

# Geodesic Length Functions and Teichmüller Spaces

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**Abstract** Given a compact orientable surface with finitely many punctures  $\Sigma$ , let  $\mathcal{S}(\Sigma)$  be the set of isotopy classes of essential unoriented simple closed curves in  $\Sigma$ . We determine a complete set of relations for a function from  $\mathcal{S}(\Sigma)$  to  $\mathbf{R}$  to be the geodesic length function of a hyperbolic metric with geodesic boundary and cusp ends on  $\Sigma$ . As a consequence, the Teichmüller space of hyperbolic metrics with geodesic boundary and cusp ends on  $\Sigma$  is reconstructed from an intrinsic  $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$  structure on  $\mathcal{S}(\Sigma)$ .

### §0. Introduction

Let  $\Sigma = \Sigma_{g,r}^s$  be a compact oriented surface of genus  $g$  with  $r$  boundary components and  $s$  punctures, i.e., a surface of signature  $(g, r, s)$ . The Teichmüller space of isotopy classes of hyperbolic metrics with geodesic boundary and cusp ends on  $\Sigma$  is denoted by  $T_{g,r}^s = T(\Sigma)$  and the isotopy classes of essential simple closed unoriented curves in  $\Sigma$  is denoted by  $\mathcal{S} = \mathcal{S}(\Sigma)$ . A simple loop in  $\Sigma$  is called *parabolic* if it is homotopic into an end of  $\Sigma$ . The set of isotopy classes of essential parabolic simple loops in  $\Sigma$  is denoted by  $P(\Sigma)$ . For each  $m \in T(\Sigma)$  and  $\alpha \in \mathcal{S}(\Sigma)$ , let  $l_m(\alpha)$  be the length of the geodesic representing  $\alpha$  if  $\alpha \notin P(\Sigma)$  and let  $l_m(\alpha) = 0$  if  $\alpha \in P(\Sigma)$ . The goal of the paper is to characterize the geodesic length function  $l_m$  in terms of an intrinsic  $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$  structure on  $\mathcal{S}(\Sigma)$ .

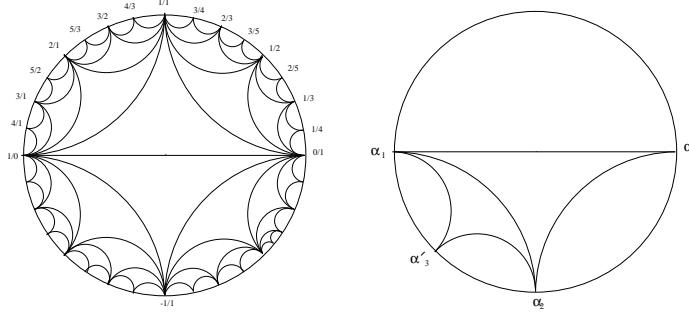


Figure 1

**Theorem 1.** For surface  $\Sigma_{g,r}^s$  of negative Euler number, a function  $f : \mathcal{S}(\Sigma_{g,r}^s) \rightarrow \mathbf{R}$  is a geodesic length function if and only if  $f|_{\mathcal{S}(\Sigma')}$  is a geodesic length function for each incompressible subsurface  $\Sigma' \cong \Sigma_{1,1}^0, \Sigma_{0,r}^s$  ( $r+s=4$ ) in  $\Sigma_{g,r}^s$ . Furthermore, geodesic length functions on  $\mathcal{S}(\Sigma_{1,r}^s)$  ( $r+s=1$ ) and  $\mathcal{S}(\Sigma_{0,r}^s)$  ( $r+s=4$ ) are characterized by two polynomial equations (in  $\cosh(f/2)$ ) in the  $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$  structure on  $\mathcal{S}$ .

Recall that a subsurface  $\Sigma' \subset \Sigma$  is *incompressible* if each essential loop in  $\Sigma'$  is still essential in  $\Sigma$ . Given two isotopy classes  $\alpha$  and  $\beta$  in  $\mathcal{S}(\Sigma)$ , the *geometric intersection number* between  $\alpha, \beta$ , denoted by  $I(\alpha, \beta)$  is  $\min\{|a \cap b| \mid a \in \alpha \text{ and } b \in \beta\}$  where  $|a \cap b|$  is the number of points in  $a \cap b$ .

Given a surface  $\Sigma$ , let  $\mathcal{S}'(\Sigma)$  be the set of isotopy classes of essential, non-boundary parallel non parabolic simple loops in  $\Sigma$ . For surfaces  $\Sigma = \Sigma_{1,r}^s$  ( $r+s=1$ ) and  $\Sigma_{0,r}^s$  ( $r+s=4$ ), it is well known that there exists a bijection  $\pi : \mathcal{S}'(\Sigma) \rightarrow \mathbf{Q}P^1 (= \hat{\mathbf{Q}})$  so that  $p'q - pq' = \pm 1$  if and only if  $I(\pi^{-1}(p/q), \pi^{-1}(p'/q')) = 1$  (for  $\Sigma_{1,r}^s$ ) and 2 (for  $\Sigma_{0,r}^s$ ). See figure 1. We say that three classes  $\alpha_1, \alpha_2, \alpha_3$  in  $\mathcal{S}'(\Sigma)$  form an *ideal triangle* if they correspond to the vertices of an ideal triangle in the modular relation under the map  $\pi$ .

For the rest of the paper, we introduce the *trace function*  $t_m(\alpha) = 2\cosh l_m(\alpha)/2$  from  $\mathcal{S}(\Sigma)$  to  $\mathbf{R}_{\geq 2}$ . We will deal with the trace function  $t_m$  instead of  $l_m$ .

**Theorem 2.** (a) For surface  $\Sigma_{1,r}^s$ ,  $r+s=1$  with  $b$  as the isotopy class of the boundary loop or the parabolic loop, a function  $t : \mathcal{S} \rightarrow \mathbf{R}_{\geq 2}$  is a trace function if and only if the following hold.

$$(1) \quad \prod_{i=1}^3 t(\alpha_i) = \sum_{i=1}^3 t^2(\alpha_i) + t(b) - 2 \quad \text{and}$$

$$t(\alpha_3)t(\alpha'_3) = \sum_{i=1}^2 t^2(\alpha_i) + t(b) - 2$$

where  $(\alpha_1, \alpha_2, \alpha_3)$  and  $(\alpha_1, \alpha_2, \alpha'_3)$  are distinct ideal triangles in  $\mathcal{S}'$ .

(b) For surface  $\Sigma_{0,r}^s$ ,  $r+s=4$ , let  $b_1, b_2, b_3, b_4$  be four isotopy classes of simple loops represented by the boundary components and the parabolic loops, a function  $t : \mathcal{S} \rightarrow \mathbf{R}_{\geq 2}$  is a trace function if and only if for each ideal triangle  $(\alpha_1, \alpha_2, \alpha_3)$  so that  $(\alpha_i, b_j, b_k)$  bounds a  $\Sigma_{0,3}^0$  in  $\Sigma_{0,r}^s$  the following hold.

$$(2) \quad \prod_{i=1}^3 t(\alpha_i) = \sum_{i=1}^3 t^2(\alpha_i) + \sum_{j=1}^4 t^2(b_j) + \prod_{j=1}^4 t(b_j) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^4 f(\alpha_i)f(b_j)f(b_k) - 4 \quad \text{and}$$

$$f(\alpha_3)f(\alpha'_3) = \sum_{i=1}^2 t^2(\alpha_i) + \sum_{j=1}^4 t^2(b_j) + \prod_{j=1}^4 t(b_j) + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^4 f(\alpha_i)f(b_j)f(b_k) - 4$$

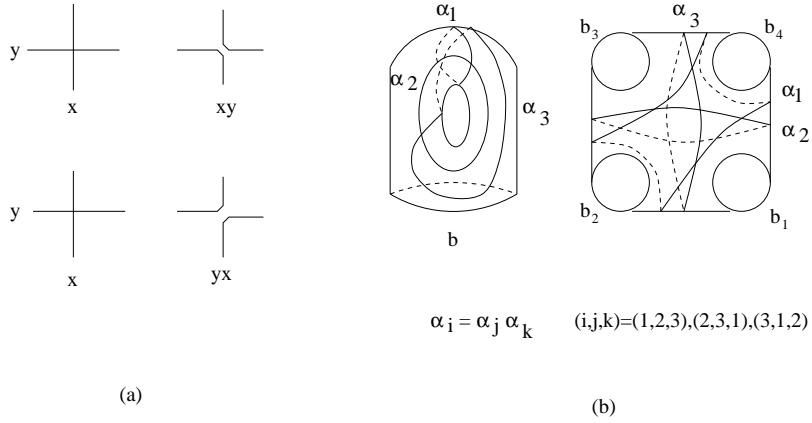
where  $(\alpha_1, \alpha_2, \alpha'_3)$  and  $(\alpha_1, \alpha_2, \alpha_3)$  are two distinct ideal triangles in  $\mathcal{S}'$ .

Part (a) of theorem 2 was a result of Fricke-Klein [FK] and Keen [Ke].

Thurston's compactification of the Teichmüller space  $T(\Sigma)$  (see [Bo1], [FLP], [Th]) uses the embedding  $\tau : T(\Sigma) \rightarrow \mathbf{R}^{\mathcal{S}(\Sigma)}$  sending  $m$  to  $l_m$ . Theorems 1 and 2 gives a complete description of the image of the embedding.

The modular relation on  $\mathcal{S}$  is derived from an intrinsic combinatorial structure on  $\mathcal{S}$  as. If two simple closed curves  $a$  and  $b$  intersect at one point transversely (resp.  $\alpha, \beta \in \mathcal{S}(\Sigma)$  with  $I(\alpha, \beta) = 1$ ), we denote it by  $a \perp b$  (resp.  $\alpha \perp \beta$ ); if two simple closed curves  $a$  and  $b$  intersect at two points of different signs transversely and  $I([a], [b]) = 2$ , we denote it by  $a \perp_0 b$ . In this case, we denote the relation between their isotopy classes by  $[a] \perp_0 [b]$ . Suppose  $x$  and  $y$  are two arcs in  $\Sigma$  so that  $x$  intersects  $y$  transversely at one point. Then the *resolution* of  $x \cap y$  from  $x$  to  $y$  is defined as follows. Take any orientation on  $x$  and use the orientation on  $\Sigma$  to determine an

orientation on  $y$ . Now resolve the intersection point  $x \cap y$  according to the orientations (see figure 2(a)). If  $a \perp b$  or  $a \perp_0 b$ , we define  $ab$  to be the curve obtain by resolving intersection points in  $a \cap b$  from  $a$  to  $b$ . We define  $\alpha\beta = [ab]$  where  $a \in \alpha, b \in \beta$  with  $|a \cap b| = I(\alpha, \beta)$ . It follows from the definition that  $\alpha\beta \perp$  (resp.  $\perp_0$ )  $\alpha, \beta$  if  $\alpha \perp \beta$  (resp.  $\alpha \perp_0 \beta$ ). Furthermore,  $\alpha(\beta\alpha) = (\alpha\beta)\alpha = \beta$ . For surface  $\Sigma = \Sigma_{1,r}^s$  ( $r+s=1$ ) and  $\Sigma_{0,r}^s$  ( $r+s=4$ ), three elements  $\alpha_1, \alpha_2, \alpha_3$  in  $\mathcal{S}'(\Sigma)$  form an ideal triangle if and only if  $\alpha_1 \perp \alpha_2$  and  $\alpha_3 = \alpha_1\alpha_2$  or  $\alpha_2\alpha_1$ . In particular the two distinct ideal triangles in theorem 2 are  $(\alpha_1, \alpha_2, \alpha_1\alpha_2)$  and  $(\alpha_1, \alpha_2, \alpha_2\alpha_1)$ .



Right-hand orientation on the front face

Figure 2

The relations (1) and (2) come from trace identities for  $SL(2, \mathbf{R})$  matrices. Note that the second part of relations (1), (2) shows that  $t(\alpha_3)$  and  $t(\alpha'_3)$  are the two roots of the quadratic equation (in  $t(\alpha_3)$ ) in the first part of the relations. Thus we obtain two more relations as follows.

$$t(\alpha_1\alpha_2) + t(\alpha_2\alpha_1) = t(\alpha_1)t(\alpha_2) \quad \text{where } \alpha_1 \perp \alpha_2 \text{ and}$$

$$t(\alpha_1\alpha_2) + t(\alpha_2\alpha_1) = t(\alpha_1)t(\alpha_2) - t(b_i)t(b_j) - t(b_k)t(b_l) \quad \{i, j, k, l\} = \{1, 2, 3, 4\}$$

where  $\alpha_1 \perp_0 \alpha_2$  and  $(\alpha_1\alpha_2, b_i, b_j)$  bounds a  $\Sigma_{0,3}^0$ .

The main part of the proof of theorems is to show that these relations are sufficient. To prove this, we use induction on  $|\Sigma_{g,r}^s| = 3g + r + s$ . There are two key ingredients involved in the proof: a gluing lemma and an iteration process.

For simplicity, we describe the gluing lemma for a compact surface  $\Sigma = X \cup Y$  where  $X$  and  $Y$  are compact incompressible subsurfaces so that  $X \cap Y \cong \Sigma_{0,3}^0$  (see figure 3 (b), (c)). Let the three boundary components of  $X \cap Y$  be  $a_1, a_2$  and  $a_3$ . Then the gluing lemma states that for each hyperbolic metric  $m_X$  and  $m_Y$  on  $X$  and  $Y$  respectively so that  $a_i$  are geodesics in both metrics with  $l_{m_X}(a_i) = l_{m_Y}(a_i)$  ( $i=1,2,3$ ), there is a hyperbolic metric  $m$  in  $\Sigma$  unique up

to isotopy so that the restriction of  $m$  to  $X$  is isotopic to  $m_X$  and the restriction of  $m$  to  $Y$  is isotopic to  $m_Y$ .

The iteration process is derived as follows. Given a function  $t$  on  $\mathcal{S}(\Sigma)$  satisfying the relations (1) and (2), using the gluing lemma and the induction hypothesis, one constructs a hyperbolic metric on the surface so that  $t$  and the trace of the metric coincide on  $\mathcal{S}(X) \cup \mathcal{S}(Y)$ . To show that these two functions are the same on all simple closed curves, we observe that the second part of the relations (1) and (2) indicates that the value of  $t$  at  $\beta\alpha$  is determined by the values of  $t$  on  $\alpha$ ,  $\beta$ ,  $\alpha\beta$  and  $b'_i$ 's. By iterated use of the relations together with the multiplicative structure on  $\mathcal{S}$ , we show that these two functions are the same.

By the work of Thurston, the degenerations of hyperbolic metrics become measured laminations and the corresponding projective limits of geodesic length functions become geometric intersection numbers. Thus, relations (1) and (2) degenerate to universal relations for the geometric intersection numbers. It is shown in [Lu2] that these degenerated equations determine Thurston's measured lamination spaces and Thurston's compactification of the Teichmüller spaces.

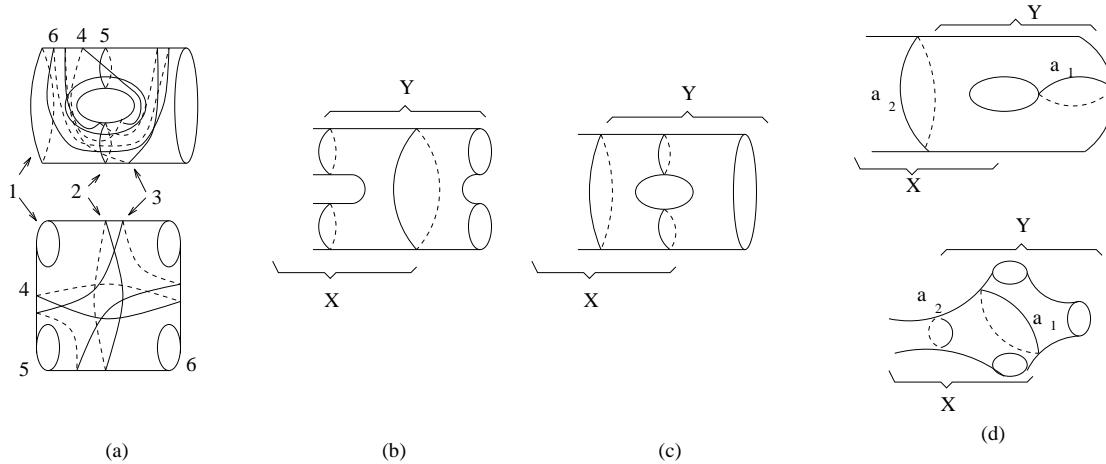


Figure 3

As another consequence of theorem 1, we consider finite dimensional embeddings of the Teichmüller spaces. Given a subset  $F$  of  $\mathcal{S}(\Sigma_{g,r}^s)$ , let  $\pi_F : T(\Sigma_{g,r}^s) \rightarrow \mathbf{R}^F$  be the map  $\pi_F(m) = t_m|_F$ . It is well known from the work of Fricke-Klein [FK] that there exists a finite set  $F$  so that  $\pi_F$  is an embedding. The work of Okumura [Ok1], Schmutz [Sc], Seppälä-Sorvali [SS], Sorvali [So] show that there exists a set  $F$  consisting of  $N$  ( $N=6g+3r+2s-6$  if  $r > 0$  and  $N=6g+2s-5$  if  $r = 0$ ) elements so that  $\pi_F$  is an embedding. This number  $N$  is necessary the minimal number by a result of Wolpert [Wo] in case  $r = 0$ . We shall indicate a proof of the existence of such set  $F$  for compact surface with boundary below. By theorem 2 and the gluing lemma, it is easy to show that hyperbolic metrics on  $\Sigma_{0,4}^0$  and  $\Sigma_{1,2}^0$  are determined by the geodesic lengths of six curves

as shown in figure 3(a). Now each compact oriented surface with boundary and Euler number smaller than  $-2$  is obtained from  $\Sigma_{0,4}^0$  and  $\Sigma_{1,2}^0$  by repeated use of gluing along 3-holed spheres (see figure 3(b), (c)). Furthermore, one of the subsurface used in the gluing (surface Y) is either  $\Sigma_{0,4}^0$  or  $\Sigma_{1,2}^0$ . Thus, each time the Euler number of the resulting surface changes by  $-1$  and the number of curves needed to determine the hyperbolic metric increases by 3 (the curves 3, 4, 6 in figure 3(a) are the needed ones and the curves 1, 2, 5 are in the subsurface X).

