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ZERO DISSIPATION LIMIT OF THE COMPRESSIBLE HEAT-CONDUCTING NAVIER-STOKES EQUATIONS IN THE PRESENCE OF THE SHOCK*

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Abstract The zero dissipation limit of the compressible heat-conducting Navier–Stokes equations in the presence of the shock is investigated. It is shown that when the heat conduction coefficient κ and the viscosity coefficient ε satisfy $\kappa = O(\varepsilon)$, $\frac{\kappa}{\varepsilon} \geq c > 0$, as $\varepsilon \to 0$ (see (1.3)), if the solution of the corresponding Euler equations is piecewise smooth with shock wave satisfying the Lax entropy condition, then there exists a smooth solution to the Navier–Stokes equations, which converges to the piecewise smooth shock solution of the Euler equations away from the shock discontinuity at a rate of ε. The proof is given by a combination of the energy estimates and the matched asymptotic analysis introduced in [3].

Key words Zero dissipation limit, Navier–Stokes equations, shock waves

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1 Introduction

We study the zero dissipation limit of the solution of the full Navier–Stokes system which reads in the Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon\Big(\dfrac{u_x}{v}\Big)_x, \\ \Big(e + \dfrac{u^2}{2}\Big)_t + (pu)_x = \kappa\Big(\dfrac{\theta_x}{v}\Big)_x + \varepsilon\Big(\dfrac{uu_x}{v}\Big)_x, \end{cases} \tag{1.1}$$

where the functions $v(x,t) > 0, u(x,t)$, and $\theta(x,t) > 0$ represent the specific volume, the velocity, and the absolute temperature of the gas, respectively. And $p = p(v,\theta)$ is the pressure,

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$e = e(v, \theta)$ is the internal energy, $\varepsilon > 0$ is the viscosity constant and $\kappa > 0$ is the coefficient of the heat conduction. Here we consider the perfect gas case, that is,

$$p = \frac{R\theta}{v}, \qquad e = \frac{R\theta}{\gamma - 1} + \text{const}, \tag{1.2}$$

where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent.

The study of the vanishing viscosity limit of viscous flow is an important problem in the theory of the compressible fluid flow. When the solution of the inviscid flow is smooth, the zero dissipation limit can be investigated through classical scaling method. However, the inviscid compressible solution is, in general, discontinuous like shock wave and its investigation is much more complicated. For the hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx},$$

Goodman and Xin [3] proposed a matched asymptotic expansion method to study the zero dissipation limit in the presence of shock wave. For the vanishing viscosity limit of shock wave of the isentropic Navier–Stokes equation, whose viscosity matrix is only semi-positive definite, Hoff and Liu [4] first considered the case of single shock wave, where the initial data are discontinuous so that both initial layer and shock layer are included in the solution. Recently, Wang [15] applied the idea of [3] to study the isentropic Navier–Stokes equations in the case of a finite number of shock waves, whereas the initial value of the Navier–Stokes equation is well chosen so that there is no initial layer.

In this article, we consider the vanishing viscosity limit of shock wave for the full Navier–Stokes equations (1.1). Motivated by [3], we first construct an approximate solution to the inviscid shock wave by matching the inner and outer expansion solutions and the higher order correction, then we use the scaling method to transform the zero dissipation limiting to the time asymptotic stability of the approximate solution. Note that the leading order of the inner solution in the approximate solution is the shock profile for the Navier–Stokes equations (1.1). So we can use the energy method in [7] to show that, for a given piecewise smooth shock solution of the Euler system, there exists a smooth approximate solution for the Navier–Stokes system (1.1) such that it converges to the piecewise smooth shock solution away from the shock discontinuity at a rate of ε as ε tends to zero. The precise statement of our main result is the following.

As in [6], where the vanishing viscosity limit of rarefaction wave was investigated, we assume

$$\begin{cases} \kappa = O(\varepsilon) & \text{as} \quad \varepsilon \to 0; \\ \mu \doteq \dfrac{\kappa}{\varepsilon} \geq c > 0 & \text{for some positive constant } c, \quad \text{as} \quad \varepsilon \to 0. \end{cases} \tag{1.3}$$

The assumption (1.3) is reasonable because, when we consider the compressible Navier–Stokes system (1.1) as a derivation from the Bolzmann equation through Chapman–Enskog expansion, the heat conducting coefficient κ and the viscosity coefficient ε satisfy (1.3).

Denote the total energy function E by $E = \frac{R\theta}{\gamma-1} + \frac{u^2}{2}$, we have

$$\theta = \frac{\gamma - 1}{R}\Big(E - \frac{u^2}{2}\Big), \tag{1.4}$$

and

$$p = p(v, u, E) = \frac{(\gamma-1)E}{v} - \frac{(\gamma-1)u^2}{2v}. \tag{1.5}$$

Then, the system (1.1) can be written as

$$\begin{pmatrix} v \\ u \\ E \end{pmatrix}_t + A \begin{pmatrix} v \\ u \\ E \end{pmatrix}_x = \varepsilon \left[B \begin{pmatrix} v \\ u \\ E \end{pmatrix}_x \right]_x, \tag{1.1$'$}$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & -\frac{(\gamma-1)u}{v} & \frac{\gamma-1}{v} \\ -\frac{pu}{v} & p - \frac{(\gamma-1)u^2}{v} & \frac{(\gamma-1)u}{v} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{v} & 0 \\ 0 & \frac{(1-\nu)u}{v} & \frac{\nu}{v} \end{pmatrix},$$

and $\nu = \frac{\kappa(\gamma-1)}{\varepsilon R} = \frac{(\gamma-1)\mu}{R}$, where μ is defined in (1.3).

By a direct computation, the eigenvalues of the matrix A are

$$\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{\gamma p}{v}}. \tag{1.6}$$

The corresponding Euler equation of the system (1.1) or (1.1)$'$ is

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ E_t + (pu)_x = 0. \end{cases} \tag{1.7}$$

It is well known that the first and the third characteristic field of (1.7) are genuinely nonlinear and the second field is linearly degenerate (see [13]).

When we consider the initial shock discontinuity, i.e.,

$$(v, u, E)(x, t=0) = (v^0, u^0, E^0)(x), \tag{1.8}$$

the system (1.7) admits a single shock solution $(v_0, u_0, E_0)(x,t)$ (we assume it is 3-shock, without loss of generality) and the following four items are satisfied:

(1) $(v_0, u_0, E_0)(x,t)$ is a distribution solution of (1.7), (1.8).

(2) There exists a smooth shock curve $x = s(t)$, $0 \le t \le T$, such that $(v_0, u_0, E_0)(x,t)$ is smooth up to the curve $x = s(t)$, and the left and right limits of $(v_0, u_0, E_0)(x,t)$ and its derivatives at $x = s(t)$ exist.

(3) Across the shock $x = s(t)$, the Rankine-Hugoniot condition holds:

$$\begin{cases} \dot{s}(v_0^- - v_0^+) = u_0^+ - u_0^-, \\ \dot{s}(u_0^- - u_0^+) = p_0^- - p_0^+, \\ \dot{s}(E_0^- - E_0^+) = p_0^- u_0^- - p_0^+ u_0^+. \end{cases} \tag{1.9}$$

(4) Lax entropy condition holds:

$$0<\lambda_3^+<\dot{s}<\lambda_3^-. \tag{1.10}$$

In (1.9) and (1.10), $\dot{s}=\frac{\mathrm{d}}{\mathrm{d}t}s(t)$ and $v_0^-=v_0(s(t)-0,t),\ v_0^+=v_0(s(t)+0,t)$, etc.

We also assume that there exist two positive constants c_* and c^* such that the initial data (1.8) satisfy

$$0<c_*<v^0(x),\theta^0(x)<c^*, \tag{1.11}$$

where $\theta^0(x)=\frac{\gamma-1}{R}(E^0-\frac{(u^0)^2}{2})$. Then we have

Theorem 1.1 Suppose that (1.7), (1.8) admit a 3-shock solution $(v_0,u_0,E_0)(x,t)$ up to time T satisfying

$$\sum_{k=1}^{7}\int_0^T\int_{x\neq s(t)}|\partial_x^k(v_0,u_0,E_0)(x,t)|^2\mathrm{d}x\mathrm{d}t<\infty. \tag{1.12}$$

Let $\gamma\in(1,2]$, then there exist positive constants ε_0 and ε_1 such that, if $\varepsilon\in(0,\varepsilon_0]$ and

$$(\gamma-1)|v_0^+(t)-v_0^-(t)|<\varepsilon_1, \qquad \forall\ t\in[0,T], \tag{1.13}$$

the system (1.1) or (1.1)$'$ under the assumption (1.3) admits a unique smooth solution $(v^\varepsilon,u^\varepsilon,$ $E^\varepsilon)$ (x,t) satisfying

$$\sup_{0\le t\le T}\int|(v^\varepsilon,u^\varepsilon,E^\varepsilon)(x,t)-(v_0,u_0,E_0)(x,t)|^2\mathrm{d}x\le C\varepsilon^\alpha, \qquad \forall\ \alpha\in(\frac{2}{3},1), \tag{1.14}$$

and

$$\sup_{0\le t\le T}\sup_{|x-s(t)|\ge h}|(v^\varepsilon,u^\varepsilon,E^\varepsilon)(x,t)-(v_0,u_0,E_0)(x,t)|\le C_h\varepsilon, \qquad \forall\ h>0, \tag{1.15}$$

where the constants C and C_h are independent of ε, but C_h depends on h.

Remark 1.1 The convergence rate ε in (1.15) is optimal and Theorem 1 is also valid for the case that the system (1.7) has a finite number of noninteracting piecewise smooth shock waves.

Notations In this article, we use the notations c,C to represent the generic constants which are independent of the variables ε,x, and t, except for additional explanations. And we use $\|\cdot\|$ to denote the usual $L_2(\mathbf{R})$ norm, and $\|\cdot\|_{H^i}$ to denote the norm of Sobolev space $H^i(\mathbf{R})$ $(i=1,2,3,\cdots)$. We use $O(1)$ to denote the uniform bounded constant and $o(1)$ to denote the infinitesimal small quantity in some limit process.

2 Approximate Solutions

Motivated by [3], we use the matched asymptotic expansion method to construct the approximate solutions to the inviscid piecewise smooth solution $(v_0,u_0,E_0)(x,t)$.

2.1 Outer and inner expansion and the matching condition

Away from the 3-shock location $x=s(t)$, we give an outer expansion

$$(v^\varepsilon,u^\varepsilon,E^\varepsilon)(x,t)\sim(v_0,u_0,E_0)(x,t)+\varepsilon(v_1,u_1,E_1)(x,t)+\varepsilon^2(v_2,u_2,E_2)(x,t)+\cdots. \tag{2.1}$$

Substituting (2.1) into (1.1)′ and equating the coefficients of power ε yield

$$O(1): \quad \begin{cases} v_{0t} - u_{0x} = 0, \\ u_{0t} + p_{0x} = 0, \\ E_{0t} + (p_0 u_0)_x = 0; \end{cases} \tag{2.2}$$

$$O(\varepsilon): \quad \begin{cases} v_{1t} - u_{1x} = 0, \\ u_{1t} + [\nabla p_0 \cdot (v_1, u_1, E_1)]_x = \Big(\dfrac{u_{0x}}{v_0}\Big)_x, \\ E_{1t} + [\nabla (p_0 u_0) \cdot (v_1, u_1, E_1)]_x = \nu \Big(\dfrac{E_{0x}}{v_0}\Big)_x + (1-\nu)\Big(\dfrac{u_0 u_{0x}}{v_0}\Big)_x; \end{cases} \tag{2.3}$$

$$O(\varepsilon^2): \quad \begin{cases} v_{2t} - u_{2x} = 0, \\ u_{2t} + [\nabla p_0 \cdot (v_2, u_2, E_2)]_x = \Big(\dfrac{u_{1x}}{v_0} - \dfrac{u_{0x}}{v_0^2} v_1\Big)_x - \dfrac{1}{2}[\nabla^2 p_0 \cdot (v_1, u_1, E_1)^2]_x, \\ E_{2t} + [\nabla (p_0 u_0) \cdot (v_2, u_2, E_2)]_x = \nu \Big(\dfrac{E_{1x}}{v_0} - \dfrac{E_{0x}}{v_0^2} v_1\Big)_x \\ \quad + (1-\nu)\Big(\dfrac{(u_0 u_1)_x}{v_0} - \dfrac{u_0 u_{0x}}{v_0^2} v_1\Big)_x - \dfrac{1}{2}[\nabla^2 (p_0 u_0) \cdot (v_1, u_1, E_1)^2]_x; \end{cases} \tag{2.4}$$

$\cdots$.

