

A DILOGARITHM IDENTITY ON MODULI SPACES OF CURVES

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To Michael Freedman on the occasion of his sixtieth birthday

ABSTRACT. We establish an identity for compact hyperbolic surfaces with or without boundary whose terms depend on the dilogarithms of the lengths of simple closed geodesics in all 3-holed spheres and 1-holed tori in the surface.

1. INTRODUCTION

1.1. Statement of results. In [6], McShane established a remarkable identity for the lengths of simple closed geodesics in hyperbolic surfaces with cusp ends. Since then there have been many generalizations of McShane's identity, for example, to hyperbolic surfaces with geodesic boundaries [8], [10] and cone singularities [10]. Mirzakhani also found fantastic applications of these identities to the computation of the volumes of moduli spaces of bordered Riemann surfaces. There has been much research since then towards finding a McShane type identity for closed hyperbolic surfaces. In [7] and [10], McShane and Tan et al established such an identity for closed hyperbolic surfaces of genus 2. However, the techniques used there do not generalize as they depend crucially on the fact that every genus 2 surface admits a hyperelliptic involution. The goal of this paper is to establish a McShane type identity for simple closed geodesics on *any* closed hyperbolic surface. Our result for the genus 2 case is different from that given in [7] or [10]. The generalization of our identity to surfaces with cusps or geodesic boundary also differ from those in [6] or [8]. This seems to suggest that there are possibilities of producing many different McShane type identities for hyperbolic surfaces. We expect that the identity found here will have applications towards the study of the moduli space of curves.

The identity that we produce involves the dilogarithm of the lengths of simple closed geodesics in all 1-holed tori and 3-holed spheres in the surface. Our work is motivated by [6], [8], [10] and [1]. In [1], Bridgeman considers compact hyperbolic surfaces with non-empty geodesic boundary and geodesic paths starting and ending at the boundary. Our approach is similar to that of [1] in two aspects. First, we consider the unit tangent bundle instead of the surface itself, and second the identity obtained involves dilogarithm functions. In fact, we use Bridgeman's work in producing the main identity. The main idea in arriving at our identity is also closely related to the interpretation and proof of McShane's identity in [10].

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In this paper, we consider oriented surfaces. For a hyperbolic surface F , a compact embedded subsurface $\Sigma \subset F$ is said to be *geometric* if the boundaries of Σ are geodesic and *proper* if the inclusion map $i : \Sigma \rightarrow F$ is injective. Furthermore call a surface *simple* if it is a 3-holed sphere or one hole torus (both of them have Euler characteristic -1). Our main result is the following:

Theorem 1.1. *Let F be a closed hyperbolic surface of genus $g \geq 2$. There exist functions f and g involving the dilogarithm of the lengths of the simple closed geodesics in a 3-holed sphere or 1-holed torus, such that*

$$\sum_P f(P) + \sum_T g(T) = 8\pi^2(g-1) \quad (1)$$

where the first sum is over all properly embedded geometric 3-holed spheres $P \subset F$, the second sum is over all properly embedded geometric 1-holed tori $T \subset F$.

The definitions of the functions f and g in the identities are given in §2. The right-hand-side in (1) is the volume of the unit tangent bundle over the surface F .

Remarks:

- (1) Each T in the second summand can be cut along simple closed geodesics into a 3-holed sphere. These 3-holed spheres are not properly embedded.
- (2) Bridgman's identity [1] does not extend to closed hyperbolic surfaces without boundary. Nonetheless, the terms involved in our identity are similar to those of Bridgman's in the sense that they involve the Roger's dilogarithm function.
- (3) Our identity can be thought of as a hybrid of both the McShane and Bridgman identities. It can also be thought of as an identity on the moduli space \mathcal{M}_g of curves, rather than the Teichmüller space \mathcal{T}_g , as the mapping class group has the effect of permuting the terms in the summands.
- (4) The theorem can be extended to hyperbolic surfaces with geodesic boundary and cusp ends. The expression is more complicated though. See theorem 1.2 below.
- (5) We have been informed by G. McShane that he and D. Calegari have recently obtained results similar to theorem 1.1.

1.2. Basic idea of the proof. The key idea is to decompose the unit tangent bundle $S(F)$ of a closed hyperbolic surface F according to, and indexed by, the properly embedded geometric 1-holed tori and 3-holed spheres in F . The decomposition is measure theoretic in the sense that we will ignore a measure zero set in $S(F)$. Here is the way to produce the decomposition. For a unit tangent vector $v \in S(F)$, consider the unit speed geodesic rays $g_v^+(t)$ and $g_v^-(t)$ ($t \geq 0$) determined by $\pm v$. If the vector v is generic, then both rays will self intersect transversely by the ergodicity of the geodesic flow. This vector v will determine a canonical graph $G(v)$ as follows. Consider the path $A_t = g_v^-([0, t]) \cup g_v^+([0, t])$ for $t > 0$ obtained by letting the geodesic rays g_v^- and g_v^+ grow at equal speed from time 0 to t . Let $t_1 > 0$ be the smallest positive number so that A_{t_1} is not a simple arc. Say $g_v^+(t_1) \in g_v^-([0, t_1]) \cup g_v^+([0, t_1])$. Next, let $t_2 \geq t_1$ be the next smallest time so that $g_v^-(t_2) \in g_v^-([0, t_2]) \cup g_v^+[0, t_1]$.

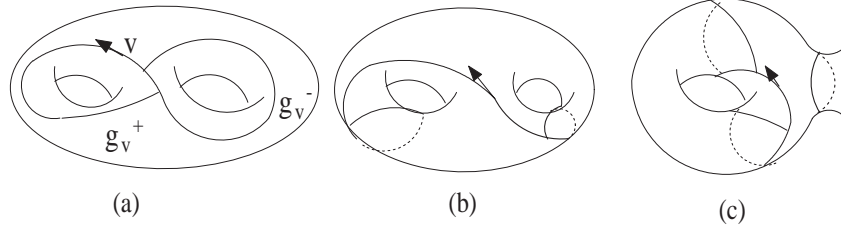


FIGURE 1. creation of spine

The union $g_v^-[0, t_2] \cup g_v^+[0, t_1]$ is the graph, denoted by $G(v)$ associated to v . Its Euler characteristic is -1 . The graph $G(v)$ is contained in a unique properly embedded geometric subsurface $\Sigma(v)$ which is either a 1-holed torus or a 3-holed sphere in F . Furthermore either the graph $G(v)$ is a deformation retract of $\Sigma(v)$, or $\Sigma(v)$ is a 1-holed torus so that $\Sigma(v) - G(v)$ is a union of two annuli (figure 1(c)). By abuse of notation, we will say in this case that $G(v)$ is also a spine for $\Sigma(v)$. In this way, we produce a decomposition of the unit tangent bundle $S(F)$. Namely, generically, each vector $v \in S(F)$ is in a unique geometric 1-holed torus T or a 3-holed sphere P so that $G(v)$ is a spine for the subsurface. It remains to calculate for a simple hyperbolic surface Σ the volume of the set of all unit tangent vectors v in Σ so that $G(v)$ is a spine for Σ . It turns out the volume of this set can be explicitly calculated using the dilogarithm and the lengths of simple closed geodesics in Σ .

1.3. Extension to non-closed hyperbolic surfaces. The identity can be extended to finite area hyperbolic surfaces $F_{g,n}^r$ of genus g with n geodesic boundary components and r cusps, by modifying the function f for 3-holed spheres P whose boundaries become peripheral.

Let $\mathcal{F}_{g,n}^r$ denote the set of all marked hyperbolic structures a surface $\Sigma_{g,n}^r$ of genus g with n boundary components and r punctures so that the boundaries are geodesics and punctures are cusps. Let $F_{g,n} = F_{g,n}^0$ and $\mathcal{F}_{g,n} = \mathcal{F}_{g,n}^0$.

We have:

Theorem 1.2. *Let $F \in \mathcal{F}_{g,n}^0$ be a hyperbolic surface with geodesic boundaries so that its Euler characteristic is strictly less than -1 . There exist functions $\hat{f}, \bar{f}, f : \mathcal{F}_{0,3} \rightarrow \mathbb{R}_+$, $g : \mathcal{F}_{1,1} \rightarrow \mathbb{R}_+$ such that*

$$\sum_{\hat{P}} \hat{f}(\hat{P}) + \sum_{\bar{P}} \bar{f}(\bar{P}) + \sum_P f(P) + \sum_T g(T) = 4\pi^2(2g - 2 + n) \quad (2)$$

where the first sum is over all properly embedded geometric pairs of pants $\hat{P} \subset F$ with exactly one boundary component in ∂F , the second is over all properly embedded geometric pairs of pants $\bar{P} \subset F$ with exactly two boundary components in ∂F , the third sum is over all properly embedded geometric pairs of pants $P \subset F$ such that $\partial P \cap \partial F = \emptyset$, the fourth sum is over all properly embedded geometric one holed tori $T \subset F$.

Furthermore, if lengths of k boundary components of $F_{g,n}$ tend to zero, then each term and each summation in (2) converge. The limit is the identity for all hyperbolic surfaces $F_{g,n-k}^k$ of genus g with $n - k$ geodesic boundary and k cusps.

The right hand side of (2) is the volume of the unit tangent bundle over F .

1.4. Plan of the paper. In section 2, we define the functions f, g, \hat{f}, \bar{f} in (1) and (2). In section 3, we describe how to decompose the unit tangent bundle $S(F)$ of the surface F by showing how each $v \in S(F)$ generates a spine for a simple subsurface $\Sigma \subset F$. In section 4, for simple subsurfaces $\Sigma \subset F$, we identify the subset of the unit tangent vectors in $S(\Sigma)$ which generate spines for Σ with a subset of $S(\mathbb{H}^2)$. In section 5, we derive the formula for the measure of the set studied in section 4, thereby giving the formulas for f, g, \hat{f} and \bar{f} . Finally, in the appendix, we first give an interpretation of the pentagon relation for the dilogarithm function in terms of lengths of right-angled hyperbolic pentagons and then explain why different rules for generating the spines $G(v)$ will result in the same theorem 1.1.

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2. DEFINITIONS OF THE FUNCTIONS

In this section we define f, g, \hat{f}, \bar{f} in the identities (1) and (2).

2.1. Dilogarithm and Roger's dilogarithm functions. We first recall the dilogarithm function Li_2 and the Roger's dilogarithm function \mathcal{L} . See [9] for more details.

The dilogarithm function Li_2 is defined for $|z| < 1$ by the Taylor series

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (3)$$

so that for $x \in \mathbf{R}$ with $x < 1$, $\text{Li}_2(x) = -\int_0^x \frac{\log(1-z)}{z} dz$.

The Rogers \mathcal{L} -function is defined by

$$\mathcal{L}(x) = \text{Li}_2(x) + \frac{1}{2} \log(|x|) \log(1-x) \quad (4)$$

so that $\mathcal{L}'(z) = -\frac{1}{2} \left(\frac{\log(1-z)}{z} + \frac{\log(z)}{1-z} \right)$ and $\mathcal{L}(0) = 0$. It satisfies $\mathcal{L}(x) + \mathcal{L}(1-x) = \pi^2/6$ for $0 < x < 1$. The fundamental identity which characterizes the function $\mathcal{L}(x)$ is the following pentagon relation, for $x, y \in (0, 1)$,

$$\mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(1-xy) + \mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}. \quad (5)$$

A geometric interpretation of (5) in terms of the lengths of right-angle hexagon is given in the appendix.

2.2. Length invariants of 3-holed spheres. Let $P \in \mathcal{F}_{0,3}$ be a hyperbolic 3-holed sphere with geodesic boundaries L_1, L_2, L_3 . For $\{i, j, k\} = \{1, 2, 3\}$, let M_i be the shortest geodesic arc between L_j and L_k , and B_i the shortest non-trivial geodesic arc from L_i to itself. Note that M_i and B_i are orthogonal to ∂P . See figure 2. We define:

- l_i to be the length of L_i .
- m_i to be the length of M_i .
- p_i to be the length of B_i .

Note that P is decomposed into two right-angled hyperbolic hexagons with cyclically ordered side-lengths $\{\frac{l_1}{2}, m_3, \frac{l_2}{2}, m_1, \frac{l_3}{2}, m_2\}$ by cutting along the M_i . Furthermore, cutting along each B_i decomposes the two hexagons into 2 right-angled pentagons. See figure 2.

