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Discrete heat kernel determines discrete Riemannian metric

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ABSTRACT

The Laplace–Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from the eigenvalues and eigenfunctions of the Laplace–Beltrami operator determines the Riemannian metric. This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete heat kernel and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other. Given a Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace–Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge lengths can be determined by the edge weights unique up to a scaling using the variational approach.

The constructive proof leads to a computational algorithm that finds the unique metric on a triangle mesh from a discrete Laplace–Beltrami operator matrix.

1. Introduction

Laplace–Beltrami operator plays a fundamental role in Riemannian geometry [26]. Discrete Laplace–Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications, including mesh parameterization, segmentation, reconstruction, compression, re-meshing and so on [16,24,31]. Laplace–Beltrami operator is determined by the Riemannian metric. The heat kernel can be constructed from the eigenvalues and eigenfunctions of the Laplace–Beltrami operator; conversely, it fully determines the Riemannian metric (uniquely up to a scaling). In this work, we prove the *discrete analogy* to this fundamental fact for surface case, that *the discrete heat kernel and the discrete Riemannian metric are mutually determined by each other.*

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1.1. Motivation

The Laplace–Beltrami operator on a Riemannian manifold plays an fundamental role in Riemannian geometry. The spectrum of its eigenvalues encodes the Riemannian metric information, the nodal lines of its eigenfunctions reflects the intrinsic symmetry. Especially, the heat kernel composed by both eigenvalues and eigenfunctions fully determines the Riemannian metric.

The above theorems from Riemannian geometry have been applied in a broad range of engineering applications. The eigenfunctions corresponding to the zero eigenvalue are called harmonic functions, which have been applied for mesh parameterizations in graphics fields, such as thorough surveys can be found in [9] and [15]. Spectrum has been applied as shape-DNA [21] for surfaces or solids; Eigenfunctions are applied for global intrinsic symmetry detection [19]; Heat Kernel Signatures are applied for shape analysis and comparison in [27]. More detailed survey for the applications of spectrum theory can be found in [31].

All these algorithms have the advantages from Laplace–Beltrami operator theory, which are intrinsic to the Riemannian metric, independent of embedding, invariant under isometric transformation, stable under small perturbation, and robust to geometric and topological noises.

All the applications above are based on the fundamental theorem (see Theorem 2.2) that the heat kernel fully determines the Riemannian metric, or the eigenvalues and eigenfunctions of the Laplace–Beltrami operator partially determine the Riemannian metric. These results have been proven only for *smooth manifolds*. However, all the computations are on *discrete meshes*. Therefore, it is important to prove the discrete analogy of Theorem 2.2, that discrete heat kernel (or equivalently, Laplace–Beltrami operator matrix) determines the discrete Riemannian metric (see the Main Theorem 3.5). This motivates the current work. To the best of our knowledge, this work is the first one to fill the huge gap and ensure the rigor for all these existing computational algorithms in real applications.

1.2. Discretizations of Laplace–Beltrami operator

In real applications, a smooth metric surface is usually represented as a triangulated mesh. The manifold heat kernel is estimated from the discrete Laplace operator. There are many ways to discretize the Laplace–Beltrami operator.

The most well-known and widely-used discrete formulation of Laplace operator over triangulated meshes is the so-called *cotangent scheme*, which was originally introduced in [8,20]. Xu [30] proposed several simple discretization schemes of Laplace operators over triangulated surfaces, and established the theoretical analysis on convergence. Wardetzky et al. [29] proved the theoretical limitation that the discrete Laplacians cannot satisfy all natural properties, thus, explained the diversity of existing discrete Laplace operators. A family of operations were presented by extending more natural properties into the existing operators. Reuter et al. [21] computed a discrete Laplace operator using the finite element method, and exploited the isometry invariance of the Laplace operator as shape fingerprint for object comparison. Belkin et al. [1] proposed the first discrete Laplacian that pointwise converges to the true Laplacian as the input mesh approximates a smooth manifold better. Dey et al. [7] employed this mesh Laplacian and provided the first convergence to relate the discrete spectrum with the true spectrum, and studied the stability and robustness of the discrete approximation of Laplace spectra.

1.3. Discrete curvature flow

The proof for the correspondence between the discrete Laplace–Beltrami matrix and the discrete metric uses the Legendre duality principle [18] (Lemma 4.3 in this work), which is similar to the discrete curvature flow theory. Legendre duality principle can be formulated as follows. Given a convex function $\phi : \Omega \to \mathbb{R}$ defined on a convex domain Ω, $\nabla\phi(x)$ denotes the gradient at the point $x \in \Omega$. Then $x \to \nabla\phi(x)$ has one-to-one correspondence, x and $\nabla\phi(x)$ are Legendre dual of each other.

All the existing discrete surface curvature flow theories are based on Legendre duality principle. In discrete surface curvature flow, there are different ways to discretize conformal transformation. Thurston [28] introduced circle packing method. Colin de Verdiere [6] established the first variational principle for circle packing and proved Thurston's existence of circle packing metrics. Chow and Luo [5] generalized Colin de Verdiere's work and introduced the discrete Ricci flow and discrete Ricci energy on surfaces. The algorithmic was later implemented and applied for surface parameterization [13,12]. Circle pattern was proposed by Bowers and Hurdal [4], and has been proven to be a minimizer of a convex energy by Bobenko and Springborn [3]. An efficient circle pattern algorithm was developed by Kharevych et al. [14]. Discrete Yamabe flow was introduced by Luo in [17]. In a recent work of Springborn et al. [25], the Yamabe energy is explicitly given by using the Milnor-Lobachevsky function.

In all above works, the discrete conformal factor and the discrete Gaussian curvature form the Legendre dual pair. All the proofs are to construct a convex energy defined on the discrete conformal factor, the gradient of the energy is the discrete curvature. If the space of all admissible conformal factor functions is convex, then by Legendre duality, the correspondence between the conformal factor and the curvature is one-to-one.