In the above expressions, we have used the notations $p_0 = p(v_0, u_0, E_0)$, $\nabla p_0 = \nabla p(v_0, u_0, E_0)$, etc.

Near the 3-shock location $x = s(t)$, we give an inner expansion by

$$(v^\varepsilon, u^\varepsilon, E^\varepsilon)(x,t) \sim (V_0, U_0, \mathcal{E}_0)(\xi, t) + \varepsilon (V_1, U_1, \mathcal{E}_1)(\xi, t) + \varepsilon^2 (V_2, U_2, \mathcal{E}_2)(\xi, t) + \cdots, \tag{2.5}$$

where ξ is the scaled variable defined by

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t, \varepsilon), \tag{2.6}$$

where $\delta(t,\varepsilon)$ has an expansion

$$\delta(t, \varepsilon) = \delta_0(t) + \varepsilon \delta_1(t) + \varepsilon^2 \delta_2(t) + \cdots. \tag{2.7}$$

Substituting (2.5)–(2.7) into (1.1)′, we have

$$O(\varepsilon^{-1}): \quad \begin{cases} -\dot{s} V_{0\xi} - U_{0\xi} = 0, \\ -\dot{s} U_{0\xi} + P_{0\xi} = \Big(\dfrac{U_{0\xi}}{V_0}\Big)_\xi, \\ -\dot{s} \mathcal{E}_{0\xi} + (P_0 U_0)_\xi = \nu \Big(\dfrac{\mathcal{E}_{0\xi}}{V_0}\Big)_\xi + (1-\nu)\Big(\dfrac{U_0 U_{0\xi}}{V_0}\Big)_\xi; \end{cases} \tag{2.8}$$

$$O(1): \quad \begin{cases} -\dot{s} V_{1\xi} - U_{1\xi} = -V_{0t} - \dot{\delta}_0 V_{0\xi}, \\ -\dot{s} U_{1\xi} + [\nabla P_0 \cdot (V_1, U_1, \mathcal{E}_1)]_\xi = \Big(\dfrac{U_{1\xi}}{V_0} - \dfrac{U_{0\xi}}{V_0^2} V_1\Big)_\xi - U_{0t} - \dot{\delta}_0 U_{0\xi}, \\ -\dot{s} \mathcal{E}_{1\xi} + [\nabla (P_0 U_0) \cdot (V_1, U_1, \mathcal{E}_1)]_\xi = \nu \Big(\dfrac{\mathcal{E}_{1\xi}}{V_0} - \dfrac{\mathcal{E}_{0\xi}}{V_0^2} V_1\Big)_\xi \\ \quad + (1-\nu)\Big(\dfrac{(U_0 U_1)_\xi}{V_0} - \dfrac{U_0 U_{0\xi}}{V_0^2} V_1\Big)_\xi - \mathcal{E}_{0t} - \dot{\delta}_0 \mathcal{E}_{0\xi}; \end{cases} \tag{2.9}$$

$$O(\varepsilon):\quad \begin{cases} -\dot{s}V_{2\xi}-U_{2\xi}=-V_{1t}-\dot{\delta}_1V_{0\xi}-\dot{\delta}_0V_{1\xi},\\ -\dot{s}U_{2\xi}+[\nabla P_0\cdot(V_2,U_2,\mathcal{E}_2)]_\xi=\Big[\dfrac{U_{2\xi}}{V_0}+\dfrac{U_{0\xi}}{V_0^3}V_1^2-\dfrac{U_{0\xi}}{V_0^2}V_2-\dfrac{U_{1\xi}}{V_0^2}V_1\Big]_\xi\\ -U_{1t}-\dot{\delta}_0U_{1\xi}-\dot{\delta}_1U_{0\xi}-\dfrac{1}{2}[\nabla^2P_0\cdot(V_1,U_1,\mathcal{E}_1))^2]_\xi,\\ -\dot{s}\mathcal{E}_{2\xi}+[\nabla(P_0U_0)\cdot(V_2,U_2,\mathcal{E}_2)]_\xi=\nu\Big[\dfrac{\mathcal{E}_{2\xi}}{V_0}+\dfrac{\mathcal{E}_{0\xi}}{V_0^3}V_1^2-\dfrac{\mathcal{E}_{0\xi}}{V_0^2}V_2-\dfrac{\mathcal{E}_{1\xi}}{V_0^2}V_1\Big]_\xi\\ +(1-\nu)\Big[\dfrac{U_1U_{1\xi}+(U_0U_2)_\xi}{V_0}-\dfrac{(U_0U_1)_\xi}{V_0^2}V_1-\dfrac{U_0U_{0\xi}}{V_0^2}V_2+\dfrac{U_0U_{0\xi}}{V_0^3}V_1^2\Big]_\xi\\ -\mathcal{E}_{1t}-\dot{\delta}_0\mathcal{E}_{1\xi}-\dot{\delta}_1\mathcal{E}_{0\xi}-\dfrac{1}{2}[\nabla^2(P_0U_0)\cdot(V_1,U_1,\mathcal{E}_1)^2]_\xi; \end{cases} \tag{2.10}$$

$\cdots$,

where we have used the notations $P_0=p(V_0,U_0,\mathcal{E}_0)$, $\nabla P_0=(\nabla p)(V_0,U_0,\mathcal{E}_0)$, etc.

The above inner expansion is supposed to be valid in a small zone of size $O(\varepsilon)$ around $x=s(t)$.

The outer expansion and the inner expansion are expected to be valid in the matching zone, where both $|\xi|\to\infty$ and $|x-s(t)|$ small are true. Therefore, they must agree with each other there. Expressing the outer expansion solution in terms of ξ and using Taylor series to equate the coefficients of power ε in outer and inner expansions, we get the following matching condition as $\xi\to\pm\infty$,

$$\begin{aligned} &(V_0,U_0,\mathcal{E}_0)(\xi,t)=(v_0,u_0,E_0)(s(t)\pm0,t)+o(1),\\ &(V_1,U_1,\mathcal{E}_1)(\xi,t)=(v_1,u_1,E_1)(s(t)\pm0,t)+(\xi-\delta_0)(v_{0x},u_{0x},E_{0x})(s(t)\pm0,t)+o(1),\\ &(V_2,U_2,\mathcal{E}_2)(\xi,t)=(v_0,u_0,E_0)(s(t)\pm0,t)+(\xi-\delta_0)(v_{1x},u_{1x},E_{1x})(s(t)\pm0,t)\\ &\quad-\delta_1(v_{0x},u_{0x},E_{0x})(s(t)\pm0,t)+\frac{1}{2}(\xi-\delta_0)^2(v_{0xx},u_{0xx},E_{0xx})(s(t)\pm0,t)+o(1), \end{aligned} \tag{2.11}$$

etc.

We note that the matching condition (2.11) requires that the inner functions $(V_i,U_i,\mathcal{E}_i)(\xi,t)$ have algebraic growth rate $|\xi|^i$ $(i=0,1,2,\cdots)$ as $\xi\to\pm\infty$.

2.2 Viscous shock profiles

This section is devoted to the traveling wave solution of $(1.1)'$, which has the form

$$(v,u,E)(x,t)=(V,U,\mathcal{E})(\xi,t),\qquad \xi=\frac{x-s(t)}{\varepsilon},$$

which satisfies

$$\begin{cases} -\dot{s}V'-U'=0,\\ -\dot{s}U'+P'=(\dfrac{U'}{V})',\\ -\dot{s}\mathcal{E}'+(PU)'=\nu\Big(\dfrac{\mathcal{E}'}{V}\Big)'+(1-\nu)\Big(\dfrac{UU'}{V}\Big)', \end{cases} \tag{2.12}$$

and

$$\lim_{\xi\to\pm\infty}(V,U,\mathcal{E})(\xi,t)=(v_\pm,u_\pm,E_\pm)(t), \tag{2.13}$$

where $\dot{s}=\frac{\mathrm{d}}{\mathrm{d}t}s(t)$, $'=\frac{\mathrm{d}}{\mathrm{d}\xi}$, $P=p(V,U,\mathcal{E})$ and $v_\pm(t)>0,u_\pm(t),\theta_\pm(t)=\frac{\gamma-1}{R}(E_\pm-\frac{u_\pm^2}{2})(t)>0$,

$s(t)$ are the functions of t and satisfy the Rankine–Hugoniot condition

$$\begin{cases} \dot{s}(v_+ - v_-) = -(u_+ - u_-), \\ \dot{s}(u_+ - u_-) = p_+ - p_-, \\ \dot{s}(E_+ - E_-) = p_+u_+ - p_-u_-. \end{cases} \tag{2.14}$$

For 3-shock wave, Lax entropy condition holds:

$$0 < \lambda_3^+ < \dot{s} < \lambda_3^-,$$

or equivalently,

$$\dot{s}(t) > 0, \quad v_+(t) > v_-(t), \quad u_+(t) < u_-(t), \quad \forall \ t \in [0, T]. \tag{2.15}$$

Integrating (2.12) over $[\xi, \pm\infty)$ gives

$$\begin{cases} -\dot{s}V - U = -(\dot{s}v_\pm + u_\pm), \\ -\dot{s}U + P = \dfrac{U'}{V} + (-\dot{s}u_\pm + p_\pm), \\ -\dot{s}\mathcal{E} + PU = \nu\dfrac{\mathcal{E}'}{V} + (1-\nu)\dfrac{UU'}{V} + (-\dot{s}E_\pm + p_\pm u_\pm). \end{cases} \tag{2.16}$$

Thus, we have

$$\begin{cases} \dfrac{\dot{s}V'}{V} = -\Big[P + \dot{s}^2(V - \dfrac{b_1}{\dot{s}^2})\Big], \\ \dfrac{\mu\Theta'}{\dot{s}V} = -\Big[\dfrac{R}{\gamma-1}\Theta - \dfrac{\dot{s}^2}{2}(V - \dfrac{b_1}{\dot{s}^2})^2 + \dfrac{b_1^2}{2\dot{s}^2} - b_2\Big], \\ U = -\dot{s}V + (\dot{s}v_\pm + u_\pm), \end{cases} \tag{2.17}$$

where $\Theta = \frac{R}{\gamma-1}(\mathcal{E} - \frac{U^2}{2})$, $b_1 = p_\pm + \dot{s}^2 v_\pm$, and $b_2 = \frac{R}{\gamma-1}\theta_\pm + p_\pm v_\pm + \frac{\dot{s}^2 v_\pm^2}{2}$.

Regarding (v_-, u_-, E_-) and $\dot{s}$ as parameters, we denote the traveling wave solution $(V, U, \mathcal{E})(\xi, t)$ and (v_+, u_+, E_+) by

$$(V, U, \mathcal{E})(\xi, t) = (V, U, \mathcal{E})(\xi; v_-, u_-, E_-, \dot{s}),$$

$$(v_+, u_+, E_+)(t) = (v_+, u_+, E_+)(v_-, u_-, E_-, \dot{s}).$$

Similar to [7], the following lemma holds.