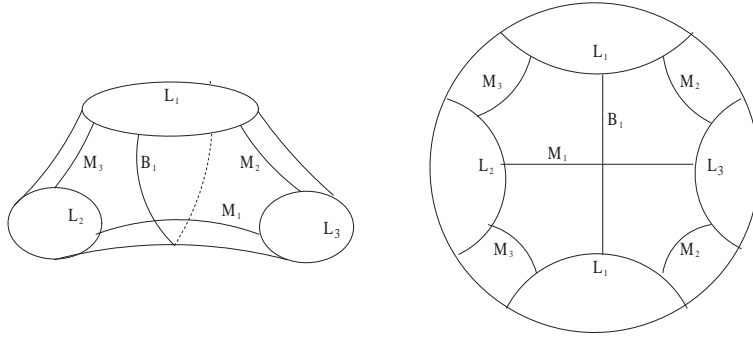


FIGURE 2. 3-holed sphere

The sine and cosine rules for right angled hexagons and pentagons say that for $\{i, j, k\} = \{1, 2, 3\}$,

$$\frac{\sinh m_i}{\sinh(l_i/2)} = \frac{\sinh m_j}{\sinh(l_j/2)} = \frac{\sinh m_k}{\sinh(l_k/2)} \quad (6)$$

$$\cosh m_i \sinh(l_j/2) \sinh(l_k/2) = \cosh(l_i/2) + \cosh(l_j/2) \cosh(l_k/2) \quad (7)$$

$$\cosh(p_k/2) = \sinh(l_i/2) \sinh m_j \quad (8)$$

In particular, all lengths m_i, p_i can be expressed in terms of l_1, l_2 and l_3 .

2.3. Length invariants of 1-holed tori. Let $T \in \mathcal{F}_{1,1}$ be a hyperbolic 1-holed torus with boundary component C . For any non-boundary parallel simple closed geodesic A on T , cutting T along A gives a hyperbolic pair of pants P_A with boundary geodesics C, A^+ and A^- , see figure 3. Let

- c be the length of C
- a be the length of A
- m_A be the shortest distance between C and A^+ in P_A (or A^-)
- p_A be the length of the shortest non-trivial geodesic arc from C to C in P_A
- q_A be the length of the shortest non-trivial path from A^+ to A^- in P_A .

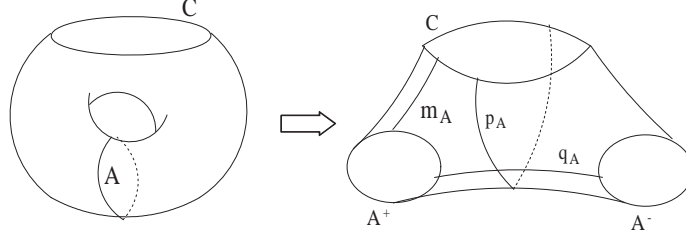


FIGURE 3. cutting 1-holed torus into a 3-holed sphere

2.4. The main functions and the identity. The functions f and g in theorem 1.1 are given as follows. We first define the lasso function $La(l, m)$ to be

$$La(l, m) = \mathcal{L}(y) - \mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) \quad (9)$$

where $x = e^{-l}$ and $y = \tanh^2(m/2)$. The three terms above appear in (5).

Now for $P \in \mathcal{F}_{0,3}$ with length invariants l_i , m_i and p_i , as given in §2.2, and $1 \leq i, j \leq 3$, we define

$$f(P) := 4\pi^2 - 8 \left\{ \sum_{i=1}^3 \left(\mathcal{L}\left(\frac{1}{\cosh^2(m_i/2)}\right) + \mathcal{L}\left(\frac{1}{\cosh^2(p_i/2)}\right) \right) + \sum_{i \neq j} La(l_i, m_j) \right\} \quad (10)$$

$$= 8 \left[\sum_{i \neq j} \left(\mathcal{L}\left(\frac{1-x_i}{1-x_i y_j}\right) - \mathcal{L}\left(\frac{1-y_j}{1-x_i y_j}\right) \right) - \sum_{k=1}^3 \left(\mathcal{L}(y_k) + \mathcal{L}\left(\frac{1}{\cosh^2(p_k/2)}\right) \right) \right] \quad (11)$$

$$= 4 \sum_{i \neq j} \left[2\mathcal{L}\left(\frac{1-x_i}{1-x_i y_j}\right) - 2\mathcal{L}\left(\frac{1-y_j}{1-x_i y_j}\right) - \mathcal{L}(y_j) - \mathcal{L}\left(\frac{(1-y_j)^2 x_i}{(1-x_i)^2 y_j}\right) \right] \quad (12)$$

where $x_i = e^{-l_i}$, $y_i = \tanh^2(m_i/2)$ and by (8), $\frac{1}{\cosh^2(p_k/2)} = \frac{(1-y_i)^2 x_i}{(1-x_i)^2 y_j}$ for $\{i, j, k\} = \{1, 2, 3\}$.

For $T \in \mathcal{F}_{1,1}$ with boundary geodesic C , we define

$$g(T) := 4\pi^2 - 8 \sum_A \left(\mathcal{L}\left(\frac{1}{\cosh^2(p_A/2)}\right) + 2La(a, m_A) \right) \quad (13)$$

where the sum is taken over all non-boundary parallel simple closed geodesics A on T and c , a , p_A and m_A are defined as in §2.3. A further simplification of $g(T)$ is obtained recently, see [4] for details.

Theorem 2.1. ([4])

$$g(T) = \sum_A \left\{ 4\pi^2 - 8 \left[2\mathcal{L}\left(\frac{1}{\cosh^2(m_A/2)}\right) + \mathcal{L}\left(\frac{1}{\cosh^2(q_A/2)}\right) + \mathcal{L}\left(\frac{1}{\cosh^2(p_A/2)}\right) + 2La(c/2, m_A) + 2La(a, m_A) \right] \right\} \quad (14)$$

where the sum is taken over all non-boundary parallel simple closed geodesics A on T , and c , a , p_A , m_A and q_A are defined in §2.3.

Identities (11),(12),(13) and (14) put the main identity (1) in theorem 1.1 as a sum over all homotopy classes of essential embedded 3-holed spheres in the surface F . At this moment, we are not able to reconcile the two different expressions in (12) and (14). The function $\mathcal{L}(\frac{1}{\cosh^2(x/2)})$ was first introduced and used by Bridgeman [1].

For the identity (2) in Theorem 1.2, the functions \hat{f} and \bar{f} are defined using the lasso function $La(l, m)$ and the function $f(P)$ as follows:

$$\hat{f}(P) := f(P) + 8(\mathcal{L}(\frac{1}{\cosh^2(p_1/2)}) + La(l_2, m_3) + La(l_3, m_2))$$

where $\partial P \cap \partial F = L_1$ and

$$\begin{aligned} \bar{f}(P) := & f(P) + 8(\mathcal{L}(\frac{1}{\cosh^2(p_1/2)}) + \mathcal{L}(\frac{1}{\cosh^2(p_2/2)}) + \mathcal{L}(\frac{1}{\cosh^2(m_3/2)})) \\ & + 8(La(l_2, m_3) + La(l_3, m_2) + La(l_3, m_1) + La(l_1, m_3)) \end{aligned}$$

where $\partial P \cap \partial F = L_1 \cup L_2$.

Remark. The expressions $f(P)$, $g(T)$, $\hat{f}(P)$ and $\bar{f}(P)$ defined above are still valid if P or T are hyperbolic surfaces with some cusp ends. Namely, if some l_i or c tend to 0 (which imply the corresponding m_j 's and p_i 's tend to infinity), the functions f, g, \bar{f}, \hat{f} converge to well defined limit functions. If we use these limit functions in (2), then (2) becomes the identity for finite area hyperbolic surfaces F with geodesic boundary and cusp ends. In this case, the right-hand-side of (2) is the volume of the unit tangent bundle of F and the left-hand-side is the sum over all hyperbolic 3-holed spheres P and 1-holed torus T where P may have cusp ends. For simplicity, we omit the details here. Some details, including an identity for the cusped torus can be found in [4].

3. DECOMPOSING THE UNIT TANGENT BUNDLE OF THE SURFACE

Suppose F is a compact hyperbolic surface with or without boundary so that if $\partial F \neq \emptyset$, then ∂F consists of geodesics. Let $S(F)$ be the unit tangent bundle of F and μ be the measure on $S(F)$ invariant under the geodesic flow so that $\mu(S(F)) = -4\pi^2\chi(F)$.

We will produce a decomposition of $S(F)$ as follows. Given a vector $v \in S(F)$, let g_v^+ and g_v^- be the geodesic rays determined by v and $-v$. By the ergodicity of the geodesic flow, for generic choice of v with respect to μ , we may assume that

- (1) if $\partial F = \emptyset$, each geodesic ray g_v^- and g_v^+ is not simple and intersects every closed geodesic,
- (2) if $\partial F \neq \emptyset$, each geodesic ray g_v^- and g_v^+ intersects ∂F .

Indeed, the set X of all v 's in $S(F)$ satisfying (1) is invariant under the geodesic flow. Furthermore, the set X has positive μ -measure. It follows that $\mu(S(F) - X) = 0$. To see (2), we apply the ergodicity of the geodesic flow to the metric double of F across the boundary of F .

In the sequel, we will focus only on these generic vectors v .

Given a generic vector $v \in S(F)$, we define an associated graph $G(v)$ to v as follows.

Let $t_1 > 0$ be the smallest number so that the geodesic segment $g_v^-[0, t_1] \cup g_v^+[0, t_1]$ either intersects ∂F or intersects itself. Say for simplicity that this occurs

in the ray g_v^+ . This means $g_v^-([0, t_1]) \cup g_v^+([0, t_1])$ is a simple path in F so that $g_v^+(t_1)$ is in ∂F or in $g_v^-([0, t_1]) \cup g_v^+([0, t_1])$. Next, let $t_2 \geq t_1$ be the smallest number so that $g_v^-(t_2)$ is either in ∂F or in $g_v^-([0, t_2]) \cup g_v^+([0, t_1])$. The associated graph $G(v)$ is defined to be the connected component of $g_v^-([0, t_2]) \cup g_v^+([0, t_1]) \cup \partial F$ which contains $g_v^+(0)$. In particular, $G(v) = g_v^-([0, t_2]) \cup g_v^+([0, t_1])$ if $\partial F = \emptyset$. By the construction, the Euler characteristic of $G(v)$ is always -1 . We also define $G(v)^\circ := g_v^-([0, t_2]) \cup g_v^+([0, t_1])$, with the orientation induced from v . Note that $G(v)^\circ = G(v)$ if $G(v) \cap \partial F = \emptyset$, otherwise, it is a strict subset of $G(v)$. See figure 1 for closed surfaces and figure 4 for surfaces with non-empty boundary.

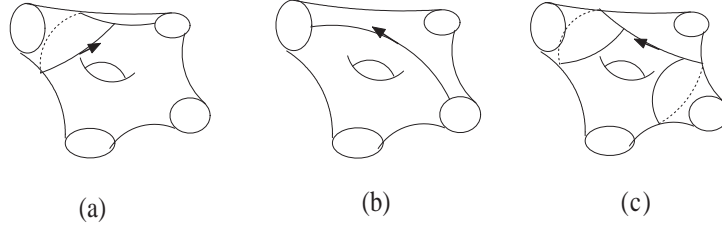


FIGURE 4. creation of spine

Recall that for a hyperbolic surface F , a compact embedded subsurface $S \subset F$ is said to be *geometric* if the boundaries of S are geodesic and *proper* if the inclusion map $i : S \rightarrow F$ is injective. Furthermore a surface is simple if it is a 3-holed sphere or 1-holed torus.

Proposition 3.1. *The graph $G(v)$ is contained in a unique geometric embedded simple surface $\Sigma(v)$.*

Proof. Cutting F open along $G(v)$, we obtain a (possibly disconnected) surface whose metric completion \hat{F} is a (possibly disconnected) compact hyperbolic surface with convex boundary. The boundary of \hat{F} consists of simple closed geodesics (corresponding to components of ∂F not in $G(v)$) and piecewise simple geodesic loops (corresponding to $G(v)$).

If $\hat{\gamma}$ is a piecewise simple geodesic loop in $\partial\hat{F}$, it is freely homotopic to a simple closed geodesic γ in \hat{F} which is a component of the boundary of the convex core $core(\hat{F})$ of \hat{F} . Furthermore $\hat{\gamma}$ and γ are disjoint by convexity. Therefore, $\hat{\gamma}$ and γ bound a convex annulus exterior to $core(\hat{F})$ and $G(v)$ is disjoint from $core(\hat{F})$. The subsurface $\Sigma(v) \subset F$ is the union of these convex annuli bounded by $\hat{\gamma}$ and γ . The Euler characteristic of $\Sigma(v)$ is -1 by the construction. The surface $\Sigma(v)$ is unique. Indeed, if $\Sigma' \neq \Sigma \subset F$ is a simple geometric subsurface so that $G(v) \subset \Sigma'$, then Σ' has a boundary component say B which intersects one of the boundaries γ of Σ transversely. Therefore, B must intersect the other boundary $\hat{\gamma}$ of the convex annulus described earlier. Hence it intersects $G(v)$ which contradicts $G(v) \subset \Sigma'$. \square

Note that topologically a regular neighborhood $N(G(v))$ of the graph $G(v)$ is either the 3-holed sphere $F_{0,3}$ or the 1-holed torus $F_{1,1}$. In the case that $N(G(v)) \cong F_{0,3}$ so that two boundary components of $N(G(v))$ are freely homotopic, then $\Sigma(v) \cong F_{1,1}$ and $\Sigma(v) - G(v)$ consists of two annuli (see figure 1(c)). In this case, $G(v)$ is not a deformation retract of $\Sigma(v)$. In all other cases, $\Sigma(v)$ is isotopic to $N(G(v))$ so that $\Sigma(v)$ deformation retracts to $G(v)$.

