Justification of Diffusion limit for the Boltzmann Equation with a non-trivial Profile

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Abstract

Under the diffusion scaling and a scaling assumption on the microscopic component, a non-classical fluid dynamic system was derived in [3] that is related to the system of ghost effect derived in [41] in different settings. This paper aims to justify this limit system for a non-trivial background profile with slab symmetry. The result reveals not only the diffusion phenomena in the temperature and density, but also the flow of higher order in Knudsen number due to the gradient of the temperature. Precisely, we show that the solution to the Boltzmann equation converges to a diffusion wave with decay rates in both Knudsen number and time.

Keywords: Boltzmann equation, Knudsen number, diffusive scaling, diffusion wave

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Contents

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1 Introduction

Consider the Boltzmann equation with slab symmetry under the diffusive scaling

$$\varepsilon\partial_t f^\varepsilon + \xi_1 f^\varepsilon_x = \frac{1}{\varepsilon}Q(f^\varepsilon, f^\varepsilon), \quad (t,x,\xi)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^3. \tag{1.1}$$

Here $f^\varepsilon(t,x,\xi)\geq 0$ is the distribution density of particles at (t,x) with velocity ξ, $Q(f,f)$ is the collision operator which is a non-local bilinear operator in the velocity variable with a kernel determined by the physics of particle interaction. For monatomic gas, the rotational invariance of the particle leads to the collision operator $Q(f,f)$ as a bilinear collision operator in the form of, cf. [5]:

$$Q(f,g)(\xi) \equiv \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+}\Big(f(\xi')g(\xi'_*)+f(\xi'_*)g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\Big)B(|\xi-\xi_*|,\theta)d\xi_* d\Omega,$$

with θ being the angle between the relative velocity and the unit vector Ω. Here $\mathbb{S}^2_+ = \{\Omega\in\mathbb{S}^2 : (\xi-\xi_*)\cdot\Omega\geq 0\}$. The conservation of momentum and energy gives the following relation between velocities before and after collision:

$$\begin{cases} \xi' = \xi - [(\xi-\xi_*)\cdot\Omega]\,\Omega, \\ \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\Omega]\,\Omega. \end{cases}$$

In this paper, we will consider the two basic models, i.e., the hard sphere model and the hard potential with angular cut-off, for which the collision kernel $B(|\xi-\xi_*|,\theta)$ takes the form of

$$B(|\xi-\xi_*|,\theta) = |(\xi-\xi_*,\Omega)|,$$

and

$$B(|\xi-\xi_*|,\theta) = |\xi-\xi_*|^{\frac{p-5}{p-1}}b(\theta), \quad b(\theta)\in L^1([0,\pi]),\ p\geq 5,$$

respectively. Here, p is the index in potential of the inverse power law that is proportional to r^{1-p} with r being the distance between two particles.

Motivated by [41], the following macroscopic and microscopic decomposition with scalings was introduced in [3]:

$$f^\varepsilon = M_{[\rho^\varepsilon,\varepsilon u^\varepsilon,\theta^\varepsilon]} + \varepsilon G^\varepsilon. \tag{1.2}$$

Here $M_{[\rho^\varepsilon,\varepsilon u^\varepsilon,\theta^\varepsilon]}$ is the local Maxwellian and G^ε is the microscopic component. Moreover, the local Maxwellian $M_{[\rho^\varepsilon,\varepsilon u^\varepsilon,\theta^\varepsilon]}$ is defined by the five conserved quantities, that is, the mass density $\rho^\varepsilon(t,x)$, momentum density $m^\varepsilon(t,x)=\varepsilon\rho^\varepsilon(t,x)u^\varepsilon(t,x)$ and energy density $e^\varepsilon(t,x)+\frac{1}{2}|\varepsilon u^\varepsilon(t,x)|^2$ given by

$$\begin{cases} \rho^\varepsilon(t,x) \equiv \int_{\mathbb{R}^3} f^\varepsilon(t,x,\xi)d\xi, \\ m^\varepsilon_i(t,x) \equiv \int_{\mathbb{R}^3}\psi_i(\xi)f^\varepsilon(t,x,\xi)d\xi \text{ for } i=1,2,3, \\ \left[\rho^\varepsilon\left(e^\varepsilon+\frac{\varepsilon^2}{2}|u^\varepsilon|^2\right)\right](t,x) \equiv \int_{\mathbb{R}^3}\psi_4(\xi)f^\varepsilon(t,x,\xi)d\xi, \end{cases} \tag{1.3}$$

as

$$M \equiv M_{[\rho^\varepsilon,\varepsilon u^\varepsilon,\theta^\varepsilon]}(t,x,\xi) \equiv \frac{\rho^\varepsilon(t,x)}{\sqrt{(2\pi R\theta^\varepsilon(t,x))^3}}\exp\left(-\frac{|\xi-\varepsilon u^\varepsilon(t,x)|^2}{2R\theta^\varepsilon(t,x)}\right). \tag{1.4}$$

Here $\psi_\alpha(\xi)$ are the collision invariants:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for} \quad i=1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases}$$

satisfying

$$\int_{\mathbb{R}^3} \psi_j(\xi) Q(h,g) d\xi = 0, \quad \text{for} \quad j = 0,1,2,3,4.$$

Here, θ^ε is the temperature related to the internal energy e^ε by $e^\varepsilon = \frac{3}{2} R\theta^\varepsilon$ with R being the gas constant, and $\varepsilon u^\varepsilon$ is the bulk velocity. Note that even though u^ε is of higher order, it is the scaled velocity that appears in the equations for the macroscopic variables ρ^ε and θ^ε.

The Boltzmann equation is a fundamental equation in statistical physics for rarefied gas which describes the time evolution of particle distribution. There has been tremendous progress on the mathematical theories for the Boltzmann equation with ε being a fixed constant, such as the global existence of weak (renormalized) solution for large data in [11] and classical solutions as small perturbations of equilibrium states (Maxwellian) in [20, 34, 42] and the references therein, etc.

On the other hand, the study on the hydrodynamic limit of Boltzmann equation is important and challenging. For this, it is well known that the classical works of Hilbert, Chapman-Enskog reveal the relation of the Boltzmann equation to the classical systems of fluid dynamics through asymptotic expansions with respect to the Knudsen number. For the hydrodynamic limit of Boltzmann equation to the compressible Euler system, we refer [2, 3] for the formal derivation. If the Euler system is assumed to have smooth solution, this hydrodynamic limit is proved rigorously in [43, 6] with and without initial layer respectively.

However, it is well known that the compressible Euler system develops singularity in finite time even for sufficiently smooth initial data. The Riemann problem is the basic problem to the compressible Euler system, and its solution turns out to be fundamental in the theory of hyperbolic conservation laws because it not only captures the local and global behavior of solutions but also reveals the effect of nonlinearity in the structure of the solutions. There are three basic wave patterns for the Euler system, that is, shock wave, rarefaction wave, and contact discontinuity. For the hydrodynamic limit of the Boltzmann equation in the setting of Riemann solutions, we refer [23, 24, 25, 44, 45].

Under the diffusive scaling, usually, the density function $f^\varepsilon(t,x,\xi)$ is set as a perturbation of a global Maxwellian $M_{[1,0,1]}$, i.e.

$$f^\varepsilon(t,x,\xi) = M_{[1,0,1]} + M_{[1,0,1]} \Big(\varepsilon f_1(t,x,\xi) + \cdots + \varepsilon^n f_n^\varepsilon(t,x,\xi) \Big). \tag{1.5}$$

There has been extensive study on the hydrodynamic limit $\varepsilon \to 0$ of the Boltzmann equation to the incompressible Navier-Stokes-Fourier system, for example, to justify the DiPerna-Lions' renormalized solution in [11] of the Boltzmann equation to the Leray-Hopf weak solutions of the incompressible Navier-Stokes-Fourier system. For this, Bardos-Golse-Levermore [2] first studied this problem under certain a priori assumption. Recently, a breakthrough was achieved by Golse-Raymond in [17] which established a proof of such limit for certain class of collision kernels. After that, some progress was made for more general collision kernels, cf. [30]. In fact, there are also a lot of important contributions on this problem over the years, see [15, 18, 31, 32, 38, 39, 40] and the references therein.

In the framework of classical solutions to the incompressible Navier-Stokes-Fourier system, it was proved in [9] that one can find a Boltzmann solution $f^\varepsilon(t,x,\xi)$ such that f_2^ε is of order ε^2, but it is not clear about the amplitude f_2^ε at the initial time. Later, the Navier-Stokes-Fourier limit was proved for $f^\varepsilon(0,x,\xi)$ with small data in [4]. Recently, Guo in [21] justified the diffusive expansion (1.5) when $f_1(0,x,\xi)$ has small amplitude while $f_i^\varepsilon(0,x,\xi)$ can have arbitrarily large amplitude for $i \geq 2$ in a torus. This work was later generalized to some other settings, cf. [33, 28, 29]. Moreover, based on the $L^2 - L^\infty$ estimate, Esposito-Guo-Kim-Marra [13] proved the hydrodynamic limit of the rescaled Boltzmann equation to the incompressible Navier-Stokes-Fourier system in a bounded domain if the initial data is small.

Notice that all the results under the diffusive scaling mentioned above are either about large perturbation of vacuum or small perturbation of a global Maxwellian. A natural question to ask is how about the perturbation of a non-trivial profile. The purpose of this paper is to study this problem in the setting of (1.2).

In fact, under the assumption (1.2), when $\varepsilon \to 0$, formally we have

$$f^\varepsilon = M_{[\rho^\varepsilon,\varepsilon u^\varepsilon,\theta^\varepsilon]} + \varepsilon G^\varepsilon \to M_{[\rho,0,\theta]}, \tag{1.6}$$

which shows that in the macroscopic level, only the unknown limit functions ρ, θ survive because the macroscopic velocity is zero. However, as shown in [3] and will be recalled in the next section, the equations of ρ and θ are actually closely related to the scaled velocity u. Indeed, this diffusive scaling induces diffusion phenomenon for both the temperature θ and density ρ, and the non-zero gradient of temperature induces a non-trivial flow in the higher order along the same direction. In this paper, we will construct such diffusion wave and study the hydrodynamic limit of the rescaled Boltzmann equation to such a diffusion wave global in time.

The rest of the paper will be organized as follows. The construction of the diffusion wave and the main theorem will be given in the next section. We will reformulate the problem and derive some a priori estimates in Section 3. Based on the a priori estimates, the main theorem will be proved in Section 4.

Notations: Throughout this paper, the positive generic constants that are independent of ε are denoted by $c, C, C_i (i = 1, 2, 3, \cdots)$. And we will use $\|\cdot\|$ to denote the standard $L_2(\mathbb{R}; dz)$ norm, and $\|\cdot\|_{H^i}$ $(i = 1, 2, 3, \cdots)$ to denote the standard Sobolev $H^i(\mathbb{R}; dz)$ norm with $z = x$ or y. Sometimes, we also use $O(1)$ to denote a uniform bounded constant independent of ε.

2 Construction of Profile and the Main Result

We will drop the superscript ε in the case of no confusion for simple notation. The inner product of $h,\ g$ in $L^2_\xi(\mathbb{R}^3)$ with respect to a given Maxwellian $\tilde{M}$ is defined by:

$$\langle h, g\rangle_{\tilde{M}} \equiv \int_{\mathbb{R}^3} \frac{1}{\tilde{M}} h(\xi) g(\xi) d\xi,$$

when the integral is well defined. If $\tilde{M}$ is the local Maxwellian M, with respect to this inner product, the macroscopic space is spanned by the following five pairwise orthogonal functions

$$\begin{cases} \chi_0(\xi) \equiv \dfrac{1}{\sqrt{\rho}} M, \\ \chi_i(\xi) \equiv \dfrac{\xi_i - \varepsilon u_i}{\sqrt{R\theta\rho}} M \quad \text{for} \quad i = 1, 2, 3, \\ \chi_4(\xi) \equiv \dfrac{1}{\sqrt{6\rho}} (\dfrac{|\xi - \varepsilon u|^2}{R\theta} - 3) M, \\ \langle \chi_i, \chi_j\rangle = \delta_{ij},\ \ i, j = 0, 1, 2, 3, 4. \end{cases}$$

Using these functions, we define the macroscopic projection P_0 and microscopic projection P_1 as follows:

$$\begin{cases} P_0 h \equiv \displaystyle\sum_{j=0}^{4} \langle h, \chi_j\rangle \chi_j, \\ P_1 h \equiv h - P_0 h. \end{cases}$$

The projections P_0 and P_1 are orthogonal:

$$P_0 P_0 = P_0, P_1 P_1 = P_1, P_0 P_1 = P_1 P_0 = 0.$$

A function $h(\xi)$ is called microscopic or non-fluid if

$$\int h(\xi)\psi_j(\xi)d\xi = 0,\ \ j = 0, 1, 2, 3, 4.$$

Under this decomposition, the solution $f(t, x, \xi)$ of the Boltzmann equations satisfies

$$P_0 f = M,\ \ P_1 f = \varepsilon G,$$

and the Boltzmann equation becomes

$$(\varepsilon M + \varepsilon^2 G)_t + \xi_1 (M + \varepsilon G)_x = 2Q(M, G) + \varepsilon Q(G, G),$$

which is equivalent to the following fluid-type system for the fluid components (see [34] and [36] for details):

$$\begin{cases} \varepsilon\rho_t + (\varepsilon\rho u_1)_x = 0, \\ \varepsilon(\varepsilon\rho u_1)_t + (\varepsilon^2\rho u_1^2 + p)_x = -\varepsilon\displaystyle\int \xi_1^2 G_x d\xi, \\ \varepsilon(\varepsilon\rho u_i)_t + (\varepsilon^2\rho u_1 u_i)_x = -\varepsilon\displaystyle\int \xi_1\xi_i G_x d\xi,\ \ i = 2, 3, \\ \varepsilon\big[\rho(e + \dfrac{|\varepsilon u|^2}{2})\big]_t + \big[\varepsilon\rho u_1(e + \dfrac{|\varepsilon u|^2}{2}) + \varepsilon p u_1\big]_x = -\varepsilon\displaystyle\int \frac{1}{2}\xi_1|\xi|^2 G_x d\xi, \end{cases} \tag{2.1}$$

or more precisely,

$$\begin{cases} \varepsilon\rho_t + (\varepsilon\rho u_1)_x = 0, \\ \varepsilon(\varepsilon\rho u_1)_t + (\varepsilon^2\rho u_1^2 + p)_x = \dfrac{4}{3}\varepsilon(\mu(\theta)\varepsilon u_{1x})_x - \varepsilon\displaystyle\int \xi_1^2\Theta_x d\xi, \\ \varepsilon(\varepsilon\rho u_i)_t + (\varepsilon^2\rho u_1 u_i)_x = \varepsilon(\mu(\theta)\varepsilon u_{ix})_x - \varepsilon\displaystyle\int \xi_1\xi_i\Theta_x d\xi,\ \ i = 2, 3, \\ \varepsilon\big[\rho(e + \frac{|\varepsilon u|^2}{2})\big]_x + \big[\varepsilon\rho u_1(e + \dfrac{|\varepsilon u|^2}{2}) + \varepsilon p u_1\big]_x = \varepsilon(\kappa(\theta)\theta_x)_x \\ \qquad + \dfrac{4}{3}\varepsilon(\varepsilon^2\mu(\theta)u_1 u_{1x})_x + \displaystyle\sum_{i=2}^{3}\varepsilon(\varepsilon^2\mu(\theta)u_i u_{ix})_x - \varepsilon\displaystyle\int \frac{1}{2}\xi_1|\xi|^2\Theta_x d\xi, \end{cases} \tag{2.2}$$

together with an equation for the non-fluid component G:

$$\varepsilon^2 G_t + P_1(\xi_1 M_x) + \varepsilon P_1(\xi_1 G_x) = L_M G + \varepsilon Q(G, G), \tag{2.3}$$

where

$$G = L_M^{-1}(P_1(\xi_1 M_x)) + \Theta,$$

and

$$\Theta = L_M^{-1}(\varepsilon^2 G_t + \varepsilon P_1(\xi_1 G_x) - \varepsilon Q(G, G)).$$

Here L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(M, h) + Q(h, M),$$

and the null space N of L_M is spanned by the macroscopic variables:

$$\chi_j,\ \ j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant $\sigma_0(\rho, u, \theta) > 0$ such that for any function $h(\xi) \in N^\perp$, see [19],

$$\langle h, L_M h\rangle \le -\sigma_0\langle \nu(|\xi|)h, h\rangle,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0<\nu_0<\nu(|\xi|)<c(1+|\xi|)^\beta,$$

for some positive constants ν_0, c and $0<\beta\leq 1$.

In the above presentation, we normalize the gas constant R to be $\frac{2}{3}$ for simplicity so that $e=\frac{3}{2}R\theta=\theta$ and $p=R\rho\theta=\frac{2}{3}\rho\theta$. Notice also that the viscosity coefficient $\mu(\theta)>0$ and the heat conductivity coefficient $\kappa(\theta)>0$ are smooth functions of the temperature θ. And the following relation holds between these two functions, [8, 19],

$$\kappa(\theta)=\frac{15}{4}R\mu(\theta)=\frac{5}{2}\mu(\theta), \tag{2.4}$$

after taking $R=\frac{2}{3}$. It should be pointed out that (2.4) is crucially used in the following analysis. In fact, in our analysis, it is required that

$$\inf_\theta \kappa(\theta)>\frac{5}{4}\sup_\theta \mu(\theta)$$

for all θ under consideration. By (2.4), it is known that the above condition holds provided that the variation of the temperature is suitably small.

Now we are in a position to derive the limit equations for (ρ,u,θ) in the diffusive limit (1.6) formally. As [3], we assume that

$$p^\varepsilon=\text{const}+O(1)\varepsilon^2, \tag{2.5}$$

then, as $\varepsilon\to 0$, $(2.2)_1, (2.2)_2$ and $(2.2)_4$ yields formally that

$$\begin{cases} p=\text{const},\\ \rho_t+(\rho u_1)_x=0,\\ (\rho u_1)_t+(\rho u_1^2)_x+P^*_x=\frac{4}{3}(\mu(\theta)u_{1x})_x,\\ (\rho\theta)_t+(\rho u_1\theta+pu_1)_x=(\kappa(\theta)\theta_x)_x, \end{cases} \tag{2.6}$$

where P^* is unknown function. The equation (2.6) reveals how the zero order function ρ,θ depend on the scaled velocity even though the macroscopic velocity tends to zero.