Lemma 2.1 There exists a traveling wave solution $(V, U, \mathcal{E})(\xi, t)$, unique up to a shift, solving (2.12), (2.13) with the properties:

$$\dot{s} > 0, \quad V_\xi > 0, \quad U_\xi = -\dot{s}V_\xi < 0, \quad \Theta_\xi < 0, \ \Big|\frac{\Theta_\xi}{V_\xi}\Big| \le C(\gamma - 1);$$

$$|(V, U, \Theta) - (v_\pm, u_\pm, \theta_\pm)| \le C\mathrm{e}^{-c|\xi|},$$

$$|V_\xi|, |V_{\xi\xi}|, |\Theta_{\xi\xi}| \le C|v_+ - v_-|\mathrm{e}^{-c|\xi|}, \quad |\Theta_\xi| \le C(\gamma-1)|v_+ - v_-|\mathrm{e}^{-c|\xi|}, \quad \text{as } \xi \to \pm\infty;$$

$$\frac{\partial(V, U, \mathcal{E})}{\partial(v_-, u_-, E_-)} - I_{3\times3} = O(1)\mathrm{e}^{-c|\xi|}, \quad \frac{\partial(V, U, \mathcal{E})}{\partial \dot{s}} = O(1)\mathrm{e}^{-c|\xi|}, \quad \text{as } \xi \to -\infty;$$

$$\frac{\partial(V, U, \mathcal{E})}{\partial(v_-, u_-, E_-)} - \frac{\partial(v_+, u_+, E_+)}{\partial(v_-, u_-, E_-)} = O(1)\mathrm{e}^{-c|\xi|}, \ \frac{\partial(V, U, \mathcal{E})}{\partial \dot{s}} - \frac{\partial(v_+, u_+, E_+)}{\partial \dot{s}} = O(1)\mathrm{e}^{-c|\xi|},$$

$$\text{as } \xi \to +\infty;$$

where $I_{3\times 3}$ represents the identity matrix of the third order.

2.3 The outer and the inner expansion solutions

Let $(v_0, u_0, E_0)(x,t)$ be the 3-shock solution described in Theorem 1 and $(V_0, U_0, \mathcal{E}_0)(\xi,t)$ be the traveling wave solution $(V, U, \mathcal{E})(\xi, t)$ in Lemma 2.1 with the end states

$$(v_\pm, u_\pm, E_\pm)(t) = (v_0, u_0, E_0)(s(t) \pm 0, t).$$

Because the shift in Lemma 2.1 can be absorbed in $\delta(t,\varepsilon)$, we let it be zero.

We shall construct the next order terms $(v_1, u_1, E_1)(x,t)$, $(V_1, U_1, \mathcal{E}_1)(\xi, t)$ and $\delta_0(t)$ by studying the linear hyperbolic system (2.3), where the coefficients have jump across the shock curve $x = s(t)$, and the ODE system (2.9) of ξ. The boundary values of (2.3) on the shock location $x = s(t)$ and (2.9) on $\xi = \pm\infty$ are related to the matching condition $(2.11)_2$. Once the boundary values for the systems (2.3) and (2.9) are known and the initial value for (2.3) is given, the corresponding solutions are immediately obtained. So the remaining problem is how to determine the boundary values for (2.3) and (2.9) by the previous terms $(v_0, u_0, E_0)(x,t)$, $(V_0, U_0, \mathcal{E}_0)(\xi,t)$ and the matching condition $(2.11)_2$.

We first consider the system (2.9). Since $(V_1, U_1, \mathcal{E}_1)(\xi,t)$ tends to infinity with order of $O(|\xi|)$ as $|\xi| \to \infty$ due to $(2.11)_2$, we introduce a new variable $(\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)(\xi,t)$ by

$$(V_1, U_1, \mathcal{E}_1)(\xi,t) = (\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)(\xi,t) + (D_1, D_2, D_3)(\xi,t),$$

where

$$(D_1, D_2, D_3)(\xi,t) = \xi(v_{0x}, u_{0x}, E_{0x})(s(t) \pm 0, t), \qquad \text{if} \quad \pm\,\xi \geq 1,$$

so that

$$\begin{aligned}(\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)(\xi,t) = {}& (v_1, u_1, E_1)(s(t) \pm 0, t) - \delta_0(t)(v_{0x}, u_{0x}, E_{0x})(s(t) \pm 0, t)\\ &+ o(1) \qquad \text{as} \quad \xi \to \pm\infty,\end{aligned} \tag{2.18}$$

and

$$\begin{cases} -\dot{s}\tilde{V}_{1\xi} - \tilde{U}_{1\xi} = -\dot{\delta}_0 V_{0\xi} + h_1(\xi,t),\\ -\dot{s}\tilde{U}_{1\xi} + [\nabla P_0 \cdot (\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)]_\xi = \Big(\dfrac{\tilde{U}_{1\xi}}{V_0} - \dfrac{U_{0\xi}}{V_0^2}\tilde{V}_1\Big)_\xi - \dot{\delta}_0 U_{0\xi} + h_2(\xi,t),\\ -\dot{s}\tilde{\mathcal{E}}_{1\xi} + [\nabla (P_0 U_0) \cdot (\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)]_\xi = \nu\Big(\dfrac{\tilde{\mathcal{E}}_{1\xi}}{V_0} - \dfrac{\mathcal{E}_{0\xi}}{V_0^2}\tilde{V}_1\Big)_\xi\\ \qquad + (1-\nu)\Big(\dfrac{(U_0\tilde{U}_1)_\xi}{V_0} - \dfrac{U_0 U_{0\xi}}{V_0^2}\tilde{V}_1\Big)_\xi - \dot{\delta}_0 \mathcal{E}_{0\xi} + h_3(\xi,t), \end{cases} \tag{2.19}$$

where

$$\begin{aligned} h_1(\xi,t) &= \dot{s}D_{1\xi} + D_{2\xi} - V_{0t}\\ &= \dot{s}v_{0x}(s(t)\pm 0, t) + u_{0x}(s(t)\pm 0,t) - [v_0(s(t)\pm 0,t)]_t\\ &\quad - [V_0 - v_0(s(t)\pm 0,t)]_t = O(1)\mathrm{e}^{-c|\xi|}, \qquad \text{as } |\xi| \to \infty,\\ h_2(\xi,t) &= \dot{s}D_{2\xi} - [\nabla P_0 \cdot (D_1, D_2, D_3)]_\xi - U_{0t} + \Big(\frac{D_{2\xi}}{V_0} - \frac{U_{0\xi}}{V_0^2}D_1\Big)_\xi\\ &= \dot{s}u_{0x}(s(t)\pm 0,t) - \nabla p_0 \cdot (v_{0x}, u_{0x}, E_{0x})(s(t)\pm 0, t)\\ &\quad - [u_0(s(t)\pm 0,t)]_t + O(1)\mathrm{e}^{-c|\xi|}\\ &= O(1)\mathrm{e}^{-c|\xi|}, \qquad \text{as } |\xi| \to \infty, \end{aligned}$$

$$
\begin{aligned}
h_3(\xi,t) = {}& \dot{s}D_{3\xi} - [\nabla(P_0U_0)\cdot(D_1,D_2,D_3)]_\xi - \mathcal{E}_{0t} + \nu\Big(\frac{D_{3\xi}}{V_0} - \frac{U_{0\xi}}{V_0^2}D_1\Big)_\xi \\
& +(1-\nu)\Big(\frac{(U_0D_2)_\xi}{V_0} - \frac{U_0U_{0\xi}}{V_0^2}D_1\Big)_\xi = O(1)\mathrm{e}^{-c|\xi|}, \qquad \text{as } |\xi|\to\infty.
\end{aligned}
$$

Here we have used (2.17) and Lemma 2.1.

Integrating the system (2.19) over $[0,\xi]$ yields

$$
\begin{cases}
-\dot{s}\tilde{V}_1 - \tilde{U}_1 = -\dot{\delta}_0V_0 + H_1(\xi,t) + C_1(t), \\
-\dot{s}\tilde{U}_1 + \nabla P_0\cdot(\tilde{V}_1,\tilde{U}_1,\tilde{\mathcal{E}}_1) = \dfrac{\tilde{U}_{1\xi}}{V_0} - \dfrac{U_{0\xi}}{V_0^2}\tilde{V}_1 - \dot{\delta}_0U_0 + H_2(\xi,t) + C_2(t), \\
-\dot{s}\tilde{\mathcal{E}}_1 + \nabla(P_0U_0)\cdot(\tilde{V}_1,\tilde{U}_1,\tilde{\mathcal{E}}_1) = \nu\Big(\dfrac{\tilde{\mathcal{E}}_{1\xi}}{V_0} - \dfrac{\mathcal{E}_{0\xi}}{V_0^2}\tilde{V}_1\Big) \\
\quad +(1-\nu)\Big(\dfrac{(U_0\tilde{U}_1)_\xi}{V_0} - \dfrac{U_0U_{0\xi}}{V_0^2}\tilde{V}_1\Big) - \dot{\delta}_0\mathcal{E}_0 + H_3(\xi,t) + C_3(t),
\end{cases}
\tag{2.20}
$$

where $H_i(\xi,t) = \int_0^\xi h_i(\zeta,t)\mathrm{d}\zeta, i = 1,2,3$, and the integral constants $C_i(t)$ $(i = 1,2,3)$ will be determined later.

Let $(\tilde{v}_1^\pm, \tilde{u}_1^\pm, \tilde{E}_1^\pm) = \lim\limits_{\xi\to\pm\infty}(\tilde{V}_1,\tilde{U}_1,\tilde{\mathcal{E}}_1)(\xi,t)$. From (2.18), $\lim\limits_{\xi\to\pm\infty}(\tilde{V}_{1\xi},\tilde{U}_{1\xi},\tilde{\mathcal{E}}_{1\xi})(\xi,t) = (0,0,0)$. Then, letting $\xi\to\pm\infty$ in (2.20) implies

$$
M^\pm\begin{pmatrix}\tilde{v}_1^\pm \\ \tilde{u}_1^\pm \\ \tilde{E}_1^\pm\end{pmatrix} = \begin{pmatrix}-\dot{\delta}_0v_0^\pm + H_1^\pm(t) + C_1(t) \\ -\dot{\delta}_0u_0^\pm + H_2^\pm(t) + C_2(t) \\ -\dot{\delta}_0E_0^\pm + H_3^\pm(t) + C_3(t)\end{pmatrix},
\tag{2.21}
$$

where

$$
M^\pm = \begin{pmatrix}
-\dot{s} & -1 & 0 \\
-\dfrac{p_0^\pm}{v_0^\pm} & -\dfrac{(\gamma-1)u_0^\pm}{v_0^\pm} - \dot{s} & \dfrac{\gamma-1}{v_0^\pm} \\
-\dfrac{p_0^\pm u_0^\pm}{v_0^\pm} & p_0^\pm - \dfrac{(\gamma-1)(u_0^\pm)^2}{v_0^\pm} & \dfrac{(\gamma-1)u_0^\pm}{v_0^\pm} - \dot{s}
\end{pmatrix},
\tag{2.22}
$$

$H_1^\pm(t) = \int_0^{\pm\infty} h_1(\zeta,t)\mathrm{d}\zeta$, and $v_0^\pm = \lim\limits_{\xi\to\pm\infty} V_0(\xi,t) = v_0(s(t)\pm 0,t))$, etc. It is straight forward to compute

$$
\det M^\pm = \dot{s}(-\dot{s}^2 + \frac{\gamma p_0^\pm}{v_0^\pm}) \neq 0,
$$

due to the Lax entropy condition (1.10). There are four unknown quantities $\dot{\delta}_0$ and $C_i(t), i = 1,2,3$, in the formula (2.21).

On the other hand, from (2.18), $(\tilde{v}_1^\pm, \tilde{u}_1^\pm, \tilde{E}_1^\pm)$ can be expressed by $(v_1^\pm, u_1^\pm, E_1^\pm) = (v_1,u_1,E_1)$ $(s(t)\pm 0,t)$, i.e.,

$$
\begin{cases}
\tilde{v}_1^\pm = v_1^\pm - \delta_0v_{0x}^\pm, \\
\tilde{u}_1^\pm = u_1^\pm - \delta_0u_{0x}^\pm, \\
\tilde{E}_1^\pm = E_1^\pm - \delta_0E_{0x}^\pm,
\end{cases}
\tag{2.23}
$$

where, and in the sequel, $v_{0x}^\pm = v_{0x}(s(t)\pm 0,t)$, etc. We turn to the linear hyperbolic system (2.3) for (v_1,u_1,E_1). Because the coefficients of the system (2.3) have jump across the shock curve

$x = s(t)$, we have to study it in $\Omega_+ = \{(x,t); x > s(t), t \geq 0\}$ and $\Omega_- = \{(x,t); x < s(t), t \geq 0\}$, respectively.