The corollary below strength their result to conclude that the image of the embedding is an explicit semi-analytic set. Okumura [Ok2] has also obtained the result for  $s = r = 0$  using a different method. The semi-analytic property in the corollary also follows from the work of Brumfiel [Br], Morgan-Shalen [MS], and Helling [He].

**Corollary.** (a) For surface  $\Sigma_{g,r}^s$  of negative Euler number and  $r > 0$ , there exists a finite subset  $F$  in  $\mathcal{S}(\Sigma_{g,r}^s)$  consisting of  $6g + 3r + 2s - 6$  elements so that the map  $\pi_F : T(\Sigma_{g,r}^s) \rightarrow \mathbf{R}^F$  is a real analytic embedding onto an open subset which is defined by a finite set of explicit real analytic inequalities in the coordinates of  $\pi_F$ .

(b) For surface  $\Sigma_{g,0}^s$  of negative Euler number, there exists a finite subset  $F$  of  $\mathcal{S}(\Sigma_{g,0}^s)$  consisting of  $6g + 2s - 5$  elements so that  $\pi_F : T(\Sigma_{g,0}^s) \rightarrow \mathbf{R}^F$  is an embedding whose image in  $\mathbf{R}^F$  is defined by one real analytic equation and finitely many explicit real analytic inequalities in the coordinates of  $\pi_F$ .

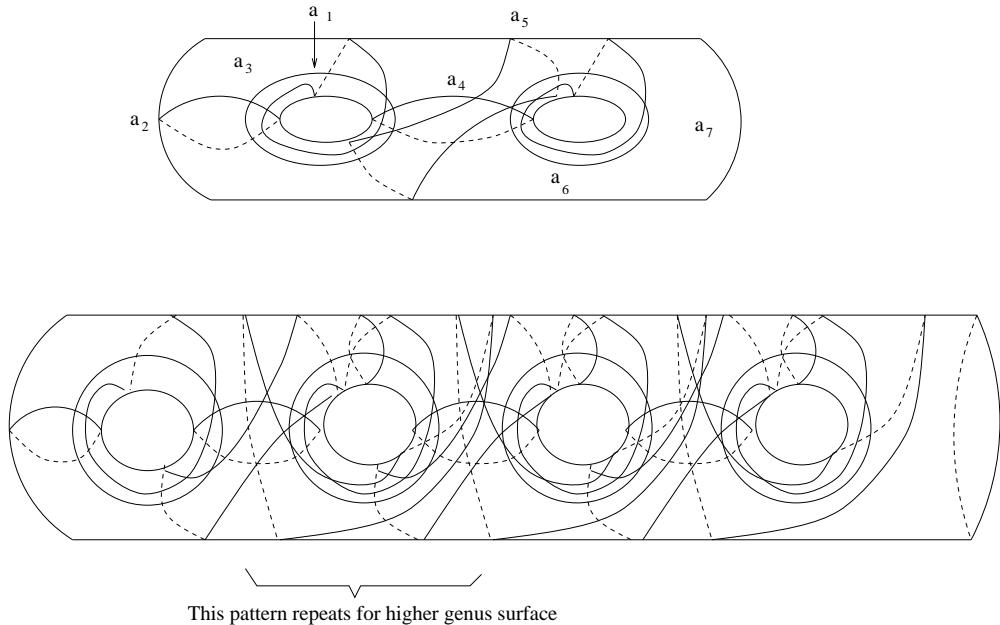


Figure 4

The inequalities and the equation in the corollary are given by functions which are obtained from the coordinates of  $\pi_F$  by a finite number of algebraic operations (summation, multiplication, and division over the rationals) and the square root operation.

Some examples of the collection  $F$  and the images of the Teichmüller spaces are as follows. For  $\Sigma_{2,0}^0$ , take  $F = \{[a_1], [a_2], [a_3], [a_4], [a_5], [a_6], [a_7]\}$  as in figure 4. Then the map  $\pi_F$  is an embedding with image  $\pi_F(T_{2,0}^0) = \{(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \in \mathbf{R}_{>2}^7 \mid t_8 > 2, t_9 > 2, t_8 = t_6 t_7 t_9 - t_6^2 - t_7^2 - t_9^2 + 2, \text{ where } t_8 = t_1 t_2 t_3 - t_1^2 - t_2^2 - t_3^2 + 2, \text{ and } (2 + t_2^2 + t_8) t_9^2 + 2 t_2 (t_4 + t_5) t_9 + 2 t_2^2 + t_4^2 + t_5^2 + t_8^2 + t_2^2 t_8 - t_4 t_5 t_8 - 4 = 0\}$ .

The explicit equations and inequalities in the corollary for the surface  $\Sigma_{1,r}^s$  ( $r + s = 1$ ) are as follows. For  $\Sigma_{1,1}^0$  (resp.  $\Sigma_{1,0}^1$ ), Keen [Ke] proved that one takes  $F = \{[a_1], [a_2], [a_3]\}$  to be an ideal triangle and the image  $\pi_F(T(\Sigma_{1,1}^0))$  is  $\{(t_1, t_2, t_3) \in \mathbf{R}_{>2}^3 \mid \text{formula (3) holds}\}$ .

$$(3) \quad t_1 t_2 t_3 > t_1^2 + t_2^2 + t_3^2.$$

$(\pi_F(T(\Sigma_{1,0}^1))) = \{(t_1, t_2, t_3) \in \mathbf{R}_{>2}^3 \mid t_1 t_2 t_3 = t_1^2 + t_2^2 + t_3^2\}$ . For  $\Sigma_{0,r}^s$  with  $r + s = 4$ , we take the collection  $F$  to be the isotopy classes of six curves  $b_1, b_2, b_3, a_{12}, a_{23}$ , and  $a_{31}$  where  $[a_{ij}]$  forms an ideal triangle and  $(a_{ij}, b_i, b_j)$  bounds a  $\Sigma_{0,3}$ . Then  $\pi_F$  is an embedding whose image  $\pi_F(T(\Sigma_{0,4}^0))$  is given by  $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}) \in \mathbf{R}_{>2}^6 \mid \text{so that formula (4) holds}\}$ .

$$(4) \quad t_{12} t_{23} t_{31} > t_{12}^2 + t_{23}^2 + t_{31}^2 + t_1^2 + t_2^2 + t_3^2 + t_{12} t_1 t_2 + t_{23} t_2 t_3 + t_{31} t_3 t_1 + 2 t_1 t_{23} + 2 t_2 t_{31} + 2 t_3 t_{12} + 2 t_1 t_2 t_3.$$

The organization of the paper is as follows. In §1, we prove a gluing lemma and recall basic facts on discrete subgroups of  $SL(2, \mathbf{R})$  and the spin structures on surfaces. We prove theorem 2 in §2. In §3, we establish a proposition on the multiplicative structure on  $\mathcal{S}$ . Theorem 1 is proved in §4. In §5, we discuss applications. In the main body of the paper (§2, §3, and §4) we shall treat hyperbolic metrics without cusp ends in order to reduce the length of the paper. No new ideas are needed for metrics with cusps. The proofs of the theorems 1 and 2 for metrics with cusps ends will be discussed briefly in §5.3.

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## §1. Preliminaries on Discrete Subgroups of $SL(2, \mathbf{R})$

We prove a gluing lemma in §1.1. Basic facts about discrete representations of surface groups into  $SL(2, \mathbf{R})$  and spin structures on surfaces will be recalled in §1.2 - 1.4.

We shall use the following notations throughout the paper. Let  $\Sigma_{g,r} = \Sigma_{g,r}^0$ ;  $\Sigma_g = \Sigma_{g,0}^0$ ; and  $T_{g,r} = T_{g,r}^0$ . We use  $cl(X)$  and  $int(X)$  to denote the closure and the interior of a submanifold  $X$ . The isotopy class of a simple loop  $a$  is denoted by  $[a]$  and the isotopy class of a hyperbolic metric  $d$  is denoted by  $[d]$ . If  $f : \mathcal{S} \rightarrow \mathbf{R}$  is a function and  $a$  is a simple loop, we define  $f(a)$  to be  $f([a])$ . In particular,  $I(a, b) = I([a], [b]) = I(a, [b])$ . A regular neighborhood of a submanifold  $X$  is denoted by  $N(X)$ . Regular neighborhoods are always assumed to be small. All intersections of curves are assumed to be transverse.

An RC-function (compass and ruler constructible function) in variables  $x_1, \dots, x_n$  is a function obtained from  $1, x_1, \dots, x_n$  by a finite number of algebraic operations and the square root operation.

The set of RC-functions is closed under algebraic operations and compositions. Note that  $|x| = \sqrt{x^2}$  is an RC-function. An RC-function is continuous in its natural domain and is analytic away from its singular set.

### 1.1. A gluing lemma

First some definitions and conventions. A surface  $\Sigma$  is oriented and connected which is either  $\Sigma_{g,r}^s$  or obtained from  $\bar{\Sigma} = \Sigma_{g,r}^s$  by removing some boundary components. Each boundary component of  $\bar{\Sigma}$  is called a boundary component of  $\Sigma$ . A hyperbolic metric with geodesic boundary and cusp ends on  $\Sigma$  is a hyperbolic metric whose completion is a hyperbolic metric on  $\bar{\Sigma}$  with geodesic boundary and cusp ends. Two hyperbolic metrics are *isotopic* if there is an isometry between them which is isotopic to the identity. The Teichmüller space of hyperbolic metrics with geodesic boundary and cusp ends on  $\Sigma$  is denoted by  $T(\Sigma)$ . It is canonically isomorphic to  $T(\bar{\Sigma})$ .

A subsurface  $X$  of  $\Sigma$  is *incompressible* if the inclusion map induces a monomorphism in fundamental groups. If the subsurface is compact, then it is incompressible if and only if each boundary component of  $X$  is essential in  $\Sigma$ . A *good incompressible* subsurface is an incompressible subsurface whose interior is a component of the complement of a finite union of disjoint, pairwise non-parallel, non-boundary parallel, non-parabolic simple closed curves in  $\Sigma$ . For instance, if  $s$  is a non-separating simple closed curve in  $\Sigma$ , then  $\Sigma - s$  is a good incompressible subsurface but  $\Sigma - N(s)$  is not. If  $X$  is an incompressible subsurface of negative Euler number, then  $\text{int}(X)$  is isotopic to a good incompressible subsurface. For a good incompressible subsurface  $X$  of  $\Sigma$ , we define the *restriction map*  $R_X = R_X^\Sigma : T(\Sigma) \rightarrow T(X)$  as follows. Given  $[d] \in T(\Sigma)$ , there is a homeomorphism  $h$  of  $\Sigma$  isotopic to the identity so that the frontier of  $X$ ,  $\text{cl}(X) - \text{int}(X)$ , is a union of geodesics in the pull back metric  $h^*(d)$ . We define  $R_X([d])$  to be  $[h^*(d)|_X]$ . It follows from elementary hyperbolic geometry and topology of surfaces that  $R_X$  is well defined (see [CB], or [Bu]). Furthermore, it follows from the definition that if  $X$  is good incompressible in  $Y$  and  $Y$  is good incompressible in  $Z$ , then  $R_X^Z = R_X^Y R_Y^Z$ . The restriction map is in general not onto. For instance, if we take  $X$  to be the complement of a non-separating simple closed curve in a surface  $\Sigma$  with negative Euler number, then  $R_X$  is not onto.

**Lemma 1.** (Gluing along a 3-holed sphere) *Suppose  $X$  and  $Y$  are two good incompressible subsurfaces of  $\Sigma$  whose union is  $\Sigma$  so that either (1)  $X \cap Y \cong \Sigma_{0,3}$ , or (2)  $Y \cong \Sigma_{1,1}$  and  $X \cap Y \cong \Sigma_{1,1} - s$  where  $s$  is a non-separating simple closed curve in  $\text{int}(Y)$  (see figure 3(b), (c), (d)), or (3)  $X \cap Y \cong \Sigma_{0,2}^1$  with the punctured end in  $\Sigma_{0,2}^1$  being a punctured end of  $\Sigma$ . Then for any two elements  $m_X \in T(X)$  and  $m_Y \in T(Y)$  with  $R_{X \cap Y}(m_X) = R_{X \cap Y}(m_Y)$ , there exists a unique element  $m \in T(\Sigma)$  so that  $R_X(m) = m_X$  and  $R_Y(m) = m_Y$ .*

**Proof.** To show the existence, let  $d_X \in m_X$  (resp.  $d_Y \in m_Y$ ) be a representative so that  $d_X|_{X \cap Y}$  (resp.  $d_Y|_{X \cap Y}$ ) has geodesic boundary and cusp ends, i.e.,  $R_{X \cap Y}([d_X]) = [d_X|_{X \cap Y}]$  (resp.  $R_{X \cap Y}([d_Y]) = [d_Y|_{X \cap Y}]$ ). Let  $h : X \cap Y \rightarrow X \cap Y$  be an isometry from  $d_X|_{X \cap Y}$  to  $d_Y|_{X \cap Y}$  which is isotopic to the identity map. By the assumption on  $X$  and  $Y$ , we can extend  $h$  to a homeomorphism  $g$  of  $X$  which is isotopic to the identity. Define a hyperbolic metric  $d$  on  $\Sigma$  with geodesic boundary and cusp ends as follows:  $d|_X = g^*(X)$  and  $d|_Y = Y$ . It follows from the definition that  $R_X([d]) = [d_X]$  and  $R_Y([d]) = [d_Y]$ . The uniqueness follows from the fact that an analytic automorphism of a complex structure on  $\text{int}(\Sigma_{0,3})$  which preserves each end is the identity map.  $\square$

## 1.2. Monodromy representations and spin structures

Given a hyperbolic metric  $d$  with geodesic boundary and cusp ends on  $\Sigma$ , its monodromy is a discrete faithful representation  $\rho : \pi_1(\Sigma) \rightarrow \text{PSL}(2, \mathbf{R})$  unique up to  $\text{PGL}(2, \mathbf{R}) = \text{GL}(2, \mathbf{R})/\{\pm I\}$  conjugation so that there is an isometric embedding  $h$  from the universal cover  $\tilde{\Sigma}$  with the pull back metric into the hyperbolic plane  $\mathbf{H}$  satisfying  $h(\gamma(x)) = \rho(\gamma)(h(x))$  for all  $x \in \tilde{\Sigma}$  and  $\gamma \in \pi_1(\Sigma)$ . Isotopic metrics have the same  $\text{PGL}(2, \mathbf{R})$  conjugacy class of monodromies. If the isometric embedding  $h$  is orientation preserving (resp. reversing), we say the monodromy  $\rho$  is orientation preserving (resp. reversing). Thus each  $m \in T(\Sigma)$  gives rise to two  $\text{PSL}(2, \mathbf{R})$  conjugacy classes of monodromy representations: one preserving the orientation and the other reversing the orientation. Let  $R(\Sigma)$  be the set of all such monodromy representations with the topology induced by algebraic convergence of representations. Then  $R(\Sigma)$  has two connected components corresponding to the two orientations. Each of the component is a trivial principal  $\text{PSL}(2, \mathbf{R})$  bundle over  $T(\Sigma)$  (see [Go1], [Har], [MSi] for details). Each representation  $\rho \in R(\Sigma)$  can be lifted to a representation  $\tilde{\rho} : \pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbf{R})$  (see [Be]) and there are exactly  $2^N$  such lifting where  $N = 2g$  if  $\Sigma$  has signature  $(g, 0, 0)$  and  $N = 2g + r + s - 1$  if  $\Sigma$  has signature  $(g, r, s)$  ( $r + s > 0$ ). Given a lifting  $\tilde{\rho}$  of  $\rho$ , all other liftings are obtained as follows. Let  $\{\gamma_1, \dots, \gamma_N\}$  be a set of generators for  $\pi_1(\Sigma)$  and  $I$  a subset of  $\{1, \dots, N\}$ . Then all other liftings are  $\tilde{\rho}_I$  where  $\tilde{\rho}_I(\gamma_i) = \tilde{\rho}(\gamma_i)$  if  $i \in I$  and  $\tilde{\rho}_I(\gamma_i) = -\tilde{\rho}(\gamma_i)$  if  $i \notin I$ . Let  $\tilde{R}(\Sigma)$  be the set of all liftings of the monodromies with the algebraic convergent topology. The representation space  $\tilde{R}(\Sigma)$  has  $2^{N+1}$  many connected components. These components are classified into two types according to the orientation of the representations in  $R(\Sigma)$ . Each component corresponds to a spin structure on the surface. We shall recall briefly spin structures. Let  $U\Sigma$  be the unit tangent bundle over the surface  $\Sigma$  with  $S^1$  as a fiber. A *spin structure* on  $\Sigma$  is a two-fold covering space of  $U\Sigma$  so that the  $S^1$ -fiber does not lift. Since two-fold covering spaces correspond to index-two subgroups of the fundamental groups, a spin structure is the same as an epimorphism  $\eta : \pi_1(U\Sigma) \rightarrow \mathbf{Z}_2 = \{\pm 1\}$  (as a multiplicative group) so that  $\eta(S^1) = -1$ . Since  $\mathbf{Z}_2$  is abelian, the epimorphism  $\eta$  is induced by an epimorphism (still denoted by)  $\eta : H_1(U\Sigma, \mathbf{Z}_2) \rightarrow \mathbf{Z}_2$  with  $\eta(S^1) = -1$ . Given a smooth immersed curve  $c$  in  $\Sigma$ , let  $\bar{c}$  be the unit tangent vectors of  $c$  in  $U\Sigma$ . We define  $\eta(c)$  to be  $\eta([\bar{c}])$ . For instance, if  $c$  bounds a disc, then  $\eta(c) = -1$  and if  $c$  is null homotopic with exactly one self intersection (a figure eight), then  $\eta(c) = 1$ .

Johnson in [Jo] provides an algorithm to calculate  $\eta(c)$  which we summarize as follows.

