= -\left[\frac{R^2}{2(\gamma-1)\eta^2}\right]_\tau W^2 + \frac{R\nu}{V\eta^2}W_y^2,\end{aligned}$$

$$I_4(\Psi, W, \Phi_y, W_y) = \left[-\left(\frac{R}{\eta}\right)_y + \frac{R}{\eta^2}U_\tau \right]\Psi W - \left(\frac{R\nu}{V\eta^2}\right)_y WW_y$$
$$+\frac{(\gamma-1)\nu}{V\eta^2}U_y\Psi W_y + \Big[\frac{(\gamma-1)\nu}{V\eta^2}\Big]_y U_y\Psi W + \frac{R\nu}{V^2\eta^2}\Theta_y\Phi_y W, \tag{3.23}$$

and $(\cdots)_y$ denotes the conservative terms which disappear after integration with respect to y over $\mathbf{R}$.

From (3.18), we have

$$c\left(\Phi^2 + \Psi^2 + \frac{W^2}{\gamma-1}\right) \leqq I_1 \leqq C\left(\Phi^2 + \Psi^2 + \frac{W^2}{\gamma-1}\right), \tag{3.24}$$

for some positive constants c and C.

Now we estimate $I_i,\ i = 2, 3, 4$ term by term. Note that

$$\begin{aligned}
\eta_\tau &= \left(P - \frac{U_y}{V}\right)_\tau \\
&= \left(P^{S_3} - \frac{U_y^{S_3}}{V^{S_3}}\right)_\tau + \left(P^{R_1} - \frac{U_y^{R_1}}{V^{R_1}}\right)_\tau + Q_4 \\
&= -s_3^2 U_y^{S_3} + \left(P^{R_1} - \frac{U_y^{R_1}}{V^{R_1}}\right)_\tau + Q_4,
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
Q_4 &= (P - P^{R_1} - P^{S_3})_\tau - \Big(\frac{U_y}{V} - \frac{U_y^{R_1}}{V^{R_1}} - \frac{U_y^{S_3}}{V^{S_3}}\Big)_\tau \\
&= R\Big(\frac{\Theta_\tau}{V} - \frac{\Theta_\tau^{R_1}}{V^{R_1}} - \frac{\Theta_\tau^{S_3}}{V^{S_3}}\Big) - R\Big(\frac{\Theta V_\tau}{V^2} - \frac{\Theta^{R_1}V_\tau^{R_1}}{V^{R_1}} - \frac{\Theta^{S_3}V_\tau^{S_3}}{V^{S_3}}\Big) \\
&\quad -\Big(\frac{U_{y\tau}}{V} - \frac{U_{y\tau}^{R_1}}{V^{R_1}} - \frac{U_{y\tau}^{S_3}}{V^{S_3}}\Big) + \Big[\frac{U_yV_\tau}{V^2} - \frac{U_y^{R_1}V_\tau^{R_1}}{(V^{R_1})^2} - \frac{U_y^{S_3}V_\tau^{S_3}}{(V^{S_3})^2}\Big] \\
&:= \sum_{i=1}^4 Q_{4i}.
\end{aligned} \tag{3.26}$$

By the definition of the superposition wave profile (2.15) and (2.16), we have

$$\begin{aligned}
Q_{41} &= R\Theta_\tau^{R_1}\left(\frac{1}{V} - \frac{1}{V^{R_1}}\right) + R\Theta_\tau^{S_3}\left(\frac{1}{V} - \frac{1}{V^{S_3}}\right) + \frac{\gamma-1}{V}d_{3\tau} \\
&\quad -\frac{\gamma-1}{V}\big(UU_\tau - U^{R_1}U_\tau^{R_1} - U^{S_3}U_\tau^{S_3}\big) \\
&= R\Theta_\tau^{R_1}\left(\frac{1}{V} - \frac{1}{V^{R_1}}\right) + R\Theta_\tau^{S_3}\left(\frac{1}{V} - \frac{1}{V^{S_3}}\right) + \frac{\gamma-1}{V}d_{3\tau} \\
&\qquad -\frac{\gamma-1}{V}\big[U_\tau^{R_1}(U - U^{R_1}) + U_\tau^{S_3}(U - U^{S_3})\big] - \frac{\gamma-1}{V}Ud_{2\tau} \\
&= O(1)\varepsilon\Big[|(\Theta_t^{R_1}, U_t^{R_1})||(V^{S_3} - v_*, U^{S_3} - u_*)| + |(\Theta_t^{R_1}d_1, U_t^{R_1}d_2)|
\end{aligned}$$

$$
\begin{aligned}
&+|(\Theta_t^{S_3}, U_t^{S_3})||(V^{R_1}-v_*, U^{R_1}-u_*, d_1, d_2)| + |(d_{1t}, d_{2t})|\Big] \\
&= O(1)\varepsilon.
\end{aligned}
$$

On the other hand, we can compute that

$$
\begin{aligned}
Q_{43} &= -U_{y\tau}^{R_1}\left(\frac{1}{V}-\frac{1}{V^{R_1}}\right) - U_{y\tau}^{S_3}\left(\frac{1}{V}-\frac{1}{V^{S_3}}\right) - \frac{d_{2y\tau}}{V} \\
&= O(1)\varepsilon^2\Big[|U_{xt}^{R_1}||U^{S_3}-u_*| + |U_{xt}^{S_3}||(U^{R_1}-u_*, d_2)| + |U_{xt}^{R_1} d_2| + |d_{2xt}|\Big] \\
&= O(1)\varepsilon.
\end{aligned}
$$

Similar estimations hold for Q_{42} and Q_{44}. Therefore, we have that

$$
Q_4 = O(1)\varepsilon. \tag{3.27}
$$

Substituting (3.27) and the fact

$$
\begin{aligned}
\left(P^{R_1} - \frac{U_y^{R_1}}{V^{R_1}}\right)_\tau &= O(1)\varepsilon|(V_t^{R_1}, \Theta_t^{R_1})| + O(1)\varepsilon^2|(U_{xt}^{R_1}, U_x^{R_1} V_t^{R_1})| \\
&= O(1)\varepsilon,
\end{aligned}
$$

into (3.25) implies that

$$
\eta_\tau = -s_3^2 U_y^{S_3} + O(1)\varepsilon. \tag{3.28}
$$

Similarly, we have

$$
\eta_y = s_3 U_y^{S_3} + O(1)\varepsilon. \tag{3.29}
$$

Then by (3.14), (3.18) and (3.28), we have

$$
\begin{aligned}
&\frac{\gamma-1}{\eta}U_y - \left(\frac{V}{2\eta}\right)_\tau = \frac{2\gamma-3}{2\eta}U_y + \frac{V}{2\eta^2}\eta_\tau = \frac{1}{2\eta^2}[(2\gamma-3)\eta U_y + V\eta_\tau] \\
&= \frac{1}{2\eta^2}\Big[(2\gamma-3)(b_1 - s_3^2 V^{S_3}) - s_3^2 V^{S_3}\Big]U_y^{S_3} \\
&\quad + \frac{2\gamma-3}{2\eta^2}\Big[\eta U_y - (b_1 - s_3^2 V^{S_3})U_y^{S_3}\Big] + \frac{1}{2\eta^2}\Big[V\eta_\tau - (-s_3^2 U_y^{S_3})V^{S_3}\Big] \\
&= \frac{1}{2\eta^2}\Big[(2\gamma-3)(b_1 - s_3^2 V^{S_3}) - s_3^2 V^{S_3}\Big]U_y^{S_3} + O(1)\varepsilon \\
&= \frac{1}{2\eta^2}\Big[b_1 - 2(\gamma-1)(b_1 - s_3^2 V^{S_3})\Big]|U_y^{S_3}| + O(1)\varepsilon,
\end{aligned} \tag{3.30}
$$

where we have used the wave interaction estimates and the estimates of the hyperbolic waves in Lemma 2.3.

