Continuity of the Volume of Simplices in Classical Geometry

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Abstract

It is proved that the volume of spherical or hyperbolic simplices, when considered as a function of the dihedral angles, can be extended continuously to degenerated simplices.

§1. Introduction

1.1. It is well known that the area of a spherical or a hyperbolic triangle can be expressed as an affine function of the inner angles by the Gauss-Bonnet formula. In particular, the area considered as a function of the inner angles can be extended continuously to degenerated spherical or hyperbolic triangles. The purpose of the paper is to show that the continuous extension property holds in any dimension. Namely, if a sequence of spherical (or hyperbolic) n-simplices has the property that their corresponding dihedral angles at codimension-2 faces converge, then the volumes of the simplices converge. Note that if we consider the area as a function of the three edge lengths of a triangle, then there does not exist any continuous extension of the area to all degenerated triangles. For instance, a degenerated spherical triangle of edge lengths $0, \pi, \pi$ is represented geometrically as the intersection of two great circles at the north and the south poles. However, its area depends on the intersection angle of these two geodesics and cannot be defined in terms of the lengths. This 2-dimensional simple phenomenon still holds in high dimension for both spherical and hyperbolic simplices.

To state our result, let us introduce some notations. Given an n-simplex with vertices $v_1, ..., v_{n+1}$, the i-th codimension-1 face is defined to be the (n-1)-simplex with vertices $v_1,...,v_{i-1},v_{i+1},..., v_{n+1}$. The dihedral angle between the i-th and j-th codimension-1 faces is denoted by a_{ij}. As a convention, we define $a_{ii} = \pi$ and call the symmetric matrix $[a_{ij}]_{(n+1)\times(n+1)}$ the *angle matrix* of the simplex. It is well known that the angle matrix $[a_{ij}]_{(n+1)\times(n+1)}$ determines the simplex up to isometry in spherical and hyperbolic geometry.

Let $\mathbf{R}^{m\times m}$ be the space of all real $m \times m$ matrics. Our main result is the following.

Theorem 1.1. *Let $X_n(1)$ and $X_n(-1) \subset \mathbf{R}^{(n+1)\times(n+1)}$ be the spaces of angle matrices of all n-dimensional spherical and hyperoblic simplices respectively. The volume function $V : X_n(k) \to \mathbf{R}$ can be extended continuously to the closure of $X_n(k)$ in $\mathbf{R}^{(n+1)\times(n+1)}$ for $k = 1, -1$.*

Note that both spaces $X_n(1)$ and $X_n(-1)$ are fairly explicitly known. Topologically, both of them are homeomorphic to the Euclidean space of dimension $n(n + 1)/2$. We do not know if Theorem 1.1 can be generalized to convex polytopes of the same combinatorial type in the 3-sphere or the hyperbolic 3-space.

The proof of the theorem for spherical simplieces is quite simple. It is an easy consequence of the continuity of the function which sends a semi-positive definite symmetric matrix to its square root. The case of the hyperbolic simplices is more subtle. It uses the continuity of the square roots of semi-positive definite symmetric matrices and the following property of hyperbolic simplices. We use $B_R(x)$ to denote the ball of radius R centered at x.

Theorem 1.2. *For any $\epsilon > 0$ and any $r > 0$, there is $R = R(\epsilon, r, n)$ so that for any hyperbolic n-simplex σ, if $x \in \sigma$ is a point whose distance to each totally geodesic codimenison-1 hypersurface containing a codimension-1 face is at most r, then the volume of $\sigma - B_R(x)$ is at most ϵ.*

Recall that the *center* and the *radius* of a simplex are defined to be the center and the radius of its inscribed ball. The raduis of a hyperbolic n-simplex is well known to be uniformaly bounded from above. Applying Theorem 1.2 to the center of the n-simplex, we conclude that for any $\epsilon > 0$, there is $R = R(\epsilon)$ so that the volume of $\sigma - B_R(c)$ is less than ϵ for any hyperbolic n-simplex σ with center c.

Recent work of [MY] produces an explicit formula expressing volume of spherical and hyperbolic tetrahedra in terms of the dihedral angles using dilogarithmic function. It is not clear if Theorem 1.1 in dimension 3 follows from their explicit formula.

1.2. Using the work of Aomoto [Ao] and Vinberg [Vi], one may express the volume of a simplex in terms of an integral related to the Gaussian distribution (see (2.3) and (2.7)). To state Theorem 1.1 in terms of matrices, let us introduce some notations. For an $n \times n$ matrix A, we use $ad(A)$ to denote the adjacency matrix of A. The transpose of A is denoted by A^t. The ij-th entry of A is denoted by A_{ij}. We use $A > 0$ to denote the condition that all entries in A are positive. Evidently, if a matrix A is positive definite, or $ad(A) > 0$, then the following function F is well defined,

$$F(A) = \sqrt{|det(ad(A))|} \int_{R^n_{\geq 0}} e^{-x^t ad(A) x} dx \tag{1.1}$$

where $x \in \mathbf{R}^n$ is a column vector, $\mathbf{R}_{\geq 0}$ is the set of all non-negative numbers and dx is the Euclidean volume form. Theorem 1.1 is equivalent to the following,

Theorem 1.3. *Let $\mathcal{X}_n = \{A \in \mathbf{R}^{n\times n} | \ A^t = A,$ all $A_{ii} = 1,$ A is positive definite$\}$ and let $\mathcal{Y}_n = \{A \in \mathbf{R}^{n\times n} | \ A^t = A,$ all $A_{ii} = 1,$ $ad(A) > 0,$ $detA < 0,$ and all principal $(n-1)\times(n-1)$ submatrices of A are positive definite$\}$. Then the function $F : \mathcal{X}_n \cup \mathcal{Y}_n \to \mathbf{R}$ can be extended continuously to the closure of $\mathcal{X}_n \cup \mathcal{Y}_n$ in $\mathbf{R}^{n\times n}$.*

We don't know a proof of Theorem 1.3 without using hyperbolic geometry (i.e., Theorem 1.2).

1.3. The paper is organized as follows. In §2, we recall the basic set up and the Gram matrices of simplices. Also, we prove Theorem 1.1 for spherical simplices. In §3, we prove

Theorem 1.1 for hyperbolic simplices assuming Theorem 1.2. We prove Theorem 1.2 in section §4.

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§2. **Preliminaries on Spherical and Hyperbolic Simplices**

We recall some of the basic material related to the spherical and hyperbolic simplices in this section. In particular, we will recall the Gram matrices, the dual simplex and the volume formula. We also give a proof of Theorem 1.1 for spherical simplices. Here are the conventions and notations. Let $\mathbf{R}^m$ denote the m-dimensional real vector space whose elements are column vectors. A diagonal matrix with diagonal entries $a_{11}, ..., a_{nn}$ will be denoted by $diag(a_{11}, ..., a_{nn})$. A diagonal matrix is *positive* if all diagonal entries are positive. The Kronecker delta is denoted by δ_{ij}. The standard inner product in $\mathbf{R}^m$ is denoted by $(x, y) = x^t y$. The length of a vector $x \in \mathbf{R}^m$ is denoted by $|x| = \sqrt{(x, x)}$. We use $dx = dx_1 dx_2 ... dx_m$ to denote the Euclidean volume element in $\mathbf{R}^m$ and $\mathbf{R}^m_{\geq 0}$ to denote the set $\{(x_1, ..., x_m) \in \mathbf{R}^m | x_i \geq 0 \text{ for all } i\}$.

We will make a use of the continuity of the square root of symmetric semi-positive definite matrix. Recall that if A is a symmetric semi-positive definite matrix, then its square root $\sqrt{A}$ is the symmetric semi-positive definite matrix so that it commutes with A and its square is A. It is well known that the square root matrix is unique. Furthermore, the square root operation, considered as a self map defined on the space of all symmetric semi-positive definite matrices, is continuous (theorem 6.2.37 in [HJ]).

2.1. **Gram Matrices of Spherical Simplices**

Let $\mathbf{R}^{n+1}$ be the Euclidean space with the standard inner product. The sphere S^n is $\{x \in \mathbf{R}^{n+1} | (x, x) = 1\}$. A spherical n-simplex σ^n has vertices $v_1, ..., v_{n+1}$ in S^n so that the vectors $v_1, ..., v_{n+1}$ are linearly independent. The codimension-1 face of σ^n opposite v_i is denoted by σ^n_i. Let d_{ij} be the spherical distance between v_i and v_j and a_{ij} be the dihedral angle between the codimension-1 faces σ^n_i and σ^n_j for $i \neq j$. Define $d_{ii} = 0$ and $a_{ii} = \pi$. Then the *Gram matrix* of σ^n is defined to be the matrix $G = [\cos(d_{ij})] = [(v_i, v_j)]$ and the *angle Gram matrix* of the the simplex is the matrix $G^* = [-\cos(a_{ij})]$. Note that both of them are symmetric with diagonal entries being 1. The following is a well known fact.

