

Large-time Behavior of Solutions to the Inflow Problem of Full Compressible Navier-Stokes Equations

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Abstract

Large-time behavior of solutions to the inflow problem of full compressible Navier-Stokes equations is investigated on the half line $\mathbf{R}_+ = (0, +\infty)$. The wave structure which contains four waves: the transonic(or degenerate) boundary layer solution, 1-rarefaction wave, viscous 2-contact wave and 3-rarefaction wave to the inflow problem is described and the asymptotic stability of the superposition of the above four wave patterns to the inflow problem of full compressible Navier-Stokes equations is proven under some smallness conditions. The proof is given by the elementary energy analysis based on the underlying wave structure. The main points in the proof are the degeneracies of the transonic boundary layer solution and the wave interactions in the superposition wave.

Key words: compressible Navier-Stokes equations, inflow problem, boundary layer solution, rarefaction wave, viscous contact wave

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1 Introduction

In this paper, we consider an initial-boundary-value problem for full compressible Navier-Stokes equations in *Eulerian* coordinates on the half line $\mathbf{R}_+ = (0, +\infty)$

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = (\mu u_x)_x, \\ [\rho(e + \frac{1}{2}u^2)]_t + [\rho u(e + \frac{1}{2}u^2) + pu]_x = (\kappa \theta_x + \mu u u_x)_x \end{cases} \quad x > 0, t > 0, \quad (1.1)$$

where $\rho(t, x) > 0$, $u(t, x)$, $\theta(t, x) > 0$, $p(t, x) > 0$ and $e(t, x) > 0$ represent the mass density, the velocity, the absolute temperature, the pressure, and the specific internal energy of the gas respectively and $\mu > 0$ is the coefficient of viscosity, $\kappa > 0$ is the

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coefficient of heat conduction. Here we assume that both μ and κ are positive constants. Let $v = \frac{1}{\rho} (> 0)$ and s denote the specific volume and the entropy of the gas, respectively. Then by the second law of thermodynamics, we have for the ideal polytropic gas

$$p = Rv^{-1}\theta = Av^{-\gamma} \exp\left(\frac{\gamma-1}{R}s\right), \quad e(v, \theta) = \frac{R}{\gamma-1}\theta, \quad (1.2)$$

where $\gamma > 1$ denotes the adiabatic exponent of gas, and A and R are positive constants.

We consider the initial-boundary-value problem (1.1) with the initial values

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \rightarrow (\rho_+, u_+, \theta_+) \text{ as } x \rightarrow +\infty, \quad \inf_{x \in \mathbf{R}_+} (\rho_0, \theta_0)(x) > 0 \quad (1.3)$$

where $\rho_+ > 0$, u_+ and $\theta_+ > 0$ are given constants.

As pointed out by [15], the boundary conditions to the half space problem (1.1) can be proposed as one of the following three cases:

Case I. outflow problem (negative velocity on the boundary):

$$u(t, x)|_{x=0} = u_- < 0, \quad \theta(t, x)|_{x=0} = \theta_-. \quad (1.4)_1$$

Case II. impermeable wall problem (zero velocity on the boundary):

$$u(t, x)|_{x=0} = 0, \quad \theta(t, x)|_{x=0} = \theta_-. \quad (1.4)_2$$

Case III. inflow problem (positive velocity on the boundary):

$$u(t, x)|_{x=0} = u_- > 0, \quad \rho(t, x)|_{x=0} = \rho_-, \quad \theta(t, x)|_{x=0} = \theta_-. \quad (1.4)_3$$

Here all the $\rho_- > 0$, u_- and $\theta_- > 0$ in (1.4) are prescribed constants and of course we assume that the initial values (1.3) and the boundary conditions (1.4) satisfy the compatibility condition at the origin. Notice that in Cases I and II, the density ρ_- on the boundary $\{x = 0\}$ could not be given, but in Case III, ρ_- must be imposed due to the well-posedness theory of the hyperbolic equation (1.1)₁.

In the present paper, we are concerned with the large-time behavior of the solutions to the inflow problem (Case III) of the full compressible Navier-Stokes equations (1.1), (1.3) and (1.4)₃. The large-time behavior of the solutions to the compressible Navier-Stokes equations (1.1) is closely related to the corresponding Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \left[\rho \left(e + \frac{u^2}{2} \right) \right]_t + \left[\rho u \left(e + \frac{u^2}{2} \right) + pu \right]_x = 0. \end{cases} \quad (1.5)$$

The Euler system (1.5) is a typical example of the hyperbolic conservation laws. It is well-known that the main feature of the solutions to the hyperbolic conservation laws is the formation of the shock wave no matter how smooth the initial values are. The Euler system (1.5) contains three basic wave patterns, that is, two nonlinear waves, called shock wave and rarefaction wave and one linear wave called contact discontinuity in the solutions to the Riemann problem. The above three dilation invariant wave solutions and their linear superpositions in the increasing order of characteristic speed, i.e., Riemann

solutions, govern both local and large-time behavior of solutions to the Euler system and so govern the large-time behavior of the solutions to the compressible Navier-Stokes equations (1.1).

There have been a large amount of literature on the large-time behavior of solutions to the Cauchy problem of the compressible fluid system (1.1) towards the viscous version of the basic wave patterns. We refer to [1], [2], [5], [7], [8], [11], [13], [14], [16], [20], [23], [24] and some references therein. All these works show that the large-time behavior of the solutions to the Cauchy problem is basically governed by the Riemann solutions to its corresponding hyperbolic system.

Recently, the initial-boundary value problem of (1.1) attracts increasing interest because it has more physical meanings and of course produces some new mathematical difficulties due to the boundary effect. Not only basic wave patterns but also a new wave, which is called boundary layer solution (BL-solution for brevity) [15], may appear in the IVBP case. Matsumura [15] proposes a criterion on the question when the BL-solution forms to the isentropic Navier-Stokes equations, where the entropy of the gas is assumed to be constant and the equation (1.1)₃ for the energy conservation is neglected. The argument in [15] for the isentropic Navier-Stokes equations can also be applied to the full Navier-Stokes equations (1.1), see [3] for details. Consider the Riemann problem to the Euler equations (1.5), where the initial right state of the Riemann data is given by the far field state (ρ_+, u_+, θ_+) in (1.3), and the left end state (ρ_-, u_-, θ_-) is given by the all possible states which are consistent with the boundary condition (1.4) at $\{x = 0\}$. Note that to the outflow problem, ρ_- can not be prescribed and is free on the boundary. On one hand, when the left end state is uniquely determined so that the value at the boundary $\{x = 0\}$ of the solution to the Riemann problem is consistent with the boundary condition, we expect that no BL-solution occurs. On the other hand, if the value of the solution to the Riemann problem on the boundary is not consistent with the boundary condition for any admissible left end state, we expect a BL-solution which compensates the gap comes up. Such BL-solution could be constructed by the stationary solution to Navier-Stokes equations. The existence and stability of the BL-solution (to the inflow or outflow problems, to the isentropic or full Navier-Stokes equations) are studied extensively by many authors, see [3], [4], [6], [10], [15] [18], [21], [25], etc.

Now we review some recent works on the large-time behavior of the solutions to the inflow problem of the full Navier-Stokes equation (1.1), (1.3), (1.4)₃ by Huang-Li-Shi [3] and Qin-Wang [21]. In [21], we rigorously prove the existence (or non-existence) of BL-solution to the inflow problem (1.1), (1.3), (1.4)₃ when the right end state (ρ_+, u_+, θ_+) belongs to the subsonic, transonic and supersonic regions respectively. When $(\rho_\pm, u_\pm, \theta_\pm)$ both belong to the subsonic region, the BL-solution is expected and the stability of this BL-solution and its superposition with the 3-rarefaction wave is proved under some smallness assumptions in [3]. The stability of the superposition of the subsonic BL-solution, the viscous 2-contact wave and 3-rarefaction wave is shown in [21] under the condition that the amplitude of BL-solution and the contact wave is small enough but the amplitude of the rarefaction wave is not necessarily small. The stability of the single viscous contact wave is also obtained in [21] if the contact wave is weak enough. It should be remarked that the subsonic BL-solution decays exponentially with respect to $\xi = x - \sigma_- t$, which is good enough to get the desired estimates. When the boundary value (ρ_-, u_-, θ_-) belongs to the supersonic region, there is no BL-solution. Thus the large-time behavior of the

solution is expected to be same as that of the Cauchy problem and the stability of the 3-rarefaction waves is also given in [3].

In the present paper, we are interested in the stability of wave patterns to the inflow problem (1.1), (1.3) and (1.4)₃ when (ρ_-, u_-, θ_-) belongs to the transonic region. In this case, a new wave structure which contains four waves: the transonic(or degenerate) BL-solution, 1-rarefaction wave, viscous 2-contact wave and 3-rarefaction wave, occurs. Due to the fact that the first characteristic speed on the boundary is coincident with the speed of the moving boundary in the transonic BL-solution case, the nonlinear waves in the first characteristic field may appear, which is quite different from the regime that (ρ_-, u_-, θ_-) belongs to the subsonic region in our previous result [21], where the waves in the first characteristic field must be absent. Here we just assume that the 1-rarefaction wave appear in the first characteristic field. Correspondingly, some new mathematical difficulties occur due to the degeneracy of the transonic BL-solution and its interactions with other wave patterns in the superposition wave. In particular, the transonic boundary layer solution is attached with 1-rarefaction wave for all time, so the interaction of these two waves should be carefully treated in the stability analysis.

Because the system (1.1) we consider is in one dimension of the space variable x , it is convenient to use the following Lagrangian coordinate transformation:

$$(t, x) \Rightarrow \left(t, \int_{(0,0)}^{(t,x)} \rho(\tau, y) dy - \rho u(\tau, y) d\tau \right).$$

Thus the system (1.1) can be transformed into the following moving boundary problem of Navier-Stokes equations in the Lagrangian coordinates [18]:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu \left(\frac{u_x}{v} \right)_x, & t > 0, x > \sigma_- t, \\ \left(\frac{R}{\gamma-1} \theta + \frac{1}{2} u^2 \right)_t + (pu)_x = \kappa \left(\frac{\theta_x}{v} \right)_x + \mu \left(\frac{u u_x}{v} \right)_x, \\ (v, u, \theta)(0, x) = (v_0, u_0, \theta_0)(x) \rightarrow (v_+, u_+, \theta_+), \quad \text{as } x \rightarrow +\infty, \\ (v, u, \theta)(t, x = \sigma_- t) = (v_-, u_-, \theta_-), \quad u_- > 0 \end{cases} \quad (1.6)$$

where $\sigma_- := -\frac{u_-}{v_-} < 0$ is the speed of the moving boundary.

In order to fix the moving boundary $x = \sigma_- t$, we introduce a new variable $\xi = x - \sigma_- t$. Then we have the half-space problem

$$\begin{cases} v_t - \sigma_- v_\xi - u_\xi = 0, \\ u_t - \sigma_- u_\xi + p_\xi = \mu \left(\frac{u_\xi}{v} \right)_\xi, & t > 0, \xi \in \mathbf{R}_+, \\ \left(\frac{R}{\gamma-1} \theta + \frac{1}{2} u^2 \right)_t - \sigma_- \left(\frac{R}{\gamma-1} \theta + \frac{1}{2} u^2 \right)_\xi + (pu)_\xi = \kappa \left(\frac{\theta_\xi}{v} \right)_\xi + \mu \left(\frac{u u_\xi}{v} \right)_\xi, \\ (v, u, \theta)(t = 0, \xi) = (v_0, u_0, \theta_0)(\xi) \rightarrow (v_+, u_+, \theta_+) \quad \text{as } \xi \rightarrow +\infty, \\ (v, u, \theta)(t, \xi = 0) = (v_-, u_-, \theta_-), \quad u_- > 0. \end{cases} \quad (1.7)$$

Given the right end state (v_+, u_+, θ_+) , we can define the following wave curves in the phase space (v, u, θ) with $v > 0$ and $\theta > 0$.