As a consequence, we have produced the following decomposition of the unit tangent bundle $S(F)$. Given a simple geometric subsurface Σ in F , let

$$W(\Sigma) = \{v \in S(F) | G(v) \subset \Sigma\}.$$

Then by proposition 3.1, we have the following decomposition

$$S(F) = \mathbf{Z} \bigsqcup_{\mathbf{P}} \bigsqcup_{\mathbf{T}} \mathbf{W}(\mathbf{P}) \bigsqcup_{\mathbf{T}} \mathbf{W}(\mathbf{T})$$

where \mathbf{Z} is a set of measure zero and the union is over all simple geometric 3-holed spheres P and 1-holed tori T .

Take the μ measure of the above decomposition, we obtain the main identities (1) and (2) in Theorems 1.1 and 1.2

$$\mu(S(F)) = \sum_P \mu(W(P)) + \sum_T \mu(W(T)). \quad (15)$$

The focus of the rest of the paper is to calculate the volume of $W(\Sigma)$ for simple surfaces Σ .

We end this section with a related simpler decomposition of $S(F)$ indexed by the set of all simple closed geodesics. For simplicity, we assume that F is a closed hyperbolic surface. Given a generic unit tangent vector v , the geodesic ray g_v^+ intersects itself. Let $t_1 > 0$ be the first time so that $g_v^+(t_1) \in g_v^+([0, t_1))$, say $g_v^+(t_1) = g_v^+(t_2)$ for some $0 \leq t_2 < t_1$. Then $g_v^+|_{[t_2, t_1]}$ is a simple loop freely homotopic to a simple closed geodesic s in F . Denote $g_v^+([t_2, t_1])$ by $Lop(v)$. For any given simple closed geodesic s in F , let $U(s) = \{v \in S(F) | Lop(v) \cong s\}$. Then we obtain a decomposition $S(F) = \mathbf{Z}' \bigsqcup_s U(s)$ where the disjoint union is indexed by the simple closed geodesics s and $\mu(\mathbf{Z}') = 0$. The associated identity is $\mu(S(F)) = \sum_s \mu(U(s))$. However, we are not able to calculate $\mu(U(s))$. It is not clear if $\mu(U(s))$ depends only on the length of s and the topology of F .

4. IDENTIFYING THE SETS IN THE DECOMPOSITION

We will investigate the sets $W(P)$ and $W(T)$ by studying their complements in $S(P)$ and $S(T)$. We will decompose the complementary sets into a disjoint union of sets, and identify each with subsets of $S(\mathbb{H}^2)$ in this section so that the computation of their volume can be carried out in §5.

For simplicity, we will deal with closed hyperbolic surfaces F . The modification for surfaces with non-empty boundary is easy, see §5.3. For a generic unit tangent vector $v \in S(F)$, $G(v)$ is a graph lying in a simple geometric surface Σ of F . In particular, $v \in S(\Sigma)$. Now for $v \in S(\Sigma)$, let $G_\Sigma(v)$ be the associated graph of v in Σ . Then by definition,

$$W(\Sigma) = \{v \in S(\Sigma) | G(v) = G_\Sigma(v)\}.$$

To calculate $\mu(W(\Sigma))$, we will focus on the complement

$$V(\Sigma) = S(\Sigma) - W(\Sigma) = \{v \in S(\Sigma) | G_\Sigma(v) \cap \partial\Sigma \neq \emptyset\}.$$

To this end, recall that $G_\Sigma(v) = g_v^+([0, t_1]) \cup g_v^-([0, t_2]) \cup B$ where B consists of some boundary components of Σ (B could be the empty set). Let $G_\Sigma^+(v)$ and $G_\Sigma^-(v)$ be the geodesic paths $g_v^+|_{[0, t_1]}$ and $g_v^-|_{[0, t_2]}$ defined in §3. By definition,

$$V(\Sigma) = \{v \in S(\Sigma) | G_\Sigma^+(v) \text{ or } G_\Sigma^-(v) \text{ is a simple arc ending at } \partial\Sigma\}.$$

There are two cases which can occur. Namely either both $G_{\Sigma}^{+}(v)$ and $G_{\Sigma}^{-}(v)$ are simple arcs ending at $\partial\Sigma$ or exactly one of them ends at $\partial\Sigma$.

These two cases will be discussed separately in the case Σ is a 3-holed sphere in the subsections §4.2 and §4.3 below, and in §4.5 in the case Σ is a 1-holed torus. We will first recall some facts in §4.1 about convex hyperbolic surfaces.

4.1. Preliminaries on convex surfaces. Suppose X is a compact connected surface with a hyperbolic metric so that ∂X consists of convex curves. Then, unless X is simply connected, each component of ∂X is an essential loop in X homotopic to a geodesic. As a convention, we will identify the universal cover \tilde{X} of X with a convex subset of \mathbb{H}^2 .

The following notation and conventions will be used. For a hyperbolic surface Y , a *geodesic path* is a map $s : [a, b] \rightarrow Y$ satisfying the geodesic equation so that $s'(t) \in S(Y)$. We are mainly interested in geodesic paths whose end points are in ∂Y . A geodesic path s is called a *geodesic loop* if $s(a) = s(b)$. A *simple* geodesic path or loop satisfies the condition that $s|_{(a,b)}$ is an injective map. Two paths $\alpha_i : ([a_i, b_i], \{a_i, b_i\}) \rightarrow (X, \partial X)$, $i = 0, 1$, are *homotopic*, denoted by $\alpha_0 \cong \alpha_1$ if there is a homotopy $H : ([0, 1] \times [0, 1], \{0, 1\} \times [0, 1]) \rightarrow (X, \partial X)$ so that $H(t, i) = \alpha_i(a_i + t(b_i - a_i))$ for $i = 0, 1$ and all t . Two loops α_i , $i = 0, 1$, with the same base point $p = \alpha_i(a_i) = \alpha_i(b_i)$ which are *relatively homotopic with respect to p* will be denoted by $\alpha_0 \cong \alpha_1 \text{ rel}\{p\}$.

The main technical result in this subsection is the following:

Proposition 4.1. *Suppose X is a compact non-simply connected hyperbolic surface with convex boundary.*

- (1) *If X is a topological annulus, then any geodesic path s in X joining different boundary components of X is simple;*
- (2) *If $s \cong t$ are two geodesic paths in X joining different boundary components of X and t is simple, then s is simple;*
- (3) *If $p \in \partial X$ and $s : ([0, a], \{0, a\}) \rightarrow (X, \{p\})$ is a geodesic path so that $s \cong t \text{ rel}\{p\}$ and t is a simple loop, then s is a simple loop.*

Proof. We will need the following simple lemma whose proof is omitted.

Lemma 4.2. *Suppose γ is a hyperbolic isometry of \mathbb{H}^2 with axis A and g is a geodesic intersecting A transversely. Then $\gamma^n(g) \cap g = \emptyset$ for all $n \in \mathbf{Z} - \{0\}$.*

To see (1), let c be the unique simple closed geodesic in X . Then s must intersect c in X . Lifting s and c to the universal cover and using the above lemma, we see that any two distinct lifts of s in \tilde{X} are disjoint. Thus s is simple.

To see (2), suppose otherwise there exist two distinct lifts s_1 and s_2 of $s : [0, d] \rightarrow X$ in \tilde{X} so that the interiors of s_1 and s_2 intersect. Let s join boundary components a and b of X and \tilde{a}_i and \tilde{b}_i be the lifts of a and b so that $s_i(0) \in \tilde{a}_i$ and $s_i(d) \in \tilde{b}_i$. Since $s \cong t$, by the homotopy lifting theorem, there exist two distinct lifts t_1 and t_2 of t in \tilde{X} so that t_i joins \tilde{a}_i to \tilde{b}_i .

We claim that interiors of t_1 and t_2 intersect. This in turn contradicts the fact that t is simple.

To see the claim, first, we note that \tilde{a}_1 is disjoint from \tilde{a}_2 . For otherwise, $s_2 = \gamma^n(s_1)$ for a deck transformation element γ corresponding to the boundary a of X . Furthermore, due to convexity both s_1 and s_2 intersect the axis of γ . Thus by the lemma above, s_1 is disjoint from s_2 which contradicts the assumption.

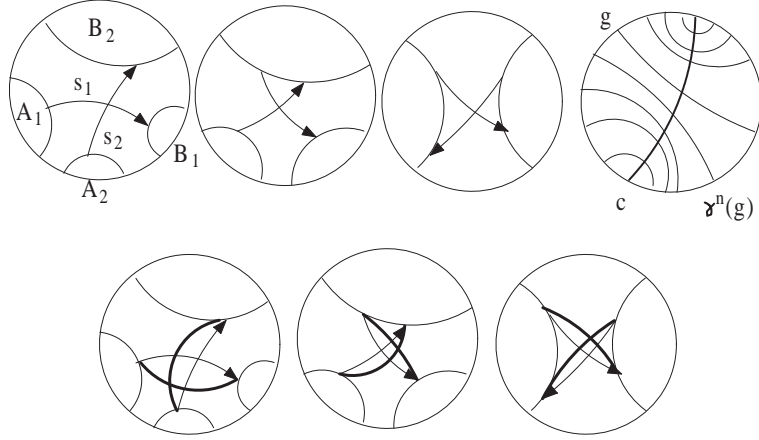


FIGURE 5. lifting and disjointness

By the same argument we see that \tilde{b}_1 is disjoint from \tilde{b}_2 . Since $a \cap b = \emptyset$ by assumption, we see that $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ consists of four distinct convex curves in \tilde{X} . Let A_1, A_2, B_1, B_2 be the four disjoint half-spaces in \mathbb{H}^2 bounded by these four convex curves. Let S_∞^1 be the circle at infinity of the hyperbolic plane. Then $s_1 \cap s_2 \neq \emptyset$ is equivalent to saying that $\overline{A_1} \cap S_\infty^1$ and $\overline{B_1} \cap S_\infty^1$ are in the different components of $S_\infty^1 - \overline{A_2} \cup \overline{B_2}$. This in turn implies that interiors t_1 and t_2 intersect. Thus part (2) holds.

The proof of part (3) is similar to that of (2). Suppose the result is false. Then there exist two distinct lifts $s_1, s_2 : [0, d] \rightarrow \tilde{X}$ of $s : ([0, d], \{0, d\}) \rightarrow (X, \{p\})$ so that $s_1(d_1) = s_2(d_2)$ for some $d_1, d_2 \in (0, d)$. Let t_1, t_2 be the lifts of t so that the end points of t_i are the same as that of s_i (by the homotopy lifting property). We claim that the interior of t_1 intersects the interior of t_2 . This would produce a contradiction to the assumption on t .

To see this, let $s_i(0) \in \tilde{a}_i$ and $s_i(d) \in \tilde{b}_i$ where \tilde{a}_i and \tilde{b}_i are lifts of the same boundary a of X . By the same argument as above, we see that $\tilde{a}_1 \cap \tilde{a}_2 = \emptyset$ and $\tilde{b}_1 \cap \tilde{b}_2 = \emptyset$. However, it is possible that $\tilde{a}_1 = \tilde{b}_2$ and $\tilde{a}_2 = \tilde{b}_1$. If $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ are pairwise disjoint, then the same argument as above shows that the claim holds. In the other cases, $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ consists of 2 or 3 geodesics, the same argument again shows that the interiors of t_1 and t_2 intersect since s_1 and s_2 are geodesics and t_i and s_i have the same end points. See figure 5. \square

4.2. Vectors v in $V(P)$ so that $G_P^+(v)$ and $G_P^-(v)$ are simple arcs ending at ∂P . We begin by recalling the beautiful work of M. Bridgeman [1] relevant to our setting. Given a compact hyperbolic surface X with geodesic boundary and a (not necessarily simple) geodesic path $\alpha : ([0, a], \{0, a\}) \rightarrow (X, \partial X)$ so that $\alpha'(0)$ and $\alpha'(a)$ are perpendicular to ∂X , let

$$H(\alpha) = \{s'(t)|s : ([0, b], \{0, b\}) \rightarrow (X, \partial X) \text{ geodesic, so that } s \cong \alpha\}.$$

Theorem 4.3. (Bridgeman) *The measure $\mu(H(\alpha))$ of $H(\alpha)$ is $4\mathcal{L}(\frac{1}{\cosh^2(l(\alpha)/2)})$ where $l(\alpha)$ is the length of α .*

Calegari gave a very nice short and elegant proof of this in [2]. If we use α^{-1} to denote the reversed path $\alpha^{-1}(t) = \alpha(a - t)$, then the measures of $H(\alpha^{-1})$ and $H(\alpha)$ are the same. In Bridgeman's work, he considered unoriented paths, i.e., the elements in $H(\alpha) \cup H(\alpha^{-1})$ and showed that its measure is $8\mathcal{L}(\frac{1}{\cosh^2(l(\alpha)/2)})$. For simplicity, we use $H(\alpha^{\pm 1})$ to denote $H(\alpha) \cup H(\alpha^{-1})$.