Let's compute the Jacobi matrix A_0 of (2.3). We have

$$A_0 = \begin{pmatrix} 0 & -1 & 0 \\ -\dfrac{p_0}{v_0} & -\dfrac{(\gamma-1)u_0}{v_0} & \dfrac{\gamma-1}{v_0} \\ -\dfrac{p_0u_0}{v_0} & p_0 - \dfrac{(\gamma-1)u_0^2}{v_0} & \dfrac{(\gamma-1)u_0}{v_0} \end{pmatrix}. \tag{2.24}$$

The eigenvalues of A_0 on $x = s(t) \pm 0$ are

$$\lambda_1^\pm = -\sqrt{\frac{\gamma p_0^+}{v_0^\pm}}, \quad \lambda_2^\pm = 0, \quad \lambda_3^\pm = \sqrt{\frac{\gamma p_0^\pm}{v_0^\pm}}. \tag{2.25}$$

When we consider (2.3) in the region Ω_+. From the Lax entropy condition (1.10), all eigenvalues $\lambda_i^+, i = 1,2,3$ are less than the slope $\dot{s}$ of the boundary $x = s(t)$. That is all characteristics are incoming into the boundary and it is not necessary to impose the boundary values on $x = s(t)$. So (v_1^+, u_1^+, E_1^+) can be uniquely determined by the initial data.

When we consider the region Ω_-, the situation is subtle. The third wave is incoming on the boundary since $\lambda_3^- > \dot{s}$. However, the eigenvalues λ_1^-, λ_2^- are less than $\dot{s}$ so that the corresponding waves are outgoing on the boundary. Let $\beta_j^- = l_j^-(v_1^-, u_1^-, E_1^-)^t$, $i = 1,2,3$, where l_j^- are the left eigenvectors of $A_0(v_0^-, u_0^-, E_0^-)$. From (2.21) and (2.23), we have

$$(\beta_1^-, \beta_2^-) = (G_1, G_2)(t)\beta_3^- + (G_3, G_4)(t), \tag{2.26}$$

and

$$\dot{\delta}_0 + G_5(t)\delta_0 = G_6(t)\beta_3^- + G_7(t), \tag{2.27}$$

where $G_i(t), i = 1, \cdots, 7$, are smooth known functions. The theory of linear hyperbolic system [9], [12] implies that the problem (2.3) and (2.26) in the domain Ω_- has a solution smooth up to the shock location $x = s(t)$ if the initial data are chosen to satisfy the compatibility conditions at $x = s(0)$. Thus, we solve the hyperbolic equation (2.3) for $(v_1, u_1, E_1)(x,t)$. Then, we solve the ordinary differential equation (2.27) with $\delta_0(0) = 0$ to get $\delta_0(t)$ in terms of the incoming wave β_3^-. Again using (2.21), we obtain the integral constants $C_i(t), i = 1,2,3$. Therefore, we get $(\tilde{V}_1, \tilde{U}_1, \tilde{\mathcal{E}}_1)(\xi,t)$ from (2.20) and finally obtain $(V_1, U_1, \mathcal{E}_1)(\xi,t)$. In summary, we have

Lemma 2.2 The functions $\delta_0(t), (v_1, u_1, E_1)(x,t), (V_1, U_1, \mathcal{E}_1)(\xi,t)$ constructed above satisfy

(1) $(v_1, u_1, E_1)(x,t)$ and its derivatives are uniformly continuous up to the shock $x = s(t)$ and

$$\sum_{k=0}^{5} \int_0^T \int_{x \neq s(t)} |\partial_x^k (v_1, u_1, E_1)(x,t)|^2 \mathrm{d}x\mathrm{d}t < \infty, \tag{2.28}$$

(2) $(V_1, U_1, \mathcal{E}_1)(\xi,t)$ and $\delta_0(t)$ are smooth functions, and there is a positive constant $c > 0$ such that

$$(V_1, U_1, \mathcal{E}_1)(\xi,t) = (v_1, u_1, E_1)(s(t) \pm 0, t) + (\xi - \delta_0))(v_{0x}, u_{0x}, E_{0x})(s(t) \pm 0, t) + O(1)\mathrm{e}^{-c|\xi|},$$

$$\text{as} \qquad \xi \to \pm\infty.$$

Similarly, the above procedure can be carried out to any higher order terms, in particular $\delta_1(t)$, $(v_2, u_2, E_2)(x,t)$, $(V_2, U_2, \mathcal{E}_2)(\xi, t)$ and $\delta_2(t)$, $(v_3, u_3, E_3)(x,t)$, $(V_3, U_3, \mathcal{E}_3)(\xi, t)$, with the similar estimates as in Lemma 2.2.

2.4 Construction of approximate solutions

With the outer and inner solutions determined in the previous section, we now construct the approximate solutions. Let $m(z) \in C_0^\infty(\mathbf{R})$ satisfying $0 \le m(z) \le 1$ and

$$m(z) = \begin{cases} 1 & |y| \le 1, \\ 0 & |y| \ge 2. \end{cases}$$

Choosing $\alpha \in (\frac{2}{3}, 1)$, we construct the approximate solutions as

$$\begin{pmatrix} \bar{v} \\ \bar{u} \\ \bar{E} \end{pmatrix}(x,t) = m(z)\begin{pmatrix} V_{\rm in} \\ U_{\rm in} \\ \mathcal{E}_{\rm in} \end{pmatrix}(x,t) + (1-m(z))\begin{pmatrix} v_{\rm out} \\ u_{\rm out} \\ E_{\rm out} \end{pmatrix}(x,t) + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}(x,t), \tag{2.29}$$

where

$$z = \frac{x - s(t)}{\varepsilon^\alpha},$$

$$\begin{pmatrix} V_{\rm in} \\ U_{\rm in} \\ \mathcal{E}_{\rm in} \end{pmatrix}(x,t) = \begin{pmatrix} V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 \\ U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 \\ \mathcal{E}_0 + \varepsilon \mathcal{E}_1 + \varepsilon^2 \mathcal{E}_2 + \varepsilon^3 \mathcal{E}_3 \end{pmatrix}\Big(\frac{x - s(t)}{\varepsilon} + \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2, t\Big), \tag{2.30}$$

$$\begin{pmatrix} v_{\rm out} \\ u_{\rm out} \\ E_{\rm out} \end{pmatrix}(x,t) = \begin{pmatrix} v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \\ u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 \\ E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \varepsilon^3 E_3 \end{pmatrix}(x,t), \tag{2.31}$$

and $d_i(x,t)$ $(i = 1,2,3)$ are the higher order correction terms to be determined later.

The approximate solution $(\bar{v}, \bar{u}, \bar{E})(x,t)$ defined in (2.29) satisfies the following equations:

$$\begin{cases} \bar{v}_t - \bar{u}_x = q_1(x,t) + d_{1t} - d_{2x}, \\ \bar{u}_t + \bar{p}_x = \varepsilon(\dfrac{\bar{u}_x}{\bar{v}})_x + \displaystyle\sum_{i=2}^{5} q_i(x,t) + d_{2t} - \varepsilon(\dfrac{d_{2x}}{\bar{v} - d_1})_x, \\ \bar{E}_t + (\bar{p}\bar{u})_x = \varepsilon\nu(\dfrac{\bar{E}_x}{\bar{v}})_x + \varepsilon(1-\nu)(\dfrac{\bar{u}\bar{u}_x}{\bar{v}})_x + \displaystyle\sum_{i=6}^{9} q_i(x,t) + d_{3t} - \varepsilon\nu(\dfrac{d_{3x}}{\bar{v} - d_1})_x, \end{cases} \tag{2.32}$$

where

$$q_1(x,t) = m_t(V_{\rm in} - v_{\rm out}) - m_x(U_{\rm in} - u_{\rm out}) + m((V_{\rm in})_t - (U_{\rm in})_x), \tag{2.33}$$

$$\begin{aligned} q_2(x,t) = {}& m_t(U_{\rm in} - u_{\rm out}) - \varepsilon m_x\Big(\frac{(U_{\rm in})_x}{V_{\rm in}} - \frac{(u_{\rm out})_x}{v_{\rm out}}\Big) \\ & + \{[p(\bar{v} - d_1, \bar{u} - d_2, \bar{E} - d_3)]_x - m(P_{\rm in})_x - (1-m)(p_{\rm out})_x\}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} q_3(x,t) = {}& m\Big[(P_{\rm in} - \chi(P_{\rm in}))_x - \varepsilon(\frac{(U_{\rm in})_x}{V_{\rm in}} - \chi(\frac{(U_{\rm in})_x}{V_{\rm in}}))_x\Big] \\ & + m\varepsilon^3(U_{3t} + \dot{\delta}_0 U_{3\xi} + \dot{\delta}_1 U_{2\xi} + \dot{\delta}_2 U_{1\xi} + \varepsilon\dot{\delta}_1 U_{3\xi} + \varepsilon\dot{\delta}_2 U_{2\xi} + \varepsilon^2\dot{\delta}_2 U_{3\xi}), \end{aligned} \tag{2.35}$$

$$q_4(x,t)=(1-m)\Big[(p_{\rm out}-\chi(p_{\rm out}))_x-\varepsilon(\frac{(u_{\rm out})_x}{v_{\rm out}}-\chi(\frac{(u_{\rm out})_x}{v_{\rm out}}))_x\Big], \tag{2.36}$$

$$\begin{aligned}q_5(x,t)=&-\varepsilon\Big[\frac{\bar u_x}{\bar v}-m\frac{(U_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{(u_{\rm out})_x}{v_{\rm out}}-\frac{d_{2x}}{\bar v}\Big]_x-\varepsilon\Big(\frac{d_{2x}}{\bar v}-\frac{d_{2x}}{\bar v-d_1}\Big)_x\\&+[\bar p-p(\bar v-d_1,\bar u-d_2,\bar E-d_3)]_x,\end{aligned} \tag{2.37}$$

$$\begin{aligned}q_6(x,t)=&\,m_t(\mathcal{E}_{\rm in}-E_{\rm out})-\varepsilon\nu m_x\Big(\frac{(\mathcal{E}_{\rm in})_x}{V_{\rm in}}-\frac{(E_{\rm out})_x}{v_{\rm out}}\Big)\\&-\varepsilon(1-\nu)m_x\Big(\frac{U_{\rm in}(U_{\rm in})_x}{V_{\rm in}}-\frac{u_{\rm out}(u_{\rm out})_x}{v_{\rm out}}\Big)\\&+\{[p(\bar v-d_1,\bar u-d_2,\bar E-d_3)\cdot(\bar u-d_2)]_x-m(P_{\rm in}U_{\rm in})_x-(1-m)(p_{\rm out}u_{\rm out})_x\},\end{aligned} \tag{2.38}$$

$$\begin{aligned}q_7(x,t)=&\,m\Big[(P_{\rm in}U_{\rm in}-\chi(P_{\rm in}U_{\rm in}))_x-\varepsilon\nu(\frac{(\mathcal{E}_{\rm in})_x}{V_{\rm in}}-\chi(\frac{(\mathcal{E}_{\rm in})_x}{V_{\rm in}}))_x\\&-\varepsilon(1-\nu)(\frac{U_{\rm in}(U_{\rm in})_x}{V_{\rm in}}-\chi(\frac{U_{\rm in}(U_{\rm in})_x}{V_{\rm in}}))_x\Big]\\&+m\varepsilon^3(\mathcal{E}_{3t}+\dot\delta_0\mathcal{E}_{3\xi}+\dot\delta_1\mathcal{E}_{2\xi}+\dot\delta_2\mathcal{E}_{1\xi}+\varepsilon\dot\delta_1\mathcal{E}_{3\xi}+\varepsilon\dot\delta_2\mathcal{E}_{2\xi}+\varepsilon^2\dot\delta_2\mathcal{E}_{3\xi}),\end{aligned} \tag{2.39}$$

$$\begin{aligned}q_8(x,t)=&\,(1-m)\Big[(p_{\rm out}u_{\rm out}-\chi(p_{\rm out}u_{\rm out}))_x-\varepsilon\nu(\frac{(E_{\rm out})_x}{v_{\rm out}}-\chi(\frac{(E_{\rm out})_x}{v_{\rm out}}))_x\\&-\varepsilon(1-\nu)(\frac{u_{\rm out}(u_{\rm out})_x}{v_{\rm out}}-\chi(\frac{u_{\rm out}(u_{\rm out})_x}{v_{\rm out}}))_x\Big],\end{aligned} \tag{2.40}$$

