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Monodromy groups of projective structures on punctured surfaces

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1 Introduction

The purpose of this paper is to study the monodromy groups associated to the quasi-bounded holomorphic quadratic forms on punctured surfaces. As a consequence, we obtain a natural family of symplectic structures on the Teichmüller space $T_{g,n}$ for $n > 0$. As another consequence, we show that the projective monodromy map from a class of Fuchsian equations to the representation variety is generically a local diffeomorphism.

Recall that a punctured Riemann surface S is a surface obtained from a closed Riemann surface $\bar{S}$ of genus g by removing finitely many points $\{p_1, \ldots, p_n\}$, i.e., S is a surface of finite type (g, n). A projective atlas on S is an open cover of S by coordinate charts so that the transition functions are restrictions of projective transformations of CP^1. A projective structure on S is an equivalence class of projective atlases. Since the set of all projective structures on S is naturally an affine space modeled on the space of all holomorphic quadratic forms on S, we may identify the space of all projective structures on S with the space of all holomorphic quadratic forms by the uniformization theorem. A quasi-bounded holomorphic quadratic form on S is the restriction of a meromorphic quadratic form $\phi(w)dw^2$ on $\bar{S}$ so that the order of ϕ at each puncture is greater than or equal to -2. Denote the space of all quasi-bounded quadratic forms on S by $Q_2(S)$. For each $\phi(w)dw^2 \in Q_2(S)$ and each cusp point $p \in \bar{S} - S$, the coefficient of w^{-2} in the Laurent expansion of $\phi(w)$ at p (where w is a local coordinate with $w = 0$ at p) is independent of the choice of local coordinates. We call it the residue of $\phi(w)dw^2$ at p, see Bers [Be, p. 141]. If the residue of $\phi(w)dw^2 \in Q_2(S)$ at p is $-n^2/2$ for some $n \in Z$, p is called an apparent singularity of $\phi(w)dw^2$. Suppose the residue of $\phi(w)dw^2 \in Q_2(S)$ at a cusp p is α, then the monodromy homomorphism of $\phi(w)dw^2$ takes the parabolic transformation in the deck group corresponding to p to an element in $\mathrm{PSL}(2, C)$ whose trace squared is $e^{2\pi i\sqrt{-2\alpha}} + e^{-2\pi i\sqrt{-2\alpha}} + 2$.

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Thus, at an apparent singularity p, the monodromy group element corresponding to the parabolic transformation is parabolic.

Our result is the following.

Theorem. *Let Q be the fibration over the Teichmüller space $T_{g,n}$ of surfaces S of type (g, n) whose fiber at a point S is the space of all quasi-bounded holomorphic quadratic forms without apparent singularities on S, and let $\pi: Q \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ be the monodromy map. Then the derivative $D\pi$ of π is injective from the tangent space $T_\phi Q$ to the Zariski tangent space of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ at $\pi(\phi)$. In particular, if $\pi(\phi)$ is a smooth point of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$, then π is a local diffeomorphism near ϕ.*

$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ is not a Hausdorff space in the quotient topology. The "Zariski tangent space" to $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ at an equivalence class $[\rho]$ is defined to be the cohomology group $H^1(\pi_1(S)_\rho, \Pi)$ where Π is the space of all polynomials of degree at most two and $\pi_1(S)$ acts on Π by $(P \cdot \gamma)(z) = P(\rho(\gamma)(z))/(\rho(\gamma)'(z))$. The derivative of the monodromy map from $T_\phi Q$ to the cohomology is well defined and is given by the variational formula of Earle (see Sect. 2.4). Let $\mathrm{Hom}^-(\pi_1(S), \mathrm{PSL}(2, C))$ be the open subset of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, C))$ consisting of representations whose image group does not fix a point in CP^1. Then $\mathrm{PSL}(2, C)$ acts (by conjugation) properly on $\mathrm{Hom}^-(\pi_1(S), \mathrm{PSL}(2, C))$ and the quotient $\mathrm{Hom}^-(\pi_1(S), \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ is an affine algebraic set whose Zariski tangent space is isomorphic to $H^1(\pi_1(S)_\rho, \Pi)$. See Goldman [Go1], or Gardiner and Kra [GK] for details.

Projective structures associated to quasi-bounded forms are generalizations of the cone structures in geometry. Indeed, spherical cone structures on surfaces are projective structures associated to quasi-bounded holomorphic forms whose monodromies are representations into $SO(3)$ and hyperbolic cone structures on surfaces are projective structures associated to quasi-bounded holomorphic forms whose monodromies are representations into $\mathrm{PSL}(2, R)$ and whose developing images are in the upper half plane. As a consequence of the theorem, we have,

Corollary 1 (a) *Suppose $\mathrm{SP}(g, n)$ is the space of all spherical cone structures so that none of the cone angles are $2\pi k$, $k \in Z$, on surfaces S of type (g, n) modulo isometries homotopic to the identity relative to the cusps. Then the monodromy map from $\mathrm{SP}(g, n)$ to the representation variety $\mathrm{Hom}(\pi_1(S), SO(3))/SO(3)$ is a local diffeomorphism.*

(b) *Suppose $\mathrm{HY}(g, n)$ is the space of all hyperbolic cone structures so that none of the cone angles are $2\pi k$, $k \in Z$, on surfaces S of type (g, n) modulo isometries homotopic to the identity relative to the cusps. Then the monodromy map from $\mathrm{HY}(g, n)$ to the representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, R))/\mathrm{PSL}(2, R)$ is a local diffeomorphism.*

Given n non-negative numbers $a_1, \ldots, a_n$ so that no a_i is $2k\pi$ for $k \in Z$ and $\sum_{i=1}^n a_i < 2\pi(2g + n - 2)$, the subspace $\mathrm{C}(g; a_1, \ldots, a_n)$ of $\mathrm{HY}(g, n)$ consisting of cone metrics whose cone angle at the i-th puncture is a_i is homeomorphic to the Teichmüller space $T_{g,n}$ by the uniformization theorem for hyperbolic cone metrics (See McOwen [Mc]). The space $\mathrm{C}(g; a_1, \ldots, a_n)$ has a symplectic structure coming from the corresponding subvariety of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, R))/\mathrm{PSL}(2, R)$ whose symplectic form is derived from the Poincaré duality of the first cohomology group of $\pi_1(S)$ with the Lie algebra $\mathrm{sl}(2, R)$ as coefficient module. See Iwasaki

[Iw1] and Goldman [Go1] for a detailed discussion of the symplectic form. Therefore the Teichmüller space $T_{g,n}$ considered as a parametrization of the cone metric space inherits a symplectic structure. It seems likely that Wolpert's theorem [Wo] about the Weil-Petersson symplectic form generalizes to this case. Namely, the Fenchel-Nielson twist tangent vector along a simple closed geodesic c on the surface should be dual to the geodesic length function at c via the symplectic form.