In the current work, we follow the same principle to construct a convex energy and show that the edge length (discrete metric) and the cotangent edge weight (discrete Laplace–Beltrami operator) are Legendre dual pair, and they mutually determine each other.

1.4. Contribution

The Laplace–Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from its eigenvalues and eigenfunctions determines the Riemannian metric. This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete heat kernel and the discrete Riemannian metric (uniquely up to a scaling) are mutually determined by each other.

Given a Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace–Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge lengths can be determined by the edge weights uniquely up to a scaling using the variational approach.

First, we show that the space of all possible metrics of a polyhedral surface is convex. Second, we construct a special energy defined on the metric space, such that the gradient of the energy equals to the edge weights. Third, we show the Hessian matrix of the energy is positive definite, restricted on the tangent space of the metric space, therefore the energy is convex. Finally, by the fact that the parameter on a convex domain and the gradient of a convex function defined on the domain have one-to-one correspondence, we show the edge weights determines the polyhedral metric uniquely up to a scaling.

The constructive proof leads to a computational algorithm that finds the unique metric on a triangle mesh from a discrete Laplace–Beltrami operator matrix.

1.4.1. Organization

The paper is organized as follows: Section 2 introduces the theoretical background on Laplace–Beltrami operator and heat kernel. Section 3 introduces discrete heat kernel and presents the main theorem of this work. Section 4 describes the theoretic deduction details for the proposed theorem. Numerical experiments are discussed in Section 5. Section 6 concludes the paper and gives the future work.

2. Theoretic background

In the following, we briefly introduce the theoretic background for heat kernel. For more thorough theoretic treatment, we refer readers to the differential geometry textbook [23]. For more technical details of the applications of heat kernel on geometric processing, we refer readers to [27].

2.1. Laplace–Beltrami operator

Suppose $(M,\mathbf{g})$ is a compact Riemannian manifold with a Riemannian metric $\mathbf{g}$, $u : M \to \mathbb{R}$ is a function defined on M. The Laplace–Beltrami operator computes the divergence of the gradient of the function,

$$\Delta_{\mathbf{g}} u = div \cdot grad\ u.$$

Select a local coordinate coordinates $\{x^i\}$, the Riemannian metric tensor is given by $\mathbf{g} = g_{ij} dx^i dx^j$, the inverse of (g_{ij}) is denoted as (g^{ij}), the determinant is $g = det(g_{ij})$. Then the local representation of the Laplace–Beltrami operator is

$$\Delta_{\mathbf{g}} u = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^j}\left(g^{ij}\sqrt{g}\frac{\partial u}{\partial x^i}\right).$$

The *eigenfunction* ϕ_i of $\Delta_{\mathbf{g}}$ is defined as

$$\Delta_{\mathbf{g}}\phi_i = \lambda_i\phi_i, \lambda_i \in \mathbb{R}.$$

Because $\Delta_{\mathbf{g}}$ is bounded and symmetric negative semi-definite, λ_i's are non-negative real numbers, there are countable eigenfunctions.

2.2. Heat kernel

The *heat diffusion process* on M is governed by the heat equation, let $u(x,t) : M \times \mathbb{R}^+ \to \mathbb{R}$ represent the temperature field on M at time t, then it satisfies the following heat equation

$$\Delta_{\mathbf{g}} u(x,t) = -\frac{\partial u(x,t)}{\partial t}, \tag{1}$$

with initial condition $u(x,0)$.

Definition 2.1. (Heat Kernel) The heat kernel $K(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$ is given by

$$K(x,y,t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

The solution to the heat Eq. 1 can be explicitly given by the heat kernel

$$u(x,t) = \int_M K(x,y,t)u(y,0)dy.$$

Heat kernel plays a fundamental role in geometric modeling and shape analysis [27], because heat kernel is the complete invariant of the Riemannian metric.

Suppose F: $(M_1,\mathbf{g_1}) \to (M_2,\mathbf{g_2})$ is a mapping between two Riemannian manifolds, such that F preserves geodesic distances, then we say F is an isometric map. In differential geometry, F is isometric, then the pull back metric on M_1 $F^*\mathbf{g_2}$ induced by F equals to $\mathbf{g_1}$

$$F^*\mathbf{g_2} = \mathbf{g_1}.$$

Then the following theorem shows heat kernel is the complete invariant of the Riemannian metric:

Theorem 2.2. *Let F: $(M_1,\mathbf{g_1}) \to (M_2,\mathbf{g_2})$ be a surjective map between two Riemannian manifolds. F is an isometry, $F^*\mathbf{g_2} = \mathbf{g_1}$, if and only if $K_2(F(x),F(y),t) = K_1(x,y,t)$ for any x, $y \in M_1$ and any $t > 0$.*

The main focus of the current work is to prove the discrete analogy to the fundamental relation between the heat kernel and the Riemannian metric.

3. Discrete heat kernel

In this work, we focus on discrete surfaces, namely polyhedral surfaces. For example, a triangle mesh is piecewise linearly embedded in $\mathbb{R}^3$.

Definition 3.1. (Polyhedral Surface) A Euclidean polyhedral surface is a triple $(S,T,\mathbf{d})$, where S is a closed surface, T is a triangulation of S and $\mathbf{d}$ is a metric on S, whose restriction to each triangle is isometric to a Euclidean triangle.

3.1. Discrete Laplace–Beltrami operator

The well-known cotangent edge weight [8,20] on a Euclidean polyhedral surface is defined as follows:

Definition 3.2. (Cotangent Edge Weight) Suppose $[v_i, v_j]$ is a boundary edge of M, $[v_i, v_j] \in \partial M$, then $[v_i, v_j]$ is incident with a triangle $[v_i, v_j, v_k]$, the angle opposite to $[v_i, v_j]$, at the vertex v_k, is α, then the weight of $[v_i, v_j]$ is given by $w_{ij} = \frac{1}{2}\cot\alpha$. Otherwise, if $[v_i, v_j]$ is an interior edge, the two angles opposite to it are α, β, then the weight is $w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$.