**Lemma 2. (Johnson)** (a) Suppose  $\{a_1, \dots, a_n\}$  and  $\{b_1, \dots, b_m\}$  are two collections of disjoint simple closed curves in  $\Sigma$  so that  $\sum_{i=1}^n [a_i] = \sum_{j=1}^m [b_j]$  in  $H_1(\Sigma, \mathbf{Z}_2)$ . Then  $\sum_{i=1}^n [\bar{a}_i] + n[S^1] = \sum_{j=1}^m [\bar{b}_j] + m[S^1]$  in  $H_1(U\Sigma, \mathbf{Z}_2)$ .

(b) Given  $\alpha \in H_1(\Sigma, \mathbf{Z}_2)$ , represent  $\alpha$  as  $\sum_{i=1}^n [a_i]$  in  $H_1(\Sigma, \mathbf{Z}_2)$  where  $\{a_1, \dots, a_n\}$  is a collection of disjoint simple closed curves in  $\Sigma$ . Then  $\eta^*(\alpha) = (-1)^n \prod_{i=1}^n \eta(a_i)$  is a  $\mathbf{Z}_2$ -quadratic map from  $H_1(\Sigma, \mathbf{Z}_2)$  to  $\mathbf{Z}_2$ , i.e.,  $\eta^*(\alpha + \beta) = (-1)^{<\alpha, \beta>} \eta^*(\alpha)\eta^*(\beta)$  where  $<\alpha, \beta>$  is the  $\mathbf{Z}_2$ -intersection number.

As a simple consequence, if  $\{a_1, a_2, a_3\}$  bounds a 3-holed sphere in  $\Sigma$ , then  $\eta(a_1)\eta(a_2)\eta(a_3) = -1$ ; if  $b$  is the boundary of a subsurface of signature  $(g, 1, 0)$ , then  $\eta(b) = -1$ ; and if  $a_1 \perp a_2$ , then  $\eta(a_1)\eta(a_2)\eta(a_1a_2) = 1$ .

The relationship between a lifting  $\tilde{\rho} \in \tilde{R}(\Sigma)$  of  $\rho \in R(\Sigma)$  and a spin structure is as follows. We first identify  $\text{PSL}(2, \mathbf{R})$  with  $U\mathbf{H}$  by sending an isometry  $g$  to  $g(v_0)$  where  $v_0$  is a specified element

in  $U\mathbf{H}$ . Under this identification, given a hyperbolic metric with geodesic boundary and cusp ends on  $\Sigma$  whose monodromy is  $\rho$ ,  $U\Sigma$  is canonically identified with a deformation retractor ( $U(\text{Nielsen core})$ ) of  $\text{PSL}(2, \mathbf{R})/\rho(\pi_1(\Sigma))$ . Let  $P : \text{SL}(2, \mathbf{R}) \rightarrow \text{PSL}(2, \mathbf{R})$  be the canonical projection. It is a two-fold covering map so that the  $S^1$  fiber (corresponding to  $\text{PSO}(2)$  in  $\text{PSL}(2, \mathbf{R})$ ) does not lift. Then  $P$  induces a two-fold covering map from  $\text{SL}(2, \mathbf{R})/\tilde{\rho}(\pi_1(\Sigma))$  to  $\text{PSL}(2, \mathbf{R})/\rho(\pi_1(\Sigma))$  so that the  $S^1$  fiber does not lift. Thus we have a spin structure  $\eta$  on  $\Sigma$  associated to the lifting  $\tilde{\rho}$  of  $\rho$ . A simple calculation shows that

$$(5) \quad \eta(\gamma_*) = \text{sign}(\text{tr}(\tilde{\rho}(\gamma))), \quad \gamma \in \pi_1(\Sigma)$$

where  $\gamma_*$  is the geodesic representative or a multiple of a parabolic simple closed curve in the conjugacy class of  $\gamma$ .

### 1.3. Trace identities and representations of surface groups into $SL(2, \mathbf{R})$

Given three matrices  $A_1, A_2, A_3$  in  $\text{SL}(2, \mathbf{C})$ , we have the following identities on the traces of their products (see [FK], [Go2], [Ho], or [Mag]). The basic trace identity is  $\text{tr}A_1A_2 + \text{tr}A_1^{-1}A_2 = \text{tr}A_1\text{tr}A_2$ . By iterated use of it, one obtains the following relations.

$$(6) \quad \text{tr}A_1A_2\text{tr}A_1^{-1}A_2 = \text{tr}^2A_1 + \text{tr}^2A_2 - \text{tr}[A_1, A_2] - 2.$$

$$(7) \quad \text{tr}[A_1, A_2] + 2 = \text{tr}^2A_1 + \text{tr}^2A_2 + \text{tr}^2A_1A_2 - \text{tr}A_1\text{tr}A_2\text{tr}A_1A_2.$$

$$(8) \quad \text{tr}A_1A_2A_3 + \text{tr}A_1A_3A_2 = \text{tr}A_1\text{tr}A_2A_3 + \text{tr}A_2\text{tr}A_3A_1 + \text{tr}A_3\text{tr}A_1A_2 - \text{tr}A_1\text{tr}A_2\text{tr}A_3.$$

$$(9) \quad \text{tr}A_1A_2A_3\text{tr}A_1A_3A_2 = \text{tr}^2A_1 + \text{tr}^2A_2 + \text{tr}^2A_3 + \text{tr}^2A_1A_2 + \text{tr}^2A_2A_3 + \text{tr}^2A_3A_1 + \text{tr}A_1A_2\text{tr}A_2A_3\text{tr}A_3A_1 - \text{tr}A_1\text{tr}A_2\text{tr}A_1A_2 - \text{tr}A_2\text{tr}A_3\text{tr}A_2A_3 - \text{tr}A_3\text{tr}A_1\text{tr}A_3A_1 - 4.$$

Combining formulas (8) and (9), we see that  $\text{tr}A_1A_2A_3$  and  $\text{tr}A_1A_3A_2$  are the two roots of the quadratic equation (10) below where P and Q stand for the right-hand sides of formulas (8) and (9) respectively.

$$(10) \quad x^2 - Px + Q = 0.$$

Using the basic trace relation, one obtains the following (see [Ho],[CS]).

**Lemma 3. (Fricke-Klein)** Suppose  $F_n$  is the free group on  $n$  generators  $\gamma_1, \dots, \gamma_n$ . Then for each element  $w$  in  $F_n$ , there is a polynomial  $P_w$  with integer coefficient in  $2^n - 1$  variables  $x_{i_1\dots i_k}$  with  $1 \leq i_1 < \dots < i_k \leq n$  so that for any representation  $\rho : F_n \rightarrow SL(2, \mathbf{R})$

$$\text{tr}\rho(w) = P_w(x_1, x_2, \dots, x_{i_1\dots i_k}, \dots, x_{12\dots n})$$

where  $x_{i_1\dots i_k} = \text{tr}\rho(\gamma_{i_1}\dots \gamma_{i_k})$ . Furthermore, if  $\rho_1$  and  $\rho_2$  are two representations with the same character and  $\rho_1(F_n)$  is not a solvable group, then  $\rho_1$  is conjugated to  $\rho_2$  by a  $GL(2, \mathbf{R})$  matrix.

In particular, if  $n = 2$ , then the three variables are  $\text{tr}\rho(\gamma_1)$ ,  $\text{tr}\rho(\gamma_2)$  and  $\text{tr}\rho(\gamma_1\gamma_2)$ ; if  $n = 3$ , the seven variables are  $\text{tr}\rho(\gamma_i)$ , and  $\text{tr}\rho(\gamma_i\gamma_j)$  and  $\text{tr}\rho(\gamma_1\gamma_2\gamma_3)$  where  $i, j = 1, 2, 3$  and  $i < j$ .

The discrete faithful representations of  $\pi_1(\Sigma_{0,r}^s)$  ( $r+s=3$ ) and  $\pi_1(\Sigma_{1,r}^s)$  ( $r+s=1$ ) which uniformize hyperbolic structures on  $\Sigma_{0,r}^s$  ( $r+s=3$ ) and  $\Sigma_{1,r}^s$  ( $r+s=1$ ) are as follows. See [GM], [Ke] for details.

For surface  $\Sigma_{0,r}^s$ ,  $r+s=3$ , we choose a set of geometric generators  $\gamma_1$  and  $\gamma_2$  in  $\pi_1(\Sigma_{0,r}^s)$  so that  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3 = \gamma_1\gamma_2$  are represented by simple closed curves homotopic into the three ends of  $\text{int}(\Sigma_{0,r}^s)$ .  $\Sigma_{0,2}^1$  has the puncture at the end corresponding to  $\gamma_3$  and  $\Sigma_{0,1}^2$  has the punctures at the ends corresponding to  $\gamma_2$  and  $\gamma_3$ .

**Lemma 4. (Fricke-Klein)** (a) If  $\rho \in \tilde{R}(\Sigma_{0,r}^s)$  with  $r+s=3$ , then  $\text{tr}\rho(\gamma_1)\text{tr}\rho(\gamma_2)\text{tr}\rho(\gamma_3) < 0$  and  $|\text{tr}\rho(\gamma_i)| \geq 2$  for  $i = 1, 2, 3$  so that the equality holds if and only if the corresponding end is a cusp.

(b) Given three real numbers  $t_1$ ,  $t_2$  and  $t_3$  with  $t_1t_2t_3 < 0$  and  $|t_i| > 2$  ( $i = 1, 2, 3$ ), there exist two elements  $\rho_1$  and  $\rho_2$  in  $\tilde{R}(\Sigma_{0,3})$  unique up to  $SL(2, \mathbf{R})$  conjugation so that  $\text{tr}\rho_i(\gamma_j) = t_j$  ( $i = 1, 2; j = 1, 2, 3$ ). These two representations are  $GL(2, \mathbf{R})$  conjugated and are related by  $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$ . Furthermore, if  $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $\lambda > 1$ , and  $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $c = 1$ , then  $a, b, d$  and  $\lambda$  are real analytic RC-functions of  $t_1$ ,  $t_2$  and  $t_3$  in the domain defined by  $t_1t_2t_3 < 0$  and  $|t_i| > 2$  ( $i = 1, 2, 3$ ).

(c) Given three numbers  $t_1$ ,  $t_2$  and  $t_3$  with  $t_1t_2t_3 < 0$  and  $|t_1| > 2$ ,  $|t_2| > 2$  and  $|t_3| = 2$  (resp.  $|t_1| > 2$ ,  $|t_2| = |t_3| = 2$ ), there exist two elements  $\rho_1$  and  $\rho_2$  in  $\tilde{R}(\Sigma_{0,2}^1)$  (resp.  $\tilde{R}(\Sigma_{0,1}^2)$ ) unique up to  $SL(2, \mathbf{R})$  conjugation so that  $\text{tr}\rho_i(\gamma_j) = t_j$  ( $i = 1, 2; j = 1, 2, 3$ ). These two representations are  $GL(2, \mathbf{R})$  conjugated and are related by  $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$ . Furthermore, if  $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $\lambda > 1$ , and  $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $c = 1$ , then  $a, b, d$  and  $\lambda$  are real analytic RC-functions of  $t_1$ ,  $t_2$  and  $t_3$  in the domain defined by  $t_1t_2t_3 < 0$ ,  $|t_1| > 2$  and  $|t_2| > 2$ .

(d)  $T(\Sigma_{0,0}^3)$  consists of one point.

Note that part (a) is a consequence of lemma 2 and formula (5). To find the explicit expression of  $a, b, d$  and  $\lambda$  in terms of  $t'_i$ s, see [Har], pp305.

For surface  $\Sigma_{1,r}^s$  ( $r+s=1$ ), we take a set of geometric generators  $\{\gamma_1, \gamma_2\}$  in  $\pi_1(\Sigma_{1,r}^s)$  so that they are represented by two simple closed curves  $a_1$  and  $a_2$  with  $a_1 \perp a_2$ . The multiplication  $\gamma_3 = \gamma_1\gamma_2$  is represented (in the free homotopy class) by either  $a_1a_2$  or  $a_2a_1$  depending on the orientation of the surface. The commutator  $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$  is represented by the simple closed curve  $\partial N(a_1 \cup a_2)$  homotopic into the end of  $\text{int}(\Sigma_{1,r}^s)$ .

**Lemma 5. (Fricke-Klein, Keen)** (a) If  $\rho \in \tilde{R}(\Sigma_{1,r}^s)$  with  $r+s=1$ , then  $\text{tr}\rho(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \leq -2$  so that equality holds if and only if  $s = 1$ . In particular,  $\text{tr}^2\rho(\gamma_1) + \text{tr}^2\rho(\gamma_2) + \text{tr}^2\rho(\gamma_3) - \text{tr}\rho(\gamma_1)\text{tr}\rho(\gamma_2)\text{tr}\rho(\gamma_3) \leq 0$  so that equality holds if and only if  $s = 1$ .

(b) Give three numbers  $t_i$ ,  $i = 1, 2, 3$  with  $|t_i| > 2$  and  $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 < 0$  (resp.  $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 = 0$ ), there exist two representations  $\rho_1$  and  $\rho_2$  in  $\tilde{R}(\Sigma_{1,1}^0)$  (resp.  $\tilde{R}(\Sigma_{1,0}^1)$ ) unique up

to  $SL(2, \mathbf{R})$  conjugation so that  $tr\rho_i(\gamma_j) = t_j$  ( $i = 1, 2; j = 1, 2, 3$ ). These two representations are  $GL(2, \mathbf{R})$  conjugated and are related by  $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$ . Furthermore, if  $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $\lambda > 1$ , and  $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $c = 1$ , then  $a, b, c, d$  and  $\lambda$  are real analytic RC-functions of  $t_1, t_2, t_3$ .

The first part of the lemma also follows lemma 2 and formula (5). Below is a proof of part (b) (known to J. Gilman). By lemma 3, it suffices to show the existence of  $\rho \in \tilde{R}(\Sigma_{1,r}^s)$  with  $tr(\rho(\gamma_j)) = t_j$ ,  $j = 1, 2, 3$ . We first construct three points  $A_1, A_2$  and  $A_3$  in  $\mathbf{H}$  so that their pairwise hyperbolic distance  $d(A_i, A_j)$  is determined by  $2\cosh d(A_i, A_j)/4 = |t_k|$  where  $i \neq j \neq k \neq i$ . That the pairwise distances satisfy the triangular inequalities follows from the given condition on  $t'_i$ s. Let  $h_{A_i}$  be the hyperbolic isometry which rotates by degree  $\pi$  at the point  $A_i$  (a half-turn). Then  $h_{A_i}h_{A_j}$  ( $i \neq j$ ) is a hyperbolic isometry so that the absolute value of its trace is  $|t_k|$  by the construction ( $k \neq i, j$ ). Furthermore,  $tr(h_{A_1}h_{A_2}h_{A_3})^2 = tr[h_{A_1}h_{A_2}, h_{A_3}h_{A_1}] = t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 - 2$  which is at most  $-2$ . Thus the isometry  $h_{A_1}h_{A_2}h_{A_3}$  has a fixed point  $p$  at the circle at the infinity of  $\mathbf{H}$ . By the construction, the three vertices of the triangle  $A_1, A_2$  and  $A_3$  are in the three sides of the ideal hyperbolic triangle  $\Delta$  with vertices  $p, h_{A_3}(p)$  and  $h_{A_2}h_{A_3}(p)$ . The four components of the complement of the ideal quadrilateral  $\Delta \cup h_{A_3}(\Delta)$  give rise to a Schottky condition for the group  $\langle h_{A_1}h_{A_3}, h_{A_3}h_{A_2} \rangle$ . Thus by Poincaré polyhedron theorem, the group  $\langle h_{A_1}h_{A_3}, h_{A_3}h_{A_2} \rangle$  uniformizes either  $\Sigma_{1,1}^0$  or  $\Sigma_{1,0}^1$  so that the geodesics of  $h_{A_1}h_{A_3}$  and  $h_{A_3}h_{A_2}$  are simple closed curves intersecting at one point. Let  $Y$  be the lifting of  $h_{A_1}h_{A_3}$  to  $SL(2, \mathbf{R})$  with  $t_2 tr Y > 0$  and  $X$  be the lifting of  $h_{A_3}h_{A_2}$  to  $SL(2, \mathbf{R})$  with  $t_1 tr X > 0$ . Then  $tr X = t_1$  and  $tr Y = t_2$  and  $tr(XY) = t_3$  due to the spin structure. This finishes the proof.

## §2. Proof of Theorem 2

Given a hyperbolic metric  $m$  on  $\Sigma$  and a monodromy  $\rho \in \tilde{R}(\Sigma)$  of the metric  $m$ , we have  $t_m(x) = |tr(\rho(x))|$  where  $x$  is the homotopy class of a loop.

### 2.1. Proof of theorem 2 for $\Sigma_{1,1}$

To show that condition (1) in part (a) is necessary, take three classes  $\alpha_1, \alpha_2, \alpha_3$  forming an ideal triangle in  $\mathcal{S}$ . Choose  $\gamma_1, \gamma_2 \in \pi_1(\Sigma)$  so that the homotopy classes  $\gamma_1, \gamma_2, \gamma_1\gamma_2$  and  $\gamma_1^{-1}\gamma_2$  represent  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha'_3$  respectively. If  $t_m$  is a trace function corresponding to a monodromy  $\rho \in \tilde{R}(\Sigma_{0,4})$ , then condition (1) follows from the trace identities (6), (7) and lemma 5 where  $A_i = \rho(\gamma_i)$ .

To show that condition (1) is also sufficient, we note that the modular relation implies that the value of  $t$  is determined by  $t$  on  $\{\alpha_1, \alpha_2, \alpha_3\}$  where  $\alpha_i$ 's form an ideal triangle. Now since  $t(b) \geq 2$ , by condition (1),  $t_i = t(\alpha_i)$  satisfies the inequalities in lemma 5. By lemma 5, we construct a hyperbolic metric  $m$  so that  $t_m(\alpha_i) = t_i$ . Thus,  $t = t_m$  on  $\mathcal{S}$  by the modular relation.

The proof of theorem 2 for  $\Sigma_{0,4}$  is in the same spirit, but technically is more complicated.