As another corollary of the theorem, we have,

Corollary 2 *Suppose $F_n (n > 2)$ is the space of all Fuchsian equations in $\bar{C}$ of the form* $y'' = \left[\sum_{k=1}^{n} \frac{a_k}{(z-p_k)^2} + \frac{b_k}{(z-p_k)}\right] y$ *so that*

(i) *no p_k is an apparent nor a logarithmic singularity of the equation,*

(ii) $p_1 = 0$ *and* $p_2 = 1$,

(iii) ∞ *is a regular singular point and is not an apparent nor a logarithmic singularity.*

Let $\pi: F_n \to \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ be the projective monodromy map where $\Gamma = \pi_1(C - \{p_1, \ldots, p_n\})$. Then the derivative $D\pi$ of π is injective from $T_\phi F_n$ to the Zariski tangent space of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$. In particular, if $\pi(\phi)$ is a smooth point of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$, the monodromy map is a local diffeomorphism at ϕ.

The theorem above is a generalization of the analogous result on the monodromy map associated to the holomorphic forms on closed surfaces. See Earle [Ea], Gunning [Gu], Hejhal [He], Goldman [Go2], and Hubbard [Hu] for references. The recent work of K. Iwasaki on Fuchsian equations [Iw2] is also related to the present work.

The organization of the paper is as follows. In Sect. 2, we recall the basic facts concerning projective structures and Earle's variational formula for monodromy map. In Sect. 3, we prove the theorem by estimating the solutions of the Fuchsian equations in the cusp regions of the punctured surfaces.

2 Projective structures and Earle's variational formula

The materials in this section can be found in Earle's paper [Ea]. We present them here for the sake of completeness.

2.1 Let S be a Riemann surface of genus g with n punctures (a surface of type (g, n)). Let Γ be a Fuchsian group acting on the open unit disc D such that $S = D/\Gamma$. Let $Q_2(\Gamma)$ (or $Q_2(S)$) and $B_2(\Gamma)$ (or $B_2(S)$) be the space of quasi-bounded and bounded holomorphic quadratic forms of Γ (or of S) respectively. As usual, a holomorphic quadratic form on S is represented by a holomorphic function $\phi: D \to C$ so that $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$ for all $\gamma \in \Gamma$ and $z \in D$. Indeed, the relation says that $\phi(z)dz^2$ is the pull back of a holomorphic quadratic form on S.

Given $\phi \in Q_2(\Gamma)$, let f be a meromorphic locally homeomorphic function from D to $\bar{C}$ so that the Schwarzian derivative $\{f, z\} = \phi(z)$. There is a homomorphism $\rho: \Gamma \to \mathrm{PSL}(2, C)$ of Möbius transformations such that

$$f(\gamma(z)) = \rho(\gamma) f(z) \quad \text{for all } \gamma \in \Gamma, z \in D . \tag{2.1}$$

We say (f, ρ) is a projective structure associated to ϕ and we call ρ the monodromy homomorphism associated to the projective structure (also to the form ϕ). The meromorphic functions g on D with $\{g, z\} = \phi(z)$ are precisely the functions $g = A \circ f$, $A \in \mathrm{PSL}(2, C)$. The associated monodromy homomorphism is $A \circ \rho(\gamma) \circ A^{-1}$. We say (f, ρ) and $(A \circ f, A \circ \rho \circ A^{-1})$ are equivalent. Thus, each $\phi \in Q_2(\Gamma)$ determines an equivalence class of projective structures. Conversely, given a projective structure (f, ρ) i.e., f is a meromorphic local homeomorphism so that (2.1) holds, $\{f, z\}$ is a holomorphic form on S.

2.2 In order to study the variational formula, we introduce the Teichmüller space $T_{g,n}$ of Riemann surfaces of type (g, n). The surface S is represented by the point $(D, \Gamma) \in T_{g,n}$. The Teichmüller space $T_{g,n}$ is a complex manifold of dimension $3g + n - 3$. A neighborhood of (D, Γ) in $T_{g,n}$ may be described as follows. Let W_0 be a small neighborhoold of 0 in $B_2(\Gamma)$ with respect to the Nehari norm (see [Ga]) so that for each $\phi \in W_0$, $\mu(z) = \lambda^{-2}(z)\overline{\phi(z)}$ has L_∞ norm less than 1 where $\lambda(z)|dz|$ is the Poincaré metric on D. Then $\mu(z)\, d\bar{z}/dz$ is Γ invariant and is called a harmonic Beltrami coefficient of Γ. For each $\phi \in W_0$, let $\mu(z) = \lambda^{-2}(z)\bar{\phi}(z)$ for $z \in D$ and $\mu(z) = 0$ for $z \notin D$. The collection of all such μ is denoted by W_1. Given $\mu \in W_1$, there is a unique quasi-conformal map $w = w^\mu$ of $\bar{C}$ onto itself satisfying the Beltrami equation $w_{\bar{z}} = \mu w_z$ and fixing 0, 1, ∞. Furthermore, $w^\mu(D)$ is a Jordan domain and the group of Möbius transformations $\gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1}$ is a quasi-Fuchsian group $\Gamma(\mu)$ with invariant domain $w^\mu(D)$. A neighborhood system of (D, Γ) in $T_{g,n}$ consists of the marked surfaces $(w^\mu(D), \Gamma(\mu))$ for $\mu \in W_1$. There is a standard pairing between harmonic Beltrami coefficients and bounded forms which is induced by the Petersson pairing between bounded and integrable holomorphic quadratic forms.

