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L^2-contraction of large planar shock waves for multi-dimensional scalar viscous conservation laws ☆

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Abstract

We consider a L^2-contraction (a L^2-type stability) of large viscous shock waves for the multi-dimensional scalar viscous conservation laws, up to a suitable shift by using the relative entropy methods. Quite different from the previous results, we find a new way to determine the shift function, which depends both on the time and space variables and solves a viscous Hamilton-Jacobi type equation with source terms. Moreover, we do not impose any conditions on the anti-derivative variables of the perturbation around the shock profile. More precisely, it is proved that if the initial perturbation around the viscous shock wave is suitably small in L^2-norm, then the L^2-contraction holds true for the viscous shock wave up to a suitable shift function. Note that BV-norm or the L^∞-norm of the initial perturbation and the shock wave strength can be arbitrarily large. Furthermore, as the time t tends to infinity, the L^2-contraction holds true up to a (spatially homogeneous) time-dependent shift function. In particular, if we choose some special initial perturbations, then L^2-convergence of the solutions towards the associated shock profile can be proved up to a time-dependent shift.

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Contents

1. Introduction and main results

We consider L^2-contraction properties, up to a shift, for viscous shock waves of the multi-dimensional scalar viscous conservation laws

$$\begin{cases} \partial_t u + \operatorname{div} A(u) = \Delta u, \\ u(t=0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}^+$, $x = (x_1, x') \in \mathbb{R} \times \mathbb{T}^{N-1}$ with $\mathbb{T}^{N-1} := \mathbb{R}^{N-1}/\mathbb{Z}^{N-1}$ being $N-1$ dimensional flat torus, $N \geq 2$, $u = u(t,x) \in \mathbb{R}$, and $A(u) = (A_1(u), A_2(u), \cdots, A_N(u))^t \in \mathbb{R}^N$ is a smooth vector field of N fluxes A_i $(i = 1, 2, \cdots, N)$ with A_1 being strictly convex, i.e., $A_1''(u) > 0$, $\forall\, u \in \mathbb{R}$. For the far fields of the initial value $u_0(x)$ on x_1 direction, it is assumed that

$$u_0(x) \to u_\pm, \quad \text{as } x_1 \to \pm\infty,$$

with $u_\pm$ being prescribed constants, and the periodic boundary conditions are imposed on $x' = (x_2, \cdots, x_N) \in \mathbb{T}^{N-1}$. Without loss of generality, we consider the stationary planar shock waves $U(x_1)$ satisfying the ordinary differential equation

$$\begin{cases} (A_1(U))' = U'', \quad ' = \dfrac{d}{dx_1}, \\ U(x_1) \to u_\pm, \quad \text{as } x_1 \to \pm\infty. \end{cases} \tag{1.2}$$

Here, the two end points $u_\pm$ satisfy $u_- > u_+$ by the strict convexity of A_1 and the Lax entropy condition, and the Rankine-Hugoniot condition $A_1(u_+) = A_1(u_-)$. The existence of the planar shock profile to (1.2) is well-known and the profile is unique up to a constant shift (see for example [27]). On the other hand, when the flux A periodically depend on the space variable x, the existence of planar shocks has been proved in [12].

For the multi-dimensional scalar conservation laws (1.1) with or without viscosity, Kruzhkov [31] established the well-known L^1-contraction by using Kruzhkov entropies. However, generally the Kruzhkov semi-group is not a contraction in L^p for any $p > 1$, unless the equation is linear. In this article, we are interested in the L^2-contraction (a L^2-type stability) of large viscous shock waves $U(x_1)$ in (1.2) up to a shift for the multi-dimensional scalar viscous conservation laws (1.1) under rough perturbations by using relative entropy methods. The main point in the proof is the construction of a suitable shift function $Y(t, x)$, which depends both on the time and space variables, and solves a viscous Hamilton-Jacobi type equation (1.7) with source terms. In addition, we do not introduce the anti-derivative variables on the perturbation around the shock profile, which is quite different from the previous well-known results in [16,20,21].

This work follows a program initiated in [47,42,43,34,35,33] concerning the relative entropy method for the study on the stability of inviscid shocks (or viscous shocks) for the system of hyperbolic (or viscous) conservation laws. The relative entropy method, which is purely nonlinear and allows to handle rough perturbations, was first introduced by Dafermos [11] and Diperna [13] to prove the L^2 stability and uniqueness of Lipschitzian solutions to the hyperbolic conservation laws endowed with a convex entropy. In [13], that was also used to get uniqueness of some discontinuous solutions for some particular cases, while no stability result was obtained. Later, Chen-Frid [7,8] and Chen-Frid-Li [9] used this method to prove the uniqueness and asymptotic stability of Riemann solutions to some hyperbolic conservation laws. Concerning the contraction of discontinuous solutions, the method was used by Leger [34] to show the L^2-contraction up to a shift of inviscid shocks to the scalar conservation laws (see also [1] for an extension to L^p, $1 < p < \infty$). That has been extended to the system case in [35,48] for extreme shocks, and general criteria have been developed in Kang-Vasseur [30], Serre-Vasseur [42] for possibly all shocks including intermediate characteristic fields. The relative entropy method is also an effective method for the study of asymptotic limits. One of the first usage of the method in this context is due to Yau [52] for the hydrodynamic limit of Ginzburg-Landau models. Since then, there have been many works in this context, see [2–5,15,28,36,40] and the survey paper in [47], although they are all considering the limit to a smooth (Lipschitz) limit function. Recently, the relative entropy method has been successfully applied to showing the vanishing viscosity limit of the viscous scalar conservation laws to shocks by Choi-Vasseur [10], and the zero dissipation limit of full compressible Navier-Stokes-Fourier system to contact discontinuities by Vasseur-Wang [49]. Furthermore, that has been also successfully used to prove the L^2-contraction of viscous shock profiles to the one-dimensional scalar viscous conservation laws by Kwon-Vasseur [33], up to a time-dependent shift.

This article is the first attempt to obtain the L^2-contraction of large viscous planar shock waves to the multidimensional viscous conservation laws with rough initial perturbations, based on the relative entropy method. The main difficulty is how to deal with the compressibility of the viscous shock wave without introducing the anti-derivative variables. In one-dimensional case, Kang-Vasseur [29] use the only time-dependent shift function to overcome this difficulty. However, that can not be applied to the multi-dimensional case since the perturbation may propagate along the transverse directions $x' = (x_2, \cdots, x_N) \in \mathbb{T}^{N-1}$. Instead, we here construct a spatially inhomogeneous shift function $Y(t, x)$ (defined in (1.7) below) satisfying a viscous Hamilton-

Jacobi type equation with source terms, up to which we have the L^2-contraction of the viscous shock profile to the multidimensional equation (1.1). The main issue related to the shift is to prove the global-in-time existence, and more importantly the large-time behavior of the shift function $Y(t,x)$, which is determined by a spatial-homogeneous function $m(t)$ (defined in (1.5) below). On the other hand, if we choose a special initial perturbation, we have that the special perturbation is L^2-contractive and the solution to (1.1) time-asymptotically converges to the viscous shock wave up to a time-dependent shift. Our results require the initial perturbations being suitably small in $L^2(\mathbb{R}\times\mathbb{T}^{N-1})$ but the shock strength, BV-norm or L^∞-norm of the perturbation can be arbitrarily large.

There are also many literatures concerning the stability of viscous shock wave to the viscous conservation laws in one-dimensional case, based on the weighted characteristic energy methods or point-wise Green function methods by crucially using the anti-derivative variables to the perturbation around the shock profile. In 1960s, Il'in-Oleinik [27] first proved the time-asymptotic stability of viscous shock waves to the scalar equation, see also Sattinger [45]. Goodman [16] and Matsumura-Nishihara [39] independently proved the stability of viscous shock waves to the system case under the zero mass condition on the perturbation about the shock profile. Then, Liu [37] first observed that a generic small perturbation (i.e., without zero mass condition) of a viscous shock profile produces not only a constant translation on the shock profile but also diffusion waves on the transversal fields. Furthermore, Spzessy-Xin [46] introduced the coupled diffusion waves to improve the stability result in [37] and finally removed the zero mass condition in [16]. Recently, Huang-Matsumura [22] and Liu-Zeng [38] removed the zero mass conditions in [39] in the two shocks and a single shock cases, respectively. Very recently, Vasseur-Yao [50] removed the smallness condition on the shock strength in [39] by introducing a new entropy variable. For the multi-dimensional case $N\geq 2$, Goodman [17] first proved the stability of weak shocks based on the anti-derivative variables by introducing a shift function depending on the temporal and transversal spatial variables in H^s-framework. Then Hoff-Zumbrun [20,21] improved the stability result in [17] to the large shock waves by using the detailed point-wise Green function method. Recently, Shi-Wang [44] proved nonlinear stability of the weak scalar planar shock wave with large perturbation in $\mathbb{R}^2$. For the limit to the basic wave patterns of the hyperbolic conservation laws from the various viscous conservation laws system, one can refer to [19], [23–26], [32], [51] and the references therein.

On the other hand, for L^1-type stability, Osher-Ralston [41] proved the L^1-stability of scalar viscous shock wave with some restrictions on the perturbation, while Freistühler-Serre [14] removed the restrictions in [41] and proved L^1-stability of viscous shock wave to the one-dimensional scalar conservation laws (1.1) with arbitrarily large L^1-perturbations. In 2-D case, Goodman-Miller [18] proved the L^1 stability of scalar viscous shock wave in $\mathbb{R}^2$. Note also that Brenier [6] establish a L^2 formulation for the multidimensional scalar hyperbolic conservation laws (without the viscosity term) based on a combination of level-set, kinetic and transport-collapse approximations.

For notational convenience, we will denote the spatial domain by

$$\Omega := \mathbb{R}\times\mathbb{T}^{N-1}.$$

Our first result is the following.

Theorem 1.1. *Let $U(x_1)$ be a planar shock wave defined by* (1.2)*. Then, for any fixed $t_0>0$, there exists a positive constant δ_0 such that the following holds.*

For any initial data u_0 with $\|u_0 - U\|_{L^2(\Omega)} < \delta_0$ and $u_0 \in L^\infty(\Omega)$, there exists a shift function $Y(t,x)$ such that

$$\int_\Omega |u(t,x) - U(x_1 + Y(t,x))|^2 dx$$

is non-increasing in time for $t > t_0$. In addition, there exists a positive constant $C(t_0)$ depending only on t_0 such that

$$\int_\Omega |u(t,x) - U(x_1 + Y(t,x))|^2 dx \le C(t_0) \int_\Omega |u_0(x) - U(x_1)|^2 dx, \quad \forall t > 0. \tag{1.3}$$

Furthermore, we have the following time-asymptotic behavior for the shift $Y(t,x)$:

$$\lim_{t\to\infty} \int_\Omega |U(x_1 + Y(t,x)) - U(x_1 + m(t))|^2 dx = 0, \tag{1.4}$$

where the shift $m(t)$ is defined by

$$m(t) = \frac{\int_\Omega |U'(x_1 + m(t))| Y dx}{\int_\Omega |U'(x_1 + m(t))| dx}. \tag{1.5}$$

Thus,

$$\lim_{t\to\infty} \int_\Omega |u(t,x) - U(x_1 + m(t))|^2 dx \le C(t_0) \int_\Omega |u_0(x) - U(x_1)|^2 dx.$$

The above spatially inhomogeneous shift $Y(t,x)$ can be constructed such that

$$\begin{aligned}
&\|\sqrt{|U'(\cdot + m(t))|}(Y - m(t))\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{|U'(\cdot + m(t))|}\nabla Y\|_{L^2((0,\infty)\times\Omega)} \le C\delta_0,\\
&\|\nabla Y\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\Delta Y\|_{L^2((0,\infty)\times\Omega)} \le C\delta_0,\\
&\|\nabla Y\|_{L^\infty(0,\infty;H^s_{loc}(\Omega))} + \|\Delta Y\|_{L^2(0,\infty;H^s_{loc}(\Omega))} \le C(t_0)\delta_0,
\end{aligned} \tag{1.6}$$

where $s > \frac{N}{2}$ and C is some positive constant independent of $t \in \mathbb{R}^+$ and $C(t_0)$ depending on t_0 but independent of $t \in \mathbb{R}^+$.

Several remarks are listed in the following.

Remark 1.2. Theorem 1.1 holds true for arbitrarily large shock wave and any spatial dimension $N \ge 2$. Moreover, we only assume that the L^2-perturbation $u_0 - U$ is suitably small, while the oscillations of the solution, BV-norm of the perturbation can be arbitrarily large. In addition, we do not impose any conditions on the anti-derivative variables on the perturbation of shock, which is quite different from the previous well-known results in [16,20,21].

Remark 1.3. In proof of Theorem 1.1, we will consider the shift $Y(t,x)$ as a solution of a viscous Hamilton-Jacobi type equation

$$\begin{cases} \partial_t Y - A_1'(U(Y+x_1))\partial_{x_1} Y + \sum_{i=2}^{N} A_i'(U(Y+x_1))\partial_{x_i} Y \\ \quad -A_1'(U(Y+x_1))|\nabla_x Y|^2 + w\cdot\nabla_x Y - \Delta Y \\ \quad = -(w_1 - h_M(t))\psi_M(x_1+m(t)) - h_M(t) - g(t), \\ Y|_{t=0} = 0. \end{cases} \tag{1.7}$$

Here, $w = (w_1, \cdots, w_N)$ is a vector field defined by

$$w = \varphi(t)\frac{A(u|U(Y+x_1))}{u-U(Y+x_1)}, \tag{1.8}$$

where φ is a smooth function such that $0 \le \varphi \le 1$ and

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{t_0}{2}, \\ 1 & \text{if } t > t_0, \end{cases}$$

with t_0 being the arbitrarily fixed positive time in Theorem 1.1.

Moreover, $h_M(t)$ is an average of w_1 as

$$h_M(t) := \frac{1}{2(M+1)} \int_{\mathbb{T}^{N-1}} \int_{|x_1+m(t)|\le M+1} w_1 dx,$$

where $m(t)$ is defined by (1.5), and M is suitably large positive constant, and $\psi_M(x_1)$ is a smooth cut-off function satisfying $0 \le \psi_M \le 1$ and

$$\psi_M(x_1) = \begin{cases} 1 & \text{if } x_1 \in [-M, M], \\ 0 & \text{if } x_1 \in [-(M+1), (M+1)]^c. \end{cases} \tag{1.9}$$

In addition,

$$g(t) = \int_\Omega (u - U(Y+x_1))U'(x_1+m(t))dx.$$

Remark 1.4. In Theorem 1.1, the smallness condition on $u_0 - U$ is only in $L^2(\Omega)$. In addition to Theorem 1.1, we will show that if there exists a constant $\delta_0 > 0$ such that $\|u_0 - U\|_{H^s(\Omega)} < \delta_0$ for $s > \frac{N}{2}$, then the solution u to (1.1) with the initial data u_0 satisfies the L^2-contraction for all $t > 0$, i.e.,

$$\int_\Omega |u(t,x) - U(x_1+Y(t,x))|^2 dx \le \int_\Omega |u_0(x) - U(x_1)|^2 dx, \quad t > 0, \tag{1.10}$$

where the shift $Y(t,x)$ can be constructed as a solution of the equation (1.7) with $\varphi(t)\equiv 1$ for all $t>0$, thus the shift Y satisfies the properties (1.4) and (1.6). As a consequence, we have a time-asymptotic L^2-contraction of the shock up to the spatially homogeneous shift $m(t)$, i.e.,

$$\lim_{t\to\infty}\int_\Omega |u(t,x)-U(x_1+m(t))|^2dx \le \int_\Omega |u_0(x)-U(x_1)|^2dx.$$

Remark 1.5. Thanks to the smallness condition on $\|\nabla Y\|_{L^\infty((0,\infty)\times\Omega)}$ in (1.6), the existence of $m(t)$ can be proved by the implicit function theorem through (1.5). Then it follows from (1.5) that

$$\begin{aligned} m'(t) &= \frac{\partial_t \int_\Omega |U'(x_1+m(t))| Y(t,x_1,x')dx}{\int_\Omega |U'(x_1)|dx} = \frac{\partial_t \int_\Omega |U'(x_1)| Y(t,x_1-m(t),x')dx}{\int_\Omega |U'(x_1)|dx} \\ &= \frac{\int_\Omega |U'(x_1)|\big(Y_t(t,x_1-m(t),x') - m'(t)\partial_{x_1}Y(t,x_1-m(t),x')\big)dx}{\int_\Omega |U'(x_1)|dx}. \end{aligned}$$

Therefore, by (1.7), $m(t)$ satisfies the following ODE

$$\begin{aligned} m'(t)&\int_\Omega |U'(x_1)|\big(1+\partial_{x_1}Y(t,x_1-m(t),x')\big)dx = \int_\Omega |U'(x_1+m(t))|Y_t(t,x_1,x')dx \\ &= \int_\Omega |U'(x_1+m(t))|\Big[A_1'(U(Y+x_1))\partial_{x_1}Y - \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i}Y + A_1'(U(Y+x_1))|\nabla_x Y|^2 \\ &\qquad - w\cdot\nabla_x Y + \Delta Y - (w_1 - h_M(t))\psi_M(x_1+m(t)) - h_M(t) - g(t)\Big]dx, \end{aligned} \tag{1.11}$$

with the initial value

$$m(0)=0.$$

Moreover, if we assume the initial perturbation u_0-U is sufficiently small in $H^s(\Omega)$ with $s\gg 1$, then it can be shown that the shift function $m(t)$ satisfies

$$\lim_{t\to+\infty}\frac{m(t)}{t}=0,$$

which means that $|m(t)|$ is sub-linear as $t\to+\infty$, and implies that our result is consistent with the previous ones in [17,20,21].

We here briefly present a rough idea on the construction of the shift Y as a solution of (1.7). First of all, we observe that the shift function should depend on both temporal and spatial variables, due to the compressibility of the shock profile and the wave propagation along the transversal directions $x'=(x_2,\cdots,x_N)$. In Goodman's classical paper [17], the shift function $\delta(t,x')$ depends on the time t and transversal directions x', and satisfies a diffusion equation and the zero mass condition

$$\int_{\mathbb{R}} (u(x,t) - U(x_1 + \delta(t,x')))dx_1 = 0, \quad \forall t \in \mathbb{R}^+, \forall x' \in \mathbb{T}^{N-1},$$

so that the anti-derivative variable $\int_{-\infty}^{x_1} (u(x,t) - U(x_1+\delta(t,x')))dx_1$ is well-defined in H^s-space and then the compressibility of the shock profile can be fully used. In the present paper, since we want to establish the L^2-contraction of large viscous planar shock waves $U(x_1)$ to the multidimensional viscous conservation laws (1.1), it seems that the anti-derivative method in [17] can not be used. Alternatively, we use the relative entropy method and construct the parabolic equation (1.7) for the shift function $Y(t,x)$, so that the shifted profile $V(t,x) := U(x_1 + Y(t,x))$ satisfies the viscous balance law with a transport term:

$$\begin{cases} V_t + \mathrm{div} A(V) + w \cdot \nabla V - \Delta V = G, \\ V(x, t=0) = U(x_1). \end{cases} \tag{1.12}$$

Here G is some inhomogeneous term and w is defined as (1.8) to get rid of some difficult terms caused by the compressibility of the shock profile. Using (1.12) with suitably chosen G, we can establish the L^2-contraction of the perturbation $u(t,x) - U(x_1 + Y(t,x))$ under the suitably small assumption of L^2-norm of the initial perturbation. Then the remaining issues are the global-in-time existence and large-time behavior of the shift function $Y(t,x)$, which is another difficult part of the present paper.