The discrete Laplace–Beltrami operator is constructed from the cotangent edge weight.

Definition 3.3. (Discrete Laplace Matrix) The discrete Laplace matrix $L = (L_{ij})$ for a Euclidean polyhedral surface is given by

$$L_{ij} = \begin{cases} -w_{ij}, & i \neq j \\ \sum_k w_{ik}, & i = j \end{cases}$$

Because L is symmetric, it can be decomposed as

$$L = \Phi \Lambda \Phi^T, \tag{2}$$

where $\Lambda = diag(\lambda_0, \lambda_1, \ldots, \lambda_n)$, $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, are the eigenvalues of L, and $\Phi = (\phi_0 - \phi_1 - \phi_2 - \ldots - \phi_n)$, $L\phi_i = \lambda_i \phi_i$, are the orthonormal eigenvectors, n is the number of vertices, such that $\phi_i^T \phi_j = \delta_{ij}$.

3.2. Discrete heat kernel

Definition 3.4. (Discrete Heat Kernel) The discrete heat kernel is defined as follows:

$$K(t) = \Phi exp(-\Lambda t)\Phi^T. \tag{3}$$

3.3. Main theorem

The **main theorem**, called *Global Rigidity Theorem*, in this work is as follows:

Theorem 3.5. *Suppose two Euclidean polyhedral surfaces $(S,T,\mathbf{d_1})$ and $(S,T,\mathbf{d_2})$ are given,*

$$L_1 = L_2,$$

if and only if $\mathbf{d_1}$ and $\mathbf{d_2}$ differ by a scaling.

Corollary 3.6. *Suppose two Euclidean polyhedral surfaces $(S,T,\mathbf{d_1})$ and $(S,T,\mathbf{d_2})$ are given,*

$$K_1(t) = K_2(t), \forall t > 0,$$

if and only if $\mathbf{d_1}$ and $\mathbf{d_2}$ differ by a scaling.

Proof. Note that

$$\left.\frac{dK(t)}{dt}\right|_{t=0} = -L.$$

Therefore, the discrete Laplace matrix and the discrete heat kernel mutually determine each other. □

4. Global Rigidity Theorem

The proof is based on the Legendre duality principle [18] (Lemma 4.3 in this work). Same principle has also been used in Rivin's work [22], discrete Ricci flow work [12,13] and Yamabe flow work [17].

4.1. Proof outline

The main idea for the proof is as follows. We fix the connectivity of the polyhedral surface (S,T). Suppose the edge set of (S,T) is sorted as $E = \{e_1, e_2, \ldots, e_m\}$, where $m = —E—$ is the number of edges and F denotes the face set. A triangle $[v_i, v_j, v_k] \in F$ is also denoted as $\{i,j,k\} \in F$.

By definition, a Euclidean polyhedral metric on (S,T) is given by its edge length function $d : E \to \mathbb{R}^+$. We denote a metric as $\mathbf{d} = (d_1, d_2, \ldots, d_m)$, where $d_i = d(e_i)$ is the length of edge e_i. Let

$$E_{\mathbf{d}}(2) = \{(d_1, d_2, d_3) | d_i + d_j > d_k\}$$

be the space of all Euclidean triangles parameterized by the edge lengths, where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$. In this work, for convenience, we use $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ to represent the metric, where $u_k = \frac{1}{2} d_k^2$.

Definition 4.1. (Admissible Metric Space)Given a triangulated surface (S,K), the admissible metric space is defined as

$$\Omega_u = \left\{ (u_1, u_2, u_3, \ldots, u_m) \middle| \sum_{k=1}^m u_k = m, (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_{\mathbf{d}}(2),\ \forall \{i,j,k\} \in F \right\}.$$

We show that Ω_u is a convex domain in $\mathbb{R}^m$.

Definition 4.2. (Energy) An energy $E : \Omega_u \to \mathbb{R}$ is defined as:

$$E(u_1, u_2, \ldots, u_m) = \int_{(1,1,\ldots,1)}^{(u_1,u_2,\ldots,u_m)} \sum_{k=1}^m w_k(u) du_k, \tag{4}$$

where $w_k(u)$ is the cotangent weight on the edge e_k determined by the metric u, d is the exterior differential operator.

Next we show this energy is convex in Lemma 4.10. According to the following lemma, the gradient of the energy $\nabla E(\mathbf{d}) : \Omega \to \mathbb{R}^m$

$$\nabla E : (u_1, u_2, \ldots, u_m) \to (w_1, w_2, \ldots, w_m)$$

is an embedding. Namely the metric is determined by the edge weight uniquely up to a scaling.

Lemma 4.3. (Legendre Duality) *Suppose $\Omega \subset \mathbb{R}^n$ is an open convex domain in $\mathbb{R}^n$, $h : \Omega \to \mathbb{R}$ is a strictly convex function with positive definite Hessian matrix, then $\nabla h : \Omega \to \mathbb{R}^n$ is a smooth embedding.*

Proof. If $\mathbf{p} \neq \mathbf{q}$ in Ω, let $\gamma(t) = (1-t)\mathbf{p} + t\mathbf{q} \in \Omega$ for all $t \in [0,1]$. Then $f(t) = h(\gamma(t)) : [0,1] \to \mathbb{R}$ is a strictly convex function, so that

$$\frac{df(t)}{dt} = \nabla h \bigg|_{\gamma(t)} \cdot (\mathbf{q} - \mathbf{p}).$$

Because

$$\frac{d^2 f(t)}{dt^2} = (\mathbf{q} - \mathbf{p})^T H \bigg|_{\gamma(t)} (\mathbf{q} - \mathbf{p}) > 0,$$

$$\frac{df(0)}{dt} \neq \frac{df(1)}{dt},$$

therefore

$$\nabla h(\mathbf{p}) \cdot (\mathbf{q} - \mathbf{p}) \neq \nabla h(\mathbf{q}) \cdot (\mathbf{q} - \mathbf{p}).$$

This means $\nabla h(\mathbf{p}) \neq \nabla h(\mathbf{q})$, therefore ∇h is injective.