$$\begin{aligned}q_9(x,t)=&-\varepsilon\nu\Big[\frac{\bar E_x}{\bar v}-m\frac{(\mathcal{E}_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{(E_{\rm out})_x}{v_{\rm out}}-\frac{d_{3x}}{\bar v}\Big]_x\\&-\varepsilon(1-\nu)\Big[\frac{\bar u\bar u_x}{\bar v}-m\frac{U_{\rm in}(U_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{u_{\rm out}(u_{\rm out})_x}{v_{\rm out}}\Big]_x\\&-\varepsilon\nu\Big(\frac{d_{3x}}{\bar v}-\frac{d_{3x}}{\bar v-d_1}\Big)_x+[\bar p\bar u-p(\bar v-d_1,\bar u-d_2,\bar E-d_3)\cdot(\bar u-d_2)]_x,\end{aligned} \tag{2.41}$$

where $P_{\rm in}=p(V_{\rm in},U_{\rm in},\mathcal{E}_{\rm in})$, $p_{\rm out}=p(v_{\rm out},u_{\rm out},E_{\rm out})$, and

$$\begin{aligned}\chi(P_{\rm in})=&\,P_0+\varepsilon\nabla P_0\cdot(V_1,U_1,\mathcal{E}_1)+\varepsilon^2\nabla P_0\cdot(V_2,U_2,\mathcal{E}_2)+\varepsilon^3\nabla P_0\cdot(V_3,U_3,\mathcal{E}_3)\\&+\frac12\varepsilon^2\nabla^2P_0\cdot(V_1,U_1,\mathcal{E}_1)^2+\frac12\varepsilon^3\nabla^2P_0\cdot[(V_1,U_1,\mathcal{E}_1),(V_2,U_2,\mathcal{E}_2)],\\ \chi(p_{\rm out})=&\,p_0+\varepsilon\nabla p_0\cdot(v_1,u_1,E_1)+\varepsilon^2\nabla p_0\cdot(v_2,u_2,E_2)+\varepsilon^3\nabla p_0\cdot(v_3,u_3,E_3)\\&+\frac12\varepsilon^2\nabla^2p_0\cdot(v_1,u_1,E_1)^2+\frac12\varepsilon^3\nabla^2p_0\cdot[(v_1,u_1,E_1),(v_2,u_2,E_2)],\end{aligned}$$

represent the expansions of the Taylor series of $P_{\rm in}$ and $p_{\rm out}$ with respect to ε at the base state $(V_0,U_0,\mathcal{E}_0)$, (v_0,u_0,E_0) to the order ε^3, respectively. Similar notations are used in (2.35), (2.36), (2.40) and (2.41).

Because $q_i(x,t), i=5,9$, are in conservative form, they are good terms for the stability analysis when introducing anti-derivative variables. The other terms $q_i(x,t), i\neq5,9$, though a little more complicated, can be estimated in the following Lemma 2.3 in the different zones (inner, outer and matching zones) by a tedious calculation. First, we compute

$$\partial_x^k(V_{\rm in}-v_{\rm out},U_{\rm in}-u_{\rm out},\mathcal{E}_{\rm in}-E_{\rm out})=O(1)\varepsilon^{(4-k)\alpha},\quad k=0,1,2,3, \tag{2.42}$$

on the matching zone $\{(x,t):\varepsilon^\alpha\le|x-s(t)|\le2\varepsilon^\alpha, t\in[0,T]\}$. We have

Lemma 2.3 $q_i(x,t), i\neq 5,9$ satisfy the following properties ($k=0,1,2,3$):

(1) supp q_1, supp q_3, supp $q_7\subseteq$ Inner zone: $\{(x,t):|x-s(t)|\le 2\varepsilon^\alpha, t\in[0,T]\}$,

$$\partial_x^k(q_1,q_3,q_7)(x,t)=O(1)\varepsilon^{(3-k)\alpha}. \tag{2.43}$$

(2) supp q_2, supp $q_6\subseteq$ Matching zone: $\{(x,t):\varepsilon^\alpha\le|x-s(t)|\le 2\varepsilon^\alpha, t\in[0,T]\}$,

$$\partial_x^k(q_2,q_6)(x,t)=O(1)\varepsilon^{(3-k)\alpha}. \tag{2.44}$$

(3) supp q_4, supp $q_8\subseteq$ Outer zone: $\{(x,t):|x-s(t)|\ge 2\varepsilon^\alpha, t\in[0,T]\}$,

$$\partial_x^k(q_4,q_8)(x,t)=O(1)\varepsilon^{4-k\alpha}, \tag{2.45}$$

$$\Big(\int_0^T\|\partial_x^k(q_4,q_8)\|^2\mathrm{d}t\Big)^{\frac12}=O(1)\varepsilon^{4-k\alpha}. \tag{2.46}$$

Choose the higher order terms $d_i(x,t), i=1,2,3$, by

$$\begin{cases} d_{1t}-d_{2x}=-q_1(x,t),\\ d_{2t}-\varepsilon\Big(\dfrac{d_{2x}}{\bar v-d_1}\Big)_x=-\displaystyle\sum_{i=2}^4 q_i(x,t),\\ d_{3t}-\varepsilon\nu\Big(\dfrac{d_{3x}}{\bar v-d_1}\Big)_x=-\displaystyle\sum_{i=6}^8 q_i(x,t),\\ d_1(x,t=0)=d_2(x,t=0)=d_3(x,t=0)=0.\end{cases} \tag{2.47}$$

Then, the approximate solutions $(\bar v,\bar u,\bar E)(x,t)$ satisfy the following equations:

$$\begin{cases} \bar v_t-\bar u_x=0,\\ \bar u_t+\bar p_x=\varepsilon(\dfrac{\bar u_x}{\bar v})_x+Q_{1x},\\ \bar E_t+(\bar p\bar u)_x=\varepsilon\nu(\dfrac{\bar E_x}{\bar v})_x+\varepsilon(1-\nu)(\dfrac{\bar u\bar u_x}{\bar v})_x+Q_{2x},\end{cases} \tag{2.48}$$

where

$$\begin{aligned} Q_1=&-\varepsilon\Big[\frac{\bar u_x}{\bar v}-m\frac{(U_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{(u_{\rm out})_x}{v_{\rm out}}-\frac{d_{2x}}{\bar v}\Big]\\ &-\varepsilon\Big(\frac{d_{2x}}{\bar v}-\frac{d_{2x}}{\bar v-d_1}\Big)+[\bar p-p(\bar v-d_1,\bar u-d_2,\bar E-d_3)]\\ =&O(1)\varepsilon|m_x(U_{\rm in}-u_{\rm out})|+O(1)m(1-m)\varepsilon|(V_{\rm in}-v_{\rm out})|\\ &+O(1)\varepsilon|d_1||d_{2x}|+O(1)|(d_1,d_2,d_3)|,\end{aligned} \tag{2.49}$$

$$\begin{aligned} Q_2=&-\varepsilon\nu\Big[\frac{\bar E_x}{\bar v}-m\frac{(\mathcal{E}_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{(E_{\rm out})_x}{v_{\rm out}}-\frac{d_{3x}}{\bar v}\Big]-\varepsilon\nu\Big(\frac{d_{3x}}{\bar v}-\frac{d_{3x}}{\bar v-d_1}\Big)\\ &-\varepsilon(1-\nu)\Big[\frac{\bar u\bar u_x}{\bar v}-m\frac{U_{\rm in}(U_{\rm in})_x}{V_{\rm in}}-(1-m)\frac{u_{\rm out}(u_{\rm out})_x}{v_{\rm out}}\Big]\\ &+[\bar p\bar u-p(\bar v-d_1,\bar u-d_2,\bar E-d_3)\cdot(\bar u-d_2)]\\ =&O(1)\varepsilon|m_x(\mathcal{E}_{\rm in}-E_{\rm out})|+O(1)m(1-m)\varepsilon|(V_{\rm in}-v_{\rm out},U_{\rm in}-u_{\rm out})|\\ &+O(1)\varepsilon|d_1||d_{3x}|+O(1)|(d_1,d_2,d_3)|.\end{aligned} \tag{2.50}$$

It is convenient to use a variable $\bar\theta(x,t)=\frac{\gamma-1}{R}(\bar E-\frac{\bar u^2}{2})$. With the approximate solutions defined in (2.29), we have

$$\begin{aligned}\bar\theta(x,t)=&m\Big[\Theta_0+\varepsilon\frac{\gamma-1}{R}(\mathcal{E}_1-U_0U_1)+\varepsilon^2\frac{\gamma-1}{R}(\mathcal{E}_2-\frac{U_1^2}{2}-U_0U_2)\\&+\varepsilon^3\frac{\gamma-1}{R}(\mathcal{E}_3-U_0U_3-U_1U_2)\Big]+(1-m)\Big[\theta_0+\varepsilon\frac{\gamma-1}{R}(E_1-u_0u_1)\\&+\varepsilon^2\frac{\gamma-1}{R}(E_2-\frac{u_1^2}{2}-u_0u_2)+\varepsilon^3\frac{\gamma-1}{R}(E_3-u_0u_3-u_1u_2)\Big]\\&+\frac{\gamma-1}{R}(d_3^2-\frac{d_1^2}{2})+O(1)\varepsilon^{4\alpha-\frac12},\end{aligned}\tag{2.51}$$

where $\Theta_0=\frac{\gamma-1}{R}(\mathcal{E}_0-\frac{U_0^2}{2})$ and $\theta_0=\frac{\gamma-1}{R}(E_0-\frac{u_0^2}{2})$.

Thus the approximate system (2.48) is equivalent to

$$\begin{cases}\bar v_t-\bar u_x=0,\\ \bar u_t+\bar p_x=\varepsilon(\frac{\bar u_x}{\bar v})_x+Q_{1x},\\ \bar E_t+(\bar p\bar u)_x=\kappa(\frac{\bar\theta_x}{\bar v})_x+\varepsilon(\frac{\bar u\bar u_x}{\bar v})_x+Q_{2x},\end{cases}\tag{2.48$'$}$$

From (2.47), we have the following lemma about the properties of the higher order terms $d_i(x,t),i=1,2,3$.

Lemma 2.4 The following estimates are valid for $d_i(x,t),i=2,3$:

$$\|\partial_x^k d_i(\cdot,t)\|_{L_\infty}\le C\varepsilon^{(4-k)\alpha-\frac12}\ (k=0,1,2,3),$$
$$\|d_i(\cdot,t)\|_{L_2}\le C\varepsilon^{4\alpha-\frac12},$$
$$\|\partial_x^k d_i(\cdot,t)\|_{L_2}\le C\varepsilon^{(4-k+\frac12)\alpha-\frac12},k=1,2,3.$$

The proof can be found in [3] and [15].