Our second result is on a special kind of perturbation:

Theorem 1.6. *Let $U(x_1)$ be a planar shock wave defined by (1.2). Then there exists a constant $\delta_0 > 0$ such that the following holds.*

For any function $Y_0 : \Omega \to \mathbb{R}$ with $\|Y_0\|_{L^\infty(\Omega)} < \delta_0$, there exists a solution $Y(t,x)$ to a viscous Hamilton-Jacobi equation

$$\begin{cases} \partial_t Y - A_1'(U(Y+x_1))\partial_{x_1} Y + \sum_{i=2}^{N} A_i'(U(Y+x_1))\partial_{x_i} Y - A_1'(U(Y+x_1))|\nabla_x Y|^2 - \Delta Y = 0, \\ Y|_{t=0} = Y_0. \end{cases} \tag{1.13}$$

Moreover, the solution u to (1.1) with $u_0(x) = U(x_1 + Y_0(x))$ satisfies $u(t,x) = U(x_1 + Y(t,x))$, and $Y(t,x)$ satisfies

$$\|Y\|_{L^\infty((0,\infty)\times\Omega)} + \frac{1}{C}\Big[\|\sqrt{|U'(x_1)|}(Y - c(t))\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{|U'(x_1)|}\nabla Y\|_{L^2((0,\infty)\times\Omega)}\Big] \le \delta_0, \tag{1.14}$$

with the positive constant C independent of the time $t \in \mathbb{R}^+$.

Furthermore, the perturbation $u = U(x_1 + Y(t,x))$ time-asymptotically converges towards the viscous shock wave $U(x_1)$ up to a time-dependent shift $c(t)$, i.e.,

$$\lim_{t\to\infty}\int_{\Omega} |u(t,x) - U(x_1 + c(t))|^2 dx = 0,$$

where $c(t)$ is defined by

$$c(t) = \frac{\int_\Omega |U'(x_1)| Y(t,x) dx}{\int_\Omega |U'(x_1)| dx}. \tag{1.15}$$

Remark 1.7. Notice that since $U'(x_1) \in L^2(\Omega)$ by (2.4) below, we easily see that the specific perturbation $u_0(x) = U(x_1 + Y_0(x))$ is a small L^2-perturbation of the shock profile $U(x_1)$.

Remark 1.8. In Theorem 1.6, $Y(t,x)$ is chosen to satisfy the viscous Hamilton-Jacobi equation (1.13), therefore the maximum principle for Y holds true. However, for the general perturbation case in Theorem 1.1, since the equation (1.7) has the inhomogeneous source terms, it is not obvious to get a global-in-time L^∞-bound for Y. Therefore, we shall perform the higher order localized energy estimates on Y for the general case.

The paper is organized as follows. In the next section, we derive an energy equality based on the relative entropy method, and present basic properties of the shock waves and useful inequalities, which are crucial for our analysis. We will first prove Theorem 1.6 in Section 3, whose proof is simpler than the one of Theorem 1.1, while it presents some instructive ideas to prove the general perturbation case. Section 4 is devoted to the proofs of Theorem 1.1 and the claim in Remark 1.4. There, we first prove the claim of Remark 1.4 and then Theorem 1.1. In Appendix, we present a proof on local-in-time existence of the shift as a solution to (1.7).

2. Preliminaries

In this section, we present an energy equality based on the relative entropy method, and basic properties on the viscous shock waves, and then useful inequalities, which are needed for our analysis in the following sections.

2.1. Relative entropy method

In this part, we present a useful energy equality based on the relative entropy method as follows.

Lemma 2.1. *Let u be the smooth solution of the conservation laws* (1.1)*, and V be a smooth solution of a nonlinear parabolic equation*

$$V_t + \mathrm{div} A(V) - \Delta V + w \cdot \nabla V = G, \tag{2.1}$$

where w and G are some inhomogeneous coefficient functions. Then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |u-V|^2dx+\int_\Omega |\nabla(u-V)|^2dx\\&=-\int_\Omega \Big(A(u|V)-(u-V)w\Big)\cdot\nabla V dx+\int_\Omega (u-V)Gdx.\end{aligned}\tag{2.2}$$

The remaining part is devoted to the proof of Lemma 2.1. Even though our framework is based on the L^2-norm, we here present the general case of the relative entropy $\eta(\cdot|\cdot)$ for a given entropy η. Then, we will focus on the quadratic entropy and explain why the choice of quadratic entropy is essential. Concerning the following relative entropy method, we refer to [29].

For a strictly convex entropy η of the scalar conservation laws (1.1), we define the associated relative entropy function by

$$\eta(u|v)=\eta(u)-\eta(v)-\eta'(v)(u-v),$$

and the relative flux by

$$A(u|v):=A(u)-A(v)-A'(v)(u-v).$$

Let $q(\cdot,\cdot)$ be the flux of the relative entropy defined by

$$q(u,v)=q(u)-q(v)-\eta'(v)(A(u)-A(v)),$$

where q is the entropy flux of η, i.e., $q'=\eta'A'$.

We now investigate the relative entropy between the solution u of (1.1) and the solution V of (2.1). A straightforward computation together with (1.1) and (2.1) yields that

$$\begin{aligned}\partial_t\eta(u|V)&=(\eta'(u)-\eta'(V))\partial_t u-\eta''(V)(u-V)\partial_t V\\&=\underbrace{-(\eta'(u)-\eta'(V))\mathrm{div}A(u)+\eta''(V)(u-V)\mathrm{div}A(V)}_{I}+\eta''(V)(u-V)w\cdot\nabla V\\&\quad+(\eta'(u)-\eta'(V))\Delta u-\eta''(V)(u-V)\Delta V+\eta''(V)(u-V)G.\end{aligned}$$

Since the flux part I above can be written by

$$I=-\mathrm{div}q(u,V)-\eta''(V)A(u|V)\cdot\nabla V,$$

we have

$$\begin{aligned}\partial_t\eta(u|V)&=-\mathrm{div}q(u,V)+(\eta'(u)-\eta'(V))\Delta u-\eta''(V)(u-V)\Delta V\\&\quad-\eta''(V)A(u|V)\cdot\nabla V+\eta''(V)(u-V)w\cdot\nabla V+\eta''(V)(u-V)G.\end{aligned}$$

Then, we integrate the above equality over Ω to get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \eta(u|V)dx \\
&\quad=\int_\Omega \Big((\eta'(u)-\eta'(V))\Delta u-\eta''(V)(u-V)\Delta V\Big)dx \\
&\qquad+\int_\Omega \Big(-\eta''(V)A(u|V)\cdot\nabla V+\eta''(V)(u-V)w\cdot\nabla V\Big)dx+\int_\Omega \eta''(V)(u-V)Gdx.
\end{aligned}
$$

Now, if we consider the quadratic entropy $\eta(u)=\frac{u^2}{2}$, then the parabolic term induces a positive dissipation. Therefore, we have (2.2).

2.2. *Properties of viscous shock wave $U(x_1)$*

We briefly present some well-known properties of shock profile $U(x_1)$, which are crucially used in the proofs of main results. We first mention that the shock profile $U(x_1)$ exponentially converges towards the two end points $u_\pm$. Since $A_1''>0$, it follows from (1.2) that $U(x_1)$ satisfies the compressibility condition

$$U'(x_1)<0, \tag{2.3}$$

and the R-H condition $A_1(u_+)=A_1(u_-)$ and the Lax entropy condition $A_1'(u_+)<0<A_1'(u_-)$ hold true. Thus, there exist positive constants $c_\pm$ such that

$$|U'(x_1)|\sim \exp(-c_\pm|x_1|) \quad \text{as } x_1\to\pm\infty. \tag{2.4}$$

Indeed, since

$$\frac{A_1(U)-A_1(u_\pm)}{U-u_\pm}\to A'(u_\pm) \quad \text{as } U\to u_\pm,$$

it follows from (1.2) that

$$U'=A_1(U)-A_1(u_\pm)\sim A_1'(u_\pm)(U-u_\pm) \quad \text{as } U\to u_\pm,$$

which together with the above Lax condition implies (2.4).

In addition, by the Lax entropy condition, there exists a unique state $u_*\in(u_+,u_-)$ such that

$$A_1'(u_*)=0.$$

Let $U(x_{1*})=u_*$, then it is worth noticing that the monotonicity condition (2.3) together with $A_1''>0$ implies that $|U'(x_1-x_{1*})|$ has a maximum at a unique point x_{1*}, and is increasing as $|x_1-x_{1*}|$ increases. Without loss of generality, we assume $x_{1*}=0$.

2.3. Useful inequalities

In this part, we present two lemmas associated with some weighted Poincaré type inequalities, which are used several times in the following sections.

Lemma 2.2. *Let m be any constant, and $\phi_1(x_1), \phi_2(x_1)$ any integrable functions on $\mathbb{R}$ such that $|x_1|\phi_1(x_1)$ and $|x_1|\phi_2(x_1)$ are all integrable on $\mathbb{R}$, and $\phi_1 \geq 0$, $\int_{\mathbb{R}} \phi_2 dx_1 \neq 0$, $\phi_2 \in L^2(\mathbb{R})$. Then, there exists a positive constant C independent of m (depending only on N, $\|\phi_2\|_{L^1(\mathbb{R})}$, $\|\phi_2\|_{L^2(\mathbb{R})}$, $\int_{\mathbb{R}} \phi_2 dx_1$, $\||\cdot|\phi_2(\cdot)\|_{L^1(\mathbb{R})}$) such that for any function $f : \Omega(= \mathbb{R} \times \mathbb{T}^{N-1}) \to \mathbb{R}$ satisfying $\phi_2(\cdot + m)f \in L^1(\Omega)$ and $\nabla f \in L^2(\Omega)$, the following inequality holds.*

$$\begin{aligned}&\int_\Omega f^2(x)\phi_1(x_1+m)dx \\ &\quad \leq C\int_{\mathbb{R}} (|x_1|+1)\phi_1 dx_1\Big[\Big(\int_\Omega f(x)\phi_2(x_1+m)dx\Big)^2 + \int_\Omega |\nabla f|^2 dx\Big].\end{aligned} \tag{2.5}$$

Proof. Integrating the following identity w.r.t. $y_1 \in \mathbb{R}$,

$$f(x_1,x')\phi_2(y_1+m) = f(y_1,x')\phi_2(y_1+m) + \int_{y_1}^{x_1} \partial_{x_1} f(z_1,x')dz_1\phi_2(y_1+m),$$

we have

$$\begin{aligned}&f(x_1,x')\int_{\mathbb{R}} \phi_2(y_1+m)dy_1 \\ &\quad = \int_{\mathbb{R}} f(y_1,x')\phi_2(y_1+m)dy_1 + \int_{\mathbb{R}}\int_{y_1}^{x_1} \partial_{x_1} f(z_1,x')dz_1\phi_2(y_1+m)dy_1.\end{aligned}$$

Then we have

$$\begin{aligned}&f^2(x_1,x')\Big(\int_{\mathbb{R}} \phi_2 dy_1\Big)^2 \\ &\leq 2\Big(\int_{\mathbb{R}} f(y_1,x')\phi_2(y_1+m)dy_1\Big)^2 + 2\Big(\int_{\mathbb{R}}\int_{y_1}^{x_1} \partial_{x_1} f(z_1,x')dz_1\phi_2(y_1+m)dy_1\Big)^2.\end{aligned}$$

Since $\phi_1 \geq 0$, multiplying the above inequality by $\phi_1(x_1+m)$, and then integrating w.r.t. $x = (x_1,x') \in \Omega$, we have

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}} \phi_2 dy_1\Big)^2 \int_{\Omega} f^2(x)\phi_1(x_1+m)dx \\
&\le 2\int_{\mathbb{R}} \phi_1(x_1+m)dx_1 \int_{\mathbb{T}^{N-1}} \Big(\int_{\mathbb{R}} f(y_1,x')\phi_2(y_1+m)dy_1\Big)^2 dx' \\
&\quad + 2\int_{\Omega} \Big(\int_{\mathbb{R}} \int_{y_1}^{x_1} \partial_{x_1} f(z_1,x')dz_1 \phi_2(y_1+m)dy_1\Big)^2 \phi_1(x_1+m)dx \\
&=: I_1 + I_2.
\end{aligned} \tag{2.6}
$$

Set $H(x') := \int_{\mathbb{R}} f(y_1,x')\phi_2(y_1+m)dy_1$, and $\bar{H}(x') := H(x') - \int_{\mathbb{T}^{N-1}} H(z',t)dz'$. Then, using the Poincaré inequality, we have

$$
\begin{aligned}
\int_{\mathbb{T}^{N-1}} |H(x')|^2 dx' &\le 2\int_{\mathbb{T}^{N-1}} |\bar{H}(x')|^2 dx' + 2\Big(\int_{\mathbb{T}^{N-1}} H(z')dz'\Big)^2 \\
&\le C\int_{\mathbb{T}^{N-1}} |\partial_{x'}\bar{H}(x')|^2 dx' + 2\Big(\int_{\Omega} f(x)\phi_2(x_1+m)dx\Big)^2 \\
&\le C\|\phi_2\|^2_{L^2(\mathbb{R})} \int_{\Omega} |\partial_{x'} f|^2 dx + 2\Big(\int_{\Omega} f(x)\phi_2(x_1+m)dx\Big)^2.
\end{aligned}
$$

This together with $\int_{\mathbb{R}} \phi_1(x_1+m)dx_1 = \int_{\mathbb{R}} \phi_1(x_1)dx_1$ yields

$$
I_1 \le C\|\phi_1\|_{L^1(\mathbb{R})}\Big(\|\phi_2\|^2_{L^2(\mathbb{R})}\int_{\Omega} |\nabla f|^2 dx + 4\Big(\int_{\Omega} f(x)\phi_2(x_1+m)dx\Big)^2\Big).
$$

For the estimate on I_2, since

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}} \int_{y_1}^{x_1} \partial_{x_1} f(z_1,x')dz_1 \phi_2(y_1+m)dy_1\Big)^2 \\
&\le \Big(\int_{\mathbb{R}} \|\partial_{x_1} f(\cdot,x')\|_{L^2(\mathbb{R})} |x_1-y_1|^{1/2}\phi_2(y_1+m)dy_1\Big)^2 \\
&\le \|\partial_{x_1} f(\cdot,x')\|^2_{L^2(\mathbb{R})}\|\phi_2\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} (|x_1+m|+|y_1+m|)|\phi_2(y_1+m)|dy_1 \\
&\le \|\partial_{x_1} f(\cdot,x')\|^2_{L^2(\mathbb{R})}\|\phi_2\|_{L^1(\mathbb{R})}\Big(|x_1+m|\|\phi_2\|_{L^1(\mathbb{R})} + \||\cdot|\phi_2(\cdot)\|_{L^1(\mathbb{R})}\Big),
\end{aligned}
$$

we have

$$I_2 \le C \int_{\mathbb{T}^{N-1}} \|\partial_{x_1} f(\cdot, x')\|^2_{L^2(\mathbb{R})} dx' \int_{\mathbb{R}} \big(|x_1+m| + C\big)\phi_1(x_1+m)dx$$
$$\le C\int_{\mathbb{R}} (|x_1|+1)\phi_1 dx_1 \int_{\Omega} |\nabla f|^2 dx.$$

Therefore, dividing the both sides of (2.6) by $\Big(\int_{\mathbb{R}} \phi_2 dy_1\Big)^2$, we have the desired inequality. □

Remark 2.3. On the right hand side of (2.5), we leave the coefficient $C\int_{\mathbb{R}}(|x_1|+1)\phi_1 dx_1$ explicitly in terms of ϕ_1, because that will be crucially used, for example in (4.7).

Lemma 2.4. *Let $U(x_1)$ be a planar shock wave defined by (1.2), and m any constant. Then there exists a positive constant C independent of m (depending only on $\|U'\|_{L^1(\mathbb{R})}$, $\|\sqrt{\cdot}\sqrt{|U'(\cdot)|}\|_{L^1(\mathbb{R})}$) such that for any function $\widetilde{Y}:\Omega\to\mathbb{R}$ satisfying $\int_\Omega |U'(x_1+m)|\widetilde{Y}(x)dx = 0$, and the integrals $\int_{\mathbb{R}} |U'(x_1+m)||\partial_{x_1}\widetilde{Y}|^2 dx_1$ and $\int_\Omega |U'(x_1+m)||\nabla_{x'}\widetilde{Y}|^2 dx$ be finite, the following inequality holds: $\forall x=(x_1,x')\in\Omega$,*

$$|U'(x_1+m)||\widetilde{Y}(x)|^2$$
$$\le C(|x_1+m| + |U'(x_1+m)|)\int_{\mathbb{R}} |U'(y_1+m)||\partial_{y_1}\widetilde{Y}(y_1,x')|^2 dy_1$$
$$+C|U'(x_1+m)|\int_\Omega |U'(y_1+m)|\Big(\sum_{i=2}^N |\partial_{y_i}\widetilde{Y}(y_1,\cdots,y_i,x_{i+1},\cdots,x_N)|^2\Big)dy.$$

Proof. Since $U'\in L^1(\mathbb{R})$ and

$$\int_\Omega |U'(x_1+m)|\widetilde{Y}(x)dx = 0,$$

we have

$$|U'(x_1+m)||\widetilde{Y}(x)|^2 = |U'(x_1+m)|\Big|\widetilde{Y}(x) - \frac{\int_\Omega |U'(y_1+m)|\widetilde{Y}dy}{\int_\Omega |U'(y_1+m)|dy}\Big|^2$$
$$= \frac{|U'(x_1+m)|}{\|U'\|^2_{L^1(\Omega)}}\Big|\int_\Omega |U'(y_1+m)|\big(\widetilde{Y}(x_1,x') - \widetilde{Y}(y_1,y')\big)dy\Big|^2$$
$$= \frac{|U'(x_1+m)|}{\|U'\|^2_{L^1(\Omega)}}\Big|\int_\Omega |U'(y_1+m)|\Big(\int_{y_1}^{x_1} \partial_{z_1}\widetilde{Y}(z_1,x')dz_1 \qquad (2.7)$$
$$+\sum_{i=2}^N \int_{y_i}^{x_i} \partial_{z_i}\widetilde{Y}(y_1,\cdots,y_{i-1},z_i,x_{i+1},\cdots,x_N)dz_i\Big)dy\Big|^2$$
$$\le \frac{2}{\|U'\|^2_{L^1(\Omega)}}(I_1+I_2+I_3),$$

where

$$I_1 := |U'(x_1+m)|\Big|\int_\Omega |U'(y_1+m)| \int_{-m}^{x_1} \partial_{z_1}\widetilde{Y}(z_1,x')dz_1dy\Big|^2$$
$$I_2 := |U'(x_1+m)|\Big|\int_\Omega |U'(y_1+m)| \int_{y_1}^{-m} \partial_{z_1}\widetilde{Y}(z_1,x')dz_1dy\Big|^2$$
$$I_3 := |U'(x_1+m)|\Big|\int_\Omega |U'(y_1+m)| \sum_{i=2}^N \int_{y_i}^{x_i} \partial_{z_i}\widetilde{Y}(y_1,\cdots,y_{i-1},z_i,x_{i+1},\cdots,x_N)dz_idy\Big|^2. \tag{2.8}$$