With slab symmetry, in the macroscopic level, it is more convenient to rewrite the system by using the *Lagrangian* coordinates as in the study of conservation laws. That is, consider the coordinate transformation:

$$(x,t)\to\Big(\int_{(0,0)}^{(x,t)}\rho(y,s)dy-(\rho u_1)(y,s)ds, s\Big),$$

which is still denoted as (x,t) without confusion. Denote that $v=\frac{1}{\rho}$, the system (1.1) and (2.1) in the Lagrangian coordinates become

$$\varepsilon f_t-\frac{\varepsilon u_1}{v}f_x+\frac{\xi_1}{v}f_x=\frac{1}{\varepsilon}Q(f,f), \tag{2.7}$$

and

$$\begin{cases} \varepsilon v_t-\varepsilon u_{1x}=0,\\ \varepsilon^2 u_{1t}+p_x=-\varepsilon\int\xi_1^2G_x d\xi,\\ \varepsilon^2 u_{it}=-\varepsilon\int\xi_1\xi_iG_xd\xi,\ i=2,3,\\ \varepsilon\big(e+\frac{|\varepsilon u|^2}{2}\big)_t+(\varepsilon pu_1)_x=-\varepsilon\int\frac{1}{2}\xi_1|\xi|^2G_xd\xi, \end{cases} \tag{2.8}$$

respectively. Moreover, (2.2) and (2.3) take the form

$$
\begin{cases}
\varepsilon v_t - \varepsilon u_{1x} = 0, \\
\varepsilon^2 u_{1t} + p_x = \frac{4}{3}\varepsilon^2(\frac{\mu(\theta)}{v}u_{1x})_x - \varepsilon \int \xi_1^2 \Theta_{1x} d\xi, \\
\varepsilon^2 u_{it} = \varepsilon^2(\frac{\mu(\theta)}{v}u_{ix})_x - \varepsilon \int \xi_1 \xi_i \Theta_{1x} d\xi, \ i = 2,3, \\
\varepsilon\big(e + \frac{|\varepsilon u|^2}{2}\big)_t + (\varepsilon p u_1)_x = \varepsilon(\frac{\kappa(\theta)}{v}\theta_x)_x + \frac{4}{3}\varepsilon^3(\frac{\mu(\theta)}{v}u_1 u_{1x})_x \\
\qquad + \sum_{i=2}^{3} \varepsilon^3(\frac{\mu(\theta)}{v}u_i u_{ix})_x - \varepsilon \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_{1x} d\xi,
\end{cases}
\tag{2.9}
$$

and

$$
\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} G_x + P_1(\frac{\xi_1}{v}M_x) + \varepsilon P_1(\frac{\xi_1}{v}G_x) = L_M G + \varepsilon Q(G,G), \tag{2.10}
$$

with

$$
G = L_M^{-1}(P_1(\frac{\xi_1}{v}M_x)) + \Theta_1,
$$

and

$$
\Theta_1 = L_M^{-1}\Big(\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v}G_x + \frac{\varepsilon}{v}P_1(\xi_1 G_x) - \varepsilon Q(G,G)\Big). \tag{2.11}
$$

The limiting equation (2.6) becomes

$$
\begin{cases}
p = \text{const}, \\
v_t - u_{1x} = 0, \\
u_{1t} + P_x^* = \frac{4}{3}(\frac{\mu(\theta)}{v}u_{1x})_x, \\
\theta_t + p u_{1x} = (\frac{\kappa(\theta)}{\theta}\theta_x)_x.
\end{cases}
\tag{2.12}
$$

2.1 Construction of profile

We will construct a background solution to (2.12) in this subsection. Without loss of generality, set

$$
p = \frac{2\theta}{3v} = \frac{2}{3}, \tag{2.13}
$$

that is

$$
v = \theta. \tag{2.14}
$$

Assume the boundary conditions at the far fields given by

$$
\lim_{x\to\pm\infty}(v,\theta)(x,t) = (v_\pm, \theta_\pm), \quad \text{and} \quad \frac{\theta_+}{v_+} = \frac{\theta_-}{v_-} = 1, \quad \text{with } \theta_- \neq \theta_+. \tag{2.15}
$$

Note that if $\theta_- = \theta_+$, then $v = \theta = 1, u_1 = 0$ is a trivial solution to (2.12), and the diffusive limit of the rescaled Boltzmann equation to the incompressible Navier-Stokes-Fourier system is well studied as mentioned in the introduction.

Noting (2.14), the equation $(2.12)_4$ is rewritten as

$$
\theta_t + \frac{2}{3}u_{1x} = (\frac{\kappa(\theta)}{\theta}\theta_x)_x. \tag{2.16}
$$

Substituting $(2.12)_2$ into (2.16) and noting (2.14), we have the following scalar nonlinear diffusion equation

$$\theta_t = (a(\theta)\theta_x)_x, \ a(\theta) = \frac{3\kappa(\theta)}{5\theta}, \quad \text{with} \quad \lim_{x\to\pm\infty} \theta(x,t) = \theta_\pm. \tag{2.17}$$

From [1] and [10], it is known that the nonlinear diffusion equation (2.17) admits a self-similar solution $\hat{\theta}(\eta)$ with $\eta = \frac{x}{\sqrt{1+t}}$ satisfying the boundary conditions $\hat{\theta}(\pm\infty, t) = \theta_\pm$. Furthermore, $\hat{\theta}(\eta)$ is a monotonic function. Let $\delta = |\theta_+ - \theta_-|$, then $\hat{\theta}(t,x)$ has the property that

$$\hat{\theta}_x(t,x) = \frac{O(1)\delta}{\sqrt{1+t}} e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \qquad \text{as} \ \ x \to \pm\infty. \tag{2.18}$$

Define

$$(\tilde{v}, \tilde{u}_1, \tilde{\theta}) \doteq (\hat{\theta}, a(\hat{\theta})\hat{\theta}_x, \hat{\theta})(x,t), \tag{2.19}$$

then it is easy to check that $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ satisfying (2.12) as

$$\begin{cases} \tilde{p} = \dfrac{2\tilde{\theta}}{3\tilde{v}} = \dfrac{2}{3}, \\ \tilde{v}_t - \tilde{u}_{1x} = 0, \\ \tilde{u}_{1t} + P^*_x = \dfrac{4}{3}(\dfrac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{1x})_x, \\ \tilde{\theta}_t + \tilde{p}\tilde{u}_{1x} = (\dfrac{\kappa(\tilde{\theta})}{\tilde{\theta}}\tilde{\theta}_x)_x, \end{cases} \tag{2.20}$$

where $P^* = -a(\tilde{\theta})\tilde{\theta}_t + \frac{4\mu(\tilde{\theta})}{\tilde{\theta}}(a(\tilde{\theta})\tilde{\theta}_x)_x$.

Remark 2.1 *By* (2.19) *and* (2.20), *we actually construct a diffusion wave to the limit system. On the other hand, if* $\theta_- < \theta_+$, *then* $\tilde{u}_1 = a(\hat{\theta})\hat{\theta}_x > 0$, *that is, the variation of temperature along the x-axis induces a nontrivial scaled flow along the same direction, see Figure 1. The case* $\theta_- > \theta_+$ *is similar, see Figure 2.*

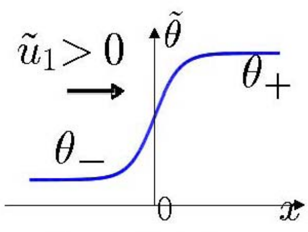

Figure 1: $0 < \theta_- < \theta_+$

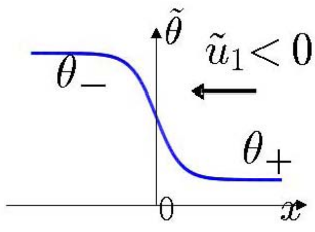

Figure 2: $\theta_- > \theta_+ > 0$

Remark 2.2 *The construction of the profile* $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ *is motivated by the viscous contact wave of compressible Navier-Stokes equations, see [22], [26] and [27]. The viscous contact wave is used to approximate the contact discontinuity for compressible Euler equation and its pressure keeps constant.*

In order to justify the hydrodynamic limit of the rescaled Boltzmann equation to the limit system (2.20), if we use the profile $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$, then some non-integrable error terms with respect to time coming from the non-fluid component for the system about perturbation. Therefore, one needs to construct another profile $(\bar{v}, \varepsilon\bar{u}, \bar{\theta})$ for the rescaled Boltzmann equation, based on $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$. For this, we require that the approximate pressure p satisfies

$$\bar{p} = \frac{2\bar{\theta}}{3\bar{v}} = \frac{2}{3} + O(1)\varepsilon^2 \doteq p_+ + O(1)\varepsilon^2. \tag{2.21}$$

Motivating by [27], we first notice that the main part of the non-fluid component in the solution G and part of Θ_1 defined in (2.11), are given by

$$w = \frac{1}{v} L_M^{-1}(P_1(\xi_1 M_x)) = \frac{1}{Rv\theta} L_M^{-1}\{P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta}\theta_x + \xi \cdot \varepsilon u_x)M]\},$$

and

$$\hat{\Theta}_1 = L_M^{-1}(\frac{\varepsilon}{v} P_1(\xi_1 w_x) - \varepsilon Q(w, w)),$$

respectively. To distinguish the leading term coming from the non-fluid component, we rewrite the Boltzmann equation (2.9) as

$$\begin{cases} \varepsilon v_t - \varepsilon u_{1x} = 0, \\ \varepsilon^2 u_{1t} + p_x = \frac{4}{3}\varepsilon^2(\frac{\mu(\theta)}{v} u_{1x})_x - \sum_{j=1}^{2} \varepsilon \int \xi_1^2 \Theta_{1x}^j d\xi, \\ \varepsilon^2 u_{it} = \varepsilon^2(\frac{\mu(\theta)}{v} u_{ix})_x - \sum_{j=1}^{2} \varepsilon \int \xi_1 \xi_i \Theta_{1x}^j d\xi, \ i = 2, 3, \\ \varepsilon\big(e + \frac{|\varepsilon u|^2}{2}\big)_t + (\varepsilon p u_1)_x = \varepsilon(\frac{\kappa(\theta)}{v}\theta_x)_x - \sum_{j=1}^{2} \varepsilon \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_{1x}^j d\xi + H_x, \end{cases} \tag{2.22}$$

with

$$\begin{aligned} \varepsilon^2 \tilde{G}_t - L_M \tilde{G} = &-\frac{1}{Rv\theta} P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta}(\theta - \bar{\theta})_x + \xi \cdot (\varepsilon u - \varepsilon \bar{u})_x)M] \\ &+ \frac{\varepsilon^2 u_1}{v} G_x - \frac{\varepsilon}{v} P_1(\xi_1 G_x) + \varepsilon Q(G, G) - \varepsilon^2 \bar{G}_t, \end{aligned} \tag{2.23}$$

where

$$\begin{cases} \bar{G} = \frac{1}{Rv\theta} L_M^{-1}\{P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta}\bar{\theta}_x + \xi \cdot \varepsilon \bar{u}_x)M]\}, \qquad \tilde{G} = G - \bar{G}, \\ H = \frac{4\varepsilon^3}{3}\frac{\mu(\theta)}{v} u_1 u_{1x} + \sum_{i=2}^3 \varepsilon^3 \frac{\mu(\theta)}{v} u_i u_{ix}, \\ \Theta_1^1 = L_M^{-1}\Big(\frac{\varepsilon}{v} P_1(\xi_1 \bar{G}_x) - \varepsilon Q(\bar{G}, \bar{G})\Big), \\ \Theta_1^2 = L_M^{-1}\Big(\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} G_x + \frac{\varepsilon}{v} P_1(\xi_1 \tilde{G}_x) - \varepsilon Q(\tilde{G}, \tilde{G}) - 2\varepsilon Q(\bar{G}, \tilde{G})\Big), \end{cases} \tag{2.24}$$

satisfying

$$\sum_{j=1}^{2} \Theta_1^j = \Theta_1 = L_M^{-1}(\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} G_x + \frac{\varepsilon}{v} P_1(\xi_1 G_x) - \varepsilon Q(G, G)).$$

Here, the function $(\bar{v}, \varepsilon\bar{u}, \bar{\theta})(x, t)$ is the profile to be constructed.

Since the velocity εu decays faster than (v, θ) in time, the leading terms in the energy equation $(2.22)_4$ are

$$\varepsilon\theta_t + \varepsilon p u_{1x} = \varepsilon(\frac{\kappa(\theta)}{v}\theta_x)_x - \varepsilon \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_{1x}^1 d\xi. \tag{2.25}$$

By the definition of Θ_1^1, it holds that

$$\begin{cases} -\varepsilon\int \frac{1}{2}\xi_1|\xi|^2\Theta_1^1 d\xi = \varepsilon^2 N_1 + \varepsilon^3 F_1, \\ N_1 = f_{11}\theta_x\bar{\theta}_x + f_{12}v_x\bar{\theta}_x + f_{13}\bar{\theta}_x^2 + f_{14}\bar{\theta}_{xx}, \\ |F_1| = O(1)[(|v_x| + |\theta_x| + |\bar{\theta}_x| + \varepsilon|u_x| + \varepsilon|\bar{u}_x|)|\bar{u}_x| + |u_x\bar{\theta}_x| + |\bar{u}_{xx}|], \end{cases} \tag{2.26}$$

where the coefficients $f_{1j}, j = 1, 2, 3, 4$ are smooth functions of $(v, \varepsilon u, \theta)$. By (2.21), it is expected that the profile $(\bar{v}, \varepsilon\bar{u}, \bar{\theta})$ for the Boltzmann equation satisfies $\bar{\theta} \cong \bar{v}$. Thus, by choosing only the leading term in (2.25), one obtains that

$$\varepsilon\theta_t = \varepsilon(a(\theta)\theta_x)_x + \frac{3\varepsilon^2}{5}N_{1x}, \tag{2.27}$$

where $a(\theta)$ is given in (2.17). Thus the leading part of (2.27) is the nonlinear diffusion equation (2.17) and an explicit solution $\hat{\theta}(\frac{x}{\sqrt{1+t}})$ is given with the boundary conditions $\hat{\theta}(\pm\infty, t) = \theta_\pm$.

To include more microscopic effect, let the profile $\bar{\theta} \approx \hat{\theta}(\frac{x}{\sqrt{1+t}}) + \varepsilon\theta^{nf}(x, t)$, where $\theta^{nf}(x, t)$ represents the part of the nonlinear diffusion wave coming from the non-fluid component. Moreover, the term $\theta^{nf}(x, t)$ in the form of $\frac{1}{\sqrt{1+t}}D_1(\frac{x}{\sqrt{1+t}})$ is from N_1 in (2.27). Note that $\theta^{nf}(x, t)$ decays faster than $\hat{\theta}(x, t)$ so that it can be viewed as a perturbation around profile $\hat{\theta}(x, t)$. To construct $\theta^{nf}(x, t)$, we linearize the equation (2.27) around $\hat{\theta}(x, t)$ and keep only the linear terms. This leads to a linear equation for $\theta^{nf}(x, t)$ from (2.27)

$$\theta^{nf}_t = (a(\hat{\theta})\theta^{nf}_x)_x + (a'(\hat{\theta})\hat{\theta}_x\theta^{nf})_x + \frac{3}{5}\hat{N}_{1x}, \tag{2.28}$$

where $\hat{N}_1 = (\hat{f}_{11} + \hat{f}_{12} + \hat{f}_{13})(\hat{\theta}_x)^2 + \hat{f}_{14}\hat{\theta}_{xx}$ with $\hat{f}_{1j} = f_{1j}(\tilde{v}, 0, \hat{\theta}), j = 1, 2, 3, 4$. Let

$$g_1(x, t) = \int_{-\infty}^{x} \theta^{nf}(x, t)dx,$$

then integrating (2.28) with respect to x yields that

$$g_{1t} = a(\hat{\theta})g_{1xx} + a'(\hat{\theta})\hat{\theta}_x g_{1x} + \frac{3}{5}\hat{N}_1. \tag{2.29}$$

Note that $\hat{N}_1$ takes the form of $\frac{1}{1+t}D_2(\frac{x}{\sqrt{1+t}})$ and satisfies

$$|\hat{N}_1| = O(1)\delta(1+t)^{-1}e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty.$$

We can check that there exists a self-similar solution $g_1(\eta), \eta = x/\sqrt{1+t}$ for (2.29) with the boundary condition $g_1(-\infty, t) = 0,\ g_1(+\infty, t) = \delta_1$. Here δ_1 satisfies $0 < \delta_1 < \delta$. Note that even though the function $g_1(x, t)$ depends on the constant δ_1, $\theta^{nf}(x, t) = g_{1x}(x, t) \to 0$ as $x \to \pm\infty$. That is, the choice of the constant δ_1 has no influence on the ansantz as long as $|\delta_1| < \delta$. From now on, we fix δ_1 so that the function $g_1(x, t)$ is uniquely determined and its derivative $g_{1x} = \theta^{nf}$ has the property

$$|\theta^{nf}| = |g_{1x}| = O(\delta)(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty.$$

Now we follow the same procedure to construct the second and third components of the velocity profile denoted by $\varepsilon\bar{u}_i,\ i = 2, 3$. That is, the leading part of the equation for εu_i coming from (2.22) is

$$\varepsilon^2 u_{it} = \varepsilon^2(\frac{\mu(\theta)}{\theta}u_{ix})_x - \varepsilon\int \xi_1\xi_i\Theta^1_{1x}d\xi. \tag{2.30}$$

For $i=2,3$, one gets

$$\begin{cases} -\varepsilon\int \xi_1\xi_i\Theta_1^1 d\xi = \varepsilon^2 N_i + \varepsilon^3 F_i, \\ N_i = f_{i1}\theta_x\bar{\theta}_x + f_{i2}v_x\bar{\theta}_x + f_{i3}\bar{\theta}_x^2 + f_{i4}\bar{\theta}_{xx}, \\ |F_i| = O(1)((|v_x|+|\theta_x|+|\bar{\theta}_x|+\varepsilon|u_x|+\varepsilon|\bar{u}_x|)|\bar{u}_x| + |u_x||\bar{\theta}_x| + |\bar{u}_{xx}|), \end{cases} \tag{2.31}$$

with smooth functions $f_{ij}, i=2,3,\ j=1,2,3,4$. Notice that the symbols N_i and F_i, $i=2,3$, used here are for the convenience of notations.

From (2.30) and (2.31), we expect that the profile $\bar{u}_i(x,t)$ takes the form of $\frac{1}{\sqrt{1+t}}h_i(\frac{x}{\sqrt{1+t}})$ and satisfies the following linear equation

$$\varepsilon^2\bar{u}_{it} = \varepsilon^2(\frac{\mu(\hat{\theta})}{\hat{\theta}}\bar{u}_{ix})_x + \varepsilon^2\hat{N}_{ix},\ i=2,3, \tag{2.32}$$

where $\hat{N}_i = (\hat{f}_{i1}+\hat{f}_{i2}+\hat{f}_{i3})(\hat{\theta}_x)^2 + \hat{f}_{i4}\hat{\theta}_{xx},\ \hat{f}_{ij} = f_{ij}(\tilde{v},0,\hat{\theta}), i=2,3, j=1,2,3,4.$

Denote

$$g_i(x,t) = \int_{-\infty}^{x} \bar{u}_i(x,t)dx,$$

then integrating (2.32) with respect to x, one has

$$g_{it} = \frac{\mu(\hat{\theta})}{\hat{\theta}} g_{ixx} + \hat{N}_i. \tag{2.33}$$

For given $\hat{\theta}$, we can check that there exists a self-similar solution $g_i(\eta)$ with $\eta = \frac{x}{\sqrt{1+t}}$ with the boundary conditions $g_i(-\infty,t)=0,\ g_i(+\infty,t)=\delta_i$, where δ_i satisfies $0<\delta_i<\delta$. As we explained before, the choice of the constant δ_i is not essential. From (2.18), we fix δ_i so that the function $g_i(x,t)$ is uniquely determined and the derivative $g_{ix}=\bar{u}_i$ $(i=2,3)$ has the following property

$$|\varepsilon\bar{u}_i| = |\varepsilon g_{ix}| = O(1)\delta\varepsilon(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4b(\theta_\pm)(1+t)}}, \quad \text{as } x\to\pm\infty,$$

where $b(\theta_\pm) = \max\{a(\theta_\pm), \frac{\mu(\theta_\pm)}{\theta_\pm}\}$.