On the other hand, the Jacobian matrix of ∇h is the Hessian matrix of h, which is positive definite. It follows that $\nabla h : \Omega \to \mathbb{R}^n$ is a smooth embedding. □

From the discrete Laplace–Beltrami operator (Eq. 2) or the heat kernel (Eq. 3), we can compute all the cotangent edge weights, then because the edge weight determines the metric, we attain the Main Theorem 3.5.

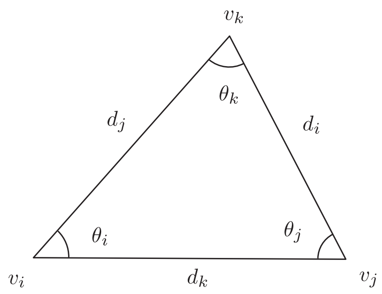

Fig. 1. A Euclidean triangle.

4.2. Rigidity on one face

In this section, we show the proof for the simplest case, a Euclidean triangle; in the next section, we generalize the proof to all types of triangle meshes.

Given a triangle $\{i,j,k\}$, three corner angles denoted by $\{\theta_i,\theta_j,\theta_k\}$, three edge lengths denoted by $\{d_i,d_j,d_k\}$, as shown in Fig. 1. In this case, the problem is trivial. Given $(w_i,w_j,w_k)=(\cot\theta_i,\cot\theta_j,\cot\theta_k)$, we can compute $(\theta_i,\theta_j,\theta_k)$ by taking the *arccot* function. Then the normalized edge lengths are given by

$$(d_i,d_j,d_k)=\frac{3}{\sin\theta_i+\sin\theta_j+\sin\theta_k}(\sin\theta_i,\sin\theta_j,\sin\theta_k).$$

Although this approach is direct and simple, it cannot be generalized to more complicated polyhedral surfaces. In the following, we use a different approach, which can be generalized to all polyhedral surfaces.

The following Lemma 4.4 is called derivative cosine law [18], which is well known in the literature [22,17,13,12,2]. Lemma 4.5 is the direct corollary of Lemma 4.4, which appeared in [17,2]. For the sake of completeness, we give the detailed proofs here.

Lemma 4.4. *Suppose a Euclidean triangle is with angles $\{\theta_i,\theta_j,\theta_k\}$ and edge lengths $\{d_i,d_j,d_k\}$, angles are treated as the functions of the edge lengths $\theta_i(d_i,d_j,d_k)$, then*

$$\frac{\partial\theta_i}{\partial d_i}=\frac{d_i}{2A} \tag{5}$$

and

$$\frac{\partial\theta_i}{\partial d_j}=-\frac{d_i}{2A}\cos\theta_k, \tag{6}$$

where A is the area of the triangle.

Proof. According to Euclidean cosine law

$$\cos\theta_i=\frac{d_j^2+d_k^2-d_i^2}{2d_jd_k}, \tag{7}$$

we take derivative on both sides with respective to d_i,

$$-\sin\theta_i\frac{\partial\theta_i}{\partial d_i}=\frac{-2d_i}{2d_jd_k}$$
$$\frac{\partial\theta_i}{\partial d_i}=\frac{d_i}{d_jd_k\sin\theta_i}=\frac{d_i}{2A}, \tag{8}$$

where $A=\frac{1}{2}d_jd_k\sin\theta_i$ is the area of the triangle. Similarly

$$\frac{\partial}{\partial d_j}\left(d_j^2+d_k^2-d_i^2\right)=\frac{\partial}{\partial d_j}(2d_jd_k\cos\theta_i)$$
$$2d_j=2d_k\cos\theta_i-2d_jd_k\sin\theta_i\frac{\partial\theta_i}{\partial d_j}$$
$$2A\frac{\partial\theta_i}{\partial d_j}=d_k\cos\theta_i-d_j=-d_i\cos\theta_k.$$

We get

$$\frac{\partial\theta_i}{\partial d_j}=-\frac{d_i\cos\theta_k}{2A}.\quad\square$$

Lemma 4.5. *In a Euclidean triangle, let $u_i=\frac{1}{2}d_i^2$ and $u_j=\frac{1}{2}d_j^2$ then*

$$\frac{\partial\cot\theta_i}{\partial u_j}=\frac{\partial\cot\theta_j}{\partial u_i}. \tag{9}$$

Proof.

$$\begin{aligned}\frac{\partial\cot\theta_i}{\partial u_j}&=\frac{1}{d_j}\frac{\partial\cot\theta_i}{\partial d_j}=-\frac{1}{d_j}\frac{1}{\sin^2\theta_i}\frac{\partial\theta_i}{\partial d_j}\\&=\frac{1}{d_j}\frac{1}{\sin^2\theta_i}\frac{d_i\cos\theta_k}{2A}=\frac{d_i^2}{\sin^2\theta_i}\frac{\cos\theta_k}{2Ad_id_j}\\&=\frac{4R^2}{2A}\frac{\cos\theta_k}{d_id_j},\end{aligned} \tag{10}$$

where R is the radius of the circumcircle of the triangle. The righthand side of Eq. 10 is symmetric with respect to the indices i and j. $\square$

In the following, we introduce a differential form. We are going to use them for proving that the integration involved in computing energy is independent of paths. This follows from the fact that the forms which are integrated are closed, and the integration domain is simply connected.