Since $|U'(x_1)|$ is decreasing in $|x_1|$, we have

$$\begin{aligned}
I_1 &= \|U'\|^2_{L^1(\Omega)}|U'(x_1+m)|\Big|\int_{-m}^{x_1} \partial_{z_1}\widetilde{Y}(z_1,x')dz_1\Big|^2 \\
&= \|U'\|^2_{L^1(\Omega)}|U'(x_1+m)|\Big|\int_0^{x_1+m} \partial_{z_1}\widetilde{Y}(z_1-m,x')dz_1\Big|^2 \\
&\le \|U'\|^2_{L^1(\Omega)}\Big|\int_0^{|x_1+m|} \sqrt{|U'(z_1)|}|\partial_{z_1}\widetilde{Y}(z_1-m,x')|dz_1\Big|^2 \\
&\le \|U'\|^2_{L^1(\Omega)}|x_1+m|\int_{\mathbb{R}} |U'(z_1+m)||\partial_{z_1}\widetilde{Y}(z_1,x')|^2dz_1.
\end{aligned} \tag{2.9}$$

Likewise, we estimate I_2 as

$$\begin{aligned}
I_2 &= |U'(x_1+m)|\Big|\int_\Omega |U'(y_1+m)| \int_{y_1+m}^{0} \partial_{z_1}\widetilde{Y}(z_1-m,x')dz_1dy\Big|^2 \\
&\le |U'(x_1+m)|\Big|\int_\Omega \sqrt{|U'(y_1+m)|} \int_0^{|y_1+m|} \sqrt{|U'(z_1)|}|\partial_{z_1}\widetilde{Y}(z_1-m,x')|dz_1dy\Big|^2 \\
&\le |U'(x_1+m)|\Big(\int_\Omega \sqrt{|y_1+m|}\sqrt{|U'(y_1+m)|}dy\Big)^2 \int_{\mathbb{R}} |U'(z_1+m)||\partial_{z_1}\widetilde{Y}(z_1,x')|^2dz_1.
\end{aligned} \tag{2.10}$$

Thanks to (2.4), we have

$$I_2 \le C|U'(x_1+m)|\int_{\mathbb{R}} |U'(z_1+m)||\partial_{z_1}\widetilde{Y}(z_1,x')|^2dz_1.$$

Since $x_i, y_i \in \mathbb{T}$ for $i \ge 2$, using the Hölder inequality twice, we have

$$
\begin{aligned}
I_3 &\le |U'(x_1+m)|\Big|\int_\Omega |U'(y_1+m)| \\
&\qquad \times \sum_{i=2}^N \Big(\int_{\mathbb{T}} |\partial_{z_i}\widetilde{Y}(y_1,\cdots,y_{i-1},z_i,x_{i+1},\cdots,x_N)|^2 dz_i\Big)^{\frac12} |x_i-y_i|^{\frac12} dy\Big|^2 \\
&\le C\|U'\|_{L^1(\mathbb{R})}|U'(x_1+m)|\int_\Omega |U'(y_1+m)| \\
&\qquad \times \Big(\sum_{i=2}^N |\partial_{y_i}\widetilde{Y}(y_1,\cdots,y_i,x_{i+1},\cdots,x_N)|^2\Big)dy. \quad \square
\end{aligned}
\tag{2.11}
$$

3. Proof of Theorem 1.6: special perturbation

In this section, we prove Theorem 1.6. A straightforward computation together with (1.13) implies that a special perturbation $u = U(x_1 + Y(t,x))$ is a solution of (1.1), since

$$
\partial_t u + \mathrm{div} A(u) - \Delta u = U'(x_1+Y)\Big(\partial_t Y - A_1'(U(Y+x_1))\partial_{x_1} Y + \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i} Y
$$
$$
- A_1'(U(Y+x_1))|\nabla_x Y|^2 - \Delta Y\Big) = 0.
$$

We now prove the existence of solutions Y to the equation (1.13). The local existence follows the same arguments as in Appendix. For global-in-time estimates, notice that the new variable $\widetilde{Y} := Y - c(t)$, $c(t)$ as in (1.15), satisfies

$$
\begin{cases}
\partial_t \widetilde{Y} - A_1'(U(Y+x_1))\partial_{x_1}\widetilde{Y} + \displaystyle\sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i}\widetilde{Y} \\
\qquad\qquad - A_1'(U(Y+x_1))|\nabla \widetilde{Y}|^2 - \Delta\widetilde{Y} = -c'(t), \\
\widetilde{Y}(t=0,x) = Y_0(x) - c(0) := \widetilde{Y}_0(x).
\end{cases}
\tag{3.1}
$$

Multiplying the above equation by $|U'(x_1)|\widetilde{Y}$, and simple computations yield that

$$
\begin{aligned}
&\partial_t\Big(|U'(x_1)|\frac{\widetilde{Y}^2}{2}\Big) \underbrace{-A_1'(U(x_1))|U'(x_1)|\partial_{x_1}\Big(\frac{\widetilde{Y}^2}{2}\Big)}_{J_1} - \Big[A_1'(U(Y+x_1)) - A_1'(U(x_1))\Big]|U'(x_1)|\widetilde{Y}\partial_{x_1}\widetilde{Y} \\
&+ \underbrace{\sum_{i=2}^N A_i'(U(Y+x_1))|U'(x_1)|\partial_{x_i}\Big(\frac{\widetilde{Y}^2}{2}\Big)}_{J_2} - A_1'(U(Y+x_1))|\nabla Y|^2|U'(x_1)|\widetilde{Y} - \mathrm{div}(|U'(x_1)|\widetilde{Y}\nabla\widetilde{Y}) \\
&+ \partial_{x_1}\Big(\partial_{x_1}|U'(x_1)|\frac{\widetilde{Y}^2}{2}\Big) + |U'(x_1)||\nabla\widetilde{Y}|^2 \underbrace{-\partial^2_{x_1x_1}|U'(x_1)|\frac{\widetilde{Y}^2}{2}}_{J_3} = 0.
\end{aligned}
\tag{3.2}
$$

Since it follows from (1.2) that the shock profile $U'(x_1)$ satisfies that

$$|U'(x_1)|'' = \Big(A_1'(U(x_1))|U'(x_1)|\Big)', \tag{3.3}$$

the summation of the two terms J_1 and J_3 can be computed by

$$J_1 + J_3 = -\partial_{x_1}\Big(A_1'(U(x_1))|U'(x_1)|\frac{\widetilde{Y}^2}{2}\Big).$$

We rewrite the term J_2 as

$$J_2 = \sum_{i=2}^{N} A_i'(U(\widetilde{Y} + x_1 + c(t)))|U'(x_1)|\partial_{x_i}\Big(\frac{(\widetilde{Y} + x_1 + c(t))^2}{2} - (x_1 + c(t))(\widetilde{Y} + x_1 + c(t))\Big),$$

setting $F_i(z) := \int_0^z A_i'(U(s))sds$ and $G_i(z) = \int_0^z A_i'(U(s))ds$ yield that

$$J_2 = \sum_{i=2}^{N} |U'(x_1)|\partial_{x_i}\Big(F_i(\widetilde{Y} + x_1 + c(t)) - (x_1 + c(t))G_i(\widetilde{Y} + x_1 + c(t))\Big),$$

which vanishes after the integration with respect to $x' \in \mathbb{T}^{N-1}$. Thus, integrating (3.2) over Ω yields that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |U'(x_1)|\frac{\widetilde{Y}^2}{2}dx + \int_\Omega |U'(x_1)||\nabla\widetilde{Y}|^2 dx \\
&= \int_\Omega \Big(A_1(U(Y + x_1)) - A_1(U(x_1))\Big)|U'(x_1)|\partial_{x_1}(\frac{\widetilde{Y}^2}{2})dx \\
&\quad + \int_\Omega A_1'(U(Y + x_1))|\nabla\widetilde{Y}|^2|U'(x_1)|\widetilde{Y}dx \\
&:= I_1 + I_2.
\end{aligned} \tag{3.4}$$

Notice that thanks to the maximum principle on the equation (1.13) as

$$\|Y\|_{L^\infty((0,\infty)\times\Omega)} \le \|Y_0\|_{L^\infty(\Omega)},$$

it holds that for any $t \ge 0$,

$$|c(t)| \le \|Y\|_{L^\infty((0,\infty)\times\Omega)} \le \|Y_0\|_{L^\infty(\Omega)},$$

which yields that

$$\|\widetilde{Y}\|_{L^\infty((0,\infty)\times\Omega)} \le \|Y\|_{L^\infty((0,\infty)\times\Omega)} + |c(t)|_{L^\infty(0,\infty)} \le 2\|Y_0\|_{L^\infty(\Omega)}.$$

Therefore, I_2 is estimated as

$$
\begin{aligned}
|I_2| &\le C\|\widetilde{Y}\|_{L^\infty}\int_\Omega |U'(x_1)||\nabla\tilde{Y}|^2 dx \\
&\le 2C\|Y_0\|_{L^\infty}\int_\Omega |U'(x_1)||\nabla\tilde{Y}|^2 dx.
\end{aligned}
$$

For the first term I_1, we use Lemma 2.4 with $m=0$ to estimate

$$
\begin{aligned}
|I_1| \le& \int_\Omega\int_0^1 |U'(x_1+\theta Y)|d\theta|Y||U'(x_1)||\partial_{x_1}\widetilde{Y}||\widetilde{Y}|dx \\
\le& C\|Y\|_{L^\infty}\Big[\int_\Omega |U'(x_1)||\partial_{x_1}\widetilde{Y}|^2 dx + \int_\Omega\int_0^1 |U'(x_1+\theta\widetilde{Y})|^2|U'(x_1)||\widetilde{Y}|^2 d\theta dx\Big] \\
\le& C\|Y_0\|_{L^\infty}\int_\Omega |U'(x_1)||\partial_{x_1}\widetilde{Y}|^2 dx \\
&+ C\|Y_0\|_{L^\infty}\int_\Omega\int_0^1 |U'(x_1+\theta\widetilde{Y})|^2\Big[(|x_1|+|U'(x_1)|)\int_{\mathbb{R}} |U'(y_1)||\partial_{y_1}\widetilde{Y}(t,y_1,x')|^2 dy_1 \\
&+ |U'(x_1)|\int_\Omega |U'(y_1)||\partial_{y'}\widetilde{Y}(t,y)|^2 dy\Big]d\theta dx \\
\le& C\|Y_0\|_{L^\infty}\int_\Omega |U'(x_1)||\nabla\widetilde{Y}|^2 dx.
\end{aligned}
$$

Taking $\|Y_0\|_{L^\infty}\ll 1$ yields that

$$
\frac{d}{dt}\int_\Omega |U'(x_1)|\frac{\widetilde{Y}^2}{2}dx + \int_\Omega |U'(x_1)||\nabla\widetilde{Y}|^2 dx \le 0. \tag{3.5}
$$

Since

$$
\int_\Omega |U'(x_1)|\frac{\widetilde{Y}_0^2}{2}dx \le 2\|\widetilde{Y}_0\|^2_{L^\infty(\Omega)}\int_\Omega |U'(x_1)|dx
$$

we complete the proof of (1.14).

With the weighted estimates (1.14), we can first show the large-time behavior of the shift $\widetilde{Y}$ and then prove the L^2 stability of viscous shock profile for the special perturbation. Set

$$
F(t) := \int |U'(x_1)|^2|\widetilde{Y}(t,x)|^2 dx.
$$

We want to show that

$$\lim_{t\to+\infty} F(t) = 0. \tag{3.6}$$

Using Lemma 2.4 with $m = 0$, and then using (3.5), we have

$$\int_0^\infty F(t)dt \le C \int_0^\infty \int_\Omega |U'(x_1)||\nabla \widetilde{Y}(t,x)|^2 dxdt \le C. \tag{3.7}$$

On the other hand, it follows from (3.5) that $F(t)$ is decreasing in time t, and therefore,

$$\int_0^t |F'(s)|ds \le F(0) - F(t) \le F(0), \ t > 0,$$

which implies that $F' \in L^1(0,+\infty)$.

Therefore, (3.6) holds true. Then we have

$$\begin{aligned}\int_\Omega |U(x_1 + Y(t,x)) - U(x_1 + c(t))|^2 dx &\le C \int_\Omega \int_0^1 |U'(x_1 + \theta Y + (1-\theta)c(t))|^2 |\widetilde{Y}|^2 d\theta dx \\ &\le C \int |U'(x_1)|^2 |\widetilde{Y}|^2 dx \to 0, \ \text{ as } \ t \to +\infty,\end{aligned}$$

which completed the proof of Theorem 1.6.

4. Proof of Theorem 1.1 and Remark 1.4: general perturbation

In this section, we present the proofs for Theorem 1.1 and the claim in Remark 1.4. Since the initial assumption (on smallness of $\|u_0 - U\|_{H^s(\Omega)}$) in Remark 1.4 is stronger than the one in Theorem 1.1, we first prove the claim in Remark 1.4 and then Theorem 1.1 in each step.

As stated in Theorem 1.1 and Remark 1.4, we aim to show that L^2-norm of the perturbation

$$u(t,x) - U(x_1 + Y(t,x))$$

is non-increasing in time.

For that, we first derive an equation on $V(t,x) := U(x_1 + Y(t,x))$. Using (1.2), (1.7) and the chain rule, we find that V satisfies the equation (2.1) with

$$G = U'(Y + x_1)\Big(w_1(1 - \psi_M(x_1 + m(t))) - h_M(t)(1 - \psi_M(x_1 + m(t))) - g(t)\Big),$$

and the initial value $V(0,x) = U(x_1)$. That is,

$$\begin{aligned}
&\partial_t V + \mathrm{div} A(V) + w \cdot \nabla V - \Delta V \\
&\quad = U'(Y + x_1)\Big(w_1(1 - \psi_M(x_1 + m(t))) - h_M(t)(1 - \psi_M(x_1 + m(t))) - g(t)\Big), \\
&V(0, x) = U(x_1).
\end{aligned} \tag{4.1}$$

Therefore, it follows from (2.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u - V|^2 dx + \int_\Omega |\nabla(u - V)|^2 dx \\
&= -\int_\Omega \Big(A(u|V) - (u - V)w\Big) \cdot \nabla V dx \\
&+ \int_\Omega (u - V)U'(Y + x_1)\Big(w_1(1 - \psi_M(x_1 + m(t))) - h_M(t)(1 - \psi_M(x_1 + m(t))) - g(t)\Big) dx.
\end{aligned} \tag{4.2}$$

4.1. A priori estimate on $u - V$

In this part, we first show a L^2-contraction of $u - V$ under an a priori assumption that ∇Y is uniformly small in $(t, x) \in (0, T) \times \Omega$ for any fixed $T > 0$. Then, in the next steps, we shall prove a global-in-time existence of Y in suitable spaces, from which the prior assumption on ∇Y is guaranteed.

We first get a L^2-contraction of $u - V$ in the case of $\varphi \equiv 1$ in (1.7) (for Remark 1.4). In the sequel, T denotes any positive constant.

Lemma 4.1. *Let Y be a solution of (1.7) with $\varphi \equiv 1$ for all $t > 0$. Assume there exists $\varepsilon_0 > 0$ suitably small such that*

$$\|\nabla Y\|_{L^\infty((0,T)\times\Omega)} < \varepsilon_0. \tag{4.3}$$

Then, for all $t \in [0, T]$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u - V)^2 dx + \int_0^T \int_\Omega |\nabla(u - V)|^2 dx dt + \int_0^T \Big(\int_\Omega (u - V)U'(x_1 + m(t)) dx\Big)^2 dt \\
&\quad \le \frac{1}{2}\int_\Omega (u_0 - U)^2 dx.
\end{aligned}$$

Proof. First of all, since $w = \frac{A(u|V)}{u-V}$, it follows from (4.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u - V|^2 dx + \int_\Omega |\nabla(u - V)|^2 dx \\
&= \int_\Omega (u - V)U'(Y + x_1)\Big(w_1(1 - \psi_M(x_1 + m(t))) - h_M(t)(1 - \psi_M(x_1 + m(t))) - g(t)\Big) dx.
\end{aligned}$$

Then, we derive the other dissipation term $\left(\int_\Omega (u-V)U'(x_1+m(t))dx\right)^2$ from the above last term related to $g(t)$ as follows:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u-V|^2 dx + \int_\Omega |\nabla(u-V)|^2 dx + \Big(\int_\Omega (u-V)U'(x_1+m(t))dx\Big)^2 \\
&= -\int_\Omega (u-V)U'(x_1+m(t))dx \int_\Omega (u-V)\big(U'(Y+x_1)-U'(x_1+m(t))\big)dx \\
&\quad + \int_\Omega (u-V)U'(Y+x_1)w_1(1-\psi_M(x_1+m(t)))dx \\
&\quad + \frac{1}{2(M+1)} \int_{|x_1+m(t)|\le M+1} w_1 dx \int_\Omega (u-V)U'(Y+x_1)(1-\psi_M(x_1+m(t)))dx \\
&:= J_1 + J_2 + J_3.
\end{aligned}
\tag{4.4}
$$

In the sequel, we often use the notation $\widetilde{Y}$ to denote $\widetilde{Y} := Y - m(t)$.

We first estimate J_1 as

$$
\begin{aligned}
|J_1| &\le \frac{1}{2}\Big(\int_\Omega (u-V)U'(x_1+m(t))dx\Big)^2 + \frac{1}{2}\Big(\int_\Omega (u-V)\big(U'(Y+x_1)-U'(x_1+m(t))\big)dx\Big)^2 \\
&\le \frac{1}{2}\Big(\int_\Omega (u-V)U'(x_1+m(t))dx\Big)^2 + \frac{1}{2}\underbrace{\Big(\int_\Omega |u-V| \int_0^1 |U''(\theta\widetilde{Y}+x_1+m(t))|d\theta|\widetilde{Y}|dx\Big)^2}_{L}.
\end{aligned}
$$

To control the second term L above, we use the following estimates

$$
\begin{aligned}
|\widetilde{Y}(t,x)| &\le \Big| Y(t,x) - \frac{\int_\Omega |U'(y_1+m(t))|Y dy}{\int_\Omega |U'(y_1+m(t))|dy} \Big| \\
&\le C\int_\Omega |U'(y_1+m(t))||Y(t,x)-Y(t,y)|dy \\
&\le C\|\nabla Y\|_{L^\infty} \int_\Omega |U'(y_1+m(t))|(|x_1+m(t)|+|y_1+m(t)|+C)dy \\
&\le C\varepsilon_0(|x_1+m(t)|+1),
\end{aligned}
\tag{4.5}
$$

where we have used the assumption $\|\nabla Y\|_{L^\infty((0,T)\times\Omega)} < \varepsilon_0$.