2.3 There is a complex vector bundle Q_2 over $T_{g,n}$ whose fiber over a point S is the space of quasi-bounded holomorphic quadratic forms on S. Given $\phi \in Q_2(\Gamma(\mu))$, there is meromorphic local homeomorphism $f: w^\mu(D) \to \bar{C}$ and $\rho: \Gamma \to \mathrm{PSL}(2, C)$ so that $\{f, z\} = \phi(z)$ in $w^\mu(D)$, and

$$f(\gamma^\mu(z)) = \rho(\gamma) f(z) \quad \text{for all } \gamma \in \Gamma \text{ and } z \in w^\mu(D) .$$

It is the dependence of the conjugacy class of ρ on (μ, ϕ) that we wish to study.

2.4 Variational formula of Earle

Consider a projective structure (f_0, ρ_0) corresponding to $\phi_0 \in Q_2(\Gamma)$. ρ_0 induces an action of Γ on the space II of all polynomials of degree at most two,

$$P \cdot \gamma = P \circ \rho_0(\gamma)(\rho_0'(\gamma))^{-1} \quad \text{for } P \in \mathrm{II},\ \gamma \in \Gamma .$$

The tangent space at ρ_0 of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))$ is the space $Z^1(\Gamma, \mathrm{II})$ of cocycles for this action. There is another way to describe the tangent space. Let V be the three dimensional solution space of the equation

$$\sigma''' + 2\phi_0\sigma' + \phi_0'\sigma = 0 \text{ in } D .$$

Then, Γ acts on V by

$$\sigma \cdot \gamma = \sigma \circ \gamma (\gamma')^{-1} .$$

The linear map $P \mapsto P \circ f_0 / f'_0$ is an isomorphism from II to V and conjugates the actions of Γ. Thus, the space $Z^1(\Gamma, V)$ of cocycles is isomorphic to $Z^1(\Gamma, \mathrm{II})$. The Zariski tangent space at $[\rho_0]$ in $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ is isomorphic to the cohomology group $H^1(\Gamma, V)$ (or equivalently $H^1(\Gamma, \mathrm{II})$). See Gardiner and Kra [GK, p. 1041].

To state the variational formula, we suppose that (μ, ϕ) depends on a complex parameter t so that

$$\mu = t\dot{\mu}, \quad \phi^t = \phi_0 + t\dot{\phi} + o(t) \in Q_2(\Gamma(\mu)) .$$

Let $w^t = w^\mu$ be the quasi-conformal homeomorphism satisfying $w_{\bar{z}} = t\dot{\mu} w_z$ and fixing 0, 1, ∞. It is known that $\dot{w}(z) = \frac{d}{dt} w^t(z)|_{t=0}$ is given by

$$\frac{1}{2\pi i} \int\int_C \dot{\mu}(\zeta) \left(\frac{1}{\zeta - z} - \frac{z}{\zeta - 1} + \frac{z-1}{\zeta} \right) d\zeta \wedge d\bar{\zeta} ,$$

$$\dot{w}_{\bar{z}}(z) = \dot{\mu}(z) ,$$

and

$$\dot{w}(z) = O(|z|^2) \text{ as } |z| \to \infty .$$

We have $\dot{\mu}(z) = \lambda^{-2}(z)\overline{\psi(z)}$ for all $z \in D$ and for some $\psi \in B_2(\Gamma)$, and $\dot{\mu}(z) = 0$ for $z \notin D$. $\dot{\phi}$ is a holomorphic function in D and is in general not in $Q_2(\Gamma)$ unless $\dot{\mu} = 0$. Indeed, $(\dot{\mu}, \dot{\phi})$ is a tangent vector of Q_2 at ϕ_0.

Find a solution f of

$$\sigma''' + 2\phi_0 \sigma' + \phi'_0 f = \dot{\phi} \quad \text{in } D .$$

Then $\dot{h}(z) = \dot{f}(z) + \dot{w}(z)$ gives rise to a cocycle $Q_\gamma \in Z^1(\Gamma, V)$ by the formula,

$$Q_\gamma = \dot{h}(\gamma(z))/\gamma'(z) - \dot{h}(z) . \tag{2.2}$$

The variational formula of Earle's states that the derivative of the monodromy map takes the tangent vector $(\dot{\mu}, \dot{\phi})$ in $T_{\phi_0} Q_2$ to the tangent vector in $H^1(\Gamma, V)$ represented by the cocycle (2.2).

Our theorem can be restated as follows,

Theorem. *If $\phi_0 \in Q_2(\Gamma)$ is a quasi-bounded holomorphic quadratic form without apparent singularities, then the derivative of the monodromy map from $T_{\phi_0} Q_2$ to the Zariski tangent space $H^1(\Gamma, V)$ of* Hom(Γ, PSL(2, C)/PSL(2, C) *at $[\phi_0]$ is injective. In particular, if the monodromy representation is a smooth point of the variety, then the monodromy map is a local diffeomorphism.*

3 Proof of the theorem

The second part of the theorem is easy. Since Γ is a free group of rank $2g + n - 1$, $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, C))/\mathrm{PSL}(2, C)$ has complex dimension $6g + 3n - 6$ at its smooth points. The dimension of Q_2 is also $6g + 3n - 6$. Thus the derivative of monodromy map is an isomorphism. Therefore, it is a local diffeomorphism.

To prove the theorem, we need to show that $Q_\gamma = \dot{h} \circ \gamma/\gamma' - \dot{h}$ being a coboundary implies $\dot{\mu} = 0$ and $\dot{\phi} = 0$. By choosing a different solution of $\sigma''' + 2\phi_0\sigma' + \phi_0'\sigma = \dot{\phi}$ in D, we may assume that $Q_\gamma = 0$, i.e.,

$$\dot{h}(\gamma(z))/\gamma'(z) = \dot{h}(z) \quad \text{for all } \gamma \in \Gamma,\ z \in D\ . \tag{3.1}$$

Thus, $\dot{h}$ defines a C^∞ vector field X of type (1, 0) on S. Our major observation is that condition (3.1) implies that the vector field X on S has polynomial growth at the cusps. To be more precise, suppose w is a local coordinate in $\bar{S}$ near p so that $w = 0$ at p, and $X = g(w)\dfrac{\partial}{\partial w}$ in the coordinate. Then, (3.1) implies

$$g(w) = O(|w||\log w|^N), \quad \text{for } |w| \text{ small} \tag{3.2}$$

where N is an integer.