Corollary 4.6. *The differential form*

$$\omega=\cot\theta_idu_i+\cot\theta_jdu_j+\cot\theta_kdu_k \tag{11}$$

is a closed 1-form.

Proof. By the above Lemma 4.5 regarding symmetry,

$$d\omega=\left(\frac{\partial\cot\theta_j}{\partial u_i}-\frac{\partial\cot\theta_i}{\partial u_j}\right)du_i\wedge du_j+\left(\frac{\partial\cot\theta_k}{\partial u_j}-\frac{\partial\cot\theta_j}{\partial u_k}\right)du_j\wedge du_k+\left(\frac{\partial\cot\theta_i}{\partial u_k}-\frac{\partial\cot\theta_k}{\partial u_i}\right)du_k\wedge du_i=0.\ \square$$

Definition 4.7. (Admissible Metric Space) Let $u_i=\frac{1}{2}d_i^2$, the admissible metric space is defined as

$$\Omega_u:=\left\{(u_i,u_j,u_k)\middle|\left(\sqrt{u_i},\sqrt{u_j},\sqrt{u_k}\right)\in E_{\mathbf{d}}(2),\ \ u_i+u_j+u_k=3\right\}.$$

Lemma 4.8. *The admissible metric space Ω_u is a convex domain in $\mathbb{R}^3$.*

Proof. Suppose $(u_i, u_j, u_k) \in \Omega_u$ and $(\tilde{u}_i, \tilde{u}_j, \tilde{u}_k) \in \Omega_u$, then from

$$\sqrt{u_i} + \sqrt{u_j} > \sqrt{u_k},$$

we get

$$u_i + u_j + 2\sqrt{u_i u_j} > u_k.$$

Define

$$\left(u_i^\lambda, u_j^\lambda, u_k^\lambda\right) = \lambda(u_i, u_j, u_k) + (1-\lambda)(\tilde{u}_i, \tilde{u}_j, \tilde{u}_k),$$

where $0 < \lambda < 1$. Then

$$\begin{aligned} u_i^\lambda u_j^\lambda &= (\lambda u_i + (1-\lambda)\tilde{u}_i)(\lambda u_j + (1-\lambda)\tilde{u}_j) \\ &= \lambda^2 u_i u_j + (1-\lambda)^2 \tilde{u}_i \tilde{u}_j + \lambda(1-\lambda)(u_i \tilde{u}_j + u_j \tilde{u}_i) \\ &\geqslant \lambda^2 u_i u_j + (1-\lambda)^2 \tilde{u}_i \tilde{u}_j + 2\lambda(1-\lambda)\sqrt{u_i u_j \tilde{u}_i \tilde{u}_j} \\ &= (\lambda\sqrt{u_i u_j} + (1-\lambda)\sqrt{\tilde{u}_i \tilde{u}_j})^2. \end{aligned}$$

It follows

$$\begin{aligned} u_i^\lambda + u_j^\lambda + 2\sqrt{u_i^\lambda u_j^\lambda} &\geqslant \lambda\left(u_i + u_j + 2\sqrt{u_i u_j}\right) \\ &\quad + (1-\lambda)\left(\tilde{u}_i + \tilde{u}_j + 2\sqrt{\tilde{u}_i \tilde{u}_j}\right) \\ &> \lambda u_k + (1-\lambda)\tilde{u}_k = u_k^\lambda. \end{aligned}$$

This shows $(u_i^\lambda, u_j^\lambda, u_k^\lambda) \in \Omega_u$. □

Similarly, we define the edge weight space as follows.

Definition 4.9. (Edge Weight Space) The edge weights of a Euclidean triangle form the edge weight space

$$\Omega_\theta = \{(\cot\theta_i, \cot\theta_j, \cot\theta_k) | 0 < \theta_i, \theta_j, \theta_k < \pi, \theta_i + \theta_j + \theta_k = \pi\}.$$

Note that

$$\cot\theta_k = -\cot(\theta_i + \theta_j) = \frac{1 - \cot\theta_i \cot\theta_j}{\cot\theta_i + \cot\theta_j}.$$

Lemma 4.10. *The energy* $E : \Omega_u \to \mathbb{R}$

$$E(u_i, u_j, u_k) = \int_{(1,1,1)}^{(u_i,u_j,u_k)} \cot\theta_i \, d\tau_i + \cot\theta_j \, d\tau_j + \cot\theta_k \, d\tau_k \qquad (12)$$

is well defined on the admissible metric space Ω_u *and is convex.*

Proof. According to Corollary 4.6, the differential form is closed. Furthermore, the admissible metric space Ω_u is a simply connected domain and the differential form is exact. Therefore, the integration is path independent, and the energy function is well defined.

Then we compute the Hessian matrix of the energy,

$$H = -\frac{2R^2}{A}\begin{bmatrix} \frac{1}{d_i^2} & -\frac{\cos\theta_k}{d_i d_j} & -\frac{\cos\theta_j}{d_i d_k} \\ -\frac{\cos\theta_k}{d_j d_i} & \frac{1}{d_j^2} & -\frac{\cos\theta_i}{d_j d_k} \\ -\frac{\cos\theta_j}{d_k d_i} & -\frac{\cos\theta_i}{d_k d_j} & \frac{1}{d_k^2} \end{bmatrix} = -\frac{2R^2}{A}\begin{bmatrix} (\boldsymbol{\eta}_i, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_i, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_i, \boldsymbol{\eta}_k) \\ (\boldsymbol{\eta}_j, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_j, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_j, \boldsymbol{\eta}_k) \\ (\boldsymbol{\eta}_k, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_k, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_k, \boldsymbol{\eta}_k) \end{bmatrix}.$$

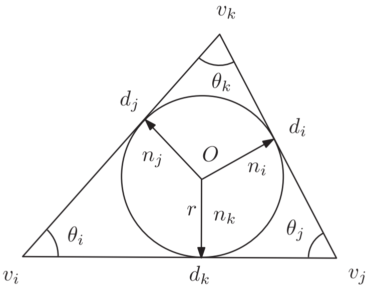

Fig. 2. The geometric interpretation of the Hessian matrix. The in circle of the triangle is centered at O, with radius r. The perpendiculars n_i, n_j and n_k are from the incenter of the triangle and orthogonal to the edge e_i, e_j and e_k respectively.