Lemma 2.1. *The Gram matrix G and the angle Gram matrix G^* of a simplex are related by the following formula*

$$G^* = DG^{-1}D \tag{2.1}$$

where D is a positive diagonal matrix.

Proof. Let $B = [v_1, ..., v_{n+1}]$ be the $(n+1) \times (n+1)$ matrix whose i-th column is the i-th vertex v_i. Then the Gram matrix G of the simplex σ^n is $B^t B$ due to the obvious formula

$v_i^t v_j = (v_i, v_j) = \cos(d_{ij})$. To relate the matrix G^* with G, we consider the *dual simplex.* First, find (n+1) independent vectors $w_1, ..., w_{n+1} \in \mathbf{R}^{n+1}$ so that

$$(v_i, w_j) = \delta_{ij}. \tag{2.2}$$

Define $v_i^* = w_i/|w_i|$. Then the dual simplex of σ^n is the spherical simplex with vertices $\{v_1^*, ..., v_{n+1}^*\}$. If we use $W = [w_1, ..., w_{n+1}]$, then (2.2) says $B^t W = Id$. In particular, $W = (B^t)^{-1}$. Thus $W^t W = (B^t B)^{-1} = G^{-1}$. However, by the formula $v_i^* = w_i/|w_i|$, we see that the Gram matrix of the dual simplex is $D(W^t W)D = DG^{-1}D$ where D is the diagonal matrix whose ii-th entry is $|w_i|^{-1}$. On the other hand, by the definition of dual simplex, the Gram matrix of the dual is exactly the same as the angle Gram matrix of σ^n. Namely, the spherical distance between v_i^* and v_j^* is $\pi - a_{ij}$. Thus (2.1) follows. QED

The volume of the simplex σ^n can be calculated as follows (see [Ao], [Vi]). For the simplex $\sigma^n \subset S^n$, let the cone in $\mathbf{R}^{n+1}$ based at the origin over σ^n be $K(\sigma^n) = \{rx \in \mathbf{R}^{n+1} | r \geq 0 \text{ and } x \in \sigma^n\}$. Note that the linear transformation $B : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ sending the vector x to Bx takes the standard basis element e_i to v_i. In particular $B(\mathbf{R}^n_{\geq 0}) = K(\sigma^n)$. Let $\mu_k = \int_0^\infty x^k e^{-x^2} dx$, i.e., $\mu_{2k} = \sqrt{\pi}(1.3....(2k-3)(2k-1)/2^{k+1}$ and $\mu_{2k+1} = 2.4....(2k-2)(2k)/2^{k+1}$. Let the volume element on S^n be ds, then the volume $V(\sigma^n)$ of the simplex σ^n is given by (see [Ao], [Vi]),

$$\begin{aligned} V(\sigma^n) &= \int_{\sigma^n} ds \\ &= \mu_n^{-1} \int_{K(\sigma^n)} e^{-(x,x)} dx \\ &= \mu_n^{-1} \int_{B(\mathbf{R}^{n+1}_{\geq 0})} e^{-(x,x)} dx \\ &= \mu_n^{-1} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{-(By,By)} |detB| dy \\ &= \mu_n^{-1} \sqrt{|detG|} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{-y^t G y} dy. \end{aligned} \tag{2.3}$$

Lemma 2.2. *Let χ be the characteristic function of the set $\mathbf{R}^{n+1}_{\geq 0}$ in $\mathbf{R}^{n+1}$, then the volume $V(\sigma^n)$ of a spherical simplex σ^n can be written as*

$$V(\sigma^n) = \mu_n^{-1} \int_{\mathbf{R}^{n+1}} e^{-(x,x)} \chi(\sqrt{G^*}(x)) dx. \tag{2.4}$$

Proof. Note that since $G = B^t B$ is positive definite, G^{-1} is again symmetric and positive definite. Let $A = \sqrt{G^{-1}}$ be the square root of G^{-1} so that A is symmetric positive definite and $AGA = Id$. Now make a change of variable $y = Az$ in (2.3) where

$z \in A^{-1}(\mathbf{R}^{n+1}_{\geq 0})$. Then, $V(\sigma^n) = \mu_n^{-1} \int_{A^{-1}(\mathbf{R}^{n+1}_{\geq 0})} e^{-(z,z)} dz$. Note that the characteristic function of $A^{-1}(\mathbf{R}^{n+1}_{\geq 0})$ is the same as the composition $\chi \circ A$. Thus the volume is

$$V(\sigma^n) = \mu_n^{-1} \int_{\mathbf{R}^{n+1}} e^{-(x,x)} \chi(A(x)) dx.$$

Finally, note that if we make a change of variable of the form $x = D(y)$ where D is a positive diagonal matrix, the integral (2.3) does not change. By lemma 1.1, we have $A = D\sqrt{G^*}D$ for a positive diagonal matrix D. Thus (2.4) holds.

2.2. **A Proof of Theorem 1.1 for Spherical Simplices**

We give a proof of Theorem 1.1 for spherical simplices in this section. Let $X_n(1)$ be the space of all angle matrices $[a_{ij}]_{(n+1)\times(n+1)}$ of spherical n-simplices where $a_{ij} = a_{ji}$ and $a_{ii} = \pi$. The map sending $[a_{ij}]$ to the angle Gram matrix $G^* = [-\cos(a_{ij})]$ is an embedding of the closure of $X_n(1)$ into the space of all semi-positive definite, symmetric matrices whose diagonal entries are 1. Thus, to prove the continuity of the volume function on $X_n(1)$, by (2.4) it suffices to show the continuity of the function $W : \mathcal{X}_n \to \mathbf{R}$ sending a matrix A to

$$W(A) = \int_{R^{n+1}} e^{-(x,x)} \chi \circ \sqrt{A}(x) dx. \tag{2.5}$$

To this end, take a sequence $\{A_m\}$ in $\mathcal{X}_n$ so that $\lim_{m\to\infty} A_m = A$ in $\mathbf{R}^{(n+1)\times(n+1)}$. To establish the existence of $\lim_{m\to\infty} W(A_m)$, we first use the fact that the function sending a semi-positive definite matrix to its square root is continuous (theorem 6.2.37 in [HJ]). In particular, $\sqrt{A_m}$ converges to $\sqrt{A}$.

Lemma 2.3. *Suppose B_m is a convergent sequence of $(n+1) \times (n+1)$ matrices so that $\lim_{m\to\infty} B_m = B$. If each row vector of B is none-zero, then the function $\chi \circ B_m$ converges almost everywhere to $\chi \circ B$ in $\mathbf{R}^{n+1}$.*

Assuming this lemma, we finish the proof as follows. Since all diagonal entries of A are 1, we conclude that no row vector in $\sqrt{A}$ is zero. Thus by the lemma, $\chi \circ \sqrt{A_m}$ converges almost everywhere to $\chi \circ \sqrt{A}$ in $\mathbf{R}^{n+1}$. Since the integrant in $W(A))$ is bounded by the integrable function $e^{-(y,y)}$, the dominant convergent theorem implies that $\lim_{m\to\infty} W(A_m)$ exists.

To prove lemma 2.3, let $R_i = \{x \in \mathbf{R}^{n+1} | x_i = 0\}$ be the coordinate planes. Then $B^{-1}(R_i)$ is a proper subspace of $\mathbf{R}^{n+1}$. Indeed, if otherwise, say for some index i, $B(\mathbf{R}^{n+1}) \subset R_i$, then the i-th row of B must be zero. This contradicts the assumption. Therefore, the Lebegue measure of $B^{-1}(R_i)$ is zero for all indices i. Now we claim for every point $x \in \mathbf{R}^{n+1} - \cup_{i=1}^{n+1} B^{-1}(R_i)$, the sequence $\chi \circ B_m(x)$ converges to $\chi \circ B(x)$. Indeed, by the assumption, $B_m(x)$ converges to $B(x) \in \mathbf{R}^{n+1} - \cup_{i=1}^{n+1} R_i$. Thus we have $\chi(B_m(x))$ converges to $\chi(B(x))$. QED

The above also produced a proof of Theorem 1.3 for the case of continuous extension of F to the closure of $\mathcal{X}_n$.