In summary, one can define the profile $(\bar{v},\varepsilon\bar{u},\bar{\theta})$ for the Boltzmann equation as follows. To satisfy the conservation of mass, one needs

$$\varepsilon\bar{v}_t - \varepsilon\bar{u}_{1x} = 0.$$

By letting $\bar{v} = \hat{\theta} + \varepsilon\theta^{nf}$, one gets

$$\varepsilon\bar{u}_1 = \varepsilon[a(\hat{\theta})\hat{\theta}_x + \varepsilon a(\hat{\theta})\theta_x^{nf} + \varepsilon a'(\hat{\theta})\hat{\theta}_x\theta^{nf}] + \frac{3\varepsilon^2}{5}\hat{N}_1. \tag{2.34}$$

However, by plugging (2.34) into the momentum equation of (2.22), we have a non-conservative term containing $\varepsilon^2\hat{N}_{1t}$. To avoid this, one defines

$$\varepsilon\bar{u}_1 = \varepsilon[a(\hat{\theta})\hat{\theta}_x + \varepsilon a(\hat{\theta})\theta_x^{nf} + \varepsilon a'(\hat{\theta})\hat{\theta}_x\theta^{nf}].$$

Similarly, to avoid the non-conservative term $(|\bar{u}|^2)_t$ in the energy equation, set

$$\tilde{\theta} = \theta^{ns} + \varepsilon\theta^{nf} - \frac{1}{2}|\varepsilon\bar{u}|^2.$$

Therefore, the profile $(\bar{v}, \varepsilon\bar{u}, \bar{\theta})$ is finally defined as:

$$\begin{cases} \bar{v} = \hat{\theta} + \varepsilon\theta^{nf}, \\ \varepsilon\bar{u}_1 = \varepsilon[a(\hat{\theta})\hat{\theta}_x + \varepsilon a(\hat{\theta})\theta^{nf}_x + \varepsilon a'(\hat{\theta})\hat{\theta}_x\theta^{nf}], \\ \varepsilon\bar{u}_i = \varepsilon g_{ix}, \ i = 2, 3, \\ \bar{\theta} = \hat{\theta} + \varepsilon\theta^{nf} - \frac{1}{2}|\varepsilon\bar{u}|^2, \end{cases} \tag{2.35}$$

where $\hat{\theta}$ is given by (2.17), θ^{nf} by (2.28) and g_i, $i = 2, 3$ by (2.33). Then a direct but tedious computation shows that

$$\begin{cases} \varepsilon\bar{v}_t - \varepsilon\bar{u}_{1x} = \frac{3\varepsilon^2}{5}\hat{N}_{1x}, \\ \varepsilon^2\bar{u}_{1t} + \bar{p}_x = \frac{4\varepsilon^2}{3}(\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1x})_x + \bar{R}_{1x}, \\ \varepsilon^2\bar{u}_{it} = \varepsilon^2(\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{ix})_x + \varepsilon^2\bar{N}_{ix} + \bar{R}_{ix}, i = 2, 3, \\ \varepsilon\big(\bar{e} + \frac{|\varepsilon\bar{u}|^2}{2}\big)_t + (\varepsilon\bar{p}\bar{u}_1)_x = \varepsilon(\frac{\kappa(\bar{\theta})}{\bar{v}}\bar{\theta}_x)_x + \bar{H}_x + \varepsilon^2\bar{N}_{1x} - \frac{2\varepsilon^2}{5}\hat{N}_{1x} + \bar{R}_{4x}, \end{cases} \tag{2.36}$$

where

$$\begin{aligned} \bar{R}_1 &= \varepsilon^2[a(\hat{\theta})\hat{\theta}_t + (a(\hat{\theta})\theta^{nf})_t] + \bar{p} - p_+ - \frac{4}{3}\varepsilon(\frac{\mu(\bar{\theta})}{\bar{v}}\varepsilon\bar{u}_{1x}) \\ &= O(1)\delta\varepsilon^2(1+t)^{-1}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \bar{R}_i &= \varepsilon[\frac{\mu(\hat{\theta})}{\hat{\theta}} - \frac{\mu(\bar{\theta})}{\bar{v}}]\varepsilon\bar{u}_{ix} + \varepsilon^2(\hat{N}_i - \bar{N}_i) \\ &= O(1)\delta\varepsilon^3(1+t)^{-3/2}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty, \ i = 2, 3, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \bar{R}_4 = &\Big[\frac{5}{3}\varepsilon(a(\hat{\theta})\hat{\theta}_x + a(\hat{\theta})\theta^{nf}_x + a'(\hat{\theta})\hat{\theta}_x\theta^{nf}) - \varepsilon\frac{\kappa(\bar{\theta})}{\bar{v}}\bar{\theta}_x\Big] \\ &+(\bar{p} - p_+)\varepsilon\bar{u}_1 + \varepsilon^2(\hat{N}_1 - \bar{N}_1) - \bar{H} \\ = &\ O(\delta)\varepsilon^3(1+t)^{-3/2}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty, \end{aligned} \tag{2.39}$$

$$\hat{N}_i = O(1)\delta(1+t)^{-1}e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty, \ i = 1, 2, 3, \tag{2.40}$$

with $c(\theta_\pm) = \max\{a(\theta_\pm), \frac{1}{2}b(\theta_\pm)\}$, $\bar{N}_i, i = 1, 2, 3$, and $\bar{H}$ are the corresponding functions defined in (2.24), (2.26) and (2.31) by substituting the variable $(v, \varepsilon u, \theta)$ by the profile $(\bar{v}, \varepsilon\bar{u}, \bar{\theta})$. Note that the decay rate of $\bar{R}_i, i = 2, 3, 4$ is of order $\varepsilon^3(1+t)^{-3/2}$. Furthermore, even though the decay rate of $\bar{R}_1$ is still $\varepsilon^2(1+t)^{-1}$, it is sufficient to obtain the desired a priori estimates through some subtle analysis coming from the intrinsic dissipation mechanism in the momentum equations as shown in the following.

Define

$$\bar{M} = \frac{\bar{v}^{-1}}{\sqrt{(2\pi R\bar{\theta})^3}} \exp\left(-\frac{|\xi - \varepsilon\bar{u}|^2}{2R\bar{\theta}}\right), \quad \bar{G}_0 = L^{-1}_{\bar{M}}\left(\frac{1}{\bar{v}}\bar{P}_1(\xi_1\bar{M}_x)\right),$$

and

$$\bar{f} = \bar{M} + \varepsilon\bar{G}_0.$$

Then it follows from (2.36) that

$$\varepsilon\bar{f}_t - \frac{\varepsilon\bar{u}_1}{\bar{v}}\bar{f}_x + \frac{1}{\bar{v}}\xi_1\bar{f}_x = L_{\bar{M}}\bar{G}_0 + \varepsilon Q(\bar{G}_0, \bar{G}_0) + \bar{R}_{\bar{f}}, \tag{2.41}$$

where

$$\bar{R}_{\bar{f}} = \varepsilon^2 \hat{B}_2(x,t,\xi)\bar{M} + \varepsilon^2 \bar{G}_{0t} - \varepsilon \frac{\varepsilon \bar{u}_1}{\bar{v}} \bar{G}_{0x} + \varepsilon \bar{P}_1\Big(\frac{\varepsilon}{\bar{v}} \bar{G}_{0x}\Big) - \varepsilon Q(\bar{G}_0, \bar{G}_0),$$

and $|\hat{B}_2(x,t,\xi)| = O(1)\delta(1+t)^{-\frac{3}{2}} e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}} |\xi|^3$, as $x \to \pm\infty$.

Remark 2.3 *From the definition of* $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ *in* (2.19) *and the definition of* $(\bar{v}, \bar{u}_1, \bar{\theta})$ *in* (2.35), *it holds that*

$$|(\bar{v} - \tilde{v}, \bar{u}_1 - \tilde{u}_1, \bar{\theta} - \tilde{\theta})(x,t)| = O(1)\delta\varepsilon(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \tag{2.42}$$

that implies that the ansantz $(\bar{v}, \bar{u}_1, \bar{\theta})$ *well approximates* $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ *when* ε *is small.*

2.2 Main result

Now we consider the system (2.9)-(2.10) with the initial data

$$(v, u, \theta)|_{t=0} = (\bar{v}, \bar{u}, \bar{\theta})(x, 0), \qquad G(x,t)|_{t=0} = \bar{G}(x, 0). \tag{2.43}$$

Then the main result in this paper can be stated as follows.

Theorem 2.4 *Let* $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ *be the profile defined in* (2.35) *with strength* $\delta = |\theta_+ - \theta_-|$. *Then there exist small positive constants* δ_0 *and* ε_0 *and a global Maxwellian* $M_* = M_{[v_*, u_*, \theta_*]}$, *such that when* $\delta \le \delta_0$ *and* $\varepsilon \le \varepsilon_0$, *the Cauchy problem* (2.9)- (2.10) *with the initial data* (2.43) *has a unique global solution* (v, u, θ, G) *satisfying, for any sufficiently small but fixed positive constant* $\vartheta > 0$,

$$\begin{cases} \|(v-\bar{v}, \varepsilon u - \varepsilon\bar{u}, \theta - \bar{\theta})(t)\|^2_{L^2_x} \le C\sqrt{\delta}\varepsilon^3(1+t)^{-1+C_0\sqrt{\delta}}, \\ \|(v-\bar{v}, \varepsilon u - \varepsilon\bar{u}, \theta - \bar{\theta})_x(t)\|^2_{L^2_x} \le C\sqrt{\delta}\varepsilon^2(1+t)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}}, \\ \|f_{xx}(t)\|^2_{L^2_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))} + \|(v-\bar{v}, \varepsilon u - \varepsilon\bar{u}, \theta - \bar{\theta})_{xx}(t)\|^2_{L^2_x} \le C\sqrt{\delta}(1+t)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}}, \\ \|(G-\bar{G})(t)\|^2_{L^2_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))} \le C\sqrt{\delta}(1+t)^{-\frac{1}{2}}, \\ \|(G-\bar{G})_x(t)\|^2_{L^2_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))} \le C\sqrt{\delta}(1+t)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}}, \end{cases} \tag{2.44}$$

that implies that

$$\begin{cases} \|(v-\bar{v}, \varepsilon u - \varepsilon\bar{u}, \theta - \bar{\theta})(t)\|_{L^\infty_x} \le C\delta^{\frac{1}{4}}\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{8}+\frac{3}{4}\vartheta}, \\ \|(v-\bar{v}, \varepsilon u - \varepsilon\bar{u}, \theta - \bar{\theta})_x(t)\|_{L^\infty_x} \le C\delta^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{4}+\vartheta}, \end{cases} \tag{2.45}$$

where C *and* C_0 *are positive constants independent of* ε *and* δ.

The following result justifies the hydrodynamic limit of the rescaled Boltzmann equation (1.1) to the diffusion wave $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ global in time.

Corollary 2.5 *Under the conditions of Theorem 2.4, from* (2.42) *and* (2.45), *it holds that*

$$\begin{cases} |(v-\tilde{v}, \theta - \tilde{\theta})(x,t)| \le C\varepsilon(1+t)^{-\frac{1}{2}} \to 0, \\ |(u_1 - \tilde{u}_1)(x,t)| \le C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{1}{2}}, \to 0, \end{cases} \quad \text{as } \varepsilon \to 0, \tag{2.46}$$

that is, the fluid part (v, u_1, θ) *of the solution of the rescaled Boltzmann equation* (1.1) *converges to the diffusion wave solution* $(\tilde{v}, \tilde{u}_1, \tilde{\theta})$ *of* (2.20) *in the sense of* (2.46) *as* $\varepsilon \to 0$, *which reveals that* v *and* θ *are diffusive.*

Remark 2.6 *The above Corollary shows that if the zero order function in* (1.5) *is not a global Maxwellian, then one has to consider the effect of diffusive wave in the diffusive limit of rescaled Boltzmann equation* (1.1)*.*

Since the scaled velocity $\tilde{u}_1$ is actually induced by the variation of temperature $\tilde{\theta}$, i.e., $\tilde{u}_1 = a(\tilde{\theta})\tilde{\theta}_x$. The following result shows that the scaled velocity u_1 is also induced by the variation of temperature θ in some sense when ε is small. From the definition of $\hat{\theta}(\eta)$ with $\eta = \frac{x}{\sqrt{1+t}}$ in (2.17) and (2.18), it can be seen that $\hat{\theta}$ is monotonic. To be definite and without loss of generality, let us assume that $\theta_- < \theta_+$, that is, $\hat{\theta}$ is monotonically increasing. Then there exists a positive constant $\eta_0 > 0$ such that

$$\hat{\theta}'(\eta) > c_{\eta_0}\delta, \qquad \text{for } \ |\eta| \le \eta_0, \tag{2.47}$$

where c_{η_0} depends on η_0 and $c_{\eta_0} \to 0$ as $\eta_0 \to +\infty$.

Corollary 2.7 *Under the conditions of Theorem 2.4 and $\theta_- < \theta_+$, for any fixed $\eta_0 > 0$, there exists a small positive constant $\varepsilon_1 = \varepsilon_1(\eta_0) \le \varepsilon_0$, such that if $\varepsilon \le \varepsilon_1$, then it follows from* (2.47) *and* (2.45) *that*

$$\begin{cases} 0 < \frac{c_{\eta_0}\delta}{C_1\sqrt{1+t}} < \frac{1}{C_1}\hat{\theta}_x \le u_1(x,t) \le C_1\hat{\theta}_x, \\ \qquad 0 < \frac{1}{2}\hat{\theta}_x \le \theta_x(x,t) \le \frac{3}{2}\hat{\theta}_x, \end{cases} \quad \text{for } |x| \le \eta_0(1+t)^{\frac{1}{2}}, \ t \ge 0, \tag{2.48}$$

that is

$$\frac{2}{3C_1}\theta_x(x,t) \le u_1(x,t) \le 2C_1\theta_x(x,t), \ \text{for } |x| \le \eta_0(1+t)^{\frac{1}{2}}, \ t \ge 0, \tag{2.49}$$

where C_1 is a suitably large positive constant depending only on $\theta_\pm$. In particular, (2.49) *implies that variation of the temperature induces a non-trivial flow of higher order in the following parabolic region*

$$\Big\{(x,t) \ : \ |x| \le \eta_0(1+t)^{\frac{1}{2}}, \ t \ge 0 \ \Big\}.$$

3 Stability Analysis

In this section, we will investigate the stability of the profile constructed in (2.36) for the Boltzmann equation (1.1). This section is organized as follows: in Section 3.1, the fluid type system (2.2) is reformulated in terms of the integrated variables; Section 3.2 is devoted to the lower order estimate, while Section 3.3 is for the derivative estimate.

3.1 Reformulated system

We now reformulate the system by introducing a scaling for the independent variables. Set

$$y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}. \tag{3.1}$$

In the following, we will also use the notations $(v,u,\theta)(\tau,y)$ and $(\bar{v},\bar{u},\bar{\theta})(\tau,y)$, etc., in the scaled independent variables. Set the perturbation around the profile $(\bar{v},\bar{u},\bar{\theta})(\tau,y)$ by

$$\phi = v - \bar{v}, \psi = \varepsilon u - \varepsilon\bar{u}, \zeta = \theta - \bar{\theta},$$

and

$$(\Phi,\Psi,\bar{W})(y,\tau)=\int_{-\infty}^{y}\Big(\phi,\ \psi,\ (\theta+\frac{|\varepsilon u|^2}{2})-(\bar\theta+\frac{|\varepsilon\bar u|^2}{2})\Big)(z,\tau)dz.$$

Then we have $(\phi,\psi)=(\Phi,\Psi)_y$ and $\zeta+\frac{1}{2}|\Psi_y|^2+\sum_{i=1}^{3}\varepsilon\bar u_i\Psi_{iy}=\bar W_y$.

Subtracting (2.36) from the equation (2.22) and integrating the reduced system yield

$$\begin{cases}\Phi_\tau-\Psi_{1y}=-\dfrac{2}{5p_+}\varepsilon^2\hat N_1,\\ \Psi_{1\tau}+p-\bar p=\dfrac{4\varepsilon}{3}\Big(\dfrac{\mu(\theta)}{v}u_{1y}-\dfrac{4}{3}\dfrac{\mu(\bar\theta)}{\bar v}\bar u_{1y}\Big)-\varepsilon\displaystyle\sum_{j=1}^{2}\int\xi_1^2\Theta_1^jd\xi-\bar R_1,\\ \Psi_{i\tau}=\varepsilon\Big(\dfrac{\mu(\theta)}{v}u_{iy}-\dfrac{\mu(\bar\theta)}{\bar v}\bar u_{iy}\Big)+\varepsilon^2(N_i-\bar N_i)+\varepsilon^3F_i-\varepsilon\displaystyle\int\xi_1\xi_i\Theta_1^2d\xi-\bar R_i,i=2,3,\\ \bar W_\tau+\varepsilon pu_1-\varepsilon\bar p\bar u_1=\Big(\dfrac{\kappa(\theta)}{v}\theta_y-\dfrac{\kappa(\bar\theta)}{\bar v}\bar\theta_y\Big)+(H-\bar H)+\varepsilon^2(N_1-\bar N_1)+\varepsilon^3F_1\\ \qquad\qquad-\varepsilon\displaystyle\int\frac{1}{2}\xi_1|\xi|^2\Theta_1^2d\xi-\bar R_4+\frac{2}{5}\varepsilon^2\hat N_1.\end{cases}\tag{3.2}$$

Since the variable $\bar W$ is the anti-derivative of the total energy, not the temperature, it is more convenient to introduce another variable

$$W=\bar W-\varepsilon\bar u_1\Psi_1.$$

It follows that

$$\zeta=W_y-Y,\ \text{with}\ Y=\frac{1}{2}|\Psi_y|^2-\varepsilon\bar u_{1y}\Psi_1+\varepsilon\bar u_2\Psi_{2y}+\varepsilon\bar u_3\Psi_{3y}.$$

Using the new variable W and linearizing the left hand side of the system (3.2) by using the formula of H in (2.24) give that

$$\begin{cases}\Phi_\tau-\Psi_{1y}=-\dfrac{3}{5}\varepsilon^2\hat N_1,\\ \Psi_{1\tau}-\dfrac{p_+}{\bar v}\Phi_y+\dfrac{2}{3\bar v}W_y=\dfrac{4}{3}\dfrac{\mu(\bar\theta)}{\bar v}\Psi_{1yy}+\dfrac{4}{3}(\dfrac{\mu(\theta)}{v}-\dfrac{\mu(\bar\theta)}{\bar v})\varepsilon u_{1y}\\ \qquad-\varepsilon\displaystyle\sum_{j=1}^{2}\int\xi_1^2\Theta_1^jd\xi+J_1+\frac{2}{3\bar v}Y-\bar R_1\doteq\frac{4}{3}\frac{\mu(\bar\theta)}{\bar v}\Psi_{1yy}+Q_1,\\ \Psi_{i\tau}=\dfrac{\mu(\bar\theta)}{\bar v}\Psi_{iyy}+\varepsilon(\dfrac{\mu(\theta)}{v}-\dfrac{\mu(\bar\theta)}{\bar v})u_{iy}+\varepsilon^2(N_i-\bar N_i)+\varepsilon^3F_i\\ \qquad-\varepsilon\displaystyle\int\xi_1\xi_i\Theta_1^2d\xi-\bar R_i\doteq\frac{\mu(\bar\theta)}{\bar v}\Psi_{iyy}+Q_i,i=2,3,\\ W_\tau+p_+\Psi_{1y}=\dfrac{\kappa(\bar\theta)}{\bar v}W_{yy}+(\dfrac{\kappa(\theta)}{v}-\dfrac{\kappa(\bar\theta)}{\bar v})\theta_y+\varepsilon^2(N_1-\bar N_1)+\varepsilon^3F_1+\dfrac{4\varepsilon}{3}\dfrac{\mu(\theta)}{v}u_{1y}\Psi_{1y}\\ \qquad+\varepsilon^3\displaystyle\sum_{i=2}^{3}[\frac{\mu(\theta)}{v}u_iu_{iy}-\varepsilon\frac{\mu(\bar\theta)}{\bar v}\bar u_i\bar u_{iy}]-\varepsilon\bar u_{1\tau}\Psi_1+J_2-\varepsilon\int\frac{1}{2}\xi_1|\xi|^2\Theta_1^2d\xi\\ \qquad+\varepsilon^2\bar u_1\sum_{j=1}^{2}\int\xi_1^2\Theta_1^jd\xi-\frac{\kappa(\bar\theta)}{\bar v}Y_y+\frac{2}{5}\varepsilon^2\hat N_1+\varepsilon\bar u_1\bar R_1-\bar R_4\\ \qquad\doteq\dfrac{\kappa(\bar\theta)}{\bar v}W_{yy}+\dfrac{2}{5}\varepsilon^2\hat N_1+Q_4,\end{cases}\tag{3.3}$$

where

$$
\begin{cases}
J_1 = \frac{\bar{p}-p_+}{\bar{v}}\Phi_y - [p-\bar{p}+\frac{\bar{p}}{\bar{v}}\Phi_y - \frac{2}{3\bar{v}}(\theta-\bar{\theta})] = O(1)(\Phi_y^2+(\theta-\bar{\theta})^2+|\varepsilon\bar{u}|^4),\\
J_2 = (p_+-p)\Psi_{1y} = O(1)(\Phi_y^2+\Psi_{1y}^2+(\theta-\bar{\theta})^2+|\varepsilon\bar{u}|^4),\\
Q_1 = \frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})u_{1y} - \varepsilon\sum_{j=1}^{2}\int \xi_1^2\Theta_1^j d\xi + J_1 + \frac{2}{3\bar{v}}Y - \bar{R}_1,\\
Q_i = \varepsilon(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})u_{iy} + \varepsilon^2(N_i-\bar{N}_i) + \varepsilon^3 F_i - \varepsilon\int \xi_1\xi_i\Theta_1^2 d\xi - \bar{R}_i,\ i=2,3,\\
Q_4 = (\frac{\kappa(\theta)}{v}-\frac{\kappa(\bar{\theta})}{\bar{v}})\theta_y + \varepsilon^2(N_1-\bar{N}_1) + \varepsilon^3 F_1 + \frac{4\varepsilon}{3}\frac{\mu(\theta)}{v}u_{1y}\Psi_{1y}\\
\qquad + \varepsilon^3\sum_{i=2}^{3}[\frac{\mu(\theta)}{v}u_iu_{iy} - \frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_i\bar{u}_{iy}] - \varepsilon\bar{u}_{1\tau}\Psi_1 + J_2 - \varepsilon\int\frac{1}{2}\xi_1|\xi|^2\Theta_1^2 d\xi\\
\qquad\qquad + \varepsilon^2\bar{u}_1\sum_{j=1}^{2}\int \xi_1^2\Theta_1^j d\xi - \frac{\kappa(\bar{\theta})}{\bar{v}}Y_y + \varepsilon\bar{u}_1\bar{R}_1 - \bar{R}_4.
\end{cases}
\tag{3.4}
$$

The equation of microscopic component $\tilde{G}$ given in (2.23) in the coordinate (y,τ) becomes

$$
\begin{aligned}
v\tilde{G}_\tau - vL_M\tilde{G} = &-\frac{1}{R\theta}P_1[\xi_1(\frac{|\xi-\varepsilon u|^2}{2\theta}\frac{1}{\varepsilon}\zeta_y + \xi\cdot\frac{1}{\varepsilon}\psi_y)M]\\
&+\varepsilon u_1 G_y - vP_1(\xi_1 G_y) + \varepsilon vQ(G,G) - v\bar{G}_\tau.
\end{aligned}
\tag{3.5}
$$

In the scaling of (3.1), the equation (2.7) reads

$$
f_\tau - \varepsilon\frac{u_1}{v}f_y + \frac{\xi_1}{v}f_y = \varepsilon L_M G + \varepsilon^2 Q(G,G). \tag{3.6}
$$

Set

$$
\tilde{f} \doteq f - \bar{f}, \tag{3.7}
$$

then from (3.6) and (2.41), we have

$$
\begin{aligned}
v\tilde{f}_\tau - \varepsilon u_1\tilde{f}_y + \xi_1\tilde{f}_y = &\varepsilon vL_M\tilde{G} + \varepsilon\big[vL_M\bar{G} - \bar{v}L_{\bar{M}}\bar{G}_0\big] + \varepsilon^2\big[vQ(G,G) - \bar{v}Q(\bar{G}_0,\bar{G}_0)\big]\\
&- \phi\bar{f}_\tau + \psi\bar{f}_y - \varepsilon v\bar{R}_{\bar{f}}.
\end{aligned}
\tag{3.8}
$$

Note that to prove the main theorem in this paper, it is sufficient to prove the following *a priori* estimate in the scaled independent variables based on the construction of the approximate profile.