It is clear from (3.2) that for all bounded forms $\phi \in B_2(\Gamma)$, $\int_S \bar{\partial}(X\phi) = 0$ (see Lemma 1 below). On the other hand, $\bar{\partial}X = \dot{\mu}\dfrac{d\bar{z}}{dz}$, thus $\int_S \dot{\mu}\phi = 0$ for all bounded forms $\phi \in B_2(\Gamma)$. This is the inner product of $\dot{\mu}$ with ϕ under the standard pairing between Beltrami differentials and bounded forms. Since $\dot{\mu}$ is a harmonic Beltrami differential, this implies $\dot{\mu} = 0$. Therefore, $\dot{w} \equiv 0$, and $\dot{h} = \dot{f}$ is a holomorphic vector field satisfying (3.2). Thus $\dot{h} = 0$, or $\dot{\phi} = 0$. This completes the proof.

3.1 Cusps and growth condition

Let $w: D \to S$ be the covering map and let $U_c = \{z \mid \operatorname{Im} z > c\}$. To every puncture $p \in \bar{S} - S$, there corresponds a parabolic element $\gamma \in \Gamma$, unique up to conjugation, with fixed point $\zeta \in \partial D (\zeta \neq 1)$, and there is a Möbius transformation A with the following properties:

(i) $A(\infty) = \zeta$, and $A^{-1} \circ \gamma \circ A$ is the translation $z \mapsto z + 2\pi$.

(ii) $A\{z \mid \operatorname{Im} z > c\} \subset D$ for some $c > 0$.

(iii) Two points z_1, z_2 of $A(U_c)$ are equivalent under Γ if and only if $z_2 = \gamma^n(z_1)$ for some integer n, and the image of $A(U_c)$ under the covering map is a deleted neighborhood of p homeomorphic to a punctured disc.

We shall call U_c a cusp half plane belonging to p (under A).

Definition. A vector field X of type (1, 0) on S is said to have polynomial growth at a cusp p if in a local coordinate w in $\bar{S}$ with $w(p) = 0$, $X = f(w)\partial/\partial w$ and

$$f(w) = O(|w||\log w|^N)$$

for some integer N as $|w| \to 0$.

If we take $w = e^{2\pi i z}$ where z is a coordinate for a cusp half plane belonging to p, and pull back X to U_c, say we obtain $g(z)\partial/\partial z$, then the polynomial growth condition for X is equivalent to

$$g(z) = O(|z|^N)$$

as $\operatorname{Im} z \to +\infty$.

Lemma 1 (Bers) *For any C^2 vector field X of type* (1, 0) *on a finite type Riemann surface S so that X has polynomial growth at all cusps, we have*

$$\int_S \bar{\partial}(\phi X) = 0$$

for all cusp forms $\phi \in B_2(S)$.

Proof. (Sketch). By Stokes's theorem, the integral is $\lim_{\varepsilon \to 0} \int_{\partial S_\varepsilon} \phi X$ where S_ε is the surface S with disc of radius ε around each cusp removed. Now, at each circle ∂S_ε of radius ε, $\phi(w) = \left(\frac{a}{w} + \text{holomorphic function} \right) dw^2$, $X = f(w) \frac{\partial}{\partial w}$, thus,

$$\left| \int_{\partial S_\varepsilon} \phi(w) X \right| \leqq 2\pi\varepsilon \cdot \frac{2|a|}{\varepsilon} \cdot \text{const.}\, \varepsilon \cdot |\log \varepsilon|^N$$

$$= \text{const.}\, \varepsilon |\log \varepsilon|^N$$

which tends to zero as ε tends to zero. Therefore, $\lim_{\varepsilon \to 0} \int_{\partial S_\varepsilon} \phi X = 0$. Another way to prove this is to lift the integration to an integration in the universal cover over a fundamental domain Ω. Bounded holomorphic forms $\phi \in B_2(\Gamma)$ have exponential decay in U_c under the pull back map $A \circ w$. By Stokes's theorem, the result follows. For detail, see Kra [Kr2, p. 587].

3.2 Our goal now is to establish the polynomial growth condition for the vector field $\dot{h}(z)\partial/\partial z$ in the cusp half plane U_c for each cusp p under the assumption that ϕ_0 has no apparent singularities. Since all our estimates will be made in $U_c = \{z \mid \operatorname{Im} z > c\}$, we are going to pull back all the functions ϕ_0, $\dot{\mu}$, w^t, ϕ^t, $\dot{\phi}$, $\dot{f}$, $\dot{w}$ to U_c by A. The corresponding functions in U_c are indexed by 1. Thus, $\Gamma_1 = A^{-1} \circ \Gamma \circ A$ denotes a Fuchsian group acting on some half space H so that $H/\Gamma_1 = S$;

$$\phi_1(z) = \phi_0(A(z))A'(z)^2 \in Q_2(\Gamma_1);$$

$$\dot{\mu}_1(z) = \dot{\mu}(A(z))\overline{A'(z)}/A'(z)$$

is a Beltrami differential for Γ_1; $w_1^t = A^{-1} \circ w^t \circ A$ is the quasi-conformal homeomorphism of $\bar{C}$ which fixes $A^{-1}(\infty)$, and satisfies the Beltrami equation

$$w_{\bar{z}} = t\dot{\mu}_1 w_z\,,$$

$$\dot{w}_1 = \frac{d}{dt} w_1^t|_{t=0} = \dot{w}(A(z))/A'(z)\,,$$

$$\partial/\partial \bar{z}\, \dot{w}_1 = \dot{\mu}_1\,,$$

$$\dot{w}_1(z) = 0(|z|^2) \text{ as } |z| \to \infty\,,$$

$$\Gamma_1^t = w_1^t \circ \Gamma_1 \circ (w_1^t)^{-1} = A^{-1} \circ \Gamma(\mu) \circ A\,,$$

$$\phi_1^t = \phi^t(A(z))A'(z)^2 \in Q_2(\Gamma_1^t)\,,$$

and

$$\dot{\phi}_1 = \frac{d}{dt}\phi_1^t|_{t=0} = \dot{\phi}(A(z))A'(z)^2$$

is holomorphic in $A^{-1}(D)$; finally $\dot{f}_1 = \dot{f}(A(z))/A'(z)$.