On the other hand, by (3.14), (3.18) and (3.29), we have

$$
\begin{aligned}
\left(\frac{1}{\eta}\right)_y &= -\frac{\eta_y}{\eta^2} = -\frac{s_3 U_y^{S_3}}{\eta^2} + O(1)\varepsilon \\
&= \frac{s_3|U_y^{S_3}|}{\eta^{\frac{3}{2}}\sqrt{b_1 - s_3^2 V^{S_3}}} + \frac{s_3|U_y^{S_3}|}{\eta^{\frac{3}{2}}}\left[\eta^{-\frac{1}{2}} - (b_1 - s_3^2 V^{S_3})^{-\frac{1}{2}}\right] + O(1)\varepsilon \\
&= \frac{s_3|U_y^{S_3}|}{\eta^{\frac{3}{2}}\sqrt{b_1 - s_3^2 V^{S_3}}} + O(1)|U_y^{S_3}||\eta - (b_1 - s_3^2 V^{S_3})| + O(1)\varepsilon \\
&= \frac{s_3|U_y^{S_3}|}{\eta^{\frac{3}{2}}\sqrt{b_1 - s_3^2 V^{S_3}}} + O(1)|U_y^{S_3}||(P^{R_1} - p_*) - \frac{U_y^{R_1}}{V^{R_1}} + Q_3| + O(1)\varepsilon \\
&= \frac{s_3|U_y^{S_3}|}{\eta^{\frac{3}{2}}\sqrt{b_1 - s_3^2 V^{S_3}}} + O(1)\varepsilon.
\end{aligned} \tag{3.31}
$$

Thus, substituting (3.30) and (3.31) into $(3.23)_2$, we have

$$
\begin{aligned}
I_2(\Psi, \Psi_y) &= \frac{1}{2}\left[b_1 - 2(\gamma - 1)(b_1 - s_3^2 V^{S_3})\right]\left(\frac{\sqrt{|U_y^{S_3}|}\Psi}{\eta}\right)^2 \\
&\quad + \frac{s_3\sqrt{|U_y^{S_3}|}}{\sqrt{b_1 - s_3^2 V^{S_3}}}\frac{\sqrt{|U_y^{S_3}|}\Psi}{\eta}\frac{\Psi_y}{\sqrt{\eta}} + \left(\frac{\Psi_y}{\sqrt{\eta}}\right)^2 + O(1)\varepsilon|(\Psi, \Psi_y)|^2.
\end{aligned} \tag{3.32}
$$

Recalling that $b_1 = p_+ + s_3^2 v_+ = p_* + s_3^2 v_*$, the inequality (3.13), the facts $v_* < v_+, \theta_+ < \Theta^{S_3} < \theta_*$ and

$$
s_3^2 \geqq \frac{\gamma p_*}{v_+} - C(\gamma - 1)|v_+ - v_*|, \quad \theta_+ \geqq \theta_* - C(\gamma - 1)|v_+ - v_*|
$$

due to (2.14) in Lemma 2.4 for some positive constant C, we can calculate that

$$
\begin{aligned}
& b_1 - 2(\gamma - 1)(b_1 - s_3^2 V^{S_3}) \\
&\quad \geqq (p_* + s_3^2 v_*) - 2(\gamma - 1)p_* \\
&\quad \geqq \left[\frac{R\theta_*}{v_*} + \frac{\gamma R\theta_*}{v_+} - C(\gamma - 1)|v_+ - v_*|\right] - 2(\gamma - 1)\frac{R\theta_*}{v_*} \\
&\quad \geqq \left[\frac{R\theta_*}{v_*} + \frac{\gamma R\theta_*}{v_+} - C(\gamma - 1)|v_+ - v_*|\right] - 2(\gamma - 1)\frac{R\theta_*}{v_+} + 2(\gamma - 1)R\theta_*\left(\frac{1}{v_+} - \frac{1}{v_*}\right) \\
&\quad \geqq (3 - \gamma)\frac{\gamma R\theta_*}{v_+} - C_1(\gamma - 1)|v_+ - v_*|,
\end{aligned}
$$

and the determinant of the quadratic term in (3.32) as

$$
\begin{aligned}
\Delta &= \frac{s_3^2|U_y^{S_3}|}{b_1 - s_3^2 V^{S_3}} - 4 \times \frac{1}{2}\Big[b_1 - 2(\gamma-1)(b_1 - s_3^2 V^{S_3})\Big] \\
&\overset{(2.13)}{=} -s_3^2 R\Theta^{S_3}(b_1 - s_3^2 V^{S_3})^{-1} + s_3^2 V^{S_3} - 2b_1 + 4(\gamma-1)(b_1 - s_3^2 V^{S_3}) \\
&= -b_1 - (b_1 - s_3^2 V^{S_3}) - s_3^2 R\Theta^{S_3}(b_1 - s_3^2 V^{S_3})^{-1} + 4(\gamma-1)(b_1 - s_3^2 V^{S_3}) \\
&\leqq -\Big[p_* + \gamma\frac{R\theta_*}{v_+} - C(\gamma-1)|v_+ - v_*|\Big] - p_+ \\
&\quad -\Big[\frac{\gamma p_*}{v_+} - C(\gamma-1)|v_+ - v_*|\Big]R\theta_+ p_*^{-1} + 4(\gamma-1)p_* \\
&= -\frac{R\theta_*}{v_*} - \gamma\frac{R\theta_*}{v_+} - (\gamma+1)\frac{R\theta_+}{v_+} + 4(\gamma-1)\frac{R\theta_*}{v_*} + C_1(\gamma-1)|v_+ - v_*| \\
&\leqq -\frac{R\theta_*}{v_+} - \gamma\frac{R\theta_*}{v_+} - (\gamma+1)\frac{R}{v_+}\big(\theta_* - C(\gamma-1)|v_+ - v_*|\big) \\
&\quad +4(\gamma-1)\frac{R\theta_*}{v_+} + C_2(\gamma-1)|v_+ - v_*| \\
&= 2(\gamma-3)\frac{R\theta_*}{v_+} + C_3(\gamma-1)|v_+ - v_*|,
\end{aligned}
$$

where we have also used the fact that

$$
b_1 = p_* + s_3^2 v_* \geqq p_* + \frac{\gamma R\theta_*}{v_+} - C(\gamma-1)|v_+ - v_*|.
$$

Thus, if $\gamma \in (1,3)$ and $(\gamma-1)|v_+ - v_*|$ is suitably small, then the coefficient of the quadratic term in (3.32) is positive, that is, $\frac{1}{2}[b_1 - 2(\gamma-1)(b_1 - s_3^2 V^{S_3})] > 0$ and the determinant of the quadratic term in (3.32) is negative, that is, $\Delta < 0$. Therefore, from (3.32), we have

$$
I_2 \geqq C^{-1}(|U_y^{S_3}|\Psi^2 + \Psi_y^2) - C\varepsilon\Psi^2, \tag{3.33}
$$

for some positive constant C provided $\varepsilon \ll 1$.