Theorem 3.1 (A priori estimate) *For any sufficiently small and fixed positive constant* $\vartheta > 0$, *there exist small positive constants* $\delta_2 > 0, \varepsilon_2 > 0$ *and a global Maxwellian* $M_* = M_{[\rho_*,u_*,\theta_*]}$ *such that if* $\delta \le \delta_2$ *and* $\varepsilon \le \varepsilon_2$, *then the Cauchy problem* (3.3), (3.5) *and* (3.8) *admits a unique smooth solution satisfying*

$$
\begin{cases}
\|(\Phi,\Psi,W)(\tau)\|_{L^\infty}^2 \le C\sqrt{\delta}\varepsilon, \quad \|(\phi,\psi,\zeta)(\tau)\|_{L_y^2}^2 \le C\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}},\\
\|(\phi,\psi,\zeta)_y(\tau)\|_{L_y^2} + \|(\phi,\psi,\zeta)_{yy}(\tau)\|_{L_y^2} + \sum_{|\alpha|=2}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\partial^\alpha\tilde{f}|^2}{M_*}d\xi dy \le C\sqrt{\delta}\varepsilon^3(1+\varepsilon^2\tau)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}},\\
\varepsilon\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\tilde{G}|^2}{M_*}d\xi dy \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{1}{2}}, \quad \sum_{|\alpha|=1}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\partial^\alpha\tilde{G}|^2}{M_*}d\xi dy \le C\sqrt{\delta}\varepsilon(1+\varepsilon^2\tau)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}},
\end{cases}
\tag{3.9}
$$

where C, C_0 *are positive constants independent of* δ *and* ε.

In the next subsection, we will work on the reformulated system (3.3) and (3.8). Since the local existence of the solution can be proved similarly as the discussion in [20] and [42], we will omit it here for brevity. To prove the global existence, it is sufficient to close the following a priori estimate:

$$\begin{aligned} N(\tau)=\sup_{0\le s\le\tau}&\Big\{\varepsilon^{-1}\|(\Phi,\Psi,W)\|_{L^\infty}^2+\varepsilon^{-2}\|(\phi,\psi,\zeta)\|_{L^2}^2+\varepsilon^{-3}\|(\phi_y,\psi_y,\zeta_y)\|_{L^2}^2\\ &+\|\int_{\mathbb{R}^3}\frac{|\tilde{G}|^2}{M_*}d\xi\|_{L^\infty}+\int_{\mathbb{R}}\int_{\mathbb{R}^3}\Big(\sum_{|\alpha|=1}\varepsilon^{-1}\frac{|\partial^\alpha\tilde{G}|^2}{M_*}+\sum_{|\alpha|=2}\varepsilon^{-3}\frac{|\partial^\alpha\tilde{f}|^2}{M_*}\Big)d\xi dy\Big\}\le\lambda_0^2, \end{aligned}\tag{3.10}$$

where λ_0 is positive small constant depending on the initial data and M_* is a global Maxwellian to be chosen later.

Before proving the a priori estimate (3.10), we list some lemmas based on the celebrated H-theorem for later use. The first one is from [16].

Lemma 3.2 *There exists a positive constant $C>0$ such that*

$$\int_{\mathbb{R}^3}\frac{\nu(|\xi|)^{-1}Q(f,g)^2}{M}d\xi\le C\Big\{\int_{\mathbb{R}^3}\frac{\nu(|\xi|)f^2}{M}d\xi\cdot\int_{\mathbb{R}^3}\frac{g^2}{M}+\int_{\mathbb{R}^3}\frac{f^2}{M}d\xi\cdot\int_{\mathbb{R}^3}\frac{\nu(|\xi|)g^2}{M}\Big\},$$

where M can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 3.2, the following three lemmas are from [35].

Lemma 3.3 *If $\theta/2<\theta_*<\theta$, then there exist two positive constants $\bar\sigma=\bar\sigma(\rho,u,\theta;\rho_*,u_*,\theta_*)>0$ and $\eta_0=\eta_0(\rho,u,\theta;\rho_*,u_*,\theta_*)>0$ such that if $|\rho-\rho_*|+|\varepsilon u-u_*|+|\theta-\theta_*|<\eta_0$, we have for $h(\xi)\in N^\perp$,*

$$-\int_{\mathbb{R}^3}\frac{hL_Mh}{M_*}d\xi\ge\bar\sigma\int_{\mathbb{R}^3}\frac{\nu(|\xi|)h^2}{M_*}d\xi,$$

where $M_*=M_{[\rho_*,u_*,\theta_*]}$ and the definition of $M=M_{[\rho,\varepsilon u,\theta]}$ can be found in (1.4).

Lemma 3.4 *Under the assumptions in Lemma 3.3, we have*

$$\begin{cases}\displaystyle\int_{\mathbb{R}^3}\frac{\nu(|\xi|)}{M}|L_M^{-1}h|^2d\xi\le\bar\sigma^{-2}\int_{\mathbb{R}^3}\frac{\nu(|\xi|)^{-1}h^2}{M}d\xi,\\ \displaystyle\int_{\mathbb{R}^3}\frac{\nu(|\xi|)}{M_*}|L_M^{-1}h|^2d\xi\le\bar\sigma^{-2}\int_{\mathbb{R}^3}\frac{\nu(|\xi|)^{-1}h^2}{M_*}d\xi,\end{cases}$$

for each $h(\xi)\in N^\perp$.

Lemma 3.5 *Under the conditions in Lemma 3.3, there exists a constant $C>0$ such that for positive constants k and λ, we have*

$$\Big|\int_{\mathbb{R}^3}\frac{g_1P_1(|\xi|^kg_2)}{M_*}d\xi-\int_{\mathbb{R}^3}\frac{g_1|\xi|^kg_2}{M_*}d\xi\Big|\le C\int_{\mathbb{R}^3}\frac{\lambda|g_1|^2+\lambda^{-1}|g_2|^2}{M_*}d\xi.$$

Note that (3.10) also gives the a priori estimates on $\|(\phi_\tau,\psi_\tau,\zeta_\tau)\|$, $\|\partial^\alpha(\phi,\psi,\zeta)\|$ and $\displaystyle\int\int\frac{|\partial^\alpha\tilde{G}|^2}{M_*}d\xi dx$ ($|\alpha|=2$). In fact, from (3.1), (2.8) and (3.10), one has

$$\begin{aligned} &\|(\phi_\tau,\psi_\tau,\zeta_\tau)\|^2\le C\|(v_\tau,\varepsilon u_\tau,\theta_\tau)\|^2+C\delta\varepsilon^3(1+\varepsilon^2\tau)^{-\frac32}\\ &\le C\Big(\|(p_y-\bar p_y,\varepsilon pu_{1y}-\varepsilon\bar p\bar u_{1y})\|^2+\|(\bar p_y,\varepsilon\bar p\bar u_{1y})\|^2+\varepsilon^2\int\int\frac{|\tilde G_y|^2+|\bar G_y|^2}{M_*}d\xi dy\Big)+C\delta\varepsilon^3(1+\varepsilon^2\tau)^{-\frac32}\\ &\le C\Big(\|(\phi_y,\psi_y,\zeta_y)\|^2+\delta\varepsilon^2\|(\phi,\psi,\zeta)\|^2+\varepsilon^2\int\int\frac{|\tilde G_y|^2}{M_*}d\xi dy\Big)+C\delta\varepsilon^3(1+\varepsilon^2\tau)^{-\frac32}\\ &\le C(\delta+\lambda_0^2)\varepsilon^3, \end{aligned}\tag{3.11}$$

where we have used the fact that

$$\int \Big(\int \xi_1^2 G_y d\xi\Big)^2 dy \le C\int\int \frac{G_y^2}{M_*} d\xi dy.$$

To derive the a priori assumption on $\|\partial^\alpha(\phi,\psi,\zeta)\|$, $(|\alpha|=2)$, we use the definition of ρ, $m=\varepsilon\rho u$ and $\rho(\theta+\frac{1}{2}|\varepsilon u|^2)$. Let $|\alpha|=2$, by (1.3), one can obtain

$$\begin{aligned}&\|\partial^\alpha(\rho,m,\rho(\theta+\frac{1}{2}|\varepsilon u|^2))\|^2\le C\int\int\frac{|\partial^\alpha f|^2}{M_*}d\xi dy\\&\le C\int\int\frac{|\partial^\alpha \tilde f|^2}{M_*}d\xi+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\le C(\lambda_0^2+\delta)\varepsilon^3.\end{aligned}\tag{3.12}$$

This yields that

$$\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\le C\int\int\frac{|\partial^\alpha\tilde f|^2}{M_*}d\xi dy+\sum_{|\beta|=1}\|\partial^\beta(\phi,\psi,\zeta)\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\le C(\lambda_0^2+\delta)\varepsilon^3.\tag{3.13}$$

Finally, one has

$$\begin{aligned}&\varepsilon^2\int\int\frac{|\partial^\alpha\tilde G|^2}{M_*}d\xi dy\le C\int\int\frac{|\partial^\alpha\tilde f|^2}{M_*}d\xi dy+C\int\int\frac{|\partial^\alpha(M-\bar M)|^2}{M_*}d\xi dy\\&\le C\int\int\frac{|\partial^\alpha\tilde f|^2}{M_*}d\xi dy+C\sum_{|\alpha|=1,2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C\delta\varepsilon^4(1+t)^{-2}\le C(\varepsilon_0+\delta)^2\varepsilon^3,\quad |\alpha|=2.\end{aligned}\tag{3.14}$$

3.2 Lower order estimate

We are now ready to derive the lower order estimate. Multiplying $(3.3)_1$ by $p_+\Phi$, $(3.3)_2$ by $\bar v\Psi_1$, $(3.3)_3$ by Ψ_i, $(3.3)_4$ by $\frac{2}{3p_+}W$ with $p_+=\frac{2}{3}$ respectively and adding all the equations, one can obtain

$$\begin{aligned}&\Big(\frac{p_+}{2}\Phi^2+\frac{1}{3p_+}W^2+\frac{\bar v}{2}\Psi_1^2+\frac{1}{2}\sum_{i=2}^3\Psi_i^2\Big)_\tau+\frac{4\mu(\bar\theta)}{3}\Psi_{1y}^2+\sum_{i=2}^3\frac{\mu(\bar\theta)}{\bar v}\Psi_{iy}^2+\frac{2\kappa(\bar\theta)}{3p_+\bar v}W_y^2\\&=\frac{2}{5}\varepsilon^2\hat N_1(-\Phi+\frac{2}{3p_+}W)+\frac{1}{2}\bar v_\tau\Psi_1^2+\bar vQ_1\Psi_1+\sum_{i=2}^3Q_i\Psi_i+\frac{2}{3p_+}WQ_4\\&-(\frac{4\mu(\bar\theta)}{3})_y\Psi_1\Psi_{1y}-\sum_{i=2}^3(\frac{\mu(\bar\theta)}{\bar v})_y\Psi_i\Psi_{iy}-(\frac{2\lambda(\bar\theta)}{3p_+\bar v})_yWW_y+(\cdots)_y.\end{aligned}\tag{3.15}$$

Here and in the sequel the notation $(\cdots)_y$ represents the term in the conservative form so that it vanishes after integration. Since it has no effect on the energy estimates, we do not write them out in detail.

Note that the term $Q_1\Psi_1$ contains $(1+t)^{-1}\Psi_1$ which can not be controlled by the dissipation from the viscosity and heat conductivity. So is the term $\tilde N_1(-\Phi+\frac{2}{3p_+}W)$. As we will see later, an intrinsic dissipation associated with the profile is derived by the diagonal method and weighted energy estimate to control the above two terms. Let us consider the equations for the conservation of the mass, the first component of velocity and energy by defining

$$V=(\Phi,\Psi_1,W)^t,$$

where $(\cdot,\cdot,\cdot)^t$ means the transpose of the vector $(\cdot,\cdot,\cdot)$. Then from (3.3), we have

$$V_\tau+A_1V_y=A_2V_{yy}+A_3,\tag{3.16}$$

where

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_+}{\bar v} & 0 & \frac{2}{3\bar v} \\ 0 & p_+ & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4\mu(\bar\theta)}{3\bar v} & 0 \\ 0 & 0 & \frac{\lambda(\bar\theta)}{\bar v} \end{pmatrix},$$

$$A_3 = (-\frac{3}{5}\varepsilon^2 \hat N_1, \quad Q_1, \quad Q_4 + \frac{2}{5}\varepsilon^2 \hat N_1)^t.$$

Direct computation shows that the eigenvalues of the matrix A_1 are $\lambda_1, 0, \lambda_3$. Here $\lambda_3 = -\lambda_1 = \sqrt{\frac{5p_+}{3\bar v}}$. The corresponding normalized left and right eigenvectors can be chosen as

$$l_1 = \sqrt{3/10}(-1, -\frac{5}{3\lambda_3}, \frac{2}{3p_+}),\ l_2 = \sqrt{2/5}(1, 0, \frac{1}{p_+}),\ l_3 = \sqrt{3/10}(-1, \frac{5}{3\lambda_3}, \frac{2}{3p_+}),$$
$$r_1 = \sqrt{3/10}(-1, -\lambda_3, p_+)^t,\ r_2 = \sqrt{2/5}(1, 0, \frac{3}{2}p_+)^t,\ r_3 = \sqrt{3/10}(-1, \lambda_3, p_+)^t,$$

such that

$$l_i r_j = \delta_{ij},\ i,j = 1,2,3, \quad LA_1R = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

with

$$L = (l_1, l_2, l_3)^t,\ R = (r_1, r_2, r_3).$$

Let

$$B = LV = (b_1, b_2, b_3),$$

then multiplying the equations (3.16) by the matrix L yields that

$$B_\tau + \Lambda B_y = LA_2RB_{yy} + 2LA_2R_yB_y + [(L_\tau + \Lambda L_y)R + LA_2R_{yy}]B + LA_3. \tag{3.17}$$

A direct computation shows that $LA_2R = A_4$ is a non-negative matrix. From (3.17), we will apply weighted energy method to derive an intrinsic dissipation. Since we have assumed that $\hat\theta_y > 0$. Let $v_1 = \frac{\hat\theta}{\theta_+}$, then $|v_1 - 1| \le C\delta$. Multiplying (3.17) by $\bar B = (v_1^n b_1, b_2, v_1^{-n} b_3)$ with a large positive integer n which will be chosen later, we have

$$\begin{aligned} &\Big(\tfrac12 v_1^n b_1^2 + \tfrac12 b_2^2 + \tfrac12 v_1^{-n} b_3^2\Big)_\tau - (\tfrac{v_1^n}{2})_\tau b_1^2 - (\tfrac{v_1^{-n}}{2})_\tau b_3^2 + \bar B_y A_4 B_y + \bar B A_{4y} B_y \\ &-\tfrac12 v_1^{n-1}(n\lambda_1 v_{1y} + v_1\lambda_{1y})b_1^2 + \tfrac12 v_1^{-n-1}(n\lambda_3 v_{1y} - v_1\lambda_{3y})b_3^2 \\ &= 2\bar B LA_2R_yB_y + \bar B[L_tR + LA_2R_{yy}]B + \bar B\Lambda L_xRB + \bar BLA_3 + (\cdots)_x. \end{aligned} \tag{3.18}$$

Let

$$E_1 = \int \Big(\frac{p_+}{2}\Phi^2 + \frac{1}{3p_+}W^2 + \frac{\bar v}{2}\Psi_1^2 + \frac12\sum_{i=2}^3 \Psi_i^2\Big)dy + \int (\frac{v_1^n}{2}b_1^2 + \frac12 b_2^2 + \frac{v_1^{-n}}{2}b_3^2)dy,$$
$$K_1 = \int (\frac{4\mu(\bar\theta)}{3}\Psi_{1y}^2 + \sum_{i=2}^3 \frac{\mu(\bar\theta)}{\bar v}\Psi_{iy}^2 + \frac{2\lambda(\bar\theta)}{3p_+\bar v}W_y^2 + B_yA_4B_y)dy.$$

Note that

$$\begin{aligned} |\int (\bar B - B)_y A_4 B_y dy &\le C\delta \int |B_y|^2 dy + C\delta^{-1}\int |\hat\theta_y|^2 |B|^2 dy \\ &\le C\varepsilon^2\delta(1+t)^{-1}E_1 + C\delta K_1 + C\delta \int |\Phi_y|^2 dy. \end{aligned} \tag{3.19}$$

Similarly, the terms in the last second line of (3.18), $\bar{B}A_{4y}B_y$, $\bar{B}LA_2R_yB_y$ and $\bar{B}[L_\tau R + LA_2R_{yy}]B$ satisfy the same estimate. For $\bar{B}\Lambda L_yRB$ and $\bar{B}LA_3$, we need to use the explicit presentation. By the choice of the characteristic matrix L and R, we have

$$\Lambda L_yR = \frac{1}{2}\lambda_{3y}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad LA_3 = \begin{pmatrix} \sqrt{\frac{2}{15}}\frac{1}{p_+}(\varepsilon^2\hat{N}_1 + Q_4) - \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3} \\ \sqrt{\frac{2}{5}}\frac{Q_4}{p_+} \\ \sqrt{\frac{2}{15}}\frac{1}{p_+}(\varepsilon^2\hat{N}_1 + Q_4) + \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3} \end{pmatrix}.$$