As shown in Fig. 2, $d_i\mathbf{n}_i + d_j\mathbf{n}_j + d_k\mathbf{n}_k = 0$,

$$\boldsymbol{\eta}_i = \frac{\mathbf{n}_i}{rd_i}, \ \boldsymbol{\eta}_j = \frac{\mathbf{n}_j}{rd_j}, \ \boldsymbol{\eta}_k = \frac{\mathbf{n}_k}{rd_k},$$

where r is the radius of the incircle of the triangle.

Suppose $(x_i, x_j, x_k) \in \mathbb{R}^3$ is a vector in $\mathbb{R}^3$, then

$$[x_i, x_j, x_k]\begin{bmatrix} (\boldsymbol{\eta}_i, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_i, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_i, \boldsymbol{\eta}_k) \\ (\boldsymbol{\eta}_j, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_j, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_j, \boldsymbol{\eta}_k) \\ (\boldsymbol{\eta}_k, \boldsymbol{\eta}_i) & (\boldsymbol{\eta}_k, \boldsymbol{\eta}_j) & (\boldsymbol{\eta}_k, \boldsymbol{\eta}_k) \end{bmatrix}\begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix} = \|x_i\boldsymbol{\eta}_i + x_j\boldsymbol{\eta}_j + x_k\boldsymbol{\eta}_k\|^2 \geqslant 0.$$

If the result is zero, then $(x_i, x_j, x_k) = \lambda(u_i, u_j, u_k), \lambda \in \mathbb{R}$. That is the null space of the Hessian matrix. In the admissible metric space Ω_u, $u_i + u_j + u_k = C (C = 3)$, then $du_i + du_j + du_k = 0$. If (du_i, du_j, du_k) belongs to the null space, then $(du_i, du_j, du_k) = \lambda(u_i, u_j, u_k)$, therefore, $\lambda(u_i + u_j + u_k) = 0$. Because u_i, u_j, u_k are positive, $\lambda = 0$. This shows the null space of Hessian matrix is orthogonal to the tangent space of Ω_u. Therefore, the Hessian matrix is positive definite on the tangent space. In summary, the energy on Ω_u is convex. □

Theorem 4.11. *The mapping* $\nabla E : \Omega_u \to \Omega_\theta$, $(u_i, u_j, u_k) \to (\cot\theta_i, \cot\theta_j, \cot\theta_k)$ *is a diffeomorphism.*

Proof. The energy $E(u_i, u_j, u_k)$ is a convex function defined on the convex domain Ω_u. According to Lemma 4.3,

$$\nabla E : (u_i, u_j, u_k) \to (\cot\theta_i, \cot\theta_j, \cot\theta_k)$$

is a diffeomorphism. □

4.3. Rigidity for the whole mesh

In this section, we consider the whole polyhedral surface.

4.3.1. Closed surfaces

Given a polyhedral surface $(S, T, \mathbf{d})$, the admissible metric space and the edge weight have been defined in Section 3 respectively.

Lemma 4.12. *The admissible metric space* Ω_u *is convex.*

Proof. For a triangle $\{i,j,k\} \in F$, define
$\Omega_u^{ijk} := \{(u_i, u_j, u_k) | (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_{\mathbf{d}}(2)\}$.

Similar to the proof of Lemma 4.8, Ω_u^{ijk} is convex. The admissible metric space for the mesh is

$$\Omega_u = \bigcap_{\{i,j,k\}\in F} \Omega_u^{ijk} \bigcap \left\{ (u_1, u_2, \ldots, u_m) \middle| \sum_{k=1}^{m} u_k = m \right\},$$

the intersection Ω_u is still convex. □

Definition 4.13. (Differential Form) The differential form ω defined on Ω_u is the summation of the differential form on each face,

$$\omega = \sum_{\{i,j,k\}\in F} \omega_{ijk} = \sum_{i=1}^{m} 2 w_i du_i,$$

where ω_{ijk} is given in Eq. 11 in Corollary 4.6, w_i is the edge weight on e_i, m is the number of edges.

Lemma 4.14. *The differential form ω is a closed 1-form.*

Proof. According to Corollary 4.6

$$d\omega = \sum_{\{i,j,k\}\in F} d\omega_{ijk} = 0. \quad \square$$

Lemma 4.15. *The energy function*

$$E(u_1, u_2, \ldots, u_m) = \sum_{\{i,j,k\}\in F} E_{ijk}(u_1, u_2, \ldots, u_m) = \int_{(1,1,\ldots,1)}^{(u_1,u_2,\ldots,u_m)} \sum_{i=1}^{n} w_i\, du_i$$

is well defined and convex on Ω_u, where E_{ijk} is the energy on the face, defined in Eq. 12.

Proof. For each face $\{i,j,k\} \in F$, the Hessian matrices of E_{ijk} is semi-positive definite, therefore, the Hessian matrix of the total energy E is semi-positive definite.