Now we calculate the term I_3 in (3.23). From the facts (3.18) and (3.28), we have

$$
\begin{aligned}
-\Big[\frac{R^2}{2(\gamma-1)\eta^2}\Big]_\tau &= \frac{R^2\eta_\tau}{(\gamma-1)\eta^3} = -\frac{R^2 s_3^2 U_y^{S_3}}{(\gamma-1)\eta^3} + O(1)\varepsilon \\
&\geqq C^{-1}\frac{|U_y^{S_3}|}{\gamma-1} - C\varepsilon,
\end{aligned}
$$

which again with the fact (3.18) yields that

$$
I_3 \geqq C^{-1}(|U_y^{S_3}|\frac{W^2}{\gamma-1} + |W_y|^2) - C\varepsilon W^2. \tag{3.34}
$$

Then, note that

$$-\left(\frac{R}{\eta}\right)_y+\frac{R}{\eta^2}U_\tau=-\frac{R}{\eta^2}\Big[U_\tau+P_y-\Big(\frac{U_y}{V}\Big)_y\Big]=-\frac{R}{\eta^2}Q_{1y}. \tag{3.35}$$

From (2.19), we can compute that

$$\begin{aligned}Q_{1y} &= \varepsilon[P-P^{R_1}-P^{S_3}-\Big(p_v^{R_1}d_1+p_u^{R_1}d_2+p_E^{R_1}d_3\Big)]_x+\varepsilon^2\Big(\frac{U_x}{V}-\frac{U_x^{R_1}}{V^{R_1}}-\frac{U_x^{S_3}}{V^{S_3}}\Big)_x\\ &= O(1)\varepsilon\Big[|(V_x^{R_1},U_x^{R_1},\mathcal{E}_x^{R_1},d_{ix})(|d_i|^2,V^{S_3}-v_*,U^{S_3}-u_*,\mathcal{E}^{S_3}-E_*)|+|d_id_{ix}|\Big]\\ &\quad+O(1)\varepsilon^2\Big[|(d_{2xx},d_{1x}d_{2x},V_x^{R_1}d_{2x},U_x^{R_1}d_{1x})|+|(U_{xx}^{R_1},U_x^{R_1}V_x^{R_1})(d_1,V^{S_3}-v_*)|\\ &\quad+|(U_{xx}^{S_3},U_x^{S_3}V_x^{S_3})(V^{R_1}-v_*,d_1)|+|(U_x^{S_3},V_x^{S_3})(U_x^{R_1},V_x^{R_1},d_{1x},d_{2x})|\Big]\\ &:= Q_{13}+Q_{14},\end{aligned} \tag{3.36}$$

where Q_{13} represents the wave interaction terms given by

$$\begin{aligned}Q_{13} &= O(1)\varepsilon\Big[|(V_x^{R_1},U_x^{R_1},\mathcal{E}_x^{R_1},d_{ix})(V^{S_3}-v_*,U^{S_3}-u_*,\mathcal{E}^{S_3}-E_*)|\Big]\\ &\quad+O(1)\varepsilon^2\Big[|(U_{xx}^{R_1},U_x^{R_1}V_x^{R_1})(d_1,V^{S_3}-v_*)|\\ &\quad+|(U_{xx}^{S_3},U_x^{S_3}V_x^{S_3})(V^{R_1}-v_*,d_1)|\\ &\quad+|(U_x^{S_3},V_x^{S_3})(U_x^{R_1},V_x^{R_1},d_{1x},d_{2x})|\Big]\\ &= O(1)\mathrm{e}^{-\frac{c|x|}{\sigma}}\mathrm{e}^{-\frac{C_h}{\sigma}},\end{aligned} \tag{3.37}$$

and Q_{14} represents the terms about the hyperbolic waves d_i $(i=1,2,3)$ defined by

$$\begin{aligned}Q_{14} &= O(1)\varepsilon\Big[|(V_x^{R_1},U_x^{R_1},\mathcal{E}_x^{R_1},d_{ix})||d_i|^2+|d_id_{ix}|\Big]\\ &\quad+O(1)\varepsilon^2\Big[|(d_{2xx},d_{1x}d_{2x},V_x^{R_1}d_{2x},U_x^{R_1}d_{1x})|\Big].\end{aligned} \tag{3.38}$$

From (3.35)–(3.38) and Lemma 2.3, we compute the first term in I_4 as

$$\Big[-(\frac{R}{\eta})_y+\frac{R}{\eta^2}U_\tau\Big]\Psi W=O(1)\varepsilon(\Psi^2+W^2).$$

Moreover, the other terms in I_4 can be estimated by Cauchy inequality and the facts (3.14), (3.29). So we can obtain

$$\begin{aligned}I_4 &\leqq \beta\left(|U_y^{S_3}|\frac{W^2}{\gamma-1}+W_y^2\right)+C_\beta(\gamma-1)|v_+\\ &\quad-v_*|\left[|U_y^{S_3}|\left(\Psi^2+\frac{W^2}{\gamma-1}\right)+W_y^2+\Phi_y^2\right]+C\varepsilon(\Psi^2+W^2),\end{aligned} \tag{3.39}$$

where and in the sequel β is a small positive constant to be determined and C_β is some positive constant depending on β.

Now we estimate the terms $\frac{V}{\eta}\Psi F_1$, and $\frac{R^2 W}{\eta^2}F_2$ on the right-hand side of (3.22). From (3.5), (3.7) and (3.18), we have

$$\left|\frac{V}{\eta}\Psi F_1\right| \leqq C\mathcal{N}(\tau)(\Phi_y^2+\Psi_y^2+\zeta^2+\psi_y^2), \tag{3.40}$$

and

$$\left|\frac{R^2 W}{\eta^2}F_2\right| \leqq C\mathcal{N}(\tau)(\Phi_y^2+\Psi_y^2+\zeta^2+\psi_y^2+\zeta_y^2). \tag{3.41}$$

Then we estimate the terms $-\frac{V}{\eta}\Psi Q_1$ and $-\frac{RW}{\eta^2}(Q_2-UQ_1)$. From (2.19) and (2.20), we have

$$\begin{aligned}\left|-\frac{V}{\eta}\Psi Q_1\right|, \left|-\frac{RW}{\eta^2}(Q_2-UQ_1)\right| &\leqq C|(\Psi,W)||(Q_{11},Q_{21})|\\ &\quad+C|(\Psi,W)||(Q_{12},Q_{22})|.\end{aligned} \tag{3.42}$$