Taking ε_0 sufficiently small such that $C\varepsilon_0 < \frac{1}{3}$, we have that for all $\theta \in [0,1]$,

$$
|\theta\widetilde{Y} + x_1 + m(t)| \ge |x_1+m(t)| - |\widetilde{Y}| \ge \frac{2|x_1+m(t)|}{3} - C,
$$

which together with (1.2) and (2.4) implies that

$$|U''(\theta\widetilde{Y}+x_1+m(t))|^{\frac{3}{2}} \le C|U'(\theta\widetilde{Y}+x_1+m(t))|^{\frac{3}{2}} \le C|U'(x_1+m(t))|. \tag{4.6}$$

Therefore, we have

$$\begin{aligned} L &\le \varepsilon_0^2 \int_\Omega |u-V|^2 |U''(\theta\widetilde{Y}+x_1+m(t))|^{\frac{3}{2}} dx \int_\Omega |U''(\theta\tilde{Y}+x_1+m(t))|^{\frac{1}{2}} (|x_1+m(t)|+C)^2 dx \\ &\le C\varepsilon_0^2 \int_\Omega |u-V|^2 |U'(x_1+m(t))| dx. \end{aligned}$$

We now use Lemma 2.2 with taking $\phi_1 = |U'|$ and $\phi_2 = U'$, to get

$$L \le C\varepsilon_0^2 \Big(\int_\Omega (u-V)U'(x_1+m(t))dx \Big)^2 + C\varepsilon_0^2 \|\nabla(u-V)\|_{L^2(\Omega)}^2.$$

For the term J_2, since

$$J_2 = \int_\Omega (u-V)U'(\widetilde{Y}+m(t)+x_1)w_1(1-\psi_M(x_1+m(t)))dx$$

we use the same estimates as the term L to get

$$|J_2| \le C \int_\Omega |u-V|^2 |U'(x_1+m(t))|^{2/3} (1-\psi_M(x_1+m(t)))dx,$$

where we have used $|w| \le C|u-V|$. Then, using Lemma 2.2 with taking $\phi_1 = |U'|^{2/3}(1-\psi_M)$ and $\phi_2 = U'$, we have

$$\begin{aligned} |J_2| &\le C\Big(\int_{\mathbb{R}} (|x_1|+1)|U'|^{2/3}(1-\psi_M)dx_1 \Big) \\ &\quad \times \Big[\Big(\int_\Omega |u-V| U'(x_1+m(t))dx \Big)^2 + \int_\Omega |\nabla(u-V)|^2 dx \Big]. \end{aligned} \tag{4.7}$$

Thanks to the definition of ψ_M (see (1.9)) and (2.4), we take M sufficiently large so that

$$\int_{\mathbb{R}} (|x_1|+1)|U'|^{2/3}(1-\psi_M)dx_1 \ll 1,$$

therefore, we have

$$|J_2| \le \frac{1}{4} \Big(\int_\Omega (u-V)U'(x_1+m(t))dx \Big)^2 + \frac{1}{4} \|\nabla(u-V)\|_{L^2(\Omega)}^2.$$

For J_3, we first see

$$|J_3| \le \frac{C}{2(M+1)} \underbrace{\int\limits_{|x_1+m(t)|\le M+1} |u-V|dx}_{J_{31}} \underbrace{\int\limits_{\Omega} |u-V||U'(Y+x_1)|(1-\psi_M(x_1+m(t)))dx}_{J_{32}}.$$

For the two positive constants c_-, c_+ in (2.4), let $c_m := \min\{c_+, c_-\}$. Then, using (2.4), we have

$$\begin{aligned}|J_{31}| &\le \sqrt{M+1}e^{\frac{c_m}{2}(M+1)}\Big(\int\limits_{0<x_1+m\le M+1} |U'(x_1+m)|^{\frac{c_m}{c_+}}|u-V|^2dx \\ &\qquad + \int\limits_{0>x_1+m\ge -(M+1)} |U'(x_1+m)|^{\frac{c_m}{c_-}}|u-V|^2dx\Big)^{\frac{1}{2}} \\ &\le C\sqrt{M+1}e^{\frac{c_\pm}{2}(M+1)}\Big(\int\limits_{\Omega} \zeta(x_1+m)|u-V|^2dx\Big)^{\frac{1}{2}},\end{aligned} \tag{4.8}$$

where

$$\zeta(x) := \begin{cases} |U'(x)|^{\frac{c_m}{c_+}}, & x>0, \\ |U'(x)|^{\frac{c_m}{c_-}}, & x<0. \end{cases}$$

Then, by Lemma 2.2 with $\phi_1 = \zeta$, $\phi_2 = U'$, we have

$$|J_{31}| \le C\sqrt{M+1}e^{\frac{c_\pm}{2}(M+1)}\Big[\Big(\int\limits_{\Omega}(u-V)U'(x_1+m(t))dx\Big)^2 + \int\limits_{\Omega}|\nabla(u-V)|^2dx\Big]^{\frac{1}{2}}.$$

Likewise, using (4.6), we have that

$$\begin{aligned}|J_{32}| &\le \Big(\int\limits_{\Omega} |u-V|^2|U'(\widetilde{Y}+x_1+m(t))|^{\frac{1}{2}}(1-\psi_M(x_1+m(t)))dx\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\int\limits_{\Omega} |U'(\widetilde{Y}+x_1+m(t))|^{\frac{3}{2}}(1-\psi_M(x_1+m(t)))dx\Big)^{\frac{1}{2}} \\ &\le \Big(\int\limits_{\Omega} |u-V|^2|U'(x_1+m(t))|^{\frac{1}{3}}(1-\psi_M(x_1+m(t)))dx\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\int\limits_{\Omega} |U'(x_1+m(t))|(1-\psi_M(x_1+m(t)))dx\Big)^{\frac{1}{2}} \\ &\le Ce^{-\frac{c_\pm}{2}M}\Big(\int\limits_{\Omega} |u-V|^2|U'(x_1+m(t))|^{\frac{1}{3}}(1-\psi_M(x_1+m(t)))dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.9}$$

Applying Lemma 2.2 with $\phi_1 = |U'|^{1/3}(1-\psi_M)$ and $\phi_2 = U'$ to (4.9), and then taking M sufficiently large such as the previous term J_2, we have

$$|J_3| \le \frac{C}{M}\Big(\int_\Omega (u-V)U'(x_1+m(t))dx\Big)^2 + \frac{C}{M}\|\nabla(u-V)\|^2_{L^2(\Omega)}.$$

Therefore, combining all estimates above together with taking small ε_0 and large M, we have

$$\frac{d}{dt}\int_\Omega (u-V)^2dx + \int_\Omega |\nabla(u-V)|^2dx + \Big(\int_\Omega (u-V)U'(x_1)dx\Big)^2 \le 0,$$

which completes the proof. □

The following Lemma provides a L^2-contraction of $u-V$ when the shift Y is a solution of (1.7).

Lemma 4.2. *For any fixed $t_0 \in (0,T)$, let Y be a solution of (1.7). Assume there exists $\varepsilon_0 > 0$ suitably small such that*

$$\|\nabla Y\|_{L^\infty((0,T)\times\Omega)} < \varepsilon_0. \tag{4.10}$$

Then, for all $t \le t_0$, there exists a constant C_0 depending on t_0 such that

$$\int_\Omega |u(t,x)-V(t,x)|^2dx \le C_0\int_\Omega |u_0(x)-U(x_1)|^2dx,$$

and for all $t \ge t_0$,

$$\frac{d}{dt}\int_\Omega (u-V)^2dx + \int_\Omega |\nabla(u-V)|^2dx + \Big(\int_\Omega (u-V)U'(x_1)dx\Big)^2 \le 0. \tag{4.11}$$

Proof. First of all, since $\varphi(t) = 1$ for all $t \ge t_0$, we have the same estimates as in Lemma 4.1, and thus complete (4.11). On the other hand, since $\varphi(t) < 1$ for all $t < t_0$, we start with (2.2):

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}(u-V)^2dx + \int_\Omega |\nabla(u-V)|^2dx \\
&= -\int_\Omega \Big(A(u|V)-(u-V)w\Big)\cdot\nabla V dx + \int_\Omega (u-V)U'(Y+x_1)w_1(1-\psi_M(x_1+m(t)))dx \\
&\quad - \int_\Omega (u-V)U'(Y+x_1)\Big(h_M(t)(1-\psi_M(x_1+m(t)))+g(t)\Big)dx := I_1+I_2+I_3.
\end{aligned} \tag{4.12}$$

Since $A(u|V) \le C|u-V|^2$, and thus $|w| \le C|u-V|$, the first term I_1 can be estimated as

$$|I_1| \le \int_\Omega |u-V|^2 |U'(Y+x_1)|(|\nabla Y|+1)dx \le C\int_\Omega |u-V|^2 dx,$$

and the second term I_2 can be estimated as

$$|I_2| \le C\int_\Omega |u-V|^2 |U'(Y+x_1)|dx \le C\int_\Omega |u-V|^2 dx.$$

Since $|h_M| \le \frac{C}{\sqrt{M+1}}\|u-V\|_{L^2(\Omega)}$ and $|g| \le C\|u-V\|_{L^2(\Omega)}$, moreover (4.6) yields

$$\Big|\int_\Omega (u-V)U'(Y+x_1)dx\Big| \le C\|u-V\|_{L^2(\Omega)},$$

we have

$$|I_3| \le C\int_\Omega |u-V|^2 dx.$$

Therefore, we can use the Gronwall inequality for $t \le t_0$, which completes the proof. □

4.2. Local existence and a prior estimates on Y

In order to complete a global-in-time L^2-contraction in the Lemmas 4.1 and 4.2, we should close a priori assumptions (4.3) and (4.10) on ∇Y. Therefore, we will prove a global-in-time existence on the shift Y in suitable spaces, on which ∇Y is uniformly small in $(t,x) \in (0,\infty)\times\Omega$. For that, we first present a local-in-time existence as follows. We postpone its proof in Appendix.

Proposition 4.3 *(Local existence). If $u_0 \in L^\infty(\Omega)$, then for any $R > 0$, there exists $T_0 \in (0, \frac{t_0}{2}]$ such that* (1.7) *has a solution Y satisfying*

$$\|\sqrt{|U'(\cdot+m(t))|}Y\|_{L^\infty(0,T_0;L^2(\Omega))} + \|\nabla Y\|_{L^\infty(0,T_0;H^s(\Omega))} + \|\Delta Y\|_{L^2(0,T_0;H^s(\Omega))} \le R, \quad (4.13)$$

where $s > \frac{N}{2}$.

In particular, if $\nabla u_0 \in H^{s-1}(\Omega)$ and $u_0 \in L^\infty(\Omega)$, there exists $T_0 > 0$ such that (1.7) *with $\varphi \equiv 1$ has a solution Y satisfying* (4.13).

To prove the global existence on the shift Y, we use the continuation argument. For that, we present the following a priori estimates.

Proposition 4.4 *(A priori estimates). Let Y be a solution of* (1.7) *with $\varphi \equiv 1$ for all $t > 0$. Assume that there exists $\varepsilon_0 > 0$ suitably small such that*

$$\|\nabla Y\|_{L^\infty(0,T;L^2(\Omega))} + \|\Delta Y\|_{L^2((0,T)\times\Omega)} \le \varepsilon_0, \tag{4.14a}$$

$$\|\nabla Y\|_{L^\infty(0,T;H^s_{loc}(\Omega))} \le \varepsilon_0, \tag{4.14b}$$

$$\|u_0 - U\|_{H^s(\Omega)} \le \varepsilon_0^{3/2}, \quad s > \frac{N}{2}. \tag{4.14c}$$

Then, there exists a positive constant C (depending only on s, N) such that

$$\|\sqrt{|U'(\cdot + m(t))|}(Y - m(t))\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla Y\|_{L^\infty(0,T;L^2(\Omega))} + \|\Delta Y\|_{L^2((0,T)\times\Omega)} \le C\varepsilon_0^{3/2}, \tag{4.15a}$$

$$\|\nabla Y\|_{L^\infty(0,T;H^s_{loc}(\Omega))} + \|\Delta Y\|_{L^2(0,T;H^s_{loc}(\Omega))} \le C\varepsilon_0^{3/2}. \tag{4.15b}$$

Proposition 4.5 *(A priori estimates). For any fixed $t_0 > 0$, let Y be a solution of (1.7). Assume that there exists $\varepsilon_0 > 0$ suitably small such that (4.14a) and (4.14b) with $s > \frac{N}{2}$, and*

$$\|u_0 - U\|_{L^2(\Omega)} \le \varepsilon_0^{3/2}, \quad u_0 \in L^\infty(\Omega). \tag{4.16}$$

Then, there exists $C > 0$ depending only on s, N and t_0 such that (4.15a) and (4.15b).

The next subsections are devoted to the proofs of Proposition 4.4 and Proposition 4.5.

4.3. *Proof of (4.15a) in Proposition 4.4 and 4.5*

We first obtain a weighted L^2 estimates for Y as in the first part of the estimate (4.15a). For that, we use the assumptions (4.14a), (4.14b) and (4.16), but do not need the higher regularity $\nabla(u_0 - U) \in H^{s-1}(\Omega)$ as in (4.14c). Notice that the assumption (4.14c) includes the condition $u_0 \in L^\infty(\Omega)$ in (4.16), thanks to the Sobolev imbedding and the L^∞-bound of the shock U.

Lemma 4.6. *Let Y be a solution of either (1.7) or (1.7) with $\varphi = 1$ for all $t > 0$. Assume (4.14a), (4.14b) and (4.16). Then, there exists a uniform-in-time constant $C > 0$ such that*

$$\int_\Omega |U'(x_1 + m(t))|(Y - m(t))^2 dx + \int_0^t \int_\Omega |U'(x_1 + m(t))||\nabla Y|^2 dx ds \le C\varepsilon_0^3, \quad \forall t \in (0, T]. \tag{4.17}$$

Proof. For notational simplification, we set $\tilde{Y} := Y - m(t)$, and then rewrite the equation (1.7) into the form:

$$\begin{aligned}
&\partial_t \widetilde{Y} - A_1'(U(Y + x_1))\partial_{x_1}\widetilde{Y} + \sum_{i=2}^N A_i'(U(Y + x_1))\partial_{x_i}\widetilde{Y} \\
&\quad - A_1'(U(Y + x_1))|\nabla_x Y|^2 + w \cdot \nabla_x Y - \Delta\widetilde{Y} \\
&\quad = -(w_1 - h_M(t))\psi_M(x_1 + m(t)) - h_M(t) - g(t) - m'(t).
\end{aligned}$$

Multiplying the above equation by $|U'(x_1+m(t))|\widetilde{Y}$, and using the same computations as in Section 3, we have that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|U'(x_1+m(t))|\widetilde{Y}^2 dx + \int_\Omega |U'(x_1+m(t))||\nabla Y|^2 dx = -\int_\Omega U''(x_1+m(t))m'(t)\frac{\widetilde{Y}^2}{2}dx\\
&+\int_\Omega (A_1'(U(Y+x_1)) - A_1'(U(x_1+m(t))))|U'(x_1+m(t))|\widetilde{Y}\partial_{x_1}\widetilde{Y}dx\\
&+\int_\Omega A_1'(U(Y+x_1))|U'(x_1+m(t))|\widetilde{Y}|\nabla Y|^2 dx - \int_\Omega \omega\cdot\nabla Y|U'(x_1+m(t))|\widetilde{Y}dx\\
&-\int_\Omega (w_1-h_M(t))\psi_M(x_1+m(t))|U'(x_1+m(t))|\widetilde{Y}dx\\
&-(h_M(t)+g(t)+m'(t))\int_\Omega |U'(x_1+m(t))|\widetilde{Y}dx := \sum_{i=1}^{6} I_i.
\end{aligned}
\tag{4.18}
$$

Since the assumption (4.14b) implies that $\|\nabla Y\|_{L^\infty((0,T)\times\Omega)} \le C\varepsilon_0 \ll 1$, it follows from (1.11) that

$$
\begin{aligned}
&|m'(t)| \le C\Big[\int_\Omega |U'(x_1+m(t))|\Big(|A_1'(U(Y+x_1))||\partial_{x_1}Y| + \sum_{i=1}^{N}|A_i'(U(Y+x_1))||\partial_{x_i}Y|\Big)dx\\
&+\int_\Omega |U'(x_1+m(t))||A_1'(U(Y+x_1))||\nabla Y|^2 dx + \int_\Omega |U'(x_1+m(t))||\omega||\nabla Y|dx\\
&+\Big|\int_\Omega |U'(x_1+m(t))|\Delta Y dx\Big| + \int_\Omega |U'(x_1+m(t))||\omega_1 - h_M(t)|\psi_M(x_1+m(t))dx\\
&+|h_M(t)| + |g(t)|\Big] := \sum_{i=1}^{7} K_i.
\end{aligned}
\tag{4.19}
$$

First, by Holder inequality, one has

$$
\begin{aligned}
K_1 &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)},\\
K_2 &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)},
\end{aligned}
$$

and

$$
K_3 \le C\|\sqrt{U'(x_1+m(t))}\nabla Y\|_{L^2(\Omega)}\|u-V\|_{L^2(\Omega)}.
$$

For K_4, integration by parts and Holder inequality give that

$$K_4 = |\int_\Omega U''(x_1+m(t))\partial_{x_1} Y dx| \le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}.$$

We use the same argument as in (4.8) with (2.4) to estimate K_6 as

$$\begin{aligned} K_6 &\le C_M \int_\Omega |u-V||U'(x_1+m(t))|^2 dx \le C_M \||U'(x_1+m(t))|(u-V)\|_{L^2(\Omega)} \\ &\le C_M\Big|\int_\Omega U'(x_1+m(t))(u-V)dx\Big| + C_M\|\nabla(u-V)\|_{L^2(\Omega)}, \end{aligned} \tag{4.20}$$

where we have used Lemma 2.2 with $\phi_1 = |U'|$ and $\phi_2 = U'$.

Likewise, we have

$$\begin{aligned} K_5 &\le C\||U'(x_1+m(t))|(u-V)\|_{L^2(\Omega)} + C|h_M(t)| \\ &\le C_M\Big|\int_\Omega U'(x_1+m(t))(u-V)dx\Big| + C_M\|\nabla(u-V)\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, we use the assumption (4.14a) and Lemma 4.1 to get

$$\begin{aligned} |m'(t)| &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}\Big(1+\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)} + \|u-V\|_{L^2(\Omega)}\Big) \\ &\quad + C\Big|\int_\Omega U'(x_1+m(t))(u-V)dx\Big| + C\|\nabla(u-V)\|_{L^2(\Omega)} \\ &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}\Big(1+\|\nabla Y\|_{L^\infty(0,T;L^2(\Omega))} + \|u-V\|_{L^\infty(0,T;L^2(\Omega))}\Big) \\ &\quad + C\Big|\int_\Omega U'(x_1+m(t))(u-V)dx\Big| + C\|\nabla(u-V)\|_{L^2(\Omega)} \\ &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)} \\ &\quad + C\Big|\int_\Omega U'(x_1+m(t))(u-V)dx\Big| + C\|\nabla(u-V)\|_{L^2(\Omega)}. \end{aligned} \tag{4.21}$$

Then, by using the fact (4.5) and Lemma 2.4, we can estimate I_1 as

$$\begin{aligned} |I_1| &\le C|m'(t)|\int_\Omega |U'(x_1+m(t))|\widetilde{Y}^2 dx \\ &\le C|m'(t)|\Big(\int_\Omega |U'(x_1+m(t))|^{\frac{3}{2}}\widetilde{Y}^4 dx\Big)^{\frac{1}{2}} \\ &\le C\varepsilon_0|m'(t)|\Big(\int_\Omega |U'(x_1+m(t))|^{\frac{3}{2}}\widetilde{Y}^2(|x_1+m(t)|^2+1)dx\Big)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon_0 |m'(t)| \|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)} \\
&\le C\varepsilon_0 \|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\Big(\int_\Omega U'(x_1+m(t))(u-V)dx\Big)^2 \\
&\quad + C\|\nabla(u-V)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

For I_2, using (4.6) and Lemma 2.4, we have

$$
\begin{aligned}
|I_2| &= \Big|\int_\Omega \int_0^1 A_1''(U(\theta\widetilde{Y}+x_1+m(t)))U'(\theta\widetilde{Y}+x_1+m(t))d\theta\, \widetilde{Y}|U'(x_1+m(t))|\widetilde{Y}\partial_{x_1}\widetilde{Y}dx\Big| \\
&\le C\int_\Omega |U'(x_1+m(t))|^{5/3}|\widetilde{Y}|^2|\partial_{x_1}\widetilde{Y}|dx \\
&\le C\|\partial_{x_1}\widetilde{Y}\|_{L^2(\Omega)}\||U'(x_1+m(t))|\,|\widetilde{Y}|^2|U'(x_1+m(t))|^{2/3}\|_{L^2(\Omega)} \\
&\le C\sup_{t\in[0,T]}\|\nabla Y\|_{L^2(\Omega)}\|\sqrt{|U'(x_1+m(t))|}\,\nabla Y\|^2_{L^2(\Omega)} \le C\varepsilon_0\|\sqrt{|U'(x_1+m(t))|}\,\nabla Y\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we have used (4.14a) in the last inequality.