From $(2.47)_1$, we get

$$d_1(x,t)=\int_0^t d_{2x}(x,\tau)\mathrm{d}\tau-\int_0^t q_1(x,\tau)\mathrm{d}\tau.$$

It follows that

$$\|\partial_x^k d_1(\cdot,t)\|_{L_\infty}\le C\varepsilon^{(3-k)\alpha-\frac12}\quad(k=0,1,2,3),$$
$$\|\partial_x^k d_1(\cdot,t)\|_{L_2}\le C\varepsilon^{(3-k+\frac12)\alpha-\frac12}\quad(k=0,1,2,3).$$

From (2.49) and (2.50), we have, for $i=1,2$,

$$\begin{aligned}&\|\partial_x^k Q_i(\cdot,t)\|_{L_\infty}\le C\varepsilon^{(3-k)\alpha-\frac12}\quad(k=0,1,2),\\&\|\partial_x^k Q_i(\cdot,t)\|_{L_2}\le C\varepsilon^{(3-k+\frac12)\alpha-\frac12}\quad(k=0,1,2),\\&\|\partial_t Q_i(\cdot,t)\|_{L_2}\le C\varepsilon^{\frac52\alpha-\frac12}.\end{aligned}\tag{2.52}$$

Lemma 2.5 The approximate solutions satisfy

(1)

$$(\bar v,\bar u,\bar E,\bar\theta)(x,t)=\begin{cases}(v_0,u_0,E_0,\theta_0)(x,t)+O(1)\varepsilon & \text{if}\ \ |x-s(t)|\ge\varepsilon^\alpha,\\(V_0,U_0,\mathcal{E}_0,\Theta_0)(x,t)+O(1)\varepsilon^\alpha & \text{if}\ \ |x-s(t)|\le 2\varepsilon^\alpha.\end{cases}\tag{2.53}$$

(2) There exist positive constants v_{**}, v^{**} and θ_{**}, θ^{**} such that

$$0 < v_{**} < \bar{v}(x,t) < v^{**}, \quad 0 < \theta_{**} < \bar{\theta}(x,t) < \theta^{**}, \quad \text{if} \ \ \varepsilon \ll 1. \tag{2.54}$$

(3) Introduce the scaled variables

$$y = \frac{x - s(t)}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon}, \tag{2.55}$$

then

$$\frac{\partial(\bar{v}, \bar{u}, \bar{E}, \bar{\theta})}{\partial y} = m(V_{oy}, U_{0y}, \mathcal{E}_{0y}) + O(1)\varepsilon, \quad \frac{\partial(\bar{v}, \bar{u}, \bar{E}, \bar{\theta})}{\partial \tau} = O(1)\varepsilon. \tag{2.56}$$

The proof of Lemma 2.5 is similar in [3].

3 Stability Analysis

Let

$$(v, u, \theta)(x,t) = (\bar{v}, \bar{u}, \bar{\theta})(x,t) + (\phi, \psi, \omega)(y, \tau), \tag{3.1}$$

where y, τ are the scaled variables defined in (2.55). Then we have

$$E - \bar{E} = \frac{R}{\gamma - 1}\omega + \frac{\psi^2}{2} + \bar{u}\psi. \tag{3.2}$$

Thus, $(\phi, \psi, \omega)(y, \tau)$ satisfies the system

$$\begin{cases} \phi_\tau - \dot{s}\phi_y - \psi_y = 0, \\ \psi_\tau - \dot{s}\psi_y + (p - \bar{p})_y = \Big(\dfrac{u_y}{v} - \dfrac{\bar{u}_y}{\bar{v}}\Big)_y - Q_{1y}, \\ \Big(\dfrac{R}{\gamma - 1}\omega + \dfrac{\psi^2}{2} + \bar{u}\psi\Big)_\tau - \dot{s}\Big(\dfrac{R}{\gamma - 1}\omega + \dfrac{\psi^2}{2} + \bar{u}\psi\Big)_y + (pu - \bar{p}\bar{u})_y \\ \qquad = \mu\Big(\dfrac{\theta_y}{v} - \dfrac{\bar{\theta}_y}{\bar{v}}\Big)_y + \Big(\dfrac{uu_y}{v} - \dfrac{\bar{u}\bar{u}_y}{\bar{v}}\Big)_y - Q_{2y}, \\ (\phi, \psi, \omega)(y, \tau = 0) = (0, 0, 0). \end{cases} \tag{3.3}$$

Introduce the anti-derivative variables

$$(\Phi, \Psi, \tilde{W})(y, \tau) = \int_{-\infty}^{y} (\phi, \psi, E - \bar{E})(y', \tau)\mathrm{d}y'. \tag{3.4}$$

Let

$$W(y, \tau) = \frac{\gamma - 1}{R}(\tilde{W} - \bar{u}\Psi). \tag{3.5}$$

Then, we have

$$\omega = W_y + \frac{\gamma - 1}{R}\Big(\bar{u}_y\Psi - \frac{1}{2}\Psi_y^2\Big). \tag{3.6}$$

Integrating (3.3) over $(-\infty, \xi]$ and linearizing the resulted system yield

$$\begin{cases} \Phi_\tau - \dot{s}\Phi_y - \Psi_y = 0, \\ \Psi_\tau - \dot{s}\Psi_y - \dfrac{\eta}{\bar{v}}\Phi_y + \dfrac{R}{\bar{v}}W_y + \dfrac{\gamma - 1}{\bar{v}}\bar{u}_y\Psi - \dfrac{\Psi_{yy}}{\bar{v}} = J_1 - Q_1, \\ \dfrac{R}{\gamma - 1}(W_\tau - \dot{s}W_y) + \eta\Psi_y - \dfrac{\mu}{\bar{v}}\Big(W_y + \dfrac{\gamma - 1}{R}\bar{u}_y\Psi\Big)_y - \dot{s}\bar{u}_y\Psi + \dfrac{\mu}{\bar{v}}\bar{\theta}_y\Phi_y \\ \qquad = J_2 - \bar{u}_\tau\Psi + \bar{u}Q_1 - Q_2, \\ (\Phi, \Psi, W)(y, \tau = 0) = (0, 0, 0), \end{cases} \tag{3.7}$$

where

$$\eta=\bar{p}-\frac{\bar{u}_y}{\bar{v}}, \tag{3.8}$$

$$J_1=\frac{p-\bar{p}}{\bar{v}}\Phi_y+\frac{\gamma-1}{2\bar{v}}\Psi_y^2-\frac{\Psi_{yy}\Phi_y}{v\bar{v}}+\frac{\bar{u}_y}{v\bar{v}^2}\Phi_y^2=O(1)(|\Phi_y|^2+|\Psi_y|^2+|W_y|^2+|\psi_y|^2), \tag{3.9}$$

$$\begin{aligned}J_2&=-\frac{\mu(\gamma-1)\Psi_y\Psi_{yy}}{R\bar{v}}-(p-\bar{p})\Psi_y-\frac{\mu\omega_y\Phi_y}{v\bar{v}}+\frac{\mu\bar{\theta}_y}{v\bar{v}^2}\Phi_y^2+\Big(\frac{u_y}{v}-\frac{\bar{u}_y}{\bar{v}}\Big)\Psi_y\\&=O(1)(|\Phi_y|^2+|\Psi_y|^2+|W_y|^2+|\psi_y|^2+|\omega_y|^2).\end{aligned} \tag{3.10}$$

We will prove that, for ε small enough, the system (3.7) has a unique solution on the time interval $0\le\tau\le\frac{T}{\varepsilon}$. Since the local existence of (3.7) is well-known, we omit it for brevity. To prove the above conclusion, we only need to verify the following a priori estimate.

Theorem 3.1 (A priori estimate) Suppose that the system (3.7) has a solution (Φ,Ψ,W) on $[0,\tau_0]$ where $\tau_0\in(0,\frac{T}{\varepsilon}]$. There exist positive constants ε_0 and ε_1 such that, if $\varepsilon\in(0,\varepsilon_0]$, $\gamma\in(1,2]$, and

$$(\gamma-1)|v_0^+(\varepsilon\tau)-v_0^-(\varepsilon\tau)|<\varepsilon_1,\qquad\forall\ \tau\in[0,\tau_0],$$

then

$$N(\tau_0)^2+\int_0^{\tau_0}\Big[\|(m|V_{0y}|)^{\frac12}(\Psi,\frac{W}{\sqrt{\gamma-1}})\|^2+\|\phi\|_{H^1}^2+\|(\psi,\omega)\|_{H^2}^2\Big](\tau)\mathrm{d}\tau\le C\varepsilon^{7\alpha-4}, \tag{3.11}$$

where

$$N(\tau_0)=\sup_{0\le\tau\le\tau_0}\left\{\|(\Phi,\Psi,\frac{W}{\sqrt{\gamma-1}})\|(\tau)+\|(\phi,\psi,\frac{\omega}{\sqrt{\gamma-1}})\|_{H^1}(\tau)\right\}. \tag{3.12}$$

In order to prove Theorem 3.1, we first do the lower order estimate.

3.1 Lower order estimate

From $(2.17)_1$, we obtain

$$P_0-\frac{U_{0y}}{V_0}=b_1-\dot{s}^2V_0. \tag{3.13}$$

Let

$$k(V_0)=(b_1-\dot{s}^2V_0)^{-1}, \tag{3.14}$$

then there exist two positive constants c and C such that

$$0<c\le k(V_0)\le C. \tag{3.15}$$

We compute

$$\begin{aligned}\eta=\bar{p}-\frac{\bar{u}_y}{\bar{v}}&=m(P_0-\frac{U_{0y}}{V_0})+(1-m)(p_0-\frac{u_{0y}}{v_0})+O(1)\varepsilon^\alpha\\&=m(b_1-\dot{s}^2V_0)+(1-m)(p_0-\frac{u_{0y}}{v_0})+O(1)\varepsilon^\alpha.\end{aligned}$$

Thus, there exist positive constants c and C such that, if $\varepsilon\ll1$,

$$0<c\le\eta\le C. \tag{3.16}$$

Multiplying $(3.7)_1$ by Φ, $(3.7)_2$ by $\frac{\bar{v}}{\eta}\Psi$, $(3.7)_3$ by $\frac{R}{\eta^2}W$, and summing the resulted equations together, we get

$$\begin{aligned}&I_1(\Phi,\Psi,W)_\tau+I_2(\Psi,\Psi_y)+I_3(W,W_y)\\&=I_4(\Psi,W,\Phi_y,W_y)+\frac{\bar{v}}{\eta}(J_1-Q_1)\Psi+\frac{R^2}{(\gamma-1)\eta^2}(J_2-\bar{u}_\tau\Psi-\bar{u}Q_1-Q_2)W+(\cdots)_y,\end{aligned} \tag{3.17}$$

where

$$I_1(\Phi,\Psi,W)=\frac{\Phi^2}{2}+\frac{\bar v}{2\eta}\Psi^2+\frac{R^2}{2(\gamma-1)\eta^2}W^2, \tag{3.18}$$

$$I_2(\Psi,\Psi_y)=\Big[\frac{\gamma-1}{\eta}\bar u_y+\dot s(\frac{\bar v}{2\eta})_y\Big]\Psi^2+(\frac{1}{\eta})_y\Psi\Psi_y+\frac{1}{\eta}\Psi_y^2, \tag{3.19}$$

$$I_3(W,W_y)=\dot s\left(\frac{R^2}{2(\gamma-1)\eta^2}\right)_y W^2+\frac{R\mu}{\bar v\eta^2}W_y^2, \tag{3.20}$$

$$\begin{aligned}I_4(\Psi,W,\Phi_y,W_y)=&\Big[(\frac{R}{\eta})_y-R\dot s\eta^{-2}\bar u_y\Big]\Psi W+(\frac{R\mu}{\bar v\eta^2})_yWW_y+\frac{(\gamma-1)\mu}{\bar v\eta^2}\bar u_y\Psi W_y\\&+(\frac{(\gamma-1)\mu}{\bar v\eta^2})_y\bar u_y\Psi W+\frac{R\mu}{\bar v^2\eta^2}\bar\theta_y\Phi_yW+(\frac{\bar v}{2\eta})_\tau\Psi^2+(\frac{R^2}{2(\gamma-1)\eta^2})_\tau W^2,\end{aligned} \tag{3.21}$$

in which $(\cdots)_y$ denotes the terms which disappear after integration with respect to y over $\mathbf{R}$.

From (3.16), we have

$$c(\Phi^2+\Psi^2+\frac{W^2}{\gamma-1})\le I_1\le C(\Phi^2+\Psi^2+\frac{W^2}{\gamma-1}), \tag{3.22}$$

for some positive constants c and C.