On one hand,

$$\begin{aligned}&\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}|(\Psi,W)||(Q_{11},Q_{21})|\mathrm{d}y\mathrm{d}\tau\\ &\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}|(\Psi,W)|\Big[\sum_{i=1}^{3}|d_i|^2+\varepsilon|(d_{2x},d_{3x})|+\varepsilon|(U_{1x}^{R_1},\Theta_x^{R_1})||(d_1,d_2,d_3)|\Big]\mathrm{d}y\mathrm{d}\tau\\ &:= I_5+I_6+I_7.\end{aligned} \tag{3.43}$$

From Lemma 2.3 and noting that $\sigma=\varepsilon^{\frac{1}{5}}$, we can compute that

$$\begin{aligned}I_5 &\leqq \int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|_{L^2(\mathrm{d}y)}\sum_{i=1}^{3}\|d_i\|_{L^2(\mathrm{d}y)}\|d_i\|_{L^\infty}\mathrm{d}\tau\\ &\leqq C_{h,T}\int_{\frac{h}{\varepsilon}}^{\tau}(\frac{\varepsilon}{\sigma})^{\frac{3}{2}}\|(\Psi,W)\|_{L^2(\mathrm{d}y)}\mathrm{d}\tau\\ &\leqq \varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|^2\mathrm{d}\tau+C_{h,T}\int_{\frac{h}{\varepsilon}}^{\tau}\frac{\varepsilon^2}{\sigma^3}\mathrm{d}\tau\\ &\leqq \varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|^2\mathrm{d}\tau+C_{h,T}\,\varepsilon^{\frac{2}{5}},\end{aligned} \tag{3.44}$$

$$\begin{aligned}I_6 &\leqq \varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|_{L^2(\mathrm{d}y)}\|(d_{2x},d_{3x})\|_{L^2(\mathrm{d}y)}\mathrm{d}\tau\\ &\leqq C_{h,T}\int_{\frac{h}{\varepsilon}}^{\tau}(\frac{\varepsilon}{\sigma})^{\frac{3}{2}}\|(\Psi,W)\|_{L^2(\mathrm{d}y)}\mathrm{d}\tau\\ &\leqq \varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|^2\mathrm{d}\tau+C_{h,T}\,\varepsilon^{\frac{2}{5}},\end{aligned} \tag{3.45}$$

and

$$
\begin{aligned}
I_7 &\leqq C_{h,T}\ \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(\mathrm{d}y)} \sum_{i=1}^{3} \|d_i\|_{L^2(\mathrm{d}y)} \mathrm{d}\tau \\
&\leqq C_{h,T}\ \frac{\varepsilon^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(\mathrm{d}y)} \mathrm{d}\tau \\
&\leqq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 \mathrm{d}\tau + C_{h,T}\ \varepsilon^{\frac{4}{5}}. \qquad (3.46)
\end{aligned}
$$

On the other hand, from (2.21), we have

$$
\begin{aligned}
&\int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)||(Q_{12}, Q_{22})| \mathrm{d}y \mathrm{d}\tau \\
&\quad \leqq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)| \mathrm{e}^{\frac{-c|x|}{\sigma}} \mathrm{e}^{-\frac{C_h}{\sigma}} \mathrm{d}y \mathrm{d}\tau \\
&\quad \leqq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 \mathrm{d}\tau + C_{h,T}\ \mathrm{e}^{-\frac{C_h}{2\sigma}}. \qquad (3.47)
\end{aligned}
$$

Integrating (3.22) with respect to τ and y, then substituting all the above estimates into (3.22) and choosing $(\gamma - 1)|v_+ - v_*|, \varepsilon, \beta$ suitably small imply (3.11) in Lemma 3.1. Thus we complete the proof of Lemma 3.1.

Now we estimate the term $\|\Phi_y\|^2$. Multiply $(3.8)_1$ by $V\Psi_y - V_y\Psi$, $(3.8)_2$ by $-V\Phi_y$, then apply ∂_y to $(3.8)_1$ and then multiply the resulting equation by Φ_y. Calculating their sums, we arrive at

$$
\begin{aligned}
\left(\frac{\Phi_y^2}{2} - V\Phi_y\Psi\right)_\tau + \eta\Phi_y^2 &= V_y\Psi_y\Psi + V\Psi_y^2 - V_\tau\Phi_y\Psi + R\Phi_y W_y \\
&\quad +(\gamma-1)U_y\Phi_y\Psi - VF_1\Phi_y + VQ_1\Phi_y + (\cdots)_y. \qquad (3.48)
\end{aligned}
$$

Integrating (3.48) with respect to τ and y over $[\frac{h}{\varepsilon}, \tau] \times \mathbf{R}$, and using a method similar to that used in obtaining Lemma 3.1, we have the following estimation for $\|\Phi_y\|^2$:

Lemma 3.2. *There exists a positive constant C independent of ε such that*

$$
\begin{aligned}
&\sup_{\frac{h}{\varepsilon} \leqq \tau_1 \leqq \tau} \|\Phi_y\|^2(\tau_1) + \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \Phi_y^2 \mathrm{d}y \mathrm{d}\tau \\
&\quad \leqq C \sup_{\frac{h}{\varepsilon} \leqq \tau_1 \leqq \tau} \|\Psi\|^2(\tau_1) + C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} (|U_y^{S_3}|\Psi^2 + \Psi_y^2 + W_y^2) \mathrm{d}y \mathrm{d}\tau \\
&\qquad + C\mathcal{N}(\tau) \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \psi_y^2 \mathrm{d}y \mathrm{d}\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \Psi^2 \mathrm{d}y \mathrm{d}\tau + C\varepsilon^{\frac{2}{5}}. \qquad (3.49)
\end{aligned}
$$

In the following, we estimate the higher order derivative terms. First we apply ∂_y to the system (3.8) to get the system for $(\phi, \psi, \zeta)(y, \tau)$:

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau - \dfrac{\eta}{V}\phi_y + \dfrac{R}{V}\zeta_y - (\dfrac{\psi_y}{V})_y - (\dfrac{\eta}{V})_y\phi + (\dfrac{R}{V})_y\zeta = F_3 - Q_{1y}, \\ \dfrac{R}{\gamma-1}\zeta_\tau + \eta\psi_y - \nu(\dfrac{\zeta_y}{V})_y - \dfrac{U_y}{V}(\eta\phi - R\zeta + \psi_y) = F_4 + UQ_{1y} - Q_{2y}, \end{cases} \tag{3.50}$$

where

$$F_3 = \left(\frac{p-P}{V}\phi - \frac{\psi_y\phi}{vV} + \frac{U_y}{vV^2}\phi^2\right)_y, \tag{3.51}$$

$$F_4 = \frac{1}{v}(\eta\phi - R\zeta + \psi_y)\left(\psi_y - \frac{U_y\phi}{V}\right) + \nu\left[(\Theta_y + \zeta_y)\left(\frac{1}{v} - \frac{1}{V}\right)\right]_y. \tag{3.52}$$