Again, we use Lemma 2.4 to estimate

$$
\begin{aligned}
|I_3| &\le C\|U'(x_1+m(t))\widetilde{Y}\|_{L^2(\Omega)}\||\nabla Y|^{1+\frac{2}{N}}\|_{L^2(\Omega)}\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)} \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla\widetilde{Y}\|_{L^2(\Omega)}\|\nabla Y\|^{1+\frac{2}{N}}_{L^{2(1+\frac{2}{N})}(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)} \\
&\le C\sup_{t\in[0,T]}\Big(\|\nabla Y\|^{\frac{2}{N}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}\|\Delta Y\|_{L^2(\Omega)} \\
&\le C\sup_{t\in[0,T]}\Big(\|\nabla Y\|^{\frac{2}{N}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\Big[\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + \|\Delta Y\|^2_{L^2(\Omega)}\Big].
\end{aligned}
$$

Using Sobolev inequality with the assumption (4.14a)-(4.14b), we have

$$
|I_3| \le C\varepsilon_0\Big(\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + \|\Delta Y\|^2_{L^2(\Omega)}\Big).
$$

For I_4, we use Gagliardo-Nirenberg interpolation to estimate

$$
\begin{aligned}
|I_4| &\le \int_\Omega |u-V||\nabla Y||U'(x_1+m(t))||\widetilde{Y}|dx \\
&\le C\||U'(x_1+m(t))|\widetilde{Y}\|_{L^2(\Omega)}\|u-V\|_{L^{\frac{2N(N-1)}{N^2-3N+4}}(\Omega)}\||\nabla Y|^{\frac{2}{N}}\|_{L^{\frac{N-1}{N-2}N}(\Omega)}\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)} \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla\widetilde{Y}\|_{L^2(\Omega)}\|u-V\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla(u-V)\|^{\frac{N-2}{N-1}}_{L^2(\Omega)} \\
&\quad \times \|\nabla Y\|^{\frac{N-2}{N(N-1)}}_{L^2(\Omega)}\|\Delta Y\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}
\end{aligned}
$$

$$\le \sup_{t\in[0,T]}\Big(\|u-V\|_{L^2(\Omega)}^{\frac{1}{N-1}}\|\nabla Y\|_{L^2(\Omega)}^{\frac{N-2}{N(N-1)}}\|\nabla Y\|_{L^\infty(\Omega)}^{1-\frac{2}{N}}\Big)$$
$$\times \|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}\|\nabla(u-V)\|_{L^2(\Omega)}^{\frac{N-2}{N-1}}\|\Delta Y\|_{L^2(\Omega)}^{\frac{1}{N-1}}.$$

Using Young inequality and Lemma 4.1 with assumptions (4.14a), (4.14b) and (4.16), we have

$$|I_4|\le C\varepsilon_0\Big(\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}^2+\|\nabla(u-V)\|_{L^2(\Omega)}^2+\|\Delta Y\|_{L^2(\Omega)}^2\Big).$$

For I_5, we use Poincaré inequality to estimate

$$\begin{aligned}|I_5| &\le C\||U'(x_1+m(t))|\widetilde{Y}\|_{L^2(\Omega)}\Big(\int_{\mathbb{T}^{N-1}}\int_{|x_1+m(t)|\le M+1}(w_1-h(t))dx\Big)^{\frac{1}{2}}\\ &\le C\|\sqrt{|U'(x_1+m(t))|}\nabla\widetilde{Y}\|_{L^2(\Omega)}\|\nabla w_1\|_{L^2(\Omega)}\\ &\le \frac{1}{4}\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}^2+C\|\nabla w_1\|_{L^2(\Omega)}^2.\end{aligned}$$

To estimate $\|\nabla w_1\|_{L^2(\Omega)}$, we notice that since

$$\frac{A(u|V)}{u-V}=(u-V)\int_0^1\int_0^1 A''(V+s\tau(u-V))\tau\, ds d\tau, \tag{4.22}$$

and then

$$\begin{aligned}\nabla\frac{A(u|V)}{u-V} &= \nabla(u-V)\int_0^1\int_0^1 A''(V+s\tau(u-V))\tau\, dsd\tau\\ &\quad+(u-V)\int_0^1\int_0^1 A'''(V+s\tau(u-V))\tau\Big(U'(Y+x_1)(\nabla Y+e_1)+st\nabla(u-V)\Big)dsd\tau,\end{aligned}$$

we have

$$\begin{aligned}\|\nabla w\|_{L^2(\Omega)} &\le C(\|\nabla(u-V)\|_{L^2(\Omega)}+(\|\nabla Y\|_{L^\infty}+1)\|(u-V)U'(Y+x_1)\|_{L^2(\Omega)}\\ &\quad+\|(u-V)\nabla(u-V)\|_{L^2(\Omega)}).\end{aligned}$$

We now use the maximum principle

$$\|u\|_{L^\infty((0,\infty)\times\Omega)}\le\|u_0\|_{L^\infty(\Omega)}. \tag{4.23}$$

Notice that if $u_0-U\in H^s(\Omega)$ with $s>\frac{N}{2}$, and thus $u_0-U\in L^\infty(\Omega)$, we have $u_0\in L^\infty(\Omega)$ thanks to $U\in L^\infty(\Omega)$. Thus, using maximum principle (4.23) and $V\in L^\infty((0,T)\times\Omega)$, we see that

$$\|(u-V)\nabla(u-V)\|_{L^2(\Omega)} \le C\|\nabla(u-V)\|_{L^2(\Omega)}.$$

It now remains to estimate $\|(u-V)U'(Y+x_1)\|_{L^2(\Omega)}$. Using (4.5), (4.6) and Lemma 2.2 with $\phi_1 = |U'|$ and $\phi_2 = U'$, we have

$$\begin{aligned}
&\|(u-V)U'(Y+x_1)\|^2_{L^2(\Omega)} \\
&\quad \le C\Big[\|(u-V)U'(x_1+m)\|^2_{L^2(\Omega)} + \|(u-V)(U'(Y+x_1)-U'(x_1+m))\|^2_{L^2(\Omega)}\Big] \\
&\quad \le C\Big[\|(u-V)U'(x_1+m)\|^2_{L^2(\Omega)} + \|(u-V)\int_0^1 U''(\theta\widetilde{Y}+x_1+m)d\theta\widetilde{Y}\|^2_{L^2(\Omega)}\Big] \\
&\quad \le C\|(u-V)\sqrt{|U'(x_1+m)|}\|^2_{L^2(\Omega)} \\
&\quad \le C\Big(\int_\Omega U'(x_1+m)(u-V)dx\Big)^2 + C\|\nabla(u-V)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.24}$$

Thus

$$|I_5| \le \frac{1}{4}\|\sqrt{|U'(x_1+m)|}\nabla Y\|^2_{L^2(\Omega)} + C\Big(\int_\Omega U'(x_1+m)(u-V)dx\Big)^2 + C\|\nabla(u-V)\|^2_{L^2(\Omega)}.$$

Notice that since $\int_\Omega |U'(x_1+m)|\widetilde{Y}dx = 0$, $I_6 = 0$.

Therefore, combining all estimates above together with Lemma 4.1 and assumptions (4.14a), (4.14b) and (4.16), we have that for all $t \in [0,T]$,

$$\begin{aligned}
&\int_\Omega |U'(x_1+m(t))|\widetilde{Y}^2 dx + \int_0^T\int_\Omega |U'(x_1+m(t))||\nabla Y|^2 dxdt \\
&\quad \le C\int_0^T\Big(\int_\Omega U'(x_1+m(t))(u-V)dx\Big)^2 dt + C\int_0^T\int_\Omega |\nabla(u-V)|^2 dxdt \\
&\qquad + C\varepsilon_0\int_0^T\int_\Omega |\Delta Y|^2 dxdt \\
&\quad \le C\varepsilon_0^3. \quad \square
\end{aligned}$$

The next lemma provides the proof of L^2 estimates on ∇Y in the estimate (4.15a).

Lemma 4.7. *Under the same assumptions as in Lemma 4.6, there exists a uniform-in-time constant $C > 0$ such that*

$$\int_\Omega |\nabla Y|^2 dx + \int_0^T\int_\Omega |\Delta Y|^2 dxds \le C\varepsilon_0^3, \quad \forall t \in [0,T]. \tag{4.25}$$

Proof. Multiplying the equation (1.7) by $-\Delta Y$ and integrating the resulting equation over Ω yield that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla Y|^2 dx + \int_\Omega |\Delta Y|^2 dx \\
&= -\int_\Omega A_1'(U(Y+x_1))\partial_{x_1} Y \Delta Y dx + \int_\Omega \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i} Y \Delta Y dx \\
&\quad - \int_\Omega A_1'(U(Y+x_1))|\nabla Y|^2 \Delta Y dx - \int_\Omega w\cdot \nabla Y \Delta Y dx \\
&\quad - \int_\Omega (w_1 - h_M(t))\psi_M(x_1+m(t))\Delta Y dx := \sum_{i=1}^5 E_i.
\end{aligned} \tag{4.26}$$

We now estimate the five terms on the right hand side of (4.26). First, integration by parts implies that

$$\begin{aligned}
E_1 &= \int_\Omega A_1'(U(Y+x_1))\partial_{x_1}\left(\frac{|\nabla Y|^2}{2}\right)dx + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)\nabla(Y+x_1)\cdot\nabla Y \partial_{x_1} Y dx \\
&= -\int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)(\partial_{x_1}Y+1)\frac{|\nabla Y|^2}{2}dx \\
&\quad + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)\sum_{i=2}^N(\partial_{x_i}Y)^2\partial_{x_1}Y dx \\
&\quad + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)(\partial_{x_1}Y)^3 dx + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)(\partial_{x_1}Y)^2 dx \\
&= -\int_\Omega A_1''(U(Y+x_1))|U'(Y+x_1)|(\partial_{x_1}Y)^2 dx \\
&\quad - \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)\partial_{x_1}Y\frac{|\nabla Y|^2}{2}dx \\
&\quad + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)\sum_{i=2}^N(\partial_{x_i}Y)^2\partial_{x_1}Y dx \\
&\quad + \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)(\partial_{x_1}Y)^3 dx - \int_\Omega A_1''(U(Y+x_1))U'(Y+x_1)\frac{1}{2}\sum_{i=2}^N(\partial_{x_i}Y)^2 dx \\
&:= -\int_\Omega A_1''(U(Y+x_1))|U'(Y+x_1)|(\partial_{x_1}Y)^2 dx + \sum_{i=1}^4 E_{1i}.
\end{aligned} \tag{4.27}$$

Using the same arguments as in previous proofs, we estimate that for each $i = 1, 2, 3$,

$$
\begin{aligned}
|E_{1i}| &\le \frac{1}{8}\int_\Omega A_1''(U(Y+x_1))|U'(Y+x_1)|(\partial_{x_1}Y)^2 dx + C\||\nabla Y|^{1+\frac{2}{N}}\|_{L^2(\Omega)}^2\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)}^2 \\
&\le \frac{1}{8}\int_\Omega A_1''(U(Y+x_1))|U'(Y+x_1)|(\partial_{x_1}Y)^2 dx \\
&\qquad + C\sup_{t\in[0,T]}\Big(\|\nabla Y\|_{L^2(\Omega)}^{\frac{4}{N}}\|\nabla Y\|_{L^\infty(\Omega)}^{2(1-\frac{2}{N})}\Big)\int_\Omega |\Delta Y|^2 dx,
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
E_{14} &= -\int_\Omega A_1''(U(Y+x_1))U'(x_1+m(t))\frac{1}{2}\sum_{i=2}^N(\partial_{x_i}Y)^2 dx \\
&\quad -\int_\Omega A_1''(U(Y+x_1))\big(U'(Y+x_1)-U'(x_1+m(t))\big)\frac{1}{2}\sum_{i=2}^N(\partial_{x_i}Y)^2 dx \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}^2 \\
&\quad -\int_\Omega A_1''(U(Y+x_1))\int_0^1 U''(\theta\widetilde{Y}+x_1+m(t))d\theta\widetilde{Y}\frac{1}{2}\sum_{i=2}^N(\partial_{x_i}Y)^2 dx
\end{aligned}
\tag{4.29}
$$

Since

$$
|U''(\theta\widetilde{Y}+x_1+m(t))\widetilde{Y}| \le C|U'(x_1+m(t))|^{2/3}\varepsilon_0(1+|x_1+m(t)|) \le C\varepsilon_0,
$$

and for each $i = 2, \cdots, N$,

$$
\int_{\mathbb{T}^{N-1}} \partial_{x_i}Y dx' = 0,
$$

we use Poincaré inequality to get

$$
E_{14} \le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|_{L^2(\Omega)}^2 + C\varepsilon_0\int_\Omega |\Delta Y|^2 dx.
$$

Similarly, since

$$
\begin{aligned}
E_2 &= -\int_\Omega \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i}\big(\frac{|\nabla Y|^2}{2}\big)dx \\
&\quad -\int_\Omega \sum_{i=2}^N A_i''(U(Y+x_1))\partial_{x_i}YU'(Y+x_1)\nabla(Y+x_1)\cdot\nabla Y dx \\
&= -\int_\Omega \sum_{i=2}^N A_i''(U(Y+x_1))\partial_{x_i}YU'(Y+x_1)\frac{|\nabla Y|^2}{2}dx \\
&\quad -\int_\Omega \sum_{i=2}^N A_i''(U(Y+x_1))\partial_{x_i}YU'(Y+x_1)\partial_{x_1}Y dx := \sum_{i=1}^2 E_{2i},
\end{aligned}
\tag{4.30}
$$

we estimate

$$
\begin{aligned}
|E_{21}| &\le C\sum_{i=2}^N \|\partial_{x_i}Y\|_{L^2(\Omega)}\||\nabla Y|^{1+\frac{2}{N}}\|_{L^2(\Omega)}\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)} \\
&\le C\|\nabla^2 Y\|_{L^2(\Omega)}\|\nabla Y\|^{1+\frac{2}{N}}_{L^{2(1+\frac{2}{N})}(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)} \\
&\le C\sup_{t\in[0,T]}\Big(\|\nabla Y\|^{\frac{2}{N}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\|\Delta Y\|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
E_{22} &= -\int_\Omega \sum_{i=2}^N A_i''(U(Y+x_1))\partial_{x_i}YU'(x_1+m(t))\partial_{x_1}Y dx \\
&\quad -\int_\Omega \sum_{i=2}^N A_i''(U(Y+x_1)))\partial_{x_i}Y\big(U'(Y+x_1)-U'(x_1+m(t))\big)\partial_{x_1}Y dx \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\int_\Omega\Big|\int_0^1 U''(\theta\widetilde{Y}+x_1+m(t))d\theta\Big||\widetilde{Y}|\sum_{i=2}^N|\partial_{x_i}Y||\partial_{x_1}Y|dx \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\varepsilon_0\int_\Omega\sqrt{|U'(x_1+m(t))|}\sum_{i=2}^N|\partial_{x_i}Y||\partial_{x_1}Y|dx \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\varepsilon_0\sum_{i=2}^N\|\partial_{x_i}Y\|_{L^2(\Omega)}\|\sqrt{|U'(x_1+m(t))|}\partial_{x_1}Y\|_{L^2(\Omega)} \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\varepsilon_0\sum_{i=2}^N\|\partial_{x_ix_i}Y\|_{L^2(\Omega)}\|\sqrt{|U'(x_1+m(t))|}\partial_{x_1}Y\|_{L^2(\Omega)} \\
&\le C\|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + C\varepsilon_0\sum_{i=2}^N\|\partial_{x_ix_i}Y\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.32}
$$

Likewise, we estimate

$$\begin{aligned}|E_3| &\le C\|\Delta Y\|_{L^2(\Omega)}\||\nabla Y|^{1+\frac{2}{N}}\|_{L^2(\Omega)}\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)}\\ &\le \|\Delta Y\|_{L^2(\Omega)}\|\nabla Y\|^{1+\frac{2}{N}}_{L^{2(1+\frac{2}{N})}(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\\ &\le \sup_{t\in[0,T]}\Big(\|\nabla Y\|^{\frac{2}{N}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\|\Delta Y\|^2_{L^2(\Omega)},\end{aligned}\tag{4.33}$$

and

$$\begin{aligned}|E_4| &\le C\int |u-V||\nabla Y||\Delta Y|dx\\ &\le \|u-V\|_{L^{\frac{2N(N-1)}{N^2-3N+4}}(\Omega)}\|\Delta Y\|_{L^2(\Omega)}\||\nabla Y|^{\frac{2}{N}}\|_{L^{\frac{N-1}{N-2}N}(\Omega)}\||\nabla Y|^{1-\frac{2}{N}}\|_{L^\infty(\Omega)}\\ &\le \|u-V\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla(u-V)\|^{\frac{N-2}{N-1}}_{L^2(\Omega)}\|\Delta Y\|_{L^2(\Omega)}\|\nabla Y\|^{\frac{N-2}{N(N-1)}}_{L^2(\Omega)}\|\Delta Y\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\\ &\le \sup_{t\in[0,T]}\Big(\|u-V\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla Y\|^{\frac{N-2}{N(N-1)}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\|\nabla(u-V)\|^{\frac{N-2}{N-1}}_{L^2(\Omega)}\|\Delta Y\|^{\frac{N}{N-1}}_{L^2(\Omega)}\\ &\le \sup_{t\in[0,T]}\Big(\|u-V\|^{\frac{1}{N-1}}_{L^2(\Omega)}\|\nabla Y\|^{\frac{N-2}{N(N-1)}}_{L^2(\Omega)}\|\nabla Y\|^{1-\frac{2}{N}}_{L^\infty(\Omega)}\Big)\Big[\|\nabla(u-V)\|^2_{L^2(\Omega)}+\|\Delta Y\|^2_{L^2(\Omega)}\Big].\end{aligned}\tag{4.34}$$

Using the same estimates as the term I_4 in the proof of Lemma 4.6, we have

$$\begin{aligned}|E_5| &\le C\|\Delta Y\|_{L^2(\Omega)}\Big(\int_{\mathbb{T}^{N-1}}\int_{|x_1+m(t)|\le M+1}(w_1-h(t))dx\Big)^{\frac{1}{2}}\\ &\le \frac{1}{8}\|\Delta Y\|^2_{L^2(\Omega)}+C\Big(\int_\Omega U'(x_1+m(t))(u-V)dx\Big)^2+C\|\nabla(u-V)\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.35}$$

Therefore, combining all estimates above together with Lemma 4.1, 4.6 and assumptions (4.14a), (4.14b) and (4.16), we have that for all $t\in[0,T]$,

$$\begin{aligned}&\int_\Omega |\nabla Y|^2dx+\int_0^T\int_\Omega |\Delta Y|^2dxdt\\ &\le C\int_0^T\int_\Omega |U'(x_1+m(t))||\nabla Y|^2dxdt+C\int_0^T\Big(\int_\Omega U'(x_1+m(t))(u-V)dx\Big)^2dt\\ &\quad +C\int_0^T\int_\Omega |\nabla(u-V)|^2dxdt\\ &\le C\varepsilon_0^3,\end{aligned}$$

which completes the proof. □

4.4. *Proof of* (4.15b) *in Proposition* 4.4

We first complete the proof of Proposition 4.4. We first recall a priori estimates in Lemma 4.1 and Lemma 4.6, 4.7 i.e.,

$$\|u-V\|_{L^\infty(0,T;L^2(\Omega))}+\|\nabla(u-V)\|_{L^2((0,T)\times\Omega)}\le C\varepsilon_0^{3/2}, \tag{4.36}$$

and

$$\|\sqrt{|U'(x_1+m)|}(Y-m)\|_{L^\infty(0,T;L^2(\Omega))}+\|\nabla Y\|_{L^\infty(0,T;L^2(\Omega))}+\|\Delta Y\|_{L^2((0,T)\times\Omega)}\le C\varepsilon_0^{3/2}. \tag{4.37}$$

In order to complete the proof of Proposition 4.4, we need to show higher-order estimates:

$$\|\nabla Y\|_{L^\infty(0,T;H^s_{loc}(\Omega))}+\|\Delta Y\|_{L^2(0,T;H^s_{loc}(\Omega))}\le C\varepsilon_0^{3/2},$$

where the constant $C>0$ depends only on s, N.