On the other hand,

$$\begin{aligned}&\frac{\gamma-1}{\eta}\bar u_y+\dot s(\frac{\bar v}{2\eta})_y\\&=(\gamma-1)[mk(V_0)+(1-m)(p_0-\frac{u_{0y}}{v_0})^{-1}+O(1)m\varepsilon^\alpha+O(1)(1-m)\varepsilon]\\&\quad\cdot(mU_{0y}+O(1)\varepsilon)+\dot s[\frac{1}{2}m(V_0k(V_0))_y+O(1)m\varepsilon^\alpha V_{0y}+O(1)\varepsilon]\\&=m[(\gamma-1)k(V_0)U_{0y}+\frac{1}{2}\dot s(V_0k(V_0))_y]+O(1)m\varepsilon^\alpha V_{0y}+O(1)\varepsilon,\end{aligned} \tag{3.23}$$

$$(\frac{1}{\eta})_y=m(k(V_0))_y+O(1)m\varepsilon^\alpha V_{0y}+O(1)\varepsilon, \tag{3.24}$$

and

$$\frac{1}{\eta}=mk(V_0)+(1-m)(p_0-\frac{u_{0y}}{v_0})^{-1}+O(1)\varepsilon^\alpha. \tag{3.25}$$

Substitute (3.23)–(3.25) into (3.19), we have

$$\begin{aligned}I_2=&m\left\{[(\gamma-1)k(V_0)U_{0y}+\frac{1}{2}\dot s(V_0k(V_0))_y]\Psi^2+(k(V_0))_y\Psi\Psi_y+k(V_0)\Psi_y^2\right\}\\&+O(1)m\varepsilon^\alpha V_{0y}\Psi^2+O(1)\varepsilon\Psi^2+[(1-m)(p_0-\frac{u_{0y}}{v_0})^{-1}+O(1)\varepsilon^\alpha]\Psi_y^2.\end{aligned} \tag{3.26}$$

We calculate that the quadric term in $\{\cdots\}$ of (3.26),

$$\begin{aligned}&[(\gamma-1)k(V_0)U_{0y}+\frac{1}{2}\dot s(V_0k(V_0))_y]\Psi^2+(k(V_0))_y\Psi\Psi_y+k(V_0)\Psi_y^2\\&=\frac{1}{2}\dot sk^2(V_0)[b_1-2(\gamma-1)k(V_0)^{-1}](\sqrt{V_{0y}}\Psi)^2-\dot s^2k^2(V_0)V_{0y}\Psi\Psi_y+k(V_0)\Psi_y^2.\end{aligned} \tag{3.27}$$

Similar to [7], we can calculate that the coefficient of the first term of (3.27)

$$\frac{1}{2}\dot sk^2(V_0)[b_1-2(\gamma-1)k(V_0)^{-1}]>0,$$

and the discriminant of (3.27)

$$\begin{aligned}\Delta &= \dot{s}^4k^4(V_0)V_{0y} - 2\dot{s}k^3(V_0)[b_1 - 2(\gamma-1)k(V_0)^{-1}]\\ &= -\dot{s}k^3(V_0)[b_1 + \dot{s}^2R\Theta_0k(V_0) + k(V_0)^{-1} - 4(\gamma-1)k(V_0)^{-1}] < 0,\end{aligned}$$

provided that $\gamma \in (1,2]$ and $(\gamma-1)|v_0^+ - v_0^-|(\varepsilon\tau)$ is small enough for all $\tau \in [0,\tau_0]$.

Thus, there exists positive constants c and $c_i, i = 1, 2, \cdots$, such that

$$\begin{aligned}I_2 &\geq mc_1(|V_{0y}|\Psi^2 + \Psi_y^2) - mc_2\varepsilon^\alpha|V_{0y}|\Psi^2 - c_3\varepsilon\Psi^2 + (1-m)c_4\Psi_y^2 - c_5\varepsilon^\alpha\Psi_y^2\\ &\geq mc(1-\varepsilon^\alpha)|V_{0y}|\Psi^2 - c\varepsilon\Psi^2 + c(1-\varepsilon^\alpha)\Psi_y^2.\end{aligned} \tag{3.28}$$

Similarly, we have

$$I_3 \geq cm(1-\varepsilon^\alpha)|V_{0y}|\frac{W^2}{\gamma-1} - c\varepsilon\frac{W^2}{\gamma-1} + c(1-\varepsilon^\alpha)W_y^2. \tag{3.29}$$

Note that

$$\begin{aligned}\Big[(\frac{R}{\eta})_y - R\dot{s}\eta^{-2}\bar{u}_y\Big]\Psi W &= [O(1)m\varepsilon^\alpha V_{0y} + O(1)\varepsilon]\Psi W\\ &\leq Cm\varepsilon^\alpha|V_{0y}|\Big(\Psi^2 + \frac{W^2}{\gamma-1}\Big) + C\varepsilon\Big(\Psi^2 + \frac{W^2}{\gamma-1}\Big).\end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.21) and using Young inequality give

$$\begin{aligned}I_4 \leq{}& \beta\Big(|V_{0y}|\frac{W^2}{\gamma-1} + W_y^2\Big) + [C_\beta(\gamma-1)|v_0^+ - v_0^-| + C\varepsilon^\alpha]\Big\{|V_{0y}|\Big(\Psi^2 + \frac{W^2}{\gamma-1}\Big) + \Phi_y^2\Big\}\\ &+C\varepsilon\Big(\Psi^2 + \frac{W^2}{\gamma-1}\Big),\end{aligned} \tag{3.31}$$

where β is a small positive constant to be chosen later and C_β is the constant only depending on β.

Next, we estimate the nonlinear terms in (3.17). First,

$$\begin{aligned}&\int_0^\tau\int_{\mathbf{R}}\frac{\bar{v}}{\eta}(J_1 - Q_1)\Psi\mathrm{d}y\mathrm{d}\tau\\ &\leq CN(\tau_0)\int_0^\tau\int_{\mathbf{R}}(\Phi_y^2 + \Psi_y^2 + W_y^2 + \psi_y^2)\mathrm{d}y\mathrm{d}\tau + \varepsilon\int_0^\tau\int_{\mathbf{R}}\Psi^2\mathrm{d}y\mathrm{d}\tau + C\varepsilon^{-1}\int_0^\tau\int_{\mathbf{R}}Q_1^2\mathrm{d}y\mathrm{d}\tau,\end{aligned} \tag{3.32}$$

and

$$\varepsilon^{-1}\int_0^\tau\int_{\mathbf{R}}Q_1^2\mathrm{d}y\mathrm{d}\tau \leq C\varepsilon^{-2}\sup_{0\leq\tau\leq\tau_0}\|Q_1(\cdot,\tau)\|^2 \leq C\varepsilon^{7\alpha-3}. \tag{3.33}$$

Similarly,

$$\begin{aligned}&\int_0^\tau\int_{\mathbf{R}}\frac{R^2}{(\gamma-1)\eta^2}(J_2 - \bar{u}_\tau\Psi - \bar{u}Q_1 - Q_2)W\mathrm{d}y\mathrm{d}\tau\\ &\leq CN(\tau_0)\int_0^\tau\int_{\mathbf{R}}(\Phi_y^2 + \Psi_y^2 + W_y^2 + \psi_y^2 + \omega_y^2)\mathrm{d}y\mathrm{d}\tau + \varepsilon\int_0^\tau\int_{\mathbf{R}}\Big(\Psi^2 + \frac{W^2}{\gamma-1}\Big)\mathrm{d}y\mathrm{d}\tau + C\varepsilon^{7\alpha-3}.\end{aligned} \tag{3.34}$$

Summarizing the above estimates, choosing β, ε and $N(\tau_0)$ suitably small and using (3.17), we get

Lemma 3.1 With the same assumptions as in Theorem 2, the following estimate is valid:

$$\begin{aligned}&\sup_{0\le\tau\le\tau_0}\Big\|(\Phi,\Psi,\frac{W}{\sqrt{\gamma-1}})\Big\|^2(\tau)+\int_0^{\tau_0}\int_{\mathbf{R}}\Big[m|V_{0y}|\Big(\Psi^2+\frac{W^2}{\gamma-1}\Big)+\Psi_y^2+W_y^2\Big]\mathrm{d}y\mathrm{d}\tau\\&\le CN(\tau_0)\int_0^{\tau_0}\int_{\mathbf{R}}(\psi_y^2+\omega_y^2)\mathrm{d}y\mathrm{d}\tau+C\varepsilon\int_0^{\tau_0}\int_{\mathbf{R}}\Big(\Psi^2+\frac{W^2}{\gamma-1}\Big)\mathrm{d}y\mathrm{d}\tau\\&\quad+C(\gamma-1)\sup_{0\le\tau\le\tau_0}|v_0^+-v_0^-|(\varepsilon\tau)\int_0^{\tau_0}\int_{\mathbf{R}}\Phi_y^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{7\alpha-3}.\end{aligned}\tag{3.35}$$

To estimate the term $\|\Phi_y\|^2$, we multiply $(3.7)_1$ by $\bar v\Psi_y-\bar v_y\Psi$, $(3.7)_2$ by $-\bar v\Phi_y$, then apply ∂_y to $(3.7)_1$ and multiply the resulting equation by Φ_y, we arrive at

$$\begin{aligned}(\frac{\Psi_y^2}{2}-\bar v\Phi_y\Psi)_\tau+\eta\Phi_y^2&=\bar v_y(\dot s\Phi_y+\Psi_y)\Psi+\bar v\Psi_y^2+\bar v_\tau\Phi_y\Psi+R\Phi_yW_y\\&\quad+(\gamma-1)\bar u_y\Phi_y\Psi-\bar vJ_1\Phi_y+\bar vQ_1\Phi_y+(\cdots)_y.\end{aligned}\tag{3.36}$$

Integrating (3.36) with respect to τ and y over $[0,\tau]\times\mathbf{R}$, and using the similar method as in Lemma 3.1, we have

Lemma 3.2 There exists a positive constant C independent of τ and ε such that

$$\begin{aligned}&\sup_{0\le\tau\le\tau_0}\|\Phi_y\|^2(\tau)+\int_0^{\tau_0}\int_{\mathbf{R}}\Phi_y^2\mathrm{d}y\mathrm{d}\tau\\&\le C\sup_{0\le\tau\le\tau_0}\|\Psi\|^2(\tau)+C\int_0^{\tau_0}\int_{\mathbf{R}}(m|V_{0y}|\Psi^2+\Psi_y^2+W_y^2)\mathrm{d}y\mathrm{d}\tau\\&\quad+CN(\tau_0)\int_0^{\tau_0}\int_{\mathbf{R}}\psi_y^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon\int_0^{\tau_0}\int_{\mathbf{R}}\Psi^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{7\alpha-3}.\end{aligned}\tag{3.37}$$

3.2 Higher order estimate

First, we apply ∂_y to the system (3.7) to get the system for $(\phi,\psi,\omega)(y,\tau)$:

$$\begin{cases}\phi_\tau-\dot s\phi_y-\psi_y=0,\\ \psi_\tau-\dot s\psi_y-\dfrac{\eta}{\bar v}\phi_y+\dfrac{R}{\bar v}\omega_y-(\dfrac{\psi_y}{\bar v})_y-(\dfrac{\eta}{\bar v})_y\phi+(\dfrac{R}{\bar v})_y\omega=J_3-Q_{1y},\\ \dfrac{R}{\gamma-1}(\omega_\tau-\dot s\omega_y)+\eta\psi_y-\mu(\dfrac{\omega_y}{\bar v})_y=J_4+\bar uQ_{1y}-Q_{2y},\end{cases}\tag{3.38}$$

where

$$J_3=\Big(\frac{p-\bar p}{\bar v}\phi-\frac{\psi_y\phi}{v\bar v}+\frac{\bar u_y}{v\bar v^2}\phi^2\Big)_y,\tag{3.39}$$

$$J_4=\Big(\frac{u_y}{v}-\frac{\bar u_y\phi}{v\bar v}\Big)\Big(\eta\phi-R\omega+\psi_y\Big)-\Big(\frac{\mu\omega_y\phi}{v\bar v}\Big)_y+\Big(\frac{\mu\bar\theta_y\phi^2}{v\bar v^2}\Big)_y-\Big(\frac{\mu\bar\theta_y\phi}{\bar v^2}\Big)_y.\tag{3.40}$$