The main result in this section is to prove:

Proposition 4.4. *Suppose P is a hyperbolic 3-holed 3-sphere with geodesic boundary components L_1, L_2, L_3 and the shortest paths joining boundary components being M_i and B_i as in §2.2. Then*

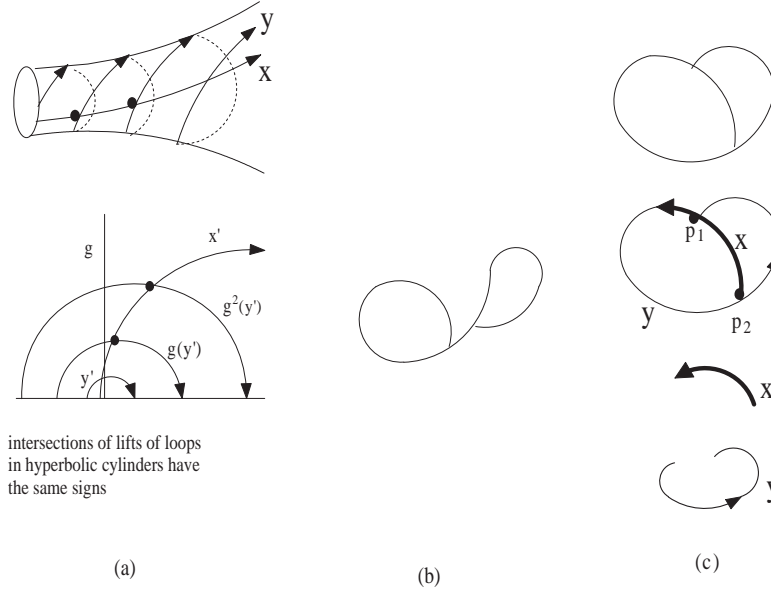
- (1) $\{v \in S(P) | G_P^+(v), G_P^-(v) \text{ both simple arcs ending at } \partial P\} \subset \cup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$.
- (2) $\cup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1})) \subset V(P)$.

Proof. To see (1), by the construction of $G_P^+(v)$ and $G_P^-(v)$, the interiors of these two simple arcs are disjoint. It follows that the geodesic path $G_P^+(v) \cup G_P^-(v)$ is a simple path with end points in ∂P . It is well known that any simple path $s : ([0, 1], \{0, 1\}) \rightarrow (P, \partial P)$ is homotopic to M_i , or B_i , or a point. The path $G_P^+(v) \cup G_P^-(v)$ cannot be homotopic to a point since it is a geodesic path. Thus the conclusion follows.

To see (2), let $s : [0, a] \rightarrow P$ be a geodesic path homotopic to M_i or B_i . If $s \cong M_i$, by proposition 4.1 (2), s is simple. Thus $s'(t) \in V(P)$. If $s \cong B_i$, we claim that there exists $b \in (0, a)$ so that $s|_{[0, b]}$ and $s|_{[b, a]}$ are simple arcs. To see this, first of all, the path s intersects M_i in exactly one point. Indeed, if there are at least two points of intersection, then there will be a lift \tilde{s} of s in the universal cover \tilde{P} so that \tilde{s} intersects two distinct lifts a_1 and a_2 of M_i . Let \tilde{B} be the lift of B_i so that both \tilde{B} and \tilde{s} start and end at the same geodesics which are lifts of L_i . Then \tilde{B} intersects a_1 and a_2 , i.e., B_i intersects M_i at two points. This is impossible. Furthermore, by topological reasons, s must intersect M_i . It follows that s intersects M_i in exactly one point, say $s(b) \in M_i$ for some $b \in (0, a)$. We claim that $\alpha := s|_{[0, b]}$ and $\beta := s|_{[b, a]}$ are both simple arcs. Indeed, let X be the surface obtained by cutting P open along M_i . Then X is a convex hyperbolic surface homeomorphic to an annulus. Both paths $\alpha|_{[0, b]}$ and $\beta|_{[b, a]}$ are geodesics in X joining different boundary components of X . Thus, by proposition 4.1, both of them are simple and both intersect the unique closed geodesic L_i in X . It is well known that if x, y are two oriented geodesics in a convex hyperbolic annulus X so that both x, y intersect the closed geodesic in X , then all intersection points between x with y have the same intersection sign. See figure 6(a) for a pictorial explanation in the universal cover. Thus all intersection points between α and β have the same sign.

We now finish the proof of (2) by showing that for any $t \in (0, a)$, $s'(t) \in V(P)$. Suppose otherwise, there exists $t_0 \in (0, a)$ so that $v = s'(t_0) \in W(P)$, i.e., the graph $G_P(v)$ does not intersect ∂P . Since $s([0, a])$ is a union of two simple arcs, the graph $G_P(v)$, considered as a sub-path $s|_{[T_1, T_2]}$ in s , is a union of two simple arcs. Since $G_P(v)$ is embedded in the planar surface P , by definition of $G_P(v)$, there are two possible embedding of $G_P(v)$ in P as shown the figure 6(b),(c).

In the first case, there are two disjoint simple loops in the graph $G_P(v)$. In this case, $G_P(v)$ cannot be a union of two simple arcs due to the disjoint simple loops. In the second case, the graph $G_P(v)$ can be expressed as a union of two simple arcs



intersections of lifts of loops
in hyperbolic cylinders have
the same signs

FIGURE 6. embedding of graphs

x and y in an essentially unique way as shown in figure 6(c). Let p_1 and p_2 be the two vertices of $G_P(v)$ and orient both arcs x and y . Then the intersection signs at p_1 and p_2 from x to y are opposite. It follows that this case does not occur in $s([0, a])$ by the calculation above. This ends the proof of (2). □

4.3. Lassos. For a hyperbolic 3-holed sphere P , it remains to identify the set $V(P) - \bigcup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$. If v is in the set, then one of $G_P^+(v)$ or $G_P^-(v)$ is a simple arc ending at ∂P and the other one is part of, or contains a loop. Thus $G_P(v)$ is a lasso (see figure 4(a)), as defined below.

Definition 4.5. (Lassos) Let X be a hyperbolic surface with geodesic boundary. A positively oriented *lasso* on X is a geodesic path

$$\alpha : [T_1, T_2] \rightarrow X$$

such that

- (1) $\alpha(T_1) \in \partial X$,
- (2) α is injective on (T_1, T_2) , and
- (3) $\alpha(T_3) = \alpha(T_2)$ for some $T_1 \leq T_3 < T_2$.

The image of α , ignoring orientation, is a lasso. A negatively oriented lasso β is a geodesic path so that $\beta(-t)$ is a positively oriented lasso. Call $\alpha(T_1)$ the *base point*, $\alpha(T_2) = \alpha(T_3)$ the *knot*, $\alpha[T_1, T_3]$ the *stem*, $\alpha|_{[T_3, T_2]}$ the *loop*, and $\alpha(\frac{T_2+T_3}{2})$ the *midpoint* of the loop of the lasso. Note that $\alpha(T_1, T_2) \cap \partial X = \emptyset$.

The midpoint of the loop $\alpha(\frac{T_2+T_3}{2})$ is diametrically opposite to the knot in the loop of a lasso. If γ is the unique oriented geodesic on X homotopic to the loop of α , then the loop of α and γ bound a hyperbolic cylinder A embedded in X . It is

easy to see by lifting to the universal cover that $\alpha(\frac{T_3+T_2}{2})$ is the point on the loop which is closest to γ on the cylinder A .

Note that if α and β are two lassos so that α is positively oriented and β is negatively oriented, then by definition $\alpha'(t) \neq \beta'(t')$ for all parameters t, t' . Furthermore, the involution map $v \rightarrow -v$ in $S(X)$ sends tangent vectors to positively oriented lassos to that of negatively oriented lassos. Thus it suffices to calculate the measure of tangents to positively oriented lassos.

Proposition 4.6. *Suppose that $\alpha : [T_1, T_2] \rightarrow \Sigma$ is a positively oriented lasso in a compact hyperbolic surface Σ with geodesic boundary, and $\alpha(T_3) = \alpha(T_2)$ is the knot of α . Then $G_\Sigma(\alpha'(t)) - \partial\Sigma = \alpha([T_1, T_2])$ if and only if $T_1 \leq t \leq \frac{T_2+T_3}{2}$.*

Proof. The midpoint $\alpha(\frac{T_2+T_3}{2})$ lies on the critical set where if we exponentiate in both directions at equal speed, we reach the knot of the lasso at the same time. For $T_1 < t < \frac{T_2+T_3}{2}$, we get the lasso α and for $\frac{T_2+T_3}{2} < t < T$, we will exponentiate in the other direction of the knot and $G_\Sigma(\alpha'(t))$ will not include the stem of α . \square

In the rest of the discussion, we assume surfaces are oriented so that their boundaries have the induced orientation. Given $v \in V(P) - \bigcup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$, the graph $G_P(v)^\circ$ (see §3) is a lasso. Since its loop is simple, it is freely homotopic to $L_i^{\pm 1}$ for some i .

For i, j, k distinct, let $W(L_i, M_j)$ be $\{v \in S(P) \mid G_P(v)^\circ \text{ is a positive lasso whose loop is homotopic to } L_i, \text{ the base point of } G_P(v) \text{ is in } L_k, \text{ and } v \notin \bigcup_{l=1}^3 (H(M_l^{\pm 1}) \cup H(B_l^{\pm 1}))\}$. Let $W(L_i^{-1}, M_j)$ be the set defined in the same way except the loop of the lasso is homotopic to L_i^{-1} . Let $\mathbf{A} : S(P) \rightarrow S(P)$ be the involution $\mathbf{A}(v) = -v$. \mathbf{A} sends vectors generating positive lassos to those generating negative lassos, and vice versa, since $G_P(v)^\circ = G_P(-v)^\circ$ with opposite orientations.

We have:

Lemma 4.7. *The set $V(P) - \bigcup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$ can be decomposed as*

$$\bigsqcup_{i \neq j} (W(L_i, M_j) \cup W(L_i^{-1}, M_j)) \bigsqcup \mathbf{A}(\bigsqcup_{i \neq j} (W(L_i, M_j) \cup W(L_i^{-1}, M_j))).$$

In particular,

$$\mu(V(P)) = 8 \sum_{i=1}^3 (\mathcal{L}(\frac{1}{\cosh^2(m_i/2)}) + \mathcal{L}(\frac{1}{\cosh^2(p_i/2)})) + 4 \sum_{i \neq j} \mu(W(L_i, M_j)).$$

Proof. The decomposition in the first sentence follows from the above discussion. We claim that $W(L_i, M_j)$ and $W(L_i^{-1}, M_j)$ are related by an isometry of P . Indeed, the hyperbolic 3-holed sphere P admits an orientation reversing isometry R so that $R|_{M_i} = id$ and R interchanges the two hexagons obtained by cutting P open along M_i 's. In particular, R reverses the orientation of each boundary component. Therefore, the derivative R_* of R sends $W(L_i, M_j)$ to $W(L_i^{-1}, M_j)$, i.e., $R_*(W(L_i, M_j)) = W(L_i^{-1}, M_j)$. In particular, $\mu(W(L_i, M_j)) = \mu(W(L_i^{-1}, M_j))$. \square

4.4. **Understanding the set $W(L_i, M_j)$.** We begin with some notation. The circle at infinity of the hyperbolic plane is denoted by S_∞^1 . Given $x \neq y \in \mathbb{H}^2 \cup S_\infty^1$, let $G[y, x]$ be the oriented geodesic from y to x . In particular, if $x \neq y \in S_\infty^1$, then $G[y, x]$ is the complete oriented geodesic determined by y, x .