For that, we will use the parabolic regularization, which provides a higher regularity estimates: for any fixed T_*,

$$\|\nabla Y\|_{L^\infty(T_*,T;H^s_{loc}(\Omega))}+\|\Delta Y\|_{L^2(T_*,T;H^s_{loc}(\Omega))}\le C(T_*)\varepsilon_0^{3/2},$$

where $C(T_*)$ is a constant independent of ε_0, if T_* does not depend on ε_0. However, we see that the life span T_0 of the local existence in Proposition 4.3 depends on the size of the above norm of Y, according to the proof of Proposition 4.3. Therefore, we will get a shaper local-in-time estimate on Y than the one of Proposition 4.3 up to any fixed time $t_0>0$.

4.4.1. *Local-in-time estimates*

We here get a local-in-time estimate. We first get higher-order estimates on $u-V$, which is used in next step. For any $1\le k\le s$, assume that there exists a constant $C>0$ such that

$$\|u-V\|_{L^\infty(0,t_0;H^{k-1}(\Omega))}+\|\nabla(u-V)\|_{L^2(0,t_0;H^{k-1}(\Omega))}\le C\varepsilon_0^{3/2}, \tag{4.38}$$

and

$$\|\nabla Y\|_{L^\infty(0,t_0;H^{k-1}(\Omega))}+\|\Delta Y\|_{L^2(0,t_0;H^{k-1}(\Omega))}\le C\varepsilon_0^{3/2}. \tag{4.39}$$

We subtract (4.1) from (1.1) to get

$$\begin{aligned}&\partial_t(u-V)+\sum_{i=1}^N\partial_{x_i}(A_i(u)-A_i(V))-w\cdot\nabla V-\Delta(u-V)\\&=-U'(Y+x_1)\Big(w_1(1-\psi_M(x_1))+h_M(t)(1-\psi_M(x_1))+g(t)\Big).\end{aligned} \tag{4.40}$$

A simple computation with (4.40) implies that for all $t \in (0,1)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla^k(u-V)|^2 dx + \int_\Omega |\nabla^{k+1}(u-V)|^2 dxds\\&\quad = \int_\Omega \nabla^{k+1}(u-V)\nabla^{k-1}\Big(\sum_{i=1}^N \partial_{x_i}(A_i(u)-A_i(V)) + w\cdot\nabla V\\&\qquad - U'(Y+x_1)\big(w_1(1-\psi_M(x_1)) + h_M(t)(1-\psi_M(x_1)) + g(t)\big)\Big)dx.\end{aligned}$$

Since

$$\partial_{x_i}(A_i(u)-A_i(V)) = A_i'(u)\partial_{x_i}(u-V) + (A_i'(u)-A'(V))\partial_{x_i}V,$$

we rewrite the terms related to the flux as

$$\begin{aligned}&\int_\Omega \nabla^{k+1}(u-V)\sum_{i=1}^N \nabla^{k-1}\partial_{x_i}(A_i(u)-A_i(V))dx\\&\quad = \int_\Omega \nabla^{k+1}(u-V)\sum_{i=1}^N\Big[\nabla^{k-1}\Big(A_i'(u)\partial_{x_i}(u-V)\Big) + \nabla^{k-1}\Big((A_i'(u)-A_i'(V))\partial_{x_i}V\Big)\Big]dx.\end{aligned}$$

Then, using Sobolev inequality, we estimate

$$\begin{aligned}&\Big|\int_\Omega \nabla^{k+1}(u-V)\sum_{i=1}^N \nabla^{k-1}\partial_{x_i}(A_i(u)-A_i(V))dx\Big|\\&\quad = \Big|\int_\Omega \nabla^{k+1}(u-V)\sum_{i=1}^N\Big[A_i'(u)\nabla^{k-1}\partial_{x_i}(u-V)\\&\qquad + \sum_{1\le l\le k-1}\binom{k-1}{l}\nabla^l A_i'(u)\nabla^{k-1-l}\partial_{x_i}(u-V)\\&\qquad + \sum_{0\le m\le k-1}\binom{k-1}{m}\nabla^m(A_i'(u)-A'(V))\nabla^{k-1-m}\partial_{x_i}V\Big]dx\Big|\\&\quad \le C\|\nabla^{k+1}(u-V)\|_{L^2(\Omega)}\Big[\|\nabla^k(u-V)\|_{L^2(\Omega)}\\&\qquad + \|u-V\|^\alpha_{H^{k-1}(\Omega)}\Big(\|\nabla u\|^\beta_{H^{s-1}(\Omega)} + \|\nabla Y\|^\gamma_{H^{k-1}(\Omega)}\Big)\Big],\end{aligned}$$

where $\alpha,\beta,\gamma\ge 1$ are some constants depending on k. Since $\nabla u_0 \in H^{s-1}(\Omega)$, applying the energy method to (1.1) together with (4.23), we have

$$\|\nabla u(t)\|^2_{H^{s-1}(\Omega)} + \int_0^t \|\nabla^2 u(s)\|^2_{H^{s-1}(\Omega)}ds \le e^{Ct}\|\nabla u_0\|^2_{H^{s-1}(\Omega)}. \tag{4.41}$$

Moreover, since (4.38) and (4.14a) yield that for all $t \le t_0$,

$$\|u - V\|^{\alpha}_{H^{k-1}(\Omega)} \le C\varepsilon_0^{\frac{3\alpha}{2}} \le C\varepsilon_0^{\frac{3}{2}},$$

and

$$\|\nabla Y\|^{\gamma}_{H^{k-1}(\Omega)} \le C,$$

we have that for all $t \le t_0$,

$$\begin{aligned}&\Big|\int_\Omega \nabla^{k+1}(u - V)\sum_{i=1}^{N}\nabla^{k-1}\partial_{x_i}(A_i(u) - A_i(V))dx\Big|\\ &\qquad \le \frac{1}{8}\|\nabla^{k+1}(u - V)\|^2_{L^2(\Omega)} + C\|\nabla^{k}(u - V)\|^2_{L^2(\Omega)} + C\varepsilon_0^3.\end{aligned}$$

Similarly, using (4.22), we have

$$\begin{aligned}&\Big|\int_\Omega \nabla^{k+1}(u - V)\nabla^{k-1}(w \cdot \nabla V)dx\Big|\\ &\le C\|\nabla^{k+1}(u - V)\|_{L^2(\Omega)}\Big[\|u - V\|^{\alpha}_{H^{k-1}(\Omega)}\Big(\|\nabla u\|^{\beta}_{H^{s-1}(\Omega)} + \|\nabla Y\|^{\gamma}_{H^{k-1}(\Omega)}\Big)\Big]\\ &\le \frac{1}{8}\|\nabla^{k+1}(u - V)\|^2_{L^2(\Omega)} + C\varepsilon_0^3.\end{aligned}$$

Likewise, we have

$$\begin{aligned}&\Big|\int_\Omega \nabla^{k+1}(u - V)\nabla^{k-1}\Big(U'(Y + x_1)w_1(1 - \psi_M(x_1))\Big)dx\Big|\\ &\qquad \le \frac{1}{8}\|\nabla^{k+1}(u - V)\|^2_{L^2(\Omega)} + C\varepsilon_0^3.\end{aligned}$$

Moreover, since

$$\begin{aligned}|h_M(t)| &\le C_M\|u - V\|_{L^2(\Omega)},\\ |g(t)| &\le C\|u - V\|_{L^2(\Omega)},\end{aligned}$$

we have

$$\begin{aligned}&\Big|\int_\Omega \nabla^{k+1}(u - V)\nabla^{k-1}\Big(U'(Y + x_1)\big(h_M(t)(1 - \psi_M(x_1)) + g(t)\big)\Big)dx\Big|\\ &\quad \le \frac{1}{8}\|\nabla^{k+1}(u - V)\|^2_{L^2(\Omega)} + C\varepsilon_0^3.\end{aligned}$$

Therefore we have

$$\frac{d}{dt}\int_\Omega |\nabla^k(u-V)|^2 dx + \int_\Omega |\nabla^{k+1}(u-V)|^2 dxds \le C\|\nabla^k(u-V)\|_{L^2(\Omega)}^2 + C\varepsilon_0^3.$$

Using (4.14c) and (4.38), we have that

$$\|\nabla^k(u-V)\|_{L^\infty(0,t_0;L^2(\Omega))} + \|\nabla^{k+1}(u-V)\|_{L^2((0,t_0)\times\Omega))} \le C\varepsilon_0^{3/2},$$

which together with (4.36) and (4.38) implies that

$$\|u-V\|_{L^\infty(0,t_0;H^s(\Omega))} + \|\nabla(u-V)\|_{L^2(0,t_0;H^s(\Omega))} < C\varepsilon_0^{3/2}. \tag{4.42}$$

We next estimate $\nabla^{k+1}Y$ as follows. A straightforward computation for (1.7) with $\varphi \equiv 1$ implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla^{k+1}Y|^2 dx + \int_\Omega |\nabla^{k+2}Y|^2 dxds \\
&= -\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y - \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i}Y \\
&\quad - A_1'(U(Y+x_1))|\nabla Y|^2 - w\cdot\nabla Y + (w_1 - h_M(t))\psi_M(x_1)\Big)dxds.
\end{aligned}$$

We use the same arguments as before, to estimate

$$\begin{aligned}
&\Big|\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y\Big)dx\Big| \\
&= \Big|\int_\Omega \nabla^{k+2}Y\Big[A_1'(U(Y+x_1))\nabla^k\partial_{x_1}Y + \sum_{1\le l\le k}\binom{k}{l}\nabla^l A_1'(U(Y+x_1))\nabla^{k-l}\partial_{x_1}Y\Big]dx\Big| \\
&\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\Big[\|\nabla^{k+1}Y\|_{L^2(\Omega)} + \|\nabla Y\|_{H^{k-1}(\Omega)}^\alpha\Big],
\end{aligned}$$

where $\alpha \ge 1$ is some constant depending on k. Thus, it follows from (4.39) that

$$\Big|\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y\Big)dx\Big| \le \frac{1}{8}\|\nabla^{k+2}Y\|_{L^2(\Omega)}^2 + C\|\nabla^{k+1}Y\|_{L^2(\Omega)}^2 + C\varepsilon_0^3.$$

Likewise, we have

$$
\begin{aligned}
&\Big|\int_\Omega \nabla^{k+2}Y\sum_{i=2}^N \nabla^k\Big(A_i'(U(Y+x_1))\partial_{x_i}Y\Big)dx\Big| \\
&\qquad \le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + C\varepsilon_0^3, \\
&\Big|\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Y+x_1))|\nabla Y|^2\Big)dxds\Big| \\
&\qquad \le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + C\varepsilon_0^3.
\end{aligned}
$$

Using (4.22) and (4.42), we estimate that for some constants $\alpha, \beta \ge 1$,

$$
\begin{aligned}
\Big|\int_\Omega \nabla^{k+2}Y\nabla^k(w\cdot\nabla Y)dx\Big| &\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\|u-V\|^\alpha_{H^k(\Omega)}\|\nabla Y\|^\beta_{H^{k-1}(\Omega)} \\
&\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\varepsilon_0^3,
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_\Omega \nabla^{k+2}Y\nabla^k\Big((w_1-h_M(t))\psi_M(x_1)\Big)dx\Big| &\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\|u-V\|_{H^k(\Omega)} \\
&\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\varepsilon_0^3.
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt}\int_\Omega |\nabla^{k+1}Y|^2dx + \int_\Omega |\nabla^{k+2}Y|^2dxds \le C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + C\varepsilon_0^3.
$$

Using (4.39) and $Y|_{t=0}=0$, we have

$$
\|\nabla^{k+1}Y\|_{L^\infty(0,t_0;L^2(\Omega))} + \|\nabla^{k+2}Y\|_{L^2((0,t_0)\times\Omega))} \le C\varepsilon_0^{3/2},
$$

which together with (4.37) and (4.39) implies that

$$
\|\nabla Y\|_{L^\infty(0,t_0;H^s(\Omega))} + \|\Delta Y\|_{L^2(0,t_0;H^s(\Omega))} \le C\varepsilon_0^{3/2}.
$$

4.4.2. Global-in-time estimates

In order to complete the proof of Proposition 4.4, we need to show global-in-time estimates:

$$
\|\nabla Y\|_{L^\infty(t_0,T;H^s_{loc}(\Omega))} + \|\Delta Y\|_{L^2(t_0,T;H^s_{loc}(\Omega))} < C\varepsilon_0^{3/2},
$$

where the positive constant $C > 0$ depends on s, N. To this end, we use the parabolic regularization.

We first get higher-order estimates on $u - V$, which is used in estimates for Y.

For any $r > 0$, we set $Q_r := (-\frac{1}{r}, 0) \times \Omega_r$, $\Omega_r := (-\frac{1}{r}, \frac{1}{r}) \times \mathbb{T}^{N-1}$. Define smooth functions ϕ_r satisfying $0 \le \phi_r \le 1$ and

$$\phi_r(t,y) = \begin{cases} 1 & \text{if } (t,x) \in Q_r, \\ 0 & \text{if } (t,x) \in Q_{r-1}^c. \end{cases}$$

For any $1 \le k \le s$, assume that

$$\|u - V\|_{L^\infty(-\frac{1}{k},0;H^{k-1}(\Omega_k))} + \|\nabla(u - V)\|_{L^2(-\frac{1}{k},0;H^{k-1}(\Omega_k))} < C\varepsilon_0^{3/2}. \tag{4.43}$$

A simple computation with (4.40) implies that for all $t \in (-\frac{1}{k}, 0)$,

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_k} \phi_{k+1}^2 |\nabla^k(u-V)|^2 dx + \int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \phi_{k+1}^2 |\nabla^{k+1}(u-V)|^2 dxds \\ &= \int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \Big[\phi_{k+1}\partial_t\phi_{k+1}|\nabla^k(u-V)|^2 - 2\phi_{k+1}\nabla\phi_{k+1}\nabla^k(u-V)\nabla^{k+1}(u-V) \\ &\quad + \nabla(\phi_{k+1}^2\nabla^k(u-V))\nabla^{k-1}\Big(\sum_{i=1}^N \partial_{x_i}(A_i(u) - A_i(V)) + w\cdot\nabla V \\ &\quad - U'(Y+x_1)\big(w_1(1-\psi_M(x_1+m(t))) + h_M(t)(1-\psi_M(x_1+m(t))) + g(t)\big)\Big)\Big]dxds. \end{aligned}$$

The assumption (4.43) yields that

$$\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \phi_{k+1}\partial_t\phi_{k+1}|\nabla^k(u-V)|^2 dxds\Big| \le C\|\nabla^k(u-V)\|_{L^2(Q_k)}^2 \le C\varepsilon_0^3,$$

and

$$\begin{aligned} &\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \phi_{k+1}\nabla\phi_{k+1}\nabla^k(u-V)\nabla^{k+1}(u-V)dxds\Big| \\ &\quad \le \frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \phi_{k+1}^2|\nabla^{k+1}(u-V)|^2 dxds + C\|\nabla^k(u-V)\|_{L^2(Q_k)}^2 \\ &\quad \le \frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \phi_{k+1}^2|\nabla^{k+1}(u-V)|^2 dxds + C\varepsilon_0^3, \end{aligned}$$

where the constants C appeared here and below depend on k.