### 2.2. Necessity of condition (2) in theorem 2

Given three classes  $\alpha_1, \alpha_2, \alpha_3$  forming an ideal triangle in  $\mathcal{S}$ , we take  $a_{ij} \in \alpha_k$ ,  $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$  so that  $|a_{ij} \cap a_{jk}| = 2$ . Without loss of generality, we may assume that  $(a_{ij}, b_i, b_j)$  bounds  $\Sigma_{0,3}$  in  $\Sigma_{0,4}$ . Choose in  $\Sigma_{0,4}$  a set of generators  $\{A_1, A_2, A_3\}$  for  $\pi_1(\Sigma_{0,4})$

as in figure 5 (multiplication of loops in  $\pi_1$  starts from left to right) so that (1) the boundary components  $b_1, b_2, b_3$ , and  $b_4$  of  $\Sigma_{0,4}$  are homotopic to representatives in  $A_1, A_2, A_3$  and  $A_1A_2A_3$  respectively; (2) the curves  $a_{12}, a_{23}$ , and  $a_{31}$  are homotopic to representatives in  $A_1A_2, A_2A_3$ , and  $A_3A_1$  respectively; and (3) the generators are symmetric with respect to a  $\mathbf{Z}_3$  action on  $\Sigma_{0,4}$  preserving  $b_4$  (figure 5(e)).

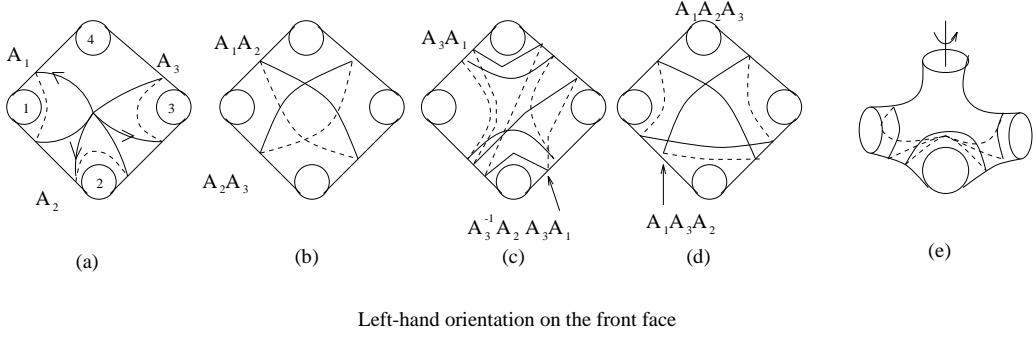


Figure 5

Given  $\rho \in \tilde{R}(\Sigma_{0,4})$  representing the monodromy of a hyperbolic metric  $m$ , we shall identify  $\rho(A)$  with  $A$  for  $A \in \pi_1(\Sigma_{0,4})$  for simplicity in this section. Thus  $A'_i$ 's are  $\text{SL}(2, \mathbf{R})$  matrices. By choosing a different lifting if necessary, we may assume that  $\text{tr}A_i < 0$  ( $i = 1, 2, 3$ ). By lemma 4,  $\text{tr}A_iA_j < 0$  ( $i \neq j$ ), and  $\text{tr}A_1A_2A_3 < 0$ . Then the first equation in condition (2) is given by trace identity (10). To see the second equation (which is the statement that  $f(\alpha_3), f(\alpha'_3)$  are the two roots in the first equation), we shall derive the equivalent equation

$$t_m(\alpha_1\alpha_2) + t_m(\alpha_2\alpha_1) = t_m(\alpha_1)t_m(\alpha_2) - t(b_1)t_m(b_2) - t_m(b_3)t_m(b_4).$$

To see this, we note that  $\alpha_1\alpha_2 = \alpha_3$  and  $\alpha_2\alpha_1$  are represented by  $A_1A_2$  and  $A_3^{-1}A_2A_3A_1$  respectively. Furthermore, by lemma 4,  $\text{tr}(A_3^{-1}A_2A_3A_1) < 0$ . Thus the above formula is a consequence of the trace identity:  $\text{tr}(A_3^{-1}A_2A_3A_1) + \text{tr}(A_1A_2) = \text{tr}(A_1)\text{tr}(A_2) + \text{tr}(A_3)\text{tr}(A_1A_2A_3) - \text{tr}(A_2A_3)\text{tr}(A_3A_1)$ .

We shall write the first equation in condition (2) (i.e., equation (10)) explicitly as follows. Let  $t_i = t_m(b_i)$  and  $t_{ij} = t_m(\alpha_k)$ . Then formulas (8) and (9) become:

$$(11) \quad -t_4 + \text{tr}(A_1A_3A_2) = t_1t_{23} + t_2t_{31} + t_3t_{12} + t_1t_2t_3.$$

$$(12) \quad -t_4\text{tr}(A_1A_3A_2) = t_1^2 + t_2^2 + t_3^2 + t_{12}^2 + t_{23}^2 + t_{31}^2 + t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{31} - t_{12}t_{23}t_{31} - 4.$$

Thus equation (10) becomes

$$(13) \quad t_4^2 + t_4(t_1t_{23} + t_2t_{31} + t_3t_{12} + t_1t_2t_3) + \sum_{i=1}^3 t_i^2 + \sum_{(i,j) \in I} t_{ij}^2 + t_i t_j t_{ij} - 4 - t_{12}t_{23}t_{31} = 0$$

where  $I = \{(1,2), (2,3), (3,1)\}$ . As a quadratic equation in  $-t_4 = \text{tr}(A_1 A_2 A_3)$ , it becomes  $x^2 - Px + Q = 0$  where  $P > 0$  and (thus)  $Q < 0$ . This implies that the equation has two real roots of different signs and  $-t_4$  is the negative root, i.e.,

$$(14) \quad t_4 = (-P + \sqrt{P^2 - 4Q})/2.$$

In particular, the number  $t_4$  is determined by the rest of the six numbers. Since  $t_4 > 2$ , we obtain the (equivalent) condition that  $-Q > 2P + 4$  which is exactly condition (4). Conversely, if  $-Q > 2P + 4$  and  $P > 0$ , then  $t_4 > 2$ .

*Remark 2.1.* We have shown that each hyperbolic metric  $m$  on  $\Sigma_{0,4}$  is determined by its lengths on six curves  $\{a_{ij}, b_1, b_2, b_3\}$ . This was first observed by Schmutz ([Sc], lemma 2).

### 2.3. Sufficiency of condition (2) in theorem 2

We use the same notations as in §2.2. Given a function  $t : \mathcal{S}(\Sigma_{0,4}) \rightarrow \mathbf{R}_{>2}$  satisfying condition (2), we note that the modular relation implies that the values of  $t$  is determined by  $t$  on  $\{\alpha_1, \alpha_2, \alpha_3, b_1, b_2, b_3, b_4\}$  where  $\alpha_i$ 's form an ideal triangle. Thus, it suffices to find  $\rho \in \tilde{R}(\Sigma_{0,4})$  so that  $\text{tr}\rho(A_i) = -t(b_i)$ ,  $\text{tr}\rho(A_i A_j) = -t(\alpha_k)$  and  $\text{tr}\rho(A_1 A_2 A_3) = -t(b_4)$ .

Let  $t_i = t(b_i)$  ( $i = 1, 2, 3, 4$ ) and  $t_{ij} = t(\alpha_k)$ , Then  $t_i, t_{ij} \in \mathbf{R}_{>2}$  and equation (13) holds. By the remark in the last paragraph, this is the same as assuming condition (4) holds for  $t_1, t_2, t_3$  and  $t_{ij}$ . We shall first construct three matrices  $A_i$  ( $i = 1, 2, 3$ ) in  $\text{SL}(2, \mathbf{R})$  so that  $\text{tr}A_i = -t_i$ ,  $\text{tr}(A_i A_j) = -t_{ij}$  and furthermore  $\text{tr}A_1 A_2 A_3 < -2$ . Then we show that  $\text{tr}(A_1 A_2 A_3) = -t_4$  and the corresponding representation  $\rho$  is in  $\tilde{R}(\Sigma_{0,4})$ .

Since conditions (13) and (4) is symmetric in  $t_{12}, t_{23}$  and  $t_{31}$  and the set of generators  $A_1, A_2$ , and  $A_3$  are also symmetric, we may assume without loss of generality that  $t_{23} = \max(t_{12}, t_{23}, t_{31})$ .

To solve  $\text{tr}A_i = -t_i$  and  $\text{tr}(A_i A_j) = -t_{ij}$ , let  $A_1 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ ,  $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $A_3 = \begin{pmatrix} -\lambda d & \lambda^{-1}b \\ \lambda c & -\lambda^{-1}a \end{pmatrix}$  be  $\text{SL}(2, \mathbf{R})$  matrices. We have,

$$A_2 A_3 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix},$$

$$A_1 A_2 = \begin{pmatrix} ax + cy & * \\ * & bz + dw \end{pmatrix},$$

$$A_3 A_1 = \begin{pmatrix} -\lambda dx + \lambda^{-1}bz & * \\ * & \lambda cy - \lambda^{-1}aw \end{pmatrix},$$

$$A_1 A_2 A_3 = \begin{pmatrix} -\lambda x & * \\ * & -\lambda^{-1}w \end{pmatrix}.$$

By the condition  $\text{tr}A_i = -t_i$  and  $\text{tr}(A_i A_j) = -t_{ij}$ , we obtain a system of quadratic and linear equations in  $a, b, c, d, x, y, z, w$  and  $\lambda$  as follows.

$$\begin{aligned}
(E1) \quad & a + d = -t_2. \\
(E2) \quad & \lambda^{-1}a + \lambda d = t_3. \\
(E3) \quad & \lambda + \lambda^{-1} = t_{23}. \\
(E4) \quad & ad - bc = 1. \\
(E5) \quad & x + w = -t_1. \\
(E6) \quad & ax + cy + bz + dw = -t_{12}. \\
(E7) \quad & -\lambda dx + \lambda cy + \lambda^{-1}bz - \lambda^{-1}aw = -t_{31}. \\
(E8) \quad & xw - yz = 1.
\end{aligned}$$

By (E3),  $\lambda$  is a positive real number not equal to 1 and is determined up to reciprocal. Let us fix  $\lambda > 1$ . By (E1) and (E2), we have  $a = -(\lambda t_2 + t_3)/(\lambda - \lambda^{-1})$  and  $d = (\lambda^{-1}t_2 + t_3)/(\lambda - \lambda^{-1})$ . Thus  $ad < 0$  and  $bc = ad - 1 < 0$ . Fix  $c = 1$ . We obtain a set of solutions in  $a, b, c, d$  and  $\lambda$  which are real analytic RC-functions in  $t'_i s$  and  $t'_{ij} s$ . We now claim that there are solutions for  $x, y, z$ , and  $w$  satisfying (E5)-(E8) in the complex number field  $\mathbf{C}$ . Indeed, by (E6) and (E7), we express  $y$  and  $z$  in terms of  $x$  and  $w$  as follows.  $y = (t_3x - \lambda^{-1}t_2w + \lambda^{-1}t_{12} - t_{31})/(c(\lambda - \lambda^{-1}))$  and

$$(E9) \quad z = (t_2\lambda x - t_3w - \lambda t_{12} + t_{31})/(b(\lambda - \lambda^{-1})).$$

Using (E5), we have  $w = -x - t_1$ . Thus,  $y = (\lambda^{-1}t_2 + t_3)x/(c(\lambda - \lambda^{-1})) + \text{const}$  and  $z = (\lambda t_2 + t_3)x/(b(\lambda - \lambda^{-1})) + \text{const}$ . Substitute these new equations and  $w = -x - t_1$  into (E8). We obtain a quadratic equation in  $x$  whose leading coefficient (after a simple calculation) is  $1/(bc) \neq 0$ . Thus there is a solution for  $x$  in  $\mathbf{C}$ . This implies the existence of solutions for  $y, z$  and  $w$  in  $\mathbf{C}$ .

We next claim that  $x, y, z$ , and  $w$  are real numbers, i.e.,  $A_1$  is in  $\text{SL}(2, \mathbf{R})$ . Indeed, the quadratic equation (in  $-t_4$ ) (13)  $x^2 - Px + Q = 0$  has two real roots of different signs. By (13), both  $\text{tr}A_1 A_2 A_3$  and  $\text{tr}A_1 A_3 A_2$  are solutions of the equation. Thus  $\text{tr}A_1 A_2 A_3$  is a real number. But  $\text{tr}A_1 A_2 A_3 = -\lambda x - \lambda^{-1}w$ . This together with equation (E5) shows that both  $x$  and  $w$  are real numbers. Thus  $y$  and  $z$  are real numbers as well.

Now by choosing a different set of solution if necessary, we may assume that  $\text{tr}A_1 A_2 A_3$  is the negative root  $-t_4$  of the equation  $t^2 - Pt + Q = 0$ , i.e.,

$$(E10) \quad \lambda x + \lambda^{-1}w = t_4.$$

Indeed, if  $\text{tr}A_1 A_2 A_3$  is the positive root, we use the new set of solution  $(A_1^{-1}, A_2^{-1}, A_3^{-1})$  to the equations  $\text{tr}X_i = -t_i$  and  $\text{tr}X_i X_j = -t_{ij}$  instead of  $(A_1, A_2, A_3)$  and use the fact that  $\text{tr}A_1^{-1} A_2^{-1} A_3^{-1} = \text{tr}A_1 A_3 A_2$ .

By the proof of above, we see that the solution  $a, b, c, d, x, y, z, w$  and  $\lambda$  are real analytic RC-functions in  $t_i$  and  $t_{ij}$  ( $i, j = 1, 2, 3$ ).

By condition (4), the negative root  $\text{tr}A_1 A_2 A_3$  is less than  $-2$ , i.e.,  $t_4 > 2$ . Thus both representations of  $\pi_1(\Sigma_{0,3})$  (in term of the pair of matrices) given by  $\langle A_1^{-1}, A_1 A_2 A_3 \rangle$  and

$\langle A_2, A_3 \rangle$  are in  $\tilde{R}(\Sigma_{0,3})$  by lemma 4. Furthermore, these two group share a common generator  $A_1^{-1}(A_1 A_2 A_3) = A_2 A_3$ . To apply the Maskit combination theorem [Ma] to amalgamate these two groups, we need to verify that the Nielsen convex cores for the two groups  $\langle A_1^{-1}, A_1 A_2 A_3 \rangle$  and  $\langle A_2, A_3 \rangle$  in  $\mathbf{H}$  lie in the different sides of the axis of  $A_2 A_3$ . The following lemma characterizes the side of the axis which contains the Nielsen core.

**Lemma 6.** Suppose  $X = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}$  and  $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  are  $SL(2, \mathbf{R})$  matrices so that  $trX < -2$ ,  $trY \leq -2$ ,  $trXY \leq -2$ . Then the side of the axis of  $X$  which contains the Nielsen convex core for the discrete group  $\langle X, Y \rangle$  is  $\{(x, y) | x > 0, y > 0\}$  if and only if  $c(\lambda - \lambda^{-1}) > 0$ .

**Proof.** Let  $trX = -t_1$ ,  $trY = -t_2$ ,  $trXY = -t_3$  with  $t_1 > 2$  and  $t_2, t_3 \geq 2$ . Then we have  $\lambda + \lambda^{-1} = t_1$ ,  $a + d = -t_2$ ,  $\lambda a + \lambda^{-1}d = t_3$ , and  $ad - bc = 1$ . We solve for  $a$  and  $d$  and obtain:  $a = (\lambda^{-1}t_2 + t_3)/(\lambda - \lambda^{-1})$  and  $d = -(\lambda t_2 + t_3)/(\lambda - \lambda^{-1})$ . The fixed points  $r_1$  and  $r_2$  of  $Y$  at the circle at the infinite of  $\mathbf{H}$  are the roots of the equation  $ct^2 + (d - a)t - b = 0$ . In particular,  $r_1 + r_2 = -(d - a)/c$  which is  $(t_1 t_2 + 2t_3)/(c(\lambda - \lambda^{-1}))$ . Since the fixed points  $r_1$  and  $r_2$  are in the Nielsen core, the result follows.  $\square$

Now to finish the proof, we verify the side condition by taking  $X = A_2 A_3$ , and  $Y = A_1$  for the group  $\langle A_1^{-1}, A_1 A_2 A_3 \rangle$ , and taking  $X = A_2 A_3$ ,  $Y = A_2^{-1}$  for  $\langle A_2, A_3 \rangle$ . Thus it suffices to show  $-zc < 0$ , or the same  $zb < 0$ .

By (E5) and (E10), we have  $x = (\lambda^{-1}t_1 + t_4)/(\lambda - \lambda^{-1})$  and  $w = -(\lambda t_1 + t_4)/(\lambda - \lambda^{-1})$ . Substitute them into (E9) and simplify it, we have,

$$bz(\lambda - \lambda^{-1})^2 = t_2 \lambda (\lambda^{-1}t_1 + t_4) + t_3 (\lambda t_1 + t_4) - (\lambda - \lambda^{-1}) \lambda t_{12} + (\lambda - \lambda^{-1}) t_{31}.$$

By (E3), we replace  $\lambda^2$  by  $\lambda t_{23} - 1$  and  $\lambda^{-1}$  by  $t_{23} - \lambda$  in the above equation and obtain,

$$bz(\lambda - \lambda^{-1})^2 = \lambda(t_1 t_3 + 2t_{31} + t_2 t_4 - t_{23} t_{12}) + (t_1 t_2 + 2t_{12} + t_3 t_4 - t_{23} t_{31}).$$

We claim that under the condition  $t_{23} = \max(t_{12}, t_{23}, t_{31})$  and equation (13) both  $t_1 t_3 + 2t_{31} + t_2 t_4 - t_{23} t_{12}$  and  $t_1 t_2 + 2t_{12} + t_3 t_4 - t_{23} t_{31}$  are negative. Indeed, since  $t_{23} = \max(t_{12}, t_{23}, t_{31})$ , and  $t_i, t_{ij}$  are at least 2, by equation (13), we have,

$$\begin{aligned} t_{12} t_{23} t_{31} &> t_1 t_3 t_{31} + t_{31}^2 + t_{23}^2 + t_2 t_{31} t_4 \\ &\geq t_1 t_3 t_{31} + 2t_{31}^2 + t_2 t_{31} t_4 \\ &= t_{31}(t_1 t_3 + 2t_{31} + t_2 t_4). \end{aligned}$$

This shows  $t_1 t_3 + 2t_{31} + t_2 t_4 - t_{23} t_{12} < 0$ . The other inequality follows by the same argument since the inequality is obtained from the previous one by interchanging the indices 2 and 3.  $\square$

The proof shows that all the entries of the matrices  $A_1, A_2, A_3$  are RC functions in  $t_i, t_{ij}$  where  $i = 1, 2, 3$ ,  $(i, j) = (1, 2), (2, 3), (3, 1)$ .