Consider the universal cover \tilde{P} of the hyperbolic 3-holed sphere P as a convex subset of \mathbb{H}^2 so that the covering map is $\Pi : \tilde{P} \rightarrow P$. We assume that \tilde{P} and P are oriented so that Π and the inclusion map $i : \tilde{P} \rightarrow \mathbb{H}^2$ are orientation preserving. Cutting P open along the shortest paths M_i 's joining L_j to L_k ($i \neq j \neq k \neq i$), we obtain two right-angled hexagons in P . Let Q be a lift of one of the hexagons in P to \tilde{P} so that Q is bounded by complete geodesics \tilde{L}_i and M_i^* with $\Pi(\tilde{L}_i) = L_i$ and $\Pi(M_i^* \cap Q) = M_i$. We choose the lift Q (of one of the hexagons) so that the cyclic order $\tilde{L}_1 \rightarrow \tilde{L}_2 \rightarrow \tilde{L}_3$ coincides with the orientation of Q . Let R_i be the hyperbolic reflection about the geodesic M_i^* . Then $\gamma_i = R_{i+2}R_{i+1}$ is the deck transformation group element so that $\gamma_i(\tilde{L}_i) = \tilde{L}_i$ and γ_i corresponds to the oriented loop L_i . The closure of the region in \mathbb{H}^2 bounded by \tilde{L}_1, \tilde{L}_2 and \tilde{L}_3 intersects the circle at infinity S_∞^1 of \mathbb{H}^2 in three disjoint intervals I_1, I_2, I_3 where I_i is disjoint from the closure of \tilde{L}_i . See figure 7(a). It is known that for $n \neq 0$

$$\gamma_i^n(I_i) \subset I_{i+1} \cup I_{i+2} \tag{16}$$

and

$$\text{end points of } \gamma_{i+1}^m(\tilde{L}_i) \text{ are in } I_i \text{ for } m > 0 \tag{17}$$

where indices are counted modulo 3. See for instance [3] for a proof.

In the rest of the subsection, we will focus on $W(L_2, M_3)$ (i.e., $i=2, j=3$). The general case of $W(L_i, M_j)$ is exactly the same.

For simplicity, we let $l := l_2$ and $m := m_3$. After conjugation by an isometry of \mathbb{H}^2 , we may assume that $\tilde{L}_2 = G[\infty, 0]$, $\tilde{L}_3 = G[e, f]$, $\tilde{L}_1 = G[c, d]$ with $0 < e < f < c < d$, and $\gamma_2(c) = 1$. Since $\gamma_2(z) = e^{-l}z$, we have $c = e^l$. Note that $I_1 = [0, e]$ in this case. By (17), $\gamma_2(\tilde{L}_1) = G[1, e^{-l}d]$ has end points in I_1 , i.e., $1 < e^{-l}d < e$. Also, since the distance between \tilde{L}_2 and \tilde{L}_1 is m and $c = e^l$, we have $d = e^l \coth^2(\frac{m}{2})$. See figure 7(b).

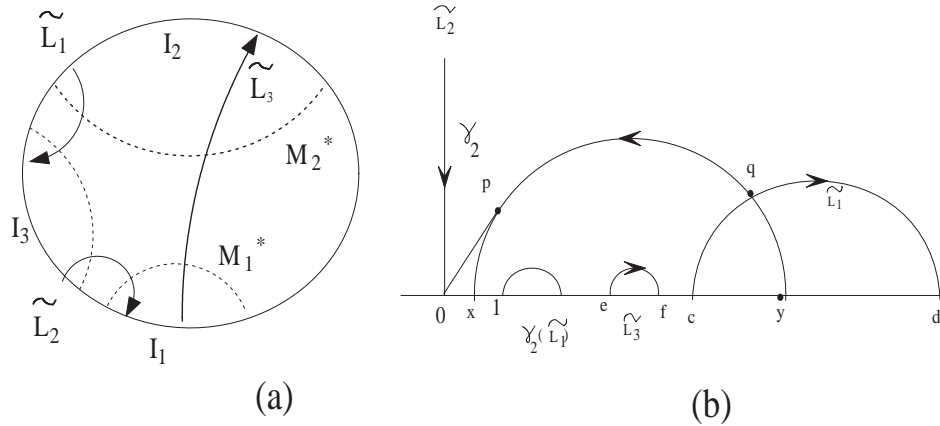


FIGURE 7. lifts of boundary

Define the subset $\Omega_{2,3}$ of $S(\mathbb{H}^2)$ as follows. Given x, y with $0 < x < 1$ and $c < y < d$, let q be the intersection point $G[y, x] \cap \tilde{L}_1$ and let p be the point on $G[y, x]$ so that the Euclidean ray Op is tangent to the semi-circle $G[y, x]$. If γ is a geodesic path and $v = \gamma'(t)$, we denote it by $v \in \gamma$. Then

$$\Omega_{2,3} = \{v \in S(\mathbb{H}^2) | v \in G[y, x], 0 < x < 1, c < y < d, \text{ and } v \in G[q, p]\}, \quad (18)$$

where $c = e^l, d = e^l \coth^2(m/2)$.

The main result in this subsection is the following:

Proposition 4.8. *Let $\Pi_* = D\Pi$ be the derivative of the universal covering map $\Pi : \tilde{P} \rightarrow P$. Then Π_* induces a bijection from $\Omega_{2,3}$ to $W(L_2, M_3)$. In particular, the volume of $W(L_2, M_3)$ is $\mu(\Omega_{2,3})$.*

Proof. We will first show that $\Pi_*(\Omega_{2,3}) \subset W(L_2, L_3)$ and then show that $\Pi_*|_{\Omega_{2,3}}$ is a bijection.

To see $\Pi_*(\Omega_{2,3}) \subset W(L_2, L_3)$, take a vector $v \in \Omega_{2,3}$ so that $v \in G[q, p] \subset G[y, x]$ as in (18).

Lemma 4.9. *Let $\tilde{\beta} = G[q, \gamma_2(q)]$ be the geodesic in \mathbb{H}^2 from $q \in \tilde{L}_1$ to $\gamma_2(q)$. Then the projection $\Pi(\tilde{\beta}) = \beta$ is a simple geodesic loop in P based at $q' = \Pi(q)$.*

Proof. By proposition 4.1, it suffices to show that $\beta \simeq \delta \text{ rel}(q')$ where δ is a simple loop at q' . Indeed, consider the shortest path $a_1 = G[q, q_1]$ from q to $q_1 \in \tilde{L}_2$. Since $\text{dist}(\tilde{L}_1, \tilde{L}_2) = \text{dist}(L_1, L_2)$, the projection $\Pi(a_1)$ is homotopic to M_3 , the shortest path from L_2 to L_1 . Thus, by proposition 4.1, $\Pi(a_1)$ is a simple arc from L_1 to L_2 . Now by the construction, $\tilde{\beta}$ and the path $a_1 * G[q_1, \gamma_2(q_1)] * \gamma_2(a_1^{-1})$ have the same end points in \mathbb{H}^2 . Thus $\beta \simeq \Pi(a_1) * \Pi(G[q_1, \gamma_2(q_1)]) * \Pi(\gamma_2(a_1)^{-1}) \text{ rel}(q')$. Since $\Pi(a_1)$ is an embedded arc whose interior is disjoint from $\Pi(G[q_1, \gamma_2(q_1)])$ ($=L_2$), by a small perturbation, the loop $\Pi(a_1) * \Pi(G[q_1, \gamma_2(q_1)]) * \Pi(\gamma_2(a_1)^{-1})$ is relatively homotopic to a simple loop δ based at q' . It follows that $\beta \simeq \delta \text{ rel}(q')$ where δ is simple. See figure 8. \square

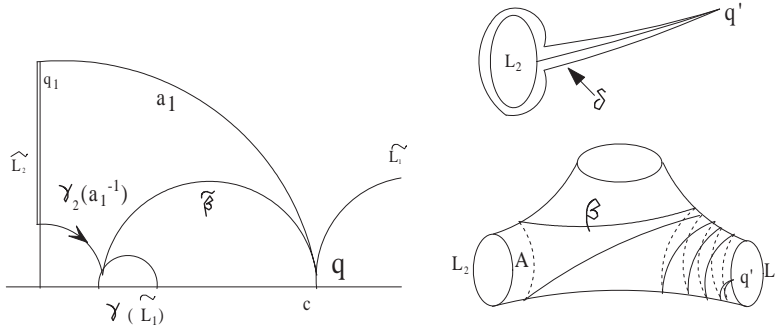


FIGURE 8. homotopic loops are simple

Since the simple loop β is disjoint from L_2 and is homotopic to L_2 , there is an annulus A in P bounded by β and L_2 . Note that A has convex boundary. The universal cover \tilde{A} of A can be identified with the convex region in \mathbb{H}^2 bounded by \tilde{L}_2 and the simple path $\cup_{n \in \mathbf{Z}} \gamma_2^n(\tilde{\beta})$. Now we show that $\Pi_*(v) \in W(L_2, M_3)$.

Consider the geodesic $\gamma(t) = \Pi_*(G[q, x])$ in P where $\gamma(0) = \Pi(q)$. For t small, by the construction, $\gamma(t)$ is in the annulus A . Since the vector v is assumed to be generic, $\gamma(t) \in \partial P$ for some $t > 0$. Thus there is the largest $T \in (0, \infty)$ so that $\gamma([0, T]) \subset A$. First $\gamma(T)$ cannot be in L_2 . Indeed, if this occurs, since A is an annulus, $\gamma|_{[0, T]} \cong M_3$. This implies that $G[q, x]$ intersects \tilde{L}_2 and contradicts $x > 0$. It follows that $\gamma(T) \in \beta$. We claim that, $\gamma|_{[0, T]}$ cannot be a simple arc. Otherwise, since A is an annulus and $\gamma|_{[0, T]}$ is an arc joining the same boundary component β of A , $\gamma|_{[0, T]} \cong \delta$ where δ is a simple geodesic arc in β . This contradicts Gauss-Bonnet theorem since there will be a bi-gon bounded by δ and $\gamma|_{[0, T]}$ in the annulus A . It follows that $\gamma|_{[0, T]}$ is not simple. Let $0 < T_1 \leq T$ be the time so that $\gamma|_{[0, T_1]}$ is a lasso based at $\Pi(q)$ inside A . The loop of this lasso is homotopic to L_2 which is the only simple closed geodesic in A . Furthermore, the mid-point of the loop of the lasso $\gamma|_{[0, T_1]}$ lifts to a point in $G[y, x]$ which is closest to the geodesic \tilde{L}_2 . Thus the midpoint of the lasso is $\Pi(p)$. Finally, the geodesic path $G_P^+(\Pi_*(v)) \cup G_P^-(\Pi_*(v))$ is not homotopic to $M_i^{\pm 1}$ and to $B_1^{\pm 1}$. Indeed, if otherwise, then a lift of this path with initial point q will end either on \tilde{L}_2 (homotopic to M_3) or \tilde{L}_3 (homotopic to M_2), or $\gamma_2(\tilde{L}_1)$, or $\gamma_3(\tilde{L}_1)$ (homotopic to B_1). All these cases contradict the assumption that $0 < x < 1$. This shows that $\Pi_*(v) \in W(L_2, M_3)$.

Next, we show that $\Pi_*|$ is onto. To see this, take a vector $v \in W(L_2, M_3)$ so that its graph $G_P(v)^o$ is a lasso based at a point q' in L_1 . We claim there is a simple geodesic loop β in P based at q' so that β intersects the lasso $G_P(v)$ only at q' and β is freely homotopic to L_2 . Indeed, cutting the surface P open along the lasso $G_P(v)$, we obtain two convex annuli. See figure 9(b), (c). One of the annulus, say A_1 , contains L_3 as a boundary component. Let q_1 and q_2 be the preimages of q' in A_1 and let c be the arc in the boundary of A_1 joining q_1 to q_2 so that c is disjoint from the preimage of L_1 .