We use the same arguments as the local-in-time estimates to get

$$
\begin{aligned}
&\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k} \nabla(\phi_{k+1}^2\nabla^k(u-V))\sum_{i=1}^{N}\nabla^{k-1}\partial_{x_i}(A_i(u)-A_i(V))dxds\Big| \\
&=\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\Big[2\phi_{k+1}\nabla\phi_{k+1}\nabla^k(u-V)+\phi_{k+1}^2\nabla^{k+1}(u-V)\Big] \\
&\quad\cdot\sum_{i=1}^{N}\Big[\nabla^{k-1}\Big(A_i'(u)\partial_{x_i}(u-V)\Big)+\nabla^{k-1}\Big((A_i'(u)-A'(V))\partial_{x_i}V\Big)\Big]dxds \\
&\le C\int_{-\frac{1}{k}}^{t}\Big[\|\nabla^k(u-V)\|_{L^2(\Omega_k)}+\Big(\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dx\Big)^{1/2}\Big] \\
&\quad\times\Big[\|\nabla^k(u-V)\|_{L^2(\Omega_k)}+\|u-V\|_{H^{k-1}(\Omega_k)}^{\alpha}\Big(\|\nabla u\|_{H^{s-1}(\Omega_k)}^{\beta}+\|\nabla Y\|_{H^{k-1}(\Omega_k)}^{\gamma}\Big)\Big]ds,
\end{aligned}
$$

where $\alpha,\beta,\gamma\ge 1$ are some constants depending on k. Thanks to (4.36), applying the parabolic regularization to (1.1) together with (4.23), we have

$$
u\in L^\infty(-\frac{1}{k},0;H^s(\Omega_k)),
$$

which together with (4.43) and (4.14b) implies that

$$
\begin{aligned}
&\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\nabla(\phi_{k+1}^2\nabla^k(u-V))\sum_{i=1}^{N}\nabla^{k-1}\partial_{x_i}(A_i(u)-A_i(V))dxds\Big| \\
&\qquad\le\frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dxds+C\varepsilon_0^3.
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
&\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\nabla(\phi_{k+1}^2\nabla^k(u-V))\nabla^{k-1}(w\cdot\nabla V)dxds\Big| \\
&\qquad\le\frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dxds+C\varepsilon_0^3,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\nabla(\phi_{k+1}^2\nabla^k(u-V))\nabla^{k-1}\Big(U'(Y+x_1)w_1(1-\psi_M(x_1))\Big)dxds\Big|\\
&\quad\le\frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dxds+C\varepsilon_0^3,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\nabla(\phi_{k+1}^2\nabla^k(u-V))\nabla^{k-1}\Big(U'(Y+x_1)\big(h_M(t)(1-\psi_M(x_1+m(t)))+g(t)\big)\Big)dxds\Big|\\
&\quad\le\frac{1}{8}\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dxds+C\varepsilon_0^3.
\end{aligned}
$$

Therefore, we get

$$
\int_{\Omega_k}\phi_{k+1}^2|\nabla^k(u-V)|^2dx+\int_{-\frac{1}{k}}^{t}\int_{\Omega_k}\phi_{k+1}^2|\nabla^{k+1}(u-V)|^2dxds<C\varepsilon_0^3.
$$

Hence we have

$$
\|\nabla^k(u-V)\|_{L^\infty(-\frac{1}{k+1},0;L^2(\Omega_{k+1}))}+\|\nabla^{k+1}(u-V)\|_{L^2((-\frac{1}{k+1},0)\times\Omega_{k+1})}<C\varepsilon_0^{3/2},
$$

which together with (4.36) and (4.43) implies that for all $0\le k\le s$,

$$
\|u-V\|_{L^\infty(-\frac{1}{k+1},0;H^k(\Omega_{k+1}))}+\|\nabla(u-V)\|_{L^2(-\frac{1}{k+1},0;H^k(\Omega_{k+1}))}<C\varepsilon_0^{3/2}. \tag{4.44}
$$

We next estimate $\nabla^{k+1}Y$ as follows. Using the same notations and arguments as before, for any $1\le k\le s$, assume that

$$
\|\nabla Y\|_{L^\infty(-\frac{1}{k+1},0;H^{k-1}(\Omega_{k+1}))}+\|\Delta Y\|_{L^2(-\frac{1}{k+1},0;H^{k-1}(\Omega_{k+1}))}<C\varepsilon_0^{3/2}. \tag{4.45}
$$

A straightforward computation for (1.7) with $\varphi\equiv 1$ implies that for all $t\in(-\frac{1}{k+1},0)$,

$$\frac{1}{2}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+1}Y|^2dx + \int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2dxds$$
$$= \int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \Big[\phi_{k+2}\partial_t\phi_{k+2}|\nabla^{k+1}Y|^2 - 2\phi_{k+2}\nabla\phi_{k+2}\nabla^{k+1}Y\nabla^{k+2}Y$$
$$- \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y - \sum_{i=2}^{N} A_i'(U(Y+x_1))\partial_{x_i}Y$$
$$- A_1'(U(Y+x_1))|\nabla Y|^2 - w\cdot\nabla Y + (w_1 - h_M(t))\psi_M(x_1+m(t))\Big)\Big]dxds.$$

We follow the same arguments as in the previous step. Again, every constant C below depends on k.

The assumption (4.45) yields that

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}\partial_t\phi_{k+2}|\nabla^{k+1}Y|^2dxds\Big| \le C\|\nabla^{k+1}Y\|^2_{L^2(Q_{k+1})} < C\varepsilon_0^3,$$

and

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}\nabla\phi_{k+2}\nabla^{k+1}Y\nabla^{k+2}Ydxds\Big|$$
$$\le \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2dxds + C\|\nabla^{k+1}Y\|^2_{L^2(Q_{k+1})}$$
$$< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2dxds + C\varepsilon_0^3.$$

For other terms related to the flux, we use Hölder inequality and Sobolev inequality together with (4.14a), to get

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y\Big)dxds\Big|$$
$$= \Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \Big[2\phi_{k+2}\nabla\phi_{k+2}\nabla^{k+1}Y + \phi_{k+2}^2\nabla^{k+2}Y\Big]$$
$$\cdot\Big[A_1'(U(Y+x_1))\nabla^k\partial_{x_1}Y + \sum_{1\le l\le k}\binom{k}{l}\nabla^l A_1'(U(Y+x_1))\nabla^{k-l}\partial_{x_1}Y\Big]dxds\Big|$$

$$\le C \int_{-\frac{1}{k+1}}^{t} \Big[\|\nabla^{k+1}Y\|_{L^2(\Omega_{k+1})} + \Big(\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2 dx\Big)^{1/2}\Big]$$
$$\cdot \Big[\|\nabla^{k+1}Y\|_{L^2(\Omega_{k+1})} + \|\nabla Y\|^{\alpha}_{H^s(\Omega_{k+1})}\Big] ds,$$

where $\alpha \ge 1$ is some constant depending on k. Thus, it follows from (4.45) and (4.14b) that

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y\Big)dxds\Big|$$
$$< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2 dxds + C\varepsilon_0^3.$$

Similarly, we have

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\sum_{i=2}^{N}\nabla^k\Big(A_i'(U(Y+x_1))\partial_{x_i}Y\Big)dxds\Big|$$
$$< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2 dxds + C\varepsilon_0^3,$$

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k\Big(A_1'(U(Y+x_1))|\nabla Y|^2\Big)dxds\Big|$$
$$< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2 dxds + C\varepsilon_0^3.$$

Using (4.22), (4.45) and (4.14b), we have

$$\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k(w\cdot\nabla Y)dxds\Big|$$
$$\le C\int_{-\frac{1}{k+1}}^{t}\Big[\|\nabla^{k+1}Y\|_{L^2(\Omega_{k+1})} + \Big(\int_{\Omega_{k+1}}\phi_{k+2}^2|\nabla^{k+2}Y|^2dx\Big)^{1/2}\Big]\|u-V\|^{\alpha}_{H^k(\Omega_{k+1})}\|\nabla Y\|_{H^s(\Omega_{k+1})}ds$$
$$< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \phi_{k+2}^2|\nabla^{k+2}Y|^2 dxds + C\varepsilon_0^3,$$

and

$$\begin{aligned}
&\Big|\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}} \nabla(\phi_{k+2}^2\nabla^{k+1}Y)\nabla^k\Big((w_1-h_M(t))\psi_M(x_1+m(t))\Big)dxds\Big| \\
&\le C\int_{-\frac{1}{k+1}}^{t}\Big[\|\nabla^{k+1}Y\|_{L^2(\Omega_{k+1})}+\Big(\int_{\Omega_{k+1}}\phi_{k+2}^2|\nabla^{k+2}Y|^2dx\Big)^{1/2}\Big]\|u-V\|_{H^k(\Omega_{k+1})}ds \\
&< \frac{1}{8}\int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}}\phi_{k+2}^2|\nabla^{k+2}Y|^2dxds + C\varepsilon_0^3.
\end{aligned}$$

Therefore, we have

$$\int_{\Omega_{k+1}}\phi_{k+2}^2|\nabla^{k+1}Y|^2dx + \int_{-\frac{1}{k+1}}^{t}\int_{\Omega_{k+1}}\phi_{k+2}^2|\nabla^{k+2}Y|^2dxds < C\varepsilon_0^3.$$

Thus,

$$\|\nabla^{k+1}Y\|_{L^\infty(-\frac{1}{k+2},0;L^2(\Omega_{k+2}))}+\|\nabla^{k+2}Y\|_{L^2((-\frac{1}{k+2},0)\times\Omega_{k+2})}<C\varepsilon_0^{3/2},$$

which together with (4.37) and (4.45) implies that

$$\|\nabla Y\|_{L^\infty(-\frac{1}{s+2},0;H^s(\Omega_{s+2}))}+\|\Delta Y\|_{L^2(-\frac{1}{s+2},0;H^s(\Omega_{s+2}))}<C\varepsilon_0^{3/2}.$$

This implies that there exists $C>0$ depending only on s, N such that

$$\|\nabla Y\|_{L^\infty(t_0,T;H^s_{loc}(\Omega))}\le C\varepsilon_0^{3/2}. \tag{4.46}$$

4.5. *Proof of* (4.15b) *in Proposition* 4.5

First of all, we use the same argument to get local-in-time estimates on Y. For any fixed $t_0>0$, and $1\le k\le s$, assume that there exists $C>0$ such that

$$\|\nabla Y\|_{L^\infty(0,\frac{t_0}{2};H^{k-1}(\Omega))}+\|\Delta Y\|_{L^2(0,\frac{t_0}{2};H^{k-1}(\Omega))}\le C\varepsilon_0^{3/2}. \tag{4.47}$$

A simple computation with (1.7) implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla^{k+1}Y|^2 dx + \int_\Omega |\nabla^{k+2}Y|^2 dxds$$
$$= -\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Y+x_1))\partial_{x_1}Y - \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i}Y$$
$$- A_1'(U(Y+x_1))|\nabla Y|^2\Big)dxds$$

Notice that $w = 0$ for all $t \le \frac{t_0}{2}$ (see (1.8)), therefore, we do not need to estimate $u - V$ unlike the proof of Proposition 4.4.

Hence, using the same arguments together with (4.47) as before, we get

$$\|\nabla^{k+1}Y\|_{L^\infty(0,\frac{t_0}{2};L^2(\Omega))} + \|\nabla^{k+2}Y\|_{L^2((0,\frac{t_0}{2})\times\Omega))} < C\varepsilon_0^{3/2},$$

which together with (4.37) and (4.47) implies that

$$\|\nabla Y\|_{L^\infty(0,\frac{t_0}{2};H^s(\Omega))} + \|\Delta Y\|_{L^2(0,\frac{t_0}{2};H^s(\Omega))} < C\varepsilon_0^{3/2}.$$

On the other hand, since the initial condition (4.16) has been used in the global-in-time estimate (4.46), we have, under the assumption (4.16), the same result as

$$\|\nabla Y\|_{L^\infty(\frac{t_0}{2},T;H^s_{loc}(\Omega))} \le C\varepsilon_0^{3/2}.$$

4.6. *Proof of Theorem 1.1*

4.6.1. *Global-in-time existence of the shift Y and contraction of the perturbation $u - V$*

First of all, Proposition 4.3 implies that

$$\|\sqrt{|U'(\cdot+m(t))|}(Y-m(t))\|_{L^\infty(0,T_0;L^2(\Omega))} \le \|\sqrt{|U'(\cdot+m(t))|}Y\|_{L^\infty(0,T_0;L^2(\Omega))} + C|m(t)|$$
$$\le C\|\sqrt{|U'(\cdot+m(t))|}Y\|_{L^\infty(0,T_0;L^2(\Omega))}.$$

Thanks to Proposition 4.5, we use the continuation argument to conclude that there exists $\delta_0 > 0$ sufficiently small such that if $\|u - U\|_{L^2(\Omega)} < \delta_0$ and $u_0 \in L^\infty(\Omega)$, then there exists C depending only on s, N such that

$$\|\sqrt{|U'(\cdot+m(t))|}(Y-m(t))\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{|U'(\cdot+m(t))|}\nabla Y\|_{L^2((0,\infty)\times\Omega)} \le C\delta_0$$
$$\|\nabla Y\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\Delta Y\|_{L^2((0,\infty)\times\Omega)} + \|\nabla Y\|_{L^\infty(0,\infty;H^s_{loc}(\Omega))} + \|\Delta Y\|_{L^2(0,\infty;H^s_{loc}(\Omega))} \le C\delta_0. \tag{4.48}$$

In particular, since the Sobolev imbedding implies that

$$\|\nabla Y\|_{L^\infty((0,\infty)\times\Omega)} \le \|\nabla Y\|_{L^\infty(0,\infty;H^s_{loc}(\Omega))},$$

it follows from Lemma 4.2 that for all $t \le t_0$, there exists a constant C_0 depending t_0 such that

$$\int_\Omega |u(t,x) - V(t,x)|^2 dx \le C_0 \int_\Omega |u_0(x) - U(x_1)|^2 dx,$$

and for all $t > t_0$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u-V)^2 dx + \int_0^\infty\int_\Omega |\nabla(u-V)|^2 dxdt + \int_0^\infty \Big(\int_\Omega (u-V)U'(x_1)dx\Big)^2 dt \\
&\quad \le \frac{1}{2}\int_\Omega (u(t_0,x) - U(x))^2 dx.
\end{aligned} \tag{4.49}$$

Likewise, thanks to Proposition 4.4 and Lemma 4.1, we have the contraction estimate (1.10) together with (4.48).

4.6.2. Large-time behavior of the shift Y

We here use the same notation $\widetilde{Y}$ as in proof of Lemma 4.6 to denote $\widetilde{Y} = Y - m(t)$.

Set

$$f(t) := \int_\Omega |U(Y + x_1) - U(x_1 + m(t))|^2 dx. \tag{4.50}$$

We want to show that

$$\lim_{t\to+\infty} f(t) = 0. \tag{4.51}$$

To this end, we show that f and f' are both integrable over $[0,\infty)$.

First of all, using the same argument as (4.5)-(4.6), and then Lemma 2.4, we estimate

$$\begin{aligned}
\int_0^\infty f(t)dt &= \int_0^\infty\int_\Omega \Big|\int_0^1 U'(\theta\widetilde{Y} + x_1 + m(t))d\theta\Big|^2 |\widetilde{Y}|^2 dxdt \\
&\le C\int_0^\infty\int_\Omega |U'(x_1 + m(t))||\nabla\widetilde{Y}|^2 dxdt.
\end{aligned}$$

Then, (4.48) yields

$$\int_0^\infty f(t)dt < \infty. \tag{4.52}$$

On the other hand, using the same arguments as in the proof of Lemma 4.6, we estimate

$$\int_0^\infty |f'(t)|dt$$

$$= \int_0^\infty \Big| \int_\Omega 2\Big(U(Y+x_1) - U(x_1+m(t))\Big)\Big(U'(Y+x_1)\partial_t Y - U'(x_1+m(t))m'(t)\Big)dx\Big| dt$$

$$= \int_0^\infty \Big| \int_\Omega 2\Big(U(Y+x_1) - U(x_1+m(t))\Big)\Big[U'(Y+x_1)\Big(A_1'(U(Y+x_1))\partial_{x_1} Y$$

$$- \sum_{i=2}^N A_i'(U(Y+x_1))\partial_{x_i} Y + A_1'(U(Y+x_1))|\nabla_x Y|^2 + w\cdot\nabla_x Y + \Delta Y$$

$$+ (w_1 - h_M(t))\psi_M(x_1+m(t)) + h_M(t) + g(t) - U'(x_1+m(t))m'(t)\Big]dx\Big| dt$$

$$\le C\int_0^\infty \Big[\|U(Y+x_1) - U(x_1+m(t))\|^2_{L^2(\Omega)} + \|\sqrt{|U'(x_1+m(t))|}\nabla Y\|^2_{L^2(\Omega)} + \|\Delta Y\|^2_{L^2(\Omega)}$$

$$+ \|\nabla(u-V)\|^2_{L^2(\Omega)} + \Big(\int_\Omega U'(x_1+m(t))(u-V)dx\Big)^2 + |h_M(t)|^2 + |g(t)|^2 + |m'(t)|^2\Big]dt.$$

Then, we use (4.48), (4.49), (4.52), (4.20) and (4.21) to get

$$\int_0^\infty |f'(t)|dt \le C.$$

Therefore, f and f' are both integrable over $[0,\infty)$, which completes (4.51).

Appendix A. Proof of Proposition 4.3

A.1. Local existence of Eq. (1.7) with $\varphi \equiv 1$

First of all, we construct approximate solutions $(Y_n)_{n\ge 0}$, following iteration scheme: Set

$$Y_0(t,x) = 0, \quad t\ge 0,\ x\in\Omega.$$

Then, for a given n-th approximate solution Y_n, we define Y_{n+1} as a solution of the linear equation

$$\partial_t Y_{n+1} - A_1'(U(Y_n+x_1))\partial_{x_1} Y_{n+1} + \sum_{i=2}^N A_i'(U(Y_n+x_1))\partial_{x_i} Y_{n+1} - A_1'(U(Y_n+x_1))|\nabla Y_n|^2$$
$$+ w_n\cdot\nabla Y_n - \Delta Y_{n+1} = -w_{n,1}\psi_M(x_1+m_n) - h_{n,M}(t)(1-\psi_M(x_1+m_n)) - g_n(t), \tag{A.1}$$

where the notations $w_n, w_{n,1}, h_{n,M}$ and m_n mean that Y_n replaces Y in those functions w, w_1, h_M and m, respectively, appeared in the Eq. (1.7) with $\varphi \equiv 1$.

We will show that for any $R > 0$, there exists $T_0 > 0$ such that

$$\|\sqrt{|U'(\cdot + m_n)|}Y_n\|_{L^\infty(0,T_0;L^2(\Omega))} + \|\nabla Y_n\|_{L^\infty(0,T_0;H^s(\Omega))} + \|\Delta Y_n\|_{L^2(0,T_0;H^s(\Omega))} \le R \quad \text{(A.2)}$$

For notational simplification, we rewrite (A.1) into a linear equation:

$$\begin{aligned}
&\partial_t Y - A_1'(U(Z+x_1))\partial_{x_1} Y + \sum_{i=2}^N A_i'(U(Z+x_1))\partial_{x_i} Y \\
&- A_1'(U(Z+x_1))|\nabla Z|^2 + w_Z \cdot \nabla Z - \Delta Y \\
&\quad = -w_{Z,1}\psi_M(x_1+m_Z) - h_{Z,M}(t)(1-\psi_M(x_1+m_Z)) - g_Z(t), \\
&Y|_{t=0} = 0,
\end{aligned} \quad \text{(A.3)}$$

where the notations $w_Z, w_{Z,1}, h_{Z,M}$ and m_Z mean that Z replaces Y in those function w, w_1, h_M and m, respectively, appeared in the Eq. (1.7) with $\varphi \equiv 1$.

Assume that for any $R > 0$, there exists $T_0 > 0$ such that

$$\|\sqrt{|U'(\cdot + m_Z)|}Z\|_{L^\infty(0,T_0;L^2(\Omega))} + \|\nabla Z\|_{L^\infty(0,T_0;H^s(\Omega))} + \|\Delta Z\|_{L^2(0,T_0;H^s(\Omega))} \le R. \quad \text{(A.4)}$$

We first estimate $\|\nabla Y\|_{L^\infty(0,T_0;H^s(\Omega))} + \|\Delta Y\|_{L^2(0,T_0;H^s(\Omega))} \le R$.