**Corollary 2.1.** For surface  $\Sigma_{0,4}$  with  $\partial\Sigma_{0,4} = b_1 \cup b_2 \cup b_3 \cup b_4$ , let  $F = \{[a_{12}], [a_{23}], [a_{31}], b_1, b_2, b_3\}$  so that  $[a_{ij}]$  forms an ideal triangle and  $(a_{ij}, b_i, b_j)$  bounds a  $\Sigma_{0,3}$ . Then the map  $\pi_F: T(\Sigma_{0,4}) \rightarrow \mathbf{R}^6$  is an embedding so that its image is given by  $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}) \in \mathbf{R}_{>2}^6 |$

formula (4) holds}. Furthermore, there exists a continuous function  $f: T(\Sigma_{0,4}) \rightarrow \tilde{R}(\Sigma_{0,4})$  sending  $m \in T(\Sigma_{0,4})$  a representation  $f(m)$  which is a lifting of a monodromy of  $m$  so that the entries of the matrices  $f(m)(\gamma)$  are real analytic RC-functions of the coordinates of  $\pi_F(m)$ , for each  $\gamma \in \mathcal{S}(\Sigma_{0,4})$ .

*Remark 2.2.* The above proof works for hyperbolic metrics with cusp ends as well since lemmas 3, 6 and Maskit combination theorem still hold. In particular, we obtain the following parametrization of the Teichmüller space of  $T_{0,0}^4$  by the geodesic lengths  $t_{12}, t_{23}$  and  $t_{31}$  (other variables  $t_1, t_2, t_3$  and  $t_4$  are 2). Take  $F = \{[a_{12}], [a_{23}], [a_{31}]\}$ . Then the image of the embedding  $\pi_F$  of  $T_{0,0}^4$  is  $\{(t_{12}, t_{23}, t_{31}) \in \mathbf{R}_{>2}^3 \mid t_{12}t_{23}t_{31} = t_{12}^2 + t_{23}^2 + t_{31}^2 + 8t_{12} + 8t_{23} + 8t_{31} + 28\}$ .

### §3. A Combinatorial Structure on the Set of Isotopy Classes of Simple Closed Curves

We introduce the following notation for convenience. If  $\alpha \perp_0 \beta$  (resp.  $\alpha \perp \beta$ ), then  $\partial N(\alpha \cup \beta)$  denotes the union of the isotopy classes of four boundary components of  $N(a \cup b)$  where  $a \in \alpha, b \in \beta$  with  $|a \cap b| = I(a, b)$  (resp.  $N(\alpha \cup \beta) = [\partial N(\alpha \cup \beta)]$ ).

The goal of this section is to prove the following proposition.

**Proposition 1.** (a) Given a set of disjoint simple closed curves and proper arcs  $\{c_1, \dots, c_n\}$  in a compact oriented surface  $\Sigma$ , let  $G_0 = \{\alpha \in \mathcal{S}(\Sigma) \mid I(\alpha, [c_i]) \leq 2$  so that for each index  $i$ , if equality holds then the two points of intersection have different signs}. Then  $\mathcal{S}(\Sigma) = \cup_{i=0}^{\infty} G_i$  where  $G_{i+1} = G_i \cup \{\alpha \mid \alpha = \beta\gamma \text{ where either (1) } \beta \perp \gamma, \text{ and } \beta, \gamma, \gamma\beta \text{ are in } G_i, \text{ or (2) } \beta \perp_0 \gamma, \text{ and } \beta, \gamma, \gamma\beta, \text{ and each component of } \partial N(\beta \cup \gamma) \text{ are in } G_i\}$ .

(b) Under the same assumption as in (a), if  $f$  is a function defined on  $\mathcal{S}(\Sigma)$  so that (1)  $f(\alpha\beta)$  is determined by  $f(\alpha)$ ,  $f(\beta)$ , and  $f(\beta\alpha)$  whenever  $\alpha \perp \beta$ , and (2)  $f(\alpha\beta)$  is determined by  $f(\alpha)$ ,  $f(\beta)$ ,  $f(\beta\alpha)$ , and  $f(\gamma_i)$  ( $i = 1, 2, 3, 4$ ) whenever  $\alpha \perp_0 \beta$  with  $\partial N(\alpha \cup \beta) = \cup_{i=1}^4 \gamma_i$ , then  $f$  is determined by  $f|_{G_0}$ .

Part (b) of the proposition follows from part (a). The proof of part (a) of the proposition is a simple application of the lemma below by induction on the number  $\max\{I(\alpha, [c_i]) \mid i = 1, \dots, n\}$  for  $\alpha \in \mathcal{S}(\Sigma)$ . This lemma is motivated by lemma 2 in [Li].

**Lemma 7.** Suppose  $a$  is a simple closed curve and  $b$  is either a simple closed curve or an arc so that either  $I(a, b) = |a \cap b| \geq 3$  or  $a$  intersects  $b$  at two points of the same intersection signs. Let  $\{c_1, \dots, c_n\}$  be a collection of disjoint simple closed curves or arcs so that  $\text{int}(b) \cap c_i = \emptyset$  for all  $i = 1, \dots, n$ . Then there exist two simple closed curves  $p_1$  and  $p_2$  in  $N(a \cup b)$  so that

(1)  $a = p_1 p_2$  where either  $p_1 \perp p_2$  or  $p_1 \perp_0 p_2$ ,

(2)  $|p_i \cap b| < |a \cap b|$ ,  $|p_2 p_1 \cap b| < |a \cap b|$ ,  $|p_i \cap c_j| \leq |a \cap c_j|$  and  $|p_2 p_1 \cap c_j| \leq |a \cap c_j|$  for  $i = 1, 2$  and  $j = 1, 2, \dots, n$ , and,

(3) if  $p_1 \perp_0 p_2$ , there are four simple closed curves  $d_1, d_2, d_3$ , and  $d_4$  isotopic to four boundary components of  $N(p_1 \cup p_2)$  so that  $|d_i \cap b| < |a \cap b|$  and  $|d_i \cap c_j| \leq |a \cap c_j|$  for  $i = 1, 2, 3, 4$ , and  $j = 1, \dots, n$ .

**Proof.** We need to consider two cases.

*Case 1.* There exist two adjacent intersection points  $x$  and  $y$  in  $b$  which have the same intersection signs (see figure 6). Let  $c$  be an arc in  $b$  joining  $x$  and  $y$  so that  $\text{int}(c) \cap a = \emptyset$ . Then the curves  $p_1$  and  $p_2$  as shown in figure 6 (with the right-hand orientation on the surface) satisfy  $p_1 \perp p_2$  and all conditions in the lemma.

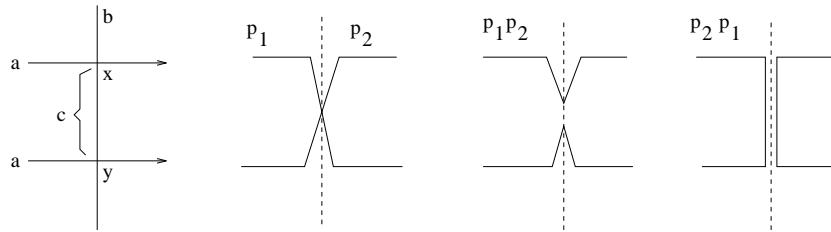


Figure 6

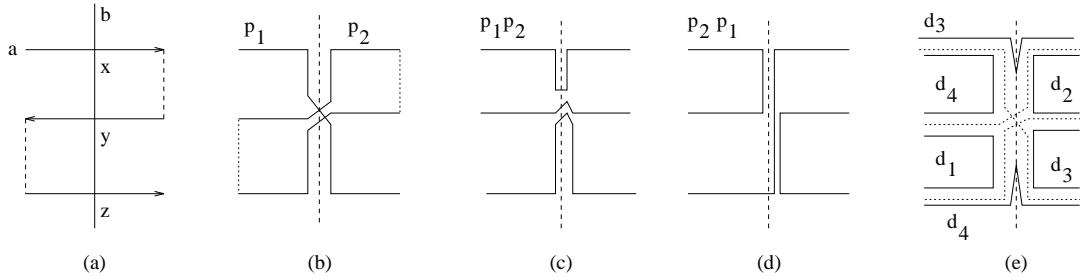


Figure 7

*Case 2.* Suppose any pair of adjacent intersection points in  $b$  has different intersection signs. Then  $|a \cap b| \geq 3$ . Take three intersection points  $x_1, x_2, x_3$  in  $b$  so that  $x, y$  and  $y, z$  are adjacent. Their intersection signs alternate. Fix an orientation on  $a$  so that the arc from  $x$  to  $y$  in  $a$  does not contain  $z$  as shown in figure 7(a). If the surface  $\Sigma$  is right hand oriented as in figure 7(a), take  $p_1$  and  $p_2$  as in figure 7(b). Then  $p_1 \perp_0 p_2$  in  $N(p_1 \cup p_2)$ . We claim that  $p_1 \perp_0 p_2$  in  $\Sigma$ . To see this, it suffices to show that  $N(p_1 \cup p_2)$  is incompressible in  $\Sigma$ . Indeed, each boundary components of  $N(p_1 \cup p_2)$  is isotopic to a simple loop  $b_i$  made by the arcs with ends  $x_i, x_{i+1}$  along  $a$  and  $d$ . Since  $|a \cap d| = I(a, d)$ , these loops  $b_i$  are essential and non-boundary parallel. Thus the claim follows. By the construction conditions (1), (2) and (3) follow from figure 7(c), (d) and (e). If  $\Sigma$  is left-hand oriented, we simply interchange  $p_1$  and  $p_2$ .  $\square$

As an application of the proposition, we show that the mapping class group is finitely generated by Dehn twists. Take  $f$  in the proposition to be the map sending  $\alpha \in \mathcal{S}(\Sigma)$  to the isotopy class of positive Dehn twist along  $\alpha$ . First of all, there are two basic relations on the Dehn twists: (1)

(braid relation)  $D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1}$  for  $\alpha \perp \beta$  and (2) (lantern relations)  $D_\alpha D_\beta D_{\alpha\beta} = D_{\partial N(\alpha \cup \beta)}$  for  $\alpha \perp_0 \beta$ . Thus by the proposition, the mapping class group is generated by elements in  $G_0$ . For all surfaces, it is easy to construct a finite set  $G_0$  so that  $G_\infty = \mathcal{S}$ . For instance, if surface  $\Sigma_{g,r}$  has  $r > 0$ , let  $\{c_1, \dots, c_n\}$  ( $n = 6g + 3r - 6$ ) be an ideal triangulation of it, i.e., a maximal collection of disjoint pairwise non-isotopic, essential arcs in  $\Sigma_{g,r}$ . Then the corresponding collection  $G_0$  in the corollary is a finite set, indeed  $|G_0| \leq 3^n$  since each  $\alpha \in CS(\Sigma)$  is determined by the n-tuple  $(I(\alpha, [c_1]), \dots, I(\alpha, [c_n]))$ .

*Remark.* The lantern relation was discovered and used by M. Dehn ([De], p333) and rediscovered independently by Johnson [Jo3]. Also the braid relation (1) implies the Artin's relation  $D_\alpha D_\beta D_\alpha = D_\alpha D_\beta D_\alpha$ .

#### §4. Thurston's Embedding of the Teichmüller Space

We prove theorem 1 for compact surface  $\Sigma_{g,r}$  in sections §4.1-4.3. In §4.4, we indicate the modification needed for non-compact surfaces. By the proof of theorem 2, it suffices to show that conditions (1) and (2) are sufficient.

##### 4.1. Reduction to the surfaces $\Sigma_{0,5}$ and $\Sigma_{1,2}$

We shall prove theorem 1 by induction on the norm  $|\Sigma_{g,r}| = 3g + r$  of a compact surface. The goal of this section is to show that theorem 1 for all surfaces follows from theorem 1 for  $\Sigma_{0,5}$  and  $\Sigma_{1,2}$ .

Given  $\Sigma = \Sigma_{g,r}$  with  $|\Sigma| \geq 5$ , and a function  $f : \mathcal{S}(\Sigma) \rightarrow \mathbf{R}$  which is a trace function on each incompressible subsurface  $\Sigma'$  of norm 4, we decompose  $\Sigma = X \cup Y$  so that  $X, Y$  are incompressible of smaller norms with  $\text{int}(X \cap Y) \cong \text{int}(\Sigma_{0,3})$  as figure 3(d). To be more precise, we take  $X = \Sigma_{0,r-1}$ ,  $Y = \Sigma_{0,4}$  if  $g = 0$  and take  $X = \Sigma_{g-1,r+2}$ ,  $Y = \Sigma_{1,1}$  if  $g \geq 1$ . Consider the restrictions  $f|_{\mathcal{S}(X)}$  and  $f|_{\mathcal{S}(Y)}$ . By the induction hypothesis we find hyperbolic metrics  $m_X$  and  $m_Y$  on  $X$  and  $Y$  respectively realizing the restrictions as the trace functions. By the gluing lemma, we construct a hyperbolic metric  $m$  on  $\Sigma$  whose restriction to  $X$  and  $Y$  are isotopic to  $m_X$  and  $m_Y$ . Thus the trace function  $t_m$  and  $f$  have the same values on  $\mathcal{S}(X) \cup \mathcal{S}(Y)$ .

The goal is to show that the above condition  $f|_{\mathcal{S}(X) \cup \mathcal{S}(Y)} = t_m|_{\mathcal{S}(X) \cup \mathcal{S}(Y)}$  implies  $f = t_m$ . To achieve this, let us rewrite the conditions (1), (2) satisfied by  $f$  and  $t_m$  as follows:

$$(1') \quad f^2(\alpha) + f^2(\beta) + f^2(\alpha\beta) - f(\alpha)f(\beta)f(\alpha\beta) - 2 + f(\partial N(\alpha \cup \beta)) = 0, \quad \text{if } \alpha \perp \beta,$$

$$(2') \quad f^2(\alpha) + f^2(\beta) + f^2(\alpha\beta) - f(\alpha)f(\beta)f(\alpha\beta) + f(\alpha)(f(\gamma_1)f(\gamma_2) + f(\gamma_3)f(\gamma_4)) + f(\beta)(f(\gamma_2)f(\gamma_3) + f(\gamma_1)f(\gamma_4)) + f(\alpha\beta)(f(\gamma_2)f(\gamma_4) + f(\gamma_1)f(\gamma_3)) + f^2(\gamma_1) + f^2(\gamma_2) + f^2(\gamma_3) + f^2(\gamma_4) + f(\gamma_1)f(\gamma_2)f(\gamma_3)f(\gamma_4) - 4 = 0, \quad \text{if } \alpha \perp_0 \beta,$$

$$(3') \quad f(\alpha\beta) + f(\beta\alpha) = f(\alpha)f(\beta), \quad \text{if } \alpha \perp \beta, \text{ and}$$

$$(4') \quad f(\alpha\beta) + f(\beta\alpha) = f(\alpha)f(\beta) - f(\gamma_1)f(\gamma_3) - f(\gamma_2)f(\gamma_4), \quad \text{if } \alpha \perp_0 \beta,$$

where  $\gamma_i$ 's are the four components of  $\partial N(\alpha \cup \beta)$  so that  $\alpha$  separates  $\{\gamma_1, \gamma_2\}$  from  $\{\gamma_3, \gamma_4\}$  and  $\beta$  separates  $\{\gamma_2, \gamma_3\}$  and  $\{\gamma_1, \gamma_4\}$ .

Note that relations (3') and (4') give rise to an iteration process. Namely, the value  $f(\alpha\beta)$  is determined by the values of  $f$  at  $\alpha$ ,  $\beta$ , and  $\beta\alpha$  if  $\alpha \perp \beta$ , and is determined by the values of  $f$  at  $\alpha$ ,  $\beta$ ,  $\beta\alpha$  and the four components of  $\partial N(\alpha \cup \beta)$  if  $\alpha \perp_0 \beta$ .

Let  $a_1, a_2$  be the simple loops in  $\partial(X \cap Y)$  which is non boundary parallel in  $\Sigma$  as in figure 3(d). Applying proposition 1 to  $f$  and to  $t_m$  with respect to the set  $\{a_2\}$ , we conclude that  $f = t_m$  follows from  $f(\alpha) = t_m(\alpha)$  where  $\alpha \perp_0 [a_2]$ . Assume that theorem 1 holds for  $\Sigma_{0,5}, \Sigma_{1,2}$ . We show  $f(\alpha) = t_m(\alpha)$  with  $\alpha \perp_0 [a_2]$  as follows. Take  $s \in \alpha$  so that  $|s \cap a_2| = 2$ . Then  $Z = Y \cup N(s)$  is an incompressible subsurface homeomorphic either to  $\Sigma_{1,2}$  or  $\Sigma_{0,5}$ . Let  $X' = X \cap Z, Y' = Y \cap Z$ . Then  $Z = X' \cup Y'$  so that  $X' \cap Y' = X \cap Y$ . Consider  $f|_{\mathcal{S}(Z)}$  and  $t_m|_{\mathcal{S}(Z)}$ . By theorem 1 for  $Z$  and the fact that  $f$  and  $t_m$  coincide on the subset  $\mathcal{S}(X') \cup \mathcal{S}(Y')$ , we conclude that  $f = t_m$  on  $\mathcal{S}(Z)$  by the gluing lemma. In particular,  $f(\alpha) = t_m(\alpha)$ .

It remains to show theorem 1 for  $\Sigma_{0,5}$  and  $\Sigma_{0,5}$ . By the same decomposition  $\Sigma = X \cup Y$  as above, it suffices to show the following two lemmas.

For simplicity, we let  $Im(\Sigma)$  be the set of all functions from  $\mathcal{S}(\Sigma)$  to  $\mathbf{R}_{>2}$  satisfying conditions (1'), (2'), (3'), and (4'). Two classes  $\alpha$  and  $\beta$  are *disjoint* if they are distinct and have disjoint representatives.