Multiplying $(3.50)_1$ by ϕ, $(3.50)_2$ by $\frac{V}{\eta}\psi$, $(3.50)_3$ by $\frac{R}{\eta^2}\zeta$, and adding all the resulted equations, we get

$$\begin{aligned} &\left(\frac{\phi^2}{2} + \frac{V}{2\eta}\psi^2 + \frac{R^2}{2(\gamma-1)\eta^2}\zeta^2\right)_\tau + \frac{1}{\eta}\psi_y^2 + \frac{\nu R}{V\eta^2}\zeta_y^2 = (\frac{V}{2\eta})_\tau\psi^2 \\ &+\left(\frac{R^2}{2(\gamma-1)\eta^2}\right)_\tau\zeta^2 - \frac{1}{V}\left(\frac{V}{\eta}\right)_y\psi\psi_y + \frac{V}{\eta}\left(\frac{\eta}{V}\right)_y\phi\psi - \frac{\nu}{V}\left(\frac{R}{\eta^2}\right)_y\zeta\zeta_y \\ &+\left[\left(\frac{R}{\eta}\right)_y - \frac{V}{\eta}\left(\frac{R}{V}\right)_y\right]\psi\zeta - \left(\frac{p-P}{V}\phi - \frac{\psi_y\phi}{vV} + \frac{U_y}{vV^2}\phi^2\right)\left(\frac{V\psi}{\eta}\right)_y \\ &+\frac{1}{v}\left(\psi_y - \frac{U_y\phi}{V}\right)(\eta\phi - R\zeta + \psi_y)\frac{R}{\eta^2}\zeta - \nu\left[(\Theta_y + \zeta_y)\left(\frac{1}{v} - \frac{1}{V}\right)\right]\left(\frac{R}{\eta^2}\zeta\right)_y \\ &+Q_{1y}\frac{V}{\eta}\psi + (UQ_{1y} - Q_{2y})\frac{R}{\eta^2}\zeta + (\cdots)_y. \end{aligned}$$

Integrating the above equation with respect to τ and y over $[\frac{h}{\varepsilon}, \tau] \times \mathbf{R}$ and using Young's inequality, we get

$$\begin{aligned} &\int_{\mathbf{R}}\left(\frac{\phi^2}{2} + \frac{V}{2\eta}\psi^2 + \frac{R^2}{2(\gamma-1)\eta^2}\zeta^2\right)\mathrm{d}y + \int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\left(\frac{1}{\eta}\psi_y^2 + \frac{\nu R}{V\eta^2}\zeta_y^2\right)\mathrm{d}y\mathrm{d}\tau \\ &\leqq (\beta + C\mathcal{N}(\tau))\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\psi_y^2 + \zeta_y^2)\mathrm{d}y\mathrm{d}\tau + C_\beta\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\phi^2 + \psi^2 + \zeta^2)\mathrm{d}y\mathrm{d}\tau \\ &+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(Q_{1y}^2 + Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau. \end{aligned} \tag{3.53}$$

Choosing β and $\mathcal{N}(\tau)$ suitably small in (3.53), we have

Lemma 3.3. *There exists a positive constant C independent ε such that*

$$\sup_{\frac{h}{\varepsilon}\leqq\tau_1\leqq\tau}\|(\phi,\psi,\frac{\zeta}{\sqrt{\gamma-1}})\|^2(\tau_1)+\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\psi_y^2+\zeta_y^2)\mathrm{d}y\mathrm{d}\tau$$
$$\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\phi^2+\psi^2+\zeta^2)\mathrm{d}y\mathrm{d}\tau+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(Q_{1y}^2+Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau. \quad (3.54)$$

Multiplying $(3.50)_1$ by $V\psi_y-V_y\psi$, $(3.50)_2$ by $-V\phi_y$, applying ∂_y to $(3.50)_1$ and then multiplying the resulting equation by ϕ_y, and calculating all their sums, after integrating, and using the same method as in getting Lemma 3.2, we have

Lemma 3.4. *There exists a positive constant C independent of ε such that*

$$\sup_{\frac{h}{\varepsilon}\leqq\tau_1\leqq\tau}\|\phi_y\|^2(\tau_1)+\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\phi_y^2\mathrm{d}y\mathrm{d}\tau$$
$$\leqq C\sup_{\frac{h}{\varepsilon}\leqq\tau_1\leqq\tau}\|\psi\|^2(\tau_1)+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\phi^2+\psi^2+\zeta^2)\mathrm{d}y\mathrm{d}\tau$$
$$+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\psi_y^2+\zeta_y^2)\mathrm{d}y\mathrm{d}\tau+C\mathcal{N}(\tau)\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\psi_{yy}^2\mathrm{d}y\mathrm{d}\tau+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}Q_{1y}^2\mathrm{d}y\mathrm{d}\tau. \quad (3.55)$$

Now we do the highest order estimate. Applying ∂_y to the system (3.50), we obtain

$$\begin{cases}\phi_{y\tau}-\psi_{yy}=0,\\ \psi_{y\tau}-\frac{\eta}{V}\phi_{yy}+\frac{R}{V}\zeta_{yy}-(\frac{\psi_y}{V})_{yy}-2(\frac{\eta}{V})_y\phi_y-(\frac{\eta}{V})_{yy}\phi\\ \quad+2(\frac{R}{V})_y\zeta_y+(\frac{R}{V})_{yy}\zeta=F_{3y}-Q_{1yy},\\ \frac{R}{\gamma-1}\zeta_{y\tau}+\eta\psi_{yy}-\nu(\frac{\zeta_y}{V})_{yy}+\eta_y\psi_y-[\frac{U_y}{V}(\eta\phi-R\zeta-\psi_y)]_y\\ \qquad=F_{4y}+(UQ_{1y})_y-Q_{2yy},\end{cases} \quad (3.56)$$

where F_3 and F_4 are defined in (3.51) and (3.52) respectively.

Multiplying $(3.56)_1$ by ϕ_y, $(3.56)_2$ by $\frac{V}{\eta}\psi_y$, $(3.56)_3$ by $\frac{R}{\eta^2}\zeta_y$, and adding all the resulting equations, after integrating and using the same method as in obtaining Lemma 3.3, we can get

Lemma 3.5. *There exists a positive constant C independent of ε such that*

$$\sup_{\frac{h}{\varepsilon}\leqq\tau_1\leqq\tau}\left\|\left(\phi_y,\psi_y,\frac{\zeta_y}{\sqrt{\gamma-1}}\right)\right\|^2(\tau_1)+\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\psi_{yy}^2+\zeta_{yy}^2)\mathrm{d}y\mathrm{d}\tau$$
$$\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\phi^2+\psi^2+\zeta^2+\phi_y^2+\psi_y^2+\zeta_y^2)\mathrm{d}y\mathrm{d}\tau+C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(Q_{1y}^2+Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau. \quad (3.57)$$