2.3. **Volume and Gram Matrices of Hyperbolic Simplices**

The (n+1)-dimensional Minkowski space $\mathbf{R}^{n,1}$ is $\mathbf{R}^{n+1}$ together with the symmetric non-singular bilinear form $< x, y >= \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} = x^t S y$ where $S = diag(1, 1, ..., 1, -1)$ is an $(n+1) \times (n+1)$ diagonal matrix. We define the hyperboloid of two sheets to be $S(-1) = \{x \in \mathbf{R}^{n,1} | < x, x >= -1\}$ and the unit sphere $S(1) = \{x \in \mathbf{R}^{n,1} | < x, x >= 1\}$. The space $S(-1)$ has two connected components. It is well known that each of them can be taken as a model for the n-dimensional hyperbolic space H^n. For simplicity, we take H^n to be the component with positive last coordinates, i.e., $H^n = S(-1) \cap \{x_{n+1} > 0\}$. Given a vector $u \in S(1)$, let $u^\perp$ be the totally geodesic codimension-1 space $\{x \in H^n | < x, u >= 0\}$. The following lemma is well known (see for instance [Vi]).

Lemma 2.4. *Suppose $u, v \in S(1) \cup S(-1)$. The following holds.*

(1) If $u, v \in H^n$, then $< u, v > \leq -1$ and the hyperbolic distance between u, v is $\cosh^{-1}(- < u, v >)$.

(2) If $u, v \in S(1)$, then $u^\perp$ intersects $v^\perp$ if and only if $| < u, v > | < 1$. In this case, the dihedral angle of the intersection $u^\perp$, $v^\perp$ in the region $\{x \in H^n | < x, u >< x, v > \geq 0\}$ is $\arccos(- < u, v >)$.

(3) If $u \in H^n$ and $v \in S(1)$, then the distance from u to $v^\perp$ is $\cosh^{-1}(\sqrt{1+ < u, v >^2})$.

A hyperbolic n-simplex σ^n has vertices $v_1, ..., v_{n+1}$ in H^n so that these vectors are linearly independent in $\mathbf{R}^{n,1}$. We denote the codimension-1 face of σ^n opposite to v_i by σ_i^n. The hyperbolic distance between v_i and v_j is denoted by d_{ij} and the dihedral angle between σ_i^n and σ_j^n is denoted by a_{ij} for $i \neq j$. As a convention, $d_{ii} = 0$ and $a_{ii} = \pi$. As in the case of spherical simplices, we define the *Gram matrix* G of σ^n to be $G = [\cosh d_{ij}] = [- < v_i, v_j >]$ and the *angle Gram matrix* of σ^n to be $G^* = [-\cos(a_{ij})]$. Note that both of these matrices are symmetric with diagonal entries ± 1.

The counterpart of lemma 2.1 holds, it is the following,

Lemma 2.5. *Suppose G and G^* are the Gram matrix and the angle Gram matrix of a hyperbolic n-simplex, then there is a positive diagonal matrix D so that*

$$G^* = -DG^{-1}D.$$

Proof. By lemma 2.4, $\cosh d_{ij} = - < v_i, v_j >= -v_i^t S v_j$. Let $B = [v_1, ..., v_{n+1}]$ be the square matrix whose i-th column is the i-th vertex v_i, then by definition the Gram matrix G is $-B^t S B$ where $S = diag(1, 1, .., 1, -1)$. To relate G^* with G, we find vectors $w_1, ..., w_{n+1}$ in $\mathbf{R}^{n,1}$ so that $< v_i, w_j >= \delta_{ij}$. Indeed, these vectors can be found by taking the matrix $W = [w_1, ..., w_{n+1}]$. The condition $< v_i, w_j >= \delta_{ij}$ translates to the equation, $B^t S W = Id$, i.e., $W = S(B^t)^{-1}$. By the construction of vertices $\{v_1, ..., v_{n+1}\}$, the bilinear form $<,>$ restricted to the codimension-1 linear space spanned by $\{v_1, ..., v_{n+1}\} - \{v_i\}$ has signature $(n-1, 1)$. This implies that $< w_i, w_i >$ is positive. Define $v_i^* = w_i/\sqrt{< w_i, w_i >}$. Then $v_i^* \in S(1)$ and $< v_i^*, v_j >= (\sqrt{< w_i, w_i >})^{-1}\delta_{ij}$. The last equation shows that v_i^* is the unit vector in $S(1)$ orthogonal to the i-th codimension-1 face σ_i^n so that $< v_i, v_i^* >> 0$. By lemma 2.4(2), the intersection angle a_{ij} between σ_i^n and σ_j^n is given by the equation $-\cos a_{ij} =< v_i^*, v_j^* >$. This shows that the Gram matrix $A = [< v_i^*, v_j^* >]$ of the vectors $\{v_1^*, ..., v_{n+1}^*\}$ is equal to the angle Gram matrix G^*. On the other hand, $v_i^* = w_i/\sqrt{< w_i, w_i >}$. Thus the Gram matrix A can be expressed as DFD where D is a diagonal matrix with positive diagonal entries and F is the Gram matrix $[< w_i, w_j >]$. By definition, $F = W^t SW$. Since $W = S(B^t)^{-1}$, we have $F = W^t SW = (B^t SB)^{-1} = -G^{-1}$. This establishes $G^* = -DG^{-1}D$. QED

Let the volume element on H^n be ds, let $K(\sigma^n) = \{rx \in \mathbf{R}^{n+1} | r \geq 0, x \in \sigma^n\}$ be the cone based at the vertex 0 spanned by the simplex σ^n in the vector space $\mathbf{R}^{n+1}$ and $dx = dx_1...dx_{n+1}$ be the Euclidean volume form in the Euclidean metric in $\mathbf{R}^{n+1}$. Then the hyperbolic volume $V(\sigma^n)$ is given by (see [Vi], p28, note the Gram matrix used in [Vi] is the angle Gram matrix in our case),

$$V(\sigma^n) = \int_{\sigma^n} ds$$

$$= \mu_n^{-1} \int_{K(\sigma^n)} e^{<x,x>} dx$$

$$= \mu_n^{-1} \int_{B(\mathbf{R}^{n+1}_{\geq 0})} e^{<x,x>} dx$$

$$= \mu_n^{-1} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{<By,By>} |detB| dy$$

$$= \mu_n^{-1} \sqrt{|detG|} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{y^t B^t SBy} dy$$

$$(2.6) \qquad = \mu_n^{-1} \sqrt{|detG|} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{-y^t Gy} dy.$$

Since the integration in (2.6) remains unchanged if we replace G by DGD for a positive diagonal matrix, by lemma 2.5, (2.6) is the same as

$$V(\sigma^n) = \mu_n^{-1} (\sqrt{|detG^*|})^{-1} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{y^t (G^*)^{-1} y} dy$$

$$(2.7) \qquad = \mu_n^{-1} \sqrt{|det(ad(G^*))|} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{-y^t ad(G^*) y} dy$$

To summary, we have

Lemma 2.6. ([Vi]) *Suppose a hyperbolic n-simplex has angle Gram matrix G^*. Then the volume of the simplex is a function of G^* given by (2.7).*

2.4. **Some Results from Matrix Pertubation Theory**

The following two results will be used frequently in the paper. See [SS], [Wi] for proofs. The first theorem states the continuous dependence of eigenvalues on the matrices.

Theorem 2.7(Ostrowski) *Let λ be an eigenvalue of A of algebraic multiplicity m. Then for any matrix norm $||.||$ and all sufficiently small $\epsilon > 0$, there is $\delta > 0$ so that if $||B - A|| \leq \delta$, the disk $\{z \in \mathbf{C} || z - \lambda| \leq \epsilon\}$ contains exactly m eigenvalues of B counted with multiplicity.*

The next theorem concerns the continuous dependence of eigenvectors on the matrices. We state the result in the form applicable to our situation. Recall that an eigenvalue of a matrix is called *simple* if it is the simple root of the characteristic polynomial.

Theorem 2.8 (see [Wi], p67) *Suppose A_m is a sequence of $n \times n$ matrices converging to B. Suppose λ is a simple eigenvalue of B and λ_m is a simple eigenvalue of A_m so that* $\lim_{m \to \infty} \lambda_m = \lambda$. *Then there exists a sequence of eigenvectors v_m of A_m associated to λ_m so that these eigenvectors converge to an eigenvector of B associated to λ.*

This theorem follows from the fact that if λ is simple eigenvalue, then the adjacency matrix $ad(B - \lambda Id)$ has rank 1 and its non-zero column vectors are the eigenvectors of B associated to λ.