For any k with $0 \le k \le s$, it follows from (A.3) that for all $t \in (0, T_0)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla^{k+1}Y|^2 dx + \int_\Omega |\nabla^{k+2}Y|^2 dxds \\
&\quad = -\int_\Omega \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Z+x_1))\partial_{x_1}Y\Big) + \nabla^{k+2}Y\nabla^k\Big(\sum_{i=2}^N A_i'(U(Z+x_1))\partial_{x_i}Y\Big) \\
&\qquad - \nabla^{k+2}Y\nabla^k\Big(A_1'(U(Z+x_1))|\nabla Z|^2\Big) + \nabla^{k+2}Y\nabla^k(w_Z \cdot \nabla Z) \\
&\qquad + \nabla^{k+2}Y\nabla^k\Big(w_{Z,1}\psi_M(x_1+m_Z)\Big) - \nabla^{k+2}Y\nabla^k\psi_M(x_1+m_Z)h_{Z,M}(t)dx \\
&:= \sum_{i=1}^6 I_i.
\end{aligned}$$

For terms related to the flux, we use Hölder inequality and Sobolev inequality together with (A.4), to get

$$
\begin{aligned}
|I_1| &= \Big|\int_\Omega \nabla^{k+2} Y\Big[A_1'(U(Z+x_1))\nabla^k\partial_{x_1}Y + \sum_{1\le l\le k}\binom{k}{l}\nabla^l A_1'(U(Z+x_1))\nabla^{k-l}\partial_{x_1}Y\Big]dx\Big|\\
&\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\Big[\|\nabla^{k+1}Y\|_{L^2(\Omega)} + \|\nabla Z\|^\alpha_{H^s(\Omega)}\|\nabla Y\|_{H^s(\Omega)}\Big]\\
&\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\Big[\|\nabla^{k+1}Y\|_{L^2(\Omega)} + R^\alpha\|\nabla Y\|_{H^s(\Omega)}\Big]\\
&\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + CR^{2\alpha}\|\nabla Y\|^2_{H^s(\Omega)},
\end{aligned}
$$

where $\alpha \ge 1$ is some constant depending on k.

Likewise, we have

$$
\begin{aligned}
|I_2| &\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + CR^{2\alpha}\|\nabla Y\|^2_{H^s(\Omega)},\\
|I_3| &\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)} + CR^{2\beta},
\end{aligned}
$$

where $\alpha, \beta \ge 1$ are some constants depending on k.

To estimate I_4, notice that

$$
\|\nabla u\|_{L^\infty(0,T_0;H^{s-1}(\Omega))} \le e^{CT_0}\|\nabla u_0\|_{H^{s-1}(\Omega)}, \quad \text{and} \quad \|u\|_{L^\infty} \le \|u_0\|_{L^\infty},
$$

which yield that

$$
\begin{aligned}
|I_4| &\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\|\nabla^k(w_Z\cdot\nabla Z)\|_{L^2(\Omega)}\\
&\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}(\|\nabla u\|_{H^{s-1}(\Omega)} + \|\nabla Z\|_{H^s(\Omega)} + 1)\|\nabla Z\|_{H^s(\Omega)}\\
&\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + CR^2(e^{CT_0} + R^2 + 1),
\end{aligned}
$$

and

$$
\begin{aligned}
|I_5| &\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}\|\nabla^k(w_{Z,1}\psi_M(x_1+m_Z))\|_{L^2(\Omega)}\\
&\le C\|\nabla^{k+2}Y\|_{L^2(\Omega)}(\|\nabla u\|_{H^{s-1}(\Omega)} + \|\nabla Z\|^\alpha_{H^s(\Omega)} + 1)\\
&\le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + C(e^{CT_0} + R^\alpha + 1).
\end{aligned}
$$

Finally, it follows from (A.4) that

$$
|m_Z| \le C\|\sqrt{|U'(\cdot+m_Z)|}Z\|_{L^\infty(0,T_0;L^2(\Omega))} \le CR,
$$

which yields

$$
|I_6| \le C|h_{Z,M}(t)|\|\nabla^{k+2}Y\|_{L^2(\Omega)}\|\nabla^k\psi_M(\cdot+m_Z)\|_{L^2(\Omega)} \le \frac{1}{8}\|\nabla^{k+2}Y\|^2_{L^2(\Omega)} + CR^2.
$$

Therefore, we have

$$\frac{d}{dt}\int_{\Omega}|\nabla^{k+1}Y|^2dx+\int_{\Omega}|\nabla^{k+2}Y|^2dxds\le C\|\nabla^{k+1}Y\|^2_{L^2(\Omega)}+C\|\nabla Y\|^2_{H^s(\Omega)}+C_R,$$

where C_R is a constant depending on R.

Then, summing the above estimates over $0\le k\le s$, we have

$$\frac{d}{dt}\|\nabla Y\|^2_{H^s(\Omega)}+\int_{\Omega}\|\nabla^2 Y\|^2_{H^s(\Omega)}ds\le C\|\nabla Y\|^2_{H^s(\Omega)}+C_R,$$

which implies

$$\|\nabla Y\|^2_{L^\infty(0,T_0;H^s(\Omega))}+\|\Delta Y\|^2_{L^2(0,T_0;H^s(\Omega))}\le C_R T_0 e^{CT_0}.$$

Hence we take T_0 to be small so that

$$\|\nabla Y\|_{L^\infty(0,T_0;H^s(\Omega))}+\|\Delta Y\|_{L^2(0,T_0;H^s(\Omega))}\le R. \tag{A.5}$$

We now estimate $\|\sqrt{|U'(\cdot+m)|}Y\|_{L^\infty(0,T_0;L^2(\Omega))}\le R$ using the above estimates (A.5). Multiplying (A.3) by $|U'(x_1+m(t))|Y$, and using the same arguments as the two terms J_1 and J_3 in (3.2), we have that

$$\begin{aligned}
&\partial_t\Big(|U'(x_1+m(t))|\frac{Y^2}{2}\Big)+U''(x_1+m(t))m'(t)\frac{Y^2}{2}\\
&-\Big[A_1'(U(Z+x_1))-A_1'(U(x_1+m(t)))\Big]|U'(x_1+m(t))|Y\partial_{x_1}Y\\
&+\sum_{i=2}^N A_i'(U(Z+x_1))|U'(x_1+m(t))|Y\partial_{x_i}Y-A_1'(U(Z+x_1))|\nabla Z|^2|U'(x_1+m(t))|Y\\
&+w_Z\cdot\nabla Z|U'(x_1+m(t))|Y-\mathrm{div}(|U'(x_1+m(t))|Y\nabla Y)\\
&+\partial_{x_1}(\partial_{x_1}|U'(x_1+m(t))|\frac{Y^2}{2})+|U'(x_1+m(t))||\nabla Y|^2\\
&=-\Big(w_{Z,1}\psi_M(x_1+m_Z)+h_{Z,M}(t)(1-\psi_M(x_1+m_Z))+g_Z(t)\Big)|U'(x_1+m(t))|Y.
\end{aligned}$$

Integrating the above equation over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega}|U'(x_1+m(t))|\frac{Y^2}{2}dx+\int_{\Omega}|U'(x_1+m(t))||\nabla Y|^2dx=-\int_{\Omega}U''(x_1+m(t))m'(t)\frac{Y^2}{2}dx\\
&\quad+\int_{\Omega}\Big(A_1'(U(Z+x_1))-A_1'(U(x_1+m(t)))\Big)|U'(x_1+m(t))|Y\partial_{x_1}Y\\
&\quad-\int_{\Omega}\sum_{i=2}^N A_i'(U(Z+x_1))|U'(x_1+m(t))|Y\partial_{x_i}Y dx
\end{aligned}$$

$$+\int_\Omega A_1'(U(Z+x_1))|\nabla Z|^2|U'(x_1+m(t))|Y\,dx-\int_\Omega w_Z\cdot\nabla Z|U'(x_1+m(t))|Y\,dx$$
$$-\int_\Omega\Big(w_{Z,1}\psi_M(x_1+m(t))+h_{Z,M}(t)(1-\psi_M(x_1+m(t)))+g_Z(t)\Big)|U'(x_1+m(t))|Y\,dx.$$

In order to control $m'(t)$, we use the same computations as in Remark 1.5 and (4.19), together with $\|\nabla Y\|_{L^\infty((0,T_0)\times\Omega)}\le R$ by (A.5). Then, we have that for all $t\in(0,T_0)$,

$$\begin{aligned}|m'(t)|\le C\Big[&\int_\Omega|U'(x_1+m(t))|\Big(|A_1'(U(Z+x_1))||\partial_{x_1}Y|+\sum_{i=1}^N|A_i'(U(Z+x_1))||\partial_{x_i}Y|\Big)dx\\&+\int_\Omega|U'(x_1+m(t))||A_1'(U(Z+x_1))||\nabla Z|^2dx+\int_\Omega|U'(x_1+m(t))||w_Z||\nabla Z|dx\\&+\int_\Omega|U'(x_1+m(t))||\Delta Y|dx+\int_\Omega|U'(x_1+m(t))|(|w_{Z,1}|+|h_{Z,M}(t)|+|g_Z(t)|)dx\Big].\end{aligned}$$

Since

$$|w_Z|\le C|u-U(Z+x_1)|\le C(\|u_0\|_{L^\infty}+\|U\|_{L^\infty}),\quad\text{and}\quad|h_{Z,M}|+|g_Z|\le C\|w_Z\|_{L^\infty},\tag{A.6}$$

we use (A.4) and (A.5) to estimate

$$|m'(t)|\le C(R+R^2)\|U'\|_{L^1(\Omega)}+C\|\Delta Y\|_{L^2(\Omega)}\|U'\|_{L^2(\Omega)}\le C(R+R^2),$$

which yields

$$\Big|\int_\Omega U''(x_1+m(t))m'(t)\frac{Y^2}{2}dx\Big|\le C(R+R^2)\int_\Omega|U'(x_1+m(t))|Y^2dx.$$

Then, we use (A.4) and (A.6) to estimate

$$\frac{d}{dt}\int_\Omega|U'(x_1+m(t))|Y^2dx+\int_\Omega|U'(x_1+m(t))||\nabla Y|^2dx\le C\int_\Omega|U'(x_1+m(t))|Y^2dx+C_R,$$

which gives

$$\|\sqrt{|U'(\cdot+m)|}Y\|_{L^\infty(0,T_0;L^2(\Omega))}\le\sqrt{C_RT_0e^{CT_0}}\le R,\quad\text{if } T_0\ll1.$$

Hence, we have shown that the sequence of approximate solutions $(Y_n)_{n\ge0}$ is uniformly bounded as (A.2). The remaining part is quite standard, so we only provide a sketch of the proof. Using the uniform estimates (A.2) and same energy estimates as above, we easily have the strong convergence of sequence $(Y_n)_{n\ge0}$ towards a limit function Y in a lower-order space $L^\infty(0,T_0;L^2(\Omega))\cap L^2(0,T_0;H^1(\Omega))$. Then, it is obvious that the limit Y is a solution of (1.7), and satisfies the estimates (4.13).

A.2. Local existence of Eq. (1.7)

For the local existence of Eq. (1.7) in the time interval of $(0, \frac{t_0}{2}]$, we just need the condition $u_0 \in L^\infty(\Omega)$ without $\nabla u_0 \in H^{s-1}(\Omega)$, because (1.7) has no terms related to w and h_M for such a time interval $(0, \frac{t_0}{2}]$, the three terms I_4, I_5 and I_6 in Section A.1 above do not appear.

References

[1] Adimurthi, Sh.S. Ghoshal, G.D. Veerappa Gowda, L^p stability for entropy solutions of scalar conservation laws with strict convex flux, J. Differ. Equ. 256 (2014) 3395–3416.

[2] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations, I. Formal derivations, J. Stat. Phys. 63 (1991) 323–344.

[3] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations, II. Convergence proofs for the Boltzmann equation, Commun. Pure Appl. Math. 46 (1993) 667–753.

[4] F. Berthelin, A. Tzavaras, A. Vasseur, From discrete velocity Boltzmann equations to gas dynamics before shocks, J. Stat. Phys. 135 (2009) 153–173.

[5] F. Berthelin, A. Vasseur, From kinetic equations to multidimensional isentropic gas dynamics before shocks, SIAM J. Math. Anal. 36 (2005) 1807–1835.

[6] Y. Brenier, L^2 formulation of multidimensional scalar conservation laws, Arch. Ration. Mech. Anal. 193 (2009) 1–19.

[7] G.Q. Chen, H. Frid, Large-time behavior of entropy solutions of conservation laws, J. Differ. Equ. 152 (1999) 308–357.

[8] G.Q. Chen, H. Frid, Uniqueness and asymptotic stability of Riemann solutions for the compressible Euler equations, Trans. Am. Math. Soc. 353 (2001) 1103–1117.

[9] G.Q. Chen, H. Frid, Y.C. Li, Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics, Commun. Math. Phys. 228 (2002) 201–217.

[10] K. Choi, A. Vasseur, Short-time stability of scalar viscous shocks in the inviscid limit by the relative entropy method, SIAM J. Math. Anal. 47 (2015) 1405–1418.

[11] C.M. Dafermos, The second law of thermodynamics and stability, Arch. Ration. Mech. Anal. 70 (1979) 167–179.

[12] A.-L. Dalibard, M.-J. Kang, Existence and stability of planar shocks of viscous scalar conservation laws with space-periodic flux, J. Math. Pures Appl. 107 (2017) 336–366.

[13] R.J. Diperna, Uniqueness of solutions to hyperbolic conservation laws, Indiana Univ. Math. J. 28 (1979) 138–188.

[14] H. Freistühler, D. Serre, L^1 stability of shock waves in scalar viscous conservation laws, Commun. Pure Appl. Math. 51 (3) (1998) 291–301.

[15] F. Golse, L. Saint-Raymond, The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, Invent. Math. 155 (2004) 81–161.

[16] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Ration. Mech. Anal. 95 (4) (1986) 325–344.

[17] J. Goodman, Stability of viscous scalar shock fronts in several dimensions, Trans. Am. Math. Soc. 311 (2) (1989) 683–695.

[18] J. Goodman, J. Miller, Long-time behavior of scalar viscous shock fronts in two dimensions, J. Dyn. Differ. Equ. 11 (2) (1999) 255–277.

[19] J. Goodman, Z.P. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, Arch. Ration. Mech. Anal. 121 (1992) 235–265.

[20] D. Hoff, K. Zumbrun, Asymptotic behavior of multidimensional scalar viscous shock fronts, Indiana Univ. Math. J. 49 (2) (2000) 427–474.

[21] D. Hoff, K. Zumbrun, Pointwise Green's function bounds for multidimensional scalar viscous shock fronts, J. Differ. Equ. 183 (2) (2002) 368–408.

[22] F.M. Huang, A. Matsumura, Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation, Commun. Math. Phys. 289 (2009) 841–861.

[23] F.M. Huang, Y. Wang, T. Yang, Hydrodynamic limit of the Boltzmann equation with contact discontinuities, Commun. Math. Phys. 295 (2010) 293–326.

[24] F.M. Huang, Y. Wang, T. Yang, Fluid dynamic limit to the Riemann solutions of Euler equations: I. Superposition of rarefaction waves and contact discontinuity, Kinet. Relat. Models 3 (2010) 685–728.

[25] F.M. Huang, Y. Wang, T. Yang, Vanishing viscosity limit of the compressible Navier-Stokes equations for solutions to Riemann problem, Arch. Ration. Mech. Anal. 203 (2012) 379–413.
[26] F.M. Huang, Y. Wang, Y. Wang, T. Yang, The limit of the Boltzmann equation to the Euler equations, SIAM J. Math. Anal. 45 (2013) 1741–1811.
[27] A.M. Il'in, O.A. Oleinik, Asymptotic behavior of the solutions of the Cauchy problem for certain quasilinear equations for large time, Mat. Sb. 51 (1960) 191–216.
[28] M.-J. Kang, A. Vasseur, Asymptotic analysis of Vlasov-type equations under strong local alignment regime, Math. Models Methods Appl. Sci. 25 (11) (2015) 2153–2173.
[29] M.-J. Kang, A. Vasseur, L^2−contraction for shock waves of scalar viscous conservation laws, Ann. Inst. Henri Poincaré (C): Anal. Non Linéaire 34 (1) (2017) 139–156.
[30] M.-J. Kang, A. Vasseur, Criteria on contractions for entropic discontinuities of systems of conservation laws, Arch. Ration. Mech. Anal. 222 (2016) 343–391.
[31] S. Kruzhkov, First-order quasilinear equations with several space variables, Math. USSR Sb. 10 (1970) 217–243.
[32] Y.S. Kwon, Asymptotic limit to shocks for scalar balance laws using relative entropy, Abstr. Appl. Anal. (2014) 690801.
[33] Y.S. Kwon, A. Vasseur, Asymptotic limit to a shock for BGK models using the relative entropy method, Nonlinearity 28 (2015) 531–543.
[34] N. Leger, L^2 stability estimates for shock solutions of scalar conservation laws using the relative entropy method, Arch. Ration. Mech. Anal. 199 (2011) 761–778.
[35] N. Leger, A. Vasseur, Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations, Arch. Ration. Mech. Anal. 201 (2011) 271–302.
[36] P.L. Lions, N. Masmoudi, From the Boltzmann equations to the equations of incompressible fluid mechanics, I, II, Arch. Ration. Mech. Anal. 158 (2001) 173–193, 195–211.
[37] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Am. Math. Soc. 56 (1985) 1–108.
[38] T.-P. Liu, Y.N. Zeng, Shock waves in conservation laws with physical viscosity, Mem. Am. Math. Soc. 234 (1105) (2015).
[39] A. Matsumura, K. Nishihara, On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas, Jpn. J. Appl. Math. 2 (1985) 17–25.
[40] N. Masmoudi, L. Saint-Raymond, From the Boltzmann equation to the Stokes-Fourier system in a bounded domain, Commun. Pure Appl. Math. 56 (2003) 1263–1293.
[41] S. Osher, J. Ralston, L^1 stability of travelling waves with applications to convective porous media flow, Commun. Pure Appl. Math. 35 (1982) 737–751.
[42] D. Serre, A. Vasseur, L^2-type contraction for systems of conservation laws, J. Éc. Polytech. Math. 1 (2014) 1–28.
[43] D. Serre, A. Vasseur, About the relative entropy method for hyperbolic systems of conservation laws, preprint, 2015.
[44] R. Shi, W. Wang, Nonlinear stability of large perturbation around the viscous shock wave for the 2D scalar viscous conservation law, Indiana Univ. Math. J. 65 (2016) 1137–1182.
[45] J. Smoller, Shock Waves and Reaction-Diffusion Equations, 2nd ed., Springer-Verlag, New York, 1994.
[46] A. Szepessy, Z. Xin, Nonlinear stability of viscous shock waves, Arch. Ration. Mech. Anal. 122 (1993) 53–103.
[47] A. Vasseur, Recent results on hydrodynamic limits, in: Handbook of Differential Equations: Evolutionary Equations, vol. IV, in: Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 323–376.
[48] A. Vasseur, Relative entropy and contraction for extremal shocks of Conservation Laws up to a shift, in: Recent Advances in Partial Differential Equations and Applications, in: Contemp. Math., vol. 666, Amer. Math. Soc., Providence, RI, 2016, pp. 385–404.
[49] A. Vasseur, Y. Wang, The inviscid limit to a contact discontinuity for the compressible Navier-Stokes-Fourier system using the relative entropy method, SIAM J. Math. Anal. 47 (6) (2015) 4350–4359.
[50] A. Vasseur, L. Yao, Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic Navier-Stokes equations with density dependent viscous coefficient, Commun. Math. Sci. 14 (2016) 2215–2228.
[51] Y. Wang, Zero dissipation limit of the compressible heat-conducting Navier-Stokes equations in the presence of the shock, Acta Math. Sci. 28B (2008) 727–748.
[52] H.T. Yau, Relative entropy and hydrodynamics of Ginzburg-Landau models, Lett. Math. Phys. 22 (1991) 63–80.