Lemma 2 (i) *σ satisfies $\sigma'''(z) + 2\phi_0(z)\sigma'(z) + \phi_0'(z)\sigma(z) = 0$ if and only if $y = \sigma(A(z))/A'(z)$ satisfies*

$$y'''(z) + 2\phi_1(z)y'(z) + \phi_1'(z)y = 0\ . \tag{3.3}$$

(ii) *$\dot{f}$ satisfies $\sigma''' + 2\phi_0\sigma' + \phi_0'\sigma = \dot{\phi}$ if and only if $\dot{f}_1 = \dot{f}\circ A/A'$ satisfies*

$$y''' + 2\phi_1 y' + \phi_1 y = \dot{\phi}_1 \tag{3.4}$$

Proof. The proof is a direct computation. Note that $\frac{d^3}{dz^3}[\sigma(A(z))/A'(z)] = \sigma'''(A(z))A'(z)^2$.

3.3 *Estimating the deformation $\dot{\phi}_1(z)$*

Lemma 3 *The deformation $\dot{\phi}_1(z)$ in U_c has an expansion*

$$\dot{\phi}_1(z) = \sum_{n=0}^{\infty}(\widetilde{a_n} + \widetilde{b_n}z + \widetilde{c_n}z^2)e^{inz} \tag{3.5}$$

where $\widetilde{c_0} = 0$.

Proof. It is well known that each $\phi(z)\in Q_2(\Gamma_1)$ has a Fourier expansion

$$\phi(z) = \sum_{n=0}^{\infty} a_n e^{inz} \text{ in } U_c$$

where $a_0 = -\operatorname{Res}_{z=p}\phi(z)$.

Now, $\phi_1^t = \phi_1 + t\dot{\phi}_1 + o(t)\in Q_2(\Gamma_1^t)$. There are two cases.

Case 1 Suppose that for all t, $w_1^t(\infty) = \infty$. Then ∞ is still a fixed point of a parabolic element $\gamma_t(z) = z + \alpha_t$ in Γ_1^t where $\lim_{t\to 0}\alpha_t = 2\pi$. Thus, each $\phi_1^t\in Q_2(\Gamma_1^t)$ has a Fourier expansion

$$\phi_1^t(z) = \sum_{n=0}^{\infty} a_n(t)e^{(2\pi/\alpha_t)inz} \tag{3.6}$$

in some half space V_t where V_t converges to U_c as t tends to zero. Taking derivative of (3.6) with respect to t at $t = 0$, we have

$$\dot{\phi}_1(z) = \sum_{n=0}^{\infty}(\widetilde{a_n} + \widetilde{b_n}z)e^{inz}\ . \tag{3.7}$$

Case 2 In general, let $b_t = w_1^t(\infty)$ ($\lim_{t\to 0} b_t = \infty$) and m_t be the Möbius transformation

$$m_t(z) = b_t z/(b_t - z)\ .$$

m_t sends b_t to ∞ and $\lim_{t\to 0} m_t(z) = z$. Let $\widetilde{w^t_n} = m_t \circ w^t_1$. Then $\widetilde{w^t_n}$ fixes ∞ and satisfies $w_{\bar z} = t\dot\mu_1 w_z$. $\tilde\Gamma^t = m_t \circ \Gamma^t_1 \circ (m_t)^{-1} = \tilde w^t \circ \Gamma_1 \circ (\tilde w^t)^{-1}$ is a quasi-Fuchsian group with ∞ as a parabolic fixed point and $\tilde\phi^t = \phi^t_1(m_t^{-1}(z))(m_t^{-1}(z)')^2 \in Q_2(\tilde\Gamma^t)$. By Case 1, we have

$$\frac{d}{dt}\tilde\phi^t(z)|_{t=0} = \sum_{n=0}^{\infty}(a_n + b_n z)e^{inz}$$

in U_c. Now, $\phi^t_1(z) = \tilde\phi^t(m_t(z))m_t'(z)^2$. Thus,

$$\dot\phi_1(z) = \frac{d}{dt}\phi^t_1(z)|_{t=0}$$

$$= \frac{d}{dt}\tilde\phi^t(z)|_{t=0} m_z'(z)^2|_{t=0} + (\tilde\phi^t)'(z)|_{t=0}\frac{d}{dt}m_t|_{t=0} m_t'(z)^2|_{t=0}$$

$$+ 2\tilde\phi^t(z)|_{t=0} m_t'|_{t=0}\frac{d}{dt}m_t'(z)|_{t=0}.$$

To figure out $\frac{d}{dt}m_t$ and $\frac{d}{dt}m_t'$, let us recall that $m_t(z) = b_t a/(b_t - z)$ and $b_t = w^t_1(\infty) = A^{-1}(w^t(A(\infty))) = A^{-1}(w^t(\zeta))$. Furthermore, A^{-1} sends ζ to ∞. Thus, $A^{-1}(z) = (cz+d)/(z-\zeta)$, $c\zeta + d \neq 0$. Let $a_t = w^t(\zeta)$. We have $\lim_{t\to 0} a_t = \zeta$ and $\frac{d}{dt}a_t|_{t=0} = \dot w(\zeta)$. Now, $b_t = A^{-1}(a_t) = (ca_t + d)/(a_t - \zeta)$, and $m_t = b_t z/(b_t - z) = (ca_t + d)z/((c-z)a_t + d + z\zeta)$. We calculate,

$$m_t'(z) = \frac{d}{dz}m_t(z) = (ca_t + d)^2/((c-z)a_t + d + z\zeta)^2,$$

$$\frac{d}{dt}m_t(z)|_{t=0} = \frac{z^2}{c\zeta + d}\dot w(\zeta), \quad m_t'(z)|_{t=0} = 1,$$

and

$$\frac{d}{dt}m_t'(z)|_{t=0} = \frac{2z}{c\zeta + d}\dot w(\zeta).$$

Substituting these into the formula for $\dot\phi_1(z)$, and noting that $(\tilde\phi^t)'(z)|_{t=0} = \frac{d}{dz}(\tilde\phi^t(z)|_{t=0})$, we obtain,

$$\dot\phi_1(z) = \sum_{n=0}^{\infty}(a_n + b_n z)e^{inz} + \left(\sum_{n=0}^{\infty} inc_n e^{inz}\right)\cdot\frac{z^2}{c\zeta + d}\cdot\dot w(\zeta)$$

$$+ 2\left(\sum_{n=0}^{\infty} c_n e^{inz}\right)\cdot\frac{2z}{c\zeta + d}\dot w(\zeta)$$

$$= \sum_{n=0}^{\infty}(\widetilde{a_n} + \widetilde{b_n} z + \widetilde{c_n} z^2)e^{inz}, \quad \widetilde{c_0} = 0.$$