- Transonic(or degenerate) boundary layer curve:

$$BL(v_+, u_+, \theta_+) := \left\{ (v, u, \theta) \left| \frac{u}{v} = -\sigma_- = \frac{u_-}{v_-}, (u, \theta) \in \Sigma(u_+, \theta_+) \right. \right\}, \quad (1.8)$$

where $(v_+, u_+, \theta_+) \in \Gamma_{trans}^+ = \{(u, \theta) | u = \sqrt{R\gamma\theta} > 0\}$ is the transonic region defined in (2.4) with positive gas velocity and $\Sigma(u_+, \theta_+)$ is the trajectory at the point (u_+, θ_+) defined in Case II of Lemma 2.1 below.

- Contact wave curve:

$$CD(v_+, u_+, \theta_+) := \{(v, u, \theta) | u = u_+, p = p_+, v \neq v_+\}, \quad (1.9)$$

- i -Rarefaction wave curve ($i = 1, 3$):

$$R_i(v_+, u_+, \theta_+) := \left\{ (v, u, \theta) \left| \lambda_i < \lambda_{i+}, u = u_+ - \int_{v_+}^v \lambda_i(\eta, s_+) d\eta, s(v, \theta) = s_+ \right. \right\}, \quad (1.10)$$

where $s_+ = s(v_+, \theta_+)$ and $\lambda_i = \lambda_i(v, s)$ is the i -th characteristic speed given in (2.2).

Our main stability result is, roughly speaking, as follows:

- Assume that $(v_-, u_-, \theta_-) \in \text{BL-R}_1\text{-CD-R}_3(v_+, u_+, \theta_+)$, that is, there exist the unique medium states $(v_*, u_*, \theta_*) \in \Gamma_{trans}^+$, (v_m, u_m, θ_m) and (v^*, u^*, θ^*) , such that $(v_-, u_-, \theta_-) \in \text{BL}(v_*, u_*, \theta_*)$, $(v_*, u_*, \theta_*) \in R_1(v_m, u_m, \theta_m)$, $(v_m, u_m, \theta_m) \in \text{CD}(v^*, u^*, \theta^*)$ and $(v^*, u^*, \theta^*) \in R_3(v_+, u_+, \theta_+)$, then the superposition of the four wave patterns: the transonic (or degenerate) BL-solution, 1-rarefaction wave, 2-viscous contact wave and 3-rarefaction wave is time-asymptotically stable provided that the wave strength $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$ is suitably small and the conditions in Theorem 2.1 hold.

This paper is organized as follows. In Section 2, after giving some preliminaries on boundary layer solution, viscous 2-contact wave, rarefaction waves and their superposition, we state our main result. In Section 3, first the wave interaction estimations are shown, then the desired energy estimates are performed and finally our main result is proven.

Notations. Throughout this paper, several positive generic constants are denoted by c, C without confusion, and $C(\cdot)$ stands for some generic constant(s) depending only on the quantity listed in the parenthesis. For function spaces, $L^p(\mathbf{R}_+)$, $1 \leq p \leq \infty$, denotes the usual Lebesgue space on \mathbf{R}_+ . $W^{k,p}(\mathbf{R}_+)$ denotes the k^{th} order Sobolev space, and if $p = 2$, we note $H^k(\mathbf{R}_+) := W^{k,2}(\mathbf{R}_+)$, $\|\cdot\| := \|\cdot\|_{L^2(\mathbf{R}_+)}$, and $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbf{R}_+)}$ for simplicity. The domain \mathbf{R}_+ will be often abbreviated without confusion.

2 Preliminaries and Main Result

It is well known that the hyperbolic system (1.5) has three characteristic speeds

$$\lambda_1(v, \theta) = -\frac{\sqrt{R\gamma\theta}}{v}, \quad \lambda_2 = 0, \quad \lambda_3(v, \theta) = \frac{\sqrt{R\gamma\theta}}{v}. \quad (2.1)$$

The first and the third characteristic field is genuinely nonlinear, which may have nonlinear waves, shock wave and rarefaction wave, while the second characteristic field is linearly degenerate, where contact discontinuity may occur.

Let

$$c(v, s) := \sqrt{-v^2 p_v(v, s)} = \sqrt{R\gamma\theta} =: c(v, \theta), \quad M(v, u, \theta) := \frac{|u|}{c(v, \theta)} \quad (2.2)$$

be the sound speed and the Mach number at the state (v, u, θ) . Correspondingly, set

$$c_+ := c(v_+, \theta_+) = \sqrt{R\gamma\theta_+}, \quad M_+ := M(v_+, u_+, \theta_+) = \frac{|u_+|}{c_+} \quad (2.3)$$

be the sound speed and the Mach number at the far field $\{x = +\infty\}$. We divide the phase space $\{(v, u, \theta) | v > 0, \theta > 0\}$ into three parts:

$$\begin{cases} \Omega_{sub} := \{(v, u, \theta) | M < 1\}, \\ \Gamma_{trans} := \{(v, u, \theta) | M = 1\}, \\ \Omega_{super} := \{(v, u, \theta) | M > 1\}. \end{cases} \quad (2.4)$$

Call them subsonic, transonic and supersonic region, respectively. Obviously, if we add the alternative condition $u > 0$ or $u \leq 0$, then we have six regions Ω_{sub}^\pm , Γ_{trans}^\pm , and Ω_{super}^\pm .

2.1 Boundary layer solution

When $(v_-, u_-, \theta_-) \in \Omega_{sub}^+ \cup \Gamma_{trans}^+$, we have

$$\lambda_1(v_-, \theta_-) = -\frac{\sqrt{R\gamma\theta_-}}{v_-} \leq -\frac{u_-}{v_-} = \sigma_- < 0, \quad (2.5)$$

hence a stationary solution $(V^b, U^b, \Theta^b)(\xi)$ to the inflow problem (1.7) is expected

$$\begin{cases} -\sigma_- V_\xi^b - U_\xi^b = 0, \\ -\sigma_- U_\xi^b + P_\xi^b = \mu \left(\frac{U_\xi^b}{V^b} \right)_\xi, \\ -\sigma_- \left(\frac{R}{\gamma-1} \Theta^b + \frac{1}{2} (U^b)^2 \right)_\xi + (P^b U^b)_\xi = \kappa \left(\frac{\Theta^b}{V^b} \right)_\xi + \mu \left(\frac{U^b U_\xi^b}{V^b} \right)_\xi, \\ (V^b, U^b, \Theta^b)(0) = (v_-, u_-, \theta_-), \quad (V^b, U^b, \Theta^b)(+\infty) = (v_+, u_+, \theta_+), \end{cases} \quad (2.6)$$

where $P^b := p(V^b, \Theta^b) = \frac{R\Theta^b}{V^b}$. We call this stationary solution $(V^b, U^b, \Theta^b)(\xi)$ the boundary layer solution (simply, BL-solution) to the inflow problem (1.7).

From the fact that $V^b(\xi) > 0$ and $u_- > 0$, then

$$u_+ > 0, \quad \frac{U^b}{V^b} = \frac{u_+}{v_+} = \frac{u_-}{v_-} = -\sigma_-. \quad (2.7)$$

Thus (2.6) is equivalent to (2.7) and the following ODE system

$$\begin{cases} (U^b)' = -\frac{\sigma_-}{\mu} V^b (U^b - u_+) + \frac{R}{\mu} \left(\Theta^b - \frac{\theta_+}{v_+} V^b \right) & ' = \frac{d}{d\xi}, \\ (\Theta^b)' = -\frac{R\sigma_-}{\kappa(\gamma-1)} V^b (\Theta^b - \theta_+) + \frac{p_+}{\kappa} V^b (U^b - u_+) + \frac{\sigma_-}{2\kappa} V^b (U^b - u_+)^2, \\ (U^b, \Theta^b)(0) = (u_-, \theta_-), \quad (U^b, \Theta^b)(+\infty) = (u_+, \theta_+), \end{cases} \quad (2.8)$$

where $p_+ := p(v_+, \theta_+)$.

We can compute that the

Now we state the existence results of the BL-solution to (2.8) while its proof has been shown in [21].

Lemma 2.1 (Existence of BL-solution) [21] *Suppose that $v_{\pm} > 0$, $u_- > 0$, $\theta_{\pm} > 0$ and let $\delta_b := |(u_+ - u_-, \theta_+ - \theta_-)|$. If $u_+ \leq 0$, then there is no solution to (2.8). If $u_+ > 0$, then there exists a suitably small constant $\delta_0 > 0$ such that if $0 < \delta^b \leq \delta_0$, then the existence and non-existence of solutions to (2.8) is divided into three cases according to the location of (u_+, θ_+) :*

Case I : $(u_+, \theta_+) \in \Omega_{sup}^+$. Then there is no solution to (2.8).

Case II : $(u_+, \theta_+) \in \Gamma_{trans}^+$. Then (u_+, θ_+) is a saddle-knot point to (2.8). Precisely, there exists a unique trajectory Σ tangent to the straight line

$$\mu u_+(u - u_+) - \kappa(\gamma - 1)(\theta - \theta_+) = 0 \quad (2.9)$$

at the point (u_+, θ_+) . For each $(u_-, \theta_-) \in \Sigma(u_+, \theta_+)$, there exists a unique solution (U^b, Θ^b) satisfying

$$U_{\xi}^b > 0, \quad \Theta_{\xi}^b > 0,$$

and

$$\left| \frac{d^n}{d\xi^n} (U^b - u_+, \Theta^b - \theta_+) \right| = O(1) \frac{\delta_b^{n+1}}{(1 + \delta_b \xi)^{n+1}}, \quad n = 0, 1, 2, \dots \quad (2.10)$$

Case III : $(u_+, \theta_+) \in \Omega_{sub}^+$. Then (u_+, θ_+) is a saddle point to (2.8). Precisely, there exists a center-stable manifold \mathcal{M} tangent to the line

$$(1 + a_2 c_2 u_+)(U^B - u_+) - a_2(\Theta^B - \theta_+) = 0$$

on the opposite directions at the point (u_+, θ_+) . Here c_2 is one of the solutions to the equation

$$y^2 + \left(\frac{M_+^2 \gamma - 1}{M_+^2 R \gamma} - \frac{\mu}{\kappa(\gamma - 1)} \right) y - \frac{\mu}{M_+^2 R \gamma \kappa} = 0$$

and $a_2 = -\frac{R}{\mu(\lambda_J^1 - \lambda_J^2)}$ with $\lambda_J^1 > 0$, $\lambda_J^2 < 0$ are the two eigenvalues of the linearized matrix of ODE (2.8). Only when $(u_-, \theta_-) \in \mathcal{M}(u_+, \theta_+)$, does there exist a unique solution $(U^b, \Theta^b) \subset \mathcal{M}(u_+, \theta_+)$ satisfying

$$\left| \frac{d^n}{d\xi^n} (U^b - u_+, \Theta^b - \theta_+) \right| = O(1) \delta_b e^{-c\xi}, \quad n = 0, 1, 2, \dots \quad (2.11)$$