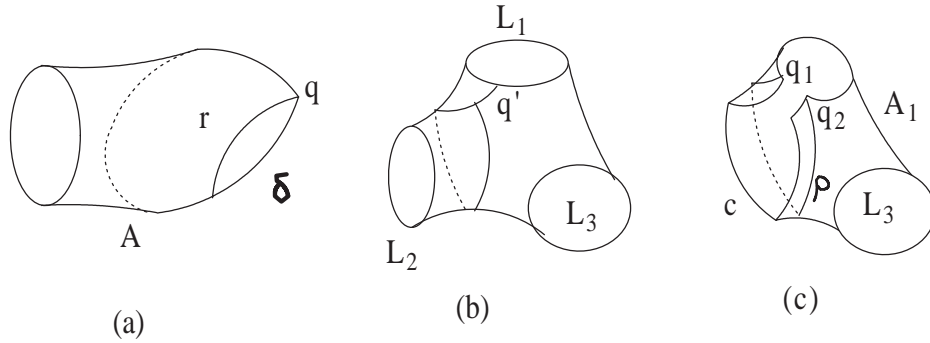


FIGURE 9. cutting surface open along lasso

Since A_1 is convex, there exists a shortest geodesic path ρ in A_1 joining q_1 to q_2 so that $\rho \cong c \text{ rel}(\{q_1, q_2\})$. Since the knot point of the lasso is a non-smooth point of c , the path ρ is different from c . The simple loop β is the quotient of ρ in P . Consider β as a loop $\beta : S^1 \rightarrow P$ and let $\alpha : \mathbf{R} \rightarrow P$ be $\alpha(t) = \beta(e^{it})$. Since β is freely homotopic to L_2 , there exists a lift $\tilde{\alpha}$ of α so that the end points of $\tilde{\alpha}$ are the same as that of \tilde{L}_2 in S_∞^1 . This lift $\tilde{\alpha}$ intersects \tilde{L}_1 at exactly one point q since $\beta \cap L_1 = \{q'\}$. Let $\tilde{\gamma}(t)$ be the geodesic starting from q which is a lift of the lasso

$G_P(v)$ and u be the unit tangent vector in $\tilde{\gamma}(t)$ which projects to v , i.e., $\Pi_*(u) = v$. We claim that $u \in \Omega_{2,3}$. Indeed, if $\tilde{\beta}$ is a lift of the geodesic path β starting at q , then $\tilde{\alpha}$ is the union $\cup_{n \in \mathbf{Z}} \gamma_2^n(\tilde{\beta})$. Let A be the annulus in P bounded by β and L_2 . Then a universal cover \tilde{A} of A is the region bounded by \tilde{L}_2 and $\cup_{n \in \mathbf{Z}} \gamma_2^n(\tilde{\beta})$. It follows that $\tilde{\gamma}(t)$ is in \tilde{A} for $t > 0$ small by the disjointness of β and the lasso. Consider the complete geodesic $G[y, x]$ which contains $\tilde{\gamma}$. First of all, $c < y < d$ since $G[y, x]$ intersects \tilde{L}_1 . Next, since $\Pi(G[y, x])$ contains the lasso $G_P(v)$, the preimages of the knot of $G_P(v)$ in $G[y, x]$ contain two points of the form $z, \gamma_2(z)$. It follows that $\gamma_2(G[y, x]) \cap G[y, x] \neq \emptyset$, i.e., $x < e^{-l}y$. But $e^{-l}y < e^{-l}d$. Thus $x < e^{-l}d$. Next, $x > 0$ since v is not in $H(M_i^{\pm 1})$. Furthermore, it is impossible for $x \in [1, e^{-l}d]$ where $G[1, e^{-l}d] = \gamma_2(\tilde{L}_1)$ since $v \notin H(B_i^{\pm 1})$. Therefore, $0 < x < 1$. By proposition 4.6, v is between q' and the midpoint of the lasso. Thus we conclude that u is between q and p . Thus $u \in \Omega_{2,3}$.

Finally, to see that $\Pi_*|_S$ is injective in $\Omega_{2,3}$, suppose that v_1, v_2 in $\Omega_{2,3}$ so that $\Pi_*(v_1) = \Pi_*(v_2)$. Let v_i be in the geodesic $G[q_i, x_i]$ in $\Omega_{2,3}$ where $q_i \in \tilde{L}_1$ and $0 < x_i < 1$. Since $\Pi_* : S(\tilde{P}) \rightarrow S(P)$ is a regular cover with deck transformation group $\pi_1(P)$, there exists a deck transformation element γ so that $\gamma(v_1) = v_2$. In particular, $\gamma(G[q_1, x_1]) = G[q_2, x_2]$. This implies that $\gamma(q_1) = q_2$. Therefore, $\gamma(\tilde{L}_1) = \tilde{L}_1$. However, the only deck transformations leaving \tilde{L}_1 invariant are γ_1^n . Therefore $\gamma_1^n(G[q_1, x_1]) = G[q_2, x_2]$. If $n \neq 0$, by (16) $\gamma_1^n(I_1) \cap I_1 = \emptyset$, we see that for $x_1 \in (0, 1) \subset I_1 = [0, e]$, then $x_2 = \gamma_1^n(x_1) \notin I_1$. Therefore, for $n \neq 0$, $\gamma_1(G[q_1, x_1])$ cannot be $G[q_2, x_2]$ where $x_1, x_2 \in (0, 1)$. This shows that $n = 0$, i.e., $v_1 = v_2$. \square

4.5. Vectors in $V(T)$ for a 1-holed torus T . Let T be a hyperbolic 1-holed torus with geodesic boundary C and $\{A\}$ the set of non-boundary parallel, simple closed geodesics on T . Then $v \in V(T)$ if and only if $G_T(v) \cap C \neq \emptyset$. In this case, cutting T along $G_T(v)^\circ$ gives a convex hyperbolic cylinder with two non-smooth, piecewise geodesic boundaries and there is a unique simple closed geodesic $A \subset T$ which is disjoint from $G(v)$. Hence $V(T)$ decomposes into the infinite disjoint union $V(T) = \bigsqcup_{\{A\}} V_A(T)$ where

$$V_A(T) = \{v \in V(T) \mid G_T(v) \cap A = \emptyset\}.$$

Let P_A be the 3-holed sphere obtained by cutting T along A and label the boundaries of P_A so that $L_1 = C$, $L_2 = A^+$, $L_3 = A^-$. Note that there is an isometric involution of P_A sending L_2 to L_3 and fixing L_1 . Then, similar to the arguments in the previous two subsections, we conclude that $V_A(T)$ is the disjoint union

$$H(B_1^{\pm 1}) \bigsqcup_{i \neq j \neq 1 \neq i} \bigsqcup (W(L_i, M_j) \cup W(L_i^{-1}, M_j)) \bigsqcup \mathbf{A} \left(\bigsqcup_{i \neq j \neq 1 \neq i} (W(L_i, M_j) \cup W(L_i^{-1}, M_j)) \right).$$

It follows, from the symmetry of P_A , that

$$\mu(V_A(T)) = \mu(H(B_1^{\pm 1})) + 8\mu(W(L_2, M_3)).$$

Using the notation from §2.2 and 2.3 that $length(B_1) = p_1 = p_A$, $l_2 = l_3 = a$ and $m_2 = m_3 = m_A$, we obtain

$$g(T) := \mu(W(T)) = \mu(S(T)) - \mu(V(T)) = 4\pi^2 - \sum_A \mu(V_A(T)).$$

Therefore,

$$g(T) = 4\pi^2 - 8 \sum_A \left(\mathcal{L}\left(\frac{1}{\cosh^2(p_A/2)}\right) + \mu(W(L_2, M_3)) \right). \quad (19)$$

This is the formula (13).

5. CALCULATING THE LASSO FUNCTION $La(l, m)$

By §4.4 and the work of Bridgeman, we see that the computation of the functions f and g reduces to the computation of $\mu(W(L_i, M_j))$ for a 3-holed sphere P , or equivalently, $\mu(\Omega_{i,j})$. We will show that the volume $\mu(W(L_i, M_j))$ depends only on the lengths l_i, m_j of L_i and M_j . The *lasso function* $La(l_i, m_j)$ is defined to be $\frac{1}{2}\mu(W(L_i, M_j))$. The goal of this section is to derive an explicit formula for $La(l, m)$.

Let us begin by recalling some well-known facts about hyperbolic geometry and the invariant measure on $S(\mathbb{H}^2)$. The invariant measure on the unit tangent bundle $S(\mathbb{H}^2)$ in local coordinates can be written as

$$\frac{2dx dy du}{(x - y)^2},$$

where $x \neq y \in \mathbf{R}$ and $u \in \mathbf{R}$ so that the oriented geodesic $\gamma(v)$ determined by $v \in S(\mathbb{H}^2)$ is $G[x, y]$ and u is the signed distance from the base point of v to the highest point in the semi-circle $G[x, y]$ (in the Euclidean plane). See figure 10 below.

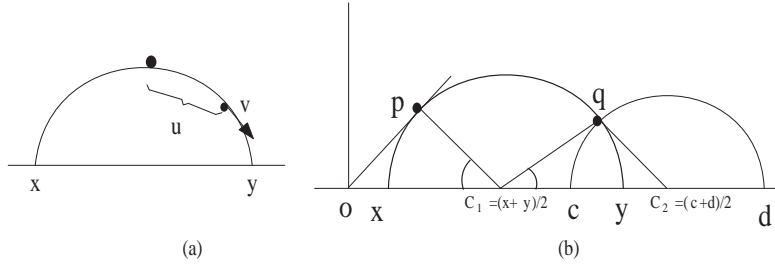


FIGURE 10. Coordinates for $S(\mathbb{H}^2)$

Let Ω be the set defined by (18) (i.e., $\Omega = \Omega_{2,3}$) where $1 < c < d$. The main result in this section shows that the volume $\mu(\Omega)$ of Ω is

$$\int_0^1 \int_c^d \frac{\ln \left| \frac{y(x-c)(x-d)}{x(y-c)(y-d)} \right|}{(x-y)^2} dy dx = 2\left(\mathcal{L}\left(\frac{d-1}{d}\right) - \mathcal{L}\left(\frac{c-1}{c}\right) + 2\mathcal{L}\left(\frac{c-1}{d-1}\right) - 2\mathcal{L}\left(\frac{c}{c-d}\right) \right)$$

where $\mathcal{L}(x)$ is the Roger's dilogarithm. The right-hand-side of the above identity will be shown in lemma 5.6 to be $2[\mathcal{L}(x) - \mathcal{L}(\frac{1-x}{1-xy}) + \mathcal{L}(\frac{1-y}{1-xy})]$ where $c = \frac{1}{x}$ and $d = \frac{1}{xy}$. By proposition 4.8, we obtain $La(l_i, m_j) = \frac{1}{2}\mu(W(L_i, M_j)) = \frac{1}{2}\mu(\Omega) = \mathcal{L}(y) - \mathcal{L}(\frac{1-x}{1-xy}) + \mathcal{L}(\frac{1-y}{1-xy})$ where $c = e^{l_i}$, $d = e^{l_i} \coth^2(\frac{m_j}{2})$ and $x = e^{-l_i}$, $y = \tanh^2(m_j/2)$. Combining with lemma 4.7 and (19), we obtain the formulas (10) and (13) in §2.

5.1. **Deriving the volume formula for Ω .** We will establish,

Proposition 5.1. *The volume of Ω is given by*

$$\int_0^1 \left(\int_c^d \frac{\ln \left| \frac{y(x-c)(x-d)}{x(y-c)(y-d)} \right|}{(y-x)^2} dy \right) dx. \quad (20)$$

Proof. We will use the following known distance formula in the hyperbolic plane. Namely, $d(e^{i\phi}, e^{-i\psi}) = \ln \cot(\phi/2) + \ln \cot(\psi/2)$ in \mathbb{H}^2 where $\phi, \psi \in (0, \pi/2)$. Let $C_1 = \frac{x+y}{2}$ be the Euclidean center of the semi-circle $G[x, y]$ and ψ and ϕ be the angles $\angle 0C_1p$ and $\angle qC_1y$ as shown in figure 10. Then by the definition of the volume form, we see that $\mu(\Omega)$ is given by

$$\int_0^1 \left(\int_c^d \frac{2 \ln \cot(\psi/2) + 2 \ln \cot(\phi/2)}{(y-x)^2} dy \right) dx = \int_0^1 \left(\int_c^d \frac{\ln[\cot^2(\psi/2) \cot^2(\phi/2)]}{(y-x)^2} dy \right) dx. \quad (21)$$

We calculate $\cot^2(\psi/2)$ and $\cot^2(\phi/2)$ using the cosine law for Euclidean triangles $\Delta 0pC_1$ and ΔC_1C_2q where $C_2 = \frac{c+d}{2}$ is the center of the semi-circle $G[c, d]$.