**Lemma 8.** *Suppose  $\alpha_1$  and  $\alpha_2$  are two disjoint elements in  $\mathcal{S}'(\Sigma_{0,5})$ . If two elements  $f$  and  $g$  in  $Im(\Sigma_{0,5})$  satisfy  $f(\alpha) = g(\alpha)$  for all  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  with  $I(\alpha, \alpha_1)I(\alpha, \alpha_2) = 0$ , then  $f = g$ .*

**Lemma 9.** *Suppose  $\alpha_1$  and  $\alpha_2$  are two disjoint elements in  $\mathcal{S}'(\Sigma_{1,2})$  so that  $\alpha_1$  is non-separating and  $\alpha_2$  is separating. If  $f$  and  $g$  are two elements in  $Im(\Sigma_{1,2})$  so that  $f(\alpha) = g(\alpha)$  for all  $\alpha \in \mathcal{S}(\Sigma_{1,2})$  with  $I(\alpha, \alpha_1)I(\alpha, \alpha_2) = 0$ , then  $f = g$ .*

#### 4.2. Proof of lemma 8

To prove lemma 8, by proposition 1, it suffices to show that  $f(\alpha) = g(\alpha)$  for  $\alpha \perp_0 \alpha_i$  for  $i = 1, 2$ . Let  $a_i \in \alpha_i$  be a representative so that  $|a_1 \cap a_2| = 0$  and let  $x \in \alpha$  so that  $x \perp_0 a_i$  for  $i = 1, 2$ . Note that if  $x' \perp_0 a_i$  for  $i = 1, 2$ , there is an orientation preserving homeomorphism  $h$  of  $\Sigma_{0,5}$  sending  $x$  to  $x'$  and preserving each  $a_i$  (since both  $N(a_1 \cup a_2 \cup x)$  and  $N(a_1 \cup a_2 \cup x')$  are strong deformation retractors for  $\Sigma_{0,5}$ ). Thus we may draw  $x$  as in figure 8(a). Let  $a, b, c, d, e$  and  $b_1, b_2, c_1, c_2, d_1$ , and  $d_2$  be curves as in figures 8(a), (b) and (c) so that each of them is either disjoint from  $a_1$  or from  $a_2$ .

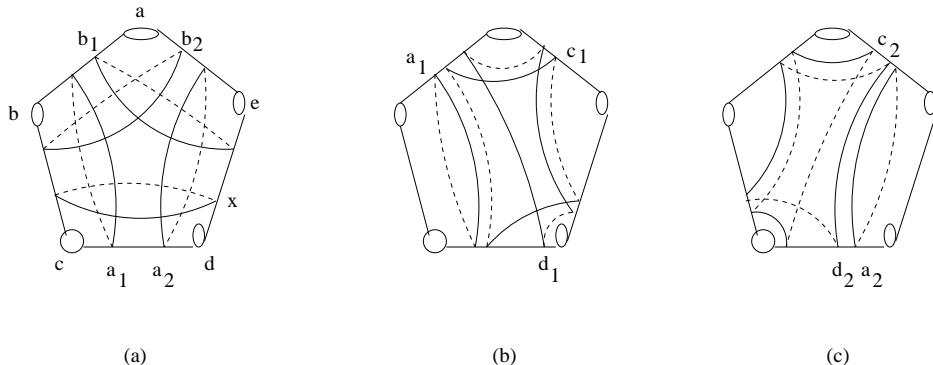


Figure 8

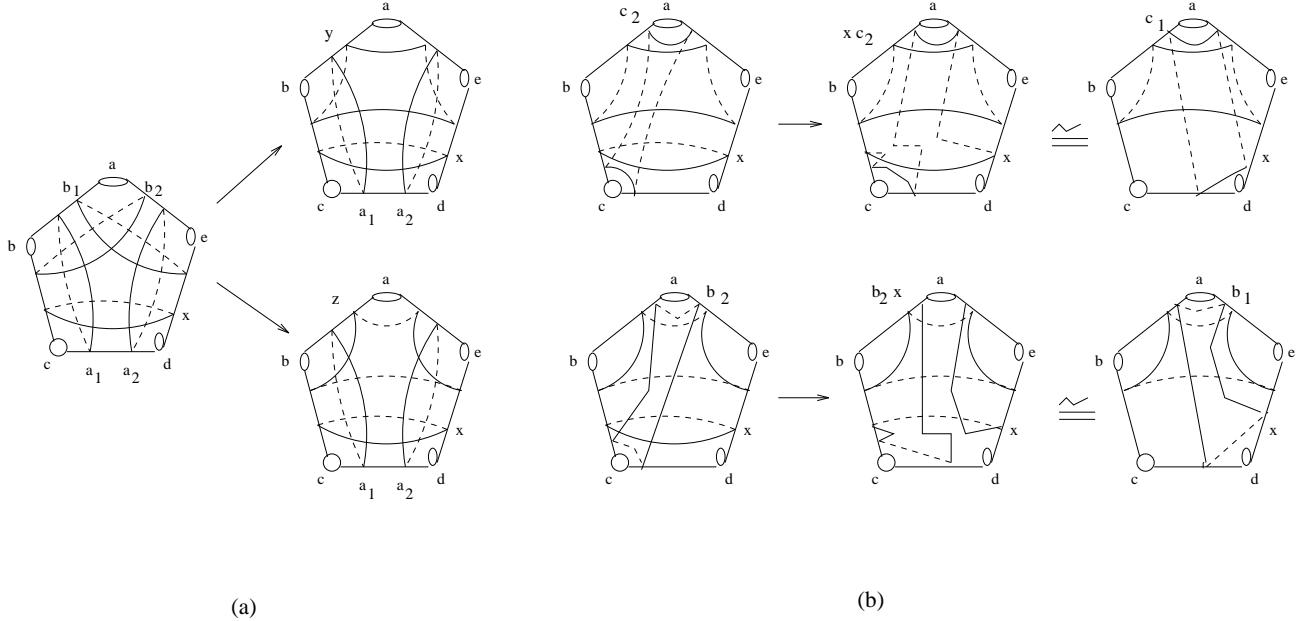


Figure 9

**Claim.** There is a rational function  $R$  so that for each  $h \in \text{Im}(\Sigma_{0,5})$ ,  $h(x) = R(h(a), h(b), \dots, h(e), h(a_1), h(a_2), h(b_1), h(b_2), h(c_1), h(c_2), h(d_1), h(d_2))$ .

It follows from the claim that  $f(\alpha) = g(\alpha)$ . This finishes the proof of lemma 8.

Before begin the proof of the claim, let us simplify the notations by making the following conventions.

(C1) The value of  $h$  at a curve  $s$  will be denoted by  $s$ .

(C2) The multiplication of two curves  $s_1$  and  $s_2$  will be denoted by  $s_1 \circ s_2$ .

(C3) Surfaces drawn in the figures have the right-hand orientation in the front face.

Let  $y = b_1 \circ b_2$  and  $z = b_2 \circ b_1$  as in figure 9(a).

Since  $b_1 \perp_0 b_2$  and  $\partial N(b_1 \cup b_2) \cong a \cup b \cup e \cup x$ , applying relation (2') in  $N(b_1 \cup b_2)$  with respect to  $b_1, b_2$  and  $y$ , we obtain:  $x^2 + a^2 + b^2 + e^2 + y^2 + b_1^2 + b_2^2 - b_1 b_2 y + abex + b_1(ae + bx) + b_2(ex + ab) + y(ax + be) - 4 = 0$ . This can be written as:

$$(15) \quad x^2 + y^2 + axy + p_1x - p_2y + p_3 = 0,$$

where  $p_j$  are some polynomials in  $a, b, c, d, e, a_i, b_i, c_i$ , and  $d_i$  (the same notations apply below) and  $p_j > 0$  for  $j = 1, 2, 3$ .

Similarly, we have,

$$(16) \quad x^2 + z^2 + axz + p_1x - p_2z + p_3 = 0.$$

Furthermore, by (4'),  $y + z = b_1 b_2 - ax - be$ , i.e.,

$$(17) \quad ax + y + z = p_4.$$

Now  $c_2 \perp_0 x$  and  $x \circ c_2 = c_1$  (see figure 9(b)). Applying the relation (2') to  $N(c_2 \cup x)$  with respect to  $c_2, x, c_1$  and using  $\partial N(c_2 \cup x) \cong a \cup c \cup d \cup y$ , we obtain  $y^2 + a^2 + c^2 + d^2 + x^2 + c_1^2 + c_2^2 - c_1 c_2 x + acdy + x(ay + cd) + c_2(ac + dy) + c_1(ad + cy) - 4 = 0$ , i.e.,

$$(18) \quad x^2 + y^2 + axy - p_5x + p_6y + p_7 = 0,$$

where  $p_5, p_6$  and  $p_7$  are positive.

Similarly, using  $d_2 \perp_0 x$  and  $d_1 \circ x = d_2$ , we obtain a relation:

$$(19) \quad x^2 + z^2 + axz - p_8x + p_9z + p_{10} = 0,$$

where  $p_8, p_9$  and  $p_{10}$  are positive.

Consider the difference of (15) and (18). We obtain,

$$(20) \quad p_{11}x - p_{12}y = p_{13},$$

where  $p_{11}$  and  $p_{12}$  are positive.

Consider (16)-(19), we obtain,

$$(21) \quad p_{14}x - p_{15}z = p_{16},$$

where  $p_{14}$  and  $p_{15}$  are positive.

Now the system of linear equations (17), (20) and (21) in variables  $x, y$  and  $z$  has a unique solution since its determinant is positive. This ends to the proof of the claim and thus finishes the proof of lemma 8.

#### 4.3. Proof of lemma 9

To prove lemma 9, by proposition 1, it suffices to show that  $f(\alpha) = g(\alpha)$  for  $\alpha \in \mathcal{S}(\Sigma_{1,2})$  with  $\alpha \perp_0 \alpha_2$  and  $\alpha \perp \alpha_1$  since there is no element  $\beta \in \mathcal{S}(\Sigma_{1,2})$  so that  $\beta \perp_0 \alpha_i$  for  $i = 1, 2$ . Fix such an  $\alpha$  for the rest of the proof. Take  $x \in \alpha, a_i \in \alpha_i, i = 1, 2$  so that  $a_1 \cap a_2 = \emptyset, x \perp a_1$  and  $x \perp_0 a_2$ .

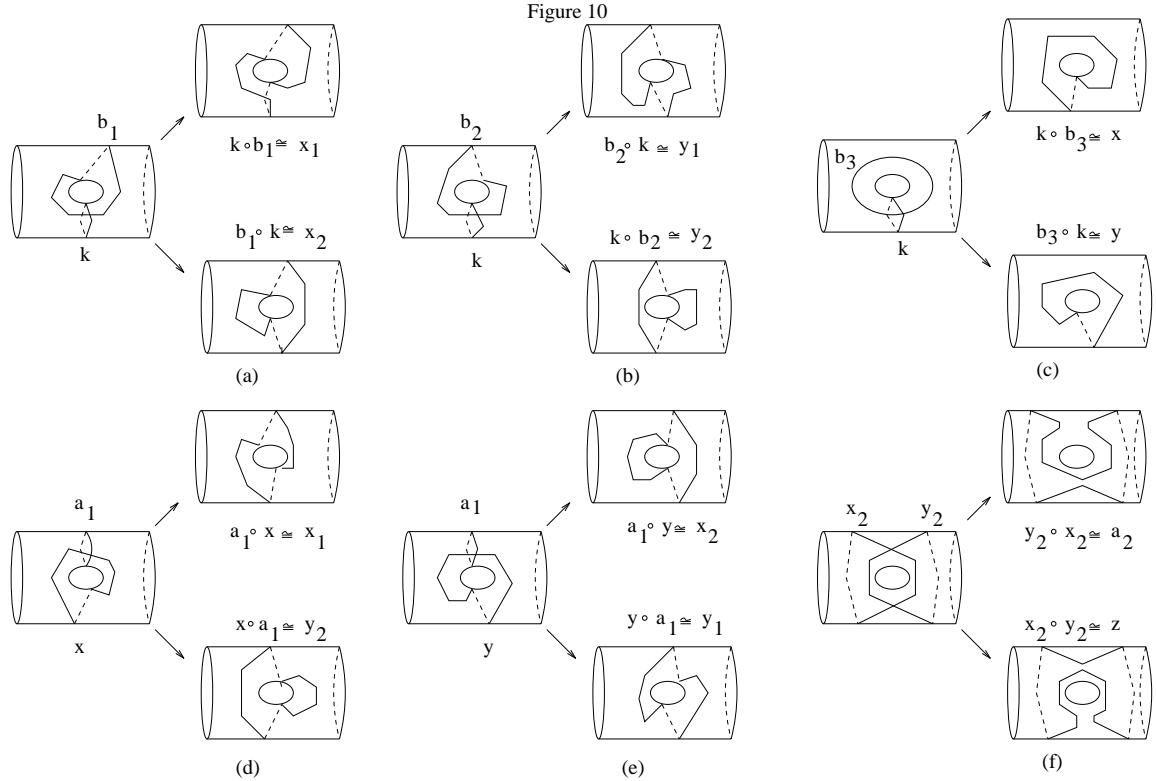
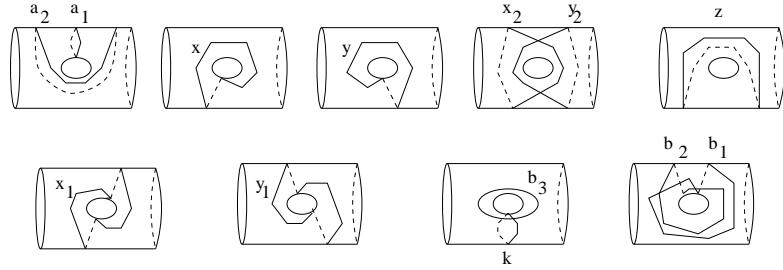
Let  $Y = \Sigma_{1,2} - a_1$  and  $X$  is the subsurface bounded by  $a_2$  containing  $a_1$ . We have  $f = g$  on the subset  $\mathcal{S}(X) \cup \mathcal{S}(Y)$ .

**Claim.** *There exists a finit set of elements  $\{\beta_1, \dots, \beta_n\}$  in  $\mathcal{S}(X) \cup \mathcal{S}(Y)$  and a function  $F$  so that for any element  $h$  in  $Im(\Sigma_{1,2})$ ,  $h(\alpha) = F(h(\beta_1), \dots, h(\beta_n))$ .*

It follows from the claim that  $f(\alpha) = g(\alpha)$ . This finishes the proof of lemma 9.

We shall adopt the same convention as in §4.2 by identifying  $h(s)$  with the simple closed curve  $s$  for the rest of the proof.

**Proof of the claim.** Since any other simple closed curve  $x'$  with  $x' \perp a_1$  and  $x' \perp_0 a_2$  is an image of  $x$  under an orientation preserving self-homeomorphism preserving  $a_1$  and  $a_2$ , we may draw  $x$  as in figure 10. Introduce a few more curves  $y, z, x_1, y_1, x_2, y_2, b_1, b_2, b_3, k$  as in figure 10. Note that the curves  $b_1, b_2, b_3$ , and  $k$  are either in  $X$  or in  $Y$ .



There are many relations among these curves as shown in figure 11.

We obtain a system of equations in  $x, y, x_1, y_1, x_2, y_2$  and  $z$  by applying formulas (1'), (2'), (3') and (4').

By figure 11(a), we have,

$$(e1) \quad x_1 + x_2 = kb_1.$$

By figure 11(b), we have,

$$(e2) \quad y_1 + y_2 = kb_2.$$

By figure 11(c), we have,

$$(e3) \quad x + y = kb_3.$$

By figure 11(d) and that  $\partial N(x \cup a_1) \subset Y$ , we have,

$$(e4) \quad x_1 + y_2 = a_1 x,$$

and,

$$(e5) \quad y_2^2 + x^2 - a_1 x y_2 = p_1.$$

By figure 11(e) and that  $\partial N(y \cup a_1) \subset Y$ , we have,

$$(e6) \quad y_1^2 + y^2 - a_1 y y_1 = p_2.$$

By figure 11(c) that  $x = k \circ b_3$  and  $\partial N(b_3 \cup k) \cong z$ , we have,

$$(e7) \quad x^2 - kb_3 x = -z + p_3.$$

By figure 11(f) and  $\partial N(x_2 \cup y_2) \cong a \cup b \cup b_3 \cup b'_3$  where  $b'_3$  is a parallel copy of  $b_3$  and  $\partial \Sigma_{1,2} = a \cup b$ , we have,

$$(e8) \quad z = x_2 y_2 + p_4.$$

Here and below,  $p'_i$ s denote some polynomials in some elements in  $\mathcal{S}(X) \cup \mathcal{S}(Y)$ .

Also from  $a_1 \perp b_3$  with  $b_3 \circ a_1 = b_2$  and  $a_1 \circ b_3 = b_1$ , we have,

$$(e9) \quad b_1 + b_2 = a_1 b_3.$$

The goal is to show that the system of equations (e1)-(e8) has a unique solution in  $x$ . Assuming this, we conclude that the claim holds.

To this end, we shall first eliminate  $x_1$ ,  $y_1$ ,  $y$  and  $z$  from the above system and show that  $x_2$  are  $y_2$  are linear functions in  $x$ .