2.2 Viscous Contact Wave

If $(v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+)$, then the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, & t > 0, x \in \mathbf{R}, \\ \left(\frac{R}{\gamma-1} \theta + \frac{1}{2} u^2 \right)_t + (pu)_x = 0, \\ (v, u, \theta)(0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0 \end{cases} \end{cases} \quad (2.12)$$

admits a contact discontinuity solution

$$(v, u, \theta)(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, t > 0, \\ (v_+, u_+, \theta_+), & x > 0, t > 0. \end{cases}$$

From [7], the viscous version of the above contact discontinuity, called viscous contact wave $(V^d, U^d, \Theta^d)(t, x)$ can be defined by

$$\begin{cases} V^d(t, x) = \frac{R\Theta^{\text{sim}}(t, x)}{p_+}, \\ U^d(t, x) = u_+ + \frac{(\gamma-1)\kappa\Theta_x^{\text{sim}}(t, x)}{\gamma\Theta^{\text{sim}}(t, x)}, \\ \Theta^d(t, x) = \Theta^{\text{sim}}\left(\frac{x}{\sqrt{1+t}}\right) + R\left(\mu - \frac{(\gamma-1)\kappa}{R\gamma}\right)\Theta_t^{\text{sim}} \end{cases} \quad (2.13)$$

where $\Theta^{\text{sim}}\left(\frac{x}{\sqrt{1+t}}\right)$ is the unique self-similar solution to the following nonlinear diffusion equation

$$\begin{cases} \Theta_t = \frac{(\gamma-1)\kappa p_+}{R^2\gamma} \left(\frac{\Theta_x}{\Theta}\right)_x, \\ \Theta(t, \pm\infty) = \theta_{\pm}. \end{cases} \quad (2.14)$$

Note that $\xi = x - \sigma_-t$, we have the following Lemma:

Lemma 2.2. [7] *The viscous contact wave $(V^d, U^d, \Theta^d)(t, x)$, $(x = \xi + \sigma_-t)$ defined in (2.13) satisfies*

- i) $\partial_{\xi}^n(\Theta^d - \theta_{\pm}) = O(1)\delta_d(1+t)^{-\frac{n}{2}} \exp\left(-\frac{C_d(\xi+\sigma_-t)^2}{1+t}\right)$, $n = 0, 1, 2, \dots$;
- ii) $U_{\xi}^d(t, \xi) = O(1)\delta_d(1+t)^{-1} \exp\left(-\frac{C_d(\xi+\sigma_-t)^2}{1+t}\right)$;
- iii) $(V^d, U^d, \Theta^d)(t, \xi = 0) - (v_-, u_-, \theta_-) = O(1)\delta_d e^{-ct}$.

where $\delta_d = |\theta_+ - \theta_-|$ is the amplitude of the viscous contact wave and $C_d, c > 0$ are constants.

Then the viscous contact wave (V^d, U^d, Θ^d) defined in (2.13) satisfies the system

$$\begin{cases} V_t^d - \sigma_-V_{\xi}^d - U_{\xi}^d = 0, \\ U_t^d - \sigma_-U_{\xi}^d + P_{\xi}^d = \mu\left(\frac{U_{\xi}^d}{V^d}\right)_{\xi}, \\ \frac{R}{\gamma-1}(\Theta_t^d - \sigma_- \Theta_{\xi}^d) + P^d U_{\xi}^d = \kappa\left(\frac{\Theta_{\xi}^d}{V^d}\right)_{\xi} + \mu\frac{(U_{\xi}^d)^2}{V^d} + H^d \end{cases} \quad t > 0, \xi \in \mathbf{R}_+, \quad (2.15)$$

where $P^d := p(V^d, \Theta^d)$ and

$$H^d = O(1)\delta_d(1+t)^{-2} \exp\left(-\frac{C_d(\xi + \sigma_-t)^2}{1+t}\right)$$

due to Lemma 2.2.

2.3 Rarefaction waves

It is well known that if $(v_-, u_-, \theta_-) \in R_i(v_+, u_+, \theta_+)$, ($i = 1, 3$), then there exist a i -rarefaction wave $(v^{r_i}, u^{r_i}, \theta^{r_i})(x/t)$ which is the global weak solution to the following Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, & t > 0, x \in \mathbf{R}, \\ \left(\frac{R}{\gamma-1}\theta + \frac{1}{2}u^2\right)_t + (pu)_x = 0, \\ (v, u, \theta)(0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0. \end{cases} \end{cases} \quad (2.16)$$

Consider the following Burgers equation

$$\begin{cases} w_t + ww_x = 0, & t > 0, x \in \mathbf{R}, \\ w_0(x) := w(0, x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q(w_+ - w_-) \int_0^x y^q e^{-y} dy, & x \geq 0. \end{cases} \end{cases} \quad (2.17)$$

Here $q \geq 14$ is a constant to be determined, and C_q is a constant such that $C_q \int_0^{+\infty} y^q e^{-y} dy = 1$. If $w_- < w_+$, then the solution to the above Burgers equation can be expressed by

$$w(t, x) = w_0(x_0(t, x)), \quad x = x_0(t, x) + w_0(x_0(t, x))t. \quad (2.18)$$

Moreover, we have

- $w(t, x) = w_-$, if $x \leq w_-t$.
- For any positive constant $\sigma_0 > 0$ and for $x \geq 0$

$$\begin{aligned} |w(t, x) - w_+| &= |w_0(x_0(t, x)) - w_+| \\ &= C_q(w_+ - w_-) \int_{x_0(t, x)}^{+\infty} y^q e^{-y} dy \\ &= C_q(w_+ - w_-) \int_{x - w_0(x_0(t, x))t}^{+\infty} y^q e^{-y} dy \\ &\leq C_q(w_+ - w_-) \int_{x - w_+t}^{+\infty} y^q e^{-y} dy \\ &\leq C_q(w_+ - w_-) e^{-\sigma_0 t}, \quad \text{if } x \geq (2\sigma_0 + w_+)t. \end{aligned} \quad (2.19)$$

Note that the estimation in (2.19) play an important role in the wave interaction estimates, which is motivated by [12] and [16].

Now the i -rarefaction wave $(V^{r_i}, U^{r_i}, \Theta^{r_i})(t, x)$ ($i = 1, 3$) to the inflow problem (1.7) can be defined by

$$\begin{cases} \lambda_i(V^{r_i}, \Theta^{r_i})(t, x) = w(1 + t, x + \sigma_-), \\ s(V^{r_i}, \Theta^{r_i})(t, x) = s_+ = s(v_+, \theta_+), \\ U^{r_i}(t, x) = u_+ - \int_{v_+}^{V^{r_i}(t, x)} \lambda_i(\eta, s_+) d\eta. \end{cases} \quad (2.20)$$

Then the i -rarefaction wave $(V^{r_i}, U^{r_i}, \Theta^{r_i})(t, x)$, ($i = 1, 3$) defined in (2.20) satisfies the system

$$\begin{cases} V_t^{r_i} - \sigma_- V_\xi^{r_i} - U_\xi^{r_i} = 0, \\ U_t^{r_i} - \sigma_- U_\xi^{r_i} + P_\xi^{r_i} = 0, \\ \left[\frac{R}{\gamma-1} \Theta^{r_i} + \frac{1}{2} (U^{r_i})^2 \right]_t - \sigma_- \left[\frac{R}{\gamma-1} \Theta^{r_i} + \frac{1}{2} (U^{r_i})^2 \right]_\xi + (P^{r_i} U^{r_i})_\xi = 0, \\ (V^{r_i}, U^{r_i}, \Theta^{r_i})(t, \xi = 0) = (v_-, u_-, \theta_-), \\ (V^{r_i}, U^{r_i}, \Theta^{r_i})(t, \xi) \rightarrow (v_+, u_+, \theta_+) \text{ as } \xi \rightarrow +\infty \end{cases} \quad (2.21)$$

where $P^{r_i} := p(V^{r_i}, \Theta^{r_i})$.

Lemma 2.3 i -rarefaction wave $(V^{r_i}, U^{r_i}, \Theta^{r_i})(t, \xi)$, ($i = 1, 3$) defined in (2.20) satisfies

- i) $U_\xi^{r_i}(t, \xi) > 0$, $(|V_\xi^{r_i}|, |\Theta_\xi^{r_i}|) \leq C U_\xi^{r_i}$;
- ii) For any p ($1 \leq p \leq \infty$), there exists a constant C_{pq} such that

$$\begin{aligned} \|(V_\xi^{r_i}, U_\xi^{r_i}, \Theta_\xi^{r_i})(t)\|_{L^p} &\leq C_p \min \{ \delta_{r_i}, \delta_{r_i}^{1/p} (1+t)^{-1+1/p} \}, \\ \|(V_{\xi\xi}^{r_i}, U_{\xi\xi}^{r_i}, \Theta_{\xi\xi}^{r_i})(t)\|_{L^p} &\leq C_p \min \{ \delta_{r_i}, \delta_{r_i}^{1/p+1/q} (1+t)^{-1+1/q} \}; \end{aligned}$$

- iii) For $\forall \sigma_0 > 0$, if $\xi \geq [-\sigma_- + \lambda_1(v_+, \theta_+) + 2\sigma_0](1+t)$, then $\left| \partial_\xi^n \{ (V^{r_1}, U^{r_1}, \Theta^{r_1})(t, \xi) - (v_+, u_+, \theta_+) \} \right| \leq C \delta_{r_1} e^{-\sigma_0 t}$, $n = 0, 1, 2, \dots$;
- iv) For $\xi \leq [-\sigma_- + \lambda_3(v_-, \theta_-)](1+t)$, $(V^{r_3}, U^{r_3}, \Theta^{r_3}) - (v_-, u_-, \theta_-) \equiv 0$;
- v) $\limsup_{t \rightarrow \infty} \sup_{\xi \in \mathbf{R}_+} |(V^{r_i}, U^{r_i}, \Theta^{r_i})(t, \xi) - (v^{r_i}, u^{r_i}, \theta^{r_i})\left(\frac{\xi}{1+t}\right)| = 0$.

Remark: The statement iii) is a direct consequence of the (2.19).