§3. **A Proof of Theorem 1.1 for Hyperbolic Simplices Assuming Theorem 1.2**

Recall that $X_n(-1)$ denotes the space of all angle matrices $[a_{ij}]$ of hyperbolic n-simplices. The map $\cos(x)$ is an embedding of $[0, \pi]$ to $[-1, 1]$. Thus the angle Gram matrix $G^* = [-\cos(a_{ij})]$ is a map which embeds the closure of $X_n(-1)$ in $\mathbf{R}^{(n+1)\times(n+1)}$ to the space of all symmetric matrics. The characterization of angle Gram matrix $[-\cos(a_{ij})]$ was known.

Lemma 3.1. ([Lu], [Mi]) *An $(n+1) \times (n+1)$ symmetric matrix A with diagonal entries being one is the angle Gram matrix of a hyperbolic n-simplex if and only if*
(3.1) all principal $n \times n$ submatrices of A are positive definite,
(3.2) $det(A) < 0$, and,
(3.3) all entries of the adjacency matrix $ad(A)$ are positive.

Let $\mathcal{Y}_{n+1}$ be the space of all real matrices satisfying conditions in lemma 3.1 and define a function $F : \mathcal{Y}_{n+1} \to \mathbf{R}$ as in (1.1). Note that by change the variable x to $D(x)$ for a positive diagonal matrix D, we see that $F(A) = F(DAD)$. Thus to establish theorem 1.1 for hyperbolic n-simplices, it suffices to prove that $F : \mathcal{Y}_{n+1} \to \mathbf{R}$ can be extended continuously to the closure $\bar{\mathcal{Y}}_{n+1}$ in $\mathbf{R}^{(n+1)\times(n+1)}$. This will be the goal in the rest of the section.

3.1. To prove Theorem 1.3 for $\mathcal{Y}_{n+1}$, take a convergent sequence of matrices $A_m \in \mathcal{Y}_{n+1}$ so that $\lim_{m\to\infty} A_m = A_\infty$ where $A_\infty \in \mathbf{R}^{(n+1)\times(n+1)}$. We will prove that $\lim_{m\to\infty} F(A_m)$ exists. Since the function $F(A) = F(DAD)$ for any positive diagonal matrix D, we will modify the sequence $\{A_m\}$ by $D_m A_m A_m$ for positive diagonal matrices D_m so that $\lim_{m\to\infty} F(D_m A_m D_m)$ converges. This will be the strategy of the proof.

By definition, all diagonal entries of A_∞ are 1. If $det(A_\infty) \neq 0$, then the signature of A_∞ is $(n, 1)$. If $det(A_\infty) = 0$, we claim that A_∞ is semi-positive definite. Indeed, by definition, all principal proper submatrices of A_∞ are semi-positive definite. This, together with $det(A_\infty) = 0$, implies that A_∞ is semi-positive definite. The proof of Theorem 1.3 uses the following lemma to perturb A_∞ and A_m to $DA_\infty D$ and $D_m A_m D_m$ for some positive diagonal matrices D and D_m so that $D_m A_m D_m$ converges to $DA_\infty D$ and all non-zero eigenvalues of $D_m A_m D_m$ and $DA_\infty D$ are simple, i.e., they are the simple roots of the characteristic polynomials.

Lemma 3.2. *Given a symmetric $n \times n$ matrix A of signature $(k, 0)$ or $(k, 1)$, and $\epsilon > 0$, there exists a positive diagonal matrix D so that $|D - Id| \leq \epsilon$ and all non-zero eigenvalues of DAD are simple.*

This is a very simple consequence of the work on multiplicative inverse eigenvalue problem (see for instance [Fr]). For completeness, we provide a simple proof of it in the appendix.

Applying this lemma, we find a positive diagonal matrix D so that $DA_\infty D$ has only simple non-zero eigenvalues and also a positive diagonal matrix D_m within distance $1/m$ of the identity matrix so that $D_m DA_m DD_m$ has distinct eigenvalues and $\lim_{m\to\infty} D_m DA_m DD_m = DA_\infty D$. Since $F(DA_m D) = F(A_m)$ for any positive diagonal matrix D, the modification of the sequence A_m to $D_m DA_m DD_m$ does not change the existence of the limit $\lim_{m\to\infty} F(A_m)$. By theorems 2.7 and 2.8, we may assume, after modifying A_m to $D_m DADD_m$, the following,

(3.4) all eigenvalues $\{\lambda_i(m) | i = 1, 2, ..., n+1\}$ of A_m are pairwise distinct, i.e.,

$$\lambda_1(m) > \lambda_2(m) > .... > \lambda_n(m) > 0 > \lambda_{n+1}(m),$$

and all non-zero eigenvaules of A_∞ are pairwise distinct.

(3.5) the limit $\lim_{m\to\infty} \lambda_i(m) = \lambda_i(\infty)$ exists for all $i = 1, ..., n+1$ where $\lambda_i(\infty)$'s are the eigenvalues of A_∞. Furthermore, either $rank(A_\infty) = n + 1$ and

$$\lambda_1(\infty) > \lambda_2(\infty) > .... > \lambda_n(\infty) > 0 > \lambda_{n+1}(\infty),$$

or $k = rank(A_\infty) \leq n$ and

$$\lambda_1(\infty) > \lambda_2(\infty) > .... > \lambda_k(\infty) > \lambda_{k+1}(\infty) = ... = \lambda_{n+1}(\infty) = 0.$$

3.2. We need the folloing canonical decomposition of matrices $A \in \mathcal{Y}_{n+1}$. Note that A^2 is symmetric and positive definite. In particular, the symmetric positive definite matrix $B = \sqrt{\sqrt{A^2}}$ exists. Furthermore, the function $B = B(A) : \mathcal{Y}_{n+1} \to \mathbf{R}^{(n+1)\times(n+1)}$ can be extended continuously to the closure $\bar{\mathcal{Y}}_{n+1}$. Suppose the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_n > 0 > -\lambda_{n+1}$. Then there exists an orthonormal matrix $U = [v_1, ..., v_{n+1}]$ whose column vectors v_i are eigenvectors of length one so that

$$A = U diag(\lambda_1, ..., \lambda_n, -\lambda_{n+1}) U^t. \tag{3.6}$$

We can recover B from (3.6) by the formula $B = U diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}) U^t$. In particular, we have

$$A = BUSU^tB \tag{3.7}$$

and

$$U^tBA^{-1}BU = S. \tag{3.8}$$

Furthermore, due to $Bv_i = \sqrt{\lambda_i} v_i$,

$$BU = [\sqrt{\lambda_1} v_1, ..., \sqrt{\lambda_n} v_n, \sqrt{\lambda_{n+1}} v_{n+1}].$$

Note that in general the matrix U is not uniquely determined by A due to the multiple eigenvalues. However, if the eigenvalues of A are pairwise distinct, then each eigenvector v_i of norm 1 is determined by the associated eigenvalue λ_i up to sign.

The geometric meaning of the decomposition (3.7) is the following,

Proposition 3.3. *Consider the hyperbolic n-simplex $\sigma = U^tB^{-1}(\mathbf{R}^{n+1}_{\geq 0}) \cap H^n$ with codimension-i faces σ_i for $i = 1, 2, ..., n+1$. The point $e_{n+1} = [0, ..., 0, 1]^t$ is in the simplex σ and the distance from e_{n+1} to the totally geodesic codimension-1 space $sp(\sigma_i)$ is at most $\cosh^{-1}(\sqrt{1 + \lambda_{n+1}})$ for all i.*

Proof. The vertices of the n-simplex $\sigma = U^tB^{-1}(\mathbf{R}^{n+1}_{\geq 0}) \cap H^n$ are $v_i = U^tB^{-1}(e_i)/ < U^tB^{-1}(e_i), U^tB^{-1}(e_i) >^{1/2}$ where $e_i = [0, .., 0, 1, 0, .., 0]^t$ is the standard basis of $\mathbf{R}^{n+1}$. To find the distance from e_{n+1} to the codimension-1 totally geodesic space $sp(\sigma_i)$, we find the normal vector to $sp(\sigma_i)$ as follows. Consider the column vectors $w_1, ..., w_{n+1}$ of $W = SU^tB$. These vectors w_i satisfy the conditions,

(3.9) $< w_i, w_i >= 1$ for all i,

(3.10) $< w_i, U^tB^{-1}(e_j) >= 0$ for $i \neq j$,

(3.11) $< w_i, U^tB^{-1}(e_i) >= 1$ for all i.