Subtracting (e1) by (e4) gives:

$$(e10) \quad x_2 = y_2 - a_1 x + kb_1,$$

and subtracting (e7) by (e8) gives:

$$(e11) \quad x_2 y_2 + x^2 - kb_3 x = p_5.$$

By (e3),  $y = kb_3 - x$  and by (e2),  $y_1 = kb_2 - y_2$ . Substitute them into (e6) and subtract the result by (e5), we obtain:

$$(e12) \quad (a_1 b_3 - 2b_2)ky_2 + (a_1 b_2 - 2b_3)kx = p_6.$$

Note that the coefficients of  $y_2$  and  $x$  in (e12) cannot be both zero since  $a_1 > 2$ . If  $a_1 b_3 - 2b_2 = 0$ , then  $x$  is determined uniquely. Suppose otherwise, then we solve  $y_2$  in terms of  $x$  and obtain,

$$(e13) \quad y_2 = p_7 x + p_8.$$

where  $p_7 = (2b_3 - a_1 b_2)/(a_1 b_3 - 2b_2)$ . From (e10), we obtain

$$(e14) \quad x_2 = (p_7 - a_1)x + p_8 + k b_1.$$

Now substitute (e13) into (e5), we obtain a quadratic equation in  $x$  as follows:

$$(e15) \quad (p_7^2 - a_1 p_7 + 1)x^2 + (2p_7 p_8 - a_1 p_8)x + p_9 = 0.$$

Substitute (e13) and (e14) into (e11) to obtain a quadratic equation in  $x$  as follows.

$$(e16) \quad (p_7^2 - a_1 p_7 + 1)x^2 + (-b_3 k + p_8(p_7 - a_1) + p_7 p_8 + k b_1 p_7)x + p_{10} = 0.$$

Subtract (e16) by (e15) to obtain a linear equation in  $x$  whose leading term is  $-kb_3 + kb_1 p_7$ . Replace  $p_7$  by  $(2b_3 - a_1 b_2)/(a_1 b_3 - 2b_2)$  and use (e9) that  $b_1 = a_1 b_3 - b_2$ , we simplify the leading coefficient to  $a_1 k(b_2^2 + b_3^2 - a_1 b_2 b_3)/(a_1 b_3 - 2b_2)$ . The number  $b_2^2 + b_3^2 - a_1 b_2 b_3$  is negative by relation (1') that  $a_1^2 + b_2^2 + b_3^2 < a_1 b_2 b_3$ . Thus we obtain a unique solution of  $x$ . This finishes the proof of lemma 9.  $\square$

#### 4.4. Proof of Theorem 1 for metrics with cups ends

We first recall theorem 2 for metrics with cups ends. Let  $\Sigma = \Sigma_{0,r}^s$  with  $r + s = 4, s < 4$ , be given with three simple closed curves  $a_{12}, a_{23}$ , and  $a_{31}$  on it satisfying  $a_{31} = a_{12}a_{23}$  and  $a_{12} \perp_0 a_{23}$ . Let  $b_i$  be four essential simple closed curves in  $\text{int}(\Sigma_{0,r}^s)$  which are homotopic into the four ends so that  $a_{ij}$ ,  $b_i$  and  $b_j$  bound a 3-holed sphere in the surface ( $i \neq j, i, j \leq 3$ ). Assume the cusp ends correspond to  $b_i$  ( $i = 1, 2, \dots, s$ ). Take the collection  $F \subset \mathcal{S}(\Sigma)$  to be the isotopy classes of  $a_{ij}$  and  $b'_i$ 's where  $i \neq j$  and  $i, j \leq 3$ . Then the same argument used in the proof of theorem 2 shows,

**Lemma 10.** *The map  $\pi_F : T(\Sigma_{0,r}^s) \rightarrow \mathbf{R}_{\geq 2}^6$  is an embedding whose image is given by  $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}) \in \mathbf{R}_{\geq 2}^6 \mid t_1 = \dots = t_s = 2, t_{s+1} > 2, \dots, t_3 > 2\}$ , so that the formula (4) holds}. Furthermore, there exists a real analytic RC-section for  $T(\Sigma_{0,r}^s)$ .*

Now to construct metrics on  $\Sigma_{g,r}^s$  with  $s > 0$ , we use the decomposition  $\Sigma_{g,r}^s = X \cup Y$  as in figure 12. The first case (1) is given by  $r > 0$ . We need to consider subcases (1.1), (1.2) and (1.3) where (1.1) corresponds to  $g > 0$ , (1.2) corresponds to  $g = 0$  and  $r + s > 5$ , and (1.3) corresponds to  $g = 0$  and  $r + s \leq 5$ . In cases (1.1), or (1.2), we choose  $X \cong \Sigma_{g,r}^{s-1}$ ,  $Y \cong \Sigma_{0,3}^1$ , and  $X \cap Y \cong \Sigma_{0,3}$ . In case (1.3) with  $r + s \leq 4$ , then it follows from theorem 2. In case (1.3) and  $r + s = 5$ , we choose  $X \cong \Sigma_{0,r-1}^{s-1}$ ,  $Y \cong \Sigma_{0,u}^v$ , where  $u + v = 4$ ,  $2 \geq v \geq 1$ , and  $X \cap Y \cong \Sigma_{0,4-v}^{v-1}$ . In the second case (2)  $r = 0$ , we need to consider subcases (2.1)  $s \geq 2$  and (2.2)  $s = 1$ . In case (2.1) that  $s \geq 2$ , if (2.1.1)  $g > 0$ , or (2.1.2)  $g = 0$  and  $s > 5$ , then  $X \cong \Sigma_{g,1}^{s-2}$ ,  $Y \cong \Sigma_{0,2}^2$ , and  $X \cap Y \cong \Sigma_{0,3}$ . If (2.1.3)  $5 \geq s \geq 2$  and  $g = 0$ , the theorem holds except for  $s = 5$  where we decompose  $\Sigma_{0,0}^5$  as a union of two  $\Sigma_{0,1}^3$  with intersection  $\Sigma_{0,2}^1$ . Finally, in case (2.2) that  $s = 1$ , it suffices to consider  $g \geq 2$ . We take  $X \cong \Sigma_{1,1}$ ,  $Y \cong \Sigma_{g,1}^1$  and  $X \cap Y \cong X - s$  where  $s$  is a non-separating simple closed curve in  $X$ .

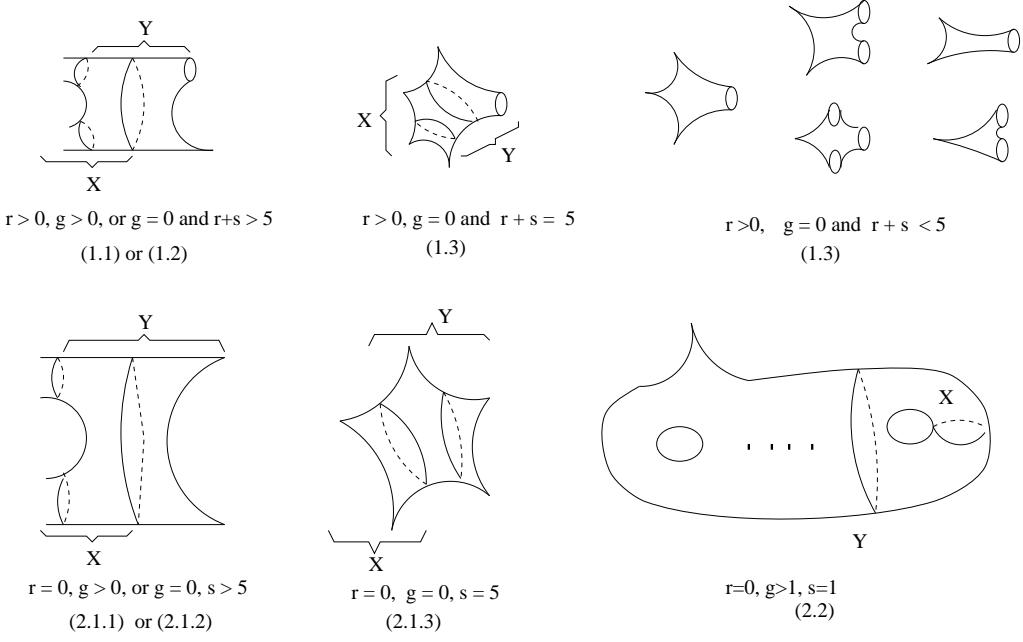


Figure 12

These give the 3-holed sphere decomposition of the surface into two subsurfaces of smaller  $|X|$  and  $|Y|$  where  $|\Sigma_{g,r}^s| = 3g + r + s$ . Note that lemmas 8 and 9 still hold for metrics with cups ends. Now by the gluing lemma, lemmas 8, 9, 10, theorem 2, the same argument used in the previous sections applies. This gives a proof of theorem 1 for metrics with cusp ends.

*Remark.* Teichmüller space is well known to be homeomorphic to a Euclidean space. This fact can also be derived from theorem 2 and lemma 1. Indeed, the gluing lemma shows that the restriction map from  $T(X \cup Y)$  to  $T(X)$  is a fiber-bundle map. The fiber can be shown to be homeomorphic to a Euclidean space by solving a simple inequality (e.g. relations (3) or (4)).

## §5. Application to Finite Dimensional Embeddings of Teichmüller Spaces

We shall prove the following stronger version of the corollary for compact surfaces by induction on  $|\Sigma_{g,r}| = 3g + r$  in this section. The proof for surfaces with cusp ends will be omitted.

**Corollary.** (a) For surface  $\Sigma_{g,r}$  of negative Euler number and  $r > 0$ , there exists a finite subset  $F$  in  $\mathcal{S}(\Sigma_{g,r})$  consisting of  $6g + 3r - 6$  elements so that the map  $\pi_F : T(\Sigma_{g,r}) \rightarrow \mathbf{R}_{>2}^F$  is an embedding onto an open subset which is defined by a finite set of real analytic RC-inequalities in the coordinates of  $\pi_F$ . Furthermore, there exists a map  $f : T(\Sigma_{g,r}) \rightarrow \tilde{R}(\Sigma_{g,r})$  so that for each  $m$  in  $T(\Sigma_{g,r})$ ,  $f(m)$  is a lifting of a monodromy of  $m$  and the entries of the matrix  $f(m)(\alpha)$  are real analytic RC-functions of  $\pi_F(m)$  for any  $\alpha \in \mathcal{S}(\Sigma_{g,r})$ .

(b) For surface  $\Sigma_{g,0}$  of negative Euler number, there exists a finite subset  $F$  of  $\mathcal{S}(\Sigma_{g,0})$  consisting of  $6g - 5$  elements so that  $\pi_F : T(\Sigma_{g,0}) \rightarrow \mathbf{R}_{>2}^F$  is an embedding whose image is defined by one real analytic RC-equation and finitely many real analytic RC-inequalities in the coordinates

of  $\tau_F$ . Furthermore, there exists a map  $f : T(\Sigma_{g,0}) \rightarrow \tilde{R}(\Sigma_{g,0})$  so that for each  $m$  in  $T(\Sigma_{g,0})$ ,  $f(m)$  is a lifting of a monodromy of  $m$  and the entries of the matrix  $f(m)(\alpha)$  are real analytic RC-functions of  $\pi_F(m)$  for any  $\alpha \in \mathcal{S}(\Sigma_{g,0})$

Note that the corollary without the statement about the lifting of monodromies follows immediately from the gluing lemma, theorems 1 and 2, and lemmas 8 and 9. To prove the full statement, we need to strengthen the gluing lemma.

In §5.1, we prove an extended version of the gluing lemma. In §5.2, we prove the corollary for  $\Sigma_{1,2}$ . The corollary for surfaces with non-empty boundary is proved in §5.3. In §5.4, we prove the corollary for closed surfaces.

### 5.1. Algebraic dependence in the gluing lemma

We begin with a parametrized version of the Jordan canonical form theorem for  $SL(2, \mathbf{R})$  matrices.

**Lemma 11.** (a) If  $A = [a_{ij}]$  in  $SL(2, \mathbf{R})$  satisfies  $|trA| > 2$  and  $a_{12}a_{21} \neq 0$ , then

$$C^{-1}AC = 1/2 \begin{pmatrix} a_{11} + a_{22} + \sqrt{(a_{11} + a_{22})^2 - 4} & 0 \\ 0 & a_{11} + a_{22} - \sqrt{(a_{11} + a_{22})^2 - 4} \end{pmatrix},$$

where

$$C = \begin{pmatrix} 2a_{12} & a_{11} - a_{22} - \sqrt{(a_{11} + a_{22})^2 - 4} \\ a_{22} - a_{11} + \sqrt{(a_{11} + a_{22})^2 - 4} & 2a_{21} \end{pmatrix}.$$

(b) For  $A = [a_{ij}]$  and  $B = [b_{ij}]$  in  $SL(2, \mathbf{R})$  with  $trA > 2$  (resp.  $trA < -2$ ),  $a_{12}a_{21} \neq 0$  and  $trABA^{-1}B^{-1} \neq 2$ , there exist four real analytic RC-functions  $c_{ij}$  in eight variables so that  $C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $\lambda > 1$  (resp.  $\lambda < -1$ ) and  $C^{-1}BC = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ ,  $|\delta| = 1$  where  $C = [c_{ij}(A, B)] \in GL(2, \mathbf{R})$ .

**Proof.** Part (a) follows by a direct calculation. Note that the matrix  $C$  is invertible since  $a_{12}a_{21} \neq 0$ . Part (b) follows from part (a). Indeed, by part (a), we may conjugate  $A$  to the required diagonal form  $A'$ . We also conjugate  $B$  by the same matrix to obtain  $B'$ . The trace of the commutator remains unchanged. Thus the new matrix  $B' = [b'_{ij}]$  has non-zero (2,1)-entry.

Now a further conjugation by the matrix  $\begin{pmatrix} \sqrt{|b'_{21}|} & 0 \\ 0 & \sqrt{|b'_{21}|^{-1}} \end{pmatrix}$  will not change matrix  $A'$  but change  $B'$  into the required form.  $\square$

We say a pair of matrices  $(A, B)$  is *normalized* if  $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  with  $|\lambda| > 1$  and the (2,1)-entry of  $B$  is 1. It follows from the normalized condition that if  $C$  is in  $GL(2, \mathbf{R})$  so that both  $(A, B)$  and  $(C^{-1}AC, C^{-1}BC)$  are normalized, then  $(A, B) = (C^{-1}AC, C^{-1}BC)$ , i.e., normalization is unique up to  $GL(2, \mathbf{R})$  conjugation. Fix a pair of elements  $(\gamma_1, \gamma_2)$  in  $\pi_1(\Sigma)$ . A representation  $\rho$  in  $\tilde{R}(\Sigma)$  is called *normalized* with respect to the pair if  $(\rho(\gamma_1), \rho(\gamma_2))$  is normalized.

A section of the natural projection from  $\tilde{R}(\Sigma)$  to  $T(\Sigma)$  is a continuous map  $f : T(\Sigma) \rightarrow \tilde{R}(\Sigma)$  so that  $f(m)$  is a lifting of a monodromy of  $m$ . Given a section  $f$ , we may produce a new section whose image lies in any given component of  $\tilde{R}(\Sigma)$  as follows. Conjugating representations in  $f(T(\Sigma))$  by the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  gives rise to a new section in a component of the opposite orientation type; and choosing a different lifting  $\rho_I$  associated to  $\rho \in f(T(\Sigma))$  for a fixed index set  $I$  (see §1.2 for the definition) gives a section in a different component of the same orientation type. We call these new sections to be the ones obtained from  $f$  by different liftings and conjugations. An *RC-section* is a section so that (1) there exists an associated finite set  $F \subset \mathcal{S}(\Sigma)$  so that the entries of the matrix  $f(m)(\alpha)$  are real analytic RC-functions of the coordinates of  $\pi_F(m)$  for all  $\alpha \in \pi_1(\Sigma)$  and (2) each representation in the image of the section is normalized with respect to a fixed pair of elements in  $\pi_1(\Sigma)$ . By lemmas 4, 5 and theorem 2, the Teichmüller spaces  $T_{0,3}$ ,  $T_{1,1}$ , and  $T_{0,4}$  have RC-sections.

For simplicity, we shall identify curves, isotopy classes of curves, and homotopy classes of curves in incompressible subsurfaces with their images in the ambient spaces without mentioning the including maps.

**Lemma 12.** (Algebraic dependence) *Let  $X$  and  $Y$  be good incompressible subsurfaces of  $\Sigma$  so that  $\Sigma = X \cup Y$  and either (1)  $X \cap Y \cong \Sigma_{0,3}$ , or (2)  $Y \cong \Sigma_{1,1}$  and  $X \cap Y = Y - s$  where  $s$  is a non-separating simple closed curve in  $\text{int}(Y)$ , or (3)  $X \cap Y \cong \Sigma_{0,2}^1$  so that the punctured end in  $\Sigma_{0,2}^1$  is a punctured end of  $\Sigma$ . If  $T(X)$  and  $T(Y)$  both have RC-sections  $f_X$  and  $f_Y$  with associated sets  $F_X$  and  $F_Y$  respectively, then  $T(\Sigma)$  has an RC-section with associated set  $F_X \cup F_Y$ .*

**Proof.** Let  $(\alpha_1, \alpha_2)$  (resp.  $(\beta_1, \beta_2)$ ) be the pair in  $\pi_1(X)$  (resp.  $\pi_1(Y)$ ) so that each representation in the image of  $f_X$  (resp.  $f_Y$ ) is normalized with respect to it. Choose two geometric generators  $\gamma_1$  and  $\gamma_2$  for  $\pi_1(X \cap Y)$  so that  $\gamma_1 \gamma_2$  is represented by the third boundary component. Then one of the three elements  $\gamma_1, \gamma_2, \gamma_1 \gamma_2$ , say  $\gamma_1$ , satisfies the condition that both subgroups  $\langle \alpha_1, \gamma_1 \rangle$  and  $\langle \beta_1, \gamma_1 \rangle$  are not solvable. Let  $\gamma_2$  be one of the remaining element. Then  $\pi_1(X \cap Y)$  is generated by  $\gamma_1$  and  $\gamma_2$ . We extend  $\{\gamma_1, \gamma_2\}$  to a minimal set of generators  $\{\gamma_1, \dots, \gamma_n\}$  for  $\pi_1(\Sigma)$  so that each  $\gamma_i$  is either in  $\pi_1(X)$  or in  $\pi_1(Y)$ .