Thus

$$\bar{B}\Lambda L_yRB = \frac{1}{2}\lambda_{3y}(v_1^n b_1^2 + v_1^{-n}b_1b_3 - v_1^n b_1b_3 - v_1^{-n}b_3^2),$$

$$\bar{B}LA_3 = \sqrt{\frac{2}{15}}\frac{1}{p_+}\varepsilon^2\hat{N}_1(v_1^n b_1 + v_1^{-n}b_3) + q_1v_1^n b_1 + q_2b_2 + q_3v_1^{-n}b_3, \tag{3.20}$$

where

$$q_1 = \sqrt{\frac{2}{15}}\frac{1}{p_+}Q_4 - \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3},\ q_2 = \sqrt{\frac{2}{5}}\frac{Q_4}{p_+},\ q_3 = \sqrt{\frac{2}{15}}\frac{1}{p_+}Q_4 + \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3}.$$

Combine (3.15), (3.18), (3.19)-(3.20), we have by choosing n sufficiently large,

$$E_{1\tau} + \frac{1}{2}K_1 + 2\int |\hat{\theta}_y|(b_1^2 + b_3^2)|dy \le C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C\delta\int \Phi_y^2 dy + I_{nf}, \tag{3.21}$$

where

$$I_{nf} = \int \bar{v}Q_1\Psi_1 dy + \int \sum_{i=2}^{3} Q_i\Psi_i dy + \int \frac{2}{3p_+}WQ_4 dy + \int (q_1v_1^n b_1 + q_2b_2 + q_3v_1^{-n}b_3)dy. \tag{3.22}$$

Here we have used the fact that

$$-\Phi + \frac{2}{3p_+}W = \sqrt{5/6}(b_1 + b_3), \tag{3.23}$$

and

$$\varepsilon^2\int |\hat{N}_1|(|b_1| + |b_3|)dy \le C\delta\int |\hat{\theta}_y|(b_1^2 + b_3^2)|dy + C\varepsilon^2\delta(1+t)^{-1},$$

and for n large enough,

$$-\frac{1}{2}v_1^{n-1}(n\lambda_1v_{1y} + 2v_1\lambda_{1y})b_1^2 + \frac{1}{2}v_1^{-n-1}(n\lambda_3v_{1y} - 2v_1\lambda_{3y})b_3^2 - \tilde{B}\Lambda L_yRB \ge 3|\hat{\theta}_y|(b_1^2 + b_3^2).$$

Even though Q_1 contains the term R_1 with the decay rate $\frac{\varepsilon^2}{1+t}$, the terms in (3.22) involving Q_1 have factor b_1 or b_3 because

$$\Psi_1 = \sqrt{3/10}\lambda_3(b_3 - b_1). \tag{3.24}$$

Thus the terms $\bar{v}Q_1\Psi_1$, $q_1v_1^n b_1$ and $q_3v_1^{-n}b_3$ can be controlled by the intrinsic dissipation on b_1 and b_3 as shown later. The estimates on the other terms involving Q_i ($i = 2,3,4$) are straightforward because from (2.37)-(2.40) and (3.4), they decay at least in the order of $\varepsilon^3(1+t)^{-3/2}$. For brevity, we only estimate $\int \bar{v}Q_1\Psi_1 dy$ and $\int q_2b_2 dy$ as follows for illustration.

Estimate on $\int \bar{v}Q_1\Psi_1 dy$:

From (3.24), we have

$$\int \bar{v}Q_1\Psi_1 dy = \sqrt{\frac{3}{10}}\int \bar{v}Q_1\lambda_3(b_3-b_1)dy. \tag{3.25}$$

Here we only consider the integral

$$I_1 = \int \bar{v}Q_1\lambda_3 b_1 dy,$$

and the other term in (3.25) can be estimated similarly. By the definition of Q_1 in (3.4), we have

$$\begin{aligned}I_1 &= \int \bar{v}\lambda_3 b_1[\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})u_{1y}+J_1+\frac{2}{3\bar{v}}Y]dy - \int \bar{v}\lambda_3 b_1\bar{R}_1 dy - \varepsilon\int \bar{v}\lambda_3 b_1\sum_{j=1}^{2}\int \xi_1^2\Theta_1^j d\xi dy\\ &= I_1^1+I_1^2+I_1^3.\end{aligned}$$

Since

$$\begin{aligned}&\int |\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})u_{1y}|\cdot|b_1|dy\\ &\le C(\delta+\lambda_0)(K_1+\|\Phi_y\|^2)+C\delta\varepsilon^2(1+t)^{-1}E_1+C(\delta+\lambda_0)\|\psi_{1y}\|^2+C\delta\varepsilon^5(1+t)^{-\frac{5}{2}},\end{aligned}$$

and

$$\int \big(|J_1|+|\frac{Y}{\bar{v}}|\big)\cdot|b_1|dy \le C(\delta+\lambda_0)(K_1+\|\Phi_y\|^2)+C\delta\varepsilon^2(1+t)^{-1}E_1+C\delta\varepsilon^5(1+t)^{-\frac{5}{2}},$$

we obtain

$$\begin{aligned}I_1^1 &\le C(\delta+\lambda_0)(K_1+\|\Phi_y\|_{L^2}^2)+C(\delta+\lambda_0)\|\psi_{1y}\|_{L^2}^2\\ &\quad + C\delta\varepsilon^2(1+t)^{-1}E_1+C\delta\varepsilon^2(1+t)^{-1}.\end{aligned} \tag{3.26}$$

On the other hand, from (2.37), we have

$$\bar{R}_1 = O(1)\delta\varepsilon^2(1+t)^{-1}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \quad \text{as } x\to\pm\infty.$$

From (2.18), $\hat{\theta}_y$ satisfies

$$|\hat{\theta}_y| = O(\delta)\varepsilon(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } x\to\pm\infty.$$

Thus, by (2.4) and the assumption on the profile, we have

$$\kappa(\theta_\pm) = \frac{5}{2}\mu(\theta_\pm) > \frac{5}{4}\mu(\theta_\pm). \tag{3.27}$$

Since $a(\theta_\pm)=\frac{3\kappa(\theta_\pm)}{5\theta_\pm}$, $b(\theta_\pm)=\max\{a(\theta_\pm),\frac{\mu(\theta_\pm)}{\theta_\pm}\}$ and $c(\theta_\pm)=\max\{a(\theta_\pm),\frac{1}{2}b(\theta_\pm)\}$, it follows from (3.27) that $a(\theta_\pm)>\frac{2}{3}c(\theta_\pm)$, which leads to

$$|I_1^2| \le \frac{1}{16}\int |\hat{\theta}_y|b_1^2 dy + C\delta\varepsilon^2(1+t)^{-1}. \tag{3.28}$$

We now estimate the integral I_1^3. Let M_* be a global Maxwellian with the state (ρ_*, u_*, θ_*) satisfying $\frac{1}{2}\theta < \theta_* < \theta$ and $|\rho-\rho_*|+|\varepsilon u-u_*|+|\theta-\theta_*| \le \eta_0$ so that Lemma 3.3 holds. Note that,

$$I_1^3 = -\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2\Theta_1^1 d\xi dy - \varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2\Theta_1^2 d\xi dy =: I_1^{31}+I_1^{32}. \tag{3.29}$$

The estimation on I_1^{31} is straightforward by using the intrinsic dissipation on b_1 and (2.24).

$$\begin{aligned}|I_1^{31}| &= |-\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}[\frac{1}{v}P_1(\xi_1\bar{G}_y)-\varepsilon Q(\bar{G},\bar{G})]d\xi dy| \\ &\le C\int |b_1|\Big(|(\varepsilon\bar{u}_y,\bar{\theta}_y)|^2+|(\varepsilon\bar{u}_{yy},\bar{\theta}_{yy}|+|(\varepsilon\bar{u}_y,\bar{\theta}_y)||(v_y,\varepsilon u_y,\theta_y)|\Big)dy \\ &\le C\delta\int|\hat{\theta}_y|b_1^2dy + C\delta\varepsilon^2(1+t)^{-1}+C\delta\|(\phi_y,\psi_y,\zeta_y)\|^2. \end{aligned}\tag{3.30}$$

The estimation on I_1^{32} is more complicated and it will be divided into five parts as follows. From (2.24), it holds that

$$\begin{aligned}I_1^{32} &= -\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}(G_\tau)d\xi dy + \varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2\frac{\varepsilon u_1}{v}L_M^{-1}(G_y)d\xi dy \\ &\quad -\varepsilon\int \bar{v}\lambda_3 b_1\frac{1}{v}\int \xi_1^2 L_M^{-1}[P_1(\xi_1\tilde{G}_y)]d\xi dy + \varepsilon^2\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}[Q(\tilde{G},\tilde{G})]d\xi dy \\ &\quad + 2\varepsilon^2\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}[Q(\tilde{G},\bar{G})]d\xi dy =: \sum_{i=1}^5 I_1^{32i}. \end{aligned}\tag{3.31}$$

For the integral I_1^{321}, one has

$$I_1^{321} = -\varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}(\tilde{G}_\tau)d\xi dy - \varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}(\bar{G}_\tau)d\xi dy =: I_1^{3211}+I_1^{3212}. \tag{3.32}$$

Note that the linearized operator L_M^{-1} satisfies that, for any $h\in N^\perp$,

$$\begin{aligned}(L_M^{-1}h)_\tau &= L_M^{-1}(h_\tau)-2L_M^{-1}\{Q(L_M^{-1}h,M_\tau)\}, \\ (L_M^{-1}h)_y &= L_M^{-1}(h_y)-2L_M^{-1}\{Q(L_M^{-1}h,M_y)\}. \end{aligned}\tag{3.33}$$

Then it follows that

$$\begin{aligned}I_1^{3211} &= -\varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2\big(L_M^{-1}\tilde{G}\big)_\tau d\xi dy - 2\varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_\tau)\}d\xi dy \\ &= -\big(\varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}\tilde{G}d\xi dy\big)_\tau + \varepsilon\int(\bar{v}\lambda_3 b_1)_\tau\int \xi_1^2 L_M^{-1}\tilde{G}d\xi dy \\ &\quad - 2\varepsilon\int \bar{v}\lambda_3 b_1\int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_\tau)\}d\xi dy. \end{aligned}\tag{3.34}$$

The Hölder inequality and Lemma 3.4 yield that

$$|\int \xi_1^2 L_M^{-1}\tilde{G}d\xi|^2 \le C\int \xi_1^4\nu(|\xi|)^{-1}M_*d\xi\cdot\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\tilde{G}|^2d\xi \le C\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi.$$

Moreover, from Lemmas 3.2-3.4, one has

$$\begin{aligned}&|\int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_\tau)\}d\xi|^2 \le C\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_\tau)\}|^2d\xi \\ &\le C\int\frac{\nu(|\xi|)^{-1}}{M_*}|Q(L_M^{-1}\tilde{G},M_\tau)|^2d\xi \le C\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\tilde{G}|^2d\xi\cdot\int\frac{\nu(|\xi|)}{M_*}|M_\tau|^2d\xi \\ &\le C(v_\tau^2+\varepsilon^2u_\tau^2+\theta_\tau^2)\int\frac{\nu(|\xi|)^{-1}}{M_*}|\tilde{G}|^2d\xi. \end{aligned}\tag{3.35}$$

Combining (3.34)-(3.35) gives that

$$\begin{aligned} I_1^{3211} \le & -\Big(\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dy\Big)_\tau + C\beta\|(\Phi_\tau,\Psi_\tau,W_\tau)\|^2 + C\delta\varepsilon^2(1+t)^{-1}E_1 \\ & + C_\beta\,\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\lambda_0^2|(\phi_\tau,\psi_\tau,\zeta_\tau)\|^2, \end{aligned} \tag{3.36}$$

where and in the sequel β is small positive constant to be chosen later and C_β is a positive constant depending on β. By the definition of $\bar{G}$ in (2.24), similar to the estimate in (3.30), one has

$$\begin{aligned} |I_1^{3212}| &= |\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}(\bar{G}_\tau) d\xi dy| \\ &\le C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}} + C\delta\|(\phi_\tau,\psi_\tau,\zeta_\tau)\|^2. \end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.32) implies that

$$\begin{aligned} I_1^{321} \le & -\Big(\varepsilon\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dy\Big)_\tau + C\beta\|(\Phi_\tau,\Psi_\tau,W_\tau)\|^2 + C\delta\varepsilon^2(1+t)^{-1}E_1 \\ & + C_\beta\,\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C(\delta+\lambda_0)\|(\phi_\tau,\psi_\tau,\zeta_\tau)\|^2 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{3.38}$$

The estimation on I_1^{32i} $(i=2,4,5)$ is straightforward by using the Cauchy inequality and Lemmas 3.2-3.4. First, it holds that

$$\begin{aligned} |I_1^{322}| \le & C\delta\varepsilon^2(1+t)^{-1}E_1 + C\lambda_0 K_1 + C\varepsilon^2\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy \\ & + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}} + C\delta\varepsilon^2\|(\phi,\psi,\zeta)_y\|^2. \end{aligned} \tag{3.39}$$

Since

$$\begin{aligned} &|\int \xi_1^2 L_M^{-1}\{Q(\tilde{G},\bar{G})\} d\xi|^2 \le C\int \frac{\nu(|\xi|)}{M_*}|L_M^{-1}\{Q(\tilde{G},\bar{G})\}|^2 d\xi \\ &\le C\int \frac{\nu(|\xi|)^{-1}}{M_*}|Q(\tilde{G},\bar{G})|^2 d\xi \le C\int \frac{\nu(|\xi|)}{M_*}|L_M^{-1}\tilde{G}|^2 d\xi\cdot\int \frac{\nu(|\xi|)}{M_*}|\bar{G}|^2 d\xi \\ &\le C|(\varepsilon\bar{u}_x,\bar{\theta}_x)|^2\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi, \end{aligned}$$

and

$$\begin{aligned} &|\int \xi_1^2 L_M^{-1}\{Q(\tilde{G},\tilde{G})\} d\xi| \le C(\int \frac{\nu(|\xi|)}{M_*}|L_M^{-1}\{Q(\tilde{G},\tilde{G})\}|^2 d\xi)^{\frac{1}{2}} \\ &\le C(\int \frac{\nu(|\xi|)^{-1}}{M_*}|Q(\tilde{G},\tilde{G})|^2 d\xi)^{\frac{1}{2}} \le C\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi, \end{aligned}$$

it follows that

$$|I_1^{324}| + |I_1^{325}| \le C(\delta+\lambda_0)\varepsilon^2\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\delta\varepsilon^2(1+t)^{-1}E_1. \tag{3.40}$$

The estimate on I_1^{323} is similar to the one for I_1^{321}. First, notice that

$$P_1(\xi_1\tilde{G}_y) = \{P_1(\xi_1\tilde{G})\}_y + \sum_{j=0}^{4}\langle \xi_1\tilde{G},\chi_j\rangle P_1(\chi_{jy}).$$

From (3.33) and Lemmas 3.2-3.4, we have

$$
\begin{aligned}
I_1^{323} &= \varepsilon \int (\frac{\bar{v}}{v}\lambda_3 b_1)_y \int \xi_1^2 L_M^{-1}[P_1(\xi_1\tilde{G})]d\xi dy - \varepsilon \int \frac{\bar{v}}{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}[\sum_{j=0}^{4}\langle \xi_1\tilde{G}, \chi_j\rangle P_1(\chi_{jy})]d\xi dy \\
&\quad - 2\varepsilon \int \frac{\bar{v}}{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}[P_1(\xi_1\tilde{G})], M_y)\}d\xi dy \\
&\le C_\beta\, \varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2) + C\lambda_0\|(\phi_y,\psi_y,\zeta_y)\|^2,
\end{aligned}
\tag{3.41}
$$

where we have used the fact that

$$
|\langle \xi_1\tilde{G}, \chi_j\rangle|^2 \le C \int \frac{\nu(|\xi|)\tilde{G}^2}{M_*} d\xi.
$$

Substituting (3.38), (3.39), (3.40) and (3.41) into (3.31) gives that

$$
\begin{aligned}
I_1^{32} \le & -\Big(\varepsilon \int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dy\Big)_\tau + C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2) \\
& + C\beta\|(\Phi,\Psi,W)_\tau\|^2 + C_\beta\, \varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy \\
& + C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}},
\end{aligned}
$$

which implies by (3.29) and (3.30) that

$$
\begin{aligned}
I_1^{3} \le & -\Big(\varepsilon \int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dy\Big)_\tau + C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2) \\
& + C\beta\|(\Phi,\Psi,W)_\tau\|^2 + C_\beta\, \varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy \\
& + C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^2(1+t)^{-1}.
\end{aligned}
\tag{3.42}
$$

And finally, (3.26), (3.28) and (3.42) yield the estimate on I_1 as follows.

$$
\begin{aligned}
I_1 \le & -\Big(\varepsilon \int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dy\Big)_\tau + C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2) \\
& + C\beta\|(\Phi,\Psi,W)_\tau\|^2 + C_\beta\, \varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy \\
& + C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^2(1+t)^{-1} + \frac{1}{16}\int |\hat{\theta}_y| b_1^2 dy,
\end{aligned}
\tag{3.43}
$$

which completes the estimate on the term $\int \bar{v}Q_1\Psi_1 dx$.

Estimate on $\int q_2 b_2 dy$:

Notice that the profile has no intrinsic dissipation on b_2. Fortunately, it holds that $q_2 = \sqrt{\frac{2}{5}\frac{Q_4}{p_+}}$ and Q_4 has the decay rate as $\varepsilon^3(1+t)^{-\frac{3}{2}}$. Thus the estimation on $\int q_2 b_2 dy$ can be directly obtained even though there is no intrinsic dissipation on b_2. For example,

$$
\begin{aligned}
&|\int \varepsilon \bar{u}_1 \bar{R}_1 b_2 dy| \le C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}, \\
&|\int\int \varepsilon^2 \bar{u}_1 b_2 \xi_1^2 \Theta_1 d\xi dy| \le C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 \\
&\qquad + C\delta\varepsilon^2 \int\int \frac{\nu(|\xi|)\tilde{G}^2}{M_*} d\xi dy + C\int\int \frac{\nu(|\xi|)}{M_*}(\tilde{G}_\tau^2 + \tilde{G}_y^2) d\xi dy + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

And the term $\varepsilon\int\int \xi_1|\xi|^2\Theta_1^2 b_2 d\xi dy$ can be estimated similarly as for I_1^{32} where the intrinsic dissipation on b_1, b_3 is not needed. Notice also that all the other terms in q_2 are of higher order. Therefore, one has

$$\begin{aligned}I_2 = \int q_2 b_2 d\xi dy &\le (\varepsilon\int\int \hat{A}(\xi,b_2)L_M^{-1}\tilde{G}d\xi dy)_\tau + C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2)\\ &+C\beta\|(\Phi,\Psi,W)_\tau\|^2 + C_\beta\,\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy\\ &+C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^2(1+t)^{-1},\end{aligned} \tag{3.44}$$

where $\hat{A}(\xi,b_2)$ is a linear function of b_2 and a polynomial function of ξ. Using (3.43), (3.44) and (3.21), we get

$$\begin{aligned}&E_{1\tau} + \Big(\int\int\varepsilon\hat{A}_1(\xi,B)L_M^{-1}\tilde{G}d\xi dy\Big)_\tau + \frac{1}{4}K_1 + \int|\hat{\theta}_y|(b_1^2+b_3^2)dy\\ &\quad\le C\delta\varepsilon^2(1+t)^{-1}E_1 + C(\lambda_0+\beta)(K_1+\|\Phi_y\|^2) + C\beta\|(\Phi,\Psi,W)_\tau\|^2\\ &\qquad+C_\beta\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy\\ &\qquad+C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^2(1+t)^{-1},\end{aligned} \tag{3.45}$$

where we have used the smallness of δ and ε_0. Here $\hat{A}_1$ is a linear function of $B=(b_1,b_2,b_3)^t$ and a polynomial function of ξ.