Similar to the proof of Lemma 4.10, the null space of the Hessian matrix H is

$$kerH = \{\lambda(d_1, d_2, \ldots, d_m), \lambda \in \mathbb{R}\}.$$

The tangent space of Ω_u at $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ is denoted by $T\Omega_u(\mathbf{u})$. Assume $(du_1, du_2, \ldots, du_m) \in T\Omega_u(\mathbf{u})$, then from $\sum_{i=1}^{m} u_i = m$, we get $\sum_{i=1}^{m} du_m = 0$. Therefore,

$$T\Omega_u(\mathbf{u}) \cap KerH = \{0\},$$

hence H is positive definite restricted on $T\Omega_u(\mathbf{u})$. So the total energy E is convex on Ω_u. □

Theorem 4.16. *The mapping on a closed Euclidean polyhedral surface ∇E: $\Omega_u \to \mathbb{R}^m, (u_1, u_2, \ldots, u_m) \to (w_1, w_2, \ldots, w_m)$ is a smooth embedding.*

Proof. The admissible metric space Ω_u is convex as shown in Lemma 4.12, the total energy is convex as shown in Lemma 4.15. According to Lemma 4.3, ∇E is a smooth embedding. □

4.3.2. Open surfaces

By the double covering technique [11], we can convert a polyhedral surface with boundaries to a closed surface. First, let $(\overline{S}, \overline{T})$ be a copy of (S, T), then we reverse the orientation of each face in $\overline{M}$, and glue two surfaces S and $\overline{S}$ along their corresponding boundary edges, the resulting triangulated surface is a closed one. We get the following corollary.

Corollary 4.17. *The mapping on a Euclidean polyhedral surface with boundaries $\nabla E : \Omega_u \to \mathbb{R}^m, (u_1, u_2, \ldots, u_m) \to (w_1, w_2, \ldots, w_m)$ is a smooth embedding.*

Surely, the cotangent edge weights can be uniquely obtained from the discrete heat kernel. By combining Theorem 4.16 and Corollary 4.17, we obtain the main Theorem 3.5, *Global Rigidity Theorem*, of this work.

5. Numerical Experiments

From above theoretic deduction, we can design the algorithm to compute discrete metric with user prescribed edge weights.

Problem. Let (S,T) be a triangulated surface, $\bar{\mathbf{w}}(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)$ are the user prescribed edge weights. The problem is to find a discrete metric $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, such that this metric $\bar{\mathbf{u}}$ induces the desired edge weight $\mathbf{w}$.

The algorithm is based on the following theorem.

Theorem 5.1. *Suppose (S, T) is a triangulated surface. If there exists an $\bar{\mathbf{u}} \in \Omega_u$, which induces $\bar{\mathbf{w}}$, then $\mathbf{u}$ is the unique global minimum of the energy*

$$E(\mathbf{u}) = \int_{(1,1,\ldots,1)}^{(u_1,u_2,\ldots,u_n)} \sum_{i=1}^{n} (\bar{w}_i - w_i) du_i. \tag{13}$$

Proof. The gradient of the energy $\nabla E(\mathbf{u}) = \bar{\mathbf{w}} - \mathbf{w}$, and since $\nabla E(\bar{\mathbf{u}}) = 0$, therefore $\bar{\mathbf{u}}$ is a critical point. The Hessian matrix of $E(\mathbf{u})$ is positive definite, the domain Ω_u is convex, therefore $\bar{\mathbf{u}}$ is the unique global minimum of the energy. □

In our numerical experiments, as shown in Fig. 3, we tested surfaces with different topologies, with different genus, with or without boundaries. All discrete polyhedral surfaces are triangle meshes scanned from real objects. Because the meshes are embedded in $\mathbb{R}^3$, they have induced Euclidean metric, which are used as the desired metric $\bar{\mathbf{u}}$. From the induced Euclidean metric, the desired edge weight $\bar{\mathbf{w}}$ can be directly computed. Then we set the initial discrete metric to be the constant metric $(1, 1, \ldots, 1)$. By optimizing the energy in Eq. 13, we can reach the global minimum, and recovered the desired metric, which differs from the induced Euclidean metric by a scaling.

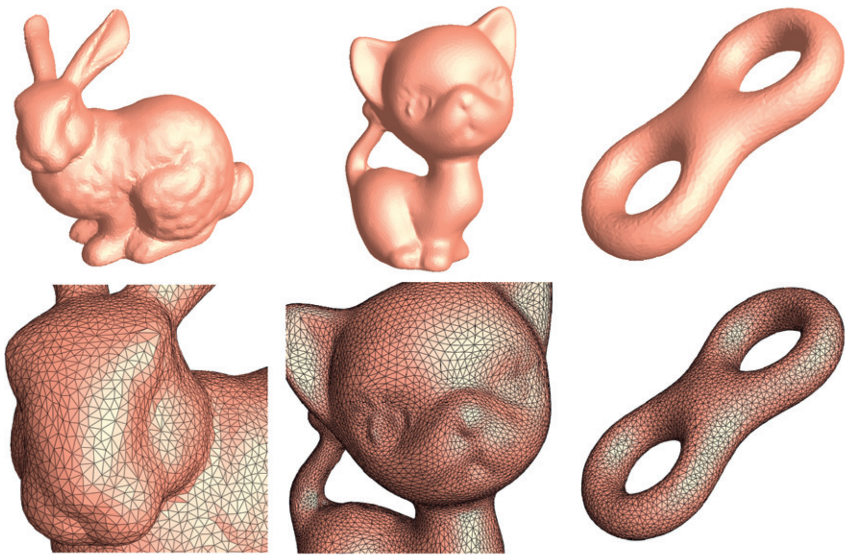

Fig. 3. Euclidean polyhedral surfaces used in the experiments.

In Fig. 3, the first row shows three examples of surfaces of genus zero, genus one, genus two, respectively, which are embedded in $\mathbb{R}^3$; the second row shows the corresponding triangulated meshing structures.

6. Conclusion and future work

This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete heat kernel and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other. We prove that the edge lengths can be determined by the edge weights unique up to a scaling using the variational approach, and design the computational algorithm that finds the unique metric on a triangle mesh from a discrete Laplace–Beltrami operator matrix.