2.4 Superposition of transonic BL-solution, 1-rarefaction wave, 2-viscous contact wave and 3-rarefaction wave

In this subsection, we consider the case that $(v_-, u_-, \theta_-) \in BL-R_1-CD-R_3(v_+, u_+, \theta_+)$, that is, there exist uniquely three medium states $(v_*, u_*, \theta_*) \in \Gamma_{trans}^+$, (v_m, u_m, θ_m) and (v^*, u^*, θ^*) such that $(v_*, u_*, \theta_*) \in BL(v_-, u_-, \theta_-)$, $(v_*, u_*, \theta_*) \in R_1(v_m, u_m, \theta_m)$, $(v_m, u_m, \theta_m) \in CD(v^*, u^*, \theta^*)$ and $(v^*, u^*, \theta^*) \in R_3(v_+, u_+, \theta_+)$. In fact, three medium states $(v_*, u_*, \theta_*) \in \Gamma_{trans}^+$, (v_m, u_m, θ_m) and (v^*, u^*, θ^*) can be expressed explicitly and uniquely by the following nine equations

$$\begin{cases} \frac{u_-}{v_-} = \frac{u_*}{v_*}, \quad u_* = \sqrt{R\gamma\theta_*}, \quad (u_-, \theta_-) \in \Sigma(u_*, \theta_*), \\ u_* = u_m - \int_{v_m}^{v_*} \sqrt{R\gamma v_+^{\gamma-1} \theta_+} \eta^{-\frac{\gamma+1}{2}} d\eta, \quad v_*^{\gamma-1} \theta_* = v_m^{\gamma-1} \theta_m, \\ u_m = u^*, \quad \frac{\theta_m}{v_m} = \frac{\theta^*}{v^*}, \\ u^* = u_+ + \int_{v^*}^{v_+} \sqrt{R\gamma v_+^{\gamma-1} \theta_+} \eta^{-\frac{\gamma+1}{2}} d\eta, \quad v^{*\gamma-1} \theta^* = v_+^{\gamma-1} \theta_+. \end{cases} \quad (2.22)$$

Define the superposition wave $(V, U, \Theta)(t, \xi)$ by

$$\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix} (t, \xi) = \begin{pmatrix} V^b + V^{r_1} + V^d + V^{r_3} \\ U^b + U^{r_1} + U^d + U^{r_3} \\ \Theta^b + \Theta^{r_1} + \Theta^d + \Theta^{r_3} \end{pmatrix} (t, \xi) - \begin{pmatrix} v_* + v_m + v^* \\ u_* + u_m + u^* \\ \theta_* + \theta_m + \theta^* \end{pmatrix} \quad (2.23)$$

where $(V^b, U^b, \Theta^b)(\xi)$ is the transonic BL-solution defined in Case II of Lemma 2.1 with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) , $(V^{r_1}, U^{r_1}, \Theta^{r_1})(t, \xi)$ is the 1-rarefaction wave defined in (2.20) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v_m, u_m, θ_m) respectively, $(V^d, U^d, \Theta^d)(t, \xi)$ is the viscous contact wave defined in (2.13) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_m, u_m, θ_m) and (v^*, u^*, θ^*) , respectively, and $(V^{r_3}, U^{r_3}, \Theta^{r_3})(t, \xi)$ is the 3-rarefaction wave defined in (2.20) with the left state (v_-, u_-, θ_-) replaced by (v^*, u^*, θ^*) .

Now we state the main result of the paper as follows.

Theorem 2.1 (Stability of superposition of four waves) *Assume that $(v_-, u_-, \theta_-) \in BL-R_1-CD-R_3(v_+, u_+, \theta_+)$. Let $(V, U, \Theta)(t, \xi)$ be the superposition of the transonic BL-solution, 1-rarefaction wave, viscous 2-contact wave and 3-rarefaction wave defined in (2.23). Then there exists a small positive constant δ_0 such that if the initial values and the wave strength $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$ satisfy*

$$\delta + \|(v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0)\|_1 \leq \delta_0. \quad (2.24)$$

the inflow problem (1.7) has a unique global-in-time solution $(v, u, \theta)(t, \xi)$ satisfying

$$\begin{cases} (v - V, u - U, \theta - \Theta)(t, \xi) \in C([0, \infty); H^1(\mathbf{R}^+)), \\ (v - V)_\xi(t, \xi) \in L^2(0, \infty; L^2(\mathbf{R}^+)), \\ (u - U, \theta - \Theta)_\xi(t, \xi) \in L^2(0, \infty; H^1(\mathbf{R}^+)). \end{cases} \quad (2.25)$$

Furthermore,

$$\lim_{t \rightarrow \infty} \sup_{\xi \in \mathbf{R}_+} |(v - V, u - U, \theta - \Theta)(t, \xi)| = 0. \quad (2.26)$$

Remark. In Theorem 2.1, we assume that $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$ is suitably small. This assumption is equivalent to the one that the amplitudes of the four waves are all suitably small. In fact, from the relations in (2.22) and the facts $U_\xi^b > 0$, $U_\xi^{r_1} > 0$, $U_\xi^{r_3} > 0$, we have

$$\begin{cases} |v_* - v_-| + |\theta_* - \theta_-| = O(1)(u_* - u_-), \\ |v_m - v_*| + |\theta_m - \theta_*| = O(1)(u_m - u_*), \\ |v_+ - v^*| + |\theta_+ - \theta^*| = O(1)(u_+ - u^*). \end{cases} \quad (2.27)$$

Thus $\delta_b = O(1)(u_* - u_-)$, $\delta_{r_1} = O(1)(u_m - u_*)$, $\delta_{r_3} = O(1)(u_+ - u^*)$. Due to $u_m = u^*$ by the contact discontinuity curve, we have if δ is small, then δ_b, δ_{r_1} and δ_{r_3} are all small. Furthermore, we have $\delta_d = |\theta^* - \theta_m| \leq \delta_b + \delta_{r_1} + \delta_{r_3} + \delta$ is small.

3 Stability Analysis

3.1 Wave interaction estimates

Recalling the definition of the superposition wave $(V, U, \Theta)(t, \xi)$ defined in (2.23), we have

$$\begin{cases} V_t - \sigma_- V_\xi - U_\xi = 0, \\ U_t - \sigma_- U_\xi + P_\xi = \mu \left(\frac{U_\xi}{V} \right)_\xi + G, \\ \frac{R}{\gamma-1} (\Theta_t - \sigma_- \Theta_\xi) + P U_\xi = \kappa \left(\frac{\Theta_\xi}{V} \right)_\xi + \mu \frac{(U_\xi)^2}{V} + H, \\ (V, U, \Theta)(t, \xi = 0) = (v_-, u_-, \theta_-) + (V^d, U^d, \Theta^d)(t, \xi = 0) - (v_m, u_m, \theta_m). \end{cases} \quad (3.1)$$

where $P := p(V, \Theta)$ and

$$\begin{cases} G = (P - P^b - P^{r_1} - P^d - P^{r_3})_\xi - \mu \left(\frac{U_\xi}{V} - \frac{U_\xi^b}{V^b} - \frac{U_\xi^d}{V^d} \right)_\xi =: G_1 + G_2, \\ H = (P U_\xi - P^b U_\xi^b - P^{r_1} U_\xi^{r_1} - P^d U_\xi^d - P^{r_3} U_\xi^{r_3}) \\ \quad - \left[\kappa \left(\frac{\Theta_\xi}{V} - \frac{\Theta_\xi^b}{V^b} - \frac{\Theta_\xi^d}{V^d} \right)_\xi + \mu \left(\frac{(U_\xi)^2}{V} - \frac{(U_\xi^b)^2}{V^b} - \frac{(U_\xi^d)^2}{V^d} \right) - H^d \right] =: H_1 + H_2. \end{cases} \quad (3.2)$$

To control the interaction terms coming from different wave patterns, we give the following lemma which will be critical in the energy estimate in Subsection 3.3.

Lemma 3.1 (Wave interaction estimates)

$$\begin{cases} \int_{\mathbf{R}_+} |V_\xi^b (V^{r_1} - v_*)| + |V_\xi^{r_1} (V^b - v_*)| d\xi = O(1) \delta^{1/8} (1+t)^{-13/16}, \\ \int_{\mathbf{R}_+} |V_\xi^b (V^d - v_m)| + |V_\xi^d (V^b - v_*)| d\xi = O(1) \delta (1+t)^{-1}, \\ \int_{\mathbf{R}_+} |V_\xi^b (V^{r_3} - v^*)| + |V_\xi^{r_3} (V^b - v_*)| d\xi = O(1) \delta^{1/8} (1+t)^{-7/8}, \\ \int_{\mathbf{R}_+} |V_\xi^d (V^{r_1} - v_m)| + |V_\xi^{r_1} (V^d - v_m)| d\xi = O(1) \delta e^{-ct}, \\ \int_{\mathbf{R}_+} |V_\xi^d (V^{r_3} - v^*)| + |V_\xi^{r_3} (V^d - v^*)| d\xi = O(1) \delta e^{-ct}, \\ \int_{\mathbf{R}_+} |V_\xi^{r_1} (V^{r_3} - v^*)| + |V_\xi^{r_3} (V^{r_1} - v_m)| d\xi = O(1) \delta e^{-ct}, \end{cases} \quad (3.3)$$

$$\begin{cases} \int_{\mathbf{R}_+} |V_\xi^b V_\xi^d| d\xi = O(1) \delta (1+t)^{-2}, & \int_{\mathbf{R}_+} |V_\xi^b V_\xi^{r_1}| d\xi = O(1) \delta (1+t)^{-1}, \\ \int_{\mathbf{R}_+} |V_\xi^b V_\xi^{r_3}| d\xi = O(1) \delta (1+t)^{-1}, & \int_{\mathbf{R}_+} |V_\xi^d V_\xi^{r_1}| d\xi = O(1) \delta e^{-ct}, \\ \int_{\mathbf{R}_+} |V_\xi^d V_\xi^{r_3}| d\xi = O(1) \delta e^{-ct}, & \int_{\mathbf{R}_+} |V_\xi^{r_1} V_\xi^{r_3}| d\xi = O(1) \delta e^{-ct}, \end{cases} \quad (3.4)$$

Proof. First we prove (3.3)₁, that is

- Interaction of transonic boundary layer solution and 1-rarefaction wave:

Since $V_\xi^{r_1} \geq 0$ and $V_\xi^b \geq 0$, we have $V^{r_1} - v_* \geq 0$ and $v_* - V^b \geq 0$. Thus we have

$$\begin{aligned}
& \int_{\mathbf{R}_+} |V_\xi^b(V^{r_1} - v_*)| + |V_\xi^{r_1}(V^b - v_*)| d\xi \\
&= 2 \left\{ \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} + \int_{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)}^{+\infty} \right\} V_\xi^{r_1}(v_* - V^b) d\xi \\
&:= J_1 + J_2.
\end{aligned} \tag{3.5}$$

Note that

$$\begin{aligned}
& -\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2} \\
&= \frac{u_-}{v_-} + \frac{\lambda_1(v_m, \theta_m)}{2} = \frac{u_*}{v_*} + \frac{\lambda_1(v_m, \theta_m)}{2} \\
&= \frac{\sqrt{R\gamma\theta_*}}{v_*} + \frac{\lambda_1(v_m, \theta_m)}{2} = -\lambda_1(v_*, \theta_*) + \frac{\lambda_1(v_m, \theta_m)}{2} \\
&= [\lambda_1(v_m, \theta_m) - \lambda_1(v_*, \theta_*)] - \frac{\lambda_1(v_m, \theta_m)}{2} \\
&\geq -\frac{\lambda_1(v_m, \theta_m)}{2} > 0.
\end{aligned}$$

Now we can compute that

$$\begin{aligned}
J_1 &= \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} V_\xi^{r_1}(v_* - V^b) d\xi \\
&= O(1) \|V_\xi^{r_1}(t)\|_{L^\infty} \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} \frac{\delta_b}{1 + \delta_b \xi} d\xi \\
&= O(1) \delta_{r_1}^{\frac{1}{8}} (1+t)^{-\frac{7}{8}} \ln(1 + \delta_b t) \\
&= O(1) \delta_{r_1}^{\frac{1}{8}} (1+t)^{-\frac{13}{16}},
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
J_2 &= \int_{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)}^{+\infty} V_\xi^{r_1}(v_* - V^b) d\xi \\
&= O(1) \delta_b (v_m - V^{r_1}(t, \xi))|_{\xi=[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} \\
&= O(1) \delta_b e^{-\sigma_0 t}.
\end{aligned} \tag{3.7}$$

due to the statement iii) in Lemma 2.3 by taking $\sigma_0 = -\frac{\lambda_1(v_m, \theta_m)}{2} > 0$. So the combination of (3.6) and (3.7) gives (3.3)₁.