Indeed, (3.9) follows from (3.7) that $W^tSW = A$ and $A_{ii} = 1$ for all i. Also (3.10) and (3.11) follow from the identity $W^tSU^tB^{-1} = BUSSU^tB^{-1} = Id$. This shows that w_i is the normal vector in the de-Sitter space $S(1)$ which is perpendicular to $sp(\sigma_i)$ so that $< w_i, U^tB^{-1}e_i >> 0$. To find the distance from e_{n+1} to the codimension-1 totally geodesic hypersurface containing a codimension-1 face, we should calculate $< w_i, e_{n+1} >$. Indeed, since $W^tSe_{n+1} = BUSSe_{n+1} = BUe_{n+1} = Bv_{n+1} = \sqrt{\lambda_{n+1}}v_{n+1}$ and the eigenvector v_{n+1} has norm 1, we obtain $| < w_i, e_{n+1} > | \leq \sqrt{\lambda_{n+1}}$. By lemma 2.4(3), we conclude that the distance from e_{n+1} to these codimension-1 faces are at most $\cosh^{-1}(\sqrt{1 + \lambda_{n+1}})$.

Finally, we need to show that e_{n+1} is in the simplex σ. This is the same as showing that all entries of the eigenvector v_{n+1} have the same sign. To this end, we need,

Lemma 3.4. *Suppose B is a symmetric $(n+1) \times (n+1)$ matrix so that all $n \times n$ principal submatrices in B are positive definite and $det(B) \leq 0$. Then no entry in the adjacent matrix $ad(B)$ is zero.*

Assuming this lemma, we prove that all entries of the eigenvector v_{n+1} have the same sign as follows. For the variable $t \in [0, \lambda_{n+1}]$, consider the matrix $C(t) = A + tId$. By definition, all $n \times n$ principal submatix of $C(t)$ are positive definite. Furthermore, $det(C(t)) \leq 0$ since the smallest eigenvalue of $C(t)$ is $t - \lambda_{n+1} \leq 0$. By the lemma, all entries $ad(C(t))_{ij}$ are non-zero. On the other hand, $ad(C(t))_{ij}$ is a polynomial in t and is positive when $t = 0$ by (3.3). Thus all $ad(C(t))_{ij} > 0$. Now for $t = \lambda_{n+1}$, the first colume of $ad(C(\lambda_{n+1}))$ is an eigenvector of A associated to $-\lambda_{n+1}$. Since this negative eigenvalue is simple, any two associated eigenvectors are multiple of each other. This ends the proof.

Now to prove lemma 3.4, we first note that $Bad(B) = det(B)Id$. Also, the positive definiteness of the principal submatrice shows that $ad(B)_{ii} > 0$ for all i. If there is an entry $ad(B)_{ij} = 0$, then $i \neq j$. Without loss of generality, let us assume that $ad(B)_{1(n+1)} = 0$. Let w be the first colume of $ad(B)$. The vector w is not the zero vector due to $ad(B)_{11} > 0$. By the assumption that the principal submatrix P obtained by removing the last row and column is positive definite, we have $w^tBw = w^tPw > 0$. On the other hand, $Bw = det(B)[1, 0, ..., 0]^t$ by definition and $w^tBw = det(B)ad(B)_{11} \leq 0$ due to $det(B) \leq 0$ and $ad(B)_{11} > 0$. This is a contradiction. QED

3.3. We now prove Theorem 1.3. Given the convergent sequence $A_m \in \mathcal{Y}_{n+1}$ as in subsection 3.1 so that (3.4) and (3.5) hold, we produce a decomposition

$$A_m = B_mU_mSU_m^tB_m \tag{3.12}$$

as in (3.7). Let k be the rank of A_∞. By theorem 2.8 and (3.4) and (3.5), we may choose eigenvectors $v_1(m), ..., v_k(m)$ of unit length for A_m associated to the simple eigenvalues $\lambda_i(m)$ for so that

$$\lim_{m \to \infty} v_i(m) = v_i(\infty)$$

exists for $i = 1, 2, ..., k$ and $v_i(\infty)$ is an eigenvector of norm 1 for A_∞. In particular, we see that the matrix

$$B_mU_m = [\sqrt{\lambda_1(m)}v_1(m), ..., \sqrt{\lambda_k(m)}v_k(m), \sqrt{\lambda_{k+1}(m)}v_{k+1}(m), ..., \sqrt{|\lambda_{n+1}(m)|}v_{n+1}(m)]$$

is converging to $[\sqrt{\lambda_1(\infty)}v_1(\infty), ..., \sqrt{\lambda_k(\infty)}v_k(\infty), 0, ..., 0]$ if $k \leq n$ or to $[\sqrt{\lambda_1(\infty)}v_1(\infty),$ $..., \sqrt{\lambda_n(\infty)}v_n(\infty), \sqrt{|\lambda_{n+1}(\infty)|}v_{n+1}(\infty)]$ for $k = n+1$ by (3.4) and (3.5).

Using (3.12), let us make a change of variable $x = B_m U_m(y)$ in

$$F(A_m) = \sqrt{|det(A_m)^{-1}|} \int_{\mathbf{R}^{n+1}_{\geq 0}} e^{x^t A_m^{-1} x} dx.$$

We obtain by (3.8),

$$F(A_m) = \int_{(B_m U_m)^{-1}(\mathbf{R}^{n+1}_{\geq 0})} e^{y^t S y} dy$$

$$(3.13) \qquad = \int_{\mathbf{R}^{n+1}} e^{<y,y>} \chi \circ (B_m U_m)(y) dy$$

where χ is the characteristic function of $\mathbf{R}^{n+1}_{\geq 0}$ in $\mathbf{R}^{n+1}$.

By the contruction, $B_m U_m$ converges to a matrix in $\mathbf{R}^{(n+1)\times(n+1)}$. We claim that the sequence of functions $\chi \circ (B_m U_m)$ converges almost everywhere in $\mathbf{R}^{n+1}$. In fact, by lemma 2.3, it suffices to verify that no row vector in $\lim_{m\to\infty} B_m U_m$ is zero. Suppose otherwise, say the i-th row is zero. Then the ii-th entry in $\lim_{m\to\infty} B_m U_m S U_m^t B_m$ is zero. But by assumption, the ii-th entry in $B_m U_m S U_m^t B_m$ is $(A_m)_{ii}$ which is always 1.

To summary, we see that the integrant in (3.13) converges almost everywhere in $\mathbf{R}^{n+1}$. To prove that the limit $\lim_{m\to\infty} F(A_m)$ exists, we will use the following well known lemma from analysis. We omit the proof.

Lemma 3.5. *Suppose $\{f_m\}$ is a sequence of integrable non-negative functions converging almost everywhere to f in $\mathbf{R}^n$. If for any $\epsilon > 0$, there exists a measureable set $E \subset \mathbf{R}^n$ so that*

(a) the restriction $f_m|_E$ converges a.e. to $f|_E$ and is dominated by an integrable function g on E, and

(b) $\int_E f_m dx \leq \epsilon$ for all integer $m \geq 1$,
then the $\lim_{m\to\infty} \int_{\mathbf{R}^n} f_m dx$ exists.

To apply this lemma, we will produce a decomposition of integral (3.13) as follows. For any $p > 0$ and $p < 1$, consider the set $\Omega_p = \{x \in \mathbf{R}^{n+1} | <x, x> \leq -p(x,x)\}$ where $(x,x) = x^t x$ is the Euclidean inner product. The intersection $\Omega_p \cap H^n$ is equal to the hyperbolic ball of radius $r = \cosh^{-1}(\sqrt{(1+p)/2p})$ centered at e_{n+1}. Indeed, we may write $(x,x) = <x,x> + 2(x, e_{n+1})^2$. Thus $<x,x> \leq -p(x,x)$ inside H^n is the same as $| <x, e_{n+1}> | \leq \sqrt{(1+p)/2p}$. By lemma 2.4(1), the claim that $\Omega_p \cap H^n = B_r(e_{n+1})$ follows. Now in the region Ω_p, the integral $\int_{\Omega_p} e^{<y,y>} \chi \circ (B_m U_m)(y) dy$ converges since the intergrant is dominated by the integrable function $e^{-p(y,y)}$. On the other hand, the integral $\int_{\mathbf{R}^{n+1}-\Omega_p} e^{<y,y>} \chi \circ (B_m U_m)(y) dy$ is the same as $\mu_n \, vol(\sigma_m - B_r(e_{n+1}))$ where $\sigma_m = (B_m U_m)^{-1}(\mathbf{R}^{n+1}_{\geq 0}) \cap H^n$ is a hyperbolic n-simplex. By proposition 3.3 and the existence of

$\lim_{m\to\infty}\lambda_{n+1}(m)$, there is a constant C independent of m so that e_{n+1} is within distance C to each codimension-1 totally geodesic surface containing a codimension-1 face of the simplex σ_m. By Theorem 1.2 and proposition 3.3, the volume $vol(\sigma_m - B_r(e_{n+1}))$ can be made arbitrary small for all n-simplices σ_m if the radius r is large. Thus, by lemma 3.5, we conclude that the limit $\lim_{m\to\infty} F(A_m)$ exists. QED

§4. **A Proof of Theorem 1.2**

We prove Theorem 1.2 in this section. Recall that $B_R(x)$ denotes the ball of radius R centered at x.