Note that K_1 does not contain the norm $\|\Phi_y\|^2$. To complete the lower order inequality, we have to estimate Φ_y. From $(3.3)_2$, we have

$$\frac{4\mu(\bar{\theta})}{3\bar{v}}\Phi_{y\tau} - \Psi_{1\tau} + \frac{p_+}{\bar{v}}\Phi_y = \frac{2}{3\bar{v}}W_y - \frac{8\mu(\bar{\theta})}{15p_+\bar{v}}\varepsilon^2\hat{N}_{1y} - Q_1. \tag{3.46}$$

Multiplying (3.46) by Φ_y yields

$$(\frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2)_\tau - (\frac{2\mu(\bar{\theta})}{3\bar{v}})_\tau\Phi_y^2 - \Phi_y\Psi_{1\tau} + \frac{p_+}{\bar{v}}\Phi_y^2 = \Big(\frac{2}{3\bar{v}}W_y - \frac{8\mu(\bar{\theta})}{15p_+\bar{v}}\varepsilon^2\hat{N}_{1y} - Q_1\Big)\Phi_y.$$

Since

$$\Phi_y\Psi_{1\tau} = (\Phi_y\Psi_1)_\tau - (\Phi_\tau\Psi_1)_y + \Psi_{1y}^2 - \frac{2}{5p_+}\varepsilon^2\hat{N}_1\Psi_{1y},$$

we can obtain

$$\Big(\int\frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2 - \Phi_y\Psi_1 dy\Big)_\tau + \int\frac{p_+}{2\bar{v}}\Phi_y^2 dy \le C\|(\Psi_{1y},W_y)\|^2 + C\delta\varepsilon^3(1+t)^{-3/2} + \int Q_1^2 dy. \tag{3.47}$$

The formula (3.4) for Q_1 and the Cauchy inequality directly yield

$$\begin{aligned}\int Q_1^2 dy &\le C(\delta+\lambda_0)(K_1+\|\Phi_y\|^2) + C\lambda_0\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\\ &\quad+C\delta\varepsilon^3(1+t)^{-3/2} + C\int|\int\xi_1^2\Theta_1 d\xi|^2 dy.\end{aligned} \tag{3.48}$$

And using Lemmas 3.2-3.4 implies

$$\int |\int \xi_1^2\Theta_1 d\xi|^2 d\xi dy \leq C\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}(|\tilde{G}_y|^2+|\tilde{G}_\tau|^2)d\xi dy + C(\delta+\lambda_0)\varepsilon^4\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy \\ + C(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\delta\varepsilon^3(1+t)^{-3/2}. \tag{3.49}$$

Substituting (3.48) and (3.49) into (3.47) yields

$$\Big(\int \frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2 - \Phi_y\Psi_1 dy\Big)_\tau + \int \frac{p_+}{4\bar{v}}\Phi_y^2 dy \\ \leq C_2K_1 + C_2\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}(|\tilde{G}_y|^2+|\tilde{G}_\tau|^2)d\xi dy + C_2(\delta+\lambda_0)\varepsilon^4\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy \\ + C_2(\delta+\lambda_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_2\delta\varepsilon^3(1+t)^{-3/2}. \tag{3.50}$$

Multiplying (3.5) by $\varepsilon^2\frac{\tilde{G}}{M_*}$, one can obtain

$$(\varepsilon^2\frac{v\tilde{G}^2}{2M_*})_\tau - \varepsilon^2\frac{v\tilde{G}}{M_*}L_M\tilde{G} = \Big\{-\frac{1}{R\theta}P_1[\xi_1(\frac{|\xi-\varepsilon u|^2}{2\theta}\frac{1}{\varepsilon}\zeta_y + \xi\cdot\frac{1}{\varepsilon}\psi_y)M] \\ + \varepsilon u_1 G_y - P_1(\xi_1 G_y) + \varepsilon v Q(G,G) - v\bar{G}_\tau\Big\}\cdot\varepsilon^2\frac{\tilde{G}}{M_*}. \tag{3.51}$$

Integrating (3.51) with respect to ξ and y and using the Cauchy inequality and Lemmas 3.2-3.4, one has

$$\Big(\varepsilon^2\int\int\frac{1}{2M_*}|\tilde{G}|^2 d\xi dy\Big)_\tau + \frac{3\bar{\sigma}}{4}\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy \\ \leq C_3\delta\varepsilon^3(1+t)^{-3/2} + C_3\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\,\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy. \tag{3.52}$$

On the other hand, since $(\Phi,\Psi,W)_\tau$ can be represented by $(\Phi,\Psi,W)_y$ and $(\Phi,\Psi,W)_{yy}$ from the equation (3.3), we can get an estimate for $(\Phi,\Psi,W)_\tau$ as follows.

$$\|(\Phi,\Psi,W)_\tau\|^2 \leq C_4(K_1+\|\Phi_y\|^2) + C_4\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_4\delta\varepsilon^3(1+t)^{-\frac{3}{2}} \\ + C_4\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}(|\tilde{G}_y|^2+|\tilde{G}_\tau|^2)d\xi dy + C_4(\delta+\lambda_0)\varepsilon^4\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy. \tag{3.53}$$

Now we can complete the lower order estimate. Since $\hat{A}_1$ is a linear function of the vector B and a polynomial of ξ, we get

$$|\varepsilon\int\int\hat{A}_1(\xi,B)L_M^{-1}\tilde{G}d\xi dy| \leq \frac{1}{4}E_1 + C\varepsilon^2\int\int\frac{|\tilde{G}|^2}{M_*}d\xi dy.$$

We choose large constants $\bar{C}_1>1$, $\bar{C}_2>1$, $\bar{C}_3>1$ and small constant β such that

$$\bar{C}_1E_1 + \bar{C}_1\varepsilon\int\int\hat{A}_1L_M^{-1}\tilde{G}d\xi dy + \bar{C}_2\int\big(\frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2 - \Phi_y\Psi_1\big)dy + \bar{C}_3\varepsilon^2\int\int\frac{|\tilde{G}|^2}{2M_*}d\xi dy \\ \geq \frac{1}{2}\bar{C}_1E_1 + \bar{C}_2\int\frac{\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2 dy + \frac{\bar{C}_3}{4}\varepsilon^2\int\int\frac{\tilde{G}^2}{M_*}d\xi dy,$$

$$
\Big(\frac{\bar{C}_1}{4}-C_2\bar{C}_2-\bar{C}_1C_1C_4\Big)K_1+\int\Big[\bar{C}_2\frac{p_+}{4\bar{v}}-(\delta+\beta+\lambda_0)\bar{C}_1C_1(1+C_4)\Big]\Phi_y^2dy
\geq\frac{\bar{C}_1}{8}K_1+\bar{C}_2\int\frac{p_+}{8\bar{v}}\Phi_y^2dy,
$$

and

$$
\frac{\bar{\sigma}}{2}\bar{C}_3-\bar{C}_1C_1C_4(\delta+\beta+\lambda_0)-C_\beta\bar{C}_1-C_2\bar{C}_2\varepsilon^2(\delta+\lambda_0)\geq\frac{\bar{\sigma}}{4}\bar{C}_3.
$$

Hence, by multiplying (3.45) by $\bar{C}_1$, (3.50) by $\bar{C}_2$, (3.52) by $\bar{C}_3$, (3.53) by $C_1(\delta+\varepsilon_0+\varepsilon_1)\bar{C}_1$ and adding all these inequalities together, we have

$$
\begin{aligned}
E_{2\tau}+K_2+\int|\hat{\theta}_y|(b_1^2+b_3^2)dy\leq{}&C_5\delta\varepsilon^2(1+t)^{-1}(E_2+1)\\
&+C_5\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}(|\tilde{G}_y|^2+|\tilde{G}_\tau|^2)d\xi dy+C_5\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2,
\end{aligned}\tag{3.54}
$$

where

$$
\begin{aligned}
E_2&=\bar{C}_1E_1+\bar{C}_1\int\int\varepsilon\hat{A}_1L_M^{-1}\tilde{G}d\xi dy+\bar{C}_2\int\big(\frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_y^2-\Phi_y\Psi_1\big)dy+\bar{C}_3\varepsilon^2\int\int\frac{|\tilde{G}|^2}{2M_*}d\xi dy,\\
K_2&=\frac{\bar{C}_1}{16}K_1+\bar{C}_2\int\frac{p_+}{16\bar{v}}\Phi_y^2dy+\|(\Phi,\Psi,W)_\tau\|^2+\frac{\bar{\sigma}}{8}\bar{C}_3\varepsilon^2\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy.
\end{aligned}
$$

3.3 Derivative estimate

In this subsection, we derive the higher order estimate for (Φ,Ψ,W). First, by the definition of Θ_1^1 in (2.24), it holds that

$$
\begin{cases}
-\varepsilon\int\xi_1^2\Theta_1^1d\xi=\varepsilon^2N_4+\varepsilon^3F_4,\\
N_4=f_{41}\theta_x\bar{\theta}_x+f_{42}v_x\bar{\theta}_x+f_{43}\bar{\theta}_x^2+f_{44}\bar{\theta}_{xx},\\
|F_4|=O(1)\big[(|v_x|+|\theta_x|+|\bar{\theta}_x|+|\varepsilon u_x|+|\varepsilon\bar{u}_x|)|\bar{u}_x|+|u_x||\bar{\theta}_x|+|\bar{u}_{xx}|\big].
\end{cases}\tag{3.55}
$$

From (2.9) and (2.36), one has

$$
\begin{cases}
\phi_\tau-\psi_{1y}=-\frac{3}{5}\varepsilon^2\hat{N}_{1y},\\
\psi_{1\tau}+(p-\bar{p})_y=\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}u_{1y}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1y})_y+Q_5,\\
\psi_{i\tau}=\varepsilon(\frac{\mu(\theta)}{v}u_{iy}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{iy})_y+Q_{4+i},i=2,3,\\
\zeta_\tau+\varepsilon pu_{1y}-\varepsilon\bar{p}\bar{u}_{1y}=(\frac{\kappa(\theta)}{v}\theta_y-\frac{\kappa(\bar{\theta})}{\bar{v}}\bar{\theta}_y)_y+Q_8,
\end{cases}\tag{3.56}
$$

where

$$
\begin{cases}
Q_5=\varepsilon^2(N_{4y}-\bar{N}_{4y})+\varepsilon^3F_{4y}-\varepsilon\int\xi_1^2\Theta_{1y}^2d\xi+(\varepsilon^2\bar{N}_{4y}-\bar{R}_{1y}),\\
Q_{4+i}=\varepsilon^2(N_{iy}-\bar{N}_{iy})+\varepsilon^3F_{iy}-\varepsilon\int\xi_1\xi_i\Theta_{1y}^2d\xi-\bar{R}_{iy},i=2,3,\\
Q_8=\frac{2}{5}\varepsilon^2\hat{N}_{1y}+\varepsilon^2(N_{1y}-\bar{N}_{1y})+\varepsilon^3F_{1y}+\frac{4}{3}\frac{\mu(\theta)}{v}\varepsilon^2u_{1y}^2+\sum_{i=2}^3\frac{\mu(\theta)}{v}\varepsilon^2u_{iy}^2\\
\qquad-\frac{1}{2}\varepsilon\int\xi_1|\xi|^2\Theta_{1y}^2d\xi+\sum_{i=1}^3\varepsilon^2u_i\int\xi_1\xi_i\Theta_{1y}d\xi-\bar{H}_y-\bar{R}_{4y}+\frac{1}{2}(|\varepsilon\bar{u}|^2)_\tau+\varepsilon\bar{p}_y\bar{u}_1,
\end{cases}
$$

and N_i, F_i $(i=1,2,3,4)$ are defined in (2.26), (2.31) and (3.55) respectively and $\bar{N}_i$ $(i=1,2,3,4)$ is the corresponding function of N_i $(i=1,2,3,4)$ by substituting the variable (v,u,θ) by the profile $(\bar{v},\bar{u},\bar{\theta})$.

We will use the convex entropy for the fluid system to obtain the first-order derivative estimates of (Φ_y,Ψ_y,W_y). Multiplying $(3.56)_2$ by ψ_1 and $(3.56)_3$ by ψ_i, one has

$$(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2)_\tau-(p-\bar{p})\psi_{1y}+\frac{4\varepsilon}{3}\Big(\frac{\mu(\theta)}{v}u_{1y}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1y}\Big)\psi_{1y}+\varepsilon(\frac{\mu(\theta)}{v}u_{iy}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{iy})\psi_{iy}=\sum_{i=1}^{3}Q_{4+i}\psi_i+(\cdots)_y.$$

Since $p-\bar{p}=\frac{2}{3}\bar{\theta}(\frac{1}{v}-\frac{1}{\bar{v}})+\frac{2\zeta}{3v}$, we obtain

$$\begin{aligned}&(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2)_\tau-\frac{2}{3}\bar{\theta}(\frac{1}{v}-\frac{1}{\bar{v}})\phi_\tau-\frac{2}{3v}\zeta\psi_{1y}+\frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1y}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{iy}^2+\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})\bar{u}_{1y}\psi_{1y}\\&+\varepsilon\sum_{i=2}^{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})\bar{u}_{iy}\psi_{iy}=\sum_{i=1}^{3}\psi_iQ_{i+4}+\frac{2\varepsilon^2}{5}\bar{\theta}(\frac{1}{v}-\frac{1}{\bar{v}})\hat{N}_{1y}+(\cdots)_x.\end{aligned}\tag{3.57}$$

Let

$$\hat{\Phi}(s)=s-1-\ln s,$$

then it holds that

$$\begin{aligned}\{\frac{2}{3}\bar{\theta}\hat{\Phi}(\frac{v}{\bar{v}})\}_\tau&=\frac{2}{3}\bar{\theta}_\tau\hat{\Phi}(\frac{v}{\bar{v}})+\frac{2}{3}\bar{\theta}(-\frac{1}{v}+\frac{1}{\bar{v}})\phi_\tau+\frac{2}{3}\bar{\theta}(-\frac{v}{\bar{v}^2}+\frac{1}{\bar{v}})\bar{v}_t+\frac{2}{3}\bar{\theta}(-\frac{1}{v}+\frac{1}{\bar{v}})\bar{v}_\tau\\&=\frac{2}{3}\bar{\theta}(-\frac{1}{v}+\frac{1}{\bar{v}})\phi_\tau-\bar{p}\hat{\Psi}(\frac{v}{\bar{v}})\bar{v}_\tau+\bar{v}\bar{p}_\tau\hat{\Phi}(\frac{v}{\bar{v}}),\end{aligned}\tag{3.58}$$

where

$$\hat{\Psi}(s)=s^{-1}-1+\ln s.$$

It is easy to check that $\hat{\Phi}(1)=\hat{\Phi}'(1)=\hat{\Psi}(1)=\hat{\Psi}'(1)=0$ and $\hat{\Phi}(s)$, $\hat{\Psi}(s)$ are strictly convex around $s=1$. Substituting (3.58) into (3.57) yields that

$$\begin{aligned}&\Big(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2+\frac{2}{3}\bar{\theta}\hat{\Phi}(\frac{v}{\bar{v}})\Big)_\tau-\frac{2}{3v}\zeta\psi_{1y}+\frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1y}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{iy}^2+\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})\bar{u}_{1y}\psi_{1y}\\&+\varepsilon\sum_{i=2}^{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})\bar{u}_{iy}\psi_{iy}=\sum_{i=1}^{3}\psi_iQ_{i+4}+\frac{2\varepsilon^2}{5}\bar{\theta}(\frac{1}{v}-\frac{1}{\bar{v}})\hat{N}_{1y}-\bar{p}\hat{\Psi}(\frac{v}{\bar{v}})\bar{v}_\tau+\bar{v}\bar{p}_\tau\hat{\Phi}(\frac{v}{\bar{v}}).\end{aligned}\tag{3.59}$$

On the other hand, multiplying $(3.56)_4$ by $\frac{\zeta}{\theta}$, it holds that

$$\frac{\zeta}{\theta}\zeta_\tau+\varepsilon(pu_{1y}-\bar{p}\bar{u}_{1y})\frac{\zeta}{\theta}=\Big(\frac{\kappa(\theta)}{v}\theta_y-\frac{\kappa(\bar{\theta})}{\bar{v}}\bar{\theta}_y\Big)_y\frac{\zeta}{\theta}+Q_8\frac{\zeta}{\theta}.\tag{3.60}$$

One can compute that

$$\frac{\zeta}{\theta}\zeta_\tau=\Big(\bar{\theta}\hat{\Phi}(\frac{\theta}{\bar{\theta}})\Big)_\tau+\bar{\theta}_\tau\hat{\Psi}(\frac{\theta}{\bar{\theta}})=\Big(\bar{\theta}\hat{\Phi}(\frac{\theta}{\bar{\theta}})\Big)_\tau+O(1)\delta\varepsilon^2(1+t)^{-1}|\zeta|^2,\tag{3.61}$$

$$\varepsilon(pu_{1y}-\bar{p}\bar{u}_{1y})\frac{\zeta}{\theta}=\frac{2\zeta}{3v}\psi_{1y}+\varepsilon(p-\bar{p})\bar{u}_{1y}\frac{\zeta}{\theta}=\frac{2\zeta}{3v}\psi_{1y}+O(1)\delta\varepsilon^2(1+t)^{-1}|(\psi,\zeta)|^2,\tag{3.62}$$

and

$$\Big(\frac{\kappa(\theta)}{v}\theta_y-\frac{\kappa(\bar\theta)}{\bar v}\bar\theta_y\Big)_y\frac{\zeta}{\theta}=(\cdots)_y-\frac{\bar\theta\kappa(\theta)}{v\theta^2}\zeta_y^2-\frac{\kappa(\theta)\bar\theta_y\zeta\zeta_y}{v\theta^2}-\frac{\bar\theta\bar\theta_y\zeta_y-|\bar\theta_y|^2\zeta}{\theta^2}\big(\frac{\kappa(\theta)}{v}-\frac{\kappa(\bar\theta)}{\bar v}\big)$$
$$\le(\cdots)_y-\frac{3}{4}\frac{\bar\theta\kappa(\theta)}{v\theta^2}\zeta_y^2+C\delta\varepsilon^2(1+t)^{-1}|(\phi,\zeta)|^2. \tag{3.63}$$

Substituting (3.61), (3.62) and (3.63) into (3.60) yields that

$$\Big(\bar\theta\hat\Phi(\frac{\theta}{\bar\theta})\Big)_\tau+\frac{2\zeta}{3v}\psi_{1y}+\frac{3}{4}\frac{\bar\theta\kappa(\theta)}{v\theta^2}\zeta_y^2\le(\cdots)_y+C\delta\varepsilon^2(1+t)^{-1}|(\phi,\zeta)|^2+|Q_8\frac{\zeta}{\theta}|. \tag{3.64}$$

Combining (3.64) and (3.59) and using Cauchy inequality, one has

$$E_{3\tau}+\frac{3}{4}K_3\le C\delta\varepsilon^2(1+t)^{-1}E_3+C\delta\varepsilon^4(1+t)^{-2}+C\varepsilon^2(1+t)^{-1}\int|\hat\theta_y|(b_1^2+b_3^2)dy$$
$$+\Big|\int\frac{2}{5}\varepsilon^2\hat N_{1y}\Big(-\frac{\bar\theta}{v\bar v}\phi+\frac{\zeta}{\theta}\Big)dy\Big|+\sum_{i=1}^4|I_{i+2}|, \tag{3.65}$$

where

$$\begin{cases}E_3=\int\Big(\frac12\sum_{i=1}^3\psi_i^2+R\bar\theta\hat\Phi(\frac{v}{\bar v})+\bar\theta\hat\Phi(\frac{\theta}{\bar\theta})\Big)dy,\\ K_3=\int\Big(\frac43\frac{\mu(\theta)}{v}\psi_{1y}^2+\sum_{i=2}^3\frac{\mu(\theta)}{v}\psi_{iy}^2+\frac34\frac{\bar\theta\kappa(\theta)}{v\theta^2}\zeta_y^2\Big)dy,\end{cases} \tag{3.66}$$

and

$$I_3=\int\varepsilon^2(N_{4y}-\bar N_{4y})\psi_1dy+\int\varepsilon^2F_{4y}\psi_1dy-\varepsilon\int\int\xi_1^2\Theta_{1y}^2\psi_1d\xi dy$$
$$I_{2+i}=\int\varepsilon^2(N_{iy}-\bar N_{iy})\psi_idy+\int\varepsilon^3F_{iy}\psi_idy-\varepsilon\int\int\xi_1\xi_i\Theta_{1y}^2\psi_id\xi dy,\quad i=2,3,$$
$$I_6=\int\varepsilon^2(N_{1y}-\bar N_{1y})\frac{\zeta}{\theta}dy+\int\varepsilon^3F_{1y}\frac{\zeta}{\theta}dy-\frac12\varepsilon\int\int\xi_1|\xi|^2\Theta_{1y}^2\frac{\zeta}{\theta}d\xi dy$$
$$+\sum_{i=1}^3\varepsilon^2u_i\int\int\xi_1\xi_i\Theta_{1y}\frac{\zeta}{\theta}d\xi dy.$$