By choosing a different lifting if necessary, we may assume that  $f_X(m)(\alpha_1)$  and  $f_Y(m)(\beta_1)$  are diagonal matrices with positive traces for  $m \in T(X \cap Y)$  ( $f_X$  and  $f_Y$  are still sections but may not be normalized any more). Now by the choice of element  $\gamma_1$ , both matrices  $f_X(m)(\gamma_1)$  and  $f_Y(m)(\gamma_1)$  have non-zero off diagonal entries for all  $m$ , and the trace of the commutator of  $f_X(m)(\gamma_1)$  and  $f_X(m)(\gamma_2)$  (resp.  $f_Y(m)(\gamma_1)$ , and  $f_Y(m)(\gamma_2)$ ) is not 2. Thus by lemma 11, we may conjugate the pair  $(f_X(m)(\gamma_1), f_X(m)(\gamma_2))$  (resp.  $(f_Y(m)(\gamma_1), f_Y(m)(\gamma_2))$ ) to the form in lemma 11(b) by a  $\text{GL}(2, \mathbf{R})$  matrix whose entries are real analytic RC-functions in the coordinates of  $\pi_{F_X}(m)$  (resp. in the coordinates of  $\pi_{F_Y}(m)$ ). This produces two sections  $g_X$  and  $g_Y$  for  $T(X)$  and  $T(Y)$  respectively so that (1) for each  $m \in T(X)$  (resp.  $m \in T(Y)$ ), the entries of the matrices  $g_X(m)(\gamma)$  (resp.  $g_Y(m)(\gamma)$ ) are real analytic RC-functions in the coordinates of  $\pi_{F_X}(m)$  (resp.  $\pi_{F_Y}(m)$ ), and (2) the matrix  $g_X(m)(\gamma_1)$  (resp.  $g_Y(m)(\gamma_1)$ ) is diagonal with (1,1)-entry bigger than one and the (2,1)-entry of  $g_X(m)(\gamma_2)$  (resp.  $g_Y(m)(\gamma_2)$ ) has absolute value one.

We may normalize the sections  $g_X$  and  $g_Y$  by choosing different lifting which changes the generator  $\rho(\gamma_2)$  to  $-\rho(\gamma_2)$  if necessary. Thus we may assume that both  $g_X$  and  $g_Y$  are normalized with respect to the pair  $(\gamma_1, \gamma_2)$ .

We now define an RC-section for  $T(\Sigma)$  as follows. By the gluing lemma, each  $m \in T(\Sigma)$  corresponds to a pair  $(m_X, m_Y) \in T(X) \times T(Y)$  so that  $R_X(m) = m_X$ ,  $R_Y(m) = m_Y$  and the restrictions of  $m_X$  and  $m_Y$  to  $X \cap Y$  are the same. The restrictions of the two representations  $g_X(m_X)$  and  $g_Y(m_Y)$  to the subgroup  $\pi_1(X \cap Y)$  uniformize the same element  $R_{X \cap Y}(m)$ . Since the pair  $(\gamma_1, \gamma_2)$  generates  $\pi_1(X \cap Y)$ , by the normalization condition for  $g_X$  and  $g_Y$ , we have  $g_X(m_X)|_{\pi_1(X \cap Y)} = g_Y(m_Y)|_{\pi_1(X \cap Y)}$ . By Maskit combination theorem (there is no need to verify the side condition since the gluing is along a 3-holed sphere), there exists a unique representation  $\rho \in \tilde{R}(\Sigma)$  so that  $\rho|_{\pi_1(X)} = g_X(m)$  and  $\rho|_{\pi_1(Y)} = g_Y(m)$ . The map from  $T(\Sigma)$  to  $\tilde{R}(\Sigma)$  sending  $m$  to  $\rho$  is a section normalized with respect to  $(\gamma_1, \gamma_2)$ . To see the RC-dependence (which also shows the continuity of the map  $m$  to  $\rho$ ), it suffices to check the condition for each generator  $\gamma_i$ . By the construction,  $\rho(\gamma_i)$  is either  $g_X(m)(\gamma_i)$  or  $g_Y(m)(\gamma_i)$ . Thus, each entry of the matrix  $\rho(\gamma_i)$  is a real analytic RC-function in the coordinates of  $\pi_{F_X \cup F_Y}(m)$ .  $\square$

### 5.2. Proof of the Corollary for $\Sigma_{1,2}$

Let  $s_7$  be an essential separating simple closed curve and  $s_1$  be a non-separating simple closed curve disjoint from  $s_7$  in  $\Sigma_{1,2}$  as in figure 13(a). We decompose  $\Sigma_{1,2}$  as a union  $X \cup Y$  where  $X$  is the compact subsurface bounded by  $s_7$  containing  $s_1$  and  $Y$  is the complement of  $s_1$ . Then  $X \cap Y$  is  $X - s_1$ . Let  $s_2, s_3$  be simple closed curves in  $X$  so that  $s_1 \perp s_2$  and  $s_3 = s_1 s_2$ ; let  $s_4, s_5, s_6, s_1^+$  and  $s_1^-$  be simple closed curves in  $Y$  so that  $s_1^+$  and  $s_1^-$  are boundary components which are identified to be  $s_1$  in  $\Sigma_{1,2}$ ,  $s_6 \subset \partial Y$  and  $s_4 \perp_0 s_7$ ,  $s_5 = s_4 s_7$ . See figures 13(b) and (c). By the gluing lemma and lemma 4, the Teichmüller space  $T(\Sigma_{1,2})$  can be identified with the subset  $\{(m_X, m_Y) \in T(X) \times T(Y) | t_{m_X}(s_1) = t_{m_Y}(s_1^+) = t_{m_Y}(s_1^-) \text{ and } t_{m_X}(s_7) = t_{m_Y}(s_7)\}$ . By lemma 5,  $m_X$  is determined by  $\pi_{F_X}(m_X) = (t_{m_X}(s_1), t_{m_X}(s_2), t_{m_X}(s_3))$  where  $F_X = \{[s_1], [s_2], [s_3]\}$ . By theorem 2,  $m_Y$  is determined by  $\pi_{F_Y}(m_Y) = (t_{m_Y}(s_1^+), t_{m_Y}(s_1^-), t_{m_Y}(s_4), t_{m_Y}(s_5), t_{m_Y}(s_6), t_{m_Y}(s_7))$ . Finally, formula (1) shows that  $t_{m_X}(s_7) = t_{m_X}(s_1)t_{m_X}(s_2)t_{m_X}(s_3) + 2 - t_{m_X}^2(s_1) - t_{m_X}^2(s_2) - t_{m_X}^2(s_3)$ . Combining these and lemma 12, we obtain the following lemma.

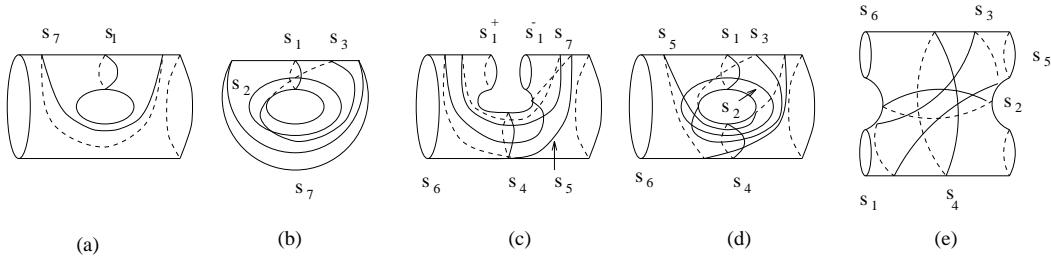


Figure 13

**Lemma 13.** *For surface  $\Sigma_{1,2}$ , let  $F$  be the collection of isotopy classes of six curves  $s_1, s_2, s_3, s_4, s_5, s_6$  as in figure 13(d). Then  $\pi_F : T(\Sigma_{1,2}) \rightarrow R^6$  is an embedding whose image is given by  $\{(t_1, t_2, t_3, t_4, t_5, t_6) \in \mathbf{R}_{>2}^6 | t_1 t_2 t_3 > t_1^2 + t_2^2 + t_3^2 \text{ and } t_4 t_5 t_7 > t_4^2 + t_5^2 + t_6^2 + t_7^2 + 2t_1^2 + 2t_2^2 + 2t_3^2\}$*

$t_1^2 t_7 + t_1 t_4 t_6 + t_1 t_5 t_6 + 2t_1 t_4 + 2t_1 t_5 + 2t_6 t_7$ , where  $t_7 = t_1 t_2 t_3 - t_1^2 - t_2^2 - t_3^2\}$ . Furthermore, there exists an RC-section for  $T(\Sigma_{1,2})$  with associated set  $F$ .

### 5.3. Proof of the Corollary for $\Sigma_{g,r}$ with $r > 0$

We prove the corollary by induction on  $|\Sigma_{g,r}| = 3g + r$  with  $r > 0$ .

For surfaces  $\Sigma_{0,3}$ ,  $\Sigma_{0,4}$ ,  $\Sigma_{1,1}$  and  $\Sigma_{1,2}$ , we have shown in the previous sections that the corollary holds. Given  $\Sigma_{g,r}$  with either  $3g + r = n > 5$  or  $(g, r) = (0, 5)$ , if  $r \geq 2$ , we decompose  $\Sigma_{g,r} = X \cup Y$  where  $X \cong \Sigma_{g,r-1}$ ,  $Y \cong \Sigma_{0,4}$  with  $X \cap Y \cong \Sigma_{0,3}$  as in figure 3(b); if  $r = 1$ , we decompose  $\Sigma_{g,r} = X \cup Y$  where  $X \cong \Sigma_{g-1,2}$ ,  $Y \cong \Sigma_{1,2}$  and  $X \cap Y \cong \Sigma_{0,3}$  as in figure 3(c). Then  $|X|$  and  $|Y|$  are less than  $|\Sigma_{g,r}|$ . By the induction hypothesis, there exists a subset  $F_X \subset \mathcal{S}(X)$  consisting of  $6g + 3r - 9$  elements so that corollary holds. Let  $F_Y \subset \mathcal{S}(Y)$  be the set  $\{[s_1], [s_2], [s_3], [s_4], [s_5], [s_6]\}$  given by theorem 2 as in figure 13(e) if  $Y \cong \Sigma_{0,4}$  and by lemma 13 as in figure 13(d) if  $Y \cong \Sigma_{1,2}$ . Let  $F = F_X \cup \{[s_2], [s_3], [s_5]\}$  consisting of  $6g + 3r - 6$  elements. We claim that the corollary holds for  $\Sigma_{g,r}$  with respect to the set  $F$ . First to show that  $\pi_F$  is an embedding, we use the gluing lemma. It follows that  $\pi_{F_X \cup F_Y}$  is an embedding. However, by the construction,  $s_1, s_4$  and  $s_6$  are in the subsurface  $X$ . Thus by the induction hypothesis,  $t_m(s_1)$ ,  $t_m(s_4)$  and  $t_m(s_6)$  are real analytic RC-functions in the coordinates of  $\pi_{F_X}(m)$ . Thus, we may drop the three elements  $[s_1], [s_4]$ , and  $[s_6]$  from the set  $F_X \cup F_Y$  without effecting the embeddedness of  $\pi_{F_X \cup F_Y}$ . Applying lemma 12 to  $F_X$  and  $F_Y$  and then dropping the three elements  $[s_1], [s_4]$  and  $[s_6]$ , we see that  $T(\Sigma_{g,r})$  has an RC-section with associated set  $F$ . Finally, we show that the image  $\pi_F(T(\Sigma_{g,r}))$  is defined by a finite set of RC-inequalities in the coordinates of  $\pi_F$ . Indeed, by the induction hypothesis,  $\pi_{F_X}(T(X))$  (resp.  $\pi_{F_Y}(T(Y))$ ) is defined by a finite set of RC-inequalities. By the gluing lemma 1, the image  $\pi_{F_X \cup F_Y}(T(\Sigma_{g,r}))$  is given by the same set of RC-inequalities for  $\pi_{F_X}(T(X))$ , together with the RC-inequalities for  $\pi_{F_Y}(T(Y))$ , and three equations expressing that the lengths of the three simple closed curves in  $\partial(X \cap Y)$  are the same in both metrics  $m_X$  and  $m_Y$ . Thus the result follows.

### 5.4. Proof of the Corollary for closed surface $\Sigma_{g,0}$ with $g \geq 2$

Given  $\Sigma_g = \Sigma_{g,0}$ , let  $Y$  be an incompressible subsurface of  $\Sigma_g$  homeomorphic to  $\Sigma_{1,1}$  with boundary  $s_1$  and let  $s_2$  be a non-separating simple closed curve in  $\text{int}(Y)$ . Set  $X = \Sigma_g - s_2$  as in figure 3(d). Thus  $\Sigma_g = X \cup Y$  and  $X \cap Y = Y - s_2$ . By the gluing lemma 1, each metric  $m \in T(\Sigma_g)$  is the same as a pair  $(m_X, m_Y) \in T(X) \times T(Y)$  with  $R_{X \cap Y}(m_X) = R_{X \cap Y}(m_Y)$ . In particular the the completion  $\bar{X}$  of  $X$  under the metric  $m_X$  has the same geodesic lengths at the two boundary components. The following lemma describes hyperbolic metrics on  $\Sigma_{0,4}$  which have the same lengths at two boundary curves.

**Lemma 14.** *Given  $\Sigma_{0,4}$  with curves  $b_i$  ( $i = 1, 2, 3, 4$ ) as boundary components, let  $a_{ij}$  ( $(i, j) = (1, 2), (2, 3), (3, 1)$ ) be simple closed curves in  $\Sigma_{0,4}$  so that  $a_{12} \perp_0 a_{23}$  and  $a_{31} = a_{12}a_{23}$  and  $b_i, b_j$  and  $a_{ij}$  bound a subsurface of signature  $(0, 3)$ . Let  $T'(\Sigma_{0,4})$  be the subspace of the Teichmüller space  $T(\Sigma_{0,4})$  defined by  $t_m(b_3) = t_m(b_4)$ , and let  $F' = \{[b_1], [b_2], [a_{12}], [a_{23}], [a_{31}]\}$ . Then  $\pi_{F'} : T'(\Sigma_{0,4}) \rightarrow \mathbf{R}_{>2}^5$  is an embedding whose image is defined by a real analytic RC-inequality in the coordinates of  $\pi_{F'}$ . Furthermore, there is an RC-section  $f : T'(\Sigma_g) \rightarrow \tilde{R}'(\Sigma_g)$  where  $\tilde{R}'(\Sigma_g)$  stands for the subset of  $\tilde{R}(\Sigma_g)$  which projects onto  $T'(\Sigma_g)$  so that the entries of  $f(m)(\gamma)$  are real analytic RC-functions in the coordinates of  $\pi_{F'}(m)$ .*

**Proof.** Given a metric  $m \in T'(\Sigma_g)$ , let  $t_i = t_m([b_i])$ ,  $i = 1, 2, 3, 4$ , and let  $t_{ij} = t_m([a_{ij}])$ ,  $(i, j) = (1, 2), (2, 3), (3, 1)$ , where  $t_3 = t_4$ . Now these  $t_i$  and  $t_{ij}$  satisfy the equation (13). Thus we obtain an equation in  $t$  ( $= t_3 = t_4$ ) below,

$$(2 + t_1 t_2 + t_{12})t^2 + (t_1 t_{31} + t_1 t_{23} + t_2 t_{31} + t_2 t_{23})t + t_1^2 + t_2^2 + t_1 t_2 t_{12} + t_{12}^2 + t_{23}^2 + t_{31}^2 - t_{12} t_{23} t_{31} - 4 = 0.$$

The coefficient of  $t^2$  is positive and the constant term is negative by (4). Thus the equation has two real roots of different signs and  $t_3$  ( $= t_4$ ) is the positive root of the equation. Thus  $t_3$  ( $= t_4$ ) is a real analytic RC-function of  $t_1, t_2, t_{12}, t_{23}$  and  $t_{31}$  which are the coordinates of  $\pi_{F'}(m)$ . This shows that  $\pi_{F'}$  is an embedding. The rest of the lemma follows by the same argument used in the proof of theorem 2.  $\square$

Let  $T'(X)$  be the subset of  $T(X)$  so that  $t_m(s_2^+) = t_m(s_2^-)$  where  $s_2^+$  and  $s_2^-$  are the boundary components of  $\bar{X}$ . Then in the proof of the corollary for  $\Sigma_{g-1,2}$  ( $\cong X$ ) in §5.3, to construct  $m \in T'(X)$ , we decompose  $X = X_1 \cup Y_1$  where  $X_1 \cong \Sigma_{g-1,1}$ ,  $Y_1 \cong \Sigma_{0,4}$  and  $X_1 \cap Y_1 \cong \Sigma_{0,3}$ . We use lemma 14 instead of theorem 2 for metrics on  $Y_1$  in the gluing process. Thus, the same argument shows that there exists a subset  $F_X \subset \mathcal{S}(\bar{X})$  consisting of  $6g - 7$  elements so that  $\pi_{F_X} : T'(X) \rightarrow \mathbf{R}_{>2}^{6g-7}$  is an embedding whose image is an open set defined by a finite set of real analytic RC-inequalities in the coordinates of  $\pi_{F_X}$ .

Let  $s_3$  and  $s_4$  be two simple closed curves in  $\text{int}(Y)$  so that  $s_3 \perp s_2$  and  $s_4 = s_2 s_3$ . Now by the gluing lemma 1, each  $m \in T(X \cup Y)$  is determined by a pair  $(m_X, m_Y) \in T'(X) \times T(Y)$  so that the restrictions of  $m_X$  and  $m_Y$  to  $X \cap Y$  are the same. The gluing condition on  $X \cap Y$  is equivalent to that  $t_{m_X}(s_2^+) = t_{m_Y}(s_2)$  and  $t_{m_X}(s_1) = t_{m_Y}(s_1)$  by lemma 4. Also lemma 5 gives the complete description of  $(t_{m_Y}(s_2), t_{m_Y}(s_3), t_{m_Y}(s_4))$ . Let  $F = F_X \cup \{[s_3], [s_4]\} \subset \mathcal{S}(\Sigma_g)$  consisting of  $6g - 5$  elements. Combining the previous facts, we obtain (1)  $\pi_F : T(\Sigma_g) \rightarrow \mathbf{R}^{6g-5}$  is an embedding, (2) the image  $\pi_F(T(\Sigma_g))$  is defined by a finite set of RC-inequalities (from those of  $\pi_{F_X}(T'(X))$  and of  $\pi_{\{[s_2], [s_3], [s_4]\}}(T(\Sigma_{1,1}))$  where we replace  $t_{m_Y}([s_2])$  by  $t_{m_X}([s_2])$ , and one real analytic RC-equation  $t_{m_Y}(s_1) = t_{m_X}(s_1)$ . Furthermore, by lemmas 13 and 14, there is an RC-section for  $T(\Sigma_g)$ .  $\square$

*Remark.* The fact that  $\pi_{F'}$  is an embedding in the lemma 10 was first proved by P. Schmutz ([Sc]).

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