We conjecture that the Main Theorem 3.5 holds for arbitrary dimensional Euclidean polyhedral manifolds, which means discrete Laplace–Beltrami operator (or equivalently the discrete heat kernel) and the discrete metric for any dimensional Euclidean polyhedral manifold are mutually determined by each other. On the other hand, we will explore the possibility to establish the same theorem for different types of discrete Laplace–Beltrami operators as in [10]. Also, we will explore further on the sufficient and necessary conditions for a given set of edge weights to be admissible.

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References

[1] M. Belkin, J. Sun, Y. Wang, Discrete Laplace operator on meshed surfaces, in: SoCG '08: Proceedings of the Twenty-fourth Annual Symposium on Computational Geometry, 2008, pp. 278–287.
[2] M. Ben-chen, C. Gotsman, G. Bunin, Conformal flattening by curvature prescription and metric scaling, Eurographics 27 (2008).
[3] A.I. Bobenko, B.A. Springborn, Variational principles for circle patterns and Koebe's theorem, Transactions of the American Mathematical Society 356 (2004) 659–689.
[4] P.L. Bowers, M.K. Hurdal, Planar conformal mapping of piecewise flat surfaces, in: Visualization and Mathematics III, Springer, Berlin, 2003, pp. 3–34.
[5] B. Chow, F. Luo, Combinatorial Ricci flows on surfaces, Journal of Differential Geometry 63 (1) (2003) 97–129.
[6] C. de Verdiere Yves, Un principe variationnel pour les empilements de cercles, Inventiones Mathematicae 104 (3) (1991) 655–669.
[7] T.K. Dey, P. Ranjan, Y. Wang, Convergence, stability, and discrete approximation of Laplace spectra, in: Proceedings of the ACM/SIAM Symposium on Discrete Algorithms (SODA) 2010, 2010, pp. 650–663.
[8] J. Dodziuk, Finite-difference approach to the Hodge theory of harmonic forms, American Journal of Mathematics 98 (1) (1976) 79–104.
[9] M.S. Floater, K. Hormann, Surface parameterization: a tutorial and survey, in: Advances in Multiresolution for Geometric Modelling, Springer, 2005, pp. 157–186.
[10] D. Glickenstein, A monotonicity property for weighted Delaunay triangulations, Discrete Computational Geometry 38 (4) (2007) 651–664.
[11] X. Gu, S.-T. Yau, Global conformal parameterization, in: Symposium on Geometry Processing, 2003, pp. 127–137.
[12] M. Jin, J. Kim, F. Luo, X. Gu, Discrete surface Ricci flow, IEEE TVCG 14 (5) (2008) 1030–1043.
[13] M. Jin, F. Luo, X. Gu, Computing surface hyperbolic structure and real projective structure, in: SPM '06: Proceedings of the 2006 ACM Symposium on Solid and Physical Modeling, 2006, pp. 105–116.
[14] L. Kharevych, B. Springborn, P. Schröder, Discrete conformal mappings via circle patterns, ACM Transactions on Graphics 25 (2) (2006) 412–438.
[15] V. Kraevoy, A. Sheffer, Cross-parameterization and compatible remeshing of 3D models, ACM Transactions on Graphics 23 (3) (2004) 861–869.
[16] B. Lévy, Laplace–Beltrami eigenfunctions towards an algorithm that "understands" geometry, in: SMI '06: Proceedings of the IEEE International Conference on Shape Modeling and Applications 2006, 2006, pp. 13.

[17] F. Luo, Combinatorial Yamabe flow on surfaces, Communications in Contemporary Mathematics 6 (5) (2004) 765–780.
[18] F. Luo, X.D. Gu, J. Dai, Variational Principles for Discrete Surfaces, International Press, Somerville, MA, 2008.
[19] M. Ovsjanikov, J. Sun, L.J. Guibas, Global intrinsic symmetries of shapes, Computer Graphics Forum 27 (5) (2008) 1341–1348.
[20] U. Pinkall, K. Polthier, Computing discrete minimal surfaces and their conjugates, Experimental Mathematics 2 (1) (1993) 15–36.
[21] M. Reuter, F.-E. Wolter, N. Peinecke, Laplace–Beltrami spectra as 'shape-DNA' of surfaces and solids, Computer Aided Design 38 (4) (2006) 342–366.
[22] I. Rivin, Euclidean structures on simplicial surfaces and hyperbolic volume, Annals of Mathematics 139 (3) (1994).
[23] R.M. Schoen, S.-T. Yau, Lectures on differential geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. I, International Press, Somerville, MA, 1994.
[24] O. Sorkine, Differential representations for mesh processing, Computer Graphics Forum 25 (4) (2006) 789–807.
[25] B. Springborn, P. Schröder, U. Pinkall, Conformal equivalence of triangle meshes, ACM Transactions on Graphics 27 (3) (2008) 1–11.
[26] S. Rosenberg, The Laplacian on a Riemannian manifold, Number 31 in London Mathematical Society Student Texts, Cambridge University Press, 1998.
[27] J. Sun, M. Ovsjanikov, L.J. Guibas, A concise and provably informative multi-scale signature based on heat diffusion, Computer Graphics Forum 28 (5) (2009) 1383–1392.
[28] W.P. Thurston, Geometry and Topology of Three-Manifolds, Lecture Notes at Princeton University, 1980.
[29] M. Wardetzky, S. Mathur, F. Kälberer, E. Grinspun, Discrete Laplace operators: no free lunch, in: Proceedings of the fifth Eurographics symposium on Geometry processing, Eurographics Association, 2007, pp. 33–37.
[30] G. Xu, Discrete Laplace–Beltrami operators and their convergence, Computer Aided Geometric Design 21 (8) (2004) 767–784.
[31] H. Zhang, O. van Kaick, R. Dyer, Spectral mesh processing, Computer Graphics Forum 29 (6) (2010) 1865–1894.