In the estimate of (3.65), we have used the estimate like

$$|\int(\varepsilon^2\bar N_{4y}-\bar R_{1y})\psi_1dy|=|\int(\varepsilon^2\bar N_{4y}-\bar R_{1y})\Psi_{1y}dy|\le C\int|(\varepsilon^2\bar N_{4yy}-\bar R_{1yy})|(|b_1|+|b_3|)dy$$
$$\le C\delta\varepsilon^4(1+t)^{-2}+C\varepsilon^2(1+t)^{-1}\int|\hat\theta_y|(b_1^2+b_3^2)dy.$$

Now, we calculate the terms on the right hand side of (3.65). Firstly, a direct calculation yields

$$-\frac{\bar\theta}{v\bar v}\phi+\frac{\zeta}{\theta}=\frac{1}{\bar\theta}(\frac{2}{3p_+}W_y-\Phi_y)+O(1)\Big[|(\phi,\psi,\zeta)|^2+|\varepsilon\bar u_{1y}\Psi_1|+\delta\varepsilon(1+t)^{-1}\Big],$$

and thus by combining (3.23) and (3.24), it holds that

$$|\int\frac25\varepsilon^2\hat N_{1y}\Big(-\frac{\bar\theta}{v\bar v}\phi+\frac{\zeta}{\theta}\Big)dy|$$
$$\le C\delta\varepsilon^2(1+t)^{-1}E_3+C\delta\varepsilon^4(1+t)^{-2}+C\int\varepsilon^2[|(\hat N_{1y}\frac{1}{\bar\theta})_y|+|\hat N_{1y}\varepsilon\bar u_{1y}|]\cdot|(b_1,b_3)|dy$$
$$\le C\delta\varepsilon^2(1+t)^{-1}E_3+C\delta\varepsilon^4(1+t)^{-2}+C\varepsilon^2(1+t)^{-1}\int|\hat\theta_y|(b_1^2+b_3^2)dy. \tag{3.67}$$

By using the definition of N_4 and F_4 in (3.55), one can obtain

$$
\begin{aligned}
&|\int \varepsilon^2(N_{4y}-\bar{N}_{4y})\psi_1 dy+\int \varepsilon^3 F_{4y}\psi_1 dy| \le \frac{1}{32}K_3+\int \varepsilon^4(N_4-\bar{N}_4)^2dy+\varepsilon^6F_4^2dy\\
&\le \frac{1}{32}K_3+C\delta\|\phi_y\|^2+C\delta\varepsilon^2(1+t)^{-1}E_3+C\delta\varepsilon^4(1+t)^{-2}. \qquad (3.68)
\end{aligned}
$$

And by using Lemma 3.2-3.4, one has that

$$
\begin{aligned}
&|\varepsilon\int\int \xi_1^2\Theta_{1y}^2\psi_1 d\xi dy| \le \frac{1}{32}K_3+C\varepsilon^2\int|\int \xi_1^2\Theta_1^2 d\xi|^2dy\\
&\le \frac{1}{32}K_3+C\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|(\tilde{G}_y,\tilde{G}_\tau)|^2d\xi dy+C\delta\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C\delta\varepsilon^2(1+t)^{-1}E_3\\
&\quad +C\delta\varepsilon^4(1+t)^{-2}+C\varepsilon^4\Big[\delta(1+t)^{-1}+\|\int\frac{|\tilde{G}|^2}{M_*}d\xi\|_{L^\infty}\Big]\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy. \qquad (3.69)
\end{aligned}
$$

Combining (3.68) and (3.69) yields that

$$
\begin{aligned}
I_3 &\le \frac{1}{32}K_3+C\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|(\tilde{G}_y,\tilde{G}_\tau)|^2d\xi dy+C\delta\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C\delta\varepsilon^2(1+t)^{-1}E_3\\
&\quad +C\delta\varepsilon^4(1+t)^{-2}+C\varepsilon^4\Big[\delta(1+t)^{-1}+\|\int\frac{|\tilde{G}|^2}{M_*}d\xi\|_{L^\infty}\Big]\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy. \qquad (3.70)
\end{aligned}
$$

Similarly, I_4, I_5, I_6 can be controlled by the right hand side of (3.70). Substituting (3.67) and (3.70) into (3.65) gives that

$$
\begin{aligned}
E_{3\tau}+\frac{1}{2}K_3 &\le C_6\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|(\tilde{G}_y,\tilde{G}_\tau)|^2d\xi dy+C_6\delta\sum_{|\alpha|=1}\|\partial_\alpha(\phi,\psi,\zeta)\|^2\\
&\quad +C_6\delta\varepsilon^2(1+t)^{-1}E_3+C_6\delta\varepsilon^4(1+t)^{-2}+C_6\varepsilon^2(1+t)^{-1}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy\\
&\quad +C_6\varepsilon^4\Big[\delta(1+t)^{-1}+\|\int\frac{|\tilde{G}|^2}{M_*}d\xi\|_{L^\infty}\Big]\int\int\frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dy. \qquad (3.71)
\end{aligned}
$$

Note that the norm $\|\phi_y\|$ is not included in K_3 (see (3.66)). To complete the first-order derivative estimate, we follow the same way as to estimate Φ_y in the previous section. By using the $(3.56)_1$, we can rewrite the equation $(3.56)_2$ as

$$
\begin{aligned}
\frac{4}{3}\frac{\mu(\bar{\theta})}{\bar{v}}\phi_{y\tau}-\psi_{1\tau}-(p-\bar{p})_y &= -\frac{4\varepsilon^2}{5}\frac{\mu(\bar{\theta})}{\bar{v}}\hat{N}_{1yy}-\frac{4}{3}(\frac{\mu(\bar{\theta})}{\bar{v}})_y\psi_{1y}\\
&\quad -\frac{4\varepsilon}{3}\Big[(\frac{\mu(\theta)}{v}u_{1y}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1y})\Big]_y+\varepsilon\int\xi_1^2\Theta_{1y}d\xi+\bar{R}_{1y}. \qquad (3.72)
\end{aligned}
$$

Multiplying the equation (3.72) by ϕ_y, one has

$$
\begin{aligned}
&\frac{2}{3}(\frac{\mu(\bar{\theta})}{\bar{v}}\phi_y^2)_\tau-\frac{2}{3}(\frac{\mu(\bar{\theta})}{\bar{v}})_\tau\phi_y^2-\psi_{1\tau}\phi_y-(p-\bar{p})_y\phi_y \qquad (3.73)\\
&=\Big\{-\frac{4\varepsilon^2}{5}\frac{\mu(\bar{\theta})}{\bar{v}}\hat{N}_{1yy}-\frac{4}{3}(\frac{\mu(\bar{\theta})}{\bar{v}})_y\psi_{1y}-\frac{4\varepsilon}{3}[(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}})u_{1y}]_y+\varepsilon\int\xi_1^2\Theta_{1y}d\xi+\bar{R}_{1y}\Big\}\phi_y.
\end{aligned}
$$

Since

$$
-(p-\bar{p})_y=\frac{\bar{p}}{\bar{v}}\phi_y-\frac{2}{3\bar{v}}\zeta_y+(\frac{p}{v}-\frac{\bar{p}}{\bar{v}})v_y-\frac{2}{3}(\frac{1}{v}-\frac{1}{\bar{v}})\theta_y,
$$

and

$$\phi_y\psi_{1\tau}=(\phi_y\psi_1)_\tau-(\phi_\tau\psi_1)_y+\psi_{1y}^2-\frac{3\varepsilon^2}{5}\hat N_{1y}\psi_{1y},$$

integrating (3.73) with respect to y and using the Cauchy inequality yield

$$\begin{aligned}\Big(\int\frac{2\mu(\bar\theta)}{3\bar v}\phi_y^2-\phi_y\psi_1dy\Big)_\tau+\int\frac{\bar p}{2\bar v}\phi_y^2dy\le C_7K_3+C_7\delta\varepsilon^2(1+t)^{-1}E_3+C_7\delta\varepsilon^5(1+t)^{-\frac52}\\+C_7(\delta+\lambda_0)\varepsilon\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C_7\delta\|\partial_\tau(\phi,\psi,\zeta)\|^2+C\varepsilon^2\int|\int\xi_1^2\Theta_{1y}d\xi|^2dy,\end{aligned}\tag{3.74}$$

where we have used the fact that

$$\int|(\frac{p}{v}-\frac{\bar p}{\bar v})v_y-\frac23(\frac1v-\frac1{\bar v})\theta_y||\phi_y|dy\le\frac18\|\phi_y\|^2+C\delta\varepsilon^2(1+t)^{-1}E_3+CK_3.$$

It follows from (2.24) and Lemmas 3.2-3.3 that

$$\begin{aligned}\varepsilon^2\int|\int\xi_1^2\Theta_{1y}d\xi|^2dy\le C\delta\varepsilon^5(1+t)^{-\frac52}+C\delta\varepsilon^4(1+t)^{-1}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\\+C\delta\varepsilon^2\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+CJ_3,\end{aligned}\tag{3.75}$$

where

$$\begin{aligned}J_3\doteq\Bigg[\varepsilon^2\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde G|^2d\xi dy+\varepsilon^4(\delta+\lambda_0)\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde G|^2d\xi dy\\+\varepsilon^4\Big(\delta\varepsilon(1+t)^{-1}+\int\int\frac{|\tilde G_y|^2}{M_*}d\xi dy+\int\int\frac{|\tilde G_{yy}|^2}{M_*}d\xi dy\Big)\int\int\frac{\nu(|\xi|)|\tilde G|^2}{M_*}d\xi dy\Bigg].\end{aligned}\tag{3.76}$$

To estimate $(\phi,\psi,\zeta)_\tau$, we use (3.56) to obtain

$$\begin{aligned}\|\partial_\tau(\phi,\psi,\zeta)\|^2\le C_8(K_3+\|\phi_y\|^2)+C_8\delta\varepsilon^2(1+t)^{-1}E_3+C_8\delta\varepsilon^5(1+t)^{-\frac52}\\+C_8\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C_8J_3.\end{aligned}\tag{3.77}$$

Thus we choose large constants $\bar C_4$ and $\bar C_5$ so that

$$\bar C_4E_3+\bar C_5\int(\frac{2\mu(\bar\theta)}{3\bar v}\phi_y^2-\phi_y\psi_1)dy\ge\frac{\bar C_4}{2}E_3+\bar C_5\int\frac{\mu(\bar\theta)}{3\bar v}\phi_y^2dy,$$

and

$$\frac12\bar C_4-\bar C_5C_7-C_8\ge\frac18\bar C_4,\quad \bar C_5\int\frac{\bar p}{2\bar v}\phi_y^2dy-C_8\|\phi_y\|^2\ge\frac{\bar C_5}{4}\int\frac{\bar p}{\bar v}\phi_y^2dy.$$

Let

$$E_4=\bar C_4\varepsilon^{-2}E_3+\bar C_5\varepsilon^{-2}\int(\frac{2\mu(\bar\theta)}{3\bar v}\phi_y^2-\phi_y\psi_1)dy,$$

$$K_4=\frac18\bar C_4\varepsilon^{-2}K_3+\frac{\bar C_5}{4}\varepsilon^{-2}\int\frac{\bar p}{\bar v}\phi_y^2dy+\varepsilon^{-2}\|(\phi_\tau,\psi_\tau,\zeta_\tau)\|^2.$$

Then from (3.71), (3.74), (3.75) and (3.77), we have the following estimate on the (ϕ,ψ,ζ)

$$E_{4\tau}+K_4\le C_9\delta\varepsilon^2(1+t)^{-1}E_4+C_9\delta\varepsilon^2(1+t)^{-2}+C_9\varepsilon^{-2}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2$$
$$+C_9(1+t)^{-1}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy+C_9\varepsilon^{-2}J_3, \tag{3.78}$$

where J_3 is defined in (3.76).

Define

$$E_5=E_4+\varepsilon\int\int\frac{|\tilde{G}|^2}{2M_*}d\xi dy,\quad K_5=K_4+\frac{\bar{\sigma}}{4}\varepsilon\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi dy. \tag{3.79}$$

Then from (3.78) and (3.52), one has

$$E_{5\tau}+K_5\le C_{10}\delta\varepsilon^2(1+t)^{-1}E_5+C_{10}\delta\varepsilon^2(1+t)^{-\frac{3}{2}}+C_{10}\varepsilon^{-2}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2$$
$$+C_{10}(\delta+\lambda_0)\varepsilon^2\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2d\xi dy+C_{10}\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2d\xi dy$$
$$+C_{10}(1+t)^{-1}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy.$$

Next we derive the higher order derivative estimate. Applying ∂_y to (3.56) yields that

$$\begin{cases}\phi_{y\tau}-\psi_{1yy}=-\frac{3}{5}\varepsilon^2\hat{N}_{1yy},\\ \psi_{1y\tau}+\frac{p_+}{\bar{v}}\zeta_{yy}-\frac{p_+}{\bar{v}}\phi_{yy}=\frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v}u_{1y}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1y})_{yy}+Q_9,\\ \psi_{iy\tau}=\varepsilon(\frac{\mu(\theta)}{v}u_{iy}-\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{iy})_{yy}+Q_{8+i},i=2,3,\\ \zeta_{y\tau}+p_+\psi_{1yy}=(\frac{\kappa(\theta)}{v}\theta_y-\frac{\kappa(\bar{\theta})}{\bar{v}}\bar{\theta}_y)_y+\frac{2}{5}\varepsilon^2\hat{N}_{1y}+Q_{12},\end{cases} \tag{3.80}$$

where

$$Q_9=\frac{p-\bar{p}_+}{v}\phi_{yy}+(\frac{p}{v}-\frac{\bar{p}}{v})\phi_{yy}+O(1)(|\bar{v}_{yy}|\cdot|(\phi,\zeta)|+|\phi\zeta_{yy}|)$$
$$-\frac{4}{3}\Big(\frac{1}{v^2}v_y\theta_y-\frac{1}{\bar{v}^2}\bar{v}_y\bar{\theta}_y\Big)+\frac{4}{3}\Big(\frac{\theta}{v^3}v_y^2-\frac{1}{\bar{\theta}^3}\bar{v}_y^2\Big)-\varepsilon\int\xi_1^2\Theta_{1yy}d\xi-\bar{R}_{1yy},$$
$$Q_{i+8}=-\varepsilon\int\xi_1\xi_i\Theta_{1yy}d\xi-\bar{R}_{iyy},i=2,3,$$
$$Q_{12}=-\varepsilon\bar{u}_{1yy}(p-\bar{p})-\varepsilon(p_yu_y-\bar{p}_y\bar{u}_y)+(p_+-\bar{p})\psi_{1yy}+Q_{13y}$$
$$-\frac{1}{2}\varepsilon\int\xi_1|\xi|\Theta_{1yy}^2d\xi+\sum_{i=1}^3\varepsilon^2(u_i\int\xi_1\xi_i\Theta_{1y}d\xi)_y,$$
$$Q_{13}=\frac{4}{3}\frac{\mu(\theta)}{v}\varepsilon^2u_{1y}^2+\varepsilon^2\sum_{i=2}^3\frac{\mu(\theta)}{v}u_{iy}^2-\bar{H}_{1y}-\bar{R}_{4y}+\frac{1}{2}(|\varepsilon\bar{u}|^2)_\tau+\varepsilon\bar{p}_y\bar{u}_1.$$

Multiplying $(3.80)_1$ by $p_+\phi_y$, $(3.80)_2$ by $\bar{v}\psi_{1y}$, $(3.80)_3$ by ψ_{iy}, $(3.80)_4$ by ζ_y, we have

$$\begin{aligned}
&\Big[\int \big(\frac{p_+}{2}\phi_y^2+\frac{\bar{v}}{2}\psi_{1y}^2+\sum_{i=2}^{3}\psi_{iy}^2+\frac{1}{2}\zeta_y^2\big)dy\Big]_\tau+\frac{3}{4}\int\Big[\frac{4\mu(\theta)}{3v}\psi_{1yy}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{iyy}^2+\frac{\kappa(\theta)}{v}\zeta_{yy}^2\Big]dy\\
&\le C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi)\|^2+C\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3\\
&\qquad+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}+C\delta\varepsilon^6(1+t)^{-3}+C\varepsilon^2\int\Big|\int|\xi|^3\Theta_{1y}d\xi\Big|^2dy\\
&\le C\delta\varepsilon(1+t)^{-\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi)\|^2+C\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3\\
&\qquad+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}+C\delta\varepsilon^5(1+t)^{-\frac{5}{2}}+C\delta\varepsilon^2\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+CJ_3,
\end{aligned}\tag{3.81}$$

where we have used (3.75) in the last inequality.

Let

$$E_6=\int\Big[\frac{p_+}{2}\phi_y^2+\frac{\bar{v}}{2}\psi_{1y}^2+\sum_{i=2}^{3}\psi_{iy}^2+\frac{1}{2}\zeta_y^2\Big]dy,\quad K_6=\int\Big[\frac{4\mu(\theta)}{3v}\psi_{1yy}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{iyy}^2+\frac{\kappa(\theta)}{v}\zeta_{yy}^2\Big]dy,$$

then (3.81) implies

$$\begin{aligned}
E_{6\tau}+\frac{1}{2}K_6&\le C_{11}\delta\varepsilon(1+t)^{-\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C_{11}\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2+C_{11}\delta\varepsilon^5(1+t)^{-\frac{5}{2}}\\
&+C_{11}\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3+C_{11}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}+C_{11}\delta\varepsilon^2\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C_{11}J_3.
\end{aligned}\tag{3.82}$$

To get the estimate on ϕ_{yy}, we use the momentum equation $(2.8)_2$. Applying ∂_y on $(2.8)_2$, it holds that

$$\psi_{1y\tau}+(p-\bar{p})_{yy}+\varepsilon\bar{u}_{1y\tau}+\bar{p}_{yy}=-\varepsilon\int\xi_1^2G_{yy}d\xi.\tag{3.83}$$

Note that

$$(p-\bar{p})_{yy}=-\frac{p}{v}\phi_{yy}+\frac{2}{3v}\zeta_{yy}-\frac{1}{v}(p-\bar{p})\bar{v}_{yy}-\frac{\phi}{v}\bar{p}_{yy}-\frac{2v_y}{v}(p-\bar{p})_y-\frac{2\bar{p}_y}{v}\phi_y,$$

then multiplying (3.83) by $-\phi_{yy}$ and integrating the reduced equation with respect to y give that

$$\begin{aligned}
&(-\int\psi_{1y}\phi_{yy}dy)_\tau+\int\frac{p}{2v}\phi_{yy}^2dy\\
&\le C_{12}K_6+C_{12}\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi,\psi,\zeta)_y\|^2+C_{12}\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2+C_{12}\delta\varepsilon^5(1+t)^{-\frac{5}{2}}\\
&\qquad+C_{12}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}+C_{12}\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_{yy}|^2d\xi dy.
\end{aligned}\tag{3.84}$$

To estimate $(\phi,\psi,\zeta)_{y\tau}$ and $(\phi,\psi,\zeta)_{\tau\tau}$, we also use the original fluid-type equation (2.8). Here we only consider the term $\int\psi_{1y\tau}^2dy$ because the other terms can be estimated similarly. It follows from $(2.8)_2$ that

$$\psi_{1y\tau}=-(p-\bar{p})_{yy}-\varepsilon\bar{u}_{1y\tau}-\bar{p}_{yy}-\varepsilon\int\xi_1^2G_{yy}d\xi.\tag{3.85}$$

By (3.85) and using the Cauchy inequality, it holds that

$$\|\psi_{1y\tau}\|^2 \le C_{13}(K_6 + \|\phi_{yy}\|^2) + C_{13}\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi,\psi,\zeta)_y\|^2 + C_{13}\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2 + C_{13}\delta\varepsilon^5(1+t)^{-\frac{5}{2}} + C_{13}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}} + C_{13}\varepsilon^2\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_{yy}|^2 d\xi dy. \tag{3.86}$$

Let $\bar{C}_6$ and $\bar{C}_7$ be suitably large constants, then it follows from (3.82), (3.84) and (3.86) that

$$\bar{C}_7\Big(\bar{C}_6E_6 - \int\psi_{1y}\phi_{yy}dy\Big)_\tau + \sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 \le C_{14}\delta\varepsilon(1+t)^{-\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_{14}\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi,\psi,\zeta)\|^2 + C_{14}\delta\varepsilon^5(1+t)^{-\frac{5}{2}} + C_{14}\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3 + C_{14}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}} + C_{14}J_3, \tag{3.87}$$

where J_3 is defined in (3.76).