Lemma 5.2. *Suppose the lengths of a Euclidean triangle are l, m, n so that the angle facing the edge of length l is θ . Then*

$$\cot^2(\theta/2) = \frac{(m+n+l)(m+n-l)}{(m+l-n)(n+l-m)}.$$

Indeed, by the cosine law that $\cos(\theta) = \frac{m^2+n^2-l^2}{2mn}$, we obtain

$$\cot^2(\theta/2) = \frac{1 + \cos(\theta)}{1 - \cos(\theta)} = \frac{(m+n)^2 - l^2}{l^2 - (m-n)^2} = \frac{(m+n+l)(m+n-l)}{(m+l-n)(n+l-m)}.$$

For the angle ψ , the triangle $\Delta 0pC_1$ is right-angled. By taking $\theta = \psi$, $n = \frac{y-x}{2}$, $m = \frac{x+y}{2}$ and $l = \sqrt{m^2 - n^2}$, we obtain $\cos(\psi) = \frac{y-x}{y+x}$ and

$$\cot^2\left(\frac{\psi}{2}\right) = \frac{y}{x}. \quad (22)$$

For the angle ϕ , the edge lengths of the Euclidean triangle ΔqC_1C_2 are $n = \frac{y-x}{2}$, $m = \frac{c+d-x-y}{2}$, and $l = \frac{d-c}{2}$ so that ϕ is facing the edge of length l . Now using

$$\begin{aligned} l + m + n &= \frac{d-c+y-x+c+d-x-y}{2} = d-x, \\ m + n - l &= \frac{c+d-x-y+y-x-d+c}{2} = c-x, \\ l + n - m &= \frac{d-c+y-x-c-d+x+y}{2} = y-c, \\ l + m - n &= \frac{d-c+c+d-x-y-y+x}{2} = d-y, \end{aligned}$$

we obtain that

$$\cot^2(\phi/2) = \left| \frac{(x-c)(x-d)}{(y-c)(y-d)} \right|. \quad (23)$$

Putting (22), (23) into (21), we obtain

$$2 \ln \cot(\psi/2) + 2 \ln \cot(\phi/2) = \ln[\cot^2(\psi/2) \cot^2(\phi/2)] = \ln \left| \frac{y(x-c)(x-d)}{x(y-c)(y-d)} \right|. \quad (24)$$

□

5.2. Evaluation of the integral (20). The evaluation of the integral is similar to the work in [1]. Recall the Roger's dilogarithm \mathcal{L} is defined by $\mathcal{L}(0) = 0$ and $2\mathcal{L}'(x) = \frac{\ln|x|}{x-1} - \frac{\ln|x-1|}{x}$, for $x < 1$.

Proposition 5.3. *If $d > c > 1$, then*

$$\int_0^1 \left(\int_c^d \frac{\ln \left| \frac{y(x-c)(x-d)}{x(y-c)(y-d)} \right|}{(y-x)^2} dy \right) dx = 2[\mathcal{L}\left(\frac{d-1}{d}\right) - \mathcal{L}\left(\frac{c-1}{c}\right) + 2\mathcal{L}\left(\frac{c-1}{d-1}\right) - 2\mathcal{L}\left(\frac{c}{c-d}\right)]. \quad (25)$$

Proof. To simplify notation, we use

$$R = \left| \frac{y(x-c)(x-d)}{x(y-c)(y-d)} \right|$$

and the integral (25) can be written as $\int_0^1 \int_c^d \frac{\ln R}{(x-y)^2} dy dx$. For simplicity, we drop the constant term in the indefinite integrals in the lemma below.

Lemma 5.4.

$$\begin{aligned} \int \frac{\ln R}{(x-y)^2} dy &= \frac{\ln \left| \frac{(x-c)(x-d)}{x} \right|}{x-y} + \left(\frac{1}{x-y} - \frac{1}{x} \right) \ln |y| \\ &+ \left(\frac{1}{x} - \frac{1}{x-c} - \frac{1}{x-d} \right) \ln |y-x| + \left(-\frac{1}{x-y} + \frac{1}{x-c} \right) \ln |y-c| + \left(-\frac{1}{x-y} + \frac{1}{x-d} \right) \ln |y-d|. \end{aligned}$$

Proof. Using integration by parts, we obtain

$$\begin{aligned} \int \frac{\ln R}{(x-y)^2} dy &= \int \ln R d\left(\frac{1}{x-y}\right) \\ &= \frac{\ln R}{x-y} - \int \frac{d \ln R}{x-y} \\ &= \frac{\ln R}{x-y} + \int \frac{dy}{y-x} \left(\frac{1}{y} - \frac{1}{y-c} - \frac{1}{y-d} \right) \\ &= \frac{\ln R}{x-y} + \int \left(\frac{1}{(y-x)y} - \frac{1}{(y-x)(y-c)} - \frac{1}{(y-x)(y-d)} \right) dy. \quad (26) \end{aligned}$$

Now using the integral formula that for $a \neq b$,

$$\int \frac{dy}{(y-a)(y-b)} = \frac{1}{(a-b)} [\ln |y-a| - \ln |y-b|],$$

we can write (26) as

$$\begin{aligned} &\frac{\ln \left| \frac{(x-c)(x-d)}{x} \right|}{x-y} + \frac{\ln \left| \frac{y}{(y-c)(y-d)} \right|}{x-y} \\ &+ \frac{1}{x} (\ln |y-x| - \ln |y|) - \frac{1}{x-c} (\ln |y-x| - \ln |y-c|) - \frac{1}{x-d} (\ln |y-x| - \ln |y-d|) \\ &= \frac{\ln \left| \frac{(x-c)(x-d)}{x} \right|}{x-y} + \left(\frac{1}{x-y} - \frac{1}{x} \right) \ln |y| \\ &+ \left(\frac{1}{x} - \frac{1}{x-c} - \frac{1}{x-d} \right) \ln |y-x| + \left(-\frac{1}{x-y} + \frac{1}{x-c} \right) \ln |y-c| + \left(-\frac{1}{x-y} + \frac{1}{x-d} \right) \ln |y-d|. \end{aligned}$$

□

Lemma 5.5. *Let $W(x) = \int_c^d \frac{\ln R}{(x-y)^2} dy$. Then*

$$W(x) = \left(\frac{\ln \left| \frac{x-d}{d} \right|}{x} - \frac{\ln \left| \frac{x}{d} \right|}{x-d} \right) - \left(\frac{\ln \left| \frac{x-c}{c} \right|}{x} - \frac{\ln \left| \frac{x}{c} \right|}{x-c} \right) + 2 \left(\frac{\ln \left| \frac{x-c}{x-d} \right|}{x-d} - \frac{\ln \left| \frac{x-d}{x-c} \right|}{x-c} \right).$$

Proof. By lemma 5.4, we can write $W(x)$ as

$$\begin{aligned} & \left(\frac{1}{x-d} - \frac{1}{x-c} \right) \ln \left(\left| \frac{(x-c)(x-d)}{x} \right| \right) + \left(\frac{1}{x-d} - \frac{1}{x} \right) \ln |d| - \left(\frac{1}{x-c} - \frac{1}{x} \right) \ln |c| \\ & + \left(\frac{1}{x} - \frac{1}{x-c} - \frac{1}{x-d} \right) (\ln |x-d| - \ln |x-c|) + \left(-\frac{1}{x-d} + \frac{1}{x-c} \right) \ln |d-c| \\ & - \lim_{y \rightarrow c} \left(-\frac{1}{x-y} + \frac{1}{x-c} \right) \ln |y-c| \\ & + \lim_{y \rightarrow d} \left(-\frac{1}{x-y} + \frac{1}{x-d} \right) \ln |y-d| - \left(-\frac{1}{x-c} + \frac{1}{x-d} \right) \ln |c-d|. \end{aligned} \quad (27)$$

Now both limits appearing in (27) are zero since $\lim_{t \rightarrow 0} t \ln |t| = 0$. Thus, by rewriting (27) after regrouping according to $\frac{1}{x}$, $\frac{1}{x-c}$ and $\frac{1}{x-d}$, we obtain,

$$\begin{aligned} W(x) &= \frac{1}{x} (-\ln |d| + \ln |c| + \ln |x-d| - \ln |x-c|) \\ &+ \frac{1}{x-c} (-\ln |x-c| - \ln |x-d| + \ln |x| - \ln |c| - \ln |x-d| + \ln |x-c| + \ln |d-c| + \ln |d-c|) \\ &+ \frac{1}{x-d} (\ln |x-c| + \ln |x-d| - \ln |x| + \ln |d| - \ln |x-d| + \ln |x-c| - \ln |d-c| - \ln |d-c|) \\ &= \frac{1}{x} (\ln \left| \frac{x-d}{d} \right| - \ln \left| \frac{x-c}{c} \right|) + \frac{1}{x-c} (\ln \left| \frac{x}{c} \right| - 2 \ln \left| \frac{x-d}{c-d} \right|) + \frac{1}{x-d} (-\ln \left| \frac{x}{d} \right| + 2 \ln \left| \frac{x-c}{d-c} \right|) \\ &= \left(\frac{\ln \left| \frac{x-d}{d} \right|}{x} - \frac{\ln \left| \frac{x}{d} \right|}{x-d} \right) - \left(\frac{\ln \left| \frac{x-c}{c} \right|}{x} - \frac{\ln \left| \frac{x}{c} \right|}{x-c} \right) + 2 \left(\frac{\ln \left| \frac{x-c}{x-d} \right|}{x-d} - \frac{\ln \left| \frac{x-d}{x-c} \right|}{x-c} \right). \end{aligned}$$

□

Now to finish the proof of proposition 5.3, following [1], we introduce the following function for $a \neq b$

$$J(x, a, b) = 2\mathcal{L}\left(\frac{x-b}{a-b}\right)$$

so that

$$J'(x, a, b) = \frac{dJ(x, a, b)}{dx} = \frac{\ln \left| \frac{x-b}{a-b} \right|}{x-a} - \frac{\ln \left| \frac{x-a}{b-a} \right|}{x-b}.$$

By lemma 5.5, it follows that

$$W(x) = J'(x, 0, d) - J'(x, 0, c) + 2J'(x, d, c).$$

Therefore, by the construction the double integral,

$$\begin{aligned} & \int_0^1 \int_c^d \frac{\ln R}{(x-y)^2} dy dx \\ &= \int_0^1 W(x) dx = J(1, 0, d) - J(0, 0, d) - J(1, 0, c) + J(0, 0, c) + 2J(1, d, c) - 2J(0, d, c) \end{aligned}$$

But $J(0, 0, k) = 2\mathcal{L}(1)$, it follows that

$$\int_0^1 \int_c^d \frac{\ln R}{(x-y)^2} dy dx = 2\left(\mathcal{L}\left(\frac{d-1}{d}\right) - \mathcal{L}\left(\frac{c-1}{c}\right) + 2\mathcal{L}\left(\frac{c-1}{c-d}\right) - 2\mathcal{L}\left(\frac{c}{c-d}\right)\right).$$

□

To express the volume in terms of the lengths l and m , we take $c = e^l$ and $d = e^l \coth^2(m/2)$. Then $\frac{c-1}{c} = 1 - e^{-l}$, $\frac{d-1}{d} = 1 - e^{-l} \tanh^2(m/2)$, $\frac{c-1}{c-d} = (e^{-1} - 1) \sinh^2(m/2)$, and $\frac{c}{c-d} = -\sinh^2(m/2)$. Thus using (25), we see the volume $\mu(\Omega)$ in this case is

$$2[\mathcal{L}(1 - e^{-l} \tanh^2(m/2)) - \mathcal{L}(1 - e^{-l}) + 2\mathcal{L}((e^{-1} - 1) \sinh^2(m/2)) - 2\mathcal{L}(-\sinh^2(m/2))].$$

We now establish the identity (9) for the lasso function from (25).

Lemma 5.6. *Suppose $c = \frac{1}{s}$ and $d = \frac{1}{st}$ in proposition 5.3 where $1 < s, t < 1$. Then*

$$\mathcal{L}\left(\frac{d-1}{d}\right) - \mathcal{L}\left(\frac{c-1}{c}\right) + 2\mathcal{L}\left(\frac{c-1}{c-d}\right) - 2\mathcal{L}\left(\frac{c}{c-d}\right) = \mathcal{L}(t) - \mathcal{L}\left(\frac{1-s}{1-st}\right) + \mathcal{L}\left(\frac{1-t}{1-st}\right). \quad (28)$$

Proof. We have $\frac{d-1}{d} = 1 - st$, $\frac{c-1}{c} = 1 - s$, $\frac{c-1}{c-d} = \frac{-r}{1-r}$ where $r = \frac{t(1-s)}{t-1}$ and $\frac{c}{c-d} = -\frac{t}{1-t}$. Note the Roger's dilogarithm satisfies $\mathcal{L}(1-u) = \pi^2/6 - \mathcal{L}(u)$ and $\mathcal{L}(-\frac{u}{1-u}) = -\mathcal{L}(u)$ for $0 < u < 1$. It follows that $\mathcal{L}(\frac{c-1}{c-d}) = \mathcal{L}(\frac{-r}{1-r}) = -\mathcal{L}(r) = -\mathcal{L}(\frac{t(1-s)}{1-st})$ and $\mathcal{L}(\frac{c}{c-d}) = \mathcal{L}(-\frac{t}{1-t}) = -\mathcal{L}(t)$.