3.4 Suppose the residue of ϕ_1 at p is not $-n^2/2$ for $n \in Z$

Lemma 4 *Let V_1 be the three dimensional space of holomorphic solutions to the equation*

(3.8) $$y''' + 2\phi_1 y' + \phi'_1 y = 0 \quad \text{in } U_c .$$

If $y \in V_1$ has $y(z+2\pi) - y(z) = O(|z|^N)$ as $\operatorname{Im} z \to +\infty$ *for some integer N, then*

$$y(z) = 0(|z|^N) \text{ as } \operatorname{Im} z \to +\infty .$$

Proof. Let the Fourier expansion of ϕ_1 in U_c be $\sum_{n=0}^{\infty} a_n e^{inz}$ where $-a_0$ is the residue of ϕ_1 at p. We try to find a solution of (3.8) of the form

$$y(z) = \sum_{n=0}^{\infty} \alpha_n e^{i(n+\rho)z}, \; \alpha_0 \neq 0 .$$

We have,

$$y'(z) = \sum_{n=0}^{\infty} i(n+\rho)\alpha_n e^{i(n+\rho)z}$$

and

$$y'''(z) = \sum_{n=0}^{\infty} -i(n+\rho)^3 \alpha_n e^{i(n+\rho)z} .$$

Substituting these together with the Fourier expansions of ϕ_1 and ϕ'_1 into (3.8), we obtain the recurrence relation,

(3.9) $$(n+\rho)[(n+\rho)^2 - 2a_0]\alpha_n = \sum_{j=0}^{n-1} (n+2\rho+j) a_{n-j}\alpha_j .$$

When $n = 0$, (3.9) reduces to

$$\rho(\rho^2 - 2a_0)\alpha_0 = 0 .$$

Since $\alpha_0 \neq 0$, $\rho = 0$ or $\rho = \pm\sqrt{2a_0}$. Indeed, the above equation is the indicial equation of (3.8) and 0, $\pm\sqrt{2a_0}$ are the exponents. Having determined ρ, we now use (3.9) to determine α_n where α_0 can be any nonzero number. Since $a_0 \neq n^2/2$ for any $n \in Z$, $\pm\sqrt{2a_0}$ are not integers. Since the a_n's are the coefficients of the convergent power series, a Cauchy majorants argument applies to α_n. Thus $\sum_{n=0}^{\infty} \alpha_n e^{inz}$ converges in U_c. There are now two cases.

Case 1 $\pm\sqrt{2a_0}$ are not half integers. Then the differences of the three exponents are never integers. Thus there are three linearly independent convergent series solutions

$$y_1(z) = \sum_{n=0}^{\infty} \alpha_n e^{inz} ,$$

$$y_2(z) = e^{i\sqrt{2a_0}z} \sum_{n=0}^{\infty} \beta_n e^{inz} ,$$

$$y_3(z) = e^{-i\sqrt{2a_0}z} \sum_{n=0}^{\infty} \gamma_n e^{inz} ,$$

where $\alpha_0, \beta_0, \gamma_0$ are not zero.

To prove the assertion, suppose

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 y_1 + e^{i\sqrt{2a_0 z}} g_2(z) + e^{-i\sqrt{2a_0 z}} g_3(z)$$

satisfies $y(z + 2\pi) - y(z) = O(|z|^N)$ as $\operatorname{Im} z \to +\infty$. By the choice of g_2, g_3, $\lim_{\operatorname{Im} z \to +\infty} g_2(z) = c_2\beta_0$ and $\lim_{\operatorname{Im} z \to +\infty} g_3(z) = c_3\gamma_0$. Clearly, $y_1(z) = O(1)$ as $\operatorname{Im} z \to +\infty$. Thus, we may assume $c_1 = 0$. Now,

$$y(z + 2\pi) - y(z) = e^{i\sqrt{2a_0 z}}(e^{2\pi i\sqrt{2a_0}} - 1)g_2(z) + e^{-i\sqrt{2a_0 z}}(e^{-2\pi i\sqrt{2a_0}} - 1)g_3(z) .$$

By the assumption on a_0 ($\neq n^2/2$ for $n \in Z$), $e^{\pm 2\pi i\sqrt{2a_0}} \neq 1$. Choose a ray L in U_c of the form $x = ky$ so that as $\operatorname{Im} z \to +\infty$ along L, one of $e^{\pm i\sqrt{2a_0 z}}$ tends to infinity exponentially in $\operatorname{Im} z$ and the other tends to zero exponentially in $\operatorname{Im} z$. We conclude that one of c_2 or c_3 must be zero. Therefore $y(z) =$ const.$(y(z + 2\pi) - y(z))$. The result follows.

Case 2 $\sqrt{2a_0}$ is a half integer. Then (3.8) has three linearly independent solutions of the form,

$$y_1(z) = \sum_{n=0}^{\infty} \alpha_n e^{inz} ,$$

$$y_2(z) = e^{i\sqrt{2a_0 z}} \sum_{n=0}^{\infty} \beta_n e^{inz} ,$$

$$y_3(z) = e^{-i\sqrt{2a_0 z}} \sum_{n=0}^{\infty} \gamma_n e^{inz} + czy_2(z) ,$$

where α_0, β_0, γ_0 are not zero and c is a constant.

The above argument still works in this case since the last term in $y_3(z)$ is a linear polynomial in z times $y_2(z)$.

3.5 Growth of a solution of the inhomogeneous equation

Lemma 5 *For any $\dot{\phi}_1(z)$ on U_c of the form*

$$\dot{\phi}_1(z) = \sum_{n=0}^{\infty} (\tilde{a}_n + \tilde{b}_n z + \tilde{c}_n z^2)e^{inz}, \ \tilde{c}_0 = 0 ,$$

there is a solution $g(z)$ of the equation

$$y''' + 2\phi_1 y' + \phi'_1 y = \dot{\phi}_1$$

so that $g(z) = \sum_{n=0}^{\infty}(\alpha_n + \beta_n z + \gamma_n z^2)e^{inz}$ in U_c. In particular, $g(z) = 0(|z|^2)$ as $\operatorname{Im} z \to +\infty$.