Then we prove (3.3)₂:

- Interaction of transonic boundary layer solution and viscous 2-contact wave:

$$\begin{aligned}
& \int_{\mathbf{R}_+} |V_\xi^b(V^d - v_m)| + |V_\xi^d(V^b - v_*)| d\xi \\
&= \left\{ \int_0^{-\frac{\sigma_- t}{2}} + \int_{-\frac{\sigma_- t}{2}}^{+\infty} \right\} |V_\xi^b(V^d - v_m)| + |V_\xi^d(V^b - v_*)| d\xi \\
&:= J_3 + J_4.
\end{aligned}$$

We calculate

$$\begin{aligned}
J_3 &= \int_0^{-\frac{\sigma-t}{2}} |V_\xi^b(V^d - v_m)| + |V_\xi^d(V^b - v_*)| d\xi \\
&= O(1)\delta_d \int_0^{-\frac{\sigma-t}{2}} \exp\left(-\frac{C_d(\xi + \sigma-t)^2}{1+t}\right) d\xi \\
&= O(1)\delta_d e^{-ct}.
\end{aligned} \tag{3.8}$$

Also, we have

$$\begin{aligned}
J_4 &= \int_{-\frac{\sigma-t}{2}}^{+\infty} |V_\xi^b(V^d - v_m)| + |V_\xi^d(V^b - v_*)| d\xi \\
&:= J_4^1 + J_4^2.
\end{aligned}$$

We can estimate

$$\begin{aligned}
J_4^1 &= \int_{-\frac{\sigma-t}{2}}^{\infty} |V_\xi^b(V^d - v_m)| d\xi \\
&= O(1)\delta_d \delta_b^2 (1 + \delta_b t)^{-2} \int_{-\frac{\sigma-t}{2}}^{\infty} \exp\left(-\frac{C_d(\xi + \sigma-t)^2}{1+t}\right) d\xi \\
&= O(1)\delta_d (1+t)^{-3/2} \int_{-\infty}^{\infty} \exp(-C_d \eta^2) d\eta \\
&= O(1)\delta_d (1+t)^{-3/2},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
J_4^2 &= \int_{-\frac{\sigma-t}{2}}^{\infty} |V_\xi^d(V^b - v_*)| d\xi \\
&= O(1)\delta_d \delta_b (1 + \delta_b t)^{-1} (1+t)^{-1/2} \int_{-\frac{\sigma-t}{2}}^{\infty} \exp\left(-\frac{C_d(\xi + \sigma-t)^2}{1+t}\right) d\xi \\
&= O(1)\delta_d (1+t)^{-1}.
\end{aligned} \tag{3.10}$$

Thus we proved (3.3)₂.

Now we compute (3.3)₃:

- Interaction of transonic boundary layer solution and 3-rarefaction wave:

$$\begin{aligned}
&\int_{\mathbf{R}_+} |V_\xi^b(V^{r_3} - v^*)| + |V_\xi^{r_3}(V^b - v_*)| d\xi \\
&= \int_{[-\sigma_+ + \lambda_3(v^*, \theta^*)](1+t)}^{\infty} V_\xi^b(v^* - V^{r_3}) + V_\xi^{r_3}(V^b - v_*) d\xi \\
&= O(1)\delta_b (1 + \delta_b t)^{-1} \\
&= O(1) \min\{\delta, (1+t)^{-1}\} \\
&= O(1)\delta^{\frac{1}{8}} (1+t)^{-\frac{7}{8}}.
\end{aligned} \tag{3.11}$$

where in the first equality we have used the fact iv) in Lemma 2.3.

Then we verify (3.3)₄:

- Interaction of 1-rarefaction wave and viscous 2-contact wave:

First we have

$$\begin{aligned}
& \int_{\mathbf{R}_+} |V_\xi^d(v_m - V^{r_1})| d\xi \\
&= \left\{ \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} + \int_{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)}^{+\infty} \right\} |V_\xi^d(v_m - V^{r_1})| d\xi \\
&:= J_5 + J_6.
\end{aligned}$$

Then we can compute

$$\begin{aligned}
J_5 &= \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} |V_\xi^d(v_m - V^{r_1})| d\xi \\
&= O(1)\delta_d(1+t)^{-\frac{1}{2}} \int_0^{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} \exp\left(-\frac{C_d(\xi + \sigma_- t)^2}{1+t}\right) d\xi \\
&= O(1)\delta_d e^{-ct},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
J_6 &= \int_{[-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)}^{+\infty} |V_\xi^d(v_m - V^{r_1})| d\xi \\
&= O(1)\delta_d \sup_{\xi \geq [-\sigma_- + \frac{\lambda_1(v_m, \theta_m)}{2}](1+t)} (v_m - V^{r_1}(t, \xi)) = O(1)\delta_d e^{-ct}.
\end{aligned} \tag{3.13}$$

Similarly, we can estimate the interaction term

$$\int_{\mathbf{R}_+} |V_\xi^{r_1}(V^d - v_m)| d\xi = O(1)\delta_d e^{-ct}. \tag{3.14}$$

So (3.3)₄ is verified.

For (3.3)₅, that is

- Interaction of 3-rarefaction wave and viscous 2-contact wave, which can be done similarly as (3.3)₄, we omit the details for simplicity.

Finally, we prove (3.3)₆:

- Interaction of 1-rarefaction wave and 3-rarefaction wave:

Since $V_\xi^{r_1} \geq 0$, $V_\xi^{r_3} \leq 0$ and the facts iii) and iv) in Lemma 2.3, one has

$$\begin{aligned}
& \int_{\mathbf{R}_+} |V_\xi^{r_1}(V^{r_3} - v^*)| + |V_\xi^{r_3}(V^{r_1} - v_m)| d\xi \\
&= 2 \int_{[-\sigma_- + \lambda_3(v^*, \theta^*)](1+t)}^{+\infty} V_\xi^{r_1}(v^* - V^{r_3}) d\xi \\
&= O(1)\delta_{r_1} e^{-ct} = O(1)\delta e^{-ct}.
\end{aligned} \tag{3.15}$$

Thus we justified (3.3). The proof of (3.4) can be done similarly, but the decay rates with respect to the time t may be higher. Therefore, we complete the proof of the wave interaction estimates in Lemma 3.1. \blacksquare

With the wave interaction estimation Lemma 3.1 in hand, we have the following Lemma:

Lemma 3.2.

$$\begin{aligned} \|G(t)\|_{L^1} + \|H(t)\|_{L^1} &= O(1)\delta^{\frac{1}{8}}(1+t)^{-\frac{13}{16}}, \\ \|G(t)\| + \|H(t)\| &= O(1)\delta(1+t)^{-1}. \end{aligned} \quad (3.16)$$

Proof. We can compute

$$\begin{aligned} G_1 &= |(P - P^b - P^{r_1} - P^d - P^{r_3})_\xi| \\ &= O(1)|V_\xi^b|(|V^{r_1} - v_*| + |V^d - v_m| + |V^{r_3} - v^*|) \\ &\quad + O(1)|V_\xi^d|(|V^b - v_*| + |V^{r_1} - v_m| + |V^{r_3} - v^*|) \\ &\quad + O(1)|V_\xi^{r_1}|(|V^b - v_*| + |V^d - v_m| + |V^{r_3} - v^*|) \\ &\quad + O(1)|V_\xi^{r_3}|(|V^b - v_*| + |V^{r_1} - v_m| + |V^d - v^*|). \end{aligned} \quad (3.17)$$

Thus by the wave interaction estimation Lemma 3.1, we have

$$\|G_1\|_{L^1} = O(1)\delta^{\frac{1}{8}}(1+t)^{-\frac{13}{16}}.$$

Similarly, $\|H_1\|_{L^1} = O(1)\delta^{\frac{1}{8}}(1+t)^{-\frac{13}{16}}$ can be obtained.

Now we estimate $\|G_2\|_{L^1}$ and $\|H_2\|_{L^1}$. Note that in G_2 , besides the wave interaction terms, there are the error terms due to the i -rarefaction waves ($i = 1, 3$). So we can write G_2 as

$$\begin{aligned} G_2 &= -\mu \left(\frac{U_\xi}{V} - \frac{U_\xi^b}{V^b} - \frac{U_\xi^d}{V^d} - \sum_{i=1,3} \frac{U_\xi^{r_i}}{V^{r_i}} \right)_\xi - \mu \left(\sum_{i=1,3} \frac{U_\xi^{r_i}}{V^{r_i}} \right)_\xi \\ &:= G_{21} + G_{22}. \end{aligned}$$

Since the wave interaction terms G_{21} can be verified similarly as G_1 , we only compute the error terms G_{22} due to rarefaction waves.

$$\begin{aligned} \|G_{22}\|_{L^1} &= O(1) \sum_{i=1,3} (\|U_{\xi\xi}^{r_i}\|_{L^1} + \|(U_\xi^{r_i}, V_\xi^{r_i})\|^2) \\ &= O(1)\delta^{\frac{1}{8}}(1+t)^{-\frac{13}{16}} \end{aligned}$$

if we choose $q \geq 14$ in Lemma 2.3.

In H_2 , besides the wave interaction terms and the error terms due to the i -rarefaction waves ($i = 1, 3$), there exists the error terms H^d due to the viscous 2-contact wave. We can compute that

$$\begin{aligned} \|H^d\|_{L^1} &= O(1)\delta_d(1+t)^{-2} \int_{\mathbf{R}_+} \exp\left(-\frac{C_d(\xi + \sigma_- t)^2}{1+t}\right) d\xi \\ &= O(1)\delta(1+t)^{-\frac{3}{2}}. \end{aligned}$$

The estimation of $\|G\|$ and $\|H\|$ can be done similarly, thus the details are omitted. ■

3.2 Reformulation of the Problem

Put the perturbation $(\phi, \psi, \vartheta)(t, \xi)$ around the superposition wave $(V, U, \Theta)(t, \xi)$ by

$$(\phi, \psi, \vartheta)(t, \xi) = (v, u, \theta)(t, \xi) - (V, U, \Theta)(t, \xi), \quad (3.18)$$

then by (1.7) and (3.1), the system for the perturbation $(\phi, \psi, \vartheta)(t, \xi)$ becomes

$$\begin{cases} \phi_t - \sigma_- \phi_\xi - \psi_\xi = 0, \\ \psi_t - \sigma_- \psi_\xi + (p - P)_\xi = \mu \left(\frac{u_\xi}{v} - \frac{U_\xi}{V} \right)_\xi - G, & t > 0, \xi > 0, \\ \frac{R}{\gamma-1} (\vartheta_t - \sigma_- \vartheta_\xi) + (pu_\xi - PU_\xi) = \kappa \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right)_\xi + \mu \left(\frac{(u_\xi)^2}{v} - \frac{(U_\xi)^2}{V} \right) - H, \\ (\psi_0, \psi_0, \vartheta_0)(\xi) := (\phi, \psi, \vartheta)(0, \xi) \rightarrow (0, 0, 0), \text{ as } \xi \rightarrow +\infty, \\ (\phi, \psi, \vartheta)(t, \xi = 0) = (V^d, U^d, \Theta^d)(t, \xi = 0) - (v_m, u_m, \theta_m). \end{cases} \quad (3.19)$$

Define the solution space $\mathbf{X}(0, T)$ to the above system by

$$\mathbf{X}(0, T) := \left\{ (\phi, \psi, \vartheta)(t, \xi) \mid (\phi, \psi, \vartheta) \in C([0, T]; H^1), \phi_\xi \in L^2(0, T; L^2), \right. \\ \left. (\psi_\xi, \vartheta_\xi) \in L^2(0, T; H^1), N(T) =: \sup_{0 \leq t \leq T} \|(\phi, \psi, \vartheta)(t)\|_1 \leq \varepsilon_0 \right\}, \quad (3.20)$$

Here $\varepsilon_0 \leq \frac{1}{4} \min \left\{ \inf_{\mathbf{R}_+ \times \mathbf{R}_+} V(t, \xi), \inf_{\mathbf{R}_+ \times \mathbf{R}_+} \Theta(t, \xi) \right\}$ is a suitably small and positive constant to be determined.