To close the a priori argument, we need to estimate the non-fluid component $\partial^\alpha\tilde{G}, |\alpha| = 1, 2$. Applying ∂_y on (3.5), we have

$$v\tilde{G}_{y\tau} - vL_M\tilde{G}_y = -v_y\tilde{G}_\tau + v_yL_M\tilde{G} + 2Q(M_y,\tilde{G}) - \Big\{\frac{1}{R\theta}P_1[\xi_1(\frac{|\xi-\varepsilon u|^2}{2\theta}\frac{1}{\varepsilon}\zeta_y + \xi\cdot\frac{1}{\varepsilon}\psi_y)M]\Big\}_y + \Big\{\varepsilon u_1G_y - P_1(\xi_1G_y) + \varepsilon vQ(G,G) - v\bar{G}_\tau\Big\}_y. \tag{3.88}$$

Multiplying (3.88) by $\frac{\tilde{G}_y}{M_*}$, then integrating the reduced equation with respect to ξ and y and using the Cauchy inequality and Lemmas 3.2-3.4, we have

$$\Big(\int\int\frac{v|\tilde{G}_y|^2}{2M_*}d\xi dy\Big)_\tau + \frac{3\bar{\sigma}}{4}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy \le C\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2 d\xi dy + C_3\delta\varepsilon^3(1+t)^{-5/2} + C_3\varepsilon^{-2}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\delta(1+t)^{-1}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\varepsilon^{-2}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^6 + C_3\sum_{|\alpha|=1}\|\partial^\alpha(v,u,\theta)\|^2_{L^\infty}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy.$$

Similarly, we can obtain the estimate for $\tilde{G}_\tau$. Hence, one obtains that

$$\Big(\sum_{|\alpha|=1}\int\int\frac{v|\partial^\alpha\tilde{G}|^2}{2M_*}d\xi dy\Big)_\tau + \frac{3\bar{\sigma}}{4}\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2 d\xi dy \le C\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2 d\xi dy + C_3\delta\varepsilon^3(1+t)^{-5/2} + C_3\varepsilon^{-2}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\delta(1+t)^{-1}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\varepsilon^{-2}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^6 + C_3\sum_{|\alpha|=1}\|\partial^\alpha(v,u,\theta)\|^2_{L^\infty}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy. \tag{3.89}$$

Finally, we need the highest order estimate to control $\sum_{|\alpha|=2}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha \tilde{G}|^2 d\xi dy$ and $\int \psi_{1y}\phi_{yy}dy$ in (3.87). To estimate $\int \psi_{1y}\phi_{yy}dy$, it is sufficient to study the a priori estimate for $\sum_{|\alpha|=2}\int\int \frac{v|\partial^\alpha \tilde{f}|^2}{2M_*}d\xi dy$ due to (3.12) and (3.13). Applying $\partial^\alpha,\ |\alpha|=2$ to (3.8), one obtains that

$$
\begin{aligned}
&v\partial^\alpha \tilde{f}_\tau - \varepsilon v L_M \partial^\alpha \tilde{G} - \varepsilon u_1 \partial^\alpha \tilde{f}_y + \xi_1 \partial^\alpha \tilde{f}_y = -\partial^\alpha v \tilde{f}_\tau + \varepsilon \partial^\alpha u_1 \tilde{f}_y - \sum_{|\beta|=1}\Big[\partial^{\alpha-\beta} v \partial^\beta \tilde{f}_\tau - \varepsilon \partial^{\alpha-\beta} u_1 \partial^\beta \tilde{f}_y\Big] \\
&\quad + \varepsilon \partial^\alpha \big[v L_M \bar{G} - \bar{v} L_{\bar{M}} \bar{G}_0\big] + \varepsilon^2 \partial^\alpha \big[vQ(G,G) - \bar{v}Q(\bar{G}_0,\bar{G}_0)\big] + \partial^\alpha \Big[-\phi \bar{f}_\tau + \psi \bar{f}_y - \varepsilon v \bar{R}_{\bar{f}}\Big]. \quad (3.90)
\end{aligned}
$$

Multiplying (3.90) by $\frac{\partial^\alpha \tilde{f}}{M_*}$, integrating the reduced equation with respect to ξ and y and using the Cauchy inequality and Lemmas 3.2-3.5, similar to the argument used in [24], one gets that

$$
\begin{aligned}
&\Big(\sum_{|\alpha|=2}\int\int \frac{v|\partial^\alpha \tilde{f}|^2}{2M_*}d\xi dy\Big)_\tau + \frac{3\bar{\sigma}}{4}\sum_{|\alpha|=2}\varepsilon^2 \int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha \tilde{G}|^2 d\xi dy \\
&\le C_3\delta\varepsilon^5(1+t)^{-5/2} + C_3(\delta+\eta_0+\lambda_0^{\frac14})\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^{\frac{10}{3}} \\
&\quad + C_3\delta\varepsilon(1+t)^{-\frac12}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_3\delta\varepsilon^3(1+t)^{-\frac32}\|(\phi,\psi,\zeta)\|^2 \\
&\quad + \frac{C_3}{\lambda_0}\Big[\delta^2\varepsilon^4(1+t)^{-2} + \sum_{|\alpha|=1}^{2}\|\partial^\alpha(\phi,\psi,\zeta)\|^4\Big]\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy \\
&\quad + C_3(\delta+\eta_0+\lambda_0^{\frac14})\varepsilon^2\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}_y|^2 d\xi dy. \quad (3.91)
\end{aligned}
$$

Choose large constants $\bar{C}_8 > 1$ and $\bar{C}_9 > 1$ such that

$$
\begin{aligned}
E_7 &= \frac{\bar{C}_8\bar{C}_7}{\varepsilon^3}\Big(\bar{C}_6E_6 - \int \psi_{1y}\phi_{yy}dy\Big) + \frac{1}{\varepsilon}\sum_{|\alpha|=1}\int\int \frac{v|\partial^\alpha \tilde{G}|^2}{2M_*}d\xi dy + \frac{\bar{C}_9}{\varepsilon^3}\sum_{|\alpha|=2}\int\int \frac{v|\partial^\alpha \tilde{f}|^2}{2M_*}d\xi dy \\
&\ge \frac{c_1}{\varepsilon^3}\Big(\|(\phi,\psi,\zeta)_y\|^2 + \sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + \sum_{|\alpha|=2}\int\int \frac{|\partial^\alpha \tilde{f}|^2}{M_*}d\xi dy\Big) \\
&\qquad + \frac{c_1}{\varepsilon}\sum_{|\alpha|=1}\int\int \frac{|\partial^\alpha \tilde{G}|^2}{2M_*}d\xi dy - C\delta(1+t)^{-\frac32}.
\end{aligned}
$$

Let

$$
K_7 = \frac{\bar{C}_8}{4\varepsilon^3}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + \frac{\bar{\sigma}}{4\varepsilon}\sum_{1\le|\alpha|\le2}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha \tilde{G}|^2 d\xi dy.
$$

Then from (3.87), (3.89) and (3.91), one obtains that

$$
\begin{aligned}
&E_{7\tau} + K_7 \le C\delta\varepsilon^2(1+t)^{-5/2} + C\frac{1}{\varepsilon^3}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^{\frac{10}{3}} + C\delta(1+t)^{-\frac32}\|(\phi,\psi,\zeta)\|^2 \\
&\quad + C\Big[\delta(1+t)^{-\frac12} + \frac{1}{\varepsilon}\|(\phi,\psi)\|\cdot\|(\phi_y,\psi_y,\zeta_y)\|\Big]\sum_{|\alpha|=1}\frac{1}{\varepsilon^2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 \quad (3.92) \\
&\quad + C\Big[\delta\varepsilon(1+t)^{-1} + \sum_{|\alpha|=1}^{2}\big(\frac{1}{\varepsilon}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + \varepsilon\int\int \frac{|\partial^\alpha \tilde{G}|^2}{M_*}d\xi dy\big)\Big]\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy.
\end{aligned}
$$

From (3.78), (3.79) and (3.92) and using the smallness of δ, λ_0 and ε, we have

$$(E_4+E_7)_\tau+\frac{1}{2}(K_4+K_7)\leq C\delta\varepsilon^2(1+t)^{-1}E_4+C\delta\varepsilon^2(1+t)^{-2}+C(1+t)^{-1}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy$$
$$+C\Big[\delta\varepsilon(1+t)^{-1}+\sum_{|\alpha|=1}^{2}\big(\frac{1}{\varepsilon}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+\varepsilon\int\int\frac{|\partial^\alpha\tilde{G}|^2}{M_*}d\xi dy\big)\Big]\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi dy, \quad (3.93)$$

and

$$(E_5+E_7)_\tau+\frac{1}{2}(K_5+K_7)$$
$$\leq C\delta\varepsilon^2(1+t)^{-1}E_5+C\delta\varepsilon^2(1+t)^{-\frac{3}{2}}+C(1+t)^{-1}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy. \quad (3.94)$$

4 The Proof of Main Result

For a suitable large constant $\bar{C}_9$, by combining (3.54) and (3.94) and using the smallness of δ, λ_0 and ε, we have

$$E_{8\tau}+K_8\leq C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1}E_8+C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1},$$

where

$$E_8=\bar{C}_9E_2+E_5+E_7,\qquad K_8=\frac{1}{4}(K_2+K_5+K_7)+\int|\hat{\theta}_y|(b_1^2+b_3^2)dy.$$

Note that

$$E_8\geq\|(\Phi,\Psi,W)\|^2+\Big\{\frac{c_2}{\varepsilon^2}\|(\phi,\psi,\zeta)\|^2+\varepsilon\int\int\frac{v|\tilde{G}|^2}{M_*}d\xi dy\Big\}$$
$$+\Big\{\frac{c_1}{\varepsilon^3}\Big(\|(\phi,\psi,\zeta)_y\|^2+\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+\sum_{|\alpha|=2}\int\int\frac{|\partial^\alpha f|^2}{M_*}d\xi dy\Big)$$
$$+\frac{c_1}{\varepsilon}\sum_{|\alpha|=1}\int\int\frac{|\partial^\alpha\tilde{G}|^2}{2M_*}d\xi dy-C\delta(1+\varepsilon^2\tau)^{-\frac{3}{2}}\Big\},$$

$$K_8\geq\sum_{|\beta|=1}\|\partial^\beta(\Phi,\Psi,W)\|^2+c_2\Big\{\frac{1}{\varepsilon^2}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+\varepsilon\int\int\frac{\nu(|\xi|)}{2M_*}|\tilde{G}|^2d\xi dy\Big\}$$
$$+\Big\{\frac{c_1}{\varepsilon^3}\sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+\frac{1}{\varepsilon}\sum_{|\alpha|=1,2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2d\xi dy\Big\}+\frac{1}{2}\int|\hat{\theta}_y|(b_1^2+b_3^2)dy,$$

and

$$\varepsilon^2E_7\leq C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-\frac{3}{2}}+C(K_4+K_7),\ \text{ and } \varepsilon^2(E_5+E_7)\leq C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-\frac{3}{2}}+CK_8. \quad (4.1)$$

Then the Gronwall inequality yields that

$$E_8\leq C\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}},\qquad \int_0^\tau K_8ds\leq C\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \quad (4.2)$$

Hence, it holds that

$$\|(\Phi,\Psi,W)\|^2\leq C\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \quad (4.3)$$

Multiplying (3.94) by $(1+\varepsilon^2\tau)$ gives

$$
\begin{aligned}
&[(1+\varepsilon^2\tau)(E_5+E_7)]_\tau + \frac{1}{2}(1+\varepsilon^2\tau)(K_5+K_7)\\
&\le C\varepsilon^2(E_5+E_7) + C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-\frac{1}{2}} + C_9\int |\hat{\theta}_y|(b_1^2+b_3^2)dy. \qquad (4.4)
\end{aligned}
$$

Integrating (4.4) with respect to τ and using (4.2) and (4.1), one has that

$$
\begin{aligned}
&(1+\varepsilon^2\tau)(E_5+E_7) + \int_0^\tau \frac{1}{2}(1+\varepsilon^2 s)(K_5+K_7)ds\\
&\le C\varepsilon^2\int_0^\tau (E_5+E_7)ds + C\sqrt{\delta}(1+\varepsilon^2\tau)^{\frac{1}{2}}\\
&\le C\sqrt{\delta}(1+\varepsilon^2\tau)^{\frac{1}{2}} + C\int_0^\tau K_8 ds \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{\frac{1}{2}},
\end{aligned}
$$

which yields

$$
(E_5+E_7) \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{1}{2}}. \qquad (4.5)
$$

In particular, one has

$$
\varepsilon\int\int \frac{|\tilde{G}|^2}{M_*}d\xi dy \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{1}{2}}. \qquad (4.6)
$$

On the other hand, multiplying (3.94) by $(1+\varepsilon^2\tau)^{\frac{1}{2}}$, it holds

$$
\int_0^\tau (1+\varepsilon^2 s)^{\frac{1}{2}}(K_5+K_7)ds \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \qquad (4.7)
$$

Multiplying (3.93) by $(1+\varepsilon^2\tau)$ and using (4.5) and (4.1), one can obtain

$$
\begin{aligned}
&[(1+\varepsilon^2\tau)(E_4+E_7)]_\tau + \frac{1}{2}(1+\varepsilon^2\tau)(K_4+K_7) \le C\varepsilon^2(E_4+E_7) + C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1}\\
&\quad + C_9\int |\hat{\theta}_y|(b_1^2+b_3^2)dy + C\varepsilon(1+\varepsilon^2\tau)^{\frac{1}{2}}\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy\\
&\quad \le C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1} + CK_8 + C(1+\varepsilon^2\tau)^{\frac{1}{2}}K_5. \qquad (4.8)
\end{aligned}
$$

Integrating (4.8) with respect to τ and using (4.2) and (4.7), one has

$$
(E_4+E_7) \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}, \qquad \int_0^\tau (1+\varepsilon^2 s)(K_4+K_7)ds \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \quad (4.9)
$$

Therefore, it holds

$$
\|(\phi,\psi,\zeta)(\tau)\|^2 \le C\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}. \qquad (4.10)
$$

Multiplying (3.92) by $(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}$ with $\vartheta>0$ in Theorem 3.1 and using (3.14), (4.1), (4.7), (4.9) and the smallness of δ, one has

$$
\begin{aligned}
&[(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}E_7]_\tau = (\frac{3}{2}-\vartheta)(1+\varepsilon^2\tau)^{\frac{1}{2}-\vartheta}\varepsilon^2 E_7 + (1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}E_{7\tau}\\
&\le C\delta K_8 + C(1+\varepsilon^2\tau)^{\frac{1}{2}-\vartheta}(K_4+K_7) + C\delta(1+\varepsilon^2\tau)\sum_{|\alpha|=1}\frac{1}{\varepsilon^2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\\
&\quad + \frac{C}{\varepsilon^3}(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^{\frac{10}{3}} + \frac{C}{\varepsilon^3}(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}\|(\phi,\psi,\zeta)\|\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^3\\
&\quad + C\varepsilon(1+\varepsilon^2\tau)^{\frac{1}{2}-\vartheta+C_0\sqrt{\delta}}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dy + C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1-\vartheta}\\
&\le C\delta K_8 + C\delta(1+\varepsilon^2\tau)(K_4+K_7) + C(1+\varepsilon^2\tau)^{\frac{1}{2}-\vartheta+C_0\sqrt{\delta}}K_5 + C\delta\varepsilon^2(1+\varepsilon^2\tau)^{-1-\vartheta} \qquad (4.11)\\
&\quad + \frac{C}{\varepsilon^3}(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^{\frac{10}{3}} + \frac{C}{\varepsilon^3}(1+\varepsilon^2\tau)^{\frac{3}{2}-\vartheta}\|(\phi,\psi,\zeta)\|\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^3.
\end{aligned}
$$

By using (3.11) and (4.9), one can get

$$
\begin{aligned}
&\frac{1}{\varepsilon^3}\int_0^\tau(1+\varepsilon^2 s)^{\frac{3}{2}-\vartheta}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^{\frac{10}{3}}ds + \frac{1}{\varepsilon^3}\int_0^\tau(1+\varepsilon^2 s)^{\frac{3}{2}-\vartheta}\|(\phi,\psi,\zeta)\|\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^3 ds\\
&\le \frac{1}{\varepsilon}\int_0^\tau(1+\varepsilon^2 s)^{\frac{3}{2}-\vartheta}\Big[(1+\varepsilon^2 s)^{-\frac{2}{3}+\frac{2}{3}C_0\sqrt{\delta}} + (1+\varepsilon^2 s)^{-1+C_0\sqrt{\delta}}\Big]\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 ds\\
&\le \frac{1}{\varepsilon}\int_0^\tau(1+\varepsilon^2 s)^{\frac{1}{2}}\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 ds \le C\varepsilon\int_0^\tau(1+\varepsilon^2 s)^{\frac{1}{2}}K_4 ds \le C\sqrt{\delta}\varepsilon(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}, \quad (4.12)
\end{aligned}
$$

provided that $C_0\sqrt{\delta}\le\vartheta$. Thus integrating (4.11) over $[0,\tau]$ and using (4.2), (4.7), (4.9) and (4.12) yield that

$$
E_7 \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}},
$$

which immediately implies

$$
\begin{aligned}
&\frac{1}{\varepsilon^3}\Big(\|(\phi,\psi,\zeta)_y\|^2 + \sum_{|\alpha|=2}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + \sum_{|\alpha|=2}\int\int\frac{|\partial^\alpha\tilde{f}|^2}{M_*}d\xi dy\Big)\\
&\qquad + \frac{1}{\varepsilon}\sum_{|\alpha|=1}\int\int\frac{|\partial^\alpha\tilde{G}|^2}{2M_*}d\xi dy \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{3}{2}+\vartheta+C_0\sqrt{\delta}}. \qquad (4.13)
\end{aligned}
$$

Proof of Theorem 3.1: Combining (4.3), (4.6), (4.13) and (4.10) and using the Sobolev inequality, it holds that

$$
\|(\Phi,\Psi,W)\|_{L^\infty}^2 \le C\|(\Phi,\Psi,W)\|\Big(\|(\phi,\psi,\zeta)\| + \delta\varepsilon(1+\varepsilon^2\tau)^{-\frac{1}{2}}\Big) \le C\sqrt{\delta}\varepsilon, \qquad (4.14)
$$

and

$$
\|\int\frac{|\tilde{G}|^2}{M_*}d\xi\|_{L^\infty} \le \Big(\int\int\frac{|\tilde{G}|^2}{M_*}d\xi dy\Big)^{\frac{1}{2}}\Big(\int\int\frac{|\tilde{G}_y|^2}{M_*}d\xi dy\Big)^{\frac{1}{2}} \le C\sqrt{\delta}(1+\varepsilon^2\tau)^{-\frac{1}{2}}. \qquad (4.15)
$$

Therefore, (4.6), (4.10), (4.13), (4.14) and (4.15) verify the a priori assumption (3.10) if we choose $\lambda_0=\delta^{\frac{1}{8}}$. Hence, the proof of Theorem 3.1 is completed. □

Proof of Theorem 2.4: The proof of (2.44) can be obtained directly from (3.9) by using the transformation (3.1) of the scaled variables (y, τ) and the original variables (x, t). By combining (2.44) and Sobolev inequality, (2.45) can be derived immediately. Thus the proof of Theorem 2.4 is completed. □

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