Remark. This is analogous to the following result for regular singular differential equations. Suppose 0 is a regular singular point of a holomorphic differential equation $L(y) = 0$ of order n where $L(y) = y^{(n)} + a_1(z)y^{(n-1)} + \dots + a_n(z)y$ so that the roots of the indicial equation are nonintegers. Then for any meromorphic function ϕ having a pole at 0 of order $\geqq -n$, there is a (single valued) holomorphic function f in a neighborhood of 0 so that $L(f) = \phi$.

Proof. Again suppose $\phi_1(z) = \sum_{n=0}^{\infty} a_n e^{inz}$ in U_c. Suppose a solution is of the form

$$g(z) = \sum_{n=0}^{\infty} (\alpha_n + \beta_n z + \gamma_n z^2) e^{inz} .$$

One calculates,

$$g'(z) = \sum_{n=0}^{\infty} ((\beta_n + 2\gamma_n z) + in(\alpha_n + \beta_n z + \gamma_n z^2)) e^{inz}$$

$$= \sum_{n=0}^{\infty} (in\alpha_n + \beta_n + (in\beta_n + 2\gamma_n) z + in\gamma_n z^2) e^{inz}$$

$$g'' = \sum_{n=0}^{\infty} (((in\beta_n + 2\gamma_n) + 2in\gamma_n z) + in(in\alpha_n + \beta_n + (in\beta_n + 2\gamma_n) z + in\gamma_n z^2)) e^{inz}$$

$$= \sum_{n=0}^{\infty} ((-n^2\alpha_n + 2in\beta_n + 2\gamma_n) + (-n^2\beta_n + 4in\gamma_n) z + (-n^2\gamma_n) z^2) e^{inz}$$

$$g''' = \sum_{n=0}^{\infty} ((-n^2\beta_n + 4in\gamma_n) - 2n^2\gamma_n z + (-in^3\alpha_n - 2n^2\beta_n + 2in\gamma_n)$$

$$+ (-in^3\beta_n - 4n^2\gamma_n) z - in^3\gamma_n z^2) e^{inz}$$

$$= \sum_{n=0}^{\infty} ((-in^3\alpha_n - 3n^2\beta_n + 6in\gamma_n) + (-in^3\beta_n - 6n^2\gamma_n) z - in^3\gamma_n z^2) e^{inz} .$$

Now, substituting these into the equation, one obtains

$$-in(n^2 - 2a_0)\gamma_n + i \sum_{j=0}^{n-1} (n-j) a_{n-j} \gamma_j = \widetilde{c_n} \tag{3.10}$$

$$-in(n^2 - 2a_0)\beta_n + (4ia_0 - 6n^2)\gamma_n + 2 \sum_{j=0}^{n-1} a_{n-j}(ij\beta_j + 2\gamma_j)$$

$$+ i \sum_{j=0}^{n-1} (n-j) a_{n-j} \beta_j = \widetilde{b_n} \tag{3.11}$$

and

$$-in(n^2 - 2a_0)\alpha_n + (2a_0 - 3n^2)\beta_n + 6i\gamma_n + 2 \sum_{j=0}^{n-1} a_{n-j}(ij\alpha_j + \beta_j)$$

$$+ \sum_{j=0}^{n-1} (n-j) a_{n-j} \alpha_j = \widetilde{a_n} . \tag{3.12}$$

The recurrence relation (3.10) for $n = 0$ is valid since $\widetilde{c_0} = 0$ by the assumption. To determine all γ_n, we use (3.11) for $n = 0$ to obtain $\gamma_0 = \widetilde{b_0}/4ia_0$. Since $a_0 \neq n^2/2$ for all integers n, (3.10) determine γ_n. Furthermore, a Cauchy majorants argument shows that $\sum_{n=0}^{\infty} \gamma_n e^{inz}$ converges for $\operatorname{Im} z > c$. By the choice of γ_0, (3.11) is now valid for $n = 0$. To determine β_n, we use (3.12) for $n = 0$ to obtain $\beta_0 = \widetilde{a_0}/2a_0 - 6i\gamma_0$. Now the recurrence relation (3.11) together with the known values γ_n determines all β_n. Since $\sum_{n=0}^{\infty} a_n e^{inz}$ and $\sum_{n=0}^{\infty} \gamma_n e^{inz}$ are both convergent for $\operatorname{Im} z > c$, a Cauchy majorants argument implies $\sum_{n=0}^{\infty} \beta_n e^{inz}$ converges. Finally, to determine α_n, we choose α_0 to be any complex number since Lemma 4 provides a solution y_1 which may be added to any solution g. (3.12) for $n = 0$ is valid due to the choice of γ_0 and β_0. The rest of α_n's are determined by (3.12). Again $\sum_{n=0}^{\infty} \alpha_n e^{inz}$ converges in U_c by a Cauchy majorants argument. Thus, we obtain a solution $g(z)$ of polynomial growth of the inhomogeneous equation $y''' + 2\phi_1 y' + \phi_1 y = \dot{\phi}_1$.

3.6 We can now conclude the proof of the polynomial growth condition for $\dot{h}_1(z) = \dot{f}_1(z) + \dot{w}_1(z)$ in U_c. Consider the parabolic transformation $\gamma_0(z) = z + 2\pi$. We have $\dot{h}_1(\gamma_0(z)) - \dot{h}_1(z) = 0$ since $\gamma'_0(z) = 1$. Thus, $\dot{f}_1(z + 2\pi) - \dot{f}_1(z) = -(\dot{w}_1(z + 2\pi) - \dot{w}_1(z))$. It is well known that the potential $\dot{w}_1(z) = \dot{w}(Az)/A'(z)$ of a Beltrami differential has growth $O(|z|^2)$ as $|z| \to \infty$. Thus $\dot{f}_1(z + 2\pi) - \dot{f}_1(z) = O(|z|^2)$ as $\operatorname{Im} z \to +\infty$. Consider the solution g in Lemma 5 of polynomial growth. Since g and $\dot{f}_1$ both satisfy the same inhomogeneous equation (3.4), $y = \dot{f}_1 - g$ satisfies the homogeneous equation (3.8). Furthermore, $y(z + 2\pi) - y(z)$ has polynomial growth in U_c. By Lemma 4, $y(z) = \dot{f}_1(z) - g(z) = O(|z|^N)$ as $\operatorname{Im} z \to +\infty$. Hence, $\dot{f}_1(z) = O(|z|^N)$ as $\operatorname{Im} z \to +\infty$. This implies $\dot{h}_1(z) = O(|z|^N)$ as $\operatorname{Im} z \to +\infty$. Thus, the proof is completed.