Since the proof for the local existence of the solution to the system (3.19) is standard, the details are omitted. To prove Theorem 2.1, it is sufficient to prove the following *a priori* estimate by combining the local existence of the solution and the continuation process.

Proposition 3.1 (A priori estimate) *Let $(\phi, \psi, \vartheta) \in \mathbf{X}(0, T)$ be a solution to the system (3.19) in the time interval $[0, T]$ with suitably small ε_0 , and the conditions in Theorem 2.1 hold. Then there exist a positive constant C independent of T such that*

$$\begin{aligned} & \|(\phi, \psi, \vartheta)(t)\|_1^2 + \int_0^t [\|\phi_\xi(\tau)\|^2 + \|(\psi_\xi, \vartheta_\xi)(\tau)\|_1^2] d\tau \\ & + \int_0^t \|\sqrt{(U_\xi^b, U_\xi^{r1}, U_\xi^{r3})}(\phi, \vartheta)(\tau)\|^2 d\tau \leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right). \end{aligned} \quad (3.21)$$

3.3 Energy estimates

To prove Proposition 3.1, we need the following several lemmas. First we give the following boundary estimates whose proof can be found in [21].

Lemma 3.3 (Boundary Estimates)[21] *There exists the positive constant C such that*

for any $t > 0$,

$$\begin{aligned}
& \int_0^t |(\phi, \psi, \vartheta)(\tau, 0)|^2 d\tau \leq C\delta, \\
& \int_0^t (|\psi\psi_\xi| + |\vartheta\vartheta_\xi|)(\tau, 0) d\tau \leq C\delta + C\delta \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|_1^2 d\tau, \\
& \int_0^t (|\phi_\tau\psi| + \phi_\xi^2)(\tau, 0) d\tau \leq C\delta + \epsilon \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau + C_\epsilon \int_0^t \|\psi_\xi(\tau)\|^2 d\tau, \\
& \int_0^t (|\psi_\tau\psi_\xi| + \psi_\xi^2)(\tau, 0) d\tau \leq C\delta + \epsilon \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau + C_\epsilon \int_0^t \|\psi_\xi(\tau)\|^2 d\tau, \\
& \int_0^t (|\vartheta_\tau\vartheta_\xi| + \vartheta_\xi^2)(\tau, 0) d\tau \leq C\delta + \epsilon \int_0^t \|\vartheta_{\xi\xi}(\tau)\|^2 d\tau + C_\epsilon \int_0^t \|\vartheta_\xi(\tau)\|^2 d\tau,
\end{aligned}$$

where $\epsilon > 0$ is a constant to be determined and C_ϵ is the constant depending on ϵ .

Lemma 3.4 *Let $(\phi, \psi, \vartheta) \in \mathbf{X}(0, T)$ be a solution to the system (3.19) for some positive T and suitably small $\varepsilon_0 > 0$, and the conditions in Theorem 2.1 hold. Then there exist a positive constant C such that*

$$\begin{aligned}
& \|(\phi, \psi, \vartheta)(t)\|_1^2 + \int_0^t \|\phi_\xi(\tau)\|^2 + \|(\psi_\xi, \vartheta_\xi)(\tau)\|_1^2 d\tau \\
& + \int_0^t \|\sqrt{(U_\xi^b, U_\xi^{r1}, U_\xi^{r3})}(\phi, \vartheta)(\tau)\|^2 d\tau \\
& \leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right) + C\delta^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\
& + C\delta \int_0^t \int_{\mathbf{R}_+} (1 + \tau)^{-1} \exp\left(-\frac{C_d(\xi + \sigma_- \tau)^2}{1 + \tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau. \tag{3.22}
\end{aligned}$$

Proof. Step 1. Define

$$\Phi(\eta) := \eta - \ln \eta - 1. \tag{3.23}$$

Under the a priori assumption, there exist a positive constant C such that

$$C^{-1}\eta^2 \leq \Phi(\eta) \leq C\eta^2. \tag{3.24}$$

Let

$$\begin{aligned}
E & := R\Theta\Phi\left(\frac{v}{V}\right) + \frac{1}{2}\psi^2 + \frac{R}{\gamma-1}\Theta\Phi\left(\frac{\theta}{\Theta}\right), \\
F & := \sigma_- E + (P-p)\psi + \mu\left(\frac{u_\xi}{v} - \frac{U_\xi}{V}\right)\psi + \kappa\left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V}\right)\frac{\vartheta}{\theta}. \tag{3.25}
\end{aligned}$$

Then a complicated but direct computation gives

$$E_t - F_\xi + \frac{\mu\Theta}{v\theta}\psi_\xi^2 + \frac{\kappa\Theta}{v\theta^2}\vartheta_\xi^2 + P(U_\xi^b + U_\xi^{r1} + U_\xi^{r3}) \left[\gamma\Phi\left(\frac{v}{V}\right) + \Phi\left(\frac{\theta V}{v\Theta}\right) \right] = Q, \tag{3.26}$$

where

$$Q = -PU_\xi^d \left[\gamma\Phi\left(\frac{v}{V}\right) + \Phi\left(\frac{\theta V}{v\Theta}\right) \right] - \left(G\psi + H\frac{\vartheta}{\theta} \right)$$

$$\begin{aligned}
& + \left[\frac{\mu U_\xi \phi \psi_\xi}{vV} + \frac{2\mu U_\xi \vartheta \psi_\xi}{v\theta} + \frac{\kappa \Theta \Theta_\xi \phi \vartheta_\xi}{vV\theta^2} + \kappa \frac{\Theta_\xi \vartheta \vartheta_\xi}{v\theta^2} - \frac{\mu(U_\xi)^2 \phi \vartheta}{vV\theta} - \frac{\kappa(\Theta_\xi)^2 \phi \vartheta}{vV\theta^2} \right] \\
& + \left[\kappa \left(\frac{\Theta_\xi}{V} \right)_\xi + \mu \frac{(U_\xi)^2}{V} + H \right] \left[(\gamma - 1) \Phi \left(\frac{v}{V} \right) + \Phi \left(\frac{\theta}{\Theta} \right) - \frac{\vartheta^2}{\Theta\theta} \right] \\
& =: \sum_{i=1}^{i=4} Q_i. \tag{3.27}
\end{aligned}$$

Integrating (3.26) over $[0, t] \times \mathbf{R}_+$ yields

$$\begin{aligned}
& \|(\phi, \psi, \vartheta)\|^2 + \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau + \int_0^t \|\sqrt{(U_\xi^b, U_\xi^{r1}, U_\xi^{r3})}(\phi, \vartheta)(\tau)\|^2 d\tau \\
& \leq C\|(\phi_0, \psi_0, \vartheta_0)\|^2 + C \int_0^t |F(\tau, \xi = 0)| d\tau + \sum_{i=1}^{i=4} I_i, \tag{3.28}
\end{aligned}$$

where $I_i = O(1) \int_0^t \int_{\mathbf{R}_+} Q_i d\xi d\tau$.

From the boundary estimates in Lemma 3.3, we have

$$\int_0^t |F(\tau, \xi = 0)| d\tau \leq C\delta + C\delta \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|_1^2 d\tau. \tag{3.29}$$

We can compute that

$$I_1 \leq C\delta \int_0^t \int_{\mathbf{R}_+} (1 + \tau)^{-1} \exp\left(-\frac{C_d(\xi + \sigma_- \tau)^2}{1 + \tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau \tag{3.30}$$

and

$$\begin{aligned}
I_2 & \leq C \int_0^t \|(\psi, \vartheta)(\tau)\|_{L^\infty} (\|G(\tau)\|_{L^1} + \|H(\tau)\|_{L^1}) d\tau \\
& \leq C\delta^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{13}{16}} \|(\psi_\xi, \vartheta_\xi)(\tau)\|^{\frac{1}{2}} \|(\psi, \vartheta)(\tau)\|^{\frac{1}{2}} d\tau \\
& \leq \epsilon \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau + C_\epsilon \delta^{\frac{1}{6}} \left(1 + \int_0^t (1 + \tau)^{-\frac{13}{12}} \|(\psi, \vartheta)(\tau)\|^2 d\tau \right) \tag{3.31}
\end{aligned}$$

where and in the sequel $\epsilon > 0$ is a small constant to be determined and C_ϵ is the positive constant depending on ϵ .

Now we calculate I_3 . By Cauchy inequality, we have

$$I_3 \leq \epsilon \int_0^t \|(\psi_\xi, \vartheta_\xi)\|^2 d\tau + C_\epsilon \int_0^t \int_{\mathbf{R}_+} |(U_\xi, \Theta_\xi)|^2 \cdot |(\phi, \vartheta)|^2 d\xi d\tau. \tag{3.32}$$

By Lemma 2.1-Lemma 2.3, one has

$$|(U_\xi, \Theta_\xi)|^2 \leq C \left[\delta^{\frac{1}{2}} (1 + t)^{-\frac{3}{2}} + \frac{\delta^4}{(1 + \delta\xi)^4} + \delta (1 + t)^{-1} \exp\left(-\frac{C_d(\xi + \sigma_- t)^2}{1 + t}\right) \right]. \tag{3.33}$$

By the techniques in [19]

$$\begin{aligned}
|f(t, \xi)| & = |f(t, \xi = 0) + \int_0^\xi f_\xi(t, \xi) d\xi| \\
& \leq |f(t, \xi = 0)| + \sqrt{\xi} \|f_\xi\|,
\end{aligned}$$

we have

$$\begin{aligned}
& \int_0^t \int_{\mathbf{R}_+} \frac{\delta^4}{(1+\delta\xi)^4} |(\phi, \vartheta)|^2 d\xi d\tau \\
& \leq C\delta^3 \int_0^t |(\phi, \vartheta)(\tau, \xi=0)|^2 d\tau + C \int_0^t \left[\|(\phi_\xi, \vartheta_\xi)\|^2 \int_{\mathbf{R}_+} \frac{\delta^4 \xi}{(1+\delta\xi)^4} d\xi \right] d\tau \\
& \leq C\delta \left(1 + \int_0^t \|(\phi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau \right). \tag{3.34}
\end{aligned}$$

Substituting (3.33) and (3.34) into (3.32) yields

$$\begin{aligned}
I_3 & \leq C(\epsilon + \delta) \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau + C\delta \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \\
& \quad + C\delta + C\delta^{\frac{1}{2}} \int_0^t (1+\tau)^{-\frac{3}{2}} \|(\phi, \vartheta)(\tau)\|^2 d\tau \\
& \quad + C\delta \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{C_d(\xi+\sigma-\tau)^2}{1+\tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau. \tag{3.35}
\end{aligned}$$

Then we have

$$I_4 = O(1) \int_0^t \int_{\mathbf{R}_+} |(\Theta_{\xi\xi}, V_\xi^2, U_\xi^2, \Theta_\xi^2, H)| |(\phi, \vartheta)|^2 d\xi d\tau. \tag{3.36}$$

So I_4 can be estimated similarly as I_2 and I_3 .