Thus the left-hand-side of (28) is

$$\begin{aligned} & \mathcal{L}(1-st) - \mathcal{L}(1-s) - 2\mathcal{L}\left(\frac{t(1-s)}{1-st}\right) + 2\mathcal{L}(t) \\ &= \pi^2/6 - \mathcal{L}(st) - \pi^2/6 + \mathcal{L}(s) - 2\mathcal{L}\left(\frac{t(1-s)}{1-st}\right) + 2\mathcal{L}(t) \\ &= \mathcal{L}(s) - \mathcal{L}(st) + 2\mathcal{L}(t) - 2\mathcal{L}\left(\frac{t(1-s)}{1-st}\right) \end{aligned}$$

Using a variation of the pentagon relation (5) that

$$\mathcal{L}(xy) - \mathcal{L}(x) - \mathcal{L}(y) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right) = 0,$$

we can write the above as

$$= \mathcal{L}(t) + \mathcal{L}\left(\frac{s(1-t)}{1-st}\right) - \mathcal{L}\left(\frac{t(1-s)}{1-st}\right).$$

Since $\frac{s(1-t)}{1-st} = 1 - \frac{1-s}{1-st}$ and $\mathcal{L}(1-u) = \pi^2/6 - \mathcal{L}(u)$, the above equation is $\mathcal{L}(t) - \mathcal{L}(\frac{1-s}{1-st}) + \mathcal{L}(\frac{1-t}{1-st})$. □

Corollary 5.7. *(Equation (11) for $f(P)$) Suppose P is a hyperbolic 3-holed sphere of boundary lengths l_i 's so that the lengths of M_i are m_i and B_i are p_i . Let $x_i = e^{-l_i}$ and $y_i = \tanh^2(m_i/2)$. Then*

$$\begin{aligned} f(P) &= 8\left[\sum_{i \neq j} \left(\mathcal{L}\left(\frac{1-x_i}{1-x_i y_j}\right) - \mathcal{L}\left(\frac{1-y_j}{1-x_i y_j}\right)\right) - \sum_{k=1}^3 \left(\mathcal{L}(y_k) + \mathcal{L}\left(\frac{1}{\cosh^2(p_k/2)}\right)\right)\right] \\ &= 4 \sum_{i \neq j} \left[2\mathcal{L}\left(\frac{1-x_i}{1-x_i y_j}\right) - 2\mathcal{L}\left(\frac{1-y_j}{1-x_i y_j}\right) - \mathcal{L}(y_j) - \mathcal{L}\left(\frac{(1-y_j)^2 x_i}{(1-x_i)^2 y_j}\right)\right] \end{aligned}$$

Proof. Recall that by definition and lemma 4.7, $f(P) = \mu(W(P)) = \mu(S(P)) - \mu(V(P)) = 4\pi^2 - [\sum_{i=1}^3 (\mu(H(M^{\pm 1})) + \mu(H(B^{\pm 1}))) + 4 \sum_{i \neq j} \mu(W(L_i, M_j))]$. It follows that

$$f(P) = 4\pi^2 - 8 \left[\sum_{i=1}^3 \left(\mathcal{L}\left(\frac{1}{\cosh^2(m_i/2)}\right) + \mathcal{L}\left(\frac{1}{\cosh^2(p_i/2)}\right) \right) + \sum_{i \neq j} La(l_i, m_j) \right].$$

Using $\mathcal{L}\left(\frac{1}{\cosh^2(m_i/2)}\right) = \mathcal{L}(1 - y_i) = \pi^2/6 - \mathcal{L}(y_i)$ and lemma 5.6, we can write the above as

$$4\pi^2 - 8 \left[\sum_{i=1}^3 \left(\mathcal{L}(1 - y_i) + \mathcal{L}\left(\frac{1}{\cosh^2(p_i/2)}\right) \right) + \sum_{i \neq j} \left(\mathcal{L}(y_i) - \mathcal{L}\left(\frac{1 - x_i}{1 - x_i y_j}\right) + \mathcal{L}\left(\frac{1 - y_j}{1 - x_i y_j}\right) \right) \right].$$

Since $\mathcal{L}(1 - y_i) + \mathcal{L}(y_i) = \pi^2/6$, and $\frac{1}{\cosh^2(p_i/2)} = \frac{(1 - y_j)^2 x_i}{(1 - x_i)^2 y_j}$ by (8), the above equation is equivalent to the identity in the corollary. \square

5.3. Surfaces with boundary. Let F be a hyperbolic surface with non-empty geodesic boundary such that the Euler characteristic $\chi(F) < -1$. As in §4, for a generic unit tangent vector $v \in S(F)$, $G(v)$ is a graph contained in an embedded simple geometric subsurface Σ of F , except that now, $G(v) \cap \partial F$ may not be empty. Again, as in §4, we need to calculate $\mu(W(\Sigma))$, where

$$W(\Sigma) = \{v \in S(\Sigma) | G(v) = G_\Sigma(v)\}.$$

When $\Sigma \cap \partial F = \emptyset$, then the computation of $\mu(W(\Sigma))$ is exactly the same as in §4. This occurs when Σ is a 1-holed torus T (since $\chi(F) < -1$), or when it is a 3-holed sphere P for which $\partial P \cap \partial F = \emptyset$. It remains to compute $\mu(W(P))$ when P is an embedded geometric 3-holed sphere for which $\partial P \cap \partial F$ consists of either one or two components.

Let L_1, L_2, L_3 be the boundary components of P . We first consider the case where $\partial P \cap \partial F$ has one component, which we may take to be L_1 by convention. We also use the shorthand notation

$$W(L_i^{\pm 1}, M_j) = W(L_i, M_j) \bigsqcup W(L_i^{-1}, M_j). \quad (29)$$

We see from the definition of $W(P)$ that in this case, besides spines $G(v)$ for P which do not intersect ∂P , $G(v)$ is also a spine for P when

$$v \in H(B_1^{\pm 1}) \cup W(L_2^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_2) \cup \mathbf{A}(W(L_2^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_2)).$$

It follows that for such P ,

$$\hat{f}(P) := \mu(W(P)) = f(P) + 8 \left(\mathcal{L}\left(\frac{1}{\cosh^2 p_1/2}\right) + La(l_2, m_3) + La(l_3, m_2) \right) \quad (30)$$

The remaining case is when $\partial P \cap \partial F$ has two components, which we may take to be L_1 and L_2 by convention. Now, besides spines $G(v)$ for P which do not intersect ∂P , $G(v)$ is also a spine for P when

$$\begin{aligned} v \in & H(B_1^{\pm 1}) \cup H(B_2^{\pm 1}) \cup H(M_3) \\ & \cup W(L_2^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_2) \cup W(L_1^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_1) \\ & \cup \mathbf{A}(W(L_2^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_2) \cup W(L_1^{\pm 1}, M_3) \cup W(L_3^{\pm 1}, M_1)). \end{aligned}$$

It follows that for such P ,

$$\begin{aligned} \bar{f}(P) := \mu(W(P)) = f(P) + 8\{ & \mathcal{L}\left(\frac{1}{\cosh^2 p_1/2}\right) + \mathcal{L}\left(\frac{1}{\cosh^2 p_2/2}\right) + \mathcal{L}\left(\frac{1}{\cosh^2 m_3/2}\right) \\ & + La(l_2, m_3) + La(l_3, m_2) + La(l_1, m_3) + La(l_3, m_1)\} \end{aligned}$$

Theorem 1.2 now follows.

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Appendix

A1. Pentagon relations for dilogarithm and hyperbolic pentagons The following simple property was discovered during our study of the Roger’s dilogarithm. It puts the pentagon relations in the perspective of lengths of hyperbolic right-angled pentagons.

Proposition 5.8. *Suppose l_1, \dots, l_5 are the lengths of the five sides of a hyperbolic right-angled pentagon. Then*

$$\sum_{i=1}^5 \mathcal{L}(\tanh^2(l_i)) = \pi^2/2,$$

and

$$\sum_{i=1}^5 \mathcal{L}\left(\frac{1}{\cosh^2(l_i)}\right) = \pi^2/3.$$

In fact, each of the above is equivalent to the pentagon relation (5) for the Roger’s dilogarithm.

Proof. We assume all pentagons are right-angled in the sequel. We begin with the sine law for pentagons. Suppose the edges e_1, \dots, e_5 in the hyperbolic pentagon are cyclically labelled so that the length of e_i is l_i . Then the sine law for pentagon says that $\cosh^2(l_i) = \sinh(l_{i+2})\sinh(l_{i+3})$. Let $s_i = \sinh^2(l_i)$, then the sine law

says $s_i + 1 = s_{i+2}s_{i+3}$. Let $x_i = \tanh^2(l_i) = \frac{s_i}{s_i+1}$ and let $x = x_1$, $y = x_3$. Then $x_5 = \frac{1-x}{1-xy}$, $x_4 = \frac{1-y}{1-xy}$ and $x_2 = 1 - xy$ by the relations $s_i + 1 = s_{i+2}s_{i+3}$.

Now the pentagon relation for the Roger's dilogarithm $\mathcal{L}(t)$ says for $x, y \in (0, 1)$,

$$\mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(1 - xy) + \mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}.$$

The five variables inside $\mathcal{L}(t)$ are exactly $\tanh^2(l_i)$ by the above calculation. Thus the first identity follows. Since $\mathcal{L}\left(\frac{1}{\cosh^2(x)}\right) = \pi^2/6 - \mathcal{L}(\tanh^2(x))$, the second equation follows. □

A2. Using different rules to generate $G(v)$

For a generic unit tangent vector $v \in S(F)$, we gave a somewhat arbitrary rule to define the graph $G(v)$ in §3 (generating the geodesic at equal speed in both forwards and backwards direction until we obtain intersections), from which we obtained the decomposition of the unit tangent bundle $S(F)$ which gave rise to the identities in Theorems 1.1 and 1.2. The main advantage of our choice was that for generic vectors $v \in S(F)$, $G(v) = G(-v)$ so that in the computation of the measures $\mu(W(\Sigma))$ for geometrically embedded simple surfaces in §4, we were able to exploit the symmetry in our computations. In particular, in the computation of the measure of the set of vectors $v \in S(\Sigma)$ which generated lassos, we just doubled the measure of the vectors which generated the positively oriented lassos. A natural question which arises is whether we get different identities if we use a different rule for generating $G(v)$. As an example, a fairly natural choice would be a forward first rule, that is, to generate $g^+(v)$ until the first point of intersection, after which we generate $g^-(v)$ until the next point of intersection, thereby producing a graph $G(v)$ as in §3. More generally, we may generate $g^+(v)$ and $g^-(v)$ at different fixed constant speeds to obtain $G(v)$.

It is clear that the homotopy type of $G(v)$ may be different for different rules, hence, we would obtain a different decomposition of the unit tangent bundle $S(F)$. We claim here that nonetheless, the resulting identities obtained are all the same. The main observation is that the measure of the complementary set $V(\Sigma)$ of vectors which do not generate spines for a simple surface $\Sigma \subset F$ are the same, for different rules.

We give a brief explanation here. Recall from Lemma 4.7 that $v \in V(P)$ if $v \in H(M_i)$ or $H(B_i)$, $i = 1, 2, 3$, or v or $-v \in W(L_i^{\pm 1}, M_j)$, $1 \leq i \neq j \leq 3$. Furthermore, the sets are disjoint.

There is no problem with $H(M_i)$ and $H(B_i)$, the sets are the same whatever rules we use to define $G(v)$ and so they have the same measures. The issue arises in the sets $W(L_i^{\pm 1}, M_j)$, $1 \leq i \neq j \leq 3$, which depend on the rule used to define $G(v)$. More specifically, suppose that $\alpha : [T_1, T_2] \rightarrow P$ is a positively oriented lasso on P with base point on L_1 and with a positive loop around L_2 such that $\alpha(T_3) = \alpha(T_2)$ for some $T_1 < T_3 < T_2$ (cf definition 4.5). Then, if we use the original rule for generating $G(v)$, $v = \alpha'(t) \in W(L_2, M_3)$ (that is, $G(v) = \alpha$) if and only if $T_1 < t < \frac{T_3+T_2}{2}$ and $-v$ generates $-\alpha$ if and only if v generates α . However, for example, if we use the forward first rule instead, than $v = \alpha'(t) \in W(L_2, M_3)$ (that is, $G(v) = \alpha$) if and only if $T_1 < t < T_3$, while $-v$ generates $-\alpha$ if and only if $v = \alpha'(t)$ with $T_1 < t < T_2$.

The main observation is that when we sum over the measure of all v and $-v$ which generate either α or $-\alpha$, it is given by $(T_3 + T_2) - 2T_1$, which is the same as for the first rule. If we let $W^-(L_2, M_3)$ be the set of vectors $v \in S(P)$ generating lassos with negative orientation and base point at L_1 and loop homotopic to L_2 , then it follows that the measure of $W(L_2, M_3) \cup W^-(L_2, M_3)$ is the same for both rules. In fact, the same argument shows that any consistently applied rule gives the same measure for $W(L_2, M_3) \cup W^-(L_2, M_2)$, the extra measure in one set is compensated by the deficit in the other. It follows that $f(P)$ depends only on the lengths L_1, L_2 and L_3 . A similar argument holds for $g(T)$, $\bar{f}(P)$ and $\hat{f}(P)$.

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