Theorem 1.2. *For any $\epsilon > 0$ and $r > 0$, there exists a positive number $R = R(\epsilon, r, n)$ so that for any hyperbolic n-simplex σ, if $x \in \sigma$ is a point whose distance to each totally geodesic hyperplane containing a codimension-1 face is at most r, then the volume of $\sigma - B_R(x)$ is at most ϵ.*

The theorem will follow from a sequence of propositions and lemmas on hyperbolic simplices. To begin with, we fix the notations and conventions as follow. The projective disk model of H^n is denoted by $D^n = \{(x_1, ..., x_n) \in \mathbf{R}^n | \sum_{i=1}^n x_i^2 < 1\}$. The compact closure of D^n is denoted by $\bar{D^n}$ which is the compactification of the hyperbolic space by adding the ideal points. The hyperbolic distance in H^n or D^n will be denoted by d. If $\{v_1, ..., v_k\}$ is a set of points in $\bar{D^n}$, the convex hull of it will be denoted by $C(v_1, ..., v_k)$. The *volume* of $C(v_1, ..., v_{n+1})$ in D^n, denoted by $vol(C(v_1, ..., v_k))$, is the hyperbolic volume of $C(v_1, ..., v_{n+1}) \cap D^n$. If $v_1, ..., v_{n+1}$ are pairwise distinct, we call $\sigma = C(v_1, ..., v_{n+1})$ a *generalized n-simplex* in $\bar{D^n}$. Its i-th codimension-1 face, denoted by σ^i is $C(v_1, ..., v_{i-1}, v_{i+1}, ..., v_{n+1})$. A generalized n-simplex is said to be *non-degenerated* if it has positive volume. Evidently, a generalized n-simplex $C(v_1, ..., v_{n+1})$ in $\bar{D^n}$ is non-degenerated if and only if the vectors $\{v_1, ..., v_{n+1}\}$ are linearly independent in $\mathbf{R}^{n+1}$. The *center* and the *radius* of a non-degenerated generalized n-simplex are defined to be the center and the radius of its inscribed ball. Given a finite set $X \in \bar{D^n}$ so that X contains at least two points, the smallest complete totally geodesic submanifold containing X in its closure is deonted by $sp(X)$. For a measurable subset X of H^n, or D^n, we use $vol(X)$ to denote the volume of the set. If X lies in a totally geodesic submanifold of dimension-k H^k in H^n, we use $vol_k(X)$ to denote the volume of X in the subspace H^k.

4.1. We will establish the following propositions and lemmas in order to prove Theorem 1.2.

The first proposition generalizes a result of Ratcliffe.

Proposition 4.1. (see [Ra], theorem 11.3.2) *Suppose $\sigma_m = C(v_1(m), ..., v_{n+1}(m))$ is a sequence of generalized n-simplices in D^n so that $\lim_{m\to\infty} v_i(m) = u_i$ exists in $\bar{D^n}$ for all $i = 1, ..., n+1$ and either $\{u_1, ..., u_{n+1}\}$ contains at least three points or $\{u_1, ..., u_{n+1}\}$ consists of two distinct points $\{p, q\}$ so that both sets $\{i | u_i = p\}$ and $\{i | u_j = q\}$ contain more than one point. Then $\lim_{m\to\infty} vol(\sigma_m) = vol(C(u_1, ..., u_{n+1}))$.*

Note that Ratcliffe proved the proposition when $C(u_1, ..., u_{n+1})$ is a non-degenerated generalized n-simplex. (In [Ra], a non-degenerated generalized simplex in our sense is called a generalized n-simplex.) However, if one exams his proof carefully in ([Ra], p527-529), the non-degeneracy condition is never used. Ratcliffe in fact already proved the proposition under the assumption that $\{u_1, ..., u_{n+1}\}$ are pairwise distinct. Thus, it suffices to prove the proposition in the case that the number of elements in $\{u_1, ..., u_{n+1}\}$ is at most n and is at least 2 as specified in the proposition. This will be proved in subsection 4.3.

Proposition 4.2. *For any $\epsilon > 0$, there is a number $\delta > 0$ so that if the radius of the inscribed ball of a hyperbolic n-simplex is less than δ, the volume of the simplex is less than ϵ.*

Lemma 4.3. *For any $\delta > 0$ and $r > 0$, there exists $R = R(\delta, r, n)$ so that for any hyperbolic n-simplex σ of radius at least δ, if $x \in \sigma$ is a point whose distance to each codimension-1 totally geodesic surface containing a codimension-1 face is at most r, then $d(x, c) \leq R$ where c is the center of σ.*

Finally, we recall the following useful lemma of Thurston,

Lemma 4.4. *Given a generalized hyperbolic n-simplex $\sigma = C(v_1, ..., v_{n+1})$ where $n \geq 2$, let $\tau = C(v_1, ..., v_n)$ be a codimension-1 face of σ, then*

$$vol_n(\sigma) \leq 1/(n-1) vol_{n-1}(\tau).$$

See [Thu], chapter 6, or [Ra], p518-528, especially p528 for a proof.

4.2. **A Proof of Theorem 1.2**

Assuming the results above, we finish the proof of Theorem 1.2 as follows. Suppose otherwise that Theorem 1.2 is not true. Then there are $\epsilon_0 > 0$, $r_0 > 0$, a sequence of hyperbolic n-simplices σ_m, and a point $x_m \in \sigma_m$ so that,

(4.1) The distance of x_m to the totally geodesic codimension-1 surface containing each codimension-1 face of σ_m is at most r_0, and,

(4.2) $Vol(\sigma_m - B_m(x_m)) \geq \epsilon_0$.

By proposition 4.2 and condition (4.2), we may assume that the radius r_m of σ_m is at least $\delta_0 > 0$ for all m. By lemma 4.3 for δ_0 and r_0, we find a constant R_0 so that $d(x_m, c_m) \leq R_0$ for all m where c_m is the center of the simplex σ_m. In particular, $B_{m-R_0}(c_m) \subset B_m(x_m)$. This implies $\sigma_m - B_m(x_m) \subset \sigma_m - B_{m-R_0}(c_m)$ and

$$vol(\sigma_m - B_{m-R_0}(c_m)) \geq \epsilon_0, \tag{4.3}$$

for all m.

In the projective disk model D^n, we put the center c_m to the Euclidean center 0 of D^n. By taking a subsequence if necessary, we may assume that $\sigma_m = C(v_1(m), ..., v_{n+1}(m))$ where the limit $\lim_{m \to \infty} v_i(m) = u_i$ exists in $\bar{D^n}$.

Lemma 4.5. *Suppose* $\sigma_m = C(v_1(m), ..., v_{n+1}(m))$ *is a sequence of hyperbolic n-simplices with center 0 in the projective model* D^n *so that the limit* $\lim_{m\to\infty} v_i(m) = u_i$ *exists in* $\bar{D^n}$ *for all i. If* $\liminf_{m\to\infty} vol(\sigma_m) > 0$*, then either* $\{u_1, ..., u_{n+1}\}$ *consists of at least three points, or* $\{u_1, ..., u_{n+1}\} = \{p, q\}$*,* $p \neq q$*, so that both sets* $\{i|u_i = p\}$ *and* $\{j|u_j = q\}$ *contain at least two points.*

To prove this lemma, suppose otherwise, there are two possibilities. In the first possibility, $\{u_1, ..., u_{n+1}\}$ consists of one point $\{p\}$. Then for all m large, the points $v_i(m)$ are close to p in the Euclidean metric in $\bar{D^n}$. If p is in D^n, then the volume of σ_m tends to zero which contradicts the assumption. If p is in S^{n-1}, then σ_m cannot have the center to be 0 for m large. In the second possibility, we may assume that $u_2 = ... = u_{n+1} \neq u_1$. In this case, consider the codimension-1 face $\sigma_m^1 = C(v_2(m), ..., v_{n+1}(m))$. This (n-1)-simplex is close to u_2 for m large in the Euclidean metric. Since the face is tangent to 0, it follows that $u_2 = ... = u_{n+1} = 0$. This implies that the (n-1)-dimensional volume $vol_{n-1}(\sigma_m^1)$ tends to zero. By Thurston's inequality lemma 4.4, this implies that the volume of σ_m tends to zero. This is again a contradiction. QED

Thus, by proposition 4.1 and (4.2), the simplex $\sigma = \sigma(u_1, ..., u_{n+1})$ has positive volume. This implies that σ is a non-degenerated n-simplex in D^n whose center is 0. Let χ_m and χ be the characteristic functions of σ_m and σ in $\bar{D^n}$. Then by definition, the function χ_m converges almost everywhere to χ in D^n. Furthermore, by proposition 4.1, the integral $\int_{D^n} \chi_m dv$ converges to $\int_{D^n} \chi dv$ where dv is the hyperbolic volume element in D^n. By Fatou's lemma (see for instance [Roy], p86, problem 9), this implies that for any ball of radius R centered at 0, $vol(\sigma_m - B_R(0))$ converges to $vol(\sigma - B_R(0))$. Choose R so large that $vol(\sigma - B_R(0)) \leq \epsilon_0/2$. Then for m large, we have $vol(\sigma_m - B_R(0)) < \epsilon_0$. But this contradicts (4.3) for m large. QED

4.3. A Proof of Proposition 4.1.

By the work of Ratcliffe [Ra], it suffices to show the proposition in two cases. In the first case, the number of elements in the set $\{u_1, ..., u_{n+1}\}$ is between 3 and n. In the second case, $\{u_1, u_2, ..., u_{n+1}\}$ consists of two elements $\{p, q\}$, $p \neq q$, so that both sets $\{i|u_i = p\}$ and $\{j|u_j = q\}$ contain at least two points. The goal is to show that $\lim_{m->\infty} vol(\sigma_m) = 0$ in both cases.

The proposition holds for $n = 2$. Indeed, in this case, u_1, u_2, u_3 are pairwise distinct. Thus the result was proved by Ratcliffe. Assume from now on that $n \geq 3$.

First of all, we claim

Claim. If $u_i = u_j$ for $i \neq j$ so that u_i is in D^n, then $\lim_{m->\infty} vol(\sigma_m) = 0$.

Indeed, by lemma 4.4, we can estimate $vol(\sigma_m) \leq 1/(n-1)! vol_1(v_i(m), v_j(m))$. Now $vol_1(v_i(m), v_j(m)) = d(v_i(m), v_j(m))$ tends to $d(u_i, u_j) = 0$.

By this claim, we may assume from now on that if $u_i = u_j$, $i \neq j$, then $u_i \in S^{n-1}$.

By the assumption on $\{u_1, ..., u_{n+1}\}$, we may choose four points, say u_1, u_2, u_3, u_4 so that $u_1 = u_2$ and either $u_3 = u_4 \neq u_1$, or $\{u_1, u_2, u_3, u_4\}$ consists of three points. By lemma 4.4, we have $vol(\sigma_m) \leq 1/((n-1)...4.3) vol_3(C(v_1(m), v_2(m), v_3(m), v_4(m)))$. This implies that it suffices to prove the proposition for $n = 3$ which we will assume.

To prove the proposition, there are two cases to be considered: case 1, all u_i's are in S^2, and case 2, some u_i's are in D^3.

In the first case that all u_i's are in S^2, let $w_1(m), ..., w_4(m)$ be four points in S^2 so that $v_1(m), v_3(m)$ lie in the geodesic from $w_1(m)$ to $w_3(m)$ and $v_2(m), v_4(m)$ lie in the geodesic from $w_2(m)$ to $w_4(m)$. We choose $w_1(m)$ to be the end point in the ray from $v_3(m)$ to $v_1(m)$ and $w_2(m)$ similarly. By the construction, we still have $\lim_{m\to\infty} w_i(m) = u_i$ for $i = 1, 2, 3, 4$. Furthermore, by the construction $C(w_1(m), ..., w_4(m))$ contains the tetrahedron $C(v_1(m), v_2(m), v_3(m), v_4(m))$. In particular,

$$vol(C(v_1(m), ..., v_4(m))) \leq vol(C(w_1(m), ..., w_4(m))).$$

Now, the volume of the ideal tetrahedra $C(w_1(m), ..., w_4(m))$ can be calculated from the cross ratio of the four vertices $w_1(m), ..., w_4(m)$. To be more precise, by [Th], the volume of an ideal hyperbolic tetrahedron with vertices $z_1, z_2, z_3, z_4 \in \mathbf{C}$ depends continuously on the cross raio $[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$. In particular, if the cross ratio tends to 0, 1, or ∞, then the volume tends to 0. In our case, by the assumption, we see that the cross ratio of $(w_1(m), w_2(m), w_3(m), w_4(m))$ tends to the cross ration of u_1, u_2, u_3, u_4 which is 0, 1, or ∞. Thus the volume $vol_3(C(v_1(m), v_2(m), v_3(m), v_4(m)))$ tends to 0.

In the second case that one of the points of $\{u_1, u_2, u_3, u_4\}$ is in D^3, by the above claim, we may assume that $u_1 = u_2$ is in S^2. Furthermore, by the claim, we may assume that $u_3 \neq u_4$ and $u_3 \in D^3$. Note that $u_4 \neq u_1$. Let $w_1(m), ..., w_4(m)$ be four points in S^2 constructed as in the previous paragraph. By the construction $C(w_1(m), ..., w_4(m))$ contains the tetrahedron $C(v_1(m), v_2(m), v_3(m), v_4(m))$. Furthermore, we have $\lim_m w_1(m) = \lim_m w_2(m) = u_1$, and $\lim_m w_3(m) = w_3$ and $\lim_m w_4(m) = w_4$ both exist so that the cross ration of $\{u_1, u_1, w_3, w_4\}$ is 0, 1, or ∞. Thus by case 1, we see that the volume of $C(w_1(m), ..., w_4(m))$ tends to zero. This in turn implies that the volume of $C(v_1(m), ..., v_4(m))$ tends to zero. This finishes the proof.

4.4. **A Proof of Proposition 4.2**

Suppose otherwise, there is $\epsilon_0 > 0$, a sequence of hyperbolic n-simplices $\sigma_m = C(v_1(m), ..., v_{n+1}(m))$ with center 0 in D^n so that the radius of σ_m is at most $1/m$ and its volume $vol(\sigma_m) \geq \epsilon_0$. By taking a subsequence if necessary, we may assume that the limit $\lim_{m\to\infty} v_i(m) = u_i$ exists in $\bar{D^n}$. Then by lemma 4.5 and proposition 4.1, we conclude that $\sigma = C(u_1, ..., u_{n+1})$ is a non-degenerate generalized n-simplex. In particular, these vectors $u_1, ..., u_{n+1}$ are linearly independent in $\mathbf{R}^{n+1}$. On the other hand, since the radius of σ_m tends to zero, we see that all codimension-1 totally geodesic surfaces $sp\{u_1, ..., u_{i-1}, u_{i+1}, ..., u_{n+1}\}$ contain 0. This is impossible for a non-degenerated simplex. QED

4.4. **A Proof of Lemma 4.3**

Suppose otherwise, there exist $\delta_0 >$, $r_0 > 0$, a sequence of n-simplices $\{\sigma_m | m \in \mathbf{Z}_{\geq 1}\}$ of radius at least δ_0, and a point $x_m \in \sigma_m$ so that

(4.4) x_m is within r_0 distance to each codimension-1 totally geodesic surface containing a codimension-1 face of σ_m, and,

(4.5) $d(x_m, c_m) \geq m$ where c_m is the center of σ_m.

Let us put the center c_m of σ_m to be the origin 0 of D^n. By chosing a subsequence if necessary, we may assume that $\sigma_m = C(v_1(m), ..., v_{n+1}(m))$ so that $\lim_{m\to\infty} v_i(m) = u_i$ exists in $\bar{D}^n$, and $\lim_{m\to\infty} x_m = x$ also exists in $\bar{D}^n$. Since the radius of σ_m is bounded away from zero, we apply lemma 4.5 and proposition 4.1 to conclude that the simplex $\sigma = C(u_1, ..., u_{n+1})$ is non-degenerated whose center is 0. Since $d(x_m, 0) \geq m$, it follows that x has to be one of the vertex, say u_1 of σ. Now consider the upper-half space model U^n for the hyperbolic space so that $u_1 = x$ is the infinity and the totally geodesic codimension-1 surface containing $u_2, ..., u_{n+1}$ is the unit upper hemi-sphere $S_+^{n-1} = \{(t_1, ..., t_n) \in \mathbf{R}^n | \sum_{i=1}^n t_i^2 = 1, t_n > 0\}$. Let the center of the simplex σ in this model be C and the point of the shortest distance to C in S_+^{n-1} be P. We claim that the angle $\angle PCu_1$ at C is at least $\pi/2$. This follows from the Gauss-Bonnet theorem. Let U^2 be the unique 2-dimensional hyperbolic plane containing C and u_1 so that U^2 is perpendicular to S_+^{n-1}. Let $Q = [0, ..., 0, 1]^t$ be the north pole in S_+^{n-1}. Then by the construction, $Q \in U^2$ and $P \in U^2$ due to the orthogonality. If $P = Q$, then the angle $\angle PCu_1$ is π. The claim follows. If otherwise, consider the hyperbolic quadrilateral $QPCu_1$ in U^2. The angle of the quadrilateral at Q, P and u_1 are $\pi/2$, $\pi/2$ and 0 respectively. On the other hand, since C is the center of the simplex σ, the complete geodesic from u_1 to C intersects the hemi-sphere S_+^{n-1} at some point, say R. Thus the quadrilateral $QPCu_1$ is inside the hyperbolic triangle Δu_1QR whose inner angles are $\pi/2, 0, \theta$. In particular, the area of this triangle is less than $\pi/2$ by the Gauss-Bonnet formula. This implies that the area of the quadrilateral $QPCu_1$ is at most $\pi/2$. By Gauss-Bonnet formula, we conclude that the angle $\angle PCu_1$ at C is at least $\pi/2$.

On the other hand, we will derive from (4.4) and (4.5) that the angle $\angle PCu_1$ is strictly less than $\pi/2$. Thus we arrive a contradiction. To see this, let P_m be the point in the totally geodesic codimension-1 surface $sp(C(v_2(m), ..., v_{n+1}(m)))$ which is closest to the center c_m of σ_m. By the construction, the limit of the angle $\angle P_mc_mx_m$ is equal to $\angle PCu_1$. To estimate the angle $\angle P_mc_mx_m$, consider the two-dimensional totally geodesic plane D_m which contains c_m and x_m so that D_m is perpendicular to $sp(C(v_2(m), ..., v_{n+1}(m)))$. By the construction P_m is in the plane D_m. Let R_m be the point in $sp(C(v_2(m), ..., v_{n+1}(m))$ of the shortest distance to x_m. Then we again have R_m is in D_m. Consider the quadrilateral $P_mR_mx_mc_m$ in the plane D_m. The angles at the vertices P_m and R_m are $\pi/2$. The distances $d(c_m, P_m) \geq \delta_0$, $d(x_m, R_m) \leq r_0$ and $d(c_m, x_m) \geq m$. Thus, as m becomes large, the quadrilateral is tending to a right angled hyperbolic triangle with one vertex at infinity (corresponding to R_m and x_m). There is an edge of the triangle having finite length which is at least δ_0 (corresponding to the edge between c_m and P_m). The accue angle at the end point of this finite length edge is at most $\theta = \arcsin(1/\cosh(\delta_0)) < \pi/2$ by the cosine law. Thus, as m tends to infinity, the angle $\angle P_mc_mx_m$ tends to a number less than or equal to θ. In particular, the angle $\angle P_mc_mx_m$ is strictly less than $\pi/2$ for m large. This contradicts the previous conclusion. QED

Appendix, A Proof of Lemma 3.2

We give a proof of the following lemma used in the paper.

Lemma 3.2. *Given a symmetric $n \times n$ matrix A of signature $(k, 0)$ or $(k, 1)$, and $\epsilon > 0$, there exists a positive diagonal matrix D so that $|D - Id| \leq \epsilon$ and all non-zero eigenvalues of DAD are simple.*

Proof. For of all, it suffices to find a positive diagonal matrix D so that all non-zero eigenvalues of DAD are simple. This is due to the fact from algebraic geometry that an algebraic subvariety in $\mathbf{R}^m$ is either the whole space or has zero Lebesgue measure. By [Fr], the set of all diagonal matrices D so that DAD has a non-simple non-zero eigenvalue forms an algebraic variety X in $\mathbf{R}^m$. Thus, as long as $X \neq \mathbf{R}^m$, we can pick D in $\mathbf{R}^m$ within ϵ distance to $[1, ..., 1]^t$ so that $D \notin X$.

Next, we claim that it suffices to prove the lemma for $n \times n$ matrix A so that $det(A) \neq 0$. Indeed, if $k = rank(A)$, due to the fact that A is diagonal, A has exactly k non-zero eigenvalues counted with multiplicity and A has a non-singular principal $k \times k$ submatrix B formed by $i_1, ..., i_k$-th rows and columns of A. For simplicity, we assume that B is formed by the first k rows and columns of A. Then by the result for non-singular symmetric matrix, we find a positive diagonal matrix $D_1 = diag(a_1, ..., a_k)$ so that $D_1 B D_1$ has k distinct eigenvalues. Consider the $n \times n$ matrix $D(t) = diag(a_1, ..., a_k, t, t, ..., t)$ where $t > 0$. For t small, by theorem 2.7, the eigenvalues of $D(t)AD(t)$ are close to the eigenvalues of $D_1 B D_1$ and 0. Since $D_1 B D_1$ has k distinct non-zero eigenvalues, this implies that $D(t)AD(t)$ has k distinct non-zero eigenvalue for t small.

Finally, we prove the lemma for non-singular matrices using induction on the size of the matrix. The result clearly holds for 1×1 and 2×2 matrices. Suppose A is a non-singular $n \times n$ matrix for $n \geq 3$. Let B the principal submatrix of A obtained by removing the last column and the last row. Then the signature of B is either $(n-2, 1)$, $(n-1, 0)$ or $(n-2, 0)$. By the induction hypothesis and the argument in the previous paragraph, we find a positive diagonal matrix $D_1 = diag(a_1, ..., a_{n-1})$ so that all $(n-1)$-eigenvalues of B are distinct. Let us denote the eigenvalues of B by $\lambda_1 > ... > \lambda_{n-1}$. Now consider the positive diagonal matrix $D(t) = diag(a_1, ..., a_{n-1}, t)$ for $t > 0$. For t small, the eigenvalues $\mu_1(t) \geq \mu_2(t) \geq ... \geq \mu_n(t)$ of $D(t)AD(t)$ is close to $\{\lambda_1, ..., \lambda_{n-1}, 0\}$. We claim that for t small $D(t)AD(t)$ has n distinct eigenvalues. Indeed, if B is non-singular, i.e., the set $\{\lambda_1, ..., \lambda_{n-1}, 0\}$ consists of n distinct elements, then for $t > 0$ small, $\mu_i(t) \neq \mu_j(t)$ for $i \neq j$. If B is singular, then B is semi-positive definite of rank $n-2$. Furthermore, this implies that A has signature $(n-1, 1)$. In particular, $D(t)AD(t)$ has a negative eigenvalue, i.e., $\mu_n(t) < 0$. Now for t small, we conclude that $\mu_1(t), ..., \mu_{n-1}(t)$ are positive and are close to the set of $n-1$ distinct numbers $\{\lambda_1, ..., \lambda_{n-2}, 0\}$ where $\lambda_i > 0$. This implies that for $t > 0$ small, the eigenvalues $\mu_1(t), ..., \mu_{n-1}(t)$ are positive and pairwsie distinct. Since the smallest eigenvalue $\mu_n(t) < 0$, we conclude that $D(t)AD(t)$ has n distinct eigenvalues.

References

[Ao] Aomoto, Kazuhiko: Analytic structure of Schlfli function. Nagoya Math J. 68 (1977), 1–16.

[Fr] Friedland, Shmuel: On inverse multiplicative eigenvalue problems for matrices. Linear Algebra and Appl. 12 (1975), no. 2, 127–137.

[HJ] Horn, Roger A.; Johnson, Charles R.: Topics in matrix analysis. Cambridge University Press, Cambridge, 1991.

[Lu] Luo, Feng: On a problem of Fenchel. Geom. Dedicata 64 (1997), no. 3, 277–282.

[Mi] Milnor, John: The Schlaefli differential equality. In Collected papers, vol. 1. Publish or Perish, Inc., Houston, TX, 1994.

[MY] Murakami June; Yano Masakazu: On the volume of a hyperbolic and spherical tetrahedron, http://www.f.waseda.jp/murakami/papers/tetrahedronrev3.pdf

[Ra] Ratcliffe, John G.: Foundations of hyperbolic manifolds. Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[Ro] Royden, H. L.: Real analysis. The Macmillan Co., New York; Collier-Macmillan Ltd., London 1963.

[SS] Stewart, G. W.; Sun, Ji Guang: Matrix perturbation theory. Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, 1990.

[Th] Thurston, W.: Geometry and topology, http://www.msri.org/publications/books /gt3m/

[Vi] Vinberg, E. B.: Volumes of non-Euclidean polyhedra. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 2(290), 17–46; translation in Russian Math. Surveys 48 (1993), no. 2, 15–45

[Wi] Wilkinson, J. H.: The algebraic eigenvalue problem. Monographs on Numerical Analysis. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1988.

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