Combining (3.29), (3.30), (3.31), (3.32), (3.35) and (3.36), and then choosing δ and ϵ suitably small yield that

$$\begin{aligned}
& \|(\phi, \psi, \vartheta)(t)\|^2 + \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau + \int_0^t \|\sqrt{(U_\xi^b, U_\xi^{r1}, U_\xi^{r3})}(\phi, \vartheta)(\tau)\|^2 d\tau \\
& \leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|^2 + \delta^{\frac{1}{8}} \right) + C\delta^{\frac{1}{8}} \int_0^t \|(\phi_\xi, \psi_{\xi\xi}, \vartheta_{\xi\xi})(\tau)\|^2 d\tau \\
& \quad + C\delta^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\
& \quad + C\delta \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{C_d(\xi+\sigma-\tau)^2}{1+\tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau. \tag{3.37}
\end{aligned}$$

Step 2. Differentiating (3.19)₁ w.r.t. ξ and multiplying it by $\frac{\phi_\xi}{v^2}$ yield

$$\left(\frac{\phi_\xi^2}{2v^2} \right)_t - \sigma_- \left(\frac{\phi_\xi^2}{2v^2} \right)_\xi + \frac{u_x \phi_\xi^2}{v^3} - \frac{\phi_\xi \psi_{\xi\xi}}{v^2} = 0. \tag{3.38}$$

Multiplying (3.19)₂ by $\frac{\phi_\xi}{v}$ gives

$$\begin{aligned}
& \left(\frac{\phi_\xi \psi}{v} \right)_t - \left(\frac{\phi_t \psi}{v} \right)_\xi + \frac{(p-P)_\xi \phi_\xi}{v} \\
& = -\frac{U_\xi \phi_\xi \psi}{v^2} + \frac{V_\xi \psi \psi_\xi}{v^2} + \sigma_- \frac{\phi_\xi \psi_\xi}{v} + \mu \left(\frac{u_\xi}{v} - \frac{U_\xi}{V} \right)_\xi \frac{\phi_\xi}{v} - G \frac{\phi_\xi}{v}. \tag{3.39}
\end{aligned}$$

$\mu \times (3.38) - (3.39)$ gives

$$\begin{aligned}
& \left(\frac{\mu\phi_\xi^2}{2v^2} - \frac{\phi_\xi\psi}{v} \right)_t - \left(\frac{\sigma_-\mu\phi_\xi^2}{2v^2} - \frac{\phi_t\psi}{v} \right)_\xi - \frac{p_v}{v}\phi_\xi^2 \\
&= \frac{U_\xi\phi_\xi\psi}{v^2} - \frac{V_\xi\psi\psi_\xi}{v^2} - \sigma_-\frac{\phi_\xi\psi_\xi}{v} + \mu\frac{V_\xi\phi_\xi\psi_\xi}{v^3} - \mu\frac{U_\xi\phi_\xi^2}{v^3} + \mu\left(\frac{U_\xi\phi}{vV}\right)_\xi\frac{\phi_\xi}{v} \\
& \quad + \frac{p_\theta\phi_\xi\vartheta_\xi}{v} + \frac{V_\xi(p_v - P_V)\phi_\xi}{v} + \frac{\Theta_\xi(p_\theta - P_\Theta)\phi_\xi}{v} + G\frac{\phi_\xi}{v}.
\end{aligned} \tag{3.40}$$

Integrating (3.40) over $[0, t] \times \mathbf{R}_+$, using the boundary estimations in Lemma3.3 and choosing δ suitably small yield

$$\begin{aligned}
& \|\phi_\xi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \\
& \leq C \left(\|(\psi_0, \phi_{0\xi})\|^2 + \delta^{\frac{1}{6}} \right) + C\delta^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\
& \quad + \int_0^t \left\{ C \left(\delta^{\frac{1}{8}} + \epsilon \right) \|(\psi_{\xi\xi}, \vartheta_{\xi\xi})(\tau)\|^2 + C_\epsilon \|\psi_\xi(\tau)\|^2 \right\} d\tau \\
& \quad + C\delta \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{C_d(\xi+\sigma-\tau)^2}{1+\tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau.
\end{aligned} \tag{3.41}$$

Step 3. Multiplying (3.19)₂ by $-\psi_{\xi\xi}$, then

$$\left(\frac{\psi_\xi^2}{2} \right)_t - \left(\psi_t\psi_\xi - \frac{\sigma_-}{2}\psi_\xi^2 \right)_\xi + \mu\frac{\psi_{\xi\xi}^2}{v} = \left[(p - P)_\xi + \frac{\mu v_\xi\psi_\xi}{v^2} + \mu\left(\frac{U_\xi\phi}{vV}\right)_\xi + G \right] \psi_{\xi\xi} \tag{3.42}$$

Integrating (3.42) over $[0, t] \times \mathbf{R}_+$ yields

$$\begin{aligned}
& \|\psi_\xi(t)\|^2 + \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau \\
& \leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right) + C\delta^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\
& \quad + \int_0^t \left\{ C \left(\delta^{\frac{1}{8}} + \epsilon \right) \|(\psi_{\xi\xi}, \vartheta_{\xi\xi})(\tau)\|^2 + C_\epsilon \|\psi_\xi(\tau)\|^2 \right\} d\tau \\
& \quad + C\delta \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{C_d(\xi+\sigma-\tau)^2}{1+\tau}\right) (\phi^2 + \vartheta^2) d\xi d\tau
\end{aligned} \tag{3.43}$$

where we use the following estimate

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}_+} |\phi_\xi\psi_\xi\psi_{\xi\xi}| d\xi d\tau & \leq \int_0^t \|\phi_\xi(\tau)\| \|\psi_{\xi\xi}(\tau)\| \|\psi_\xi(\tau)\|_{L^\infty} d\tau \\
& \leq \int_0^t \|\phi_\xi(\tau)\| \|\psi_{\xi\xi}(\tau)\|^{\frac{3}{2}} \|\psi_\xi(\tau)\|^{\frac{1}{2}} d\tau \\
& \leq \epsilon \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau + C_\epsilon \epsilon_0^4 \int_0^t \|\psi_\xi(\tau)\|^2 d\tau.
\end{aligned} \tag{3.44}$$

Multiplying (3.19)₃ by $-\vartheta_{\xi\xi}$, then

$$\frac{R}{\gamma-1} \left[\left(\frac{\vartheta_\xi^2}{2} \right)_t - \left(\vartheta_t\vartheta_\xi - \frac{\sigma_-}{2}\vartheta_\xi^2 \right)_\xi \right] + \frac{\kappa}{v}\vartheta_{\xi\xi}^2$$

$$= \left[(pu_\xi - PU_\xi) + \frac{\kappa v_\xi \vartheta_\xi}{v^2} + \kappa \left(\frac{\Theta_\xi \phi}{vV} \right)_\xi - \mu \left(\frac{(u_\xi)^2}{v} - \frac{(U_\xi)^2}{V} \right) + H \right] \vartheta_{\xi\xi}. \quad (3.45)$$

Integrating (3.45) over $[0, t] \times \mathbf{R}_+$ yields

$$\begin{aligned} & \|\vartheta_\xi(t)\|^2 + \int_0^t \|\vartheta_{\xi\xi}(\tau)\|^2 d\tau \\ & \leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right) + C\delta^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\ & \quad + \int_0^t \left\{ C \left(\delta^{\frac{1}{8}} + \epsilon \right) \|(\psi_{\xi\xi}, \vartheta_{\xi\xi})(\tau)\|^2 + C_\epsilon \|\vartheta_\xi(\tau)\|^2 \right\} d\tau \\ & \quad + C\delta \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp \left(-\frac{C_d(\xi + \sigma - \tau)^2}{1+\tau} \right) |(\phi, \vartheta)|^2 d\xi d\tau, \end{aligned} \quad (3.46)$$

where we use the following estimate

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}_+} |\phi_\xi \vartheta_\xi \vartheta_{\xi\xi}| + |\psi_\xi^2 \vartheta_{\xi\xi}| d\xi d\tau \\ & \leq \epsilon \int_0^t \|(\psi_{\xi\xi}, \vartheta_{\xi\xi})(\tau)\|^2 d\tau + C_\epsilon \epsilon_0^4 \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau. \end{aligned} \quad (3.47)$$

Combining (3.37), (3.41), (3.43) and (3.46) and choosing δ , ϵ and ϵ_0 suitably small, we can complete the proof of Lemma 3.4. ■

Now to close the a priori estimates, the remaining thing is to compute the last term in the right-hand side of (3.22) which comes from the viscous contact wave. Here we use the method of the heat kernel estimation invented in [2].

Lemma 3.5.[2] *Suppose that $h(t, \xi)$ satisfies*

$$h \in L^\infty(0, T; L^2(\mathbf{R}_+)), \quad h_\xi \in L^2(0, T; L^2(\mathbf{R}_+)), \quad h_t - \sigma_- h_\xi \in L^2(0, T; H^{-1}(\mathbf{R}_+)) \quad (3.48)$$

then

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp \left(-\frac{2a(\xi + \sigma - \tau)^2}{1+\tau} \right) h^2 d\xi d\tau \\ & \leq C_a \left\{ \|h(0, \cdot)\|^2 + \int_0^t \left[h^2(\tau, 0) + \|h_\xi(\tau, \cdot)\|^2 + \langle h_\tau - \sigma_- h_\xi, (w^a)^2 h \rangle_{H^{-1} \times H^1} \right] d\tau \right\} \end{aligned} \quad (3.49)$$

where

$$w^a(t, \xi) = -(1+t)^{-\frac{1}{2}} \int_{\xi + \sigma - t}^\infty \exp \left(-\frac{ay^2}{1+t} \right) dy, \quad (3.50)$$

and $a > 0$ is a constant to be determined.

Based on Lemma 3.5, we have the desired estimates in the following Lemma.

Lemma 3.6 *There exist a uniform constant $C > 0$ such that if δ and ϵ_0 are small enough, then we have*

$$\int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp \left(-\frac{C_d(\xi + \sigma - \tau)^2}{1+\tau} \right) |(\phi, \psi, \vartheta)|^2 d\xi d\tau$$

$$\leq C \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right) + C \delta^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau. \quad (3.51)$$

Proof. Step 1. First, let

$$h = P\phi + \frac{R}{\gamma - 1} \vartheta \quad (3.52)$$

in Lemma 3.4. Then we only need to control the last term of (3.49) on the right hand side.