Multiplying $(3.38)_1$ by ϕ, $(3.38)_2$ by $\frac{\bar v}{\eta}\psi$, $(3.38)_3$ by $\frac{R}{\eta^2}\omega$, and adding all the resulting equations, we get

$$\begin{aligned}&\Big(\frac{\phi^2}{2}+\frac{\bar v}{2\eta}\psi^2+\frac{R^2}{2(\gamma-1)\eta^2}\omega^2\Big)_\tau+\frac{1}{\eta}\psi_y^2+\frac{\mu R}{\bar v\eta^2}\omega_y^2\\&=\Big[\Big(\frac{\bar v}{2\eta})_y\Big)_\tau-\dot s\Big(\frac{\bar v}{2\eta}\Big)_y\Big]\psi^2+\Big[\Big(\frac{R^2}{2(\gamma-1)\eta^2}\Big)_\tau-\dot s\Big(\frac{R^2}{2(\gamma-1)\eta^2}\Big)_y\Big]\omega^2-\frac{1}{\bar v}\Big(\frac{\bar v}{\eta}\Big)_y\psi\psi_y\\&\quad+\frac{\bar v}{\eta}\Big(\frac{\eta}{\bar v}\Big)_y\phi\psi+\Big[\Big(\frac{R}{\eta}\Big)_y-\frac{\bar v}{\eta}\Big(\frac{R}{\bar v}\Big)_y\Big]\psi\omega-\Big(\frac{p-\bar p}{\bar v}\phi-\frac{\psi_y\phi}{v\bar v}+\frac{\bar u_y}{v\bar v^2}\phi^2\Big)\Big(\frac{\bar v\psi}{\eta}\Big)_y\end{aligned}$$

$$+\Big(\frac{u_y}{v}-\frac{\bar u_y\phi}{v\bar v}\Big)(\eta\phi-R\omega+\psi_y)\frac{R}{\eta^2}\omega+\Big(\frac{\mu\omega_y\phi}{v\bar v}\Big)\Big(\frac{R}{\eta^2}\omega\Big)_y-\Big(\frac{\mu\bar\theta_y\phi^2}{v\bar v^2}\Big)\Big(\frac{R}{\eta^2}\omega\Big)_y$$
$$+\Big(\frac{\mu\bar\theta_y\phi}{\bar v^2}\Big)\Big(\frac{R}{\eta^2}\omega\Big)_y+Q_{1y}\frac{\bar v}{\eta}\psi+(\bar uQ_{1y}-Q_{2y})\frac{R}{\eta^2}\omega+(\cdots)_y.$$

Integrating the above equation with respect to τ and y over $[0,\tau]\times\mathbf{R}$ and using Young inequality, we have

$$\int_{\mathbf{R}}\Big(\frac{\phi^2}{2}+\frac{\bar v}{2\eta}\psi^2+\frac{R^2}{2(\gamma-1)\eta^2}\omega^2\Big)\mathrm{d}y+\int_0^\tau\int_{\mathbf{R}}\Big(\frac{1}{\eta}\psi_y^2+\frac{\mu R}{\bar v\eta^2}\omega_y^2\Big)\mathrm{d}y\mathrm{d}\tau$$
$$\le(\beta+CN(\tau_0))\int_0^\tau\int_{\mathbf{R}}(\psi_y^2+\omega_y^2)\mathrm{d}y\mathrm{d}\tau+C_\beta\int_0^\tau\int_{\mathbf{R}}(\phi^2+\psi^2+\omega^2)\mathrm{d}y\mathrm{d}\tau$$
$$+C\int_0^\tau\int_{\mathbf{R}}(Q_{1y}^2+Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau, \tag{3.41}$$

where β is a small positive constant to be determined later and C_β is the constant only depending on β. From (2.54), we have

$$\int_0^\tau\int_{\mathbf{R}}(Q_{1y}^2+Q_{2y}^2)\mathrm{d}y\mathrm{d}\tau\le\int_0^T\int_{\mathbf{R}}(Q_{1x}^2+Q_{2x}^2)\mathrm{d}x\mathrm{d}t\le C\varepsilon^{5\alpha-1}. \tag{3.42}$$

Inserting (3.42) into (3.41) and choosing β and $N(\tau_0)$ suitably small, we arrive at

Lemma 3.3 There exists a positive constant C independent of τ and ε such that

$$\sup_{0\le\tau\le\tau_0}\Big\|(\phi,\psi,\frac{\omega}{\sqrt{\gamma-1}})\Big\|^2(\tau)+\int_0^{\tau_0}\int_{\mathbf{R}}(\psi_y^2+\omega_y^2)\mathrm{d}y\mathrm{d}\tau$$
$$\le C\int_0^{\tau_0}\int_{\mathbf{R}}(\phi^2+\psi^2+\omega^2)\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{5\alpha-1}. \tag{3.43}$$

Multiplying $(3.38)_1$ by $\bar v\psi_y-\bar v_y\psi$, $(3.38)_2$ by $-\bar v\phi_y$, applying ∂_y to $(3.38)_1$ and then multiplying the resulting equation by ϕ_y, calculating all their sums, and using the same method as in Lemma 3.2, we have

Lemma 3.4 There exists a positive constant C independent of τ and ε such that

$$\sup_{0\le\tau\le\tau_0}\|\phi_y\|^2(\tau)+\int_0^{\tau_0}\int_{\mathbf{R}}\phi_y^2\mathrm{d}y\mathrm{d}\tau$$
$$\le C\sup_{0\le\tau\le\tau_0}\|\psi\|^2(\tau)+C\int_0^{\tau_0}\int_{\mathbf{R}}(\phi^2+\psi^2+\omega^2)\mathrm{d}y\mathrm{d}\tau$$
$$+C\int_0^{\tau_0}\int_{\mathbf{R}}(\psi_y^2+\omega_y^2)\mathrm{d}y\mathrm{d}\tau+CN(\tau_0)\int_0^{\tau_0}\int_{\mathbf{R}}\psi_{yy}^2\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{5\alpha-1}. \tag{3.44}$$

Now, we do the highest order estimate. Applying ∂_y to the system (3.38), we obtain

$$\begin{cases}\phi_{y\tau}-\dot s\phi_{yy}-\psi_{yy}=0,\\ \psi_{y\tau}-\dot s\psi_{yy}-\dfrac{\eta}{\bar v}\phi_{yy}+\dfrac{R}{\bar v}\omega_{yy}-(\dfrac{\psi_y}{\bar v})_{yy}-2(\dfrac{\eta}{\bar v})_y\phi_y-(\dfrac{\eta}{\bar v})_{yy}\phi\\ \qquad+2(\dfrac{R}{\bar v})_y\omega_y+(\dfrac{R}{\bar v})_{yy}\omega=J_{3y}-Q_{1yy},\\ \dfrac{R}{\gamma-1}(\omega_{y\tau}-\dot s\omega_{yy})+\eta\psi_{yy}-\mu(\dfrac{\omega_y}{\bar v})_{yy}+\eta_y\psi_y=J_{4y}+(\bar uQ_{1y})_y-Q_{2yy},\end{cases} \tag{3.45}$$

where J_3 and J_4 are defined by (3.39) and (3.40), respectively. Multiplying $(3.45)_1$ by ϕ_y, $(3.45)_2$ by $\frac{\bar{v}}{\eta}\psi_y$, $(3.45)_3$ by $\frac{R}{\eta^2}\omega_y$, adding all the resulting equations, and using the same method as in Lemma 3.3, we get

Lemma 3.5 There exists a positive constant C independent of τ and ε such that

$$\begin{aligned}&\sup_{0\le\tau\le\tau_0}\Big\|(\phi_y,\psi_y,\frac{\omega_y}{\sqrt{\gamma-1}})\Big\|^2(\tau)+\int_0^{\tau_0}\int_{\mathbf{R}}(\psi_{yy}^2+\omega_{yy}^2)\mathrm{d}y\mathrm{d}\tau\\ &\le C\int_0^{\tau_0}\int_{\mathbf{R}}(\phi^2+\psi^2+\omega^2+\psi_y^2+\omega_y^2)\mathrm{d}y\mathrm{d}\tau+C\varepsilon^{5\alpha-1}.\end{aligned}\tag{3.46}$$

Note that from (3.6),

$$\int_0^{\tau_0}\int_{\mathbf{R}}\omega^2\mathrm{d}y\mathrm{d}\tau\le C\int_0^{\tau_0}\int_{\mathbf{R}}(m|V_{0y}|\Psi^2+\Psi_y^2+W_y^2)\mathrm{d}y\mathrm{d}\tau+C\varepsilon\int_0^{\tau_0}\int_{\mathbf{R}}\Psi^2\mathrm{d}y\mathrm{d}\tau.\tag{3.47}$$

Combining Lemmas 3.1–3.5 and (3.47), we finally get

$$\begin{aligned}&N(\tau_0)^2+\int_0^{\tau_0}\Big[\Big\|(m|V_{0y}|)^{\frac12}(\Psi,\frac{W}{\sqrt{\gamma-1}})\Big\|^2+\|\phi\|_{H^1}^2+\|(\psi,\omega)\|_{H^2}^2\Big](\tau)\mathrm{d}\tau\\ &\le C\varepsilon\int_0^{\tau_0}\Big\|(\Psi,\frac{W}{\sqrt{\gamma-1}})\Big\|^2\mathrm{d}\tau+C\varepsilon^{7\alpha-3},\end{aligned}\tag{3.48}$$

under the smallness condition (1.13). Then Gronwall inequality yields Theorem 3.1.

4 The Proof of Theorem 1.1

From Theorem 3.1, we can easily prove Theorem 1.1. The existence and uniqueness of the solution of (1.1) is presented in Theorem 1.1. We only need to check (1.14) and (1.15) in the following. First,

$$\begin{aligned}&\sup_{0\le t\le T}\int_{\mathbf{R}}|(v,u,E)(x,t)-(\bar{v},\bar{u},\bar{E})(x,t)|^2\mathrm{d}x\\ &\le C\varepsilon\sup_{0\le\tau\le\frac{T}{\varepsilon}}\int_{\mathbf{R}}|(\phi,\psi,\frac{\omega}{\sqrt{\gamma-1}})(y,\tau)|^2\mathrm{d}y\le C\varepsilon^{7\alpha-3}.\end{aligned}\tag{4.1}$$

From Lemma 2.5, we have

$$\begin{aligned}&\sup_{0\le t\le T}\int_{\mathbf{R}}|(\bar{v},\bar{u},\bar{E})(x,t)-(v_0,u_0,E_0)(x,t)|^2\mathrm{d}x\\ &\le\sup_{0\le t\le T}\int_{|x-s(t)|\le\varepsilon^\alpha}|(\bar{v},\bar{u},\bar{E})(x,t)-(v_0,u_0,E_0)(x,t)|^2\mathrm{d}x\\ &\quad+\sup_{0\le t\le T}\int_{|x-s(t)|\ge\varepsilon^\alpha}|(\bar{v},\bar{u},\bar{E})(x,t)-(v_0,u_0,E_0)(x,t)|^2\mathrm{d}x\\ &\le C\varepsilon^\alpha.\end{aligned}\tag{4.2}$$

Thus, (4.1) and (4.2) yield (1.14). Second, Theorem 2 shows

$$\begin{aligned}&\sup_{0\le t\le T}\sup_{x\in\mathbf{R}}|(v,u,E)(x,t)-(\bar{v},\bar{u},\bar{E})(x,t)|\\ &\le C\sup_{0\le\tau\le\frac{T}{\varepsilon}}\sup_{y\in\mathbf{R}}|(\phi,\psi,\omega)(y,\tau)|\le C\sup_{0\le\tau\le\frac{T}{\varepsilon}}\|(\phi,\psi,\omega)(\cdot,\tau)\|_{H^1}\\ &\le C\varepsilon^{\frac{7\alpha-4}{2}}\le C\varepsilon,\end{aligned}\tag{4.3}$$

if we take $\alpha \in [\frac{6}{7}, 1)$.

Therefore, for any $h > 0$, we can choose ε suitably small such that, if $\varepsilon^\alpha \le h$, Lemma 2.5 yields

$$\begin{aligned} &\sup_{0\le t\le T}\ \sup_{|x-s(t)|\ge h} |(\bar{v}, \bar{u}, \bar{E})(x,t) - (v_0, u_0, E_0)(x,t)| \\ \le &\sup_{0\le t\le T}\ \sup_{|x-s(t)|\ge \varepsilon^\alpha} |(\bar{v}, \bar{u}, \bar{E})(x,t) - (v_0, u_0, E_0)(x,t)| \le C\varepsilon. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) gives (1.15). This completes the proof of Theorem 1.1.

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