Note that from (3.3),

$$\begin{aligned}\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\zeta^2\mathrm{d}y\mathrm{d}\tau &= \int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\Big[W_y-\frac{\gamma-1}{R}\Big(\frac{\Psi_y^2}{2}-U_y\Psi\Big)\Big]^2\mathrm{d}y\mathrm{d}\tau\\ &\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(\Psi_y^2+W_y^2)\mathrm{d}y\mathrm{d}\tau + C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\big(|U_y^{R_1}|^2+|U_y^{S_3}|^2+|d_{2y}|^2\big)|\Psi|^2\mathrm{d}y\mathrm{d}\tau\\ &\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(|U_y^{S_3}|\Psi^2+\Psi_y^2+W_y^2)\mathrm{d}y\mathrm{d}\tau + C\varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}\Psi^2\mathrm{d}y\mathrm{d}\tau. \end{aligned}\tag{3.58}$$

Combining Lemmas 3.1–3.5 *and the estimate* (3.58), *we can get*

$$\begin{aligned}&\mathcal{N}(\tau)+\int_{\frac{h}{\varepsilon}}^{\tau}\Bigg[\Bigg\|\sqrt{|U_y^{S_3}|}\Bigg(\Psi,\frac{W}{\sqrt{\gamma-1}}\Bigg)\Bigg\|^2+\|\phi\|_{H^1}^2+\|(\psi,\zeta)\|_{H^2}^2\Bigg](\tau)\mathrm{d}\tau\\ &\quad\leqq C\varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\|(\Psi,W)\|^2\mathrm{d}\tau + C\varepsilon^{\frac{2}{5}} + C\int_{\frac{h}{\varepsilon}}^{\tau}\int_{\mathbf{R}}(Q_{1y}^2+Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau, \end{aligned}\tag{3.59}$$

under the smallness condition of $(\gamma-1)|v_+-v_*|$.

Now it is sufficient to estimate the terms $\int_{\frac{h}{\varepsilon}}^{\tau}\|(Q_{1y},Q_{2y})\|^2\mathrm{d}\tau$. From (3.36), (3.37) and (3.38), we have

$$\begin{aligned}\int_{\frac{h}{\varepsilon}}^{\tau}\|Q_{1y}\|^2\mathrm{d}\tau &\leqq C\int_{\frac{h}{\varepsilon}}^{\tau}\Big[\|Q_{13}\|^2+\|Q_{14}\|^2\Big]\mathrm{d}\tau\\ &\leqq C_{h,T}\Big(\mathrm{e}^{-\frac{C_h}{\sigma}}+\frac{\varepsilon^4}{\sigma^5}\Big)\\ &\leqq C_{h,T}\,\varepsilon^3. \end{aligned}\tag{3.60}$$

Similarly, we can calculate the term for Q_{2y}. Substituting (3.60) and the similar estimations for Q_{2y} into (3.59), we can get

$$\begin{aligned}&\mathcal{N}(\tau)+\int_{\frac{h}{\varepsilon}}^{\tau}\Big[\|\sqrt{|U_y^{S_3}|}(\Psi,W)\|^2+\|\phi\|_{H^1}^2+\|(\psi,\zeta)\|_{H^2}^2\Big](\tau)\mathrm{d}\tau\\ &\quad\leqq C\varepsilon\int_{\frac{h}{\varepsilon}}^{\tau}\Bigg\|\Bigg(\Psi,\frac{W}{\sqrt{\gamma-1}}\Bigg)\Bigg\|^2\mathrm{d}\tau + C\varepsilon^{\frac{2}{5}}. \end{aligned}\tag{3.61}$$

Applying Gronwall's inequality to (3.61) closes the a priori estimate (3.10) and gives the proof of Proposition 3.1.

Finally, by Sobolev's inequality, we have

$$\|(\phi,\psi,\zeta)(\tau,\cdot)\|_{L^\infty}\leqq C\|(\phi,\psi,\zeta)(\tau,\cdot)\|_{H^1}\leqq C_{h,T}\,\varepsilon^{\frac{1}{5}}.$$

This, combined with the fact (2.17) completes the proof of Theorem 1.

Appendix

Proof of Lemma 2.3. The proof of the statement (1) in Lemma 2.3 will be given by the energy method to the diagonal system (2.9).

Multiplying $(2.9)_1$ by D_1 and integrating the resulting equation over $[h,t]\times\mathbf{R}$ with $t\in(h,T)$ imply

$$\begin{aligned}
&\int_{\mathbf{R}}\frac{D_1^2}{2}(x,t)\mathrm{d}x+\int_h^t\int_{\mathbf{R}}\lambda_{1x}^{R_1}\frac{D_1^2}{2}\mathrm{d}x\mathrm{d}t\\
&\quad=\int_h^t\int_{\mathbf{R}}(b_{12}B_1+b_{13}B_2)D_1\mathrm{d}x\mathrm{d}t+\int_h^t\int_{\mathbf{R}}a_{12}V_x^{R_1}D_1D_2\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_h^t\int_{\mathbf{R}}a_{13}V_x^{R_1}D_1D_3\mathrm{d}x\mathrm{d}t\\
&\quad\leqq C\int_h^t\int_{\mathbf{R}}D_1^2\mathrm{d}x\mathrm{d}t+C\int_h^t\int_{\mathbf{R}}|(B_1,B_2)|^2\mathrm{d}x\mathrm{d}t+\beta\int_h^t\int_{\mathbf{R}}V_x^{R_1}D_1^2\mathrm{d}x\mathrm{d}t\\
&\qquad+C_\beta\int_h^t\int_{\mathbf{R}}V_x^{R_1}|(D_2,D_3)|^2\mathrm{d}x\mathrm{d}t\\
&\quad\leqq C\int_h^t\int_{\mathbf{R}}D_1^2\mathrm{d}x\mathrm{d}t+C_{h,T}\frac{\varepsilon^2}{\sigma}+\beta\int_h^t\int_{\mathbf{R}}V_x^{R_1}D_1^2\mathrm{d}x\mathrm{d}t\\
&\qquad+C_\beta\int_h^t\int_{\mathbf{R}}V_x^{R_1}(D_2,D_3)^2\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{A.1}$$

where we have used the uniform boundedness of the given functions a_{ij}, b_{ij} and the fact that

$$\begin{aligned}
&\int_t^T\int_{\mathbf{R}}|(B_1,B_2)|^2\mathrm{d}x\mathrm{d}t\\
&\quad\leqq C\int_t^T\int_{\mathbf{R}}\varepsilon^2\Big[|(U_{xx}^{R_1},\Theta_{xx}^{R_1})|^2+|(V_x^{R_1},U_x^{R_1},\Theta_x^{R_1})|^4\Big]\mathrm{d}x\mathrm{d}t\\
&\quad\leqq C_{h,T}\frac{\varepsilon^2}{\sigma}.
\end{aligned}\tag{A.2}$$

Choosing β suitably small and using Gronwall's inequality gives

$$\begin{aligned}
&\int_{\mathbf{R}}D_1^2(x,t)\mathrm{d}x+\int_h^t\int_{\mathbf{R}}V_x^{R_1}D_1^2\mathrm{d}x\mathrm{d}t\leqq C_{h,T}\frac{\varepsilon^2}{\sigma}\\
&\qquad+C_T\int_h^t\int_{\mathbf{R}}V_x^{R_1}|(D_2,D_3)|^2\mathrm{d}x\mathrm{d}t,\quad\forall t\in[h,T],
\end{aligned}\tag{A.3}$$

where C_T is independent of h.

Now we first multiply $(2.9)_2$ by $(V^{R_1})^N D_2$ with N being a sufficiently large positive constant to be determined, and integrating the resulting equation over

$[t, T] \times \mathbf{R}$ with $t \in (h, T)$, then we can get

$$
\begin{aligned}
&-\int_{\mathbf{R}} \frac{(V^{R_1})^N D_2^2}{2}(x,t)\mathrm{d}x - \int_t^T \int_{\mathbf{R}} N|\lambda_1^{R_1}|(V^{R_1})^{N-1} V_x^{R_1} D_2^2 \mathrm{d}x\mathrm{d}t \\
&= \int_t^T \int_{\mathbf{R}} (b_{22}B_1 + b_{23}B_2)(V^{R_1})^N D_2 \mathrm{d}x\mathrm{d}t + \int_t^T \int_{\mathbf{R}} a_{22} V_x^{R_1} (V^{R_1})^N D_2^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \int_t^T \int_{\mathbf{R}} a_{23} V_x^{R_1} (V^{R_1})^N D_2 D_3 \mathrm{d}x\mathrm{d}t \\
&\geqq -C \int_t^T \int_{\mathbf{R}} (V^{R_1})^N D_2^2 \mathrm{d}x\mathrm{d}t - C \int_t^T \int_{\mathbf{R}} (V^{R_1})^N |(B_1, B_2)|^2 \mathrm{d}x\mathrm{d}t \\
&\quad - C \int_t^T \int_{\mathbf{R}} V_x^{R_1} (V^{R_1})^N (D_2^2 + D_3^2) \mathrm{d}x\mathrm{d}t \\
&\geqq -C_{h,T} \int_t^T \int_{\mathbf{R}} (V^{R_1})^N D_2^2 \mathrm{d}x\mathrm{d}t - C_{N,h,T} \frac{\varepsilon^2}{\sigma} - C \int_t^T \int_{\mathbf{R}} V_x^{R_1} (V^{R_1})^N D_3^2 \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{A.4}
$$

where we have also used the uniform boundedness of the given functions a_{ij}, b_{ij} and the fact (A.2).

Thus we have

$$
\begin{aligned}
&\int_{\mathbf{R}} \frac{(V^{R_1})^N D_2^2}{2}(x,t)\mathrm{d}x + \int_t^T \int_{\mathbf{R}} N|\lambda_1^{R_1}|(V^{R_1})^{N-1} V_x^{R_1} D_2^2 \mathrm{d}x\mathrm{d}t \\
&\leqq C_{h,T} \int_t^T \int_{\mathbf{R}} (V^{R_1})^N D_2^2 \mathrm{d}x\mathrm{d}t + C_{N,h,T} \frac{\varepsilon^2}{\sigma} + C \int_t^T \int_{\mathbf{R}} V_x^{R_1} (V^{R_1})^N D_3^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{A.5}
$$

Applying Gronwall's inequality to (A.5) gives

$$
\begin{aligned}
&\int_{\mathbf{R}} (V^{R_1})^N D_2^2(x,t)\mathrm{d}x + \int_t^T \int_{\mathbf{R}} N V_x^{R_1} (V^{R_1})^N D_2^2 \mathrm{d}x\mathrm{d}t \\
&\leqq C_{N,h,T} \frac{\varepsilon^2}{\sigma} + C_T \int_t^T \int_{\mathbf{R}} V_x^{R_1} (V^{R_1})^N (D_2^2 + D_3^2) \mathrm{d}x\mathrm{d}t, \quad \forall t \in [h, T]
\end{aligned} \tag{A.6}
$$

where C_T is independent of h.

Now we multiply $(2.9)_3$ by $(V^{R_1})^N D_3$ and integrate the resulting equation over $[t, T] \times \mathbf{R}$ with $t \in (h, T)$ to get

$$
\begin{aligned}
&-\int_{\mathbf{R}} \frac{(V^{R_1})^N D_3^2}{2}(x,t)\mathrm{d}x - \int_t^T \int_{\mathbf{R}} \lambda_{1x}^{R_1} \frac{(V^{R_1})^N D_3^2}{2} \mathrm{d}x\mathrm{d}t \\
&\quad - \int_t^T \int_{\mathbf{R}} N|\lambda_1^{R_1}|(V^{R_1})^{N-1} V_x^{R_1} D_3^2 \mathrm{d}x\mathrm{d}t \\
&= \int_t^T \int_{\mathbf{R}} (b_{32}B_1 + b_{33}B_2)(V^{R_1})^N D_3 \mathrm{d}x\mathrm{d}t + \int_t^T \int_{\mathbf{R}} a_{32} V_x^{R_1} (V^{R_1})^N D_2 D_3 \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$+\int_t^T\int_{\mathbf{R}} a_{33}V_x^{R_1}(V^{R_1})^N D_3^2\mathrm{d}x\mathrm{d}t \geqq -C\int_t^T\int_{\mathbf{R}}(V^{R_1})^N D_3^2\mathrm{d}x\mathrm{d}t$$
$$-C\int_t^T\int_{\mathbf{R}}(V^{R_1})^N V_x^{R_1}(D_2^2+D_3^2)\mathrm{d}x\mathrm{d}t - C_{N,h,T}\frac{\varepsilon^2}{\sigma}, \tag{A.7}$$

where again we have used the uniform boundedness of the given functions a_{ij}, b_{ij} and the fact (A.2).

Then we arrive at

$$\int_{\mathbf{R}}\frac{(V^{R_1})^N D_3^2}{2}(x,t)\mathrm{d}x + \int_t^T\int_{\mathbf{R}} N|\lambda_1^{R_1}|(V^{R_1})^{N-1}V_x^{R_1}D_3^2\mathrm{d}x\mathrm{d}t$$
$$\leqq C\int_t^T\int_{\mathbf{R}}(V^{R_1})^N D_3^2\mathrm{d}x\mathrm{d}t + C\int_t^T\int_{\mathbf{R}}(V^{R_1})^N V_x^{R_1}(D_2^2+D_3^2)\mathrm{d}x\mathrm{d}t + C_{N,h,T}\frac{\varepsilon^2}{\sigma}. \tag{A.8}$$

Now applying Gronwall's inequality to (A.8) implies

$$\int_{\mathbf{R}}(V^{R_1})^N D_3^2(x,t)\mathrm{d}x + \int_t^T\int_{\mathbf{R}} N(V^{R_1})^N V_x^{R_1}D_3^2\mathrm{d}x\mathrm{d}t$$
$$\leqq C_{N,h,T}\frac{\varepsilon^2}{\sigma} + C_T\int_t^T\int_{\mathbf{R}}(V^{R_1})^N V_x^{R_1}(D_2^2+D_3^2)\mathrm{d}x\mathrm{d}t, \quad \forall t\in[h,T]. \tag{A.9}$$

Combining (A.3), (A.6) and (A.9) and choosing N large enough, we can get

$$\int_{\mathbf{R}}(D_1^2+D_2^2+D_3^2)(x,t)\mathrm{d}x + \int_h^T\int_{\mathbf{R}} V_x^{R_1}(D_1^2+D_2^2+D_3^2)\mathrm{d}x\mathrm{d}t$$
$$\leqq C_{h,T}\frac{\varepsilon^2}{\sigma}, \quad \forall t\in[h,T]. \tag{A.10}$$

This completes the proof of case $k=0$ in claim (1) of Lemma 2.3. The other cases $k=1,2$ can be considered similarly to the differentiated system with respect to x by k times; we omit the details for brevity.

Now we prove the claim (2) of Lemma 2.3. We will also consider the diagonalized system (2.9). If $x>\lambda_{1+}t$, then from Lemma 2.2, we have

$$D_{1t}+\lambda_1^{R_1}D_{1x} = -\lambda_{1x}^{R_1}D_1 + b_{12}B_1 + b_{13}B_2 + a_{12}V_x^{R_1}D_2 + a_{13}V_x^{R_1}D_3$$
$$= O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}. \tag{A.11}$$

Define the characteristic curve $X_1=X_1(s)$ passing through the point (t,x) by

$$\begin{cases} \dfrac{\mathrm{d}X_1(s)}{\mathrm{d}s} = \lambda_1^{R_1}(s, X_1(s)), \\ X_1(s=t) = x. \end{cases} \tag{A.12}$$

Denote the solution of (A.12) by $X_1(s;t,x)$. Then

$$X_1(s;t,x) = x - \int_s^t \lambda_1^{R_1}(s, X_1(s))\mathrm{d}s, \quad \text{if} \quad h\leqq s\leqq t.$$

Thus,

$$\begin{aligned} X_1(s;t,x)-\lambda_{1+}s &= x+|\lambda_{1+}|t+\int_t^s |\lambda_1^{R_1}|\mathrm{d}s+|\lambda_{1+}|(s-t) \\ &= x+|\lambda_{1+}|t+\int_t^s (|\lambda_1^{R_1}|-|\lambda_{1+}|)\mathrm{d}s \\ &\geqq x+|\lambda_{1+}|t \\ &= x-\lambda_{1+}t>0, \quad \text{if} \quad h\leqq s\leqq t. \end{aligned} \tag{A.13}$$

Now along this curve $X_1 = X_1(s; t, x)$, the equation (A.11) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}D_1(s, X_1(s;t,x)) = O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|X_1(s;t,x)-\lambda_{1+}s|}{\sigma}}.$$

Integrating the above equation over $[h, t]$ gives

$$\begin{aligned} D_1(t,x) &= O(1)\frac{1}{\sigma}\int_h^t \mathrm{e}^{-\frac{|X_1(s;t,x)-\lambda_{1+}s|}{\sigma}}\mathrm{d}s \\ &= O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}, \quad \text{if} \ \ x>\lambda_{1+}t. \end{aligned} \tag{A.14}$$

From the equation $(2.9)_2$ and Lemma 2.2-(3), we have that if $x > \lambda_{1+}t$, then

$$\begin{aligned} D_{2t} &= b_{22}B_1+b_{23}B_2+a_{22}V_x^{R_1}D_2+a_{23}V_x^{R_1}D_3 \\ &= O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}. \end{aligned} \tag{A.15}$$

Integrating the above equation over $[t, T]$ gives

$$D_2(t,x) = O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}, \quad \text{if} \ \ x>\lambda_{1+}t. \tag{A.16}$$

Applying Lemma 2.2-(3) to the equation $(2.9)_3$, we can obtain that if $x > \lambda_{1+}t$, then

$$\begin{aligned} D_{3t}+\lambda_3^{R_1}D_{3x} &= -\lambda_{3x}^{R_1}D_3+b_{32}B_1+b_{33}B_2+a_{32}V_x^{R_1}D_2+a_{33}V_x^{R_1}D_3 \\ &= O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}. \end{aligned} \tag{A.17}$$

Define the characteristic curve $X_3 = X_3(s)$ passing through the point (t, x) by

$$\begin{cases} \dfrac{\mathrm{d}X_3(s)}{\mathrm{d}s} = \lambda_3^{R_1}(s, X_3(s)), \\ X_3(s=t) = x. \end{cases} \tag{A.18}$$

Denote the solution of (A.18) by $X_3(s; t, x)$. Then

$$X_3(s;t,x) = x+\int_t^s \lambda_3^{R_1}(s, X_3(s))\mathrm{d}s.$$

Thus

$$
\begin{aligned}
X_3(s;t,x) - \lambda_{1+}s &= x + \int_t^s \lambda_3^{R_1}(s, X_3(s))\mathrm{d}s + |\lambda_{1+}|s \\
&\geqq x + |\lambda_{1+}|s \\
&\geqq x + |\lambda_{1+}|t \\
&= x - \lambda_{1+}t > 0, \quad \text{if} \quad t \leqq s \leqq T. \qquad \text{(A.19)}
\end{aligned}
$$

Now along this curve $X_3 = X_3(s;t,x)$, the equation (A.17) can be written as

$$
\frac{\mathrm{d}}{\mathrm{d}s} D_3(s, X_3(s;t,x)) = O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|X_3(s;t,x)-\lambda_{1+}s|}{\sigma}}.
$$

Integration over $[t, T]$ gives

$$
\begin{aligned}
D_3(t,x) &= O(1)\frac{1}{\sigma}\int_t^T \mathrm{e}^{-\frac{|X_3(s;t,x)-\lambda_{1+}s|}{\sigma}}\mathrm{d}s \\
&= O(1)\frac{1}{\sigma}\mathrm{e}^{-\frac{|x-\lambda_{1+}t|}{\sigma}}, \quad \text{if} \quad x > \lambda_{1+}t. \qquad \text{(A.20)}
\end{aligned}
$$

Thus the combination of (A.14), (A.16) and (A.20) implies the point-wise estimates of d_i $(i = 1, 2, 3)$ in the assertion (2). The corresponding estimations of D_{ix} $(i = 1, 2, 3)$ can be obtained similarly by applying ∂_x to the diagonalized system (2.9). We omit the details for brevity. By using the relation (2.7), we have

$$
(d_1, d_2, d_3)_x = (R^{R_1})_x(D_1, D_2, D_3) + R^{R_1}(D_1, D_2, D_3)_x,
$$

where $R^{R_1} = (r_1^{R_1}, r_2^{R_1}, r_3^{R_1})$ and $r_i^{R_1} = r_i^{R_1}(V^{R_1}, U^{R_1}, s_\pm)$, $(i = 1, 2, 3)$. Then we have the corresponding pointwise estimates of d_{ix} $(i = 1, 2, 3)$ in the assertion (2). Therefore, we complete the proof of Lemma 2.3.

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