We have from the energy equation (3.19)₃,

$$\begin{aligned} h_t - \sigma_- h_\xi &= (P - p)\psi_\xi + U_\xi(P - p) + (P_t - \sigma_- P_\xi)\phi \\ &\quad + \kappa \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right)_\xi + \mu \left(\frac{(u_\xi)^2}{v} - \frac{(U_\xi)^2}{V} \right) - H. \end{aligned} \quad (3.53)$$

Thus

$$\begin{aligned} &\int_0^t \langle h_\tau - \sigma_- h_\xi, (w^a)^2 h \rangle_{H^{-1} \times H^1} d\tau \\ &= -\kappa \int_0^t \left[\left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right) (w^a)^2 h \right] (\tau, 0) d\tau - \kappa \int_0^t \int_{\mathbf{R}_+} \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right) [(w^a)^2 h]_\xi d\xi d\tau \\ &\quad + \int_0^t \int_{\mathbf{R}_+} \left[(P - p)\psi_\xi + U_\xi(P - p) + (P_t - \sigma_- P_\xi)\phi \right. \\ &\quad \left. + \mu \left(\frac{(u_\xi)^2}{v} - \frac{(U_\xi)^2}{V} \right) - H \right] (w^a)^2 h d\xi d\tau. \end{aligned} \quad (3.54)$$

Notice that

$$\begin{aligned} \|w^a(t)\|_{L^\infty} &\leq C_a, \quad w_\xi^a = (1 + t)^{-\frac{1}{2}} \exp \left(-\frac{a(\xi + \sigma_- t)^2}{1 + t} \right), \quad |w_t^a - \sigma_- w_\xi^a| \leq C_a (1 + t)^{-1}, \\ |P_t - \sigma_- P_\xi| &\leq C \left\{ U_\xi^b + U_\xi^{r_1} + U_\xi^{r_3} + \delta (1 + t)^{-1} \exp \left(-\frac{C_d(\xi + \sigma_- t)^2}{1 + t} \right) \right\}, \end{aligned} \quad (3.55)$$

thus to control terms on the right hand side of (3.54), we only consider the term $(w^a)^2 (P - p)h\psi_\xi$. By using the mass equation (3.19)₁ and the momentum equation (3.19)₂ again, we have

$$\begin{aligned} (w^a)^2 (P - p)h\psi_\xi &= \frac{(w^a)^2 [\gamma P\phi - (\gamma - 1)h] h (\phi_t - \sigma_- \phi_\xi)}{v} \\ &= \frac{\gamma P (w^a)^2 h}{2v} \left[(\phi^2)_t - \sigma_- (\phi^2)_\xi \right] - \frac{(\gamma - 1)(w^a)^2 h^2}{v} (\phi_t - \sigma_- \phi_\xi) \\ &= \left(\frac{\gamma P (w^a)^2 h \phi^2 - 2(\gamma - 1)(w^a)^2 \phi h^2}{2v} \right)_t \\ &\quad - \sigma_- \left(\frac{\gamma P (w^a)^2 h \phi^2 - 2(\gamma - 1)(w^a)^2 \phi h^2}{2v} \right)_\xi \\ &\quad - \frac{\gamma P h \phi^2 - 2(\gamma - 1)\phi h^2}{v} w^a (w_t^a - \sigma_- w_\xi^a) - \frac{\gamma (w^a)^2 \phi^2 h}{2v} (P_t - \sigma_- P_\xi) \\ &\quad + \frac{\gamma P (w^a)^2 h \phi^2 - 2(\gamma - 1)(w^a)^2 \phi h^2}{2v^2} (\psi_\xi + U_\xi) \end{aligned}$$

$$+ \frac{(w^a)^2[4(\gamma-1)h - \gamma P\phi]\phi}{2v}(h_t - \sigma_- h_\xi). \quad (3.56)$$

Now the terms in the right hand side of (3.56) can be estimated directly and in particular, we have

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}_+} \frac{\gamma P(w^a)^2 h \phi^2 - 2(\gamma-1)(w^a)^2 \phi h^2}{2v^2} \psi_\xi d\xi d\tau \\ & \leq C \int_0^t \int_{\mathbf{R}_+} |\psi_\xi| (|\phi|^3 + |\vartheta|^3) d\xi d\tau \\ & \leq C \int_0^t \|(\phi, \vartheta)\|_{L^\infty}^2 \|\psi_\xi\| \|(\phi, \vartheta)\| d\tau \\ & \leq C \varepsilon_0^2 \int_0^t \|(\phi_\xi, \psi_\xi, \vartheta_\xi)(\tau)\|^2 d\tau. \end{aligned} \quad (3.57)$$

The other terms can be controlled by the similar procedure as Step 1 of Lemma 3.4. Thus the combination of the above estimates and Lemma 3.5 yield

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{2a(\xi+\sigma_-\tau)^2}{1+\tau}\right) \left(P\phi + \frac{R}{\gamma-1}\vartheta\right)^2 d\xi d\tau \\ & \leq C_a \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}}\right) + C_a \delta^{\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau \\ & \quad + C_a(\delta + \varepsilon_0) \int_0^t \int_{\mathbf{R}_+} (1+\tau)^{-1} \exp\left(-\frac{C_d(\xi+\sigma_-\tau)^2}{1+\tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau. \end{aligned} \quad (3.58)$$

Step 2. Let

$$W^A(t, \xi) := -(1+t)^{-1} \int_{\xi+\sigma_-\tau}^\infty \exp\left(-\frac{Ay^2}{1+t}\right) dy, \quad (3.59)$$

where $A > 0$ is a constant to be determined.

Then

$$W_\xi^A = (1+t)^{-1} \exp\left(-\frac{A(\xi+\sigma_-\tau)^2}{1+t}\right), \quad |W_t^A - \sigma_- W_\xi^A| \leq C_A(1+t)^{-\frac{3}{2}}. \quad (3.60)$$

From the fact $p - P = \frac{R\vartheta - P\phi}{v}$, we have

$$\frac{(R\vartheta - P\phi)_\xi}{v} - \frac{v_\xi(R\vartheta - P\phi)}{v^2} = -(\psi_t - \sigma_- \psi_\xi) + \mu \left(\frac{u_\xi}{v} - \frac{U_\xi}{V}\right)_\xi - G. \quad (3.61)$$

Multiplying (3.61) by $W^A(R\vartheta - P\phi)$ implies

$$\begin{aligned} & \left(\frac{W^A(R\vartheta - P\phi)^2}{2v}\right)_\xi - \frac{W_\xi^A(R\vartheta - P\phi)^2}{2v} - \frac{W^A v_\xi(R\vartheta - P\phi)^2}{2v^2} \\ & = -W^A \left[(\psi_t - \sigma_- \psi_\xi) - \mu \left(\frac{u_\xi}{v} - \frac{U_\xi}{V}\right)_\xi + G \right] (R\vartheta - P\phi). \end{aligned} \quad (3.62)$$

Note that

$$\begin{aligned}
W^A(\psi_t - \sigma_- \psi_\xi)(R\vartheta - P\phi) &= \{W^A\psi(R\vartheta - P\phi)\}_t - \sigma_- \{W^A\psi(R\vartheta - P\phi)\}_\xi \\
&\quad - \psi(R\vartheta - P\phi)(W_t^A - \sigma_- W_\xi^A) \\
&\quad - W^A\psi\{(R\vartheta - P\phi)_t - \sigma_-(R\vartheta - P\phi)_\xi\}, \\
&= (\gamma - 1) \left\{ (P - p)u_\xi + \kappa \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right)_\xi + \mu \left(\frac{(u_\xi)^2}{v} - \frac{(U_\xi)^2}{V} \right) - H \right\} \\
&\quad - \gamma P\psi_\xi - (P_t - \sigma_- P_\xi)\phi
\end{aligned} \tag{3.63}$$

and

$$\gamma PW^A\psi\psi_\xi = \frac{\gamma}{2}(PW^A\psi^2)_\xi - \frac{\gamma}{2}PW_\xi^A\psi^2 - \frac{\gamma}{2}P_\xi W^A\psi^2, \tag{3.64}$$

we have

$$-\frac{W_\xi^A}{2v} \{(R\vartheta - P\phi)^2 + \gamma v P\psi^2\} = -\{W^A\psi(R\vartheta - P\phi)\}_t - E_\xi^A + Q^A, \tag{3.65}$$

where

$$\begin{aligned}
E^A &:= \frac{W^A(R\vartheta - P\phi)^2}{2v} + \frac{\gamma}{2}PW^A\psi^2 - \mu W^A(R\vartheta - P\phi) \left(\frac{u_\xi}{v} - \frac{U_\xi}{V} \right) \\
&\quad - \sigma_- W^A\psi(R\vartheta - P\phi) - (\gamma - 1)\kappa W^A\psi \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right),
\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}
Q^A &:= \frac{W^A v_\xi (P - p)^2}{2} + (W_t^A - \sigma_- W_\xi^A)(R\vartheta - P\phi)\psi - W^A G(R\vartheta - P\phi) \\
&\quad + W^A\psi \left\{ (\gamma - 1) \left[(P - p)u_\xi + \mu \left(\frac{u_\xi^2}{v} - \frac{(U_\xi)^2}{V} \right) - H \right] - (P_t - \sigma_- P_\xi)\phi + \frac{\gamma P_\xi \psi}{2} \right\} \\
&\quad - \mu \{W^A(R\vartheta - P\phi)\}_\xi \left(\frac{u_\xi}{v} - \frac{U_\xi}{V} \right) - (\gamma - 1)\kappa (W^A\psi)_\xi \left(\frac{\theta_\xi}{v} - \frac{\Theta_\xi}{V} \right).
\end{aligned} \tag{3.67}$$

First, we have

$$\left| \int_0^t E^A(\tau, 0) d\tau \right| \leq C_A \delta + C_A \delta \int_0^t \|(\psi_\xi, \vartheta_\xi)(\tau)\|_1^2 d\tau. \tag{3.68}$$

The estimations of the terms concerned with W^A are similar to those in Step 1 while the other terms are similar to those of Step 1 in the proof of Lemma 3.4. Thus integrating (3.65) over $[0, t] \times \mathbf{R}_+$ yields

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}_+} (1 + \tau)^{-1} \exp\left(-\frac{A(\xi + \sigma_- \tau)^2}{1 + \tau}\right) \{(R\vartheta - P\phi)^2 + \psi^2\} d\xi d\tau \\
&\leq C_A \left(\|(\phi_0, \psi_0, \vartheta_0)\|_1^2 + \delta^{\frac{1}{6}} \right) + C_A \delta^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{13}{12}} \|(\phi, \psi, \vartheta)(\tau)\|^2 d\tau
\end{aligned}$$

$$+ C_A(\delta + \varepsilon_0) \int_0^t \int_{\mathbf{R}_+} (1 + \tau)^{-1} \exp\left(-\frac{C_d(\xi + \sigma - \tau)^2}{1 + \tau}\right) |(\phi, \vartheta)|^2 d\xi d\tau. \quad (3.69)$$

Step 3. Combining (3.58) and (3.69), then choosing $A = 2a = C_d$ and setting δ, ε_0 suitably small, we can complete the proof of Lemma 3.6. ■

Proof of Proposition 3.1. Choosing δ, ε_0 suitably small in Lemmas 3.4 and Lemma 3.6, then using Gronwall inequality yield Proposition 3.1. ■

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