3.7 The proof of the corollary 2

For any Fuchsian equation $y''(w) = -\frac{1}{2}\phi(w)y(w)$ on $\bar{C}$ with more than two singularities, we may lift the equation to the universal cover $D \to S = \bar{C} - \{\text{poles of } \phi(w)\}$. Let $w: D \to S$ be the universal covering map, and let $g(z) = y(w(z))$ be the pull back function. Then we have,

$$g''(z) = -\tfrac{1}{2}\phi(w(z))w'(z)^2 g(z) + \frac{w''}{w'} g'(z) \quad \text{in } D\,. \tag{3.13}$$

Since $w'(z) \neq 0$ in D, we may find a branch of $(w')^{1/2}$ in D. Now the function $f(z) = g(z)(w')^{-1/2}$ satisfies the equation

$$f''(z) = -\tfrac{1}{2}(\phi(w(z))w'(z)^2 - \{w, z\})f\,. \tag{3.14}$$

This shows that the (projective) monodromy group of the Fuchsian equation $y''(w) = -\frac{1}{2}\phi(w)y(w)$ is the same as the monodromy group of the quasi-bounded holomorphic quadratic form $(\phi(w(z))w'(z)^2 - \{w, z\})dz^2 \in Q_2(S)$. Therefore, the result follows from the theorem because the map sending a Fuchsian equation $y''(w) = -\frac{1}{2}\phi(w)y(w)$ in F_n to the quasi-bounded holomorphic quadratic form $\phi(w(z))w'(z)^2 - \{w, z\}$ is a diffeomorphism.

3.8 Questions

There are several questions arise from above considerations.

Given a projective structure (f_0, ρ_0) corresponding to a quasi-bounded holomorphic form $\phi_0 \in Q_2(\Gamma)$, an (holomorphic) Eichler integral f associated to a quasi-bounded form $\dot{\phi}$ is a solution of the equation $\sigma''' + 2\phi_0\sigma' + \phi'_0\sigma = \dot{\phi}$ in D.

An Eichler integral $\dot{f}$ induces an Eichler cohomology class $[Q_\gamma] \in H^1(\Gamma, V)$ by (2.2) (with $\dot{w} = 0$) where V is the space of all solutions of $\sigma''' + 2\phi_0 \sigma' + \phi'_0 \sigma = 0$, i.e., $V = \{P \circ f_0 / f'_0,\ P \in \text{II}\}$. If $\phi_0 = 0$, these notions coincide with the classical definition of Eichler integral and Eichler cohomology class. A cohomology class $[P_\gamma \circ f_0 / f'_0] \in H^1(\Gamma, V)$ is called real if P_γ is a real coefficient quadratic polynomial for each $\gamma \in \Gamma$; and P_γ is called spherical if P_γ satisfies $\overline{P_\gamma\left(\frac{1}{\bar{z}}\right)} = - P_\gamma(z) \frac{1}{z^2}$ for all γ; i.e., $P_\gamma(z) = a_\gamma z^2 + 2b_\gamma z - \bar{a}_\gamma$ where b_γ is a real number. It is well known that the isomorphism from the Lie algebra $\mathscr{G}$ of PSL(2, C) to II given by $u \in \mathscr{G} \mapsto \lim_{t \to 0}(e^{tu}(z) - z)/t = p(z) \in \text{II}$ conjugates the adjoint action of PSL(2, C) on $\mathscr{G}$ and the action of PSL(2, C) on II by $P \cdot \gamma = P \circ \gamma / \gamma'$. Under this isomorphism, real polynomials corresponds to the Lie algebra of PSL(2, R) and spherical polynomials corresponds to the Lie algebra of $SO(3)$ in PSL(2, C).

Question 1 Suppose $\phi_0 \in Q_2(\Gamma)$ has no apparent singularities and corresponds to a hyperbolic cone structure (f_0, ρ_0), i.e., $f_0 : D \to H = \{z \mid \operatorname{Im} z > 0\}$ and $\rho_0 : \Gamma \to \text{PSL}(2, R) \subset \text{PSL}(2, C)$. Is the Eichler cohomology class $[P_\gamma \circ f_0 / f'_0] \in H^1(\Gamma, V)$ corresponding to a bounded form $\dot{\phi} \in B_2(\Gamma)$ ever real?

Question 2 Suppose $\phi_0 \in Q_2(\Gamma)$ has no apparent singularities and corresponds to a spherical cone structure (f_0, ρ_0), i.e., $f_0 : D \to \bar{C}$ and $\rho_0 : \Gamma \to SO(3) \subset \text{PSL}(2, C)$. Is the Eichler cohomology class $[P_\gamma \circ f_0 / f'_0] \in H^1(\Gamma, V)$ corresponding to a bounded form $\dot{\phi} \in B_2(\Gamma)$ ever spherical?

Question 3 (suggested by I. Kra) Generalize the result to quasi-bounded holomorphic forms with apparent singularities. In this case, the deformation space $Q(g; n, m)$ consists of quasi-bounded forms over all Riemann surfaces of type (g, n) so that the first m cusps are exactly the set of apparent singularities. The monodromy map takes $Q(g; n, m)$ to the subvariety consisting of representations in $\text{Hom}(\Gamma, \text{PSL}(2, C))/\text{PSL}(2, C)$ which maps the loops surrounding the first m cusps to parabolic elements. Is the monodromy map locally injective?

Finally, there is also a Riemann-Hilbert type problem.

Question 4 Is the monodromy map $\pi : Q_2 \to \text{Hom}(\Gamma, \text{PSL}(2, C))/\text{PSL}(2, C)$ an onto map where Q_2 is the complex vector bundle over Teichmüller space $T_{g,n}$ whose fibers are quasi-bounded holomorphic forms and Γ is the fundamental group of a surface of